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ABSTRACT

Location Choices under Quality Uncertainty

We examine a linear city duopoly where firms choose their locations to maximize expected profits, uncertain about how consumers will assess the relative quality of their products. Equilibrium locations depend on the ratio of the expected quality superiority to the strength of horizontal differentiation. When it is small, firms locate at opposite endpoints. As it becomes larger, agglomeration around the centre also emerges as an equilibrium and, eventually, agglomeration becomes the only equilibrium.

JEL Classification: L13 and L15
Keywords: linear city, location, product differentiation and quality uncertainty

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1 Introduction

Horizontal differentiation in the context of geographical locations or in that of product characteristics has been studied in an extensive and important literature (see e.g. Gabszewicz and Thisse, 1992 for a review). Hotelling's (1929) classic model has suggested that firms' competition for the in-between consumers implies minimal differentiation. Subsequent work has stressed an opposite incentive, with firms locating away from their rivals, to relax price competition (see e.g. d'Aspremont *et al.,* 1979). While various aspects of the interaction of these two effects have been studied, including multi-dimensional contexts[1], the effect of uncertainty about a second characteristic on location decisions is not as well understood. In this paper, we show that locations depend critically on quality uncertainty. For instance, a restaurant's incentive to locate close to its rivals may depend critically on how likely it is that consumers will significantly favor one restaurant to another, for reasons other than their locations.

When quality uncertainty is relatively low, the equilibria in our model are as in the d'Aspremont *et al.* (1979) model of pure horizontal differentiation. However, as relative quality uncertainty increases, agglomeration emerges as an equilibrium outcome, with the set of equilibrium locations becoming larger for higher uncertainty. By locating at the same location as its rival, a firm risks obtaining zero profit if its quality proves inferior, but takes full advantage of its superior quality when this event occurs. Importantly, our model offers an economic rationale and characterization for agglomeration also at points other than the city center.[2]

The remainder of the paper is as follows. Section 2 sets up the model. Section 3 presents the equilibrium. Section 4 modifies the game so that firms choose locations sequentially. Section 5 concludes. The calculations underlying the locations equilibria are standard but relatively

[1] See e.g. Economides (1989), Neven and Thisse (1990), Irmen and Thisse (1998) and Ansari *et al.* (1998).

[2] The effect of uncertainty on location choices has been studied in Balvers and Szerb (1996), but in a model that suppresses price competition and, thus, obtains very different results (risk aversion drives firms away from agglomeration). *Heterogeneity* in consumers' preferences drives firms to central locations in de Palma *et al.* (1985), however, the strategic effects in their model are very different as there is no aggregate demand uncertainty.

tedious (since several cases have to be distinguished); they are contained in the Appendix.

2 The model

Consider two firms, A and B, each producing a single product in a (new) market. Firms locate their products on the $[0,1]$ line; consumers are uniformly distributed on the line segment with transportation cost quadratic in distance. A consumer located at point x on the line and purchasing a unit of firm i's product obtains surplus equal to

$$u(x,i) = R - t(x - x_i)^2 + q_i - p_i, \tag{1}$$

where x_i denotes the location of firm i, and p_i, and q_i its product's price and quality, respectively. Thus, products are differentiated both horizontally and vertically.

We assume that the basic reservation value, R, is high enough and that each consumer purchases one unit of the product, the one that offers the highest net surplus – in other words, consistent to the main body of the product differentiation literature, the market is "covered".

In order to capture the effect of relative quality uncertainty on location choices, we assume that the quality of product i, q_i, is a random variable, the realization of which firms ignore when they choose their locations. The basic game has the following stages:

1. Firms simultaneously choose the locations of their products.

2. The quality difference, $q_i - q_j$, is revealed and becomes common knowledge.

3. Firms simultaneously choose their product prices.

4. Having observed the firms' locations and the product qualities and prices, each consumer purchases the product that gives him the highest net surplus.

The structure of the model is consistent with the view that location choices have longer run characteristics than pricing. It also assumes that these locations are costly to change after certain aspects of the products (quality) become revealed to the consumers.

For reasons of tractability, we further assume that the random variable $q_i - q_j$ is uniformly distributed on the interval $[-h, h]$, $h > 0$. Implicit in this assumption is our treatment of firms as *a priori* symmetric. Note that the ratio h/t captures the importance of vertical (quality) differentiation relative to horizontal (location) differentiation. Since the quality difference is a random variable, the profits associated with each location pair are also random, at the time the location decisions are made. We assume that unit costs are zero (so that higher quality does not have to cost more per unit). Firms are risk neutral.

3 Equilibrium

We proceed backwards, looking for the subgame perfect Nash equilibrium.

3.1 Price equilibrium

Since each firm, A or B, can choose any location on the line, it is convenient to denote for the price competition part of the analysis the firms as 1 and 2, where firm 1 is to the left of 2 ($x_1 \leq x_2$). When turning to the location choices, we will then allow each firm to choose a location to the left or the right of its rival (that is, to assume the role of firm 1). Now, given the firms' locations their demand functions are as follows. Let z be the demand of firm 1 – then, firm 2 has demand $1 - z$. For $z \in (0, 1)$, demands are determined by the location of the consumer indifferent between the two products. From (1), this indifferent consumer is located at point z, with the property

$$p_1 + t(z - x_1)^2 = -q + p_2 + t(z - x_2)^2, \tag{2}$$

where $q \equiv q_2 - q_1$.

The location of the indifferent consumer depends on firms' locations and prices, as well as on the transportation cost parameter and the quality difference, that is, $z = z(p_1, p_2, x_1, x_2, q, t)$; for ease of notation we suppress the arguments of this function. For $z \in (0, 1)$, by solving (2), we obtain

$$z = \frac{x_1 + x_2}{2} + \frac{p_2 - q - p_1}{2t(x_2 - x_1)}.$$

Taking also into account the possibility that all consumers choose one of the products, the firms' profit functions are

$$\pi_1 = p_1 z \text{ and } \pi_2 = p_2(1 - z), \tag{3}$$

where

$$z = \begin{cases} 0 & if \ \frac{x_1+x_2}{2} + \frac{p_2-q-p_1}{2t(x_2-x_1)} \leq 0 \\ \frac{x_1+x_2}{2} + \frac{p_2-q-p_1}{2t(x_2-x_1)} & if \ \frac{x_1+x_2}{2} + \frac{p_2-q-p_1}{2t(x_2-x_1)} \in (0,1) \\ 1 & if \ \frac{x_1+x_2}{2} + \frac{p_2-q-p_1}{2t(x_2-x_1)} \geq 1. \end{cases} \tag{4}$$

The equilibrium prices, as functions of locations and the realized q, are as follows.[3]

Proposition 1: *For* $0 \leq x_1 \leq x_2 \leq 1$ *the equilibrium prices are*

$$p_1^* = \begin{cases} t\left[(x_2-1)^2 - (x_1-1)^2\right] - q & if \ q < q_- \\ \frac{1}{3}\left[t\left[(x_2+1)^2 - (x_1+1)^2\right] - q\right] & if \ q \in [q_-, q_+] \\ 0 & if \ q > q_+, \end{cases} \tag{5}$$

and

$$p_2^* = \begin{cases} 0 & if \ q < q_- \\ \frac{1}{3}\left[t\left[(x_1-2)^2 - (x_2-2)^2\right] + q\right] & if \ q \in [q_-, q_+] \\ q - t(x_2^2 - x_1^2) & if \ q > q_+, \end{cases} \tag{6}$$

where

$$\begin{aligned} q_- &\equiv t\left[(x_2-2)^2 - (x_1-2)^2\right] \text{ and} \\ q_+ &\equiv t\left[(x_2+1)^2 - (x_1+1)^2\right]. \end{aligned} \tag{7}$$

Proof. See Appendix A1. ■

Proposition 1 implies that there a unique price equilibrium for each pair of locations and each realization of the quality difference. It nests as special cases two extremes. When differentiation is only horizontal ($q = 0$), we recover the d'Aspremont *et al.* (1979) characterization of price equilibria. When differentiation is only vertical ($x_1 = x_2$), the firm selling the higher quality product charges a price equal to the quality difference, q, and serves all consumers.

3.2 Equilibrium locations

We now proceed to the first stage of the game, where firms choose locations. Taking as given equilibrium pricing in the second stage (for any quality realization), each firm's location should maximize its expected profit, given the location of its rival.

[3] This result modifies and extends one in Vettas (1999) to the case of arbitrary quality differences.

We denote the expected profit function of firm i by $E\pi_i(x_1, x_2)$, suppressing in the notation of the arguments the transportation cost parameter, t, and the quality difference parameter, h. The expectation is taken over the stochastic quality q. The expected profit function of firm 1 is

$$E\pi_1(x_1, x_2) = \int_{\min\{-h, q_-\}}^{q_-} \pi_1^m(x_1, x_2) dF + \int_{\max\{-h, q_-\}}^{\min\{q_+, h\}} \pi_1^c(x_1, x_2) dF, \tag{8}$$

where $F(x) = (x + h)/(2h)$ is the cumulative distribution function of the variable q (since it is uniformly distributed in $[-h, h]$). Depending on the quality realization, we can have either an interior solution (where both firms sell their products) or a corner solution (with only the high quality firm making positive sales). $\pi_1^m(x_1, x_2)$ denotes firm 1's profit in the case that only firm 1 sells its product, and $\pi_1^c(x_1, x_2)$ denotes the profit of firm 1 when both firms sell their products. Specifically, by substituting the equilibrium prices from the relevant ranges of (5) and (6) into the profit functions (3), we obtain

$$\begin{aligned} &\pi_1^m(x_1, x_2) = -q + t\left[(x_2 - 1)^2 - (x_1 - 1)^2\right] \text{ and} \\ &\pi_1^c(x_1, x_2) = \tfrac{1}{18(x_2 - x_1)t} \left(q + t\left[(x_1 + 1)^2 - (x_2 + 1)^2\right]\right)^2. \end{aligned} \tag{9}$$

The bounds of the integrals in (8) are set to be $\min\{-h, q_-\}$, $\max\{-h, q_-\}$ and $\min\{q_+, h\}$, to allow for the possibility that some type of price equilibria do not occur for some values of the parameters (x_1, x_2, h, t). If, for example, h is relatively low and the distance between the two firms' locations is relatively large, that is, q_- is low compared to $-h$, then even if q equals $-h$, both firms sell their products in equilibrium regardless of the realized value of q.

Similarly, the expected profit function of firm 2 is

$$E\pi_2(x_1, x_2) = \int_{\max\{-h, q_-\}}^{\min\{q_+, h\}} \pi_2^c(x_1, x_2) dF + \int_{\min\{q_+, h\}}^{h} \pi_2^m(x_1, x_2) dF. \tag{10}$$

Since a firm can choose to locate either at the same location, to the left, or to the right of its rival, the expected profit functions of firms A and B, over the entire range of locations, are

$$E\pi_i(x_i, x_j) = \begin{cases} E\pi_1(x_i, x_j) & if \quad x_i \leq x_j \\ E\pi_2(x_j, x_i) & if \quad x_i \geq x_j, \ i, j \in \{A, B\} \end{cases} \tag{11}$$

where $E\pi_1(\cdot, \cdot)$ and $E\pi_2(\cdot, \cdot)$, are given by (8) and (10).

A key characteristic of the functions $E\pi_1(x_1, x_2)$ and $E\pi_2(x_1, x_2)$ is that they are quasi-convex in x_1 and x_2, respectively, a property implying that they are maximized at the *extrema* of their domains.[4,5] Since $E\pi_1(x_1, x_2)$ is quasi-convex in x_1, firm 1's best response is to locate either at 0 or x_2. Similarly, $E\pi_2(x_1, x_2)$ achieves its maximum value either at point x_1 or 1. It follows that given the location of firm B, firm A's best response belongs to $\{0, x_B, 1\}$, that is, either it chooses the same location as its rival or one of the endpoints.

Specifically, the best response correspondence of firm i given the location of firm j, $i, j = A, B$ is as follows:

- For $h/t < r_1$

$$R_i(x_j) = \begin{cases} 1 & if \quad x_j \leq 1/2 \\ 0 & if \quad x_j \geq 1/2. \end{cases} \tag{12}$$

- For $h/t \in [r_1, r_2]$

$$R_i(x_j) = \begin{cases} 1 & if \quad x_j \leq 1 - x^* \\ x_j & if \quad x_j \in [1 - x^*, x^*] \\ 0 & if \quad x_j \geq x^*, \end{cases} \tag{13}$$

where

$$x^* \equiv \arg_{x \in [\frac{1}{2}, 1]} \left\{ \frac{h^2 + 3x^2(2+x)^2t^2}{54xt} = \frac{h}{4} \right\}. \tag{14}$$

- For $h/t > r_2$

$$R_i(x_j) = \{x_j\} \quad for\ all \quad x_j \in [0, 1]. \tag{15}$$

The values r_1 and r_2 are the solutions with respect to x, to $\frac{h^2+3x^2(2+x)^2t^2}{54xt} = \frac{h}{4}$ for $x = 0.5$ and for $x = 1$, respectively ($r_1 \approx 0.786$ and $r_2 \approx 2.442$). The explanation for the equality yielding the critical thresholds is as follows. The expected profit of a firm if it is located at the same point as its rival, is $E\pi_1(x, x) = \frac{h}{4}$, whereas $E\pi_1(0, x)$ can be easily shown to be equal to $\frac{h^2+3x^2(2+x)^2t^2}{54xt}$, and increasing in x. Also, the difference $E\pi_1(x, x) - E\pi_1(0, x)$ is increasing in h/t, that is, the greater the maximum quality difference, the greater the incentive for a firm to locate

[4] In principle, the fact that the profit functions are not quasi-concave poses threats for the existence of equilibria. However in the present model there always exists at least one equilibrium.

[5] The complete form of function $E\pi_1(x_1, x_2)$ is derived in Appendix A2 and the details for the proof of quasi-convexity are in Appendix A3.

at the same point as its rival. Thus, given the quasi-convexity of $E\pi_i(x_i, x_j)$ with respect to x_i and the symmetry of the profit function $E\pi_i(x_i, x_j)$ (that is, $E\pi_i(x_i, x_j) = E\pi_j(1-x_j, 1-x_i)$) a firm's best response depends on the distance from its rival. Further, given the location of firm j at a point $x \in [0.5, 1]$, the best response of firm i will be either at point 0, or at point x; location at point 1 gives lower profit than location at point 0 (see Appendix A4). For $h/t < r_1$, $E\pi_1(x, x) < E\pi_1(0, x)$, for every x, and for $h/t > r_2$, $E\pi_1(x, x) > E\pi_1(0, x)$, for every x (see Appendix A5). In the remaining case, when $h/t \in (r_1, r_2)$, if firm j locates at a point x, close enough to an endpoint firm i would choose the opposite endpoint. For example, if firm 2 locates close enough to endpoint 1, so that $x > x^*$, then $E\pi_1(0, x) > E\pi_1(x, x)$ holds and the best response of firm 1 is to locate at point 0. On the other hand, if $x \in (1 - x^*, x^*)$, then $E\pi_1(0, x) < E\pi_1(x, x)$, and the best response of firm i is to locate at the same point x as its rival. Finally, by implicit differentiation of the equality (14) that defines x^*, we observe that x^* is *strictly increasing* in h/t. Thus, a higher value of h/t increases the attractiveness of locations near the center.

The following diagrams illustrate the different cases discussed above. Figure 1, presents the case $h/t \in [r_1, r_2]$. Observe that there is a continuum of location pairs at which the two best response correspondences intersect, that is, there is a continuum of location equilibria.

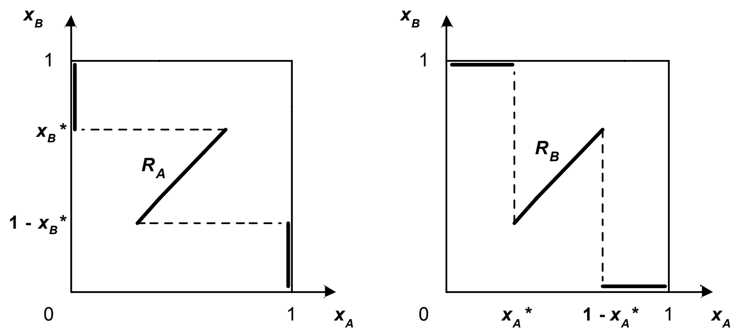

Figure 1: Best response correspondences for $h/t \in [r_1, r_2]$.

As h/t approaches r_1, x_B^* and x_A^* approach 0.5, and the best response correspondence of firm A becomes, in the limit, as in Figure 2a. Firm B's best response correspondence, in this case, is illustrated in Figure 2b. As h/t approaches r_2, x_B^* and $1 - x_A^*$ approach 1, and the best response correspondence of firm A, in the limit, becomes as shown in Figure 2c. For $h/t > r_2$, the graphs of the best response correspondences of the two firms are identical.

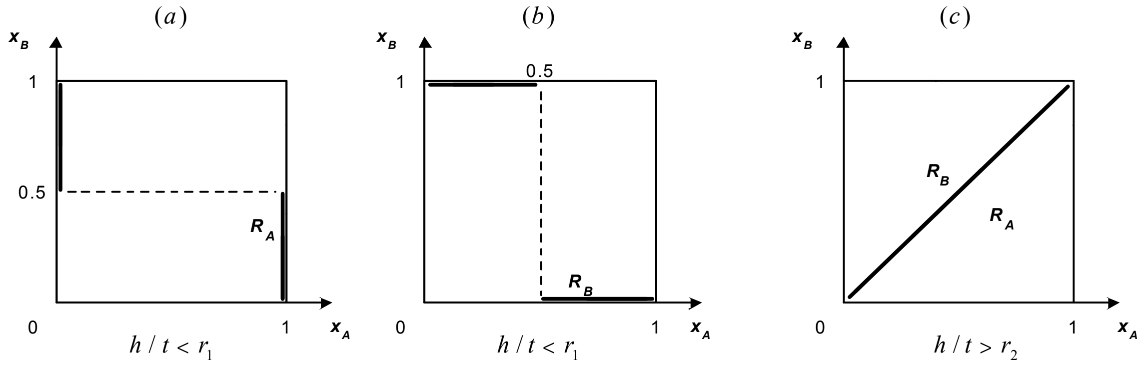

Figure 2: Best response correspondences for $h/t < r_1$ and $h/t > r_2$.

We summarize the previous analysis:

Proposition 2: *The location equilibria are as follows: (i) for $h/t < r_1$ there are only two equilibrium location pairs, the extreme points $(x_A, x_B) = (0, 1)$ and $(x_A, x_B) = (1, 0)$, (ii) for $h/t \in [r_1, r_2]$ there is a continuum of equilibria $(x_A, x_B) = (x, x)$, with $x \in [x^*, 1 - x^*]$ where x^* is given by (14), as well as the two extreme points $(x_A, x_B) = (0, 1)$ and $(x_A, x_B) = (1, 0)$, and, (iii) for $h/t > r_2$ each location pair $(x_A, x_B) = (x, x)$, with $x \in [0, 1]$ is an equilibrium.*

Thus, the firms tend to choose minimally differentiated locations if the expected gains of having a successful (high quality) product are sufficiently high. In other words, it pays to follow an aggressive product choice strategy if the expected value of the (possible) success is relatively high. Note that, when there are agglomeration equilibria, there is a continuum of these, in addition to the extreme locations equilibrium.

In models of horizontal differentiation, there are two forces, one pushing the firms close to each other and one pushing them in the opposite direction. In our formulation, which force dominates depends on the level of quality difference. When the expected quality advantage of a high quality firm is small, the price competition effect dominates and maximum horizontal differentiation is preferred. As the quality advantage one firm could gain over the other increases, a high quality firm tends to prefer locations closer to its rival. This fact is captured by the quasi-convexity of the profit functions: the expected profit $E\pi_1(x_1, x_2)$ is maximized either at point 0 or at point x_2. This, in turn, is due to the fact that the profit $\pi_1^m(x_1, x_2)$, of a firm that has quality higher enough than its rival, is increasing in x_1. Moreover, the incentive of each firm to locate at the same point as its rival depends on the location of the latter. The closer the low quality firm is to the center, the greater is the incentive of the high quality firm to choose the same location. Essentially, the high quality firm prefers to take full advantage of its quality superiority by locating at the same point as its rival, only if it the horizontal differentiation it can achieve from the rival is relatively small. Taking expectations over quality realizations, the ratio of the expected profit under agglomeration to the one when choosing different locations is increasing in h/t. Further, a higher h/t value supports a greater range of locations at which there can exist equilibria with minimum horizontal differentiation.

Note that in the simultaneous location game, expected profits are not always maximized; in particular, agglomeration equilibria are dominated in terms of profits by extreme locations equilibria if the firms could coordinate their decisions they would like to avoid them. Partly motivated by this observation, we now turn to sequential location choices.

Figure 3, below, depicts the set of locations at which there exist agglomeration equilibria. When $h/t < r_1$, there only exist equilibria with maximum horizontal differentiation. When $h/t \in [r_1, r_2]$, apart from the two equilibria with maximum differentiation, there exist equilibria with minimum differentiation also. The extent of locations at which this type of equilibrium exists increases with h/t, and for $h/t > r_2$ only equilibria with minimum differentiation exist.

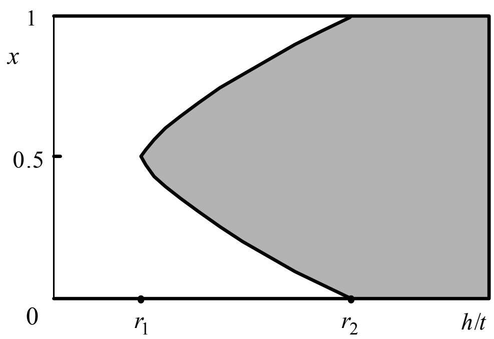

Figure 3: Equilibrium location points.

4 Sequential location choices

From the above analysis, we know that a follower's best response has to be to locate either at an endpoint or at the same point as the first mover. Unlike before, however, the firm that chooses its location first can effectively choose the location that maximizes its expected profit, knowing how its rival will respond (note that this location also maximizes the expected profit of its rival, since the expected profit functions take the same value in equilibrium). We have:

Proposition 3: *When firms choose locations sequentially, the equilibrium locations are: (i) if* $h/t < r_2$, $(x_A, x_B) = (0, 1)$ *and* $(x_A, x_B) = (1, 0)$, *and, (ii) if* $h/t \geq r_2$, $(x_A, x_B) = (x, x)$, *for any* $x \in [0, 1]$.

Proof. If either $h/t \geq r_2$ or $h/t < r_1$ the equilibrium locations are the same as in Proposition 2. Either firm wishes to locate at the same point as its rival (if $h/t \geq r_2$), or as far from each other as possible (if $h/t \leq r_1$). When $h/t \in [r_1, r_2]$, if the two firms end up located at the same point, each would have expected profit equal to $h/4$. If the two firms are located at the opposite endpoints, the expected profit of each firm would be $(t/2 + h^2/54t)$. Straightforward calculations show that $h/4 < (t/2 + h^2/54t) \iff h/t < r_2$, implying that the firms prefer locating at the opposite endpoints. ■

5 Conclusions

We have studied how quality uncertainty affects the location choices of firms. We identify the ratio of the expected quality differentiation to horizontal differentiation as the key parameter driving the results. For low values of this variable, we obtain maximum differentiation. For higher values, agglomeration equilibria also appear, around the center. Interestingly, agglomeration may occur not only at the center but also at a wider set of locations symmetrically defined around the center. This set expands as the possible quality difference increases and, after a certain threshold, it covers the entire linear city.

While we cannot claim full generality of our exact results (it is a common feature of the product differentiation literature that the equilibria may significantly depend on the model formulation), we believe the main conclusions from our analysis are more general than the model: in industries where locations (or horizontal product characteristics) have to be chosen before uncertainty concerning the relative product quality is resolved, we expect firms to tend to locate closer to their rivals relative to industries where differentiation is essentially only horizontal, in which case the incentive to relax price competition tends to keep firms away from each other.[6]

[6] Gerlach *et al.* (2004) also study the effect of quality uncertainty on horizontal differentiation. However, their focus and results are different, primarily because, in their model, firms with *ex ante* different qualities do not coexist in the market.

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Appendix

Appendix A1: Proof of Proposition 1

Standard calculations imply that the best response correspondence of firm 1 is given by

$$R_1(p_2) = \begin{cases} \text{any } a \geq 0 & if \quad p_2 \leq q - t\left(x_2^2 - x_1^2\right) \\ \frac{1}{2}\left[t\left(x_2^2 - x_1^2\right) + p_2 - q\right] & if \quad p_2 \in \left[q - t\left(x_2^2 - x_1^2\right), q - q_-\right] \\ \left[t\left((x_2-1)^2 - (x_1-1)^2\right) + p_2 - q\right] & if \quad p_2 \geq q - q_-, \end{cases}$$

where $\frac{1}{2}\left[t\left(x_2^2 - x_1^2\right) + p_2 - q\right]$ is the price p_1 that maximizes π_1 (solving $d\pi_1/dp_1 = 0$) for $z \in (0,1)$ and $\left[t\left((x_2-1)^2 - (x_1-1)^2\right) + p_2 - q\right]$ is the highest p_1 that implies $z = 1$.

When $p_2 \leq q - t\left(x_2^2 - x_1^2\right))$, for every $p_1 \geq 0$ we have $z = 0$ (by expression (4)). Consequently, every $p_1 \geq 0$ is a best response for firm 1: firm 2's price is so low that doing otherwise would imply a loss.

When $p_2 > q - t\left(x_2^2 - x_1^2\right))$, firm 1's best response implies either $z = 1$ or $z \in (0,1)$. By the previous arguments, when $p_1 < \left[t\left((x_2-1)^2 - (x_1-1)^2\right) + p_2 - q\right]$ firm 1's profit is $\pi_1 = p_1$, hence, if $\frac{1}{2}\left[t\left(x_2^2 - x_1^2\right) + p_2 - q\right] > \left[t\left((x_2-1)^2 - (x_1-1)^2\right) + p_2 - q\right]$, the best response is $\frac{1}{2}\left[t\left(x_2^2 - x_1^2\right) + p_2 - q\right]$ and $\left[t\left((x_2-1)^2 - (x_1-1)^2\right) + p_2 - q\right]$, otherwise. Since $\frac{1}{2}\left[t\left(x_2^2 - x_1^2\right) + p_2 - q\right] > \left[t\left((x_2-1)^2 - (x_1-1)^2\right) + p_2 - q\right]$ if and only if $p_2 < q - q_-$, the result follows immediately.

Similarly, the best-response correspondence of firm 2 is

$$R_2(p_1) = \begin{cases} \text{any } a \geq 0 & if \quad p_1 \leq -q - t\left((x_1-1)^2 - (x_2-1)^2\right)) \\ \frac{1}{2}\left[t\left((x_1-1)^2 - (x_2-1)^2\right) + p_1 + q\right] & if \quad p_1 \in \left[-q - t\left((x_1-1)^2 - (x_2-1)^2\right), -q + q_+\right] \\ \left[t\left(x_1^2 - x_2^2\right) + p_1 + q\right] & if \quad p_1 \geq -q + q_+, \end{cases}$$

where $\frac{1}{2}\left[t\left((x_1-1)^2 - (x_2-1)^2\right) + p_1 + q\right]$ is the price p_2 that maximizes π_2 (solving $d\pi_2/dp_2 = 0$) for $z \in (0,1)$ and $\left[t\left(x_1^2 - x_2^2\right) + p_1 + q\right]$ is the highest p_2 that implies $z = 0$. The logic is analogous to that of the previous case, for firm 1.

It remains to show that there exists a unique equilibrium pair (p_1, p_2), for every vector (x_1, x_2, q, t). First, note that the candidate price equilibria involve either $z \in (0,1)$, $z = 1$, or $z = 0$. Specifically, if $z \in (0,1)$, both firms' best responses satisfy the first-order conditions. If

$z=1$ then $p_2^*=0$ and $\lim\limits_{p_1 \searrow p_1^*} \frac{d\pi_1(p_1,p_2^*)}{dp_1} < 0$. If $z=0$, it must be that $p_1^*=0$ and $\lim\limits_{p_2 \searrow p_2^*} \frac{d\pi_2(p_1^*,p_2)}{dp_2} <$ 0.

Thus, if the equilibrium involves $z \in (0,1)$, the candidate equilibrium prices are

$$\begin{aligned} p_1^* &= \tfrac{1}{3}\left[t\left((x_2+1)^2-(x_1+1)^2\right)-q\right] \text{ and} \\ p_2^* &= \tfrac{1}{3}\left[t\left((x_1-2)^2-(x_2-2)^2\right)+q\right]. \end{aligned} \tag{16}$$

Note that, in order for both prices to be non-negative, it must be that $q \in [q_-, q_+]$.

If now the equilibrium involves $z=1$, the candidate equilibrium prices are

$$\begin{aligned} p_1^* &= t\left[(x_2-1)^2-(x_1-1)^2\right]-q \text{ and} \\ p_2^* &= 0, \end{aligned} \tag{17}$$

along with the condition $\lim\limits_{p_1 \searrow p_1^*} \frac{d\pi_1(p_1,p_2^*)}{dp_1} < 0$, where

$$\begin{aligned} \lim_{p_1 \searrow p_1^*} \frac{d\pi_1(p_1,p_2^*)}{dp_1} &= \tfrac{1}{2(x_2-x_1)}\left(q+t\left[(x_1-2)^2-(x_2-2)^2\right]\right) \\ &= \tfrac{1}{2(x_2-x_1)}(q-q_-) < 0, \end{aligned} \tag{18}$$

that is, $q < q_-$. If the equilibrium involves $z=0$, the candidate equilibrium prices are

$$\begin{aligned} p_1^* &= 0 \text{ and} \\ p_2^* &= t(x_1^2-x_2^2)+q, \end{aligned} \tag{19}$$

along with the condition $\lim\limits_{p_2 \searrow p_2^*} \frac{d\pi_2(p_1^*,p_2)}{dp_2} < 0$, where

$$\begin{aligned} \lim_{p_2 \searrow p_2^*} \frac{d\pi_2(p_1^*,p_2)}{dp_2} &= \tfrac{1}{2(x_2-x_1)}\left(-q+t\left[(x_2+1)^2-(x_1+1)^2\right]\right) \\ &= \tfrac{1}{2(x_2-x_1)}(-q+q_+) < 0, \end{aligned} \tag{20}$$

that is, $q > q_+$. Notice that the candidate equilibrium price pairs (16), (17), and (19) constitute equilibria if and only if $q \in [q_-, q_+]$, $q < q_-$, and $q > q_+$, respectively. We have, thus, proved existence and uniqueness of the price equilibrium for each set of parameters.

Appendix A2: The expected profit functions in the different cases

We derive here in detail the expected profit functions. Since the profit functions of firms 1 and 2 are symmetric in the sense that $E\pi_i(x_i,x_j) = E\pi_j(1-x_j, 1-x_i)$, we will be concerned only with firm 1.

In particular, we will study the various forms that the expected profit function given in expression (8) can take, on the basis of the values of the bounds of $\min\{-h, q_-\}$, $\max\{-h, q_-\}$ and $\min\{q_+, h\}$.

There are three cases depending on whether the ratio h/t belongs in $[0,1]$, $[1,3]$ or $[3,+\infty)$. We present here the first two cases. The last one is analyzed in Appendix A5.

For any given pair of locations, there are two critical quality levels, $q\_$ and q_+, defined in expression (7). When $q \in (q_-, q_+)$ both firms make positive sales, when $q < q_-$ only firm 1 sells, and when $q > q_+$ only firm 2 sells.

We also use the following critical values:

$$\begin{aligned} k_{1+} &= \arg_{x_1}\{q_+ = h\} = -1 + \sqrt{(x_2+1)^2 - h/t}, \\ k_{1-} &= \arg_{x_1}\{q_- = -h\} = 2 - \sqrt{(x_2-2)^2 + h/t}, \end{aligned} \tag{21}$$

and the fact that, whenever the comparison is meaningful (that is, k_{ij} is real), we have

$$\begin{aligned} &x_1 \gtrless k_{1+} \Leftrightarrow q_+ \lessgtr h \quad \text{since} \quad \tfrac{\partial q_+}{\partial x_1} < 0, \\ &x_1 \gtrless k_{1-} \Leftrightarrow q_- \gtrless -h \quad \text{since} \quad \tfrac{\partial q_-}{\partial x_1} > 0. \end{aligned} \tag{22}$$

Thus, the values of $\min\{-h, q_-\}$, $\max\{-h, q_-\}$ and $\min\{q_+, h\}$, separately, depend on the relation between x_1 and k_{1+} and k_{1-} (which are functions of the parameters h/t and x_2). Collectively, the values of all three functions, $\min\{-h, q_-\}$, $\max\{-h, q_-\}$ and $\min\{q_+, h\}$, depend on x_1 and whether $k_{1+} > k_{1-}$ or $k_{1+} \leq -k_{1-}$. We will express the relation between k_{1+} and k_{1-} in terms of x_2. The critical value regarding the relation between k_{1+} and k_{1-} is

$$\tfrac{3+h/t}{6} = \arg_{x_2}\{k_{1+} = k_{1-}\}, \tag{23}$$

where $x_2 \gtrless \frac{3+h/t}{6}$ if and only if $k_{1+} \gtrless k_{1-}$. Note that this critical value is relevant when $h/t \leq 3$ (when $h/t > 3$, then $-h < q_-$ and $q_+ < h$ since the maximum values of q_+ and q_- are $3t$ and $-3t$, respectively).

Two other critical values we use are

$$2 - \sqrt{4 - h/t} = \arg_{x_2}\{k_{1-} = 0\}, \tag{24}$$

and

$$\sqrt{1 + h/t} - 1 = \arg_{x_2}\{k_{1+} = 0\}. \tag{25}$$

Of central importance is the following set of relations between the critical values we have defined so far:

$$\sqrt{h/t} - 1 \leq 2 - \sqrt{4 - h/t} \leq \sqrt{1 + h/t} - 1 \leq \tfrac{3+h/t}{6}, \quad (when\ h/t \in [0,3]). \tag{26}$$

Figure 1, below, illustrates the qualitative form of the relations between the critical values.

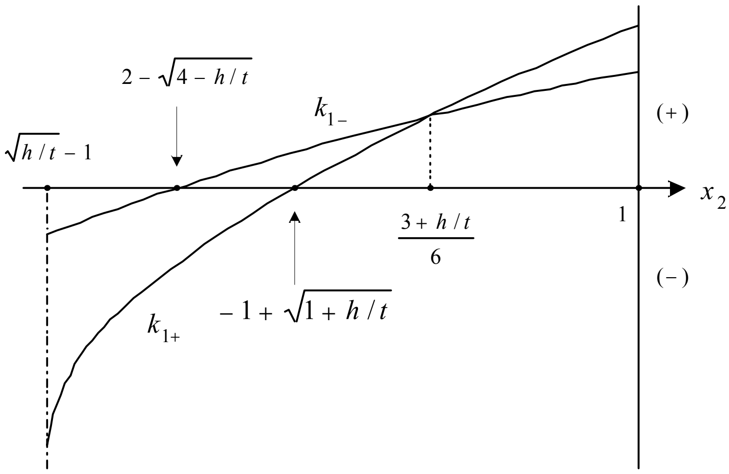

Figure 4.

We first observe that, if k_{1+} is a complex number (that is, if $x_2 < \sqrt{h/t} - 1$), then $q_+ < h$, since, by expression (7), q_+ is increasing in x_2). Note also that for $h/t < 1$, k_{1+} takes only real values.

We distinguish the following cases:

Case 1: $h/t \in [1, 3]$

In this case we have $\sqrt{h/t} - 1 \geq 0$. Thus,

$$\begin{aligned} &\text{If} \quad x_2 \leq \sqrt{h/t} - 1 \text{ then } q_+ \leq h \\ &\text{If} \quad x_2 \in \left[\sqrt{h/t} - 1, \tfrac{3+h/t}{6}\right] \text{ then } k_{1+} \leq k_{1-} \\ &\text{If} \quad x_2 \geq \tfrac{3+h/t}{6} \text{ then } k_{1-} \leq k_{1+}, \end{aligned} \tag{27}$$

Case 2: $h/t \leq 1$

In this case we have $(x_1 \geq)\, 0 \geq \sqrt{h/t} - 1 \leq 0$. Thus,

$$\begin{aligned} &\text{If} \quad x_2 \in \left[\sqrt{h/t} - 1, \tfrac{3+h/t}{6}\right] \text{ then } k_{1+} \leq k_{1-} \\ &\text{If} \quad x_2 \geq \tfrac{3+h/t}{6} \text{ then } k_{1-} \leq k_{1+}, \end{aligned} \tag{28}$$

To proceed one step ahead when $h/t \in [1,3]$ we partition the set of values of x_2 as

$$\left[0, \sqrt{h/t}-1, 2-\sqrt{4-h/t}, \sqrt{h/t+1}-1, \frac{3+h/t}{6}, 1\right], \tag{29}$$

and when $h/t \in [0,1]$ as

$$\left[0, 2-\sqrt{4-h/t}, \sqrt{h/t+1}-1, \frac{3+h/t}{6}, 1\right]. \tag{30}$$

In each partition there is a unique relation between q_+ and h, and $-h$ and q_-.

Before we write down the complete form of $E\pi_1(x_1, x_2)$ as a function of x_1, and in order to simplify the exposition, we make the following definitions that correspond to the different functional forms of $E\pi_1(x_1, x_2)$, on the basis of the whether $q_+ \leq h$ or $q_+ > h$ and whether $-h \leq q_-$ or $-h > q_-$.

When $q_+ \leq h$ and $q_- \geq -h$, we define

$$\begin{aligned} \Lambda_{11} &\equiv \int_{-h}^{q_-} \pi_1^m dF + \int_{q_-}^{q_+} \pi_1^c dF \\ &= \frac{h^2+2ht\left((x_2-1)^2-(x_1-1)^2\right)+(x_2-x_1)^2t^2\left(4(2-x_1-x_2)+(x_2+x_1)^2\right)}{4h}. \end{aligned} \tag{31}$$

When $q_+ \leq h$ and $q_- \leq -h$, we define

$$\Lambda_{12} \equiv \int_{-h}^{q_+} \pi_1^c dF = \frac{(h+q_+)^3}{108ht(x_2-x_1)}. \tag{32}$$

When $q_+ \geq h$ and $q_- \leq -h$ we define

$$\Lambda_{13} \equiv \int_{-h}^{h} \pi_1^c dF = \frac{h^2+3q_+^2}{54ht(x_2-x_1)}. \tag{33}$$

When $q_+ \geq h$ and $q_- \geq -h$ we define

$$\begin{aligned} \Lambda_{14} &\equiv \int_{-h}^{q_-} \pi_1^m dF + \int_{q_-}^{h} \pi_1^c dF \\ &= \frac{1}{108ht(x_2-x_1)}(h^3+3h^2t\,(x_1(x_1-7)-x_2(x_2-7))+3ht^2(x_2-x_1)^2 * \\ &\quad *((x_2+x_1)^2-32+22(x_2+x_1))+(x_2-x_1)^3t^3(13-x_1-x_2)(4-x_1-x_2)^2). \end{aligned} \tag{34}$$

Using these definitions, the expected profit function of firm 1 in each case is the following:

Case 1: $h/t \in [1,3]$

$$\begin{aligned} &For\ x_2 \leq \sqrt{h/t}-1 \\ &\qquad E\pi_1(x_1, x_2) = \Lambda_{11}\ for\ x_1 \in [0, x_2]. \end{aligned} \tag{35}$$

$For\ x_2 \in \left[\sqrt{h/t} - 1, (3+h/t)/6\right]$

$$\bullet\ x_2 \in \left[\sqrt{h/t} - 1, 2 - \sqrt{4 - h/t}\right] \qquad (36)$$

$$E\pi_1(x_1, x_2) = \left\{ \Lambda_{11} \quad for\ x_1 \in [0, x_2], \right.$$

$$\bullet\ x_2 \in \left[2 - \sqrt{4 - h/t}, \sqrt{h/t+1} - 1\right] \qquad (37)$$

$$E\pi_1(x_1, x_2) = \begin{cases} \Lambda_{12} & if \quad x_1 \in [0, k_{1-}] \\ \Lambda_{11} & if \quad x_1 \in [k_{1-}, x_2], \end{cases}$$

$$\bullet\ x_2 \in \left[\sqrt{h/t+1} - 1, (3+h/t)/6\right] \qquad (38)$$

$$E\pi_1(x_1, x_2) = \begin{cases} \Lambda_{13} & if \quad x_1 \in [0, k_{1+}] \\ \Lambda_{12} & if \quad x_1 \in [k_{1+}, k_{1-}] \\ \Lambda_{11} & if \quad x_1 \in [k_{1-}, x_2]. \end{cases}$$

$For\ x_2 \geq \frac{3+h/t}{6}$

$$E\pi_1(x_1, x_2) = \begin{cases} \Lambda_{13} & if \quad x_1 \in [0, k_{1-}] \\ \Lambda_{14} & if \quad x_1 \in [k_{1-}, k_{1+}] \\ \Lambda_{12} & if \quad x_1 \in [k_{1+}, x_2]. \end{cases} \qquad (39)$$

Case 2: $h/t \leq 1$

$For\ x_2 \in \left[0, \frac{3+h/t}{6}\right]$

$$\bullet\ x_2 \in x_2 \in \left[0, 2 - \sqrt{4 - h/t}\right] \qquad (40)$$

$$E\pi_1(x_1, x_2) = \left\{ \Lambda_{11} \quad for\ x_1 \in [0, x_2], \right.$$

$$\bullet\ x_2 \in \left[2 - \sqrt{4 - h/t}, \sqrt{h/t+1} - 1\right] \qquad (41)$$

$$E\pi_1(x_1, x_2) = \begin{cases} \Lambda_{12} & if \quad x_1 \in [0, k_{1-}] \\ \Lambda_{11} & if \quad x_1 \in [k_{1-}, x_2], \end{cases}$$

- $x_2 \in \left[\sqrt{h/t+1}-1, (3+h/t)/6\right]$

$$E\pi_1(x_1,x_2) = \begin{cases} \Lambda_{13} \quad if \quad x_1 \in [0, k_{1+}] \\ \Lambda_{12} \quad if \quad x_1 \in [k_{1+}, k_{1-}] \\ \Lambda_{11} \quad if \quad x_1 \in [k_{1-}, x_2]. \end{cases} \tag{42}$$

$For \; x_2 \geq \frac{3+h/t}{6}$

$$E\pi_1(x_1,x_2) = \begin{cases} \Lambda_{13} \quad if \quad x_1 \in [0, k_{1-}] \\ \Lambda_{14} \quad if \quad x_1 \in [k_{1-}, k_{1+}] \\ \Lambda_{12} \quad if \quad x_1 \in [k_{1+}, x_2]. \end{cases} \tag{43}$$

Appendix A3: The expected profit functions are quasi-convex

In this part of the Appendix we show that $E\pi_1(x_1,x_2)$ is a quasi-convex function of x_1.

Lemma 1: *$E\pi_1(x_1,x_2)$ is quasi-convex in x_1 in the relevant ranges. We distinguish the following cases:*

1. *For $h/t \in [1,3]$*

(1*a*) *$E\pi_1$ is quasi-convex for $x_2 < \sqrt{h/t}-1$.*

(1*b*) *$E\pi_1$ is quasi-convex for $x_2 \in [\sqrt{h/t}-1, 2-\sqrt{4-h/t}]$.*

2. *For $h/t \in [0,1]$*

(2) *$E\pi_1$ is quasi-convex for $x_2 \leq 2-\sqrt{4-h/t}$.*

3. *For $h/t \in [0,3]$*

(3*a*) *$E\pi_1$ is quasi-convex for $x_2 \in [2-\sqrt{4-h/t}, \sqrt{1+h/t}-1]$.*

(3*b*) *$E\pi_1$ is quasi-convex for $x_2 \in [\sqrt{1+h/t}-1, \frac{3+h/t}{6}]$.*

(3*c*) *$E\pi_1$ is quasi-convex for $x_2 \in [\frac{3+h/t}{6}, 1]$.*

Proof:

To prove each case we use the mean value theorem (which we can use because Λ_{ij}, for all i,j, are functions with continuous derivatives for $x_1 \neq x_2$), and Proposition A1 (see last part of the Appendix).

- $(1a), (1b), (2) : x_2 \in [0, 2 - \sqrt{4 - h/t}]$.

For $x_2 \in [0, 2 - \sqrt{4 - h/t}]$ and for $h/t \in [0, 3]$ the profit is given by $E\pi_1 = \Lambda_{11}$. We have that

$$\frac{\partial^3 \Lambda_{11}}{\partial x_1^3} = -\frac{6(1-x_1)t^2}{h} \leq 0 \quad , \quad \left.\frac{\partial \Lambda_{11}}{\partial x_1}\right|_{x_1 = x_2} = t(1 - x_2) > 0 \quad and \quad \left.\frac{\partial \Lambda_{11}}{\partial x_1}\right|_{x_1 = 0} \geq 0. \tag{44}$$

Specifically, regarding the last inequality, we have

$$\left.\frac{\partial \Lambda_{11}}{\partial x_1}\right|_{x_1=0} = \frac{h + (x_2 - 4)x_2 t}{h/t}, \tag{45}$$

which is minimized at the point $x_2 = 2 - \sqrt{4 - h/t}$ and $\left.\frac{\partial \Lambda_{11}}{\partial x_1}\right|_{\substack{x_1=0 \\ x_2 = 2 - \sqrt{4-h/t}}} = 0$.

The above conditions imply that $E\pi_1$ is increasing in x_1 for $x_2 \in [0, 2 - \sqrt{4 - h/t}]$. Specifically, by Proposition A1, if $E\pi_1$ is not quasi-convex, then there is $a \in (0, 2 - \sqrt{4 - h/t})$ such that $\left.\frac{\partial \Lambda_{11}}{\partial x_1}\right|_{x_1 = a} < 0$. Then, by the mean value theorem, there is $m_1 \in (0, a)$ such that $\left.\frac{\partial^2 \Lambda_{11}}{\partial x_1^2}\right|_{x_1 = m_1} < 0$, and there is $m_2 \in (a, 2 - \sqrt{4 - h/t})$ such that $\left.\frac{\partial^2 \Lambda_{11}}{\partial x_1^2}\right|_{x_1 = m_2} > 0$. Applying again the mean value theorem, since $\left.\frac{\partial^2 \Lambda_{11}}{\partial x_1^2}\right|_{x_1 = m_1} < 0$ and $\left.\frac{\partial^2 \Lambda_{11}}{\partial x_1^2}\right|_{x_1 = m_2} > 0$, there is ξ such that $\left.\frac{\partial^3 \Lambda_{11}}{\partial x_1^3}\right|_{x_1 = \xi} > 0$, which contradicts the fact that $\frac{\partial^3 \Lambda_{11}}{\partial x_1^3} \leq 0$. To prove the rest of the cases we apply the same basic argument.

Appendix A4: Best response correspondences

We here derive the best response correspondences of firms A and B, with respect to locations. At a first stage, we consider the best response of firms 1 and 2. Since the expected profit functions are symmetric, we only formally examine that of firm 1.

The fact that the expected profit function $E\pi_1(x_1, x_2)$ is quasi-convex in x_1, implies that it is maximized either at $x_1 = 0$ or at $x_1 = x_2$. We distinguish the following cases:

Case 1: For $x_2 < 2 - \sqrt{4 - h/t}$, $E\pi_1(x_1, x_2)$ is increasing in x_1 and, thus, maximized at the point $x_1 = x_2$.

Case 2: For $x_2 \in [2 - \sqrt{4 - h/t}, \sqrt{1 + h/t} - 1]$, we will show that $E\pi_1(x_1, x_2)$ is maximized at the point $x_1 = x_2$. It suffices to show that $\Lambda_{12}|_{x_1=0} - \Lambda_{11}|_{x_1=x_2} \leq 0$.

We have that $\Lambda_{12}|_{x_1=0} - \Lambda_{11}|_{x_1=x_2} = \frac{(h + x_2(2+x_2)t)^3}{108 x_2 t} - \frac{h}{4}$. We set $\Lambda_{ij}|_{x_i=y} \equiv \Lambda_{ij}^y$. Since the sign of the previous expression is ambiguous we need to rely on a more complicate argument:

$$\frac{\partial^2 \left(\Lambda_{12}^0 - \Lambda_{11}^{x_2}\right)}{\partial x_2^2} = \frac{h^3 + 3x_2^2(4 + 3x_2)ht^2 + 2x_2^3\left(448x_2 + 18x_2^2 + 5x_2^3\right)t^3}{54 x_2^3 h t} > 0. \tag{46}$$

Direct calculations imply

$$\left(\Lambda_{12}^{0}-\Lambda_{11}^{x_2}\right)|_{x_2=2-\sqrt{4-h/t}}=\begin{cases} <0 & for \quad h/t>0 \\ =0 & for \quad h/t=0, \end{cases} \tag{47}$$

and

$$\left(\Lambda_{12}^{0}-\Lambda_{11}^{x_2}\right)|_{x_2=\sqrt{1+h/t}-1}=\begin{cases} <0 & for \quad h/t>0 \\ =0 & for \quad h/t=0. \end{cases} \tag{48}$$

Since the difference is concave in x_2, (46), it attains its maximum value at either $x_2 = 2-\sqrt{4-h/t}$, or $x_2 = \sqrt{1+h/t}-1$. Since both values are non-positive, (47) and (48), the difference is non-positive for all intermediate values.

Case 3: For $x_2 \geq \sqrt{1+h/t}-1$ we will show that $E\pi_1(x_1, x_2)$ is maximized at either point 0 or point x_2, depending on the x_2. We compare $\Lambda_{13}|_{x_1=0}$ and $\Lambda_{11}|_{x_1=x_2}$.

$$\Lambda_{13}^{0}=\tfrac{h^2+3x_2^2(2+x_2)^2t}{54x_2t} \quad and \quad \Lambda_{11}^{x_2}=\tfrac{h}{4}. \tag{49}$$

We show first that the difference is monotonic in x_2, and specifically, that $\partial\left(\Lambda_{13}^{0}-\Lambda_{11}^{x_2}\right)/\partial x_2>0$.

First,

$$\frac{\partial^2\left(\Lambda_{13}^{0}-\Lambda_{11}^{x_2}\right)}{\partial x_2^2}=\frac{h^2}{27x_2^3t}+\frac{4t}{9}+\frac{tx_2}{3}>0. \tag{50}$$

Since

$$\left.\tfrac{\partial\left(\Lambda_{13}^{0}-\Lambda_{11}^{x_2}\right)}{\partial x_2}\right|_{x_2=\sqrt{1+h/t}-1}>0 \quad for \quad h/t\in[0,3], \tag{51}$$

it follows that $\partial\left(\Lambda_{13}^{0}-\Lambda_{11}^{x_2}\right)/\partial x_2>0$ for $x_2 \geq \sqrt{1+h/t}-1$.

Thus,

$$R_1(x_2)=\begin{cases} x_2 & if \quad x_2\in[0,x_2^*] \\ 0 & if \quad x_2\geq x_2^*, \end{cases} \tag{52}$$

where

$$x_2^*=\arg_{x_2}\{\Lambda_{13}=\Lambda_{11}\}=\arg_{x_2}\left\{\frac{h^2+3x_2^2(2+x_2)^2t^2}{54x_2t}=\frac{h}{4}\right\}. \tag{53}$$

The following lemma is central to the derivation of the equilibrium locations.

Lemma 2: $\frac{\partial x_2^*}{\partial(h/t)}>0$.

Proof. Define $g \equiv h/t$. The total differential of the equality $\Lambda_{13}^{0} - \Lambda_{11}^{x_2}$ is given by

$$\frac{\partial\left(\Lambda_{13}^{0} - \Lambda_{11}^{x_2}\right)}{\partial x_2} dx_2^* + \frac{\partial\left(\Lambda_{13}^{0} - \Lambda_{11}^{x_2}\right)}{\partial g} dg = 0. \tag{54}$$

By the previous arguments (see Case 3 above) we have $\partial(\Lambda_{13}^{0} - \Lambda_{11}^{x_2})/\partial x_2 > 0$. We show that $\partial(\Lambda_{13}^{0} - \Lambda_{11}^{x_2})/\partial g < 0$. $\partial(\Lambda_{13}^{0} - \Lambda_{11}^{x_2})/\partial g = (-1/4 + g/27x_2)t$, which attains its maximum value at $x_2 = \sqrt{1+g} - 1$. By direct calculation we can show that $\partial(\Lambda_{13}^{0} - \Lambda_{11}^{x_2})/\partial g|_{x_2=\sqrt{1+g}-1}$ is increasing in g and the limit at $g = 0$ is equal to $7t/4 > 0$. It follows that

$$\frac{dx_2^*}{dg} = -\frac{\partial\left(\Lambda_{13}^{0} - \Lambda_{11}^{x_2}\right)/\partial g}{\partial\left(\Lambda_{13}^{0} - \Lambda_{11}^{x_2}\right)/\partial x_2} > 0. \tag{55}$$

■

Having shown that $\frac{\partial x_2^*}{\partial(h/t)} > 0$, we can find unique values r_1 and r_2 such that $x_2^*(r_1) = 0.5$ and $x_2^*(r_2) = 1$. Specifically, $r_1 \approx 0.786$ and $r_2 \approx 2.442$.

Regarding firm 2, we have

$$R_2(x_1) = \begin{cases} 1 & if \quad x_1 \leq x_1^* \\ x_1 & if \quad x_1 \in [x_1^*, 1], \end{cases} \tag{56}$$

where

$$x_1^* = \arg_{x_1}\left\{\frac{h}{4} = \frac{h^2 + 3((2-x_1)^2 - 1)^2 t^2}{54t(1-x_1)}\right\}, \tag{57}$$

where the right-hand-side of the last expression contains profit functions of firm 2 that are symmetric to those of firm 1, given by (53).

Also, by symmetry of the expected profit functions, $x_1^* = 1 - x_2^*$. The following diagram illustrates the best responses for firms 1 and 2 for two values of the parameter h/t; the greater the value of h/t, the greater the incentive to locate at the same location as one's rival or, equivalently, the greater the measure of locations for which $R_i(x_j) = x_j$.

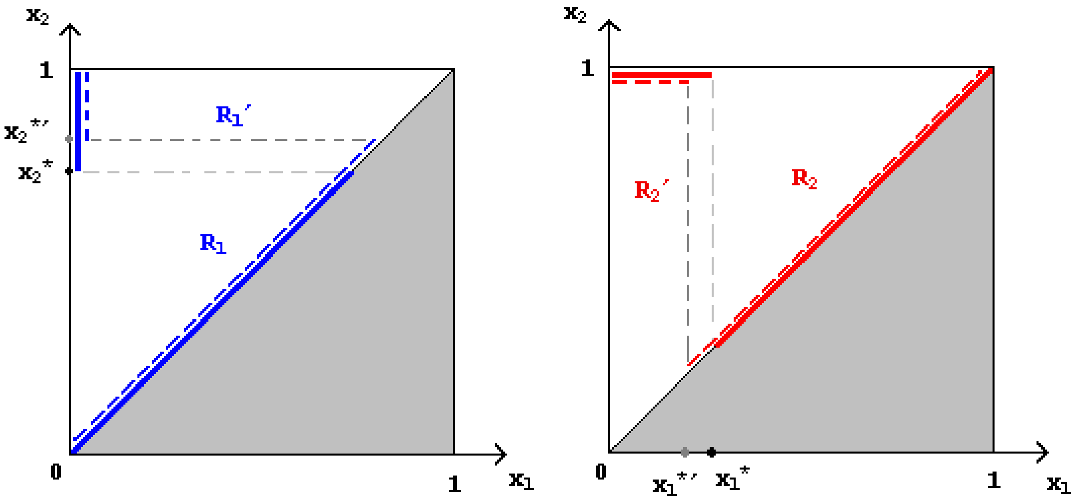

Figure 5. Best response correspondences for firms 1 and 2.

By construction, firms 1 and 2 are constrained to the space $\{(x_1, x_2) : x_1 \leq x_2,\ x_1, x_2 \in [0,1]\}$, that is, we rule out location pairs that lie in the shadowed region of the above figures. In order to derive best responses for firms A and B, we compose the best response correspondences of firms 1 and 2 (see expressions (12) – (15)).

Appendix A5: Analysis of the case where $h/t \geq 3$

We prove that, when $h/t \geq 3$, the function $E\pi_1(x_1, x_2)$ is maximized – with respect to x_1 – at $x_1 = x_2$, for any $x_2 \in [0,1]$. First, we prove that $E\pi_1(x_1, x_2)$ is quasi-convex in x_1.

Regarding the form of $E\pi_1(x_1, x_2)$, note that when $h/t \geq 3$, then $q_+ \leq h$ and $q_- \geq -h$, since $q_+ \leq 3t$, and $q_- \geq -3t$ for all x_1, x_2 (see expression (7)). It follows, by expression (8), that

$$\begin{aligned} E\pi_1(x_1, x_2) &= \int_{-h}^{q_-} \pi_1^m(x_1, x_2)dF + \int_{q_-}^{q_+} \pi_1^c(x_1, x_2)dF \\ &= \frac{1}{4h}\left[h^2 + 2((x_2-1) - (x_1-1)^2)ht + ((x_1-2)^2 + (x_2-2)^2 + 2x_2x_1)(x_2-x_1)^2t^2\right]. \end{aligned}$$

Notice that $E\pi_1(x_1, x_2)$ is a continuous functions of x_1 and has continuous derivatives (since it is a polynomial of x_1). By virtue of Proposition A2 (and the corollary that follows it – see last part of the Appendix), the following lemma ensures that the function $E\pi_1(x_1, x_2)$ is quasi-convex in x_1.

Lemma 3: *For every $x_1 \in (0,1)$, if $h/t \geq 3$, the expected profit function of firm 1 is strictly increasing in x_1 whenever it is concave in x_1.*

Proof. The first and second derivatives of $E\pi_1(x_1, x_2)$ are

$$\frac{\partial E\pi_1(x_1,x_2)}{\partial x_1} = \frac{t^2}{h}\left(\frac{h}{t}(1-x_1) - \phi_1\right),$$

where

$$\phi_1(x_1,x_2) = ((4(x_2 - x_1) - (x_2 + x_1)^2 + x_1(x_2^2 - x_1^2 + 4x_1)),$$

and

$$\frac{\partial^2 E\pi_1(x_1,x_2)}{\partial x_1^2} = \frac{t^2}{h}\left(-\frac{h}{t} + \phi_2(x_1,x_2)\right),$$

where

$$\phi_2(x_1,x_2) = (x_2(2-x_2) + 4 - 3x_1(2-x_1)).$$

Thus, if $h/t \geq \phi_2(x_1,x_2)$, $E\pi_1(x_1,x_2)$ is concave in x_1, and if $h/t > \phi_1(x_1,x_2)/(1-x_1)$, $E\pi_1(x_1,x_2)$ is strictly increasing in x_1 (given that $x_1 < 1$).

We show that $\phi_2(x_1,x_2) > \phi_1(x_1,x_2)$ when $x_1 < 1$. After some manipulations we get

$$\phi_2(x_1,x_2) - \phi_1(x_1,x_2) = 2\frac{2 - 3x_1 + 3x_1^2 - x_1^3 - x_2}{1-x_1}.$$

We set $\phi(x_1,x_2) \equiv 2 - 3x_1 + 3x_1^2 - x_1^3 - x_2$ and we will show that $\phi(x_1,x_2) > 0$ for $x_1 \in [0,1)$. First, observe that $\partial\phi/\partial x_2 < 0$ and that $\phi(1,1) = 0$. It suffices to show that $\partial\phi/\partial x_1 < 0$. Note that, $\partial^2\phi/\partial x_1^2 = 6 - 6x_1 > 0$ for $x_1 \in [0,1)$ and $\partial\phi/\partial x_1|_{x_1=1} = 0$, which implies that $\partial\phi/\partial x_1 < 0$ for $x_1 \in [0,1)$.

Since $\phi_2(x_1,x_2) > \phi_1(x_1,x_2)$, it follows that for all h/t such that $E\pi_1(x_1,x_2)$ is concave, it will also be strictly increasing. ■

We next proceed to show that, for any x_2, the best response of firm 1 is to choose location $x_1 = x_2$. In particular, we will show that the function $E\pi_1(x_1,x_2)$ is strictly increasing in x_1. The fact that $E\pi_1(x_1,x_2)$ is quasi-convex and strictly increasing at the point $x_1 = 0$, implies that it will be increasing over the $[0,x_2]$ interval. Specifically, first note that $\frac{\partial E\pi_1(0,x_2)}{\partial x_1} = \frac{h/t-(4-x_2)x_2}{h/t^2} > 0$ for $h/t \geq 3$ except for the case where $h = 3t$ and $x_2 = 1$. Note also that $\frac{\partial E\pi_1(x_2,x_2)}{\partial x_1} = t(1-x_2) > 0$ for $x_2 < 1$. Thus, except for the case where $h = 3t$ and $x_2 = 1$,

$E\pi_1(x_1, x_2)$ is an increasing function of x_1, and has a strict maximum at $x_1 = x_2$. In the remaining case where $h = 3t$ and $x_2 = 1$, given the properties of the function, it suffices to compare $E\pi_1(0, 1)$ and $E\pi_1(1, 1)$. We have

$$E\pi_1(0, 1) = \frac{2}{3}t < \frac{3}{4}t = E\pi_1(1, 1).$$

We, thus, conclude that, for all $h/t \geq 3$, the function $E\pi_1(x_1, x_2)$ attains its global maximum at $x_1 = x_2$.

Since the problem is symmetric, an analogous analysis applies to the profit of firm 2. Finally, since both firms' best response is to locate at the same point as their rival, each location pair $(x_1, x_2) = (a, a)$, $a \in [0, 1]$, is a location equilibrium.

Appendix A6: Mathematical results on quasi-convexity and monotonicity

Here we collect some general properties of quasi-convex functions that we have used in previous parts of this Appendix. First, we state:

Definition: Let $f : A \to R$, where A is a convex subset of R. We say that f is quasi-convex on A if for all $x, y \in A$ and all $t \in [0, 1]$, $f(tx + (1 - t)y) \leq \max\{f(x), f(y)\}$.

Proposition A1: Consider a differentiable function $f : A \to R$, where A is a convex subset of R. Then, f is not quasi-convex in A, if and only if there exist $a, b \in A$, $a < b$, such that $f'(a) > 0$, $f'(b) < 0$, that is, if and only if f exhibits a strict local interior maximum, at some point between a and b.

Proof.

($\Longrightarrow$) If f is not quasi-convex then, by definition, there exist points $y_1 < y_2 < y_3$ in A, such that $f(y_2) > \max\{f(y_1), f(y_3)\}$. We notice that:

Since $f(y_2) > f(y_1)$, and f is differentiable, by the mean value theorem, there is $a \in (y_1, y_2)$ such that $f'(a) = \frac{f(y_2)-f(y_1)}{y_2-y_1} > 0$.

Similarly, since $f(y_2) > f(y_3)$, there is $b \in (y_2, y_3)$ such that $f'(b) = \frac{f(y_3)-f(y_2)}{y_3-y_2} < 0$.

($\Longleftarrow$) If $f'(a) > 0$, $f'(b) < 0$ then, by continuity of f, there is $\epsilon \in (0, b - a)$ such that $f(a+\epsilon) > f(a)$ and $f(b-\epsilon) > f(b)$, which implies that $\max\{f(a+\epsilon), f(b-\epsilon)\} > \max\{f(a), f(b),$ that is, f is not quasi-convex (since $a + \epsilon, b - \epsilon \in (a, b)$). ■

Next we state the following result.[7]

[7] Thanks to Aris Dimakakos (AUEB) for help with this proof.

Proposition A2: Consider a twice differentiable function $f : A \rightarrow R$, where A is a convex subset of R. If there exist $a, b \in A : a < b$ and $f'(a) > 0$, $f'(b) < 0$, then there exist $x_1, x_2 \in (a, b)$ such that $f'(x_1) > 0$, $f''(x_1) < 0$, and $f'(x_2) < 0$, $f''(x_2) < 0$.

Proof. Since f is twice differentiable f' is continuous, and so the set $\{x : f'(x) = 0, x \in A\}$ is closed, that is, it has a minimum and a maximum point. Denote the minimum (maximum) point of this set by λ_1 (λ_2). Note that $f' > 0$ all $x \in [a, \lambda_1)$, and $f' < 0$ all $x \in (\lambda_1, b]$. Since f is twice differentiable, by the mean value theorem, there exist $x_1 \in (a, \lambda_1)$, and $x_2 \in (\lambda_1, b)$ such that $f''(x_1) = \frac{f'(\lambda_1) - f'(a)}{\lambda_1 - a} < 0$ and $f''(x_2) = \frac{f'(b) - f'(\lambda_2)}{b - \lambda_2} < 0$, respectively. ■

Finally we obtain:

Corollary: Consider a twice differentiable function $f : A \rightarrow R$, where A is a convex subset of R. If f is not quasi-convex in A, then there exist $x_1, x_2 \in (a, b)$ such that $f'(x_1) > 0$, $f''(x_1) < 0$, and $f'(x_2) < 0$, $f''(x_2) < 0$.

The corollary implies that, if f is strictly increasing in x_1 whenever it is concave in x_1, then it is quasi-convex.