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ABSTRACT

Asset Prices with Heterogenous Beliefs*

This Paper studies the dynamic behaviour of security prices in the presence of investors' heterogeneous beliefs. We provide a tractable continuous-time pure-exchange model and highlight the mechanism through which investors' differences of opinion enter into security prices. In the determination of equilibrium, we employ a representative investor with stochastic weights and solve for all economic quantities in closed form, including the perceived market prices of risk and interest rate. The basic analysis is generalized to incorporate multiple sources of risk, disagreement about non-fundamentals, and multiple investors. Other applications involving multiple goods and nominal asset pricing within monetary economies are discussed.

JEL Classification: C60, D50, D90 and G12
Keywords: asset pricing, equilibrium, heterogenous beliefs, market price of risk and survey

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1. Introduction

A pervasive feature of financial markets is the presence of differences of opinion amongst market participants. It is by now well recognized that investors' heterogeneity in beliefs plays an important role in the formation of security prices and their dynamics, and in the generation of trades amongst market participants. Recent empirical evidence strongly supports this recognition. Buraschi and Jiltsov (2002) find that differences of opinion can help explain the dynamics of option trading volume, while Pavlova and Rigobon (2003) provide empirical support for a model of international stock prices and exchange rates with heterogeneity in beliefs. Of course, it should come as no surprise that academicians widely disagree on many aspects of financial markets. In a survey of financial economists' view on the equity risk premium, Welch (2000) reports high disparity, with views ranging from 2% for the pessimists to 13% for the optimists.

There has been a steadily growing literature on models with differences of opinion amongst investors regarding some fundamental or nonfundamental aspects of a financial economy. This includes the earlier single- or multiple-period discrete-time works of Harrison and Kreps (1978), Varian (1985, 1989), Abel (1990), De Long et al. (1990), Harris and Raviv (1993), the continuous-time works of Williams (1977), Wang (1994), and the subsequent continuous-time developments of Detemple and Murphy (1994), Zapatero (1998), and Basak (2000). This article is primarily built upon the latter three works, which provide tractable continuous-time dynamic equilibrium models, cast in a Bayesian learning framework with incomplete, but symmetric information.[1] Detemple and Murphy develop a production economy with logarithmic agents having heterogeneous beliefs about the unobservable growth of the production process, while Zapatero has a pure-exchange economy with logarithmic agents having heterogeneous beliefs about the growth of the aggregate endowment process. Basak develops a similar pure-exchange economy with arbitrary utility function agents having heterogeneous beliefs about nonfundamentals. He also discusses the case of heterogeneous beliefs about the fundamental aggregate endowment process, although this is not his main focus.

This paper is a synthesis of the dynamic asset pricing implications of the presence of heterogeneous beliefs in financial markets. Our continuous-time analysis is a variant of Basak (2000), and adapts a simple, familiar security market economy to best demonstrate the critical channels through which investors' differences of opinion matter. Two investors in the economy commonly observe the aggregate consumption and risky security price processes, but have incomplete (yet symmetric) information on their dynamics. In this continuous-time setting, an investor can deduce the volatility of a process, but must estimate its mean growth rate via its conditional expectation, rationally updating his beliefs in a Bayesian fashion. This feature of the model is consistent with the common argument that mean returns are a lot harder to estimate than volatilities (Merton, 1980). The investors' inferencing heterogeneously lead them to disagree on

[1] For related continuous-time work on incomplete information (but not heterogeneous beliefs), see Detemple (1986, 1991), Dothan and Feldman (1986), Gennotte (1986), Feldman (1989), Karatzas and Xue (1991), David (1997), Yan (1998), Brennan and Xia (1999), Veronesi (1999, 2000), and Yan (2003).

the mean endowment growth rate and mean security return, and consequently on the market price of risk (or Sharpe ratio) in the economy.

The determination of equilibrium is achieved by introducing a representative investor with stochastic weights for the two investors (Cuoco and He, 1994, incomplete markets; Basak and Cuoco, 1998, restricted market participation; Basak, 2000, heterogeneous beliefs; Detemple and Serrat, 2003, liquidity constraints). The current heterogeneous beliefs analysis is unique in that the stochastic weighting (capturing investors' differing individual-specific state prices) is identified in terms of an essentially exogenous quantity, the investors' disagreement about the aggregate growth. The stochastic weighting acts as a proxy for the divergence in beliefs, making the consumption of the more optimistic investor more correlated with the uncertainty in the economy.

Our basic economic setting is shown to admit closed form expressions for all economic quantities, including the security price dynamics. Under differences of opinion, the investors price risk heterogeneously, with risk being transferred from the more pessimistic to the more optimistic investor. This transfer of risk is proportional to the extent of investors' disagreement. The equilibrium interest rate is shown to inherit additional terms, arising from the investors' extra precautionary savings motives against their disagreement and their misperceptions about aggregate growth. In the presence of investors' heterogeneous beliefs, the standard consumption CAPM-type expression obtains, with the risk premium of a security replaced by a risk-tolerance-weighted average of each investor's perceived risk premium.

In generalizations of our baseline economic setting, we demonstrate how to extend the analysis to incorporate multiple sources of risk and disagreement, and an arbitrary number of heterogeneous investors. One important variation involves investors' having heterogeneous beliefs about nonfundamentals and extraneous risk, as opposed to just fundamentals. "Nonfundamentals" are understood to be quantities other than fundamentals which only influence decisions because they are believed to affect the future. In equilibrium, nonfundamental risk is shown to matter and get priced, as a result of investors' heterogeneous beliefs about extraneous process. We also provide other valuable applications and directions for new research with differences of opinion, which are novel to this article. One economic setting, potentially useful for international applications, considers there being multiple goods in the economy and investors having heterogeneous beliefs about the growth prospects of each good supply. Another economic setting incorporates heterogeneous beliefs within a dynamic monetary economy so as to study nominal asset pricing implications.

The use of models with heterogeneous beliefs has proved fruitful in recent research studying market imperfections. In particular, in tractable asset pricing models with logarithmic preferences, investors' diverging beliefs may be used to generate trade. The works in this vein include Detemple and Murphy (1997) on portfolio constraints, Basak and Croitoru (2000) on mispricing, and Hollifield and Gallmeyer (2002) on short-sale restrictions. Finally, we remark that while we consider economic settings with fully rational Bayesian investors, we also demonstrate that an explicit analysis of the inferencing problem is not necessary in the determination and characterization of equilibrium under heterogeneous beliefs. Hence, our main methodology of incorporating heterogeneous beliefs into asset prices is equally applicable and can be adapted to other non-Bayesian inferencing rules or not fully rational beliefs. For recent continuous-time, differences of opinion models under alternative beliefs, see Kyle and Lin (2002) and Scheinkman and Xiong (2002).

The remainder of the article is organized as follows. In Section 2 we outline the economy with heterogeneous beliefs. In Section 3 we present the methodology for the determination of equilibrium and describe the equilibrium behavior of prices and investors. Section 4 discusses the generalizations to nonfundamentals, multiple sources of uncertainty and investors. Other applications and avenues for future research are reported in Section 5. Section 6 concludes and the Appendix provides all proofs.

2. Economy with Heterogeneous Beliefs

We consider a continuous-time, pure-exchange security market economy with a finite horizon $[0, T]$. There is a single consumption good which serves as the numeraire. There are two types of investors who are heterogeneous in beliefs, preferences and endowments. The continuous-time setting is constructed so as to best highlight the mechanism through which investors' differences of opinion may enter into asset pricing. Furthermore, this setting will be shown to admit closed form expressions for all equilibrium quantities in Section 3. Generalizations and applications of the basic setting are discussed in Sections 4 and 5.

2.1. Information structure and investors' perceptions

The uncertainty is represented by a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathcal{P})$ on which is defined a one-dimensional Brownian motion ω. Letting $\{\mathcal{F}_t^\omega\}$ denote the augmented filtration generated by ω, and $\mathcal{H}$ a σ-field independent of $\mathcal{F}_T^\omega$, the complete information filtration $\{\mathcal{F}_t\}$ is the augmentation of the filtration $\mathcal{H} \times \{\mathcal{F}_t^\omega\}$. The role of $\mathcal{H}$ is to allow for heterogeneity in investors' priors. Given our main focus is on characterization, we state only the most relevant technical conditions, and assume all processes and expectations introduced are well-defined without explicitly stating the required regularity conditions.

Two investors ($i = 1, 2$) commonly observe the aggregate endowment, but have incomplete (but symmetric) information on its dynamics. The exogenous aggregate endowment (or consumption supply) process $\varepsilon > 0$ follows

$$d\varepsilon(t) = \varepsilon(t)[\mu_\varepsilon(t)\, dt + \sigma_\varepsilon(t)\, d\omega(t)]. \tag{1}$$

The mean growth μ_ε is assumed to be adapted to $\{\mathcal{F}_t\}$; in particular, $\mu_\varepsilon(0)$ is $\mathcal{H}$-measurable. To keep the investors' filtering tractable, σ_ε is restricted to be $\{\mathcal{F}_t^\varepsilon\}$-adapted, where $\{\mathcal{F}_t^\varepsilon\}$ denotes the filtration generated by ε.

The investors observe ε, having the incomplete information filtration $\mathcal{F}_t^\varepsilon \subset \mathcal{F}_t$, $t \in [0,T]$. They deduce σ_ε from the quadratic variation of ε, but can only draw inferences about μ_ε. Investors have equivalent probability measures $\mathcal{P}^i$, $i = 1, 2$, also equivalent to $\mathcal{P}$, which may disagree on $\mathcal{H}$, so that investors have heterogeneous prior beliefs. Investors update their beliefs about μ_ε in a Bayesian fashion, via $\mu_\varepsilon^i(t) = E^i[\mu_\varepsilon(t)|\mathcal{F}_t^\varepsilon]$, where $E^i[\cdot]$ denotes the expectation relative to $\mathcal{P}^i$. Due to their heterogeneous priors, investors may draw different inferences about μ_ε at all times.[2] We will not provide an explicit analysis of the inferencing problem (beyond the example in Remark 1) as it turns out to not be necessary in the determination and characterization of equilibrium under heterogeneous beliefs (Section 3).

Following standard filtering theory (Liptser and Shiryaev, 2001), the innovation process ω^i induced by investor i's beliefs and filtration is

$$d\omega^i(t) \equiv \frac{1}{\sigma_\varepsilon(t)}\left[\frac{d\varepsilon(t)}{\varepsilon(t)} - \mu_\varepsilon^i(t)\,dt\right] = d\omega(t) + \frac{\mu_\varepsilon(t) - \mu_\varepsilon^i(t)}{\sigma_\varepsilon(t)}\,dt\ , \quad i = 1, 2. \tag{2}$$

The innovation process of each investor is such that given his perceived growth rate, μ_ε^i, the observed aggregate endowment obeys

$$d\varepsilon(t) = \varepsilon(t)[\mu_\varepsilon^i(t)\,dt + \sigma_\varepsilon(t)\,d\omega^i(t)], \quad i = 1, 2. \tag{3}$$

and hence indeed "agrees" with the process he observes. Furthermore, the investors' information and innovation filtrations coincide, $\{\mathcal{F}_t^\varepsilon\} = \{\mathcal{F}_t^{\omega^i}\} \equiv \{\mathcal{F}_t^i\}$. Effectively, each investor is endowed with the probability space $(\Omega, \mathcal{F}^i, \{\mathcal{F}_t^i\}, \mathcal{P}^i)$; by Girsanov's theorem, ω^i is a Brownian motion on that space. Equation (2) implies that the investors' innovations are related by

$$d\omega^2(t) = d\omega^1(t) + \bar{\mu}(t)\,dt\ , \quad \bar{\mu}(t) \equiv \frac{\mu_\varepsilon^1(t) - \mu_\varepsilon^2(t)}{\sigma_\varepsilon(t)}. \tag{4}$$

The process $\bar{\mu}$ parameterizes investors' disagreement on the mean endowment growth rate, normalized by its risk. $\bar{\mu}(t)$ is positive when investor 1 is more "optimistic", and conversely. Since $\bar{\mu}$ follows directly from the exogenous endowment process and investors' priors, with no equilibrium restrictions imposed on it, we may treat it as an exogenous parameter.

Remark 1 (Gaussian filtering example). Consider the case of the aggregate endowment process following a geometric Brownian motion, where μ_ε and σ_ε are constants. Suppose the investors both have normally distributed priors with mean $\mu_\varepsilon^i(0)$ and variance $\nu^i(0)$ as to the value of μ_ε. Then, investors' beliefs over time for the growth rate have the dynamics (Liptser and Shiryaev, 2001)

$$d\mu_\varepsilon^i(t) = \frac{\nu^i(t)}{\sigma_\varepsilon}d\omega^i(t), \quad i = 1, 2. \tag{5}$$

[2] See Morris (1995) for a convincing justification of the heterogeneous priors formulation, arguing it being fully consistent with rationality. We also note that since investors have common, and not asymmetric information, they are aware of each others' different inferences, arising from their different priors. Hence, under our heterogeneous beliefs formulation, the investors "agree to disagree".

where $\nu^i(t) = \nu^i(0)\sigma_\varepsilon^2/(\nu^i(0)t + \sigma_\varepsilon^2)$, implying

$$d\bar{\mu}(t) = -\frac{\nu^2(t)}{\sigma_\varepsilon}\bar{\mu}(t)dt + \frac{\nu^1(t) - \nu^2(t)}{\sigma_\varepsilon}d\omega^1(t). \tag{6}$$

In the special case of investors having the same prior variance, $\nu^1(0) = \nu^2(0) \equiv \nu(0)$, we obtain

$$d\bar{\mu}(t) = -\frac{\nu(t)}{\sigma_\varepsilon}\bar{\mu}(t)dt. \tag{7}$$

Hence, in this case $\bar{\mu}(t)$ is deterministic with explicit solution

$$\overline{\mu}(t) = \overline{\mu}(0)\exp\left\{-\frac{1}{\sigma_\varepsilon}\int_0^t \nu(s)ds\right\} = \overline{\mu}(0)\left(\frac{\sigma_\varepsilon^2}{\nu(0)t + \sigma_\varepsilon^2}\right)^{\sigma_\varepsilon}, \tag{8}$$

with the property that if $\overline{\mu}(0) > 0$, we get $\overline{\mu}(t) > 0$ $\forall t$: if investor 1 is initially more optimistic, he remains to be the more optimistic investor throughout.

2.2. Securities market

Trading may take place continuously in two securities, a riskless bond and a risky security, both in zero net supply. The security price dynamics satisfy

$$\begin{aligned} dB(t) &= B(t)r(t)dt & (9)\\ dS(t) &= S(t)[\mu(t)\,dt + \sigma(t)\,d\omega(t)] & (10)\\ &= S(t)[\mu^i(t)\,dt + \sigma(t)\,d\omega^i(t)], \quad i = 1, 2. & (11) \end{aligned}$$

where (11) represents the risky security dynamics as perceived by investor i. The interest rate r, perceived mean returns μ^i and volatility σ are posited to be $\{\mathcal{F}_t^\varepsilon\}$-adapted, μ to be $\{\mathcal{F}_t\}$-adapted and σ to be nonzero. All price coefficients are to be determined endogenously in equilibrium, except σ, taken as exogenous. Here, σ defines the financial contract S since this security does not pay dividends, as is standard in the literature (Karatzas and Shreve, 1998). Any security whose price is continuously resettled (e.g., a futures contract) is an example of this contract. Defining S by a dividend process does not entail any major changes to the analysis and is discussed in Section 3.4.

Investors observe the risky security price, but do not observe its mean return and so draw their own inferences, μ^i. Equation (2) and price-agreement across investors imply the following "consistency" relationship between the perceived security price mean returns:

$$\mu^1(t) - \mu^2(t) = \sigma(t)\bar{\mu}(t). \tag{12}$$

The posited dynamic completeness under the perceived price processes implies the existence of a unique state price density process for each investor ξ^i, consistent with no-arbitrage, given by

$$d\xi^i(t) = -\xi^i(t)[r(t)dt + \kappa^i(t)d\omega^i(t)]\,, \quad i = 1, 2. \tag{13}$$

where $\kappa^i(t) \equiv \sigma(t)^{-1}(\mu^i(t) - r(t))$ is the perceived the market price of risk (or the Sharpe ratio) process. The quantity $\xi^i(t,\omega)$ is interpreted as the Arrow-Debreu price per unit probability $\mathcal{P}^i$ of a unit of consumption in state $\omega \in \Omega$ at time t (with $\xi^i(0) = 1$), as perceived by investor i. Simple manipulation and consistency condition (12) implies

$$\kappa^1(t) - \kappa^2(t) = \bar{\mu}(t). \tag{14}$$

Hence under our formulation, investors disagree on the market price of risk, with the disagreement given by differences of opinion about the (normalized) mean return on the risky security or aggregate endowment.

Remark 2 (Completeness of markets). In the spirit of the standard asset pricing theory and the Arrow-Debreu tradition, our formulation assumes there exits a sufficient number of securities to allow dynamic completion of the markets relative to investors' observation filtrations. However, due to our introduction of the σ-field $\mathcal{H}$, independent of investors' filtration, an $\mathcal{H}$-measurable random variable is unobserved and hence unverifiable, so the market must be incomplete under the full information in the sense that payoffs contingent on partitions of $\mathcal{H}$ cannot be attained. In short, our formulation rules out any market incompleteness beyond the most elementary form required to sustain heterogeneous beliefs.

2.3. Investors' preferences and optimization

Investor i is endowed with an endowment stream $\varepsilon_i > 0$, where $\varepsilon_1(t)+\varepsilon_2(t) = \varepsilon(t)$, providing him with initial wealth $W_i(0) = E^i[\int_0^T \xi^i(t)\varepsilon_i(t)]/\xi^i(0)$. He then chooses a nonnegative consumption process c_i, and a portfolio process π_i, where π_i denotes the amount invested the risky security. The investor's financial wealth process W_i then follows

$$dW_i(t) = W_i(t)r(t)dt + (\varepsilon_i(t) - c_i(t))dt + \pi_i(t)(\mu^i(t) - r(t))dt + \pi_i(t)\sigma(t)d\omega^i(t), \tag{15}$$

with $W^i(T) \geq 0$. Each investor is assumed to derive time-additive, state-independent utility $u_i(c_i)$ from intertemporal consumption in $[0,T]$. The function $u_i(\cdot)$ is assumed to be three times continuously differentiable, strictly increasing, strictly concave, and to satisfy $\lim_{c\to 0} u_i'(c) = \infty$ and $\lim_{c\to\infty} u_i'(c) = 0$. An investor's dynamic optimization problem is to maximize $E^i\left[\int_0^T u_i(c_i(t))dt\right]$ subject to (15), where the expectation is taken relative to his individual information structure $(\Omega, \mathcal{F}^i, \{\mathcal{F}_t^i\}, \mathcal{P}^i)$.

Using martingale techniques (Cox and Huang, 1989; Karatzas et al., 1987), each investor's dynamic optimization problem can be restated as the following static variational one given that investor's individual-specific state price density, ξ^i:

$$\max_{c_i} E^i\left[\int_0^T u_i(c_i(t))dt\right] \tag{16}$$

$$\text{subject to} \quad E^i\left[\int_0^T \xi^i(t)c_i(t)dt\right] \leq E^i\left[\int_0^T \xi^i(t)\varepsilon_i(t)dt\right] \tag{17}$$

The necessary and sufficient conditions for optimality of the consumption streams to the above are

$$c_i(t) = I_i(y_i\xi^i(t)), \quad i = 1, 2. \tag{18}$$

where I_i is the inverse of u_i' and $y_i > 0$ is such that investor i's static budget constraint holds with equality at the optimum, i.e., y_i satisfies

$$E^i\left[\int_0^T \xi^i(t)I_i(y_i\xi^i(t))dt\right] = E^i\left[\int_0^T \xi^i(t)\varepsilon_i(t)dt\right], \quad i = 1, 2. \tag{19}$$

3. Equilibrium in the Presence of Heterogeneous Beliefs

We define equilibrium in the economy with heterogeneous beliefs as follows.[3]

Definition 1. An *equilibrium* is a price system $(r, \mu^1, \mu^2, \sigma)$ and consumption-portfolio processes (c_i, π_i) such that: (i) investors choose their optimal consumption-portfolio strategies given their perceived price processes in $(\Omega, \mathcal{F}^i, \{\mathcal{F}_t^i\}, \mathcal{P}^i)$; (ii) perceived security price processes are consistent across investors, i.e.,

$$\mu^1(t) - \mu^2(t) = \sigma(t)\bar{\mu}(t); \tag{20}$$

and (iii) good and security markets clear, i.e.,

$$\begin{aligned} c_1(t) + c_2(t) &= \varepsilon(t), && (21) \\ \pi_1(t) + \pi_2(t) &= 0, && (22) \\ W_1(t) + W_2(t) &= 0. && (23) \end{aligned}$$

We note that the equilibrium definition above requires that the price system perceived by the investors to clear the markets at a time and state, does actually clear the markets once that time has arrived and state has been revealed, i.e., investors' expectations are rational and self-fulfilled in equilibrium.

3.1. Determination of equilibrium

For analytical convenience, we introduce a representative investor with utility function

$$U(c; \lambda) \equiv \max_{c_1+c_2=c} u_1(c_1) + \lambda u_2(c_2) \tag{24}$$

[3]The unobserved mean return μ is not incorporated in the definition since it does not determine investors' policies. It may be obtained from the other model parameters via the relationship $\mu(t) = \mu^i(t) + (\mu_\varepsilon(t) - \mu_\varepsilon^i(t))\sigma(t)/\sigma_\varepsilon(t)$, which follows from the formulation in Sections 2.1–2.2.

where $\lambda > 0$ may be stochastic. As in the homogeneous beliefs case, optimality and consumption good clearing imply that the representative investor consumes the aggregate endowment, and that his marginal utility equates to investor 1's state price destiny. This leads to the investors' equilibrium state price densities and consumption allocations given by equations (26) and (29). Moreover, the equilibrium allocation must solve the representative investor's problem in (24), implying

$$\lambda(t) = \frac{u_1'(c_1(t))}{u_2'(c_2(t))} = \frac{y_1\xi^1(t)}{y_2\xi^2(t)}, \tag{25}$$

where the second equality follows from investors' optimality (18). In a homogeneous beliefs economy (Karatzas et al., 1990), investors face the same state price density, so λ is a constant, determined from investors' budget constraints. Substitution then into equations (26) and (29) fully solves for equilibrium. When investors face different state price densities, though, an extra step is required to independently identify ξ^1/ξ^2. In our heterogeneous beliefs economy, investors' individual-specific price parameters differ only due to heterogeneity in beliefs (equation (14)) and accordingly the dynamics of λ are obtained directly (equation (28)) in terms of the disagreement process $\bar{\mu}$, which is essentially exogenous as discussed previously. In this context, the random variable $\lambda(T)$, weighted by y_1/y_2, is the Radon-Nikodym derivative of investor 2's beliefs, $\mathcal{P}^2$, with respect to investor 1's beliefs, i.e., $d\mathcal{P}^2/d\mathcal{P}^1$. We note that unlike in models with market frictions, the allocations in our setting remain efficient, and the stochastic weighting does not reflect the wealth distribution in the economy (except for at the initial date).

Proposition 1 formalizes the above discussion and provides closed form expressions for the equilibrium state price densities and consumption allocations in terms of the exogenously specified primitives (preferences, beliefs, endowments).

Proposition 1. *If equilibrium exists, the investors' equilibrium state price densities are*

$$\xi^1(t) = \frac{U'(\varepsilon(t);\lambda(t))}{U'(\varepsilon(0);\lambda(0))}\ , \quad \xi^2(t) = \frac{\lambda(0)}{\lambda(t)}\frac{U'(\varepsilon(t);\lambda(t))}{U'(\varepsilon(0);\lambda(0))}\ , \tag{26}$$

where $\lambda(0)$ solves either investor's static budget constraint, i.e.,[4]

$$E^1\left[\int_0^T U'(\varepsilon(t);\lambda(t)) I_1\left(U'(\varepsilon(t);\lambda(t))\right) dt\right] = E^1\left[\int_0^T U'(\varepsilon(t);\lambda(t))\varepsilon_1(t)\, dt\right]\ , \tag{27}$$

and the stochastic weighting λ satisfies

$$\frac{d\lambda(t)}{\lambda(t)} = -\bar{\mu}(t)d\omega^1(t). \tag{28}$$

The equilibrium consumption allocations are

$$c_1(t) = I_1\left(U'(\varepsilon(t);\lambda(t))\right)\ , \quad c_2(t) = I_2\left(U'(\varepsilon(t);\lambda(t))/\lambda(t)\right)\ . \tag{29}$$

Conversely, if there exist ξ^i, λ, satisfying (26)–(28), the associated optimal policies of investors satisfy all market clearing conditions.

[4] The two investors' budget constraints are equivalent, and only determine the ratio y^1/y^2. We set $y^1 = U'(\varepsilon(0);\lambda(0))$ without loss of generality so that $\xi^1(0) = \xi^2(0) = 1$.

The representative investor has a stochastic weighting as a result of investors facing different state price densities, originating from the investors heterogeneous beliefs. As seen in equation (28), the role of the stochastic weighting is to make the consumption of the more optimistic investor more positively correlated with the uncertainty in the economy, and its volatility grows with investors' heterogeneity accordingly. The converse of Proposition 1 underscores the fact that in an economy with heterogeneous beliefs, equations (26)–(28) would fully close the model, determining the equilibrium λ and ξ^i (and hence r and κ^i) in terms of exogenous quantities. As stated in the proof of the proposition, equations (26)–(28) would be implied by good market clearing alone; financial markets clearing is guaranteed by good clearing and so yields nothing further.

Remark 3 (Existence of equilibrium). Since the equilibrium has been determined in closed form in terms of exogenous quantities, demonstrating existence would be possible under additional (many standard) technical conditions. Establishing the existence of equilibrium in this context would involve showing: (i) there exists a solution $\lambda(0)$ to (27); (ii) given ξ^i in (26)–(27), there exist optimal policies (c_i, π_i) and clear all markets; (iii) the r, μ^i and σ of the posited price dynamics in (9)–(11) exist, whose associated ξ^i satisfies (26)–(27). The latter existence, novel to current analysis, is easily verified from the explicit formulae for κ^i and r provided in Proposition 2 of Section 3.2.

3.2. Equilibrium behavior of prices and investors

In this section, we discuss further the properties of equilibrium under heterogeneous beliefs, looking at the characterization of the investors' behavior and price dynamics.

We first compare the two investors' equilibrium consumption. In the case of homogeneous preferences $u_1(\cdot) = u_2(\cdot)$, from (29) in Proposition 1, we deduce $c_1(t) > c_2(t)$ if and only if $\lambda(t) < 1$. In the homogeneous beliefs economy, this comparison is driven purely by the constant $\lambda(0)$ and hence investors' initial wealth; if investor 1 is initially more wealthy than investor 2, he consumes more in all future states and times. In the heterogeneous beliefs economy, however, this comparison is state-dependent, being also driven by the stochastic weighting-disagreement process. Even if investor 1 is initially wealthier than investor 2, there will be states and times characterized by a high $\lambda(t)$, when he consumes less than investor 2. From (28), a relatively high $\lambda(t)$ arises when investor 1's prediction of mean growth has tended to be relatively poor in the past or relatively unlucky in his prediction. In the limit $\lim_{\lambda(t)\to\infty} c_1(t) = 0$; as the disagreement weighting process becomes very high, investor 2 dominates the economy, even if he were initially (possibly much) less wealthy than investor 1.

We now turn to security price dynamics. Proposition 2 complements Proposition 1 by characterizing the investors' perceived market prices of risk (and hence consumption volatilities) and interest rate in terms of the investors' disagreement and the individual and representative investors' risk aversions and prudences.

Proposition 2. *The equilibrium perceived market prices of risk and interest rate are given by*

$$\kappa^1(t) = A(t)\varepsilon(t)\sigma_\varepsilon(t) + \frac{A(t)}{A_2(t)}\bar{\mu}(t)\ , \quad \kappa^2(t) = A(t)\varepsilon(t)\sigma_\varepsilon(t) - \frac{A(t)}{A_1(t)}\bar{\mu}(t)\ , \tag{30}$$

and

$$\begin{aligned} r(t) \ = \ & A(t)\left(\frac{A(t)}{A_1(t)}\varepsilon(t)\mu^1_\varepsilon(t) + \frac{A(t)}{A_2(t)}\varepsilon(t)\mu^2_\varepsilon(t)\right) - \frac{1}{2}A(t)P(t)\varepsilon(t)^2\sigma_\varepsilon(t)^2 \\ & + \frac{A(t)^2}{A_1(t)A_2(t)}\bar{\mu}(t)^2 - \frac{1}{2}\frac{A(t)^3}{A_1(t)^2A_2(t)^2}(P_1(t)+P_2(t))\bar{\mu}(t)^2 \\ & + \frac{A(t)^3}{A_1(t)A_2(t)}\left(\frac{P_1(t)}{A_1(t)} - \frac{P_2(t)}{A_2(t)}\right)\varepsilon(t)\sigma_\varepsilon(t)\bar{\mu}(t), \end{aligned} \tag{31}$$

where the absolute risk aversion and prudence coefficients are defined by

$$A_i(t) \equiv -\frac{u_i''(c_i(t))}{u_i'(c_i(t))}\ , \quad A(t) \equiv -\frac{U''(\varepsilon(t);\lambda(t))}{U'(\varepsilon(t);\lambda(t))} = \frac{1}{1/A_1(t)+1/A_2(t)}\ , \tag{32}$$

$$P_i(t) \equiv -\frac{u_i'''(c_i(t))}{u_i''(c_i(t))}\ , \quad P(t) \equiv -\frac{U'''(\varepsilon(t);\lambda(t))}{U''(\varepsilon(t);\lambda(t))} = \left(\frac{A(t)}{A_1(t)}\right)^2 P_1(t) + \left(\frac{A(t)}{A_2(t)}\right)^2 P_2(t)\ , \tag{33}$$

and the equilibrium λ and c_i are as in Proposition 1. The investors' equilibrium consumption volatilities are: $c_i(t)\sigma_{c_i}(t) = \kappa^i(t)/A_i(t)$.

Under homogeneous beliefs ($\bar{\mu} = 0$), investors would price risk equally, with the market price of risk given by the aggregate endowment risk ($\varepsilon\sigma_\varepsilon$) weighted by the representative investor's risk tolerance ($1/A$). Investors accordingly share risk in proportion to their risk tolerances. Under heterogeneous beliefs, however, relative to the homogeneous case, risk is transferred from the more pessimistic investor to the more optimistic. The transfer of risk is proportional to the extent of investors disagreement $\bar{\mu}$, which appears as another factor in the pricing of risk as perceived by each investor (equation (30)). It acts to reduce the price of risk of the more pessimistic investor, while increasing the optimistic investor's. The disagreement process $\bar{\mu}$, then, encourages investors to abide by the market clearing conditions under differences of opinion.

The interest rate, as in the standard homogeneous beliefs economy, is positively related to aggregate consumption growth (weighted by aggregate risk aversion) and negatively related (for decreasing absolute risk aversion (DARA) investors) to aggregate consumption risk. The second term arises since, in the face of risky future consumption, investors have a precautionary savings motive; to clear markets, the interest rate must decrease to counteract this tendency. Under heterogeneous beliefs, the standard behavior still persists (first line of equation (31)), with the risk-tolerance-weighted average perceived aggregate growth replacing the actual unobserved growth. However, heterogeneity in beliefs now yields three extra terms in the interest rate, directly dependent on the extent of investors' disagreement $\bar{\mu}$. The first extra term (the third term in (31)) arises from the discrepancy in investors' perceptions about their mean consumption growths. In this two-investor case, it turns out that the aggregate perceived consumption growth,

$\mu^1_{c_1} + \mu^2_{c_2}(= \mu_\varepsilon + \bar{\mu}^2/(A_1 + A_2))$, is unambiguously higher than the real aggregate consumption growth, μ_ε, so the interest rate must increase to reflect this perception. The fourth term in (31) decreases (for DARA) the interest rate to compensate for investors' extra precautionary savings against the disagreement risk $\bar{\mu}$ introduced into their consumption streams. The sign of the last term, arising due to the covariance of aggregate endowment and disagreement risks, depends on which investor has the higher prudence-to-risk aversion ratio and higher optimism.

From the perceived market prices of risk in Proposition 1, we may derive the risk premium for the risky security as perceived by each investor, say investor 1, as

$$\begin{aligned} \mu^1(t) - r(t) &= A(t)\varepsilon(t)\sigma_\varepsilon(t)\sigma(t) + \frac{A(t)}{A_2(t)}\overline{\mu}(t)\sigma(t) && (34) \\ &= A(t)\mathrm{cov}\left(\frac{dS(t)}{S(t)}, d\varepsilon(t)\right) - \frac{A(t)}{A_2(t)}\mathrm{cov}\left(\frac{dS(t)}{S(t)}, \frac{d\lambda(t)}{\lambda(t)}\right). && (35) \end{aligned}$$

As in the standard consumption-based CAPM (Breeden 1979, Duffie and Zame, 1989), a risky security's risk premium is positively related to the covariance of its return with the change in aggregate consumption flow. Now, additionally, the risk premium is also driven by the covariance of the security's return with the change in the disagreement weighting process. According to investor 1, the risk premium is decreasing in its covariance, since to him, a high $\lambda(t)$ is unfavorable. The well-known equity premium puzzle (Mehra and Prescott, 1985) stems from the CCAPM having just the first term in (34) or (35). Investors' heterogeneity in beliefs leads to an additional term, implying that the standard CCAPM overestimates/underestimates the equity risk premium depending on the investor's relative optimism/pessimism. Consequently, under certain conditions, the heterogeneity in beliefs will better reconcile with the data. Substituting the consistency condition (12) into (35) yields

$$\frac{A(t)}{A_1(t)}\mu^1(t) + \frac{A(t)}{A_2(t)}\mu^2(t) - r(t) = A(t)\mathrm{cov}\left(\frac{dS(t)}{S(t)}, d\varepsilon(t)\right). \tag{36}$$

Hence, we obtain an expression resembling the standard consumption CAPM, with the risk premium of a security replaced by a risk-tolerance-weighted average of each investor's perceived risk premium.

3.3. An example: the case of CRRA preferences

We assume here that investors have constant relative risk aversion (CRRA) preferences, $u_i(c) = \frac{c^{1-\gamma}}{1-\gamma}$, $\gamma > 0$, $\gamma \neq 1$, or $u_i(c) = log(c)$ (which corresponds to $\gamma = 1$), for $i = 1, 2$. For this special case, all economic quantities admit explicit formulae, and some additional unambiguous results arise.

Proposition 3. *Assume investors exhibit identical CRRA preferences. Then, in an equilibrium with heterogeneous beliefs, the state prices and consumption allocations are*

$$\xi^1(t) = \frac{\varepsilon(0)^\gamma}{(1 + \lambda(0)^{1/\gamma})^\gamma}\frac{(1 + \lambda(t)^{1/\gamma})^\gamma}{\varepsilon(t)^\gamma}, \quad \xi^2(t) = \frac{\lambda(0)\varepsilon(0)^\gamma}{(1 + \lambda(0)^{1/\gamma})^\gamma}\frac{(1 + \lambda(t)^{1/\gamma})^\gamma}{\lambda(t)\varepsilon(t)^\gamma}, \tag{37}$$

$$c_1(t) = \frac{\varepsilon(t)}{1+\lambda(t)^{1/\gamma}}, \qquad c_2(t) = \frac{\lambda(t)^{1/\gamma}\varepsilon(t)}{1+\lambda(t)^{1/\gamma}}, \tag{38}$$

where $\lambda(0)$ *and* $\lambda(t)$ *satisfy*

$$E^1\left[\int_0^T \left(1+\lambda(t)^{1/\gamma}\right)^{\gamma-1}\varepsilon(t)^{1-\gamma}dt\right] = E^1\left[\int_0^T \left(1+\lambda(t)^{1/\gamma}\right)^{\gamma}\varepsilon(t)^{-\gamma}\varepsilon_1(t)dt\right], \tag{39}$$

$$d\lambda(t) = -\lambda(t)\overline{\mu}(t)d\omega^1(t). \tag{40}$$

Hence, the perceived market prices of risk and interest rate are given by

$$\kappa^1(t) = \gamma\sigma_\varepsilon(t) + \frac{\lambda(t)^{1/\gamma}}{1+\lambda(t)^{1/\gamma}}\overline{\mu}(t), \qquad \kappa^2(t) = \gamma\sigma_\varepsilon(t) - \frac{1}{1+\lambda(t)^{1/\gamma}}\overline{\mu}(t), \tag{41}$$

$$r(t) = \gamma\left(\frac{1}{1+\lambda(t)^{1/\gamma}}\mu^1_\varepsilon(t) + \frac{\lambda(t)^{1/\gamma}}{1+\lambda(t)^{1/\gamma}}\mu^2_\varepsilon(t)\right) - \frac{1}{2}\gamma(\gamma+1)\sigma_\varepsilon(t)^2 + \frac{\gamma-1}{2\gamma}\frac{\lambda(t)^{1/\gamma}}{(1+\lambda(t)^{1/\gamma})^2}\overline{\mu}(t)^2. \tag{42}$$

As compared to the homogeneous beliefs economy, the equilibrium market price of risk as perceived by the more optimistic investor is increased, while that by the more pessimistic investor decreased. For investors more risk averse than logarithmic ($\gamma > 1$), the interest rate is increased by the presence of heterogeneous beliefs, while for investors less risk averse ($\gamma < 1$), it is decreased. A further point to note is that, in the homogeneous beliefs economy with CRRA investors, if the aggregate endowment process follows a geometric Brownian motion (i.e., $\mu_\varepsilon, \sigma_\varepsilon$ are constants), then all the endogenous equilibrium parameters, r, κ, σ_{c_i}, are constant, implying that the state price and consumption processes also follow a geometric Brownian motion. Under heterogeneous beliefs, this is not the case, as all endogenous equilibrium diffusion parameters are stochastic, driven by differences of opinion and distribution of wealth across investors.

3.4. The case of dividend-paying assets

We now consider replacing the non-dividend-paying, zero net supply, risky security, S, by a positive net supply security, paying off the aggregate endowment ε ($= \varepsilon_1 + \varepsilon_2$). Here, the perceived security price process would follow

$$dS(t) + \varepsilon(t)dt = S(t)[\mu^i(t)dt + \sigma(t)d\omega^i(t)], \qquad i = 1, 2. \tag{43}$$

Defining S by its payoff process ε in this way does not alter our analysis, and all of the expressions in the analysis so far remain valid, but σ would now be endogenously determined via a present-value formula for S. In particular, all our conclusions (Propositions 1–3) regarding heterogeneous beliefs are still valid; the main difference is that the following additional no-arbitrage pricing restriction must hold:

$$S(t) = \frac{1}{U'(\varepsilon(t),\lambda(t))}E^1\left[\int_t^T U'(\varepsilon(s),\lambda(s))\varepsilon(s)ds|\mathcal{F}^\varepsilon_t\right], \tag{44}$$

restricting the asset price in terms of its future payoff. Hence in this dividend-paying asset case, the mean return μ and volatility σ of the asset are individually determined in equilibrium.

In the special case of CRRA preferences, (44) becomes

$$S(t) = \frac{\varepsilon(t)^{\gamma}}{(1+\lambda(t)^{1/\gamma})^{\gamma}} E^{1}\left[\int_{t}^{T}(1+\lambda(s)^{1/\gamma})^{\gamma}\varepsilon(s)^{1-\gamma}ds|\mathcal{F}_{t}^{\varepsilon}\right], \tag{45}$$

where $\lambda(0), \lambda(t)$ are as in Proposition 3. Unfortunately, the stock price representation (44) in general does not admit a closed-form solution, and hence numerical techniques have to be resorted. Under certain inferencing rules, Gallmeyer and Hollifield (2000) and Buraschi and Jiltsov (2002) numerically solve (45) to analyze, amongst other things, the stock price volatility and trading volume, respectively. For the special case of logarithmic preferences ($\gamma = 1$), equation (45) does lead to an explicit expression given by

$$S(t) = (T-t)\varepsilon(t), \tag{46}$$

with dynamics $\mu^{i}(t) = \mu_{\varepsilon}^{i}(t)$, $\sigma(t) = \sigma_{\varepsilon}(t)$. Hence, for the logarithmic case, the stock price and its dynamics are not affected by heterogeneity in beliefs.

4. Generalizations: Nonfundamentals, Multiple Sources of Uncertainty and Investors

Our baseline analysis in Sections 2–3 considers there being only one source of risk, the Brownian motion ω, one risky security S and two investors. This simplicity of the set-up was for clarity and to highlight the most important insights pertaining to the implications of heterogeneous beliefs. However, for various applications, one may also want to investigate the impact of differences of opinion across different risky securities. For example, investors may have a higher level of disagreement about the prospects of some securities or asset classes than others. Moreover, it is also of interest to model multiple sources of uncertainty, so that one is able to make a distinction between different types of risks, for example, systematic versus idiosyncratic risks.

In this section, we construct economic environments in which there may be multiple sources of risk, multiple sources of disagreement, and finally, multiple investors. Since the analysis so far focused on differences of opinion about fundamental risk, to highlight another valuable application of the baseline analysis, in the first two subsections we primarily focus on nonfundamental risk.

4.1. Heterogeneous beliefs about nonfundamentals and extraneous risk

Considerable evidence suggests that fluctuations in the financial markets appear to occur even when there is no news about the market fundamentals (aggregate consumption supply, security payoff, investors preferences or endowments) and are commonly attributed to "nonfundamentals" or extraneous quantities (market psychology, consumer sentiment).[5] In general, there is little reason to believe that the investors know (and hence agree on) the true probability distribution of any observables, but perhaps an even stronger case for disagreement can be made for the less tangible types of "extraneous" processes. Following Basak (2000), we here present an adaptation of Sections 2–3 to allow for the possibility of a nonfundamental uncertainty influencing the real quantities in equilibrium (such as investors' consumption allocations). By a nonfundamental (or extraneous) uncertainty we mean uncertainty which does not affect any of the fundamentals of the economy.

In addition to the Brownian motion ω, driving the aggregate endowment process (1) as in Section 2 and representing the *fundamental uncertainty* in the economy, we assume there is another Brownian motion ω_z, driving the extraneous process (equation (47) below) and representing the *nonfundamental (or extraneous) uncertainty.* In particular, we assume there exists an extraneous process, z, that investors believe may affect some of the real quantities in the economy:

$$dz(t) = \mu_z(t)dt + \sigma_z(t)d\omega_z(t), \tag{47}$$

where μ_z is adapted to $\mathcal{H} \times \{\mathcal{F}_t^z\}$, σ_z is adapted to $\{\mathcal{F}_t^z\}$, and ω_z is independent of ω without loss of generality.

In order to clearly highlight the role of heterogenous beliefs about nonfundamentals, we consider an economic setting in which there is no incomplete information, and hence disagreement, about the aggregate endowment. So now, the investors' common observation filtration is $\{\mathcal{F}_t^{\omega,z}\}$. The fundamental uncertainty ω is assumed observable by all investors, as are $\varepsilon, \mu_\varepsilon$ and σ_ε. In addition, investors also observe the process z, but do not have complete information about its dynamics; investors may deduce σ_z from the quadratic variation of z, but must estimate μ_z via its conditional expectation, rationally updating their beliefs in a Bayesian fashion with heterogeneous prior beliefs. Under this setting, the innovation processes ω_z^i induced by investors' beliefs and filtration are $d\omega_z^i(t) = [dz(t) - \mu_z^i(t)dt]/\sigma_z(t)$, $i = 1, 2$, so that the extraneous process as perceived by investor i follows

$$dz(t) = \mu_z^i(t)\,dt + \sigma_z(t)\,d\omega_z^i(t), \quad i = 1, 2. \tag{48}$$

Moreover, the innovations of the two investors are related by

$$d\omega_z^2(t) = d\omega_z^1(t) + \bar{\mu}_z(t)\,dt\ , \quad \bar{\mu}_z(t) \equiv \frac{\mu_z^1(t) - \mu_z^2(t)}{\sigma_z(t)}, \tag{49}$$

where $\bar{\mu}_z$ parameterizes the investors' disagreement on the extraneous process z.

Since investors believe there are two relevant dimensions of uncertainty, for dynamic market completeness, we assume there to be one riskless bond and two non-dividend-paying risky securities, indexed by $n = 1, 2$, with price dynamics perceived by each investor i as

$$dS_n(t) = S_n(t)[\mu_n^i(t)dt + \sigma_n(t)d\omega(t) + \sigma_{nz}(t)d\omega_z^i(t)]\ , \quad n = 1, 2. \tag{50}$$

[5]See Cass and Shell (1983) for the seminal theoretical work on nonfundamentals (or so-called sunspots) within rational expectation equilibrium.

Common observation of the risky security price processes imply that the investors' disagreement about a security's mean return obeys

$$\mu_n^1(t) - \mu_n^2(t) = \sigma_{nz}(t)\bar{\mu}_z(t) \; , \quad n = 1, 2. \tag{51}$$

Furthermore, we may construct the perceived state price density process of each investor, ξ^i, as the process with dynamics

$$d\xi^i(t) = -\xi^i(t)[r(t)dt + \kappa^i(t)d\omega(t) + \kappa_z^i(t)d\omega_z^i(t)] \; , \quad i = 1, 2, \tag{52}$$

where the perceived *market price of fundamental risk* κ^i and *market price of nonfundamental risk* κ_z^i are given by $\sigma(t)(\kappa^i(t), \kappa_z^i(t))^\top = (\mu^i(t) - r(t)\underline{1})$, where $\mu^i \equiv (\mu_1^i, \mu_2^i)^\top$, the 2×2 volatility matrix $\sigma \equiv \{\sigma_n, \sigma_{nz}; n = 1, 2\}$, $\underline{1} \equiv (1, 1)^\top$. Some manipulation and the use of the consistency condition (51) implies

$$\begin{aligned} \kappa^1(t) - \kappa^2(t) &= 0 && (53) \\ \kappa_z^1(t) - \kappa_z^2(t) &= \bar{\mu}_z(t). && (54) \end{aligned}$$

Hence, under this formulation, both investors have the same perceived market price of fundamental risk, but may disagree on the market price of nonfundamental risk.

The definition and determination of equilibrium are as in Section 3, the case of heterogeneous beliefs about fundamentals. We here focus on the generic equilibria when the extraneous risk matters; the knife-edge case of the equilibria where extraneous risk does not matter is relegated to Remark 4.

Definition 2. An *equilibrium where extraneous risk matters* is one in which at least one of the investors' consumption processes depend with nonzero probability on the extraneous process z. This is equivalent to the condition that at least one of the investors' perceived state price densities depend with nonzero probability on the extraneous process z.

Equations (26)–(29) in Proposition 1 again fully determine equilibrium, with disagreement weighting process λ now following the dynamics $d\lambda(t)/\lambda(t) = -\bar{\mu}_z(t)d\omega_z^1(t)$, driven by the dispersion of beliefs about the extraneous process $\bar{\mu}_z$. The primary differences in the economy with heterogeneous beliefs about nonfundamentals are in the behavior of security price dynamics and the way in which risk is priced, as reported in Proposition 4.

Proposition 4. *In an equilibrium where extraneous risk matters, the perceived market prices of fundamental and nonfundamental risk are given by*

$$\begin{aligned} \kappa^1(t) &= \kappa^2(t) = A(t)\varepsilon(t)\sigma_\varepsilon(t), && (55) \\ \kappa_z^1(t) &= \frac{A(t)}{A_2(t)}\bar{\mu}_z(t), \quad \kappa_z^2(t) = -\frac{A(t)}{A_1(t)}\bar{\mu}_z(t), && (56) \end{aligned}$$

and the equilibrium interest rate by

$$r(t) = A(t)\varepsilon(t)\mu_\varepsilon(t) - \frac{1}{2}A(t)P(t)\varepsilon(t)^2\sigma_\varepsilon(t)^2 + \frac{A(t)^2}{A_1(t)A_2(t)}\overline{\mu}_z(t)^2 - \frac{1}{2}\frac{A(t)^3}{A_1(t)^2A_2(t)^2}(P_1(t)+P_2(t))\overline{\mu}_z(t)^2, \tag{57}$$

where A_i, A, P_i and P are as in Proposition 1.

The investors price fundamental risk equally, with the market price of fundamental risk given by the same expression as under homogenous beliefs. Here, the extraneous risk only appears indirectly via the representative risk aversion A. On the other hand, the investors' perceived market price of nonfundamental risk is proportional to the extent of their disagreement about the extraneous process ($\bar{\mu}_z$), with the more risk tolerant investor having the higher market price of nonfundamental risk. More importantly, under the current setting, the disentanglement of the nonfundamental risk from the fundamental risk allows us to make clearer statements about the dependencies of pertinent quantities on risk tolerances, investors' disagreement and different types of risk. For example, an important implication of the presence of the priced extraneous risk is that the volatility of each investor's state prices, $A\sqrt{(\varepsilon\sigma_\varepsilon)^2 + (\bar{\mu}_z/A_i)^2}$, is higher than the one in the benchmark economy (with no $\bar{\mu}_z/A_i$ term), for given A.

Finally, the structure of the equilibrium interest rate is similar to that in Section 3, however now the additional precautionary savings behavior is arising due to the priced nonfundamental risk, $\bar{\mu}_z$. The last term of the interest rate expression in Proposition 1, arising due to the covariance of aggregate endowment with disagreement risk, is no longer present here since the extraneous risk does not covary with aggregate endowment.

Remark 4 (Equilibria where extraneous risk does not matter). If investors have homogenous beliefs about the extraneous process, then there would be no equilibrium in which extraneous risk matters. Moreover, the equilibria in this homogenous-beliefs economy would be identical to that in the benchmark, complete-information economy. One way to construct these equilibria is to have neither risky security covary with the extraneous uncertainty, i.e. $\sigma_{nz}(t) = 0$, $n = 1, 2$. Then both risky securities must be identical and must be perceived identically by both investors ($\sigma_1(t) = \sigma_2(t)$, $\mu_1^1(t) = \mu_2^1(t) = \mu_1^2(t) = \mu_2^2(t)$), even though the investors disagree on μ_z^i. These are degenerate examples of equilibria in which extraneous risk does not matter, having $\kappa_z^1(t) = \kappa_z^2(t) = 0$. In fact, whenever extraneous risk does not matter, the equilibrium is effectively equivalent to one in the benchmark model, since $\kappa_z^1(t) = \kappa_z^2(t) = 0$.

4.2. Multiple sources of extraneous risk

We now extend the analysis of Section 4.1 to the case of $L > 1$, possibly correlated, extraneous processes driven by an L-dimensional Brownian motion process $\omega_z = (\omega_1, \ldots, \omega_L)^\top$. The extraneous process dynamics are as in equation (47) but now with the notation $z(t)$, $\mu_z(t)$ and

$\sigma_z(t)$ denoting L-dimensional vectors and an $L \times L$ matrix, respectively, where each element of $\sigma_z = \{\sigma_{z_l j}\}$ captures the covariance of an extraneous process with the jth nonfundamental uncertainty, Brownian motion ω_j. Consistent with earlier analysis, σ_z is assumed to be known to the investors. The uncertainty-information structure is as in Section 2, but with the processes z and ω replaced by its L-dimensional counterparts. Associated with each investor is an L-dimensional innovation process $\omega_z^i = (\omega_1^i, \ldots, \omega_L^i)^\top$, $i = 1, 2$, related by

$$d\omega_z^2(t) = d\omega_z^1(t) + \bar{\mu}_z(t)\, dt\ , \quad \bar{\mu}_z(t) \equiv \sigma_z(t)^{-1}(\mu_z^1(t) - \mu_z^2(t)), \tag{58}$$

where μ_z^i is the vector of mean growths of the extraneous processes, as perceived by investor i, and the L-dimensional vector $\bar{\mu}_z$ captures the disagreement between the two investors about the mean growths of the L extraneous processes.

Given that there are L dimensions of uncertainty that may affect equilibrium, for dynamic market completeness, in addition to the bond we assume there to be $L + 1$ zero net supply risky securities, with perceived price dynamics as in (50), where now σ_{nz} is an L-dimensional vector with each element representing the covariance of security S_n with the jth dimension of nonfundamental uncertainty. Given investor i's perception of price dynamics, the investors' disagreement about the risky securities' mean returns obeys

$$\mu_n^1(t) - \mu_n^2(t) = \sigma_{nz}(t)^\top \bar{\mu}_z(t), \qquad n = 1, \ldots, L + 1. \tag{59}$$

As in Section 4.1, we construct the perceived state price density of each investor i with κ_z^i now being the L-dimensional vector representing the perceived market prices of nonfundamental risk associated with the L-dimensional extraneous uncertainty. Similarly to (53)–(54), we have $\kappa^1(t) - \kappa^2(t) = 0$, $\kappa_z^1(t) - \kappa_z^2(t) = \bar{\mu}_z(t)$.

Equations (26)–(29) in Proposition 1 again fully determine equilibrium, with the disagreement weighting process λ now following the dynamics $d\lambda(t)/\lambda(t) = -\bar{\mu}_z(t)^\top d\omega_z^1(t)$, driven by the L-dimensional (normalized) dispersion of beliefs about mean returns $\bar{\mu}_z$. In Proposition 6 we characterize the equilibrium price dynamics.

Proposition 5. *In an equilibrium with L-dimensional sources of nonfundamental uncertainty, the investors' perceived market prices of risk and the interest rate are given by*

$$\kappa^1(t) = \kappa^2(t) = A(t)\varepsilon(t)\sigma_\varepsilon(t)\ , \quad \kappa_z^1(t) = \frac{A(t)}{A_2(t)}\bar{\mu}_z(t)\ , \quad \kappa_z^2(t) = -\frac{A(t)}{A_1(t)}\bar{\mu}_z(t), \tag{60}$$

and

$$\begin{aligned} r(t) \;=\; & A(t)\varepsilon(t)\mu_\varepsilon(t) - \frac{1}{2}A(t)P(t)\varepsilon(t)^2\sigma_\varepsilon(t)^2 \\ & + \frac{A(t)^2}{A_1(t)A_2(t)}\left\|\bar{\mu}_z(t)\right\|^2 - \frac{1}{2}\frac{A(t)^3}{A_1(t)^2A_2(t)^2}(P_1(t) + P_2(t))\left\|\bar{\mu}_z(t)\right\|^2, \end{aligned} \tag{61}$$

where A_i, A, P_i, P are as in Proposition 1.

These expressions have the same form as in the case of 1-dimensional nonfundamental uncertainty, the main difference being that the investors' disagreement $\bar{\mu}_z$ is now multi-dimensional, with each element capturing a different dimension of uncertainty.

4.3. More than two investors

The analysis of Sections 2–3 readily extend to the case of $N > 2$ investors. Each investor i is endowed with the income stream $\varepsilon_i(t)$ and $\varepsilon_1(t) + \ldots + \varepsilon_N(t) = \varepsilon(t)$, where $\varepsilon(t)$ is as in (1). The uncertainty-information structure is as in Section 2, where each investor is effectively endowed with the probability space $(\Omega, \mathcal{F}^i, \{\mathcal{F}^i_t\}, \mathcal{P}^i)$ and $\mathcal{F}^i \equiv \mathcal{F}^{\omega^i} = \mathcal{F}^\varepsilon$, $i = 1, \ldots, N$. We express each investor's innovation process relative to investor 1's as

$$d\omega^i(t) = d\omega^1(t) + \bar{\mu}^i(t)\, dt\ , \quad \bar{\mu}^i(t) \equiv \frac{\mu^1_\varepsilon(t) - \mu^i_\varepsilon(t)}{\sigma_\varepsilon(t)}, \qquad i = 1, \ldots, N, \tag{62}$$

where, as before, $\bar{\mu}^i$ is essentially exogenous. Given investor i's perception of the risky security price dynamics, we may also identify the disagreement of investor i relative to investor 1, about the risky security's mean return as

$$\mu^1(t) - \mu^i(t) = \sigma(t)\bar{\mu}^i(t), \qquad i = 1, \ldots, N. \tag{63}$$

Construction of the perceived state price process of investor i reveals that investors' perceived market prices of risk differ by $\bar{\mu}^i(t)$ from investor 1's, i.e.,

$$\kappa^1(t) - \kappa^i(t) = \bar{\mu}^i(t), \qquad i = 1, \ldots, N. \tag{64}$$

For an N-investor economy, the representative investor's utility is defined by

$$U(c; \lambda) \equiv \max_{c_1, \ldots, c_N} \sum_{i=1}^{N} \lambda_i u_i(c_i) \tag{65}$$

subject to $c_1 + \ldots + c_N = c$, where $\lambda \equiv (\lambda_1, \ldots, \lambda_N)$. Using the individual investor's first order conditions (18)–(19) and identifying

$$\lambda_i(t) = \frac{y_1 \xi^1(t)}{y_i \xi^i(t)}, \qquad i = 1, \ldots, N, \tag{66}$$

we may derive each investor 1's state price density from clearing in the good market as

$$\xi^i(t) = \frac{\lambda_i(0)}{\lambda_i(t)} \frac{U'(\varepsilon(t); \lambda(t))}{U'(\varepsilon(0); \lambda(0))}, \qquad i = 1, \ldots, N, \tag{67}$$

where $\lambda_i(0)$, $i = 1, \ldots, N$, satisfy the binding budget constraints of the N investors, and the weighting process of investor i relative to investor 1 is given by

$$\frac{d\lambda_i(t)}{\lambda_i(t)} = -\bar{\mu}^i(t) d\omega^1(t), \qquad i = 1, \ldots, N. \tag{68}$$

The equilibrium consumption allocations are then given by

$$c_i(t) = I_i\left(U'(\varepsilon(t); \lambda(t))/\lambda_i(t)\right), \qquad i = 1, \ldots, N. \tag{69}$$

Equations (67)–(69) fully characterize the equilibrium and determine the investor-specific state price densities, consumption allocations and their dynamics.

Proposition 5 reports the perceived market prices of risk and interest rate, revealing that the essential structure of the equilibrium in Section 3 is maintained.

Proposition 6. *In an equilibrium with N investors and heterogeneous beliefs, the investors' perceived market prices of risk and the interest rate are given by* [6]

$$\kappa^i(t) = A(t)\varepsilon(t)\sigma_\varepsilon(t) + A(t)\sum_{j\neq i}\frac{\bar{\mu}^j(t) - \bar{\mu}^i(t)}{A_j(t)}, \qquad i = 1, \ldots, N, \tag{70}$$

and

$$\begin{aligned} r(t) \;=\; & A(t)\left(\sum_{i=1}^{N}\frac{A(t)}{A_i(t)}\varepsilon(t)\mu_\varepsilon^i(t)\right) - \frac{1}{2}A(t)P(t)\varepsilon(t)^2\sigma_\varepsilon(t)^2 \\ & + A(t)^2\sum_{i=1}^{N}\frac{\bar{\mu}^i(t)}{A_i(t)}\sum_{j\neq i}\frac{\bar{\mu}^i(t) - \bar{\mu}^j(t)}{A_j(t)} - \frac{1}{2}A(t)^3\sum_{i=1}^{N}\frac{P_i(t)}{A_i(t)^2}\sum_{j\neq i}\frac{(\bar{\mu}^i(t) - \bar{\mu}^j(t))^2}{A_j(t)^2} \\ & - A(t)^3\varepsilon(t)\sigma_\varepsilon(t)\sum_{i=1}^{N}\frac{\bar{\mu}^i(t)}{A_i(t)}\sum_{j\neq i}\left(\frac{P_i(t)}{A_i(t)} - \frac{P_j(t)}{A_j(t)}\right), \end{aligned} \tag{71}$$

where A_i, P_i are as in Proposition 1, and the representative risk aversion and prudence are given by

$$\frac{1}{A(t)} = \sum_{i=1}^{N}\frac{1}{A_i(t)}, \qquad P(t) = \sum_{i=1}^{N}\left(\frac{A(t)}{A_i(t)}\right)^2 P_i(t). \tag{72}$$

An investor's perceived market price of risk is still given by the homogeneous-beliefs term ($A\varepsilon\sigma_\varepsilon$) plus an additional term arising due to heterogeneous beliefs. The additional term is proportional to a weighted average of investor i's disagreement relative to all other investors j, $(\bar{\mu}^j - \bar{\mu}^i)$, with the weight being proportional to investor i's risk tolerance. In a sense, this term is investor i's pessimism/optimism relative to the average remaining investors. As in the 2-investor case, the interest rate inherits three additional terms (third to fifth terms in (71)) due to heterogeneity in beliefs, with similar interpretations. The third term arises since investors live under differing probability spaces; it is a risk-tolerance weighted average of each investor's disagreement relative to investor 1 times that investor's disagreement relative to the average remaining investors'. Unlike in the 2-investor economy, this third term now has an ambiguous sign. The fourth term is a weighted average of each investor i's squared disagreement relative to the remaining average investor, with the weight driven by investor i's prudence and risk

[6] The notation $\sum_{j\neq i}$ is understood to mean $\sum_{j=1, j\neq i}^{N}$.

aversion. Again, this term arises due to investors' precautionary savings motive in the presence of heterogeneity in beliefs. Although the structure of the N-investor economy bears similarities to the 2-investor economy, it is a lot more complex due to the presence of the N stochastic weights λ_i, $i = 1, \ldots, N$. It would be desirable to be able to aggregate these stochastic weights into a single "aggregate weight", capturing the aggregate belief in the economy. For recent promising work in this direction, see Jouini and Napp (2003).

5. Applications and Avenues for Future Research

In this section, we investigate the implications of investors' differences of opinion in other important economic settings. To our knowledge, these settings have been little explored so far in the literature. The models that we develop are variants of the baseline analysis of Sections 2–4. Our main purpose is to demonstrate how the basic analysis can be adapted to alternative settings and to highlight the necessary critical elements in constructing models with heterogeneous beliefs. Towards this end, we do not provide a very detailed analysis or undertake the analysis in the most general possible setting.

In the first case, we consider there being two goods in the economy and investors having differences of opinion about the growth prospects of each good's aggregate supply. This setting has the obvious advantage of more realism over the single good case. But it is also amenable to various applications in an international setting, including the popular 2-good×2-country environment. For a recent analysis related to the latter setting, see Pavlova and Rigobon (2003), who model each country's preferences as logarithmic but also more generally to incorporate demand shocks within each country.

The second application incorporates heterogeneous beliefs within a dynamic monetary economy, so as to study nominal asset pricing implications. This setting is particularly of interest since almost all securities in current financial markets are denominated in nominal terms. Although the importance of incomplete information about monetary policy and nominal asset prices has long been recognized (Stulz, 1986), the modelling of differences of opinion within an asset pricing framework is novel to our analysis.

5.1. Multiple goods

Economic Set-Up. We consider an economy with two perishable goods. For this economic setting, it is natural to have multiple sources of uncertainty so that the supplies of the two goods are not perfectly correlated. Consequently, there will be similarities with the analysis of Sections 4.1–4.2, which incorporated multiple sources of risk. The primary difference here will be the presence of an additional endogenous quantity in equilibrium, namely the relative price of the second good (or the terms of trade), denoted by p. In an international 2-good×2-country type setting, with each country specializing in one good, p may be interpreted as the real exchange rate.

The first good is as in Sections 2–4, serves as the numeraire and has aggregate supply following $d\varepsilon(t) = \varepsilon(t)[\mu_\varepsilon(t)dt + \sigma_\varepsilon(t)d\omega(t)]$, where ω is an unobserved one-dimensional Brownian motion. The aggregate endowment of the second good, ε^*, is assumed to follow

$$d\varepsilon^*(t) = \varepsilon^*(t)[\mu_{\varepsilon^*}(t)dt + \sigma_{\varepsilon^*}(t)d\omega_*(t)], \tag{73}$$

where σ_{ε^*} is adapted to $\{\mathcal{F}_t^{\varepsilon^*}\}$, and the uncertainty generating the second good supply is a one-dimensional Brownian motion ω_*, assumed independent of ω for expositional clarity. The investors' common observation filtration is now $\{\mathcal{F}_t^{\varepsilon,\varepsilon^*}\}$. That is, the investors observe ε and ε^*, they may deduce σ_ε, σ_{ε^*}, but must estimate the mean growth rates μ_ε and μ_{ε^*} in a Bayesian fashion. Under this setting, the investors' innovation processes ω^i and ω_*^i are given by $d\omega^i(t) = [d\varepsilon(t)/\varepsilon(t) - \mu_\varepsilon^i(t)dt]/\sigma_\varepsilon(t)$, $d\omega_*^i(t) = [d\varepsilon^*(t)/\varepsilon^*(t) - \mu_{\varepsilon^*}^i(t)dt]/\sigma_{\varepsilon^*}(t)$, $i = 1, 2$, so that the aggregate endowment of the two goods as perceived by investor i follows

$$\begin{aligned} d\varepsilon(t) &= \varepsilon(t)[\mu_\varepsilon^i(t)dt + \sigma_\varepsilon d\omega^i(t)], & (74)\\ d\varepsilon^*(t) &= \varepsilon^*(t)[\mu_{\varepsilon^*}^i(t)dt + \sigma_{\varepsilon^*} d\omega_*^i(t)], \quad i = 1, 2. & (75)\end{aligned}$$

Here, the innovations of the two investors are related by

$$\begin{aligned} d\omega^2(t) &= d\omega^1(t) + \overline{\mu}(t)dt, \quad \overline{\mu}(t) \equiv \frac{\mu_\varepsilon^1(t) - \mu_\varepsilon^2(t)}{\sigma_\varepsilon(t)}, & (76)\\ d\omega_*^2(t) &= d\omega_*^1(t) + \overline{\mu}_*(t)dt, \quad \overline{\mu}_*(t) \equiv \frac{\mu_{\varepsilon^*}^1(t) - \mu_{\varepsilon^*}^2(t)}{\sigma_{\varepsilon^*}(t)}, & (77)\end{aligned}$$

where $\bar{\mu}$ parameterizes the investors' disagreement on the mean growth of the first good, while $\bar{\mu}_*$ on the mean growth of the second good.

Investment Opportunities. For dynamic market completeness relative to investors' observation filtrations, we posit there to be one riskless bond and two non-dividend-paying risky securities, indexed by $n = 1, 2$, with price dynamics perceived by each investor i as

$$dS_n(t) = S_n(t)[\mu_n^i(t)dt + \sigma_n(t)d\omega^i(t) + \sigma_{n*}(t)d\omega_*^i(t)], \quad n = 1, 2, \tag{78}$$

where as in Section 2, (σ_n, σ_{n*}) defines the financial contract S_n. Price-agreement across investors imply the following investors' disagreement on a security's mean return:

$$\mu_n^1(t) - \mu_n^2(t) = \sigma_n(t)\bar{\mu}(t) + \sigma_{n*}(t)\bar{\mu}_*(t), \quad n = 1, 2. \tag{79}$$

Moreover, the perceived state price density of each investor ξ^i is given by

$$d\xi^i(t) = -\xi^i(t)[r(t)dt + \kappa^i(t)d\omega^i(t) + \kappa_*^i(t)d\omega_*^i(t)] \tag{80}$$

where the perceived market prices of risk associated with the first and second good supply uncertainties κ^i and κ_*^i are given by $\sigma(t)(\kappa^i(t), \kappa_*^i(t))^\top = (\mu^i(t) - r(t)\underline{1})$, where $\mu^i \equiv (\mu_1^i, \mu_2^i)^\top$ the $2{\times}2$ volatility matrix $\sigma \equiv \{\sigma_n, \sigma_{n*}, n = 1, 2\}$. This and (79) in turn imply that investors disagree on the market prices of risk, with their disagreement given by the differences of opinion about the mean growths of the two goods:

$$\kappa^1(t) - \kappa^2(t) = \overline{\mu}(t), \tag{81}$$

$$\kappa^1_*(t) - \kappa^2_*(t) = \overline{\mu}_*(t). \tag{82}$$

Investors' Optimization. In this economy, each investor i is endowed with endowment streams $\varepsilon_i, \varepsilon_i^*$, where $\varepsilon_1(t)+\varepsilon_2(t) = \varepsilon(t)$ and $\varepsilon_1^*(t)+\varepsilon_2^*(t) = \varepsilon^*(t)$. The investor then chooses consumption processes (c_i, c_i^*) and a portfolio process $\pi_i \equiv (\pi_{i1}, \pi_{i2})^\top$, denoting the amounts invested in each security. Each investor derives utility $u_i(c_i) + v_i(c_i^*)$ from the two consumption goods. The separability in preferences is for expositional simplicity; the important case of nonseparability is straightforward to incorporate, without affecting our main points in a major way. Employing martingale techniques, each investor's dynamic optimization problem is stated as

$$\max_{c_i, c_i^*} E^i \left[\int_0^T \left[u_i(c_i(t)) + v_i(c_i^*(t)) \right] dt \right] \tag{83}$$

$$\text{subject to} \qquad E^i \left[\int_0^T \xi^i(t) \left[c_i(t) + p(t) c_i^*(t) \right] dt \right] \le E^i \left[\int_0^T \xi^i(t) \left[\varepsilon_i(t) + p(t) \varepsilon_i^*(t) \right] dt \right], \tag{84}$$

where p denotes the price of the second good. The necessary and sufficient conditions for optimality of the consumption streams are

$$c_i(t) = I_i(y_i \xi^i(t)), \quad c_i^*(t) = J_i(y_i \xi^i(t) p(t)), \tag{85}$$

where I_i, J_i are the inverses of u_i', v_i', respectively, and y_i satisfies

$$E^i \left[\int_0^T \xi^i(t) \left[c_i(t) + p(t) c_i^*(t) \right] dt \right] = E^i \left[\int_0^T \xi^i(t) \left[\varepsilon_i(t) + p(t) \varepsilon_i^*(t) \right] dt \right]. \tag{86}$$

Equilibrium. We define equilibrium as in Definition 1 with the obvious modifications, but now the main difference is the additional restriction that the second good market clears: $c_1^*(t) + c_2^*(t) = \varepsilon^*(t)$, allowing us to determine the relative price of the second good. We may introduce a representative investor, analogously to that in Section 2. Since the investors' preferences are separable over the two goods, the representative investor's utility will also be separable. In particular, as before we define $U(c;\lambda) \equiv \max u_1(c_1) + \lambda u_2(c_2)$ subject to $c_1 + c_2 = c$, and now also additionally

$$V(c^*;\lambda) \equiv \max_{c_1^* + c_2^* = c^*} v_1(c_1^*) + \lambda v_2(c_2^*) \tag{87}$$

where $\lambda > 0$ may be stochastic. Identifying $\lambda(t) = u_1'(c_1(t))/u_2'(c_2(t)) = y_1 \xi^1(t)/y_2 \xi^2(t)$, we obtain the following equilibrium conditions in terms of the representative investor's utility functions and the stochastic weighting λ, which itself is identified. Consequently, we may also derive the asset price dynamics and equilibrium consumption dynamics. For brevity, we report only the perceived market prices of risk and relative good price volatility components, and leave out the interest rate and other equilibrium mean growth rates.

Proposition 7. *In a heterogeneous-beliefs economy with two perishable goods, the investors' equilibrium state price densities and relative price of the second good are*

$$\xi^1(t) = \frac{U'(\varepsilon(t);\lambda(t))}{U'(\varepsilon(0);\lambda(0))}, \quad \xi^2(t) = \frac{\lambda(0)}{\lambda(t)}\frac{U'(\varepsilon(t);\lambda(t))}{U'(\varepsilon(0);\lambda(0))}, \tag{88}$$

$$p(t) = \frac{V'(\varepsilon^*(t);\lambda(t))}{U'(\varepsilon(t);\lambda(t))}, \tag{89}$$

where $\lambda(0)$ *satisfies (86) and* $\lambda(t)$ *satisfies*

$$\frac{d\lambda(t)}{\lambda(t)} = -\bar{\mu}(t)d\omega^1(t) - \bar{\mu}_*(t)d\omega^1_*(t). \tag{90}$$

The equilibrium consumption allocations are

$$c_1(t) = I_1(U'(\varepsilon(t);\lambda(t))), \quad c_2(t) = I_2(U'(\varepsilon(t);\lambda(t))/\lambda(t)), \tag{91}$$

$$c^*_1(t) = J_1(V'(\varepsilon(t);\lambda(t))), \quad c^*_2(t) = J_2(V'(\varepsilon(t);\lambda(t))/\lambda(t)). \tag{92}$$

Furthermore, the perceived market prices of risk and relative price volatility components are

$$\kappa^1(t) = A(t)\varepsilon(t)\sigma_\varepsilon(t) + \frac{A(t)}{A_2(t)}\bar{\mu}(t), \quad \kappa^2(t) = A(t)\varepsilon(t)\sigma_\varepsilon(t) - \frac{A(t)}{A_1(t)}\bar{\mu}(t), \tag{93}$$

$$\kappa^1_*(t) = \frac{A(t)}{A_2(t)}\bar{\mu}_*(t), \quad \kappa^2_*(t) = -\frac{A(t)}{A_1(t)}\bar{\mu}_*(t), \tag{94}$$

and

$$\sigma_p(t) = A(t)\varepsilon(t)\sigma_\varepsilon(t) + \left(\frac{A(t)}{A_2(t)} - \frac{A^*(t)}{A^*_2(t)}\right)\bar{\mu}(t), \tag{95}$$

$$\sigma_{p*}(t) = \left(\frac{A(t)}{A_2(t)} - \frac{A^*(t)}{A^*_2(t)}\right)\bar{\mu}_*(t) - A^*(t)\varepsilon^*(t)\sigma_{\varepsilon^*}(t), \tag{96}$$

where p *satisfies* $dp(t) = p(t)[\mu^i_p(t)dt + \sigma_p(t)d\omega^i(t) + \sigma_{p*}(t)d\omega^i_*(t)]$, A_i, A *are as in Proposition 1, and* $A^*_i(t) \equiv -V''(c^*_i(t))/V'(c^*_i(t))$, $1/A^*(t) = 1/A^*_1(t) + 1/A^*_2(t)$.

Due to our separable preferences assumption, the equilibrium expressions related to the first good (ξ^i, c^i, κ^i) are as in Proposition 1 of Section 3. The effect of the second good appears indirectly through the stochastic weighting $\lambda(t)$, accounting for the investors' differences of opinion, and through the initial wealth effect $\lambda(0)$. The price of the second good p (equation (89)) is given by the representative investor's marginal rate of substitution between the two goods, provided the weight assigned to each investor is stochastic. Here, the stochastic weighting $\lambda(t)$ is identified to be driven by the investors' disagreements on the mean growth rates of the two goods (equation (90)). The expressions for the perceived market prices of risk associated with the second good uncertainty, κ^i_*, (equation (94)) have a simple structure, arising due to the investors' differences of opinion about the prospects of the second good; under homogeneous beliefs, these market prices of risk would be zero.

5.2. Monetary economics and nominal pricing

Economic Set-Up. In this section, we provide a tractable framework to investigate how heterogeneous beliefs may enter into and affect nominal asset prices. We consider a dynamic monetary economy in which investors hold money for the purpose of transaction services formulated via money in the utility function. We build on the recent continuous-time nominal pricing models of Bakshi and Chen (1996) and Basak and Gallmeyer (1999), and incorporate investors' heterogeneity in beliefs about the monetary policy. In this setting, we jointly model real, financial and currency markets. The main additional feature over our earlier analysis is that the nominal quantities are endogenously determined in the model, through a transaction service role for money. As will be seen in the sequel, the real money balances, demanded by investors, act like a durable good. In this sense, as compared to the 2-good analysis of Section 5.1, we here have a second, but durable good, whose price (price of money) is determined endogenously.

The real side of the economy represented by the aggregate endowment ε of equation (1), is as in Section 2. To be able to focus on the heterogeneity in beliefs arising from monetary policies, we assume ε and its dynamics are fully observable, driven by the observed aggregate endowment uncertainty ω. The money supply process M, expressed in nominal terms (in units of money), follows

$$dM(t) = M(t)\left[\mu_M(t)dt + \sigma_M(t)d\omega_M\right] \tag{97}$$

where the monetary (or money supply) uncertainty ω_M is assumed independent of ω for expositional simplicity, and σ_M adapted to $\{\mathcal{F}_t^M\}$. Now, the investors' common observation filtration is $\{\mathcal{F}_t^{\omega,M}\}$. The investors observe the money supply M, hence may deduce σ_M, but must estimate the mean growth rate of money supply μ_M. Here, the innovative processes for each investor is given by $d\omega_M^i(t) = [dM(t)/M(t) - \mu_M^i(t)dt]/\sigma_M(t)$, $i = 1, 2$, so that the money supply as perceived by investor i follows

$$dM(t) = M(t)\left[\mu_M^i(t)dt + \sigma_M(t)d\omega_M^i\right]. \tag{98}$$

Consequently, the innovations of the two investors are related by

$$d\omega_M^2(t) = d\omega_M^1(t) + \bar{\mu}_M(t)dt, \quad \bar{\mu}_M(t) \equiv \frac{\mu_M^1(t) - \mu_M^2(t)}{\sigma_M(t)}, \tag{99}$$

where $\bar{\mu}_M$ now parameterizes investors' disagreement on the mean growth rate of the money supply. Denoting the endogenous price of money by q, its dynamics as perceived by investors follow (verified in equilibrium)

$$dq(t) = q(t)[\mu_q^i(t)dt + \sigma_q(t)d\omega(t) + \sigma_{qm}(t)d\omega_M^i(t)], \tag{100}$$

where $-\mu_q^i(t)$, consistent with the literature, is the (perceived) expected inflation rate.

Investment Opportunities. There are three securities available for trading. The first is a riskless bond, in real terms, paying off the interest rate r. The second is a risky financial security in real terms, defined by its volatility σ, as in Section 2. The third is a nominally riskless bond, paying off the nominal interest rate R over the next instant, but is risky in real terms. In particular, the perceived price dynamics of the two risky securities are given by

$$\begin{aligned} dS(t) &= S(t)[\mu^i(t)dt + \sigma(t)d\omega(t) + \sigma_m(t)d\omega^i_M(t)], & (101)\\ dB_m(t) &= B_m(t)[(\mu^i_q(t) + R(t))dt + \sigma_q(t)d\omega(t) + \sigma_m(t)d\omega^i_M(t)], \quad i = 1, 2. & (102)\end{aligned}$$

From the drift of the nominal bond price in (102), we see that the nominal interest rate R is the additional rate earned over by just holding the money (equation (100)). Hence, R represents the nominal value of the transaction services provided by holding money and qR the real value of those services. Price agreement across investors now imply

$$\mu^1(t) - \mu^2(t) = \sigma_m(t)\bar{\mu}_M(t), \quad \mu^1_q(t) - \mu^2_q(t) = \sigma_{qm}(t)\bar{\mu}_M(t). \tag{103}$$

Under this setting, each investor's perceived state price density is given by

$$d\xi^i(t) = -\xi^i(t)[r(t)dt + \kappa^i(t)d\omega(t) + \kappa^i_m(t)d\omega^i_M(t)], \tag{104}$$

where the investors' perceived market prices of risk associated with the commodity uncertainty κ^i and monetary uncertainty κ^i_m satisfy

$$\begin{pmatrix} \sigma(t) & \sigma_m(t) \\ \sigma_q(t) & \sigma_{qm}(t) \end{pmatrix} \begin{pmatrix} \kappa^i(t) \\ \kappa^i_m(t) \end{pmatrix} = \begin{pmatrix} \mu^i(t) - r(t) \\ \mu^i_q(t) + R(t) - r(t) \end{pmatrix}. \tag{105}$$

Simple manipulation implies that the investors only disagree on the monetary market price of risk:

$$\begin{aligned} \kappa^1(t) - \kappa^2(t) &= 0, & (106)\\ \kappa^1_m(t) - \kappa^2_m(t) &= \bar{\mu}_M(t). & (107)\end{aligned}$$

Investors' Optimization. The money, issued by the governing body, enters into the economy by endowing each investor with transfer M_i, where $M_1(t) + M_2(t) = M(t)$. Now, an investor chooses a consumption process c_i, a nominal money-holding process m_i and a portfolio process. To isolate the effects of heterogeneous beliefs on nominal prices, investors are assumed to derive separable utility from consumption and real money balances $u_i(c_i) + v_i(qm_i)$. Using martingale methods, an investor's dynamic optimization problem is stated as

$$\max_{c_i, m_i} E^i \left[\int_0^T \left[u_i(c_i(t)) + v_i(q(t)m_i(t)) \right] dt \right] \tag{108}$$

subject to

$$E^i \left[\int_0^T \xi^i(t) \left[c_i(t) + R(t)q(t)m_i(t) \right] dt \right] \leq E^i \left[\int_0^T \xi^i(t) \left[\varepsilon_i(t) + R(t)q(t)M_i(t) \right] dt \right], \tag{109}$$

The budget constraint (109) now additionally accounts for the value of real money holdings (Rqm_i) and the value of real money transfers (RqM_i). The necessary and sufficient conditions for optimality are

$$c_i(t) = I_i(y_i \xi^i(t)), \quad m_i(t) = J_i(y_i \xi^i(t) R(t))/q(t), \tag{110}$$

where I_i, J_i are the inverses of u_i', v_i', respectively, and y_i satisfies

$$E^i \left[\int_0^T \xi^i(t) \left[c_i(t) + R(t) q(t) m_i(t) \right] dt \right] = E^i \left[\int_0^T \xi^i(t) \left[\varepsilon_i(t) + R(t) q(t) M_i(t) \right] dt \right]. \tag{111}$$

Equilibrium. We now additionally need to impose market clearing in the money market: $m_1(t) + m_2(t) = M(t)$. Given separability of preferences, the representative investor's utility over real money balances may be defined as

$$V(qm; \lambda) \equiv \max_{m_1 + m_2 = m} v_1(qm_1) + \lambda v_2(qm_2), \tag{112}$$

with $\lambda > 0$ possibly stochastic. The representative investor's utility over consumption $U(c; \lambda)$ is as in Section 2. Identifying $\lambda(t) = u_1'(c_1(t))/u_2'(c_2(t)) = y_1 \xi^1(t)/y_2 \xi^2(t)$, we may fully determine equilibrium and its dynamics, as reported in Proposition 8. (For brevity, we leave out the interest rate, expected inflation and other endogenous mean growth rates.)

Proposition 8. *In a monetary economy with heterogeneous beliefs on the monetary policy, the investors' equilibrium state price densities ξ^i, nominal interest rate R and price of money q are*

$$\xi^1(t) = \frac{U'(\varepsilon(t); \lambda(t))}{U'(\varepsilon(0); \lambda(0))}, \quad \xi^2(t) = \frac{\lambda(0)}{\lambda(t)} \frac{U'(\varepsilon(t); \lambda(t))}{U'(\varepsilon(0); \lambda(0))}, \tag{113}$$

$$R(t) = \frac{V'(q(t)M(t); \lambda(t))}{U'(\varepsilon(t); \lambda(t))}, \tag{114}$$

$$q(t) = \frac{1}{U'(\varepsilon(t), \lambda(t))} E^i \left[\int_t^T V'(q(s)M(s); \lambda(s)) q(s) ds | \mathcal{F}_t^{\omega, M} \right], \tag{115}$$

where $\lambda(0)$ satisfies (111) and $\lambda(t)$ satisfies

$$\frac{d\lambda(t)}{\lambda(t)} = -\bar{\mu}_M(t) d\omega_M^1(t). \tag{116}$$

The equilibrium consumption and money holdings are

$$c_1(t) = I_1(U'(\varepsilon(t); \lambda(t))), \qquad c_2(t) = I_2(U'(\varepsilon(t); \lambda(t))/\lambda(t)), \tag{117}$$

$$q(t)m_1(t) = J_1(V'(q(t)M(t); \lambda(t))), \quad q(t)m_2(t) = J_2(V'(q(t)M(t); \lambda(t))/\lambda(t)). \tag{118}$$

Furthermore, the perceived market prices of risk associated with commodity uncertainty κ^i and monetary uncertainty κ_m^i are given by

$$\kappa^1(t) = \kappa^2(t) = A(t) \varepsilon(t) \sigma_\varepsilon(t), \tag{119}$$

$$\kappa_m^1(t) = \frac{A(t)}{A_2(t)} \bar{\mu}_M(t), \quad \kappa_m^2(t) = -\frac{A(t)}{A_1(t)} \bar{\mu}_M(t), \tag{120}$$

where A_i, A are as in Proposition 1.

The quantities associated with the consumption good (ξ^i, c_i, κ^i) are affected by the monetary elements only indirectly via λ, due to separable preferences and independence of commodity and money supply. The stochastic weighting λ accounts for the investors' differences of opinion about the mean monetary growth. The equilibrium nominal interest rate (114) is simply the relative price between real money balances and the commodity. Note that the equilibrium price of money q in (115) is not explicit since its time-t price $q(t)$ is the present value of its future value $q(s)$. In general, explicit characterization of q is not possible, but by further specializing the economy to logarithmic preferences it is possible to obtain q in closed form. Finally, we note the simple structure of the perceived market prices of risk associated with monetary uncertainty κ_m^i, arising solely due to heterogeneity in beliefs.

6. Conclusion

We provide a tractable continuous-time pure-exchange framework to study the asset pricing implication of the presence of heterogeneous beliefs, within a rational Bayesian setting. Equilibrium is determined in terms of a representative investor's utility function with stochastic weighting driven by the investors' disagreement about the aggregate growth. We provide a full characterization of security price dynamics in the ensuing equilibrium. The basic setting readily extends to additionally incorporate differences of opinion about nonfundamentals, multiple sources of risk, and multiple investors. Further avenues for future research are discussed. Since our conclusions in the current analysis are qualitative, it would be valuable in future work to empirically quantify and assess the economic significance of these predictions. It would also be of interest to empirically test the heterogeneous-beliefs models that are provided in this article.

Appendix: Proofs

Proof of Proposition 1. This is a variation on Karatzas et al. (1990), to incorporate heterogeneous state price densities across investors. Good market clearing (21) together with investors' optimality (18)–(19), imply (26)–(27), where we have substituted $\lambda(t) = y_1\xi^1(t)/y_2\xi^2(t)$, set $y_1 = U'(\varepsilon(0); \lambda(0))$ and made use of the fact that the inverse of $U'(c; \lambda)$ is $I(h; \lambda) = I_1(h) + I_2(h/\lambda)$. Applying Itô's Lemma to (25) and using (13)–14) identifies the weighting process λ, as reported in (28). Substitution of the equilibrium state price densities (26) into (18) yields the equilibrium consumption allocations (29). To prove the last statement, c_1 and c_2 given by (18), together with (26) imply the good market clearing in (21). The proof that the good market clearing implies the remaining two market clearing conditions (22)–(23) is a similarly modified version of that in Karatzas et al. (1990) (to account for stochastic λ). *Q.E.D.*

Proof of Proposition 2. To prove the last equality in (32), differentiate c_i in (29) with respect to ε to obtain $\partial c_i(t)/\partial\varepsilon(t) = A(t)/A_i(t)$. Then, using $\partial c_1(t)/\partial\varepsilon(t) + \partial c_2(t)/\partial\varepsilon(t) = 1$ (implied by good clearing), we deduce $1/A(t) = 1/A_1(t) + 1/A_2(t)$. To prove the last equality in (33), differentiate c_i in (29) twice with respect to ε and use $\partial c_i(t)/\partial\varepsilon(t) = A(t)/A_i(t)$ to obtain $\partial^2 c_i(t)/\partial\varepsilon^2(t) = (A(t)/A_i(t))^2 P_i(t) - (A(t)/A_i(t))P(t)$. Then, using $\partial^2 c_1(t)/\partial\varepsilon^2(t) + \partial^2 c_2(t)/\partial\varepsilon^2(t) = 0$ (implied by good clearing), we deduce $P(t)/A(t)^2 = P_1(t)/A_1(t)^2 + P_2(t)/A_2(t)^2$.

Applying Itô's Lemma to $u_i'(c_i(t)) = y_i\xi^i(t)$ in (18) and equating the diffusion and drift terms yields

$$\frac{\kappa^i(t)}{A_i(t)} = c_i(t)\sigma_{c_i}(t), \qquad i = 1, 2, \tag{121}$$

$$\frac{r(t)}{A_i(t)} = c_i(t)\mu^i_{c_i}(t) - \frac{1}{2}P_i(t)c_i(t)^2\sigma_{c_i}(t)^2, \tag{122}$$

where c_i satisfies $dc_i(t) = c_i(t)\mu^i_{c_i}(t)dt + c_i(t)\sigma_{c_i}(t)d\omega^i(t)$. Applying Itô's Lemma to both sides of good clearing $c_1(t) + c_2(t) = \delta(t)$ and equating the diffusion and drift terms (and accounting for the investors' differences in innovation processes) yields

$$c_1(t)\sigma_{c_1}(t) + c_2(t)\sigma_{c_2}(t) = \varepsilon(t)\sigma_\varepsilon(t), \tag{123}$$

$$c_1(t)\mu^1_{c_1}(t) + c_2(t)\mu^2_{c_2}(t) = c_1(t)\sigma_{c_1}(t)\frac{\mu^1_\varepsilon(t)}{\sigma_\varepsilon(t)} + c_2(t)\sigma_{c_2}(t)\frac{\mu^2_\varepsilon(t)}{\sigma_\varepsilon(t)}. \tag{124}$$

Summing (121) over investors, using (123) and $\kappa^1(t) - \kappa^2(t) = \bar{\mu}(t)$ yields the perceived market prices of risk expression (30). To obtain the equilibrium interest rate (31), sum (122) over investors, use (124), (14), and (32)–(33), and further manipulate. *Q.E.D.*

Proof of Proposition 3. Substitution of CRRA preferences into Propositions 1–2 yields the desired results. *Q.E.D.*

Proof of Proposition 4. Applying Itô's Lemma to $u_i'(c_i(t)) = y_i\xi^i(t)$, where ξ^i satisfies (52), and matching the diffusion and drift terms yields

$$\frac{\kappa^i(t)}{A_i(t)} = c_i(t)\sigma_{c_i}(t), \quad \frac{\kappa_z^i(t)}{A_i(t)} = c_i(t)\sigma_{c_i z}(t), \qquad i = 1, 2, \tag{125}$$

$$\frac{r(t)}{A_i(t)} = c_i(t)\mu_{c_i}^i(t) - \frac{1}{2}P_i(t)c_i(t)^2\sigma_{c_i}(t)^2 - \frac{1}{2}P_i(t)c_i(t)^2\sigma_{c_i z}(t)^2, \tag{126}$$

where c_i satisfies $dc_i(t) = c_i(t)\mu_{c_i}^i(t)dt + c_i(t)\sigma_{c_i}(t)d\omega(t) + c_i(t)\sigma_{c_i z}(t)d\omega_z^i(t)$. Applying Itô's Lemma to good clearing $c_1(t) + c_2(t) = \delta(t)$ yields

$$c_1(t)\sigma_{c_1}(t) + c_2(t)\sigma_{c_2}(t) = \varepsilon(t)\sigma_\varepsilon(t), \tag{127}$$

$$c_1(t)\sigma_{c_1 z}(t) + c_2(t)\sigma_{c_2 z}(t) = 0, \tag{128}$$

$$c_1(t)\mu_{c_1}^1(t) + c_2(t)\mu_{c_2}^2(t) = \varepsilon(t)\mu_\varepsilon(t) - c_2(t)\sigma_{c_2 z}(t)\bar{\mu}_z(t). \tag{129}$$

Summing (125) over investors, using (127)–(128) and (53)–(54) yields the market prices of fundamental and nonfundamental risk expressions in (55)–(56). Summing (126) over investors, using (129) and equilibrium market prices of risk (55)-(56), and further manipulating leads to the equilibrium interest rate (57). *Q.E.D.*

Proof of Proposition 5. The proof is an L-dimensional version of Proposition 4. *Q.E.D.*

Proof of Proposition 6. The proof is an N-investor version of Proposition 2. *Q.E.D.*

Proof of Proposition 7–8. The proofs are variations of Propositions 2 and 4, additionally accounting for the price of the second good. *Q.E.D.*

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