DISCUSSION PAPER SERIES

No. 4179

SPEED AND QUALITY OF COLLECTIVE DECISION-MAKING, I: IMPERFECT INFORMATION PROCESSING

Elisabeth Schulte and Hans Peter Grüner

INDUSTRIAL ORGANIZATION and PUBLIC POLICY

Centre for Economic Policy Research

www.cepr.org

Available online at: **www.cepr.org/pubs/dps/DP4179.asp**

ISSN 0265-8003

SPEED AND QUALITY OF COLLECTIVE DECISION-MAKING, I: IMPERFECT INFORMATION PROCESSING

Elisabeth Schulte, Universität Mannheim
Hans Peter Grüner, Universität Mannheim and CEPR

Discussion Paper No. 4179
February 2004

Centre for Economic Policy Research
90–98 Goswell Rd, London EC1V 7RR, UK
Tel: (44 20) 7878 2900, Fax: (44 20) 7878 2999
Email: cepr@cepr.org, Website: www.cepr.org

CEPR Discussion Paper No. 4179

January 2003

ABSTRACT

Speed and Quality of Collective Decision-Making, I: Imperfect Information Processing*

A group of P identical managers has to make a choice between N alternatives. They benefit from reaching the decision quickly. In order to learn which is the best option, the alternatives have to be compared. A manager is able to identify the better one of two alternatives only with a certain probability. This Paper compares three different hierarchy designs with respect to decision quality: two strictly-balanced hierarchies and the fastest hierarchy, which is the skip-level reporting tree proposed by Radner (1993). The latter hierarchy design is found to outperform the two others not only in terms of speed and cost but also in terms of decision quality.

JEL Classification: D23, D70, D83, L22 and P51
Keywords: bounded rationality, hierarchies and information processing

Elisabeth Schulte
Department of Economics
Universität Mannheim
68131 Mannheim
GERMANY
Email: elisabeth.schulte@vwl.uni-mannheim.de

Hans Peter Grüner
Lehrstuhl für Wirtschaftspolitik
Universität Mannheim
D-68131 Mannheim
GERMANY
Tel: (49 621) 189 1886
Fax: (49 621) 189 1884
Email: hgruener@rumms.uni-mannheim.de

For further Discussion Papers by this author see:
www.cepr.org/pubs/new-dps/dplist.asp?authorid=144072

For further Discussion Papers by this author see:
www.cepr.org/pubs/new-dps/dplist.asp?authorid=160022

*We thank summer school participants at Hebrew University, especially Timothy Van Zandt, seminar participants at the 2002 Silvaplana Workskop on Political Economy, seminar participants at the Theory Workshop and the Workshop on International Negotiations, University of Mannheim and seminar participants at the ZEI conference on Federalism, University of Bonn, for helpful comments and suggestions. Elisabeth Schulte acknowledges financial support from the Deutsche Forschungsgemeinschaft research group on the institutionalization of international negotiation system. This Paper is produced as part of a CEPR Research Network on 'Product Markets, Financial Markets and the Pace of Innovation in Europe', funded by the European Commission under the Research Training Network Programme (Contract No: HPRN-CT-2000-00061).

Submitted 18 November 2003

1 Introduction

In a seminal paper Radner (1993) studies the design of hierarchical structures when information processing takes time. Radner departs from the conventional assumption that individuals process information in¯nitely fast. This leads him to propose a hierarchical structure - which he calls a reduced tree - within which information is processed at maximum speed. The virtue of the reduced tree is that processors on all levels simultaneously process information. This reduces the delay of the entire information processing procedure. Radner's model can be applied to any information processing problem that requires repetitions of associative and commutative operations. One is the "max"-operation used in the collective decision making problem which we consider in this paper. Information processing in this case concerns the pairwise comparison of the possible alternatives and the identi¯cation of the best one.

Radner's analysis focuses on the e±cient structure of a hierarchical system. By e±ciency it is meant that given the number of processors involved the delay cannot be reduced, and at the same time the given delay cannot be achieved with a smaller number of processors. Compared to the reduced tree a hierarchy that is "strictly balanced" - in the sense that each manager's immediate subordinates are working on the next lower level only, and on a given level each manager has the same number of subordinates - is found to do worse in both, speed and the number of managers involved in processing the information.

In this paper, we add a new dimension for the evaluation of hierarchies: the quality of a decision. In Radner's original paper, which draws on a model brought forward by computer scientists (e.g. Gibbons and Rytter, 1988), it is assumed that processors work perfectly when they perform their task. But, in many real life situations individuals may make mistakes. Hence, in our analysis we study a hierarchy which is composed of agents with imperfect calculation ability. The evaluation of a hierarchy therefore focuses on three dimensions: (i) the speed as well as (ii) the cost of information processing (i.e. the number of agents involved), and (iii) the quality of the decision.

We consider the project selection example proposed by Radner (1993). It is the task of the hierarchy to select one item out of a class of N items. This corresponds to the problem of project selection in a firm where a group of managers evaluates and compares investment projects. Our mathematical analysis focuses on two measures: the probability that the best and the probability that the worst object is chosen by the hierarchy. We take these to be our measures of quality.

We compare the performance of the reduced tree to two alternative strictly balanced hierarchy designs. One is the steepest possible balanced hierarchy which we call the "2^T-Tree" and the other one is the flattest possible hierarchy: the "centralized tree". The 2^T-Tree is the strictly balanced hierarchy with the maximum number of hierarchy levels: every manager except those working on the lowest level has exactly two immediate subordinates. In the centralized tree there is one top manager and all the other agents are his immediate subordinates.

Our main result is that the reduced tree outperforms the two alternative hierarchy designs in terms of decision quality for any number of data items, any number of processors and any mistake making potential. Thus, the reduced tree dominates the balanced hierarchies not only on the dimensions speed and cost, as found by Radner (1993), but also on the third dimension decision quality. Moreover, we come to the conclusion that, given the hierarchy design, decision quality can be enhanced by adding hierarchy levels to the reduced tree as well as to the 2^T-Tree. Adding a hierarchy level has the additional effect to speed up information processing in these hierarchies. For the centralized tree, there exists an optimal number of information processing agents given the number of data items to be processed. Again, this number guarantees maximum speed given that information processing takes place in the centralized tree. Hence, there exists no trade-o® between speed and quality in hierarchy design.

The paper is structured as follows. The next section brie°y reviews the related literature. In section 3, we introduce the basic features of the three considered hierarchy designs and in Section 4 we illustrate the hierarchy's problem. We compare the proposed hierarchies with respect to decision quality in Section 5 and draw conclusions for the relationship between speed and quality in Section 6. Section 7 concludes.

2 Related Literature

This paper draws heavily on the work by Radner (1992, 1993). Assuming that information processing takes time and can be decentralized, Radner derives the minimum time in which N data items can be processed with P agents. He shows that a hierarchy is able to perform that task, namely the reduced tree which we will introduce in the next section. We take this design as given and compare its performance to that in two forms of strictly balanced hierarchies assuming that calculations involve mistakes with a certain probability.

The reduced tree is designed for one-shot problems (to which we restrict attention), i.e. there is only one set of data to be processed, or the processing of the data is ¯nished before another calculation task occurs. There are several contributions to the question how to design a hierarchy, or more generally a network of agents, when this is not the case, i.e. when new data comes in before the processing of the old set is ¯nished (e.g. Van Zandt (1997, 1998); Meagher, Orbay and Van Zandt (2001)).

Meagher and Van Zandt (1998) modify Radner's work with respect to the payment of managers and Orbay (2002) adds the frequency with which new data arrives as a new dimension to the analysis of e±cient hierarchies (which are restricted to be stationary). Prat (1997) studies hierarchies in which a manager's ability is heterogenous, i.e. some managers are able to work faster than others, and the wage a manager is paid is a function of his ability. It turns out that with these modi¯cations - except for the one made by Prat (1997) - the reduced tree is still (close to) e±cient.

To our knowledge, all previous models of information processing in hierarchies consisting of human beings treat the information processing agents more or less as machines, doing what they are programmed to do. The value added of this paper is to take into account that human beings may make mistakes.

In that regard, the problem studied in this paper has certain similarities to that in Sah and Stiglitz (1986). They study the relative performance of two economic systems, namely a hierarchy and a polyarchy, when agents make mistakes in the assessment of projects. They ¯nd that a hierarchy is less likely to accept bad projects as well as good projects than the polyarchy, because the polyarchy gives a second chance to rejected projects and the hierarchy performs a second test on accepted projects. Whereas Sah and Stiglitz assume agents to use some benchmark for the assessment of projects (which entails a mistake with a certain probability) and to undertake any number of projects they want, the agents' objective in our model is to choose exactly one object out of a given set, trying to ¯nd the best one using pairwise comparisons (which involve mistakes with a certain probability).

3 Hierarchy designs

There are N data items that have to be aggregated by P agents called managers. Managers are endowed with an inbox, a processing unit and a memory. Information processing can be done in a decentralized manner, i.e. the calculations are repetitions of associative and commutative operations on the items. Thus, one can use a hierarchy to process the information. A manager's partial result is processed by his superior in the same way as raw data. The top manager's output is the final result. In this section, we will present three possibilities to design such a hierarchy: the reduced tree, the 2^T-Tree and the centralized tree, which are interesting for the following reasons.

The reduced tree is the one that works fastest and therefore deserves special attention in a framework analyzing speed and quality in collective decision making. Balanced trees on the other hand are the most natural form to think of when delegation seems to be useful: If there is need for delegation, one could expect that the delegation occurs in form of a division of the task into equal subtasks. The same intuition should hold for further delegation. We restrict attention to two polar forms of balanced trees: the flattest one (centralized tree) and the steepest one (2^T-Tree).

In order to conduct our analysis in a meaningful manner, we have to restrict the class of hierarchies even a little bit further: We only allow for structures that are perfectly symmetric, which basically means that all agents processing raw data in a given hierarchy process the same number of raw data items, and that an enhancement of a hierarchy (by adding processors) does not change the structure of the hierarchy.

3.1 The reduced tree

For processing information in minimum time, Radner (1993) proposes the following hierarchy design:

Number the managers subsequently from 1 to P and assign $\frac{N}{P}$ objects to each manager. (If $\frac{N}{P}$ is no integer, assign the largest integer smaller than $\frac{N}{P}$ to each manager and another one to the first $N \bmod P$ ones.)

After $\frac{N}{P}$ units of time,[1] each of the P managers has reduced the information in his inbox to a single object. Assign manager i's remaining object to $(i - 1)$'s inbox for each even i. Manager $(i - 1)$ therewith becomes i's immediate superior and the

[1] As outlined above, we assume for simplicity that this is an integer.

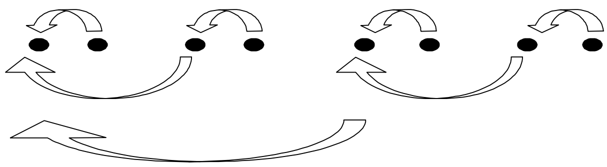

Figure 1: Construction of a reduced tree: Each even i is direct subordinate of $i-1$.

number of managers still working is reduced by half. Renumber these managers in an appropriate manner and repeat the procedure until a single manager remains (the top manager). This procedure of constructing a reduced tree is illustrated in Figure 1, and Figure 2 depicts a reduced tree. The top manager has $\log_2(P)$ immediate subordinates.[2] The N objects are reduced to one in $\frac{N}{P} + \log_2(P)$ units of time.

3.2 Strictly Balanced Trees

We now turn to the strictly balanced trees which are characterized by two properties: (i) each manager's immediate subordinates are at the next lower level and (ii) on a given level, each manager has the same number of subordinates. Deviating from Radner's definition, we extend the second property to raw data: all managers at the lowest level process the same amount of data.

We characterize two balanced tree designs: the steepest hierarchy one can build with P managers which we call "2^T-Tree" and the flattest possible balanced hierarchy which we call the "centralized tree".

[2] Again for simplicity, we assume that P is a power of 2.

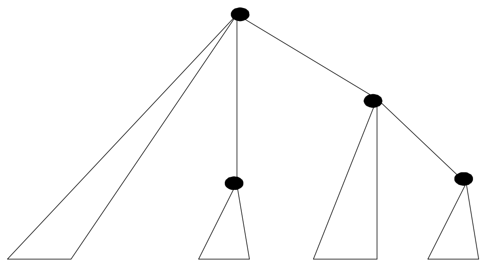

Figure 2: Reduced tree (A triangle represents a group of $\frac{N}{P}$ objects.)

3.2.1 The 2^T-Tree

The 2^T-Tree is the largest balanced hierarchy one can construct with P managers. We restrict attention to strictly balanced structures: Each manager except those working on level one has exactly two immediate subordinates. Thus, on each level, twice as much agents are working as on the next higher level. This implies that in a 2^T-Tree with P managers, there are $\frac{P+1}{2}$ managers working on level one. The number of levels in a 2^T-Tree is T. Note that in order to ensure the symmetric structure described above the 2^T-Tree requires an uneven number of managers, whereas the reduced tree works with a number of managers that is a power of 2, i.e. $P = 2^a; a \in \mathbb{N} \setminus \{0; 1\}$. To be precise, the 2^T-Tree works with exactly one manager less or $2^a - 1$ managers more than the reduced trees with the number of managers closest to it. We will take this into account when comparing these two hierarchy designs. We consider a 2^T-Tree with $2^a - 1$ managers. The number of levels T of such a tree is defined by $P = 2^a - 1 = \sum_{i=0}^{T-1} 2^i$; which gives us $T = a$.[3] A 2^T-Tree produces the final result in

[3] In the corresponding reduced tree with 2^a managers, we have $a + 1$ levels if we assume that the 2^T-Tree uses one manager less than the Radner-Tree. When looking at the next bigger 2^T-Tree, we

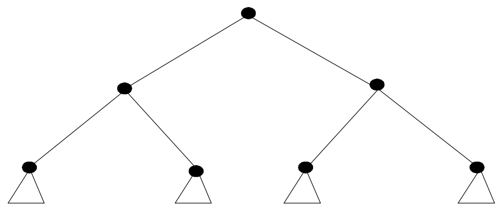

Figure 3: 2^T-Tree

$\frac{2N}{P+1} + 2(T - 1) = \frac{N}{2^{a-1}} + 2(a - 1)$ units of time.

3.2.2 The centralized tree

In the centralized tree, the $P - 1$ level one-managers process $\frac{N}{P-1}$ items each in $\frac{N}{P-1}$ units of time (we assume for simplicity that this is an integer) and send their remaining item to the top managers' inbox. Thereafter, the top manager needs again $P - 1$ units of time to produce the final decision. Thus, processing N data items takes $\frac{N}{P-1} + P - 1$ units of time in the centralized tree.

3.3 Restrictions on the parameters

In order to facilitate the comparison of the performance of the three considered hierarchy designs, we make the following assumptions:

Assumption 1: $N \bmod P = 0$.

Assumption 2: $N \bmod (P - 1) = 0$

Assumption 3: $\log_2(P) = a; a \in \mathbb{N}\setminus\{0, 1\}$:

get $T = a + 1$:

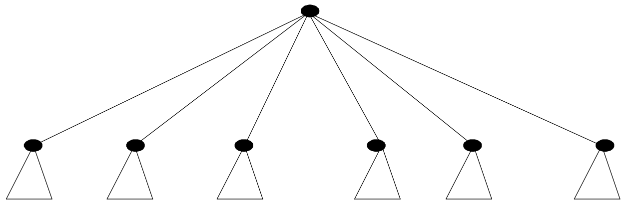

Figure 4: Centralized tree

That is, we restrict our analysis to a subset of all possible combinations of N and P. These are those that ensure that the hierarchies have the symmetric structures described above. Note that this subset is still unbounded. Taken together, all three assumptions imply that N is a multiple of the least common multiple of P and $(P - 1)$. Hence, $N \in N_m \equiv \{m; 2m; 3m; \ldots\}$, where $m = 2^a(2^a - 1)$:

4 The Model

4.1 Set-up

The hierarchy's task is to choose the best object out of a set of N alternatives, i.e. to finance the most promising project, to hire the fittest worker or to buy inputs of highest quality. The $i = 1 \ldots N$ alternatives differ only with respect to quality x_i which is not known to the agents. The vector of quality $x \in R^N$ that describes the set of alternatives is drawn according to some joint density function $\Phi(x)$. In order to learn which alternative is the best one, a manager compares the objects pairwise. He keeps the one he assesses to be the better one in his memory (in which he can store one object at most) and the other one is taken out of the set of possible alternatives. We assume that all managers have identical monotonous preferences regarding quality.

In contrast to Radner's work, we consider the case in which the managers' ability to process information is not perfect. With probability q, the agent is mistaken and accidentally deletes the better one of two objects.[4] Managers are identical with respect to calculation ability.

The processing works as follows: The objects are assigned equally and randomly to the managers' inboxes on the lowest level. In one unit of time, a manager can take an object out of his inbox into his processing unit, evaluate it and compare it to the object in his memory. He stores the object he assesses to be the better one in his memory and deletes the worse one. This assessment is correct with probability (1 ¡ q). The remaining object is sent to the immediate superior's inbox. At the beginning of the procedure, the memory is empty, such that the manager just keeps the ¯rst object and makes no mistake in this case. The top manager's output is the ¯nal decision.

4.2 Quality measures

Radner (1993) has shown that a reduced tree will come to a decision faster than any balanced hierarchy. We are concerned with the additional question which hierarchy design produces the decision of higher quality. The de¯nition of the term "quality" follows below.

We do not introduce any assumptions about the utility the hierarchy derives from a certain quality except that higher quality is better. Instead, we focus on the two extreme outcomes, which are of relevance for all continuous quality distributions

[4]Another interpretation of the assumption that agents make mistakes is that they receive an imperfect signal about which object is better suited to ¯t their needs.

©(x) and for all Von Neumann-Morgenstern utility functions: The event, that the hierarchy picks the best quality and the event that the worst quality is chosen. Since it is the hierarchy's task to ¯nd the best alternative, it is natural to de¯ne quality in terms of the probability that the best (worst) object will be the ¯nal outcome, i.e. as the probability of a success and the probability of a complete failure. Following that logic, we take the probability that the best (worst) item is chosen under the alternative hierarchy designs as measures of quality.

These measures deliver in particular a complete description of situations in which the hierarchy's task is to identify a certain object and there is only one of its kind, e.g. a murderer or a thief (whom one would like to choose as a police department - in this case he represents the best object - and avoid to choose as a recruitment team).

Since the combinatorics for choosing the best or the worst item out of the given set are the same, calculations are analogous - just the interpretation of the probability q changes. In the following, we interpret q as the individual probability of making a mistake when $q < \frac{1}{2}$ and as the complementary probability otherwise.[5] This allows us to represent the probability to choose the best and the probability to choose the worst item by the same formula given the hierarchy design.

5 Decision Quality

5.1 The reduced tree

It is useful to separate the processing task into two phases: the processing of "raw data" by the lowest level managers (level one processing) and the processing of the partial results following. In Lemma 1, we quantify the probability that the worst (best)

[5]We do not need to care about the cases $q = 0$, $q = \frac{1}{2}$ and $q = 1$, because the probability to choose the best (worst) object is the same in any hierarchy, namely 1 (0), $\frac{1}{N}$ and 1 (0) respectively.

object will still be in the set of available options after the first phase of processing N data items with P managers organized as a reduced tree, Prosposition 1 specifies the probability that the worst (best) object will finally be chosen.

Lemma 1 Consider a reduced tree. The probability that the worst (best) object stays in the set of remaining items after the level one managers have processed the initially assigned amount of data is

$$\frac{P}{N}\left(\frac{q}{1-q}\right)\left(1+(1-2q)\,q^{\frac{N}{P}-2}\right): \qquad (1)$$

Proof. Each manager has to process $\frac{N}{P}$ randomly assigned objects. Under the condition, that the worst (best) object is assigned to manager j, any of manager j's objects is the worst (best) one with probability $\frac{P}{N}$.

If it was the first or the second one to be processed, it would have to survive $\frac{N}{P}-1$ comparisons. The third one would have to survive $\frac{N}{P}-2$, the fourth $\frac{N}{P}-3$, etc. The last one would have to survive only one comparison.

Thus, the probability that a manager has kept the worst (best) object in his memory after having processed the initially assigned amount of data is $\frac{P}{N}\left(\sum_{i=1}^{\frac{N}{P}-1} q^i + q^{\frac{N}{P}-1}\right)$, which can (for $q<1$) be simplified to $\frac{P}{N}\left(\frac{q}{1-q}\right)\left(1+(1-2q)\,q^{\frac{N}{P}-2}\right)$: Q.E.D.

It is useful to define the following function:

Definition 1 $r(q;x) := \left(\frac{q}{1-q}\right)(1+(1-2q)\,q^{x-1})$.

Proposition 1 The probability that a hierarchy of P managers organized as a reduced tree chooses the worst (best) object out of a set of N alternatives is

$$p_R(q;N;P) = \frac{P}{N}\, r\left(q; \frac{N}{P}-1\right) q^{\log_2 P}; \qquad (2)$$

independently of the identity of the manager it was initially assigned to.

Proof. Call manager i's immediate superior s(i) and let $s^n(i)$ be i's n^{th} superior. Call the top manager T and let s(T) = T. Let $S_i := \{i \cup k : k = s^n(i); n \leq T\}$ be the ordered set of mangers who are working on the calculation path starting with manager i.

Suppose the worst (best) object is initially assigned to manager j situated on level l_j. He first processes the items initially assigned to him (level one processing) and keeps the object in his memory after finishing this task with probability $\frac{P}{N} r\left[q; \frac{N}{P} - 1\right]$ (see Lemma 1). Next, he has to process the items received from his subordinates. Working on level l_j he has $(l_j - 1)$ of them. Thus, he sends the object to his superior (if he has one) with probability $\frac{P}{N} r\left[q; \frac{N}{P} - 1\right] q^{l_j - 1}$. Recall the structure of the reduced tree: j's immediate superior s(j) situated on level $l_{s(j)}$ has to do another $l_{s(j)} - l_j$ calculations when receiving j's item, so he passes it to his own immediate superior with probability $q^{l_{s(j)} - l_j}$. The same is true for s(j)'s superior $s^2(j)$, etc. up to the top manager. Therefore, the probability that the item passes the whole hierarchy when initially assigned to manager j is

$$\frac{P}{N} r\left[q; \frac{N}{P} - 1\right] q^{l_j - 1} \prod_{\{k \in S_j\}} q^{l_{s(k)} - l_k} = \frac{P}{N} r\left[q; \frac{N}{P} - 1\right] q^{l_T - 1}.$$

Since the number of levels in a reduced tree with P managers is $\log_2(P) + 1$, this is equal to $\frac{P}{N} r\left[q; \frac{N}{P} - 1\right] q^{\log_2 P}$. Because j and l_j were chosen arbitrarily, this proofs Proposition 1. Q.E.D.

5.2 The 2^T-Tree

Again, we begin with level one processing and state the probability that the worst (best) object will stay in the set of available options after level one processing in a 2^T-Tree processing N items with $2^a - 1$ managers in Lemma 2. Proposition 2 specifies the probability that the worst (best) object will be the final decision.

Lemma 2 The probability that a level one-manager in a 2^T-Tree with $2^a - 1$ managers sends the worst (best) object to his superior is $\frac{2^{a-1}}{N} r(q; \frac{N}{2^{a-1}} - 1)$, if it was initially assigned to him which is the case with probability $\frac{1}{2^{a-1}}$.

Proof. There are 2^{a-1} managers working on level one. The random assignment of items implies that any manager has the worst (best) object with probability $\frac{1}{2^{a-1}}$ and that an item in his inbox is the object with probability $\frac{2^{a-1}}{N}$, given it was assigned to him. The remaining part follows analogously from the proof of Lemma 1 and applying Definition 1. Q.E.D.

Proposition 2 The probability of choosing the worst (best) item out of a set of N items with a 2^T-Tree working with $2^a - 1$ managers is

$$p_{2^T}(q; a; N) = \frac{2^{a-1}}{N} r(q; \frac{N}{2^{a-1}} - 1) q^{a-1}. \tag{3}$$

Proof. The probability that the object passes level one is $\frac{2^{a-1}}{N} r(q; \frac{N}{2^{a-1}} - 1)$ (Lemma 2). Then, the object has to pass another $a - 1$ levels after the first round of processing. On each level, a manager has to perform a single operation on it which entails an error with probability q. Q.E.D.

Note that in the 2^T-Tree the probability that the worst (best) object is chosen is again independent of the identity of the manager to whom it was initially assigned. Because of the symmetry of the structure, this can be easily seen such that a proof can be omitted.

5.3 The reduced tree versus the 2^T-Tree: Quality

Proposition 3 (i) Any reduced tree with P managers chooses the worst item with lower, and the best item with higher probability than the 2^T-Tree with $P - 1$ managers.

(ii) Any reduced tree with P managers chooses the worst as well as the best item with equal probability as a 2^T-Tree with $2P-1$ managers.

Proof. Let Assumptions 1 and 3 hold. $P = 2^a, N = c2^a, c \in \mathbb{N}\setminus\{0,1\}, a \in \mathbb{N}\setminus\{0,1\}$.

(i) To show:

$$p_R(q,P,N) - p_{2^T}(q,P-1,N) > 0 \Leftrightarrow q > \frac{1}{2}.$$

$$
\begin{aligned}
& p_R(q,P,N) - p_{2^T}(q,P-1,N) \\
= \; & \frac{2^a}{N} r\left(q,\frac{N}{2^a}-1\right) q^a - \frac{1}{N} r\left(q,\frac{N}{2^{a-1}}-1\right)(2q)^{a-1} \\
= \; & \frac{1}{N}(2q)^{a-1}\left[r\left(q,\frac{N}{2^a}-1\right)(2q) - r\left(q,\frac{N}{2^{a-1}}-1\right)\right] \\
= \; & \frac{1}{N}(2q)^{a-1}\left(\left(\frac{q}{1-q}\right)\left(1+(1-2q)q^{\frac{N}{2^a}-2}\right)(2q) - \left(\frac{q}{1-q}\right)\left(1+(1-2q)q^{\frac{N}{2^{a-1}}-2}\right)\right) \\
= \; & \frac{1}{N}(2q)^{a-1}\left(\frac{q}{1-q}\right)\left((1-2q)q^{\frac{N}{2^a}-2}(2q) - (1-2q) - (1-2q)q^{\frac{N}{2^{a-1}}-2}\right) \\
= \; & \frac{1}{N}(2q)^{a-1}\left(\frac{q}{1-q}\right)(1-2q)\left(2q^{\frac{N}{2^a}-1} - 1 - q^{\frac{N}{2^{a-1}}-2}\right) \\
= \; & \frac{1}{c2^a}(2q)^{a-1}\left(\frac{q}{1-q}\right)\left((1-2q)(2q^{c-1}-1-q^{2c-2})\right) \\
= \; & \frac{1}{c2^a}(2q)^{a-1}\left(\frac{q}{1-q}\right)\left((1-2q)(-(q^{c-1}-1)^2)\right) \\
= \; & \frac{1}{c2^a}(2q)^{a-1}\left(\frac{q}{1-q}\right)(2q-1)(q^{c-1}-1)^2.
\end{aligned}
$$

$$p_R(q,P,N) - p_{2^T}(q,P-1,N) > 0$$

$\Leftrightarrow q > \frac{1}{2}$, since all factors but $(2q-1)$ are positive $\forall q < 1$.

(ii) The second part of Proposition 3 follows directly from replacing a by $a+1$ in (3). Q.E.D.

The intuition for the second Part of Proposition 3 is simple: The 2^T-Tree with $2^{a+1}-1$ managers is the one out of which one can construct the reduced tree with 2^a managers by subsequently replacing the superior of a cadre by the first manager of that cadre, as indicated in Figure 5. Because there is the same number of agents processing on level one in the reduced tree and in the original 2^T-Tree, the first round of processing is equivalent. Moreover, since the first calculation done by any

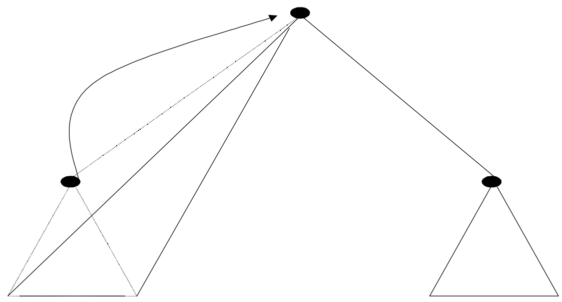

Figure 5: Reduction of a 2^T-Tree

manager does not cause a mistake, the jobs done by the eliminated superior and the new "boss" of the cadre are equivalent, so is the mistake making potential in both hierarchy designs.

Corollary 1 Adding a hierarchy level to a 2^T-Tree, i.e. increasing the number of managers from $2^a - 1$ to $2^{a+1} - 1$; enhances decision quality. Thus, obtaining the highest quality a 2^T-Tree can produce processing N items requires $P = N - 1$.

This in turn implies:

Corollary 2 An enlargement of a reduced tree, i.e. increasing the number of managers from 2^a to 2^{a+1} (adding one hierarchy level), enhances quality. Therefore, to obtain the best quality a reduced tree can produce, one should make it as large as possible, i.e. $P = \frac{N}{2}$.

The rational for this result is as follows. By adding a new hierarchy level to a reduced tree, i.e. increasing the number of managers from 2^a to 2^{a+1}, the number of calculations to be performed by each manager on level one is halved. But there is only one additional calculation along the tree until the decision is made. Because each calculation potentially entails a mistake as few calculations as possible should be performed on the best item.

For the worst object, it will become more likely to stay in the set of alternatives after level one processing when doubling the number of managers (it actually becomes roughly twice as likely, but the probability stays bounded below $\frac{1}{2}$; of course)[6]. But with an additional hierarchy level to pass the probability that it will be the final outcome given it has passed level one is reduced by more than a half, namely to a fraction $q < \frac{1}{2}$. The latter effect outweighs the former.

5.4 The centralized tree

From the previous section we know that the reduced tree outperforms the largest strictly balanced hierarchy as far as quality is concerned. We now turn to the flattest one, the centralized tree. We begin with level one processing (Lemma 3), characterize the probability that the top manager keeps the worst (best) item in Lemma 4 and specify the probability that the centralized tree will choose the worst (best) option in Proposition 4.

Lemma 3 In the centralized tree working with P managers processing N items, a level one-manager keeps the worst (best) item with probability $\frac{P-1}{N} r(q; \frac{N}{P-1} - 1)$; if it was initially assigned to him which is the case with probability $\frac{1}{P-1}$.

Proof. Analogous to the proof of Lemma 1.

---

[6] This follows from $f(x) = 2\frac{1+q^{x-2}(1-2q)}{1+q^{2x-2}(1-2q)} f(2x)$, where $f(x)$ denotes the probability that the item passes level one when x items are processed by each manager on level one. This formula can be derived from Lemma 1 using Definition 1.

Lemma 4 The top manager chooses the worst (best) alternative if it has passed the first round of processing with probability $\frac{1}{P-1} r(q; P-2)$.

Proof. Straightforward.

Both Lemmata immediately yield:

Proposition 4 The ex ante probability that a hierarchy of P managers organized as a centralized tree picks the worst (best) object out of a set of N alternatives is

$$p_C(q; N; P) = \frac{1}{N} r\left(q; \frac{N}{P-1} - 1\right) r(q; P-2). \quad (4)$$

For any number of items N to be processed, we can find a number of information processing agents P, that is optimal with respect to our measures of quality for the centralized tree.

Proposition 5 For the centralized tree, there exists a set of optimal $(N, \hat{P}(N))$-combinations. Given N, $\hat{P}(N)$ maximizes the probability that the best object is chosen and minimizes the probability that the worst object is chosen. It is given by (one of the integers closest to): $\hat{P} = \sqrt{N} + 1$.

Proof. See Appendix.

Note that this number of managers $\hat{P}$ is also the one that maximizes the speed of processing N items in the centralized tree.

5.5 The reduced tree versus the centralized tree: Quality

Proposition 6 (i) The probability that a hierarchy with P managers processing N items chooses the worst item is always greater when it is organized as a centralized tree than if it is a reduced tree. (ii) The probability that a hierarchy with P managers processing N items picks the best item is always greater when it is organized as a reduced tree than if it is a centralized tree.

Proof. See Appendix.

The intuition for this result could be the following: The processing on level one (in terms of objects processed per level one-manager) is nearly the same in both hierarchy designs (at least for large P). The advantage of the reduced tree seems to be that it processes each item more equally, whereas the centralized tree processes some items a lot and others a little. Since ex ante each item is the searched one with equal probability, it is better to processes on the items as equally as possible. This reduces the risk to process the best object a lot as well as the risk to process the worst one a little.

5.6 The 2^T-Tree versus the centralized tree

In this subsection we use a numerical example to show that there is no such clear dominance result when comparing the two balanced hierarchies as those we obtained in the previous subsections.

We compare the 2^T-Tree to the centralized tree in the following parameter constellation:

$P = 7; N = 12c; c \in \mathbb{N} \setminus \{0\}$. These are the smallest numbers for which the two designs differ and the integer restrictions hold.

In the 2^T-Tree, there are 4 level one managers processing 3c items each and the other three managers are aggregating two items each, whereas in the centralized tree, there are 6 level one managers who are aggregating 2c data items each and the top managers does 6 calculations.

We measure the relative performance of the hierarchy designs by the difference in the probability that the best (worst) object is chosen by the respective hierarchy: $D(q;c) := p_{2^T}(q;c) - p_C(q;c)$:

$$D(q;c) = \frac{1}{12c}\left(r(q;(3c-1))\,(2q)^2 - r(q;2c-1)\,r(q;5)\right):$$

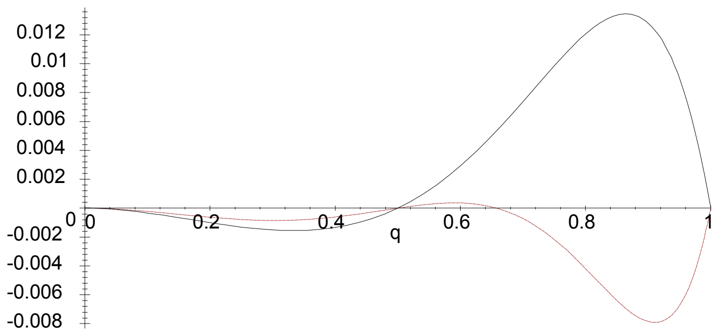

Figure 6: $D(q;2)$ (solid) and $D(q;3)$ (dashed):

Figure 6 plots the performance measure $D(q;c)$ for $c = 2$ and $c = 3$. Negative values for $D(q;c)$ indicate that the centralized tree chooses the worst (best) item with a higher probability than the 2^T-Tree. As Figure 6 shows, for some parameter constellations it is better to organize the hierarchy as a 2^T-Tree, whereas for others it might be better to use the centralized tree. The reason is that for a greater set of data to be processed, the centralized tree has a greater probability to keep the best item in the set after level one processing, because managers on level one work less in the centralized tree.

Since we already know that the reduced tree performs better than both of the balanced trees, we do not attempt to elaborate on their relative performance explicitly.

6 Speed versus Quality

The results on the decision quality produced by the di®erent hierarchy designs as well as the results concerning the comparison of the hierarchy structures seem to indicate that a hierarchy designer does not face a trade-o® between speed and quality. At least within the class of hierarchies analyzed in this paper and for our measures of decision quality, there exists no such trade-o®.

We identi¯ed the reduced tree to dominate the two considered balanced tree designs with respect to quality of decision making. Radner (1993) already showed that the reduced tree dominates the class of balanced trees with respect to speed and cost of decision making.

Moreover, within the classes of analyzed trees we found that the fastest ones also produce highest quality. Thus, we can have both: Speed and quality in collective decision making.

7 Conclusion

We have compared the performance of the reduced tree proposed by Radner (1993) to the one in two strictly balanced hierarchies (the steepest and the °attest one) when the managers' ability to process information is not perfect. It turned out that Radner's reduced tree produces decisions of higher quality for all parameter constellations.

As the reduced tree is a hierarchy in which information processing takes minimum time (Radner, 1993), there is no trade-o® between speed and quality in hierarchy design for information processing.

Moreover, we found that within the class of the analyzed hierarchy designs, the highest quality can be obtained by the largest reduced tree (which is again the fastest of its class). The largest reduced tree works with half as much agents as items to be processed.

Although we did not derive the optimal hierarchy design, we were able to present the unambiguously best of the three considered ones. More research is needed in order to characterize the set of efficient hierarchy designs on the dimensions speed, cost and quality.

There are several useful extensions of our framework. First of all, it would be desirable to consider more general measures of decision quality, not only the probability of extreme outcomes. However, one would have to introduce a specific Von Neumann-Morgenstern utility function as well as assumptions on the distribution function of quality $\Phi(x)$ in order to derive results on other measures of quality than the ones considered in this paper. Our results hold no matter which form the utility function takes or how quality is distributed.

In this paper, mistake making by agents was introduced into the analysis of decentralized information processing by restricting the calculation ability of agents in an intuitive, but rather simplifying manner. In our set-up, agents' mistakes do not depend on the intensity of difference between the two compared items. Intuitively, mistake making should depend on the task to be performed, i.e. comparing the very best to the very worst item should entail a smaller mistake making potential than the comparison of items with rather equal quality. To incorporate this consideration into our model, one could make use of probabilistic choice models, such as Luce (1959). Again, this modification would require to specify the quality distribution function $\Phi(x)$ as well as a utility function.

Finally, we assumed that agents cannot influence the individual probability of making a mistake through effort. This issue is addressed in Grüner and Schulte (2003).

Appendix

proof of Proposition 5

$$\frac{\partial p_C}{\partial P} = \frac{q^2}{(1-q)^2}(\ln q)(1-2q)q^P \left((1-2q)\,q^{\frac{N}{P-1}-5}\left(\frac{1}{N}-\frac{1}{(P-1)^2}\right) + q^{-3}\left(\frac{1}{N}-\frac{q^{\frac{N}{P-1}-P+1}}{(P-1)^2}\right)\right):$$

The first order condition for an optimum is:

$$\frac{\partial p_C}{\partial P} = 0 \;\Leftrightarrow\; P = \sqrt{N}+1 \;\Leftrightarrow\; N = (P-1)^2:$$

$$\left.\frac{\partial^2 p_c}{\partial P^2}\right|_{P=\sqrt{N}+1} = \frac{(1-2q)2q^{\sqrt{N}}(\ln q)}{(1-q)^2\left(\sqrt{N}\right)^2}\left((\ln q) + \frac{\left(1+q^{\sqrt{N}-2}(1-2q)\right)}{\sqrt{N}}\right):$$

The second factor is always negative.[7]

The first factor is negative iff $q < \frac{1}{2}$. Therefore, for $(N;P)$ combinations that satisfy $N = (P-1)^2$; p_c is maximized (minimized) for $q > (<)\frac{1}{2}$.

Q.E.D.

proof of Proposition 6

To show:

$$p_R(q;P;N) - p_C(q;P;N) > 0 \;\Leftrightarrow\; q > \tfrac{1}{2};$$

with

$$p_R(q;P;N) = \frac{P}{N}\left(\frac{q}{1-q}\right)\left(1+(1-2q)q^{P-3}\right)q^{\log_2 P};$$

$$p_C(q;P;N) = \frac{1}{N}\left(\frac{q}{1-q}\right)^2\left(1+(1-2q)q^{\frac{N}{P-1}-2}\right)\left(1+(1-2q)q^{P-3}\right);$$

$$P = 2^a; a \in \mathbb{N}\setminus\{0;1\};$$

$$N \in \{c2^a(2^a-1)\}; c \in \mathbb{N}\setminus\{0\}:$$

We prove Proposition 6 in two steps. First, we show that it holds for the upper bound of the parameter set, i.e. for all combinations $(P;N)$ with the smallest number of data to be processed. This is the set $\{(P;N) : P = 2^a; N = 2^a(2^a-1)\}$. Step 2 is to show that it holds for the whole parameter set via induction over N.

---

[7] Note that the second summand is falling in N. Therefore, it suffices to check the sign for the smallest number of $N : N = 4$. $\ln q < -\frac{(1+(1-2q))}{2} \Leftrightarrow \ln q < q-1 \Leftrightarrow \ln q + 1 < q$, which holds for $q < 1$, because for $q = 1 : \ln q + 1 = q$ and the LHS has a higher slope as the RHS $\forall q < 1$.

Step 1:

To show:

$$p_R(q;2^a;2^a(2^a-1)) - p_C(q;2^a;2^a(2^a-1)) > 0 \quad \Leftrightarrow \quad q > \tfrac{1}{2}.$$

$$\begin{aligned}
& p_R(q;2^a;2^a(2^a-1)) - p_C(q;2^a;2^a(2^a-1)) \\
= {} & \tfrac{1}{2^a-1}\left(\tfrac{q}{1-q}\right)\left(1+(1-2q)\,q^{2^a-3}\right)q^a \\
& - \tfrac{1}{2^a(2^a-1)}\left(\tfrac{q}{1-q}\right)^2\left(1+(1-2q)\,q^{2^a-2}\right)\left(1+(1-2q)\,q^{2^a-3}\right) \\
= {} & \tfrac{1}{2^a-1}\left(\tfrac{q}{1-q}\right)\left(1+(1-2q)\,q^{2^a-3}\right)\left(q^a - \tfrac{1}{2^a}\left(\tfrac{q}{1-q}\right)\left(1+(1-2q)\,q^{2^a-2}\right)\right).
\end{aligned}$$

$$\begin{aligned}
& p_R(q;2^a;2^a(2^a-1)) - p_C(q;2^a;2^a(2^a-1)) > 0 \\
\Leftrightarrow {} & q^a - \tfrac{1}{2^a}\left(\tfrac{q}{1-q}\right)\left(1+(1-2q)\,q^{2^a-2}\right) > 0 \\
\Leftrightarrow {} & (2q)^a - \left(\tfrac{q}{1-q}\right)\left(1+(1-2q)\,q^{2^a-2}\right) > 0.
\end{aligned}$$

We will show that the LHS is increasing in a for $q > \frac{1}{2}$, and decreasing in a for $q < \frac{1}{2}$. Thus, it suffices to show that the inequality holds for the smallest number for a iff $q > \frac{1}{2}$.

$$\text{LHS} := f(a) = (2q)^a - \left(\tfrac{q}{1-q}\right)\left(1+(1-2q)\,q^{2^a-2}\right).$$

$$\begin{aligned}
& f(a+1) - f(a) \\
= {} & \left((2q)^{a+1} - \left(\tfrac{q}{1-q}\right)\left(1+(1-2q)\,q^{2^{a+1}-2}\right)\right) - \left((2q)^a - \left(\tfrac{q}{1-q}\right)\left(1+(1-2q)\,q^{2^a-2}\right)\right) \\
= {} & (2q)^{a+1} - (2q)^a + \left(\tfrac{q}{1-q}\right)(1-2q)\left(q^{2^a-2} - q^{2^{a+1}-2}\right) \\
= {} & (2q)^a(2q-1) + \left(\tfrac{q}{1-q}\right)(1-2q)\,q^{2^a}\left(1-q^{2^a}\right)q^{-2} \\
= {} & (2q)^a(2q-1) + \left(\tfrac{q}{1-q}\right)(1-2q)\,q^{2^a}\left(1-q^{2^a}\right)q^{-2} \\
= {} & (1-2q)\left(\left(\tfrac{1}{(1-q)q}\right)q^{2^a}\left(1-q^{2^a}\right) - (2q)^a\right) \\
= {} & (1-2q)\left(\tfrac{q^{2^a-1}}{(1-q)}\prod_{i=1}^{a}\left(1+q^{2^{a-i}}\right)(1-q) - (2q)^a\right) \\
= {} & (1-2q)\left(q^{2^a-1}\prod_{i=1}^{a}\left(1+q^{2^{a-i}}\right) - (2q)^a\right).
\end{aligned}$$

$f(a)$ is increasing (decreasing) in a for $q > \frac{1}{2}$ $\left[q < \frac{1}{2}\right]$ if

$$\begin{aligned}
q^{2^a-1}\prod_{i=1}^{a}\left(1+q^{2^{a-i}}\right) - (2q)^a &< 0 \\
&\Leftrightarrow (2q)^a > q^{2^a-1}\prod_{i=1}^{a}\left(1+q^{2^{a-i}}\right) \\
&\Leftrightarrow a\ln(2q) > (2^a-1)\ln q + \sum_{i=1}^{a}\ln\left(1+q^{2^{a-i}}\right) \\
&\Leftarrow a\ln(2q) > (2^a-1)\ln q + a\ln 2 \\
&\Leftrightarrow a\left(\ln(2q) - \ln 2\right) > (2^a-1)\ln q \\
&\Leftrightarrow a\ln q > (2^a-1)\ln q \\
&\Leftrightarrow a < (2^a-1);
\end{aligned}$$

which holds $\forall a$.

It remains to be shown that $f(a = a^{min} = 2) > 0 \Leftrightarrow q > \frac{1}{2}$:

$$\begin{aligned}
f(2) &= (2q)^2 - \left(\frac{q}{1-q}\right)\left(1+(1-2q)q^2\right) = q(2q-1)(1-q). \\
f(2) &> 0 \Leftrightarrow q > \tfrac{1}{2}.
\end{aligned}$$

Step 2:

Let Proposition 6 hold for $P = 2^a, N = c2^a(2^a-1)$:

$$p_R(q; 2^a; c2^a(2^a-1)) - p_C(q; 2^a; c2^a(2^a-1)) > 0 \Leftrightarrow q > \frac{1}{2}.$$

Equivalently:

$$(2q)^a > \left(\frac{q}{1-q}\right)\frac{\left(1+(1-2q)q^{c2^a-2}\right)}{\left(1+(1-2q)q^{c(2^a-1)-2}\right)}\left[1+(1-2q)q^{2^a-3}\right] \Leftrightarrow q > \tfrac{1}{2}. \qquad (*)$$

Consider the next possible number of data $N = (c+1)2^a(2^a-1)$:

$$\begin{aligned}
&p_R(q; 2^a; (c+1)2^a(2^a-1)) - p_C(q; 2^a; (c+1)2^a(2^a-1)) > 0 \\
\Leftrightarrow\ &(2q)^a > \left(\frac{q}{1-q}\right)\frac{\left(1+(1-2q)q^{(c+1)2^a-2}\right)}{\left(1+(1-2q)q^{(c+1)(2^a-1)-2}\right)}\left[1+(1-2q)q^{2^a-3}\right] \\
\Leftrightarrow\ &(2q)^a > \begin{pmatrix} \left(\frac{q}{1-q}\right)\frac{\left(1+(1-2q)q^{c2^a-2}\right)}{\left(1+(1-2q)q^{c(2^a-1)-2}\right)}\left[1+(1-2q)q^{2^a-3}\right] \\ \times\frac{1+(1-2q)q^{c(2^a-1)-2}}{\left(1+(1-2q)q^{c2^a-2}\right)}\frac{\left(1+(1-2q)q^{(c+1)2^a-2}\right)}{\left(1+(1-2q)q^{(c+1)(2^a-1)-2}\right)} \end{pmatrix}.
\end{aligned}$$

A sufficient condition for the Proposition to hold is because of $(*)$:

$$\left(\frac{1+(1-2q)q^{c(2^a-1)-2}}{1+(1-2q)q^{c2^a-2}}\right)\frac{\left(1+(1-2q)q^{(c+1)2^a-2}\right)}{\left(1+(1-2q)q^{(c+1)(2^a-1)-2}\right)} \leq (\geq)\, 1 \text{ for } q > (<)\, \tfrac{1}{2}.$$

We will show that this holds in two Substeps.

Substep a: For $q < \frac{1}{2}$:

$$\left(\frac{1+(1-2q)q^{c(2^a-1)-2}}{1+(1-2q)q^{c2^a-2}}\right)\frac{\left(1+(1-2q)q^{(c+1)2^a-2}\right)}{\left(1+(1-2q)q^{(c+1)(2^a-1)-2}\right)} > 1$$

$$\Leftrightarrow \left(1+(1-2q)\,q^{c(2^a-1)-2}\right)\left(1+(1-2q)\,q^{(c+1)2^a-2}\right)$$

$$> \left(1+(1-2q)\,q^{c2^a-2}\right)\left(1+(1-2q)\,q^{(c+1)(2^a-1)-2}\right)$$

$$\Leftrightarrow (1-2q)\,q^{c(2^a-1)-2} + (1-2q)\,q^{(c+1)2^a-2} + (1-2q)^2\,q^{(c+1)2^a-2+c(2^a-1)-2}$$

$$- (1-2q)\,q^{c2^a-2} - (1-2q)\,q^{(c+1)(2^a-1)-2} - (1-2q)^2\,q^{(c+1)(2^a-1)-2+c2^a-2} > 0$$

$$\Leftrightarrow \frac{(1-2q)}{q^2}\begin{pmatrix} q^{c(2^a-1)} + q^{(c+1)2^a} + (1-2q)\,q^{(c+1)2^a+c(2^a-1)-2} \\ -q^{c2^a} - q^{(c+1)(2^a-1)} - (1-2q)\,q^{(c+1)(2^a-1)+c2^a-2} \end{pmatrix}$$

$$\Leftrightarrow \underbrace{\frac{(1-2q)}{q^2}}_{>0\,\Leftrightarrow\, q<\frac{1}{2}}\left(\begin{array}{c} \underbrace{q^{c(2^a-1)} - q^{c2^a}}_{>0} + \underbrace{q^{(c+1)2^a} - q^{(c+1)(2^a-1)}}_{>0} \\ + \underbrace{(1-2q)\left(q^{(c+1)2^a+c(2^a-1)-2} - q^{(c+1)(2^a-1)+c2^a-2}\right)}_{>0\,\Leftrightarrow\, q<\frac{1}{2}} \end{array}\right):$$

Substep b: For $q > \frac{1}{2}$:

$$\left(\frac{1+(1-2q)q^{c(2^a-1)-2}}{1+(1-2q)q^{c2^a-2}}\right)\frac{\left(1+(1-2q)q^{(c+1)2^a-2}\right)}{\left(1+(1-2q)q^{(c+1)(2^a-1)-2}\right)} < 1$$

$$\Leftrightarrow \underbrace{q^{c(2^a-1)} - q^{c2^a}}_{>0} + \underbrace{q^{(c+1)2^a} - q^{(c+1)(2^a-1)}}_{>0}$$

$$+ \underbrace{(1-2q)\left(q^{(c+1)2^a+c(2^a-1)-2} - q^{(c+1)(2^a-1)+c2^a-2}\right)}_{>0\,\Leftrightarrow\, q<\frac{1}{2}} > 0$$

$$\Leftrightarrow q^{c2^a}\left((q^{-c}-1) + q^{2^a}(q^{-c-1}-1) + q^{(c+1)2^a-2}(1-2q)(q^{-2c-1}-1)\right) > 0$$

$$\Leftrightarrow (q^{-c}-1) + q^{2^a}(q^{-c-1}-1) + q^{(c+1)2^a-2}(1-2q)(q^{-2c-1}-1) > 0.$$

Because the LHS is increasing in c:

$$\text{LHS} := g(c) = \left(q^{-c}-1\right) + q^{2^a}\left(q^{-c-1}-1\right) + q^{(c+1)2^a-2}(1-2q)\left(q^{-2c-1}-1\right);$$

$$g'(c) = -\ln q \cdot q^{-c}\left(1 + q^{2^a-1} + (2q-1)\,q^{(c+1)2^a-2}\left((2^a-2)\left(q^{-c-1}-q^{c}\right)+2q^{c}\right)\right) > 0;$$

it is sufficient to check the above inequality for $c = 1$:

$$
\begin{aligned}
& \left(q^{-1}-1\right) + q^{2^a}\left(q^{-2}-1\right) + q^{2^{a+1}-2}(1-2q)\left(q^{-3}-1\right) > 0 \\
\Leftrightarrow\quad & \left(q^{-1}-1\right) + q^{2^a}\left(q^{-1}-1\right)\left(q^{-1}+1\right) + q^{2^{a+1}-2}(1-2q)\left(q^{-3}-1\right) > 0 \\
\Leftrightarrow\quad & \left(q^{-1}-1\right)\left(1+q^{2^a}\left(q^{-1}+1\right)\right) + q^{2^{a+1}-2}(1-2q)\left(q^{-3}-1\right) > 0 \\
\Leftrightarrow\quad & q^{-1}\left[(1-q)\left(1+q^{2^a}\left(q^{-1}+1\right)\right) + q^{2^{a+1}-1}(1-2q)\left(q^{-3}-1\right)\right] > 0 \\
\Leftrightarrow\quad & (1-q)\left(1+q^{2^a-1}(1+q)\right) + q^{2^{a+1}-1}(1-2q)\left(q^{-3}-1\right) > 0 \\
\Leftrightarrow\quad & 1+q^{2^a-1}(1+q) + q^{2^{a+1}-1}\frac{(1-2q)\left(q^{-3}-1\right)}{(1-q)} > 0 \\
\Leftrightarrow\quad & 1+q^{2^a-1}(1+q) - \left(q^2+q+1\right)(2q-1)\,q^{2^{a+1}-4} > 0 \\
\Leftrightarrow\quad & 1+q^{2^a-1}(1+q) - (q+1)(2q-1)\,q^{2^{a+1}-4} - (2q-1)\,q^{2^{a+1}-2} > 0 \\
\Leftrightarrow\quad & 1-\underbrace{(2q-1)}_{<1}\underbrace{q^{2^{a+1}-2}}_{\le 1} + q^{2^a-1}(1+q)\,q^{2^a-1}\left(1-\underbrace{(2q-1)}_{<1}\underbrace{q^{2^a-3}}_{\le 1}\right) > 0.
\end{aligned}
$$

Q.E.D.

References

[1] GrÄner, Hans Peter and Schulte, Elisabeth (2003): "Speed and Quality of Collective Decision Making, II: Incentives", mimeo.

[2] Gibbons, Allen, and Wojciech Rytter (1988): E±cient Parallel Algorithms, Cambridge: Cambridge University Press.

[3] Luce, R. Duncan (1959): Individual Choice Behavior. New York: Wiley.

[4] Meagher, Kieron, Hakan Orbay and Timothy Van Zandt (2001): "Hierarchy Size and Environmental Uncertainty", CEPR Discussion Paper 2839.

[5] Meagher, Kieron and Timothy Van Zandt (1998): "Managerial Costs for one-shot decentralized Information Processing", Review of Economic Design 3: 329-345.

[6] Orbay, Hakan (2002): "Information Processing Hierarchies", Journal of Economic Theory 105: 370-407.

[7] Prat, Andrea (1997): "Hierarchies of Processors with Endogenous Capacity", Journal of Economic Theory 77: 214-222.

[8] Radner, Roy (1992): "Hierarchy: The Economics of Managing", Journal of Economic Literature 30: 1382-1415.

[9] Radner, Roy (1993): "The Organization of Decentralized Information Processing", Econometrica 61(5): 1109-1146.

[10] Sah, Raaj Kumar and Joseph E. Stiglitz (1986): "The Architecture of Economic Systems: Hierarchies and Polyarchies", The American Economic Review 76(4): 716-727.

[11] Van Zandt, Timothy (1997): "The Scheduling and Organization of Periodic Associative Computation: Essential networks", Review of Economic Design 3: 15-27.

[12] Van Zandt, Timothy (1998): "The Scheduling and Organization of Periodic Associative Computation: E±cient networks", Review of Economic Design 3: 93-127.