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**MONOTONE EQUILIBRIA IN  
BAYESIAN GAMES  
OF STRATEGIC  
COMPLEMENTARITIES**

Timothy Van Zandt and Xavier Vives

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# **MONOTONE EQUILIBRIA IN BAYESIAN GAMES OF STRATEGIC COMPLEMENTARITIES**

**Timothy Van Zandt**, INSEAD, Fontainebleau and CEPR  
**Xavier Vives**, INSEAD, Fontainebleau and CEPR

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Centre for Economic Policy Research  
90–98 Goswell Rd, London EC1V 7RR, UK  
Tel: (44 20) 7878 2900, Fax: (44 20) 7878 2999  
Email: [cepr@cepr.org](mailto:cepr@cepr.org), Website: [www.cepr.org](http://www.cepr.org)

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## ABSTRACT

### Monotone Equilibria in Bayesian Games of Strategic Complementarities\*

For Bayesian games of strategic complementarities, we provide a constructive proof of the existence of a greatest and a least Bayes-Nash equilibrium – each one in strategies monotone in type – if the payoff to a player displays increasing differences in own action and the profile of types, and if the posteriors are increasing in type with respect to first-order stochastic dominance (e.g., if types are affiliated). The result holds for multidimensional action and type spaces and also for continuous and discrete type distributions. It uses an intermediate result on monotone comparative statics under uncertainty, which implies that the extremal equilibria increase when there is a first-order stochastic dominant shift in beliefs. We provide an application to strategic information revelation in games of voluntary disclosure.

JEL Classification:

Keywords: Cournot tâtonnement, first-order stochastic dominance, incomplete information, monotone comparative statics, supermodular games and voluntary disclosure

Timothy Van Zandt  
INSEAD  
Boulevard de Constance  
77305 Fontainebleau CEDEX  
FRANCE  
Tel: (33 1) 6072 4981  
Fax: (33 1) 6074 6192  
Email: tvz@econ.insead.edu

Xavier Vives  
INSEAD  
Boulevard de Constance  
77305 Fontainebleau Cedex  
FRANCE  
Tel: (33 1) 6072 4279  
Fax: (33 1) 6074 6719  
Email: xavier.vives@insead.edu

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## Contents

1	Introduction	1
2	The Bayesian game	2
3	Cournot tatônnement and the greatest equilibrium	4
4	Existence and monotonicity of the GBR mapping	5
5	The greatest best reply to monotone strategies is monotone	5
6	Strictly monotone best replies	6
7	Summary of the main result	8
8	A counterexample for log-supermodular payoffs	9
9	The greatest equilibrium is increasing in the posteriors	11
10	Games of voluntary disclosure	12
11	Concluding remarks	16
A	Summary of lattice and comparative statics methods	17
B	Extension of comparative statics under uncertainty	18
C	Affiliation and increasing posteriors	19
	References	20

## 1 Introduction

This paper uses lattice-theoretic methods to establish, in supermodular games of incomplete information with complementarity between actions and types and with posteriors that are increasing in type with respect to first-order stochastic dominance, (i) the existence of extremal equilibria that are monotone in type and (ii) monotone comparative statics of the extremal equilibria with respect to posterior beliefs.

Existence and characterization results of pure-strategy equilibria in Bayesian games have proved difficult to obtain. Such results include the following. Milgrom and Weber (1985) (see also Aumann et al. (1982) and Radner and Rosenthal (1982)) find stringent sufficient conditions for existence, such as conditionally independent and atomless distributions for types and finite action spaces; they use atomless distributions for purification. Vives (1990, Section 6) establishes existence with general action and type spaces when payoffs are supermodular in actions; he uses a lattice fixed-point theorem. Athey (2001) shows existence of equilibria in monotone strategies when there are (a) either supermodular or log-supermodular cardinal utilities, (b) complementarity between actions and types, (c) one-dimensional action sets, and (d) one-dimensional atomless type spaces. Her methods combine monotone comparative statics (to get the monotonicity of best responses), a topological fixed-point theorem, and purification of equilibria in Bayesian games with atomless type spaces.<sup>1</sup> McAdams (2002) presents an extension to multidimensional discrete action and atomless type spaces, putting complementarity assumptions directly on interim payoffs.

On the one hand, our setting is more restrictive than Athey (2001), who also obtains results for log-supermodular payoffs (which do not imply strategic complementarities in games of incomplete information). However, by using lattice rather than topological fixed-point methods, we obtain pure-strategy equilibria by putting assumptions on primitives and without the purification via atomless type spaces; moreover, our results hold for multidimensional action and type spaces, whether discrete or continuous. Our proof, which is simple and constructive, finds the greatest and least of all pure-strategy equilibria and shows that these are in monotone strategies (even if nonmonotone equilibria also exist).

The leading version of our main result can be stated as follows. Consider any Bayesian game in which: (a) each type space is a product of measurable subsets of  $\mathbb{R}$ ; (b) each action set is a product of compact subsets of  $\mathbb{R}$ ; (c) each player's payoff has increasing differences in own action, in own action and the profile of types, and in own action and the action profile of other players; and (d) posteriors are increasing in type with respect to first-order stochastic dominance. Then there exist a greatest and a least pure-strategy Bayes-Nash equilibrium, each of which is in strategies that are monotone in type.

The reader may be accustomed to seeing “affiliated types” assumptions (e.g., “monotone likelihood ratio”, “log-supermodular density”) in the related literature. However, because

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<sup>1</sup>Note, however, that atomless type spaces are generally not sufficient for purification with continuous action sets. See Khan and Sun (1996).

we restrict attention to payoffs with increasing differences in action and type, we need only the *weaker* condition that higher types have higher posteriors when these are ordered by first-order stochastic dominance. We state in Appendix B the general comparative statics results linking increasing differences and first-order stochastic dominance that are needed for our analysis.

Identifying the greatest and least equilibria, rather than merely showing existence of one equilibrium in monotone strategies, allows us to perform comparative statics on these equilibria. In particular, we are able to show that they are increasing in the posteriors. That is, if we perturb the game such that, for each player and each type, the player's posterior beliefs about the other players shift up by first-order stochastic dominance, then the greatest and least equilibrium strategies increase for each player and each type. As an application, we generalize a result in Okuno-Fujiwara et al. (1990) on strategic revelation of information and voluntary disclosure.

The plan of the paper is as follows. In Section 2, we set up the Bayesian game and state basic maintained assumptions. Section 3 shows how (under certain assumptions) Cournot tâtonnement, starting at the greatest strategy profile and using the greatest best-reply mappings, converges to the greatest Bayes-Nash equilibrium, which is in strategies that are monotone in type. Section 4 shows existence and monotonicity of the greatest best-reply mapping. Section 5 builds on intermediate results about comparative statics under uncertainty and shows that the greatest best-reply to monotone strategies is monotone. A strict version of this result is obtained in Section 6. The pieces are then in place to state the main result in Section 7, where we also give an application to Bertrand oligopoly. Section 8 provides an example demonstrating that our approach cannot work for log-supermodular payoffs. Section 9 shows that the extremal equilibria are increasing in the posteriors. We give an application to games of voluntary disclosure in Section 10 and concluding remarks close the paper in Section 11. Appendix A provides, for completeness, some basic lattice-theoretic definitions, Appendix B presents our results on comparative statics under uncertainty, and Appendix C compares affiliation and our weaker increasing posteriors condition.

## 2 The Bayesian game

We use the following formulation of a Bayesian game:

1. The set of players is  $N = \{1, \dots, n\}$ , indexed by  $i$ .
2. The state space is  $T = T_0 \times T_1 \times \dots \times T_n$ , where  $T_0$  is residual uncertainty not observed by any player and, for  $i \in N$ ,  $T_i$  is the type space of player  $i$ .
3. The common prior on  $T$  is  $\mu$ , with marginal distribution  $\mu_i$  on  $T_i$ .
4. The action set of player  $i$  is  $A_i$ . The set of action profiles is  $A = \prod_{i \in N} A_i$ .

5. The payoff function of player  $i$  is  $u: A \times T \rightarrow \mathbb{R}$ .

Let  $T_{-i} = \prod_{j \neq i} T_j$  and  $A_{-i} = \prod_{j \neq i} A_j$ .

We impose the following restrictions, whose roles are topological and order-related.<sup>2</sup>

1. For  $i = 0, 1, \dots, n$ ,  $T_i$  is a non-empty measurable subset of a Euclidean space (inheriting its Borel field and partial order).
2. For  $i \in N$ ,  $A_i$  is a non-empty compact sublattice of a Euclidean space (inheriting its topology and lattice structure).
3. For  $i \in N$ , player  $i$ 's payoff function  $u_i$  has the following properties: (a) for all  $a \in A$ ,  $u(a, \cdot): T \rightarrow \mathbb{R}$  is measurable; (b) for a.e.  $t \in T$ ,  $u(\cdot, t): A \rightarrow \mathbb{R}$  is continuous; (c) there is an integrable  $g: T \rightarrow \mathbb{R}$  such that  $|u(\sigma(t), t)| \leq g(t)$  a.e. for all measurable  $\sigma: T \rightarrow A$ .

We use the symbol  $\geq$  for all partial orders. Expressions such as “greater than” and “increasing” mean “weakly greater than” and “weakly increasing”.

A strategy for player  $i$  is a measurable function  $\sigma_i: T_i \rightarrow A_i$ . Let  $\Sigma_i$  denote the set of equivalence classes of strategies, with equivalence defined by being equal a.e. Let  $\Sigma = \prod_{i=1}^n \Sigma_i$  denote the set of strategy profiles, and let  $\Sigma_{-i} = \prod_{j \neq i} \Sigma_j$  denote the profiles of strategies for players other than  $i$ . For notational simplicity, a strategy profile is viewed as a map from  $T$  to  $A$ , even though it does not depend on  $T_0$ . For each player, the set of strategies is also a lattice for the ordering “ $\sigma_i \geq \sigma'_i$  if and only if  $\sigma_i(t_i) \geq \sigma'_i(t_i)$  for a.e.  $t_i \in T_i$ ”. We say that a strategy  $\sigma_i \in \Sigma_i$  is *monotone* if, for a.e.  $t_i, t'_i \in T_i$  such that  $t_i \geq t'_i$ , we have  $\sigma_i(t_i) \geq \sigma_i(t'_i)$ .

Our technical assumptions assure that  $U_i(\sigma) = \int_T u_i(\sigma(t), t) d\mu$  is well-defined on  $\Sigma$  and is continuous in the topology of convergence in measure. A Bayes-Nash equilibrium is a Nash equilibrium of the game  $(N, (\Sigma_i), (U_i))$ . Let  $\beta_i: \Sigma_{-i} \rightarrow \Sigma_i$  denote player  $i$ 's best-reply correspondence:

$$(1) \quad \beta_i(\sigma_{-i}) = \arg \max_{\sigma_i \in \Sigma_i} U_i(\sigma_i, \sigma_{-i}).$$

Then a Bayes-Nash equilibrium is a strategy profile  $\sigma^*$  such that  $\sigma_i^* \in \beta_i(\sigma_{-i}^*)$  for  $i \in N$ .

**Remark 1** We also make use of the “ex post” representation of a Bayesian game and of  $\beta_i$ . In the ex ante definition of equation (1), the player chooses a strategy before observing his type in order to maximize *unconditional* expected utility. In the ex post characterization, a player observes his type and then chooses an action in order to maximize *conditional* expected utility. Let  $\Delta(T_{-i})$  be the set of probability measures on  $T_{-i}$ . Player  $i$ 's posteriors

<sup>2</sup>See Appendix A for lattice-theoretic definitions.

are given by a measurable function  $p_i: T_i \rightarrow \Delta(T_{-i})$ , where  $p_i(t_i)$  denotes  $i$ 's posteriors on  $T_{-i}$  conditional on  $t_i$ . The posteriors are consistent with and uniquely determined (up to equivalence) by the prior.<sup>3</sup> For each  $t_i \in T_i$ ,  $P_{-i} \in \Delta(T_{-i})$ , and  $\sigma_{-i} \in \Sigma_{-i}$ , let  $\varphi_i(t_i, P_{-i}, \sigma_{-i})$  be the set of actions for  $i$  that maximize  $i$ 's expected utility when his type is  $t_i$ , his posterior is  $P_{-i}$ , and the other players' strategies are  $\sigma_{-i}$ . That is,

$$(2) \quad \varphi_i(t_i, P_{-i}, \sigma_{-i}) = \arg \max_{a_i \in A_i} \int_{T_{-i}} u_i(a_i, \sigma_{-i}(t_{-i}), t_i, t_{-i}) dP_{-i}(t_{-i}).$$

Then  $\sigma_i \in \beta_i(\sigma_{-i})$  if and only if, for a.e.  $t_i \in T_i$ ,  $\sigma_i(t_i) \in \varphi_i(t_i, p_i(t_i), \sigma_{-i})$ . That is,  $\sigma_i$  is an ex ante best response to  $\sigma_{-i}$  if and only if, for a.e.  $t_i$ ,  $\sigma_i(t_i)$  is an optimal action given  $i$ 's type  $t_i$ , given  $i$ 's posteriors  $p_i(t_i)$  on  $T_{-i}$ , and given  $\sigma_{-i}$ .

Our formulation of a Bayesian is general and encompasses common and private values as well as perfect or imperfect signals. In cases of "pure" private values we have  $u_i(a, t) = v_i(a, t_i)$  (types may be correlated). For example, types are private cost parameters of the firms. A "common values" model might have  $u_i(a, t) = v_i(a, t_1 + \dots + t_n)$ , as when there is a common demand shock in an oligopoly and each firm observes one component. As an example of imperfect signals, suppose firms observe imperfectly their cost parameters. Then  $t_0$  could represent the  $n$ -vector of firms' cost parameters and  $t_i$  the private cost estimate of firm  $i$ . Not only may the cost parameters be correlated but so may the error terms in the private signals.<sup>4</sup>

### 3 Cournot tatônnement and the greatest equilibrium

If  $\beta_i(\sigma_{-i})$  has a unique-a.e. greatest element, denote it by  $\bar{\beta}_i(\sigma_{-i})$ . If  $\bar{\beta}_i(\sigma_{-i})$  is well-defined for all  $\sigma_{-i} \in \Sigma_{-i}$ , then we call  $\bar{\beta}_i: \Sigma_{-i} \rightarrow \Sigma_i$  player  $i$ 's *greatest best-reply* (GBR) mapping.

The main idea is that (under certain assumptions) Cournot tatônnement, starting at the greatest strategy profile and using the GBR mappings, converges to the greatest Bayes-Nash equilibrium, which is in strategies that are monotone. We first state this result in terms of assumptions on the GBR mapping (in Lemma 1) and then derive the assumptions from more primitive ones (in Sections 4 and 5). An analogous result, which we do not bother stating, holds for the *least* best-reply mapping and the *least* Bayes-Nash equilibrium.

**Lemma 1** *Assume the following for each player  $i$ .*

1. *The GBR mapping  $\bar{\beta}_i$  is well-defined.*

<sup>3</sup>For the existence of such posteriors, see Dellacherie and Meyer (1978, III.70 and 71).

<sup>4</sup>See Vives (1999, Section 8.1.2) for parameterized examples of the cases discussed.



2. The GBR mapping is increasing: for  $\sigma'_{-i}, \sigma_{-i} \in \Sigma_{-i}$  such that  $\sigma'_{-i} \geq \sigma_{-i}$ ,  $\bar{\beta}_i(\sigma'_{-i}) \geq \bar{\beta}_i(\sigma_{-i})$ .
3. If the strategies  $\sigma_{-i}$  are monotone, then the strategy  $\bar{\beta}(\sigma_{-i})$  is monotone.

Then there is a greatest equilibrium and it is in monotone strategies.

PROOF. Define  $\bar{\beta}: \Sigma \rightarrow \Sigma$  by  $\bar{\beta}(\sigma) = (\bar{\beta}_1(\sigma_{-1}), \dots, \bar{\beta}_n(\sigma_{-n}))$ . Since each  $\bar{\beta}_i$  is increasing, so is  $\bar{\beta}$ . By the third assumption, if  $\sigma$  is a profile of monotone strategies then so is  $\bar{\beta}(\sigma)$ .

For each player  $i$ , let  $\bar{a}_i \in A_i$  be the greatest element of  $A_i$  (which exists since  $A_i$  is a complete lattice). Let  $\sigma_i^0 \in \Sigma_i$  be the strategy that is equal a.e. to  $\bar{a}_i$ , and let  $\sigma^0 = (\sigma_1^0, \dots, \sigma_n^0)$ . Define recursively  $\sigma^k = \bar{\beta}(\sigma^{k-1})$  for  $k = 1, 2, \dots$ . Since  $\sigma^0$  is the profile of greatest strategies, we have  $\sigma^1 \leq \sigma^0$ . Since  $\bar{\beta}$  is increasing and  $\sigma^2 = \bar{\beta}(\sigma^1)$  and  $\sigma^1 = \bar{\beta}(\sigma^0)$ , we have  $\sigma^2 \leq \sigma^1$ . By induction, the sequence  $\{\sigma^k\}$  is decreasing. Thus, for each player  $i$  and for a.e.  $t_i \in T_i$ ,  $\{\sigma_i^k(t_i)\}$  is a decreasing sequence. Since every decreasing sequence in  $A_i$  converges to its infimum, it follows that  $\sigma_i^k$  converges pointwise a.e., and hence in measure, to the infimum  $\sigma_i^\infty$  of the sequence  $\{\sigma_i^k\}$ . The limit must be an equilibrium (as in Vives (1990, Theorem 5.1)) because the utility functions  $U_i$  are continuous in the topology of convergence in measure.

Furthermore, each term in the sequence  $\{\sigma^k\}$  is in monotone strategies because  $\sigma^0$  is a profile of monotone strategies and so is  $\bar{\beta}(\sigma^k)$  if  $\sigma^k$  is such a profile. The limit of a sequence of monotone strategies is also monotone.

The limit  $\sigma^\infty$  must be the greatest equilibrium, as we now show. Any other equilibrium  $\sigma$  will be smaller than the greatest strategy profile  $\sigma^0$ , that is,  $\sigma^0 \geq \sigma$ . Since  $\bar{\beta}$  is increasing, we have  $\bar{\beta}(\sigma^0) \geq \bar{\beta}(\sigma)$ . On the one hand,  $\sigma$  is a profile of best responses to  $\sigma$  since  $\sigma$  is an equilibrium; on the other,  $\bar{\beta}(\sigma)$  is the greatest best response to  $\sigma$ . Therefore,  $\bar{\beta}(\sigma) \geq \sigma$ . Combining  $\sigma^1 = \bar{\beta}(\sigma^0)$ ,  $\bar{\beta}(\sigma^0) \geq \bar{\beta}(\sigma)$ , and  $\bar{\beta}(\sigma) \geq \sigma$  yields  $\sigma^1 \geq \sigma$ . Continuing by induction,  $\sigma^k \geq \sigma$  for all  $k$  and hence  $\sigma^\infty \geq \sigma$ .  $\square$

#### 4 Existence and monotonicity of the GBR mapping

The assumptions needed for the existence and monotonicity of the GBR mapping follow from the standard theory of supermodular games.

**Proposition 1** *Assume for player  $i$  that, for a.e.  $t \in T$ ,  $u_i(\cdot, t)$  is supermodular in  $a_i$  and has increasing differences in  $(a_i, a_{-i})$ . Then, for all  $\sigma_{-i} \in \Sigma_{-i}$ ,  $\beta_i(\sigma_{-i})$  contains a greatest element; that is,  $\bar{\beta}_i(\sigma_{-i})$  is well-defined. Furthermore,  $\bar{\beta}_i$  is increasing.*

PROOF. The proof mimics that of Theorem 6.1 in Vives (1990). The main ideas there are (a) supermodularity and increasing differences are preserved by integration; (b) hence  $U_i$  is

supermodular in  $\sigma_i$  and has increasing differences in  $(\sigma_i, \sigma_{-i})$ ; (c) hence  $\beta_i$  is an increasing correspondence; (d)  $\bar{\beta}_i(\sigma_{-i})$  is well-defined as the pointwise supremum of  $\beta(\sigma_{-i})$ ; <sup>5</sup> and (e)  $\bar{\beta}_i$  is increasing because  $\beta_i$  is increasing.  $\square$

## 5 The greatest best reply to monotone strategies is monotone

We now show that the GBR to monotone strategies is monotone if, in addition to the assumptions of Proposition 1,  $u_i$  has increasing differences in  $(a_i, t)$  and a “monotone posteriors” condition is satisfied. We shall apply the monotone comparative statics results from Appendix B.

We endow  $\Delta(T_{-i})$  with the partial order of first-order stochastic dominance and assume that  $i$ 's posterior function  $p_i: T_i \rightarrow \Delta(T_{-i})$  is increasing. That is, higher types of  $i$  believe that the other players are more likely to be of higher types as well. This is implied by—but is weaker than—the more familiar assumption that  $\mu$  is affiliated (see Appendix C).

Once we fix a profile of strategies for players other than  $i$ , we can write  $i$ 's payoff as a function of only his own action and the profile of types. If the other players' strategies are increasing, then increasing differences of  $u_i$  in  $(a_i, a_{-i})$  and in  $(a_i, t)$  translate into increasing differences of the induced payoff function in  $(a_i, t)$ . According to Lemma 4, if we hold  $t_i$  fixed and let only player  $i$ 's beliefs about  $t_{-i}$  vary, then a first-order stochastic dominance shift in  $i$ 's beliefs leads to a higher optimal action by player  $i$ . Thus, when player  $i$ 's type is higher, he chooses a higher action both because of the shift in beliefs (since we assume  $p_i$  is increasing) and because the induced payoff has increasing differences in  $(a_i, t_i)$ .

**Proposition 2** *Let  $i \in N$ . Assume that:*

1.  $u_i$  is supermodular in  $a_i$ , has increasing differences in  $(a_i, a_{-i})$ , and has increasing differences in  $(a_i, t)$ ; and
2.  $p_i: T_i \rightarrow \Delta(T_{-i})$  is increasing with respect to the partial order on  $\Delta(T_{-i})$  of first-order stochastic dominance (e.g.,  $\mu$  is affiliated).

*Then, for all monotone  $\sigma_{-i} \in \Sigma_{-i}$ ,  $\bar{\beta}_i(\sigma_{-i})$  is monotone.*

**PROOF.** Fix  $\sigma_{-i} \in \Sigma_{-i}$ . Recall the ex post characterization of  $\beta_i$  in Remark 1, according to which  $\beta_i(\sigma_i)$  is the set of measurable selections of the correspondence  $t_i \mapsto \varphi_i(t_i, p(t_i), \sigma_{-i})$ ,

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<sup>5</sup>In the usual supermodular optimization theory, the existence of a greatest solution follows (as in Topkis (1978, Corollary 4.1)) by assuming the choice set to be compact in a topology finer than the interval topology and in which  $U_i$  is upper semicontinuous. With infinite type spaces, these two restrictions on the topology on  $\Sigma_i$  are inconsistent. In Vives (1990, Theorem 6.1), as in this paper,  $A_i$  is Euclidean. Then  $\Sigma_i$  is a complete lattice and the supremum of a sublattice is in the closure of the sublattice (Schaefer (1974, Proposition II.8.3)). These two properties also establish the existence of a supremum in  $\beta_i(\sigma_{-i})$ .

where  $\varphi_i$  is defined in equation (2). We show that, if  $\sigma_{-i}$  is monotone, then the objective function that defines  $\varphi_i$  is supermodular in  $a_i$  and has increasing differences in  $(a_i, t_i)$  and  $(a_i, P_{-i})$ . Therefore,  $\max \varphi_i(t_i, P_{-i}, \sigma_{-i})$  exists and is increasing in  $t_i$  and in  $P_{-i}$ . Since  $\bar{\beta}_i(\sigma_{-i})$  is equal a.e. to  $t_i \mapsto \max \varphi_i(t_i, p_i(t_i), \sigma_{-i})$  and since  $p_i$  is increasing, it follows that  $\bar{\beta}_i(\sigma_{-i})$  is increasing a.e.

If we let  $v_i(a_i, t) = u_i(a_i, \sigma_{-i}(t_{-i}), t_i, t_{-i})$  and let  $V_i(a_i, t_i, P_{-i}) = \int_{T_{-i}} v_i(a_i, t) dP_{-i}(t_{-i})$ , then  $\varphi_i(t_i, P_{-i}, \sigma_{-i}) = \arg \max_{a_i \in A_i} V_i(a_i, t_i, P_{-i})$ . The induced payoff function  $v_i(a_i, t)$  has increasing differences in  $(a_i, t)$  because  $u_i$  has increasing differences in  $(a_i, (a_{-i}, t))$  and  $\sigma_{-i}$  is increasing. It follows from Lemma 4 that  $V_i$  has increasing differences in  $(a_i, t_i)$  and in  $(a_i, P_{-i})$ . As usual, since supermodularity is preserved by integration,  $V_i$  is also supermodular in  $a_i$ .  $\square$

## 6 Strictly monotone best replies

We can strengthen the conclusion of Proposition 2 to “for all monotone  $\sigma_{-i} \in \Sigma_{-i}$ ,  $\bar{\beta}_i(\sigma_{-i})$  is *strictly* monotone” by adding some smoothness assumptions. We continue to rely on the lattice methods to obtain a weak inequality and then use differentiability to rule out equality—the inequality must then be strict.

For example, consider a choice problem  $\max_{x \in X} u(x, y)$ , where  $X$  is an interval of  $\mathbb{R}$  and  $y$  is a parameter that belongs to a partially ordered set  $Y$ . Suppose  $x^H, x^L$  are interior solutions given  $y^H, y^L \in Y$ , where  $y^H > y^L$ , and we have determined that  $x^H \geq x^L$  (e.g., using monotone comparative statics). Suppose also that  $u$  is differentiable in  $x$  and that  $\partial u / \partial x$  is strictly increasing in  $y$ . The solutions  $x^H, x^L$  must satisfy the first-order condition and so  $\partial u(x^H, y^H) / \partial x = 0$  and  $\partial u(x^L, y^L) / \partial x = 0$ . Since  $\partial u / \partial x$  is strictly increasing in  $y$ , we have  $\partial u(x^L, y^H) / \partial x > 0$ . Therefore,  $x^H \neq x^L$  and instead  $x^H > x^L$ .

This kind of argument can be applied to a single dimension of a multidimensional choice set, thereby allowing for a mix of continuous and discrete choice variables. This is our approach. We refer to the smoothness conditions needed as the “smooth case”.

**Assumption 1 (Smooth case for player  $i$ )** *The following statements hold for player  $i$ :*

1.  $A_i = A_{i1} \times A_{i2}$ , where  $A_{i1}$  is a non-empty compact interval of  $\mathbb{R}$  and  $A_{i2}$  is a non-empty compact sublattice of Euclidean space;
2.  $u_i$  is continuously differentiable in  $a_{i1}$ ;
3. for all  $t_i, P_{-i}$ , and  $\sigma_{-i}$ , the elements of  $\varphi_i(t_i, P_{-i}, \sigma_{-i})$  are such that  $a_{i1}$  is in the interior of  $A_{i1}$ .

In the smooth case for player  $i$ , a strategy  $\sigma_i$  is said to be *strictly monotone* if, for almost every  $t_i^H, t_i^L \in T_i$  such that  $t_i^H > t_i^L$ , we have  $\sigma_i(t_i^H) \geq \sigma_i(t_i^L)$  and  $\sigma_{i1}(t_i^H) > \sigma_{i1}(t_i^L)$ .

(Observe that the strict inequality is only for the dimension we have identified to satisfy the smoothness assumptions. If there are multiple such dimensions, we can apply the assumptions to each of them and thereby obtain a strict inequality for each of them.)

We are now ready for our “strict” version of Proposition 2.

**Corollary 1** *Given (a) the assumptions of Proposition 2, (b) the smooth case for player  $i$ , and (c) that  $\partial u_i / \partial a_{i1}$  is strictly increasing in  $t_i$ , it then follows, for all monotone  $\sigma_{-i} \in \Sigma_{-i}$ , that  $\bar{\beta}_i(\sigma_{-i})$  is strictly monotone.*

PROOF. (For the sake of clarity, we omit the “a.e.” qualifications in this proof.)

Let  $\sigma_{-i} \in \Sigma_{-i}$  be monotone and let  $\sigma_i = \bar{\beta}_i(\sigma_{-i})$ . Let  $t_i^H, t_i^L \in T_i$  be such that  $t_i^H > t_i^L$ . We know from Proposition 2 that  $\sigma_i(t_i^H) \geq \sigma_i(t_i^L)$ , so we only need show that  $\sigma_{i1}(t_i^H) \neq \sigma_{i1}(t_i^L)$ .

Continuing from the proof of Proposition 2,  $\sigma_i(t_i^H)$  and  $\sigma_i(t_i^L)$  are solutions to (respectively)  $\max_{a_i \in A_i} V_i(a_i, t_i^H, p_i(t_i^H))$  and  $\max_{a_i \in A_i} V_i(a_i, t_i^L, p_i(t_i^L))$ . Since  $u_i$  is continuously differentiable in  $a_{i1}$ , so is  $V_i$ . By assumption in the smooth case,  $\sigma_{i1}(t_i^H)$  and  $\sigma_{i1}(t_i^L)$  are interior. Therefore, we have the first-order conditions

$$(3) \quad \partial V_i(\sigma_i(t_i^H), t_i^H, p_i(t_i^H)) / \partial a_{i1} = 0$$

$$(4) \quad \partial V_i(\sigma_i(t_i^L), t_i^L, p_i(t_i^L)) / \partial a_{i1} = 0.$$

The next step involves substituting  $\sigma_{i2}(t_i^H)$ ,  $t_i^H$ , and  $p_i(t_i^H)$  in the left side of equation (4) and showing that this causes the expression to increase, so that

$$(5) \quad \partial V_i((\sigma_{i1}(t_i^L), \sigma_{i2}(t_i^H)), t_i^H, p_i(t_i^H)) / \partial a_{i1} > 0.$$

On the one hand, we know that  $\sigma_{i2}(t_i^H) \geq \sigma_{i2}(t_i^L)$  (a conclusion of Proposition 2),  $t_i^H > t_i^L$  (by assumption), and  $p_i(t_i^H) \geq p_i(t_i^L)$  (from the assumption that  $p_i$  is increasing). Since  $\partial u_i / \partial a_{i1}$  is strictly increasing in  $t_i$ , so is  $\partial V_i / \partial a_{i1}$ . Furthermore, we established in the proof of Proposition 2 that  $V_i$  is supermodular in  $a_i$  and has increasing differences in  $(a_i, P_{-i})$ ; therefore,  $\partial V_i / \partial a_{i1}$  is weakly increasing in  $a_{i2}$  and in  $P_{-i}$ . This establishes equation (5).

Comparing equations (3) and (5), we conclude that  $\sigma_{i1}(t_i^H) \neq \sigma_{i1}(t_i^L)$ .  $\square$

## 7 Summary of the main result

Putting together Lemma 1 and Propositions 1 and 2 yields our main result. We call games that satisfy the assumptions of Theorem 1 “monotone supermodular”.

**Theorem 1** *Assume, for each player  $i$ , that*

1.  $u_i$  is supermodular in  $a_i$ , has increasing differences in  $(a_i, a_{-i})$ , and has increasing differences in  $(a_i, t)$ ; and
2.  $p_i: T_i \rightarrow \Delta(T_{-i})$  is increasing with respect to the partial order on  $\Delta(T_{-i})$  of first-order stochastic dominance (e.g.,  $\mu$  is affiliated).

Then there exist a greatest and a least Bayes-Nash equilibrium, and each one is in monotone strategies.

PROOF. According to Proposition 1,  $\bar{\beta}_i$  is well-defined and increasing; according to Proposition 2,  $\bar{\beta}_i(\sigma_{-i})$  is monotone if  $\sigma_{-i} \in \Sigma_{-i}$  is monotone. Hence, the three assumptions of Lemma 1 are satisfied and so there exists a greatest equilibrium and it is in monotone strategies. (The same arguments apply to the least equilibrium.)  $\square$

**Corollary 2** *Given (a) the assumptions of Theorem 1, (b) the smooth case for player  $i$ , and (c) that  $\partial u_i / \partial a_{i1}$  is strictly increasing in  $t_i$ , it follows that the greatest and least Bayes-Nash equilibria are such that player  $i$ 's strategies are strictly monotone.*

PROOF. From Theorem 1, the greatest equilibrium is in monotone strategies. Player  $i$  is playing his greatest best response to a profile of monotone strategies of the other players, which according to Corollary 1 is strictly increasing in type.  $\square$

Athey (2001) obtains a result for log-supermodular payoffs, whereas we do not. However, for the case of supermodular payoffs, our result is stronger than that of Athey (2001) because:

- it covers multidimensional action and type spaces, whereas her results cover only one-dimensional type spaces;<sup>6</sup>
- it covers discrete type spaces, whereas her results cover only continuous type spaces (intervals of  $\mathbb{R}$ );
- we show that the greatest and least equilibria are in monotone strategies, whereas she shows only existence of an equilibrium in monotone strategies.

**Example 1** Consider a Bertrand multimarket oligopoly example, in which  $n$  firms compete in  $H$  interrelated product markets,  $h = 1, \dots, H$ . The firms' products are differentiated within each market. The profit function of firm  $i$  is given by  $u_i = \sum_{h=1}^H (p_{ih} -$

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<sup>6</sup>McAdams (2002) generalizes Athey's results to multidimensional action and type spaces, but for supermodular utilities the complementarity assumptions are stated directly on interim payoffs rather than on primitives.

$c_{ih})D_{ih}(p_i, p_{-i}, \theta_h)$ , where  $p_{ih}$  is  $i$ 's price for its good in market  $h$ ,  $c_{ih}$  is the random constant marginal production cost of this good, and  $\theta_h$  is a random demand shock for market  $h$ . The type of firm  $i$  is  $t_i = (c_i, s_i)$ , where  $c_i$  is the cost vector for firm  $i$  and  $s_i$  is a multidimensional signal about the vector  $\theta$ . In our notation, the vector  $\theta$  is part of  $T_0$ . The payoff  $u_i$  is supermodular in the prices and has increasing differences in  $p_i$  and  $(c_i, \theta)$  if, for example,  $D_{ih}$  is linear and increasing in  $\theta_h$  and if all the goods are gross substitutes (both across markets and across brands). If, for example,  $\theta$  and  $(c_i, s_i)_{i \in N}$  are affiliated then the increasing posteriors condition is satisfied. Then extremal equilibria will be monotone, and so prices at extremal equilibria will increase in cost and demand signals.

**Example 2** Consider the following Bertrand oligopoly with differentiated products in which firms compete in both price  $p_i$  and advertising intensity  $z_i$ . The profit to firm  $i$  is given by  $u_i = (p_i - c_i)D_i(p_i, p_{-i}, z_i) - F_i(z_i, e_i)$ , where  $c_i$  is the per-unit cost,  $D_i$  yields the demand for the product of the firm,  $F_i$  yields the cost of advertising, and  $e_i$  measures the cost-efficiency of advertising. The type of firm  $i$  is  $t_i = (c_i, e_i)$  and its action is  $a_i = (p_i, z_i)$ .

We assume that  $D_i$  is decreasing in  $p_i$ , increasing in  $p_{-i}$ , and increasing in  $z_i$ ;  $F(z_i, e_i)$  is increasing in  $z_i$  and decreasing in  $e_i$ . Then  $u_i$  is supermodular in  $a_i$  if, for  $p_i^H$  and  $p_i^L$  such that  $p_i^H > p_i^L$ ,  $D_i(p_i^L, p_{-i}, z_i) - D_i(p_i^H, p_{-i}, z_i)$  is decreasing in  $z_i$ ; this means, for example, that advertising increases demand by raising the valuations of existing consumers rather than informing new consumers of the existence of the good. Observe that  $u_i$  has increasing differences in  $(a_i, a_{-i})$  as long as, for  $p_i^H > p_i^L$  and  $z_i^H > z_i^L$ ,  $D_i(p_i^L, p_{-i}, z_i) - D_i(p_i^H, p_{-i}, z_i)$  is decreasing in  $p_{-i}$  and  $D_i(p_i, p_{-i}, z_i^H) - D_i(p_i, p_{-i}, z_i^L)$  is increasing in  $p_{-i}$ . All these conditions are satisfied (weakly) when  $D_i$  is linear in all its terms. Also,  $u_i$  has increasing differences in  $a_i$  and  $t_i$  if  $F$  has decreasing differences in  $(z_i, e_i)$ —for instance, if higher  $e_i$  decreases the marginal cost of advertising. Posteriors are increasing in type if the joint distribution of types is affiliated. Supposing also that there are natural upper bounds for  $p_i$  and  $z_i$  (e.g., that there are a choke-off price for demand and a point beyond which advertising has no further effect) and assuming continuity of  $D_i$  and  $F_i$  (but we still allow any variable to be discrete), we can apply Theorem 1. It then follows that there exist extremal equilibria and these are monotone in types; in other words, higher production cost or higher advertising efficiency induce higher prices and more advertising by firm  $i$ .

## 8 A counterexample for log-supermodular payoffs

Athey (2001) also obtains results for log-supermodular payoffs. We provide an example that shows that our approach cannot work for log-supermodular payoffs. The problem, of course, is that log-supermodularity is not preserved by integration, and a Bayesian game with log-supermodular payoffs may not have strategic complementarities. Therefore, without purification via an atomless type space, the game may not have a pure strategy equilibrium.

First, consider the following counterexample on monotone comparative statics under uncertainty. Here a payoff function is log-supermodular in the action and state, but a first-order stochastic dominance shift in the distribution of the state leads to a decrease in the optimal action. Let the set of states be  $S = \{1, 2, 3\}$  and let the set of actions be  $X = \{1, 2\}$ . The payoff function  $u: X \times S \rightarrow \mathbb{R}$  is defined in the top of the following table.

		$s$		
		1	2	3
$u(1, s)$		2	8	2
$u(2, s)$		1/2	4	4
$\log_2 u(2, s) - \log_2 u(1, s)$		-2	-1	1
$\pi^L(s)$		1/2	0	1/2
$\pi^H(s)$		0	1/2	1/2

We see that  $\log u$  has increasing differences and hence  $u$  is log-supermodular. Consider the probability measures  $\pi^L$  and  $\pi^H$  defined at the bottom of the table.  $\pi^H$  first-order stochastically dominates  $\pi^L$ , and yet the optimal action given  $\pi^L$  is  $x = 2$  whereas the optimal action given  $\pi^H$  is  $x = 1$ .

We can adapt this example to a game of incomplete information. Let the decision problem just outlined be that of player 1, such that the action  $x$  is player 1's own action  $a_1$  and the state  $s$  is player 2's action  $a_2$ . Player 1's utility depends only on the actions, with the form shown in the table, and player 1 has no private information (his type space is degenerate). Player 2's type space is  $T_2 = \{t^L, t^H\}$ , with  $\mu_2(t^L) = \mu_2(t^H) = 1/2$ . Let  $\sigma_2^L$  and  $\sigma_2^H$  be the strategies by player 2 defined by  $\sigma_2^L(t^L) = 1$ ,  $\sigma_2^L(t^H) = 3$ ,  $\sigma_2^H(t^L) = 2$ , and  $\sigma_2^H(t^H) = 3$ . Then strategy  $\sigma_2^L$  induces the distribution  $\pi^L$  over the actions, and the strategy  $\sigma_2^H$  induces the distribution  $\pi^H$  over the actions. Hence, player 1's best response to  $\sigma_2^L$  is  $a_1 = 2$ , whereas his best response to  $\sigma_2^H$  is  $a_1 = 1$ . Since  $\sigma_2^H > \sigma_2^L$ , the game does not have strategic complementarities. To construct from this the non-existence of a pure strategy equilibrium, we need only suppose that player 2 has a dominant action  $a_2 = 3$  when observing  $t^H$  and that, when observing  $t^L$ , player 2's best response to  $a_1 = 1$  is 1 and her best response to  $a_1 = 2$  is 2 (consistent with  $u_2$  being supermodular).

This example shows (a) that the assumption of atomless type spaces cannot be removed in Athey (2001) for the case of log-supermodular utility, and (b) that the assumption of supermodularity cannot be changed in this paper to log-supermodularity.

The more general message is that, whereas ordinal single-crossing properties are sufficient for existence of pure-strategy equilibria in games of complete information, we need the cardinal supermodularity and increasing differences properties in games of incomplete information because only these are preserved by integration. Hence, when relaxing these assumptions we are likely to have to resort to purification via atomless type spaces in order to obtain pure-strategy equilibria, even if we are not interested in the monotonicity of the

equilibrium strategies.

## 9 The greatest equilibrium is increasing in the posteriors

Consider two monotone supermodular games that are identical except in the posteriors. Suppose the difference between the games is a shift in the information structure such that the posteriors *increase* from  $p_i$  to  $p'_i$ , meaning that, for a.e.  $t_i \in T_i$ ,  $p'_i(t_i) \geq p_i(t_i)$ . Combining the logic of Proposition 2 and methods for comparative statics of supermodular games, we show that the greatest equilibrium increases. Specifically, we show first that the GBR mappings shift up and hence that the greatest equilibrium constructed using Cournot tatônnement is higher.

To state the result, we fix all the parameters of the game except posteriors (players, actions, types, payoffs) as presented in Section 2. Assume that, for  $i \in N$ ,  $u_i$  satisfies assumption 1 in Theorem 1. We denote posteriors  $(p_i)_{i \in N}$  by  $p$ , we let  $\mathcal{P}$  be the set of increasing posteriors, and we let  $\Gamma(p)$  be the monotone supermodular game with posteriors  $p$ .

**Proposition 3** *Consider two games  $\Gamma(p)$  and  $\Gamma(p')$  such that, for  $i \in N$ ,  $p'_i \geq p_i$ . Then the greatest equilibrium of  $\Gamma(p')$  is greater than the greatest equilibrium of  $\Gamma(p)$ .*

PROOF. Let  $\bar{\beta}_i$  and  $\bar{\beta}'_i$  be player  $i$ 's GBR mapping for the posteriors  $p_i$  and  $p'_i$ , respectively. Fix an increasing strategy profile  $\sigma_{-i} \in \Sigma_{-i}$  of the other players. Recall from the proof of Proposition 2 that  $\max \varphi_i(t_i, P_{-i}, \sigma_{-i})$  exists and is increasing in  $P_{-i}$ . Since  $\bar{\beta}_i(\sigma_{-i})$  is equal a.e. to  $t_i \mapsto \max \varphi_i(t_i, p_i(t_i), \sigma_{-i})$  and  $\bar{\beta}'_i(\sigma_{-i})$  is equal a.e. to  $t_i \mapsto \max \varphi_i(t_i, p'_i(t_i), \sigma_{-i})$ , and since  $p'_i(t_i) \geq p_i(t_i)$ , we have  $\bar{\beta}'_i(\sigma_{-i}) \geq \bar{\beta}_i(\sigma_{-i})$ .

Therefore, when we construct the greatest equilibria for the two information structures using Cournot tatônnement (as in the proof of Lemma 1), at each stage we have  $\sigma'^k \geq \sigma^k$  and then—from  $\bar{\beta}'(\sigma'^k) \geq \bar{\beta}'(\sigma^k)$  (because  $\bar{\beta}'$  is increasing) and  $\bar{\beta}'(\sigma^k) \geq \bar{\beta}(\sigma^k)$  (as shown above)—we obtain  $\sigma'^{k+1} \geq \sigma^{k+1}$ . Thus, in the limit,  $\sigma'^\infty \geq \sigma^\infty$ .  $\square$

Corollary 3 develops a strict version of Proposition 3, providing sufficient conditions for the equilibrium strategy of a particular player  $j$  to be strictly higher following a strict f.o.s.d. shift in  $j$ 's beliefs about another player  $i$  (and a weak f.o.s.d. shift for all other beliefs of player  $j$  and of other players). One possibility is that  $j$ 's action shifts up due directly to a strict complementarity between  $a_{j1}$  and  $t_i$ . The other possibility is that player  $i$ 's strategy is strictly monotone (because of strict complementarity between  $a_{i1}$  and  $t_i$ ) and there is a strict complementarity between  $a_{j1}$  and  $a_{i1}$ .

**Corollary 3** *Let  $i, j \in \{1, \dots, N\}$  with  $i \neq j$ . Given the assumptions of Proposition 3 and the smooth case for player  $j$ , assume also that either*



1.  $\partial u_j / \partial a_{j1}$  is strictly increasing in  $t_i$  or
2.  $\partial u_j / \partial a_{j1}$  is strictly increasing in  $a_{i1}$  and the smooth case holds also for player  $i$ , with  $\partial u_i / \partial a_{i1}$  strictly increasing in  $t_i$ .

Then the greatest equilibria  $\sigma'$  and  $\sigma$  of  $\Gamma(p')$  and  $\Gamma(p)$ , respectively, are such that, for a.e.  $t_j \in T_j$ , if the marginal distribution of  $p'_j(t_j)$  on  $T_i$  strictly first-order stochastically dominates that of  $p_j(t_j)$ , then  $\sigma'_{j1}(t_j) > \sigma_{j1}(t_j)$ .

PROOF. (For clarity, we omit the ‘‘a.e.’’ qualifications in this proof.)

Proposition 3 tells us that  $\sigma'_{j1}(t_j) \geq \sigma_{j1}(t_j)$ . We need to show that  $\sigma'_{j1}(t_j) \neq \sigma_{j1}(t_j)$ . The method of proof is the same as in Corollary 1.

Following first the proof of Proposition 2, we have that  $\sigma'_j(t_j)$  and  $\sigma_j(t_j)$  are solutions to (respectively)  $\max_{a_j \in A_j} V'_j(a_j, t_j, p'_j(t_j))$  and  $\max_{a_j \in A_j} V_j(a_j, t_j, p_j(t_j))$ , where

$$V'_j(a_j, t_j, P_{-j}) = \int_{T_{-j}} u_j(a_j, \sigma'_{-j}(t_{-j}), t_j, t_{-j}) dP_{-j}(t_{-j})$$

$$V_j(a_j, t_j, P_{-j}) = \int_{T_{-j}} u_j(a_j, \sigma_{-j}(t_{-j}), t_j, t_{-j}) dP_{-j}(t_{-j}).$$

As in Corollary 1, we have the first-order conditions

$$\begin{aligned} \partial V'_j(\sigma'_j(t_j), t_j, p'_j(t_j)) / \partial a_{j1} &= 0 \\ \partial V_j(\sigma_j(t_j), t_j, p_j(t_j)) / \partial a_{j1} &= 0, \end{aligned}$$

and we need to show that

$$(6) \quad \partial V'((\sigma_{j1}(t_j), \sigma'_{j2}(t_j)), t_j, p'_j(t_j)) / \partial a_{j1} > \partial V_j((\sigma_{j1}(t_j), \sigma_{j2}(t_j)), t_j, p_j(t_j)) / \partial a_{j1},$$

implying that  $\sigma'_{j1}(t_j) \neq \sigma_{j1}(t_j)$ .

Inequality (6) involves three substitutions when comparing the right-hand side with the left-hand side, which we can make one at a time. First, we substitute  $\sigma'_{j2}(t_j) \geq \sigma_{j2}(t_j)$ , which raises the value weakly because  $\partial u_j / \partial a_{j1}$  is increasing in  $a_{j2}$  ( $u_j$  is supermodular in  $a_j$ ). Then we substitute  $\sigma'_{-j} \geq \sigma_{-j}$ , which raises the value weakly because  $\partial u_j / \partial a_{j1}$  is increasing in  $a_{-j}$  ( $u_j$  has increasing differences in  $(a_j, a_{-j})$ ). Finally we substitute  $p'_j(t_j) > p_j(t_j)$ , which causes a strict rise in the value because  $\partial u / \partial a_{j1}$  is increasing in  $t_{-j}$ , strictly so under assumption 1 of the corollary, and because  $\partial u / \partial a_{j1}$  is increasing in  $a_{-j}$  and  $\sigma'_{-j}$  is increasing in  $t_{-j}$ , strictly so for  $a_{i1}$  and  $\sigma'_{i1}$  under assumption 2 of the corollary (that  $\sigma'_{i1}$  is strictly increasing was shown in Corollary 2).  $\square$

## 10 Games of voluntary disclosure

A leading application of the comparative statics result in Proposition 3 is to two-stage games in which information is revealed in the first stage. It is then important to know how the equilibria of the second stage—in particular, the players' second-stage payoffs—depend on the information structure that results from the first stage in order to understand the players' incentives to influence this information structure.

Consider the parameterized family  $\{\Gamma(p) \mid p \in \mathcal{P}\}$  of monotone supermodular Bayesian games, as defined in Section 9. Each game has a greatest equilibrium, which we denote by  $\bar{\sigma}(p)$ . Let  $W_i(p, t_i)$  be player  $i$ 's expected utility in the equilibrium  $\bar{\sigma}(p)$  of the game  $\Gamma(p)$ , conditional on  $i$ 's type being  $t_i$ .

Assume that the Bayesian games have positive externalities, meaning that  $u_i$  is increasing in  $a_{-i}$  for all  $i \in N$ . According to Proposition 3,  $\bar{\sigma}(p)$  is increasing in  $p_{-i}$ . It follows that  $W_i(p, t_i)$  is increasing in  $p_{-i}$ . That is, higher beliefs by player  $j \neq i$  lead to higher equilibrium actions, which lead to higher expected utility for player  $i$ . This is summarized in Proposition 4.

**Proposition 4** *Let  $i \in \{1, \dots, N\}$  and assume that  $u_i$  is increasing in  $a_{-i}$ . For  $p \in \mathcal{P}$  and for  $t_i \in T_i$ , let  $W_i(p, t_i)$  be player  $i$ 's expected utility in the greatest equilibrium of  $\Gamma(p)$ , conditional on being of type  $t_i$ . Then  $W_i(p, t_i)$  is increasing in  $p_{-i}$ .*

Thus, if a unique equilibrium exists or if the equilibrium selection in the second stage is of the greatest or least equilibrium, then the players' incentives in the first stage are to induce the other players to increase their beliefs.

Corollary 4 states a strict version of this result. It follows immediately from Corollary 3.

**Corollary 4** *Let  $i, j \in \{1, \dots, N\}$  with  $i \neq j$  be such that (a) the assumptions of Corollary 3 are satisfied and (b)  $u_i$  is strictly increasing in  $a_j$ . Then  $W_i(p, t_i)$  is strictly increasing in the marginal of  $p_j$  on  $T_i$ . That is, if  $p'_{-i} \geq p_{-i}$  and the marginal of  $p'_j(t_j)$  on  $T_i$  strictly first-order stochastically dominates that of  $p_j(t_j)$  for  $t_j$  in a  $p_i(t_i)$ -nonnull set of  $t_j \in T_j$ , then  $W_i((p_i, p'_{-i}), t_i) > W_i((p_i, p_{-i}), t_i)$ .*

Consider the setting in Okuno-Fujiwara et al. (1990). In the first stage, there is only information revelation. Talk is cheap: it does not affect payoffs except through the play in the second stage. However, a player's message is a statement that her type belongs to a set of types, and she cannot lie because messages are verifiable. Stated another way, for each message there is a set of types who can send that message. Let  $M_i$  be the set of messages of player  $i$ ; treat each  $m_i \in M_i$  also as the set of  $i$ 's types that can send message  $m_i$ . (We endow  $M_i$  with a  $\sigma$ -field for measurability restrictions.) Let  $M = \prod_{i \in N} M_i$ .

A first-stage strategy for player  $i$  is a measurable map  $r_i: T_i \rightarrow M_i$  such that, for a.e.  $t_i \in T_i$ , we have  $t_i \in r_i(t_i)$ . A second-stage strategy is a measurable map  $q_i: T_i \times M \rightarrow A_i$

and a second-stage belief function is a measurable map  $\pi_i: T_i \times M \rightarrow \Delta(T_{-i})$  such that, for  $t_i \in T_i$  and  $m \in M$ ,  $\pi_i(t_i, m)$  puts probability 1 on  $\prod_{j \neq i} m_j$ .

Observe that, given  $q_i$  and  $\pi_i$ , each realization  $m \in M$  of the messages induces a posterior mapping  $\pi_i(\cdot, m): T_i \rightarrow \Delta(T_{-i})$  and a strategy  $q_i(\cdot, m): T_i \rightarrow A_i$  in the second-stage game. Then  $(r_i, q_i, \pi_i)_{i \in N}$  is a *perfect Bayesian equilibrium* (PBE) if the following statements hold.

1. (Belief consistency)  $\pi_i$  is a posterior mapping given the information  $(t_i, (r_j(t_j))_{j \neq i})$ .
2. (Equilibrium in second stage) For all  $m \in M$ ,  $(q_i(\cdot, m))_{i \in N}$  is a Bayes-Nash equilibrium of the game  $\Gamma((\pi_i(\cdot; m))_{i \in N})$ .
3. (Equilibrium in first stage) For a.e.  $t_i \in T_i$ ,  $r_i(t_i)$  solves

$$\max_{\substack{m_i \in M_i \\ t_i \in m_i}} \int_{T_{-i}} u_i(q_i(t_i, m_i, r_{-i}(t_{-i})), q_{-i}(t_{-i}, m_i, r_{-i}(t_{-i})), t_i, t_{-i}) dp_i(t_{-i} | t_i).$$

Proposition 5 states that there is a fully revealing equilibrium under the following conditions.

- There are strategic complementarities and positive externalities, and there are complementarities between actions and types (assumption 1 in Proposition 5),
- For each message, there is a lowest type who can send the message (assumption 2), and for each type, there is message for which it is the lowest type (assumption 3).
- As a technicality, the following must be measurable: the “skeptical” second-stage beliefs, which conclude from each profile of messages that senders are of the lowest possible types (assumption 4); and a mapping that assigns to each type  $t_i$  a message such that  $t_i$  is the lowest type who can send the message (assumption 5).

**Proposition 5** *Assume that, for each  $i \in N$ , the following hold:*

1.  $u_i$  satisfies the assumptions of Theorem 1 and is increasing in  $a_{-i}$ ;
2. for each  $m_i \in M_i$ ,  $\min m_i$  exists;
3. for each  $t_i \in T_i$ , there exists  $m_i \in M_i$  such that  $\min m_i = t_i$ ;
4. there is a measurable map  $\pi_i^*: T_i \times M \rightarrow \Delta(T_{-i})$  such that, for  $t_i \in T_i$  and  $m \in M$ ,  $\pi_i^*(t_i, m)$  puts probability 1 on  $(\min m_j)_{j \neq i}$ ;
5. there is a measurable map  $r_i^*: T_i \rightarrow M_i$  such that  $t_i = \min r_i^*(t_i)$  for all  $t_i \in T_i$ .

Let  $q_i^*: T_i \times M \rightarrow A_i$  be such that  $q_i^*(\cdot, m)$  is the largest Bayes-Nash equilibrium in the game  $\Gamma((\pi_j^*(\cdot, m))_{j \in N})$  for each  $m \in M$ . Then  $(r_i^*, q_i^*, \pi_i^*)_{i \in N}$  is a perfect Bayesian equilibrium.

PROOF. The messages  $(r_i^*)_{i \in N}$  are fully revealing. Since the second-stage beliefs  $(\pi_i^*)_{i \in N}$  deduce (correctly, when on the equilibrium path) that a message  $m_j$  is sent by  $\min m_j$ , they satisfy belief consistency. Here  $q^*$  is defined so that  $q^*(m)$  is an equilibrium in the second stage, given  $m$ . For each message  $m$ , the second-stage game is effectively one of complete information and satisfies the assumptions of Theorem 1 (in particular, the increasing posteriors condition is satisfied trivially because posteriors are type-independent). We can apply Proposition 4 to conclude that each player would like the other players to believe he is as high a type as possible. Given the skeptical beliefs, this is achieved for type  $t_i$  by reporting a message  $m_i$  such that  $t_i = \min m_i$ . Now  $(r_i^*, q_i^*, \pi_i^*)_{i \in N}$  constitute a perfect Bayesian equilibrium.  $\square$

Okuno-Fujiwara et al. (1990) not only show existence of a fully revealing sequential equilibrium, they also provide conditions under which all sequential equilibria are fully revealing. We can do the same, with greater generality. They have unidimensional action spaces, strict concavity of payoffs (in own action), independent types, and unique interior equilibria in the second stage. All but one of their results concerns two-player games.<sup>7</sup>

Our greater generality requires two equilibrium refinements that are automatically satisfied in Okuno-Fujiwara et al. (1990). First, to apply Proposition 4 and Corollary 4, the second-stage beliefs should be monotone in type on and off the equilibrium path. The independent-types assumption in Okuno-Fujiwara et al. (1990) guarantees that beliefs are type-independent (hence trivially monotone) on and off the equilibrium path in any sequential equilibrium. In our model, if types are one-dimensional and affiliated, then for any PBE the second-stage beliefs are increasing in type for any equilibrium messages: conditioning on an equilibrium message is like conditioning on a sublattice of types, given that type spaces are one-dimensional. We have not investigated whether sequential equilibrium implies that this property holds for non-equilibrium messages; instead, we simply add this as an equilibrium refinement.

Second, whereas Okuno-Fujiwara et al. (1990) assume a unique equilibrium in any second-stage subgame, we instead require that the equilibrium selection in the second stage be of the greatest (or least) equilibrium.

**Proposition 6** *Assume that the prior distribution  $\mu$  is affiliated and that, for each  $i \in N$ :*

1.  $T_i$  is one-dimensional and finite;
2.  $p_i(t_i)$  has full support for all  $t_i \in T_i$ ;
3.  $u_i$  satisfies the assumptions of Theorem 1 and is increasing in  $a_{-i}$ ;
4. the smooth case holds for player  $i$ ;

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<sup>7</sup>The only case not covered by our results but covered in Okuno-Fujiwara et al. (1990) is an  $n$ -player strategic substitutes game with quadratic payoffs.

5. there is a player  $j \neq i$  such that the assumptions of Corollary 3 hold and  $u_i$  is strictly increasing in  $a_{j1}$ ;
6. for each  $m_i \in M_i$ ,  $\min m_i$  exists; and
7. for each  $t_i \in T_i$ , there exists  $m_i \in M_i$  such that  $\min m_i = t_i$ .

Consider a perfect Bayesian equilibrium  $(r_i^*, q_i^*, \pi_i^*)_{i \in N}$  in which (a) for  $m \in M$  not in the range of  $r^*$ ,  $\pi_i^*(t_i, m)$  is increasing in  $t_i$  for  $i \in N$  and (b)  $(q_i^*(\cdot, m))_{i \in N}$  is the greatest (or least) Bayes-Nash equilibrium in the game  $\Gamma((\pi_j^*(\cdot, m))_{i \in N})$  for each  $m \in M$ . Then, for each player  $i \in N$ ,  $r_i^*$  is fully revealing—specifically, for each type  $t_i$ ,  $t_i = \min r_i^*(t_i)$ .

Note that beliefs are skeptical on the equilibrium path, since for any equilibrium message  $m$ , the player  $j \neq i$  correctly deduces that player  $i$  is of type  $\min m_i$ .

PROOF. Suppose  $(r_i^*, q_i^*, \pi_i^*)_{i \in N}$  is a PBE that satisfies conditions (a) and (b) but is not fully revealing for player  $i$ . Let  $\bar{t}_i$  be the highest type for  $i$  that is not fully revealed in the first round; hence  $\bar{t}_i$  is being pooled with lower types. If he deviates and sends a message  $m_i$  such that  $\bar{t}_i = \min m_i$ , then the other players' posteriors about his type go up by strict first-order stochastic dominance (the assumption on full supports of posteriors rules out the case where e.g. types are perfectly correlated and hence messages have no effect on beliefs). Hence, according to Corollary 4, his second-stage payoff increases strictly. (Given the restriction on  $\pi_i^*$ , the second-stage game satisfies the assumptions in this paper.) This contradicts the assumption that  $(r_i^*, q_i^*, \pi_i^*)_{i \in N}$  is a PBE.

Suppose that, for some player  $i$  and type  $t_i$ ,  $t_i > \min r_i^*(t_i)$ . Since  $r_i^*$  is fully revealing, following message  $r_i^*(t_i)$  all other players believe with probability 1 that  $i$  is of type  $t_i$ . Then type  $\min r_i^*(t_i)$  could deviate from his message by sending instead the message  $r_i^*(t_i)$ , causing a shift in all player's beliefs from his being of type  $\min r_i^*(t_i)$  with probability 1 to his being of type  $t_i$  with probability 1. Again, according to Corollary 4, his second-stage payoff increases strictly and so  $(r_i^*, q_i^*, \pi_i^*)_{i \in N}$  is not a PBE.  $\square$

Results analogous to Propositions 5 and 6 can be obtained by replacing the assumption of positive externalities by negative externalities (each player's payoff is decreasing in the action of the other players) and replacing the “min” conditions on messages and beliefs by “max”. Then each player would like to reduce the beliefs of other players and there is a fully revealing equilibrium in which each type sends a message for which he is the highest possible type that can send the message (or, under the stricter assumptions of Proposition 6, every PBE satisfying the two refinements has this property).

## 11 Concluding remarks

For games of incomplete information with supermodular payoffs (not merely payoffs with single-crossing properties), we are able to extend various results on existence of mono-

tone pure-strategy equilibria by using quite different methods. For example, we are able to dispense with atomless type spaces, and we can easily handle multidimensional type and action spaces. Beyond such generalizations, the other value of this work is the simplicity with which the results can be obtained, in comparison to games whose payoffs are not supermodular. Furthermore, we do not merely show existence; we also show that the greatest and least equilibria are in monotone strategies. We can thereby perform comparative statics on these equilibria.

We remind the reader that these results can be applied more generally by choosing the right direction of the orderings. For example, the main results can be applied to a submodular duopoly game—meaning that  $u_i$  is supermodular in  $a_i$ , has decreasing differences in  $(a_i, a_{-i})$ , has increasing differences in  $(a_i, t_i)$ , and has decreasing differences in  $(a_i, t_{-i})$ —because changing the order of the strategy and type spaces of one player (via multiplying by  $-1$ ) transforms the submodular game into a supermodular game (Vives (1990)) with complementarity between actions and types. Similarly, if all payoffs have decreasing rather than increasing differences in actions and types, but the other assumptions of this paper hold, then we can reverse the ordering of types and apply the results of this paper. For example, under the assumptions of Theorem 1, there are greatest and least equilibria and these are *decreasing* in type (under the original ordering on types).

## A Summary of lattice and comparative statics methods

For the convenience of the reader and to fix some notation and terminology that may vary from author to author, we include a few definitions and results of lattice methods as used for monotone comparative statics. More complete treatments can be found in Topkis (1998) and Vives (1999, Chapter 2).

A binary relation  $\geq$  on a nonempty set  $X$  is a *partial order* if  $\geq$  is reflexive, transitive, and antisymmetric. An upper bound on a subset  $A \subset X$  is  $z \in X$  such that  $z \geq x$  for all  $x \in A$ . A greatest element of  $A$  is an element of  $A$  that is also an upper bound on  $A$ . Lower bounds and least elements are defined analogously. The greatest and least elements of  $A$ , when they exist, are denoted  $\max A$  and  $\min A$ , respectively. A supremum (resp., infimum) of  $A$  is a least upper bound (resp., greatest lower bound); it is denoted  $\sup A$  (resp.,  $\inf A$ ).

A *lattice* is a partially ordered set  $(X, \geq)$  in which any two elements have a supremum and an infimum. A lattice  $(X, \geq)$  is *complete* if every non-empty subset has a supremum and an infimum. A subset  $L$  of the lattice  $X$  is a *sublattice* of  $X$  if the supremum and infimum of any two elements of  $L$  belong also to  $L$ .

Let  $(X, \geq)$  and  $(T, \geq)$  be partially ordered sets. A function  $f: X \rightarrow T$  is *increasing* if, for  $x, y$  in  $X$ ,  $x \geq y$  implies that  $f(x) \geq f(y)$ .

A function  $g: X \rightarrow \mathbb{R}$  on a lattice  $X$  is *supermodular* if, all  $x, y$  in  $X$ ,  $g(\inf(x, y)) + g(\sup(x, y)) \geq g(x) + g(y)$ . It is *strictly supermodular* if the inequality is strict for all pairs

$x, y$  in  $X$  that cannot be compared with respect to  $\geq$  (i.e., neither  $x \geq y$  nor  $y \geq x$  holds). A function  $f$  is (*strictly*) *submodular* if  $-f$  is (strictly) supermodular; a function  $f$  is (*strictly*) *log-supermodular* if  $\log f$  is (strictly) supermodular.

Let  $X$  be a lattice and  $T$  a partially ordered set. The function  $g: X \times T \rightarrow R$  has (*strictly*) *increasing differences* in  $(x, t)$  if  $g(x', t) - g(x, t)$  is (strictly) increasing in  $t$  for  $x' > x$  or, equivalently, if  $g(x, t') - g(x, t)$  is (strictly) increasing in  $x$  for  $t' > t$ . Decreasing differences are defined analogously. If  $X$  is a convex subset of  $\mathbb{R}^n$  and if  $g: X \rightarrow R$  is twice-continuously differentiable, then  $g$  has increasing differences in  $(x_i, x_j)$  if and only if  $\frac{\partial^2 g(x)}{\partial x_i \partial x_j} \geq 0$  for all  $x$  and  $i \neq j$ .

Supermodularity is a stronger property than increasing differences: If  $T$  is also a lattice and if  $g$  is (strictly) supermodular on  $X \times T$ , then  $g$  has (strictly) increasing differences in  $(x, t)$ . The two concepts coincide on the product of linearly ordered sets: If  $X$  is such a lattice, then a function  $g: X \rightarrow \mathbb{R}$  is supermodular if and only if it has increasing differences in any pair of variables.

The main comparative-statics tool applied in this paper is the following.

**Lemma 2** *Let  $X$  be a compact lattice and let  $T$  be a partially ordered set. Let  $u: X \times T \rightarrow \mathbb{R}$  be a function that (a) is supermodular and continuous on the lattice  $X$  for each  $t \in T$  and (b) has increasing differences in  $(x, t)$ . Let  $\varphi(t) = \arg \max_{x \in X} u(x, t)$ . Then:*

1.  $\varphi(t)$  is a non-empty compact sublattice for all  $t$ ;
2.  $\varphi$  is increasing in the sense that, for  $t' > t$ , for  $x' \in \varphi(t')$  and  $x \in \varphi(t)$ , we have  $\sup(x', x) \in \varphi(t')$  and  $\inf(x', x) \in \varphi(t)$ ; and
3.  $t \mapsto \max \varphi(t)$  and  $t \mapsto \min \varphi(t)$  are well-defined increasing functions.

## B Extension of comparative statics under uncertainty

For monotonicity of best responses to monotone strategies, we extend the approach in Athey (2000, 2001) to our more general type and action spaces. The main idea is that we characterize when a first-order stochastic dominance shift in beliefs causes the solutions to a decision problem under uncertainty to increase. This is a straightforward generalization of classic results for univariate actions and states with differentiable and strictly concave utility (as presented e.g. by Hadar and Russell (1978)) and of the more recent results by Athey (2000, Example 2), which are also univariate but without the differentiability and strict concavity.

These comparative statics results are related to the one-dimensional results in Athey (2002) for utility functions that satisfy single-crossing properties. However, because we restrict attention to supermodular utility, we have weaker conditions on beliefs (first-order

stochastic dominant shifts rather than log-supermodular densities) and the results are simpler and apply easily to discrete and multidimensional action and state spaces.

We first state and characterize a definition of first-order stochastic dominance for general partially ordered state spaces; it is the obvious extension of first-order stochastic dominance for probability measures on  $\mathbb{R}$ .

Let  $(\Omega, \mathcal{F})$  be a measurable space and let  $\geq$  be a partial order on  $\Omega$ . A set  $E \in \mathcal{F}$  is said to be *increasing* if  $\omega \in E$ ,  $\omega' \in \Omega$ , and  $\omega' \geq \omega$  imply  $\omega' \in E$ . Let  $P^H$  and  $P^L$  be two probability measures on  $(\Omega, \mathcal{F})$ . We say that  $P^H$  first-order stochastically dominates  $P^L$  if and only if  $P^H(E) \geq P^L(E)$  for all increasing  $E \subset \mathcal{F}$ .

**Lemma 3** *The following statements are equivalent.*

1.  $P^H$  f.o.s.d.  $P^L$ .
2. For all increasing functions  $f: \Omega \rightarrow \mathbb{R}$  that are integrable with respect to  $P^H$  and  $P^L$ ,

$$\int_{\Omega} f(\omega) dP^H \geq \int_{\Omega} f(\omega) dP^L.$$

PROOF. This is a simple “bootstrapping” of the result for the case where  $\Omega = \mathbb{R}$ .

(2)  $\Rightarrow$  (1). A set  $E \in \mathcal{F}$  is increasing if and only its indicator  $\mathbf{1}_E$  is an increasing function. Then  $P^H(E) = \int \mathbf{1}_E dP^H \geq \int \mathbf{1}_E dP^L = P^L(E)$ .

(1)  $\Rightarrow$  (2). Consider the distributions  $\pi^H$  and  $\pi^L$  of the random variable  $f$  for the two probability measures  $P^H$  and  $P^L$ , respectively. We show that  $\pi^H$  f.o.s.d.  $\pi^L$ . The result then follows since, for example,  $\int f(\omega) dP^H$  is the expected value for the distribution  $\pi^H$ .

Let  $\alpha \in \mathbb{R}$ . Then  $f^{-1}([\alpha, \infty))$  and  $f^{-1}((\alpha, \infty))$  are increasing measurable sets. (For instance, let  $\omega \in f^{-1}([\alpha, \infty))$ ; then  $f(\omega) \geq \alpha$ . Let  $\omega' \in \Omega$  be such that  $\omega' \geq \omega$ ; then  $f(\omega') \geq f(\omega)$  because  $f$  is increasing. Hence,  $f(\omega') \geq \alpha$  and  $\omega' \in f^{-1}([\alpha, \infty))$ .) Therefore,  $\pi^H([\alpha, \infty)) = P^H(f^{-1}([\alpha, \infty))) \geq P^L(f^{-1}([\alpha, \infty))) = \pi^L([\alpha, \infty))$ . Similarly,  $\pi^H((\alpha, \infty)) \geq \pi^L((\alpha, \infty))$ . Therefore,  $\pi^H$  f.o.s.d.  $\pi^L$ .  $\square$

Let  $X$  be a partially ordered set and let  $u: X \times \Omega \rightarrow \mathbb{R}$  be measurable in  $\omega$ . Let  $\mathcal{M}$  be the set of probability measures on  $(\Omega, \mathcal{F})$ , partially ordered by first-order stochastic dominance. Define  $U: X \times \mathcal{M} \rightarrow \mathbb{R}$  by  $U(x, P) = \int_{\Omega} u(x, \omega) dP(\omega)$ , when well-defined.

**Lemma 4** *Assume that  $u$  has increasing differences in  $(x, \omega)$ . Then, on the domain of  $U$ ,  $U$  has increasing differences in  $(x, P)$ .*

PROOF. Let  $x^H, x^L \in X$  be such that  $x^H \geq x^L$ . Define  $h(\omega) = u(x^H, \omega) - u(x^L, \omega)$ , which is increasing in  $\omega$  because  $u$  has increasing differences in  $(x, \omega)$ . Then  $U(x^H, P) - U(x^L, P) = \int h(\omega) dP$ , which is increasing in  $P$  according to Lemma 3.  $\square$



Suppose that  $X$  is a lattice. Since supermodularity is preserved by integration,  $U$  is supermodular in  $x$  if  $u$  is supermodular in  $x$ . Therefore, we have the following corollary.

**Corollary 5** *Assume that  $u$  is supermodular in  $x$  and has increasing differences in  $(x, \omega)$ . Then  $P \mapsto \arg \max_{x \in X} U(x, P)$  is increasing in  $P$ .*

### C Affiliation and increasing posteriors

A sufficient—but not necessary—condition for the “increasing posteriors” condition is affiliation, as follows. We follow the discussion of affiliation in Milgrom and Weber (1982, Appendix). Consider a probability space  $(\Omega, \mathcal{F}, \pi)$  such that  $\Omega$  is a lattice. If  $\Omega = \mathbb{R}^k$  and  $\pi$  has a density  $f$ , then affiliation is equivalent to  $f$  being log-supermodular. The more general definition is that  $\pi$  is affiliated if and only if, for every measurable increasing set  $A, B \subset \Omega$  and every measurable sublattice  $S \subset \Omega$  (with positive measure),  $P(A \cap B | S) \geq P(A | S)P(B | S)$ .

**Lemma 5** *The measure  $\mu$  is affiliated if and only if, for all increasing sets  $A, B \subset \Omega$  and every sublattice  $S \subset \Omega$ , we have  $P(A | B \cap S) \geq P(A | B^c \cap S)$ .*

PROOF. The inequality  $P(A \cap B | S) \geq P(A | S)P(B | S)$  can be rewritten as

$$\frac{P(A \cap B | S)}{P(B | S)} \geq P(A | S)$$

or  $P(A | B \cap S) \geq P(A | S)$ . Since  $P(A | S) = P(B | S)P(A | B \cap S) + P(B^c | S)P(A | B^c \cap S)$ —that is,  $P(A | S)$  is a weighted average of  $P(A | B \cap S)$  and  $P(A | B^c \cap S)$ —it follows that  $P(A | B \cap S) \geq P(A | S)$  is equivalent to  $P(A | B \cap S) \geq P(A | B^c \cap S)$ .  $\square$

Now suppose that  $\Omega = \Omega_1 \times \Omega_2$ , where  $\Omega_1$  and  $\Omega_2$  are measurable sublattices of Euclidean space. Consider the posterior measure  $p(\omega_1)$  on  $\Omega_2$  conditional on the observation of  $\omega_1$ .

**Lemma 6** *If  $\pi$  is affiliated then, for a.e.  $\omega_1^H, \omega_1^L \in \Omega_1$  such that  $\omega_1^H > \omega_1^L$ , it follows that  $p(\omega_1^H)$  first-order stochastically dominates  $p(\omega_1^L)$ .*

PROOF. Assume first that  $\Omega$  is discrete. Let  $\omega_1^H, \omega_1^L \in \Omega_1$  have positive measure and be such that  $\omega_1^H > \omega_1^L$ . Let  $S = \{\omega_1^L, \omega_1^H\} \times \Omega_2$  and let  $B = \{\omega \in \Omega \mid \omega_1 \geq \omega_1^H\}$ . Clearly  $S$  is a sublattice and  $B$  is an increasing set. Furthermore,  $B \cap S = \{\omega_1^H\} \times \Omega_2$  and  $B^c \cap S = \{\omega_1^L\} \times \Omega_2$ . Let  $A_2 \subset \Omega_2$  be an increasing set and let  $A = \Omega_1 \times A_2$  (which is also increasing). Since  $\pi$  is affiliated,  $P(A | B \cap S) \geq P(A | B^c \cap S)$ , or  $P(\Omega_1 \times A_2 | \{\omega_1^H\} \times \Omega_2) \geq P(\Omega_1 \times A_2 | \{\omega_1^L\} \times \Omega_2)$ . This can be restated as  $P(A_2 | \omega_1^H) \geq P(A_2 | \omega_1^L)$ , which is the first-order stochastic dominance conclusion we seek.

For arbitrary (nondiscrete)  $\Omega$ , we first replace  $\omega_1^H$  and  $\omega_1^L$  in the previous argument by sublattices of  $\Omega_1$  with positive measure that are ordered (one lies entirely above the other). Then we use a standard limiting argument.  $\square$

The converse does not hold. Even if  $\Omega_1$  and  $\Omega_2$  are both subsets of  $\mathbb{R}$  and are thus one-dimensional,  $P(\cdot | \omega_1)$  and  $P(\cdot | \omega_2)$  can still be increasing even if  $\pi$  is not affiliated. Consider the following symmetric distribution (provided to us by Phil Reny):  $\Omega_1 = \Omega_2 = \{1, 2, 3\}$ , and  $\mu$  is defined as follows:

(7)

		$\omega_2$		
		1	2	3
$\omega_1$	1	1/20	1/20	1/20
	2	1/20	4/20	3/20
	3	1/20	3/20	5/20
		$\mu$		

Here  $P(\omega_2 | \omega_1)$  is increasing in  $\omega_1$  with respect to first-order stochastic dominance. However, the monotone-likelihood ratio, a known implication of affiliation, does not hold. Specifically,  $\mu(2, 2)/\mu(1, 2) > \mu(2, 3)/\mu(1, 3)$ .

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