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PUBLIC SECTOR SHADOW PRICES IN DISTORTED GENERAL EQUILIBRIUM MODELS

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ABSTRACT

In a closed competitive general equilibrium constant returns economy, consumer taxes separate consumer and producer prices of goods. Shadow prices appropriate for the evaluation of public sector projects are derived on alternative assumptions about what happens to taxes as public production changes. In some examples, equilibrium prices and shadow prices are computed, and these computations suggest both that shadow prices are quite sensitive to the precise specification of pre-existing distortions and also that partial equilibrium intuition about the relation between shadow prices and market prices may be seriously misleading. The possibility of extending the analysis to more realistic examples and models is discussed.

JEL classification: 021, 024, 321, 323

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NON-TECHNICAL SUMMARY

The objective of cost-benefit analysis is the evaluation of the production activities of the public sector. The 'social profitability' of public sector projects is calculated in a way analogous to the way that a private sector producer calculates the profitability of its activities, but in a cost-benefit appraisal, 'shadow prices' which reflect the social value of goods replace the market prices that are used in a private calculation of profitability. If we ignore complications introduced by income distributional issues, then in a perfectly competitive market, market prices and shadow prices will coincide, and cost-benefit analysis and private profitability calculation will be the same. If, however, there are market distortions, shadow prices and market prices will differ. This paper is concerned with the relationship between market prices and shadow prices when market distortions take the form of consumer taxes which cause consumers and private producers to face different prices.

The price which consumers pay for a good is one measure of social value of the good, as it measures what a consumer is willing to pay for an extra unit of the good. The price which producers face is an alternative measure of social value, since in a competitive market it is equal to the marginal cost of producing the good. In the presence of a consumer tax (or other distortion) these two measures will not coincide. One plausible reconciliation of the two is the use of a 'weighted-average' rule in which the shadow price of the good lies between the consumer and producer prices, the weights of the two market prices in the shadow price formula depending on the extent to which the impact of a change in public sector production of the good falls on consumers or on private producers.

I argue that this argument is not valid in a general equilibrium context in which the effects of distortions in other markets must be taken into account. A dual inequality theorem is used to derive a shadow price formula in a general equilibrium model and this formula is then used to calculate shadow prices in a particular small model for different sets of consumer taxes. It turns out that when there are large taxes, the shadow price of a good need not lie between the consumer and producer prices. (It also emerges that shadow prices are quite sensitive to the exact specification of the consumer taxes.)

The weakness of the 'weighted average' rule is that it assumes that the marginal costs faced by producers of one good is a proper measure of the social marginal cost of producing the good, but if there are distortions elsewhere in the economy, this will not be the case. Large agricultural subsidies, for example, will lead to expansion of the agricultural sector and will drive the market price of land above its social value. This will then raise the private marginal cost of producing other land-using goods above the social cost of production, and make the weighted-average argument invalid. The policy implication is that one should be cautious of using tools based on the logic of partial equilibrium in situations where there are significant links between markets.

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PUBLIC SECTOR SHADOW PRICES IN DISTORTED GENERAL EQUILIBRIUM MODELS

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1.0 INTRODUCTION

This paper investigates the properties of public sector shadow prices in an economy closed to foreign trade and subject to distorting consumer taxes. The tax rates are taken to be arbitrarily fixed at particular levels, though we shall see that such a statement is not unambiguous, and one of the principal questions to which this paper is addressed is whether the shadow prices which measure the social opportunity costs and the social benefits of goods produced and consumed in public sector projects are sensitive to the assumptions which are made about what happens to tax rates as the project is undertaken.

I assume that there is a single representative consumer in the economy and that any government surplus or deficit is distributed to the consumer as a lump-sum tax or transfer. This is equivalent to the assumption that the government has the ability to make lump-sum transfers between individuals and has used that ability optimally. That such an assumption sits happily with the assumption that there are arbitrarily fixed consumer taxes is not obvious, to say the least, though it is at least as defensible as the alternative assumption commonly made in the second-best literature that the government has full control over prices and taxes but no ability to distribute income directly. In the penultimate section some remarks are made on the possibility of extending the analysis presented in this paper to models with more realistic constraints on government action.

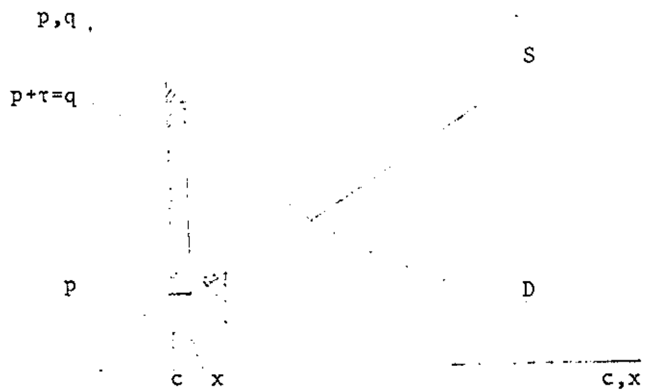

Figure 1

Figure 1 illustrates a partial equilibrium answer to the question of how to calculate the social value of a good removed from a distorted private market. Removal of one unit of the good requires consumption to fall and production to increase by amounts which together add up to one unit, and which imply equal increases in the consumer price q and the supply price p. The social opportunity cost of the good is therfore some weighted average of the price paid by consumers and the price received by suppliers (ignoring the second-order effects of changes in p and q themselves). This paper is concerned with investigation of whether a rigorous justification may be given in a general equilibrium model to this informal partial equilibrium argument, or whether the impact of distortions in the market for one good will seriously affect the calculation of the social opportunity cost of other goods.

It should be noted that the partial equilibrium argument is concerned with the case of a fixed distortion separating consumer and producer prices. A distortion which took the form of a fixed level of the producer (or consumer) price would, in the partial equilbrium approach imply that the social cost of the good would be given by the consumer (or producer) price; while a distortion in a market in which the supply curve (or demand curve) was horizontal would imply that the social cost of the good was the producer (or consumer) price. It is worth asking whether those arguments can be made in general equilibrium, but the main part of this paper does not address such cases.

The technique adopted here is the following. A general equilibrium is described in a model with many goods and many factors. The equilibrium is perturbed by a small change in the level of public production, and the impact on the level of real income of the consumer is derived, using the properties of the consumer's compensated demand functions and of the competitive supply functions. A mathematical characterization of shadow prices emerges. It should be clear from this that the shadow prices are defined as the prices that guide an arbitrarily small change in public production, and are not necessarily the shadow prices that would support the optimal level of public production.

Some general properties of these shadow prices are derived, but the precise way in which the shadow prices are affected by the pre-existing distortions is not transparent. We therefore proceed to compute equilibrium prices and quantities in a three-good, three-factor general equilibrium model, for different numerical values of the parameters of the production and utility functions, and then compute the values of the shadow prices associated with these equilibria, on various assumptions about the constancy of tax rates as the equilibrium is perturbed by a public production project.

The limitations of the analysis presented here should be noted at the outset.

(a) The examples which are presented here are from a three-factor, three-good model, but the techniques described could as easily be applied to a model with any number of goods and the same number of factors, and with significant but straightforward modifications to a model with more factors than goods. The model with more goods than factors, however, has quite different properties, some of which are discussed in a separate paper (Smith (1984)).

(b) The value of conclusions derived from arbitrary numerical examples is limited, and it would be desirable to recompute these examples in models of more realistic scale calibrated on real data, but this step has not yet been taken.

(c) It is unrealistic to suppose that all distortions in an economy are outside the government's control; but it is easy to modify the argument to allow for optimal choice of some tax rates. It may also be more realistic to have some of the distortions be expressed as fixed prices or quantities rather than as fixed tax rates and the derivation of shadow prices in such cases is quite different in detail from the principal case discussed in this paper (though the general point that optimal policy is sensitive to the precise specification of distortions is reinforced by the fact that these cases are so different from the case of fixed tax rates).

(d) The model considered here is one of "second-best" optimality because the government is assumed to have no control over commodity tax rates, although it does have the ability to redistribute income. An alternative way to model a second-best economy in which an optimising government is faced by irremovable distortions is to give the government full control over tax rates but no power to effect lump-sum redistribution, but this case is only briefly considered here. Diewert (1983) compares the properties and derivation of shadow prices in these two different kinds of second-best economies, in a model that is rather more general than the one used here.

(e) The model discussed here is of a closed economy, and introducing international trade would make for significant differences. If the economy is modelled as a small open economy in which all goods are traded at fixed world prices, the analysis of the effects of distortions is rather simple (see Smith (1982,section 7). On the other hand having some non-traded goods (as in Diewert (1983)) or making world prices endogenous gives rise to models similar in general to the one considered here but more complex in detail.

2.0 THE THEORY

Consider a competitive economy in which n goods are produced, using m factors of production, under constant returns to scale. There is a single consumer in the economy. Equilibrium is described by

$$c(q,u) = x(p,v^x) + g \qquad (1)$$

$$v = v^x + v^g \qquad (2)$$

$$q_i = p_i(1+t_i) \qquad i = 1,\ldots,n \qquad (3)$$

where c is the vector of compensated demand functions of the single consumer, x is the vector of competitive net supply functions, g is the vector of net supplies of goods by the public sector, all these being n-dimensional vectors. v is the vector of total supplies of factors available to the economy (assumed to be exogenous), v^x is the vector of factors used by the competitive producers, and v^g is the vector of factors used by the public sector, all these vectors being m-dimensional. u is the real income of the consumer. The functions c are homogeneous of degree 0 in q. The functions x are homogeneous of degree 0 in p and because of constant returns satisfy $px(p,v^x) = w(p,v^x)v^x$, where $w(p,v^x)$ are the competitive prices of the factors. The t are the *ad valorem* consumer tax rates.

In this formulation of the equilibrium, it is implicit that any surplus or deficit of the public sector appears in the consumer's budget constraint as lump-sum income. This is made explicit by letting τ be the vector with ith entry t_ip_i, so that we can write total consumer expenditure as

$$qc = \tau c + px + pg$$

$$= \tau c + wv^x + pg$$

$$= wv + [\tau c + pg - wv^g],$$

which shows that consumer income is the sum of factor earnings and of the net surplus of the public sector, that surplus consisting of the net revenue from consumer taxes and the net profit at competitive producer prices of public sector production.

Consider a perturbation of the equilibrium by changes dt, dg, dv^g, dv in the exogenous variables. The total differential of (1) is

$$Cdq + c_u du = Xdp + Zdv^x + dg \qquad (4)$$

where C and X are respectively the consumption and production substitution matrices $[\partial c_i/\partial q_j]$ and $[\partial x_i/\partial p_j]$ and Z is the matrix $[\partial x_i/\partial v_k]$, which implies

$$c_u du = (X - CT)dp - CPdt + Zdv + (dg - Zdv^g) \qquad (5)$$

where $CT = [(\partial c_i/\partial q_j)(1+t_j)]$ and $CP = [(\partial c_i/\partial q_j)p_j]$.

The matrix X - CT is singular, for

$$(X - CT)p = Xp - Cq = 0$$

from the homogeneity of x and c in their respective price variables, and we see that p is the right-hand eigenvector corresponding to the zero eigenvalue. Let k be a corresponding left-hand eigenvector, so that

$$k(X - CT) = 0 \tag{6}$$

<u>Assumption 1</u>: X - CT is of rank n-1

This assumption is made simply to rule out uninteresting complications. It follows from it that k is unique up to scalar multiplication.

Now pre-multiply (5) by k to give

$$(kc_u)du = -kCPdt + [kdg - w^k dv^g] + w^k dv \tag{7}$$

where

$$w^k = kZ = \{\sum_{i=1}^{n} k_i \partial x_i(p,v^x)/\partial v_j\}$$

Let us, for the moment, assume that $kc_u > 0$. It then follows from (7) that <u>we can interpret the k as shadow prices for goods and the w as factor shadow prices</u>: for if the tax rates t and the factor endowment vector v are constant while the public production project described by $\{dg, dv^g\}$ is implemented, then the effect of that project on welfare is indicated by its profitability calculated at these shadow prices. The factor shadow prices measure the value at goods shadow prices of the private-sector marginal products Z of the factors. Note that w^k measures the welfare effect of exogenous factor growth also. Then it is also clear what is the interpretation of (kc_u): it is the cost, at shadow prices, of an increase in welfare. If (kc_u) were non-positive, it would be possible to raise consumer welfare at no social cost, which suggests that this possibility can be ruled out by a suitable assumption about the rationality of policy in the initial equilibrium.

A formal proof of all of the above is required; and in that proof the economic interpretation of k as a shadow price vector emerges more clearly, as does the economic interpretation of the positivity of kc_u. It is convenient to replace (4) by a weak inequality, so permitting changes which create excess supply of goods. The proof uses Motzkin's dual inequality theorem (see Mangasarian (1969, p. 34)) which states that <u>either</u> $Ex \gg 0$, $Fx \geq 0$, $Gx = 0$ has a solution (where the matrix E is non-vacuous) <u>or</u> $y^1E + y^2F + y^3G = 0$, $y^1 > 0$, $y^2 \geq 0$ has a solution, but not both.

<u>Assumption</u> 2: There does not exist a $du > 0$ and a dp such that

$$c_u du \leq (X - CT)dp \qquad (8)$$

Informally, this assumption is that the original equilibrium is optimal at the given tax rates, factor endowments and public production levels. An equilibrium from which it was possible without any change in the exogenous variables except possibly disposal of excess supply to make the consumer better off and to remain in equilibrium would be suboptimal (and almost certainly unstable too), so this is a weak assumption.

Proposition 1: Assumption 2 is satisfied if and only if there exists a positive scalar λ and a non-negative vector k such that

$$\lambda = kc_u$$

and

$$0 = k(X - CT)$$

Proof

Direct application of Motzkin's theorem with $E = (1\ 0)$, $F = (-c_u\ (X - CT))$

(The argument above can be modified by removing the possibility of free disposal and replacing the inequality in (8) by equality. The effect of this modification is to remove the conclusion that k is non-negative, but to leave unchanged the conclusion that kc_u is positive.)

Proposition 2: For given dt, dv, dg, and dv^g, there exist du > 0 and dp satisfying

$$c_u du \leq (X - CT)dp - CPdt + Zdv + (dg - Zdv^g) \qquad (9)$$

if and only if

$$-kCPdt + w^k dv + [kdg - w^k dv^g] > 0$$

Proof

There exist du, dp such that

$$du > 0$$

$$-c_u du + (X - CT)dp - CPdt + Zdv + (dg - Zdv^g) \geq 0$$

if and only if there exist du, dp, z such that

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} du \\ z \\ dp \end{pmatrix} \gg \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -c_u & \begin{matrix} -CPdt + Zdv \\ + (dg - Zdv^g) \end{matrix} & X - CT \end{pmatrix} \begin{pmatrix} du \\ z \\ dp \end{pmatrix} \geq 0$$

if and only if by Motzkin's theorem, there do not exist $\lambda \geq 0$, $\mu \geq 0$, $k \geq 0$, with at least one of λ, μ strictly positive, such that

$$\lambda = kc_u$$

$$0 = k(X - CT)$$

$$-\mu = -kCPdt + w^k dv + [kdg - w^k dv^g]$$

But since we know from Proposition 1 that the first two of the equalities above are satisfied with $\lambda > 0$ and $k \geq 0$, the third equation can fail to be satisfied with $\mu \geq 0$ if and only if the right side is positive. Hence result.

This proposition gives rigorous justification to the earlier derivation of the shadow price vectors, but none of the above gives a clear indication of what will be the relation of the shadow prices to the market prices p, q, w, except that as all taxes go to zero so that p and q coincide, it is easily seen that k also coincides with p and q, this being the obvious property that shadow prices coincide with market prices when there are no distortions. It is also easy to show that in a two-good model, shadow prices are related to consumer and producer prices by a weighted-average formula. But before we investigate the relation between market prices and shadow prices when the distortions are non-zero and there are more than two goods, let us consider alternative assumptions about how prices and taxes change as public production is changed.

One alternative is to suppose that the taxes are defined as excise taxes and that it is the excise rates which are held constant. In this case, equation (5) is replaced by

$$c_u du = (X - C)dp - Cd\tau + Zdv + (dg - Zdv^g) \qquad (10)$$

where τ is the vector of excise tax rates. In order to consider the effects of a change in public production with these tax rates fixed, we need to specify a numeraire for the price vector p, for the tax rates are only economically meaningful relative to a numeraire. Let us therefore set $p_1 = 1$.

The matrices X and C are both singular, but let us assume that X - C is non-singular. Let h^1 be the first row of the inverse matrix $(X - C)^{-1}$. Pre-multiply (10) by h^1 to give

$$(h^1 c_u)du = -h^1 Cd\tau - w^1 dv + [h^1 dg - w^1 dv^g] \qquad (11)$$

using the fact that $dp_1 = 0$, and writing $w1 = h^1 Z$. An argument almost identical to the one used above to establish the positivity of (kc_u) will establish that $(h^1 c_u)$ is a positive number and that h^1 is a non-negative vector. Now it seems that the vector h^1 is the shadow price vector for goods while the factor shadow prices are given by **w** . Formal proofs along the lines of those above are not given because the arguments are so similar.

But what if we had chosen a different numeraire? Then, the same argument would have gone through with h^i, the ith row of $(X - C)^{-1}$, emerging as the shadow price vector for goods and $w^i = h^i Z$ as the factor shadow price vector.

It is easily seen that the h^i are different, for the rows of $(X - C)^{-1}$ must be linearly independent. Now to the question of what is the relation of k to p and q must be added the question of what is the relation of the h^i to these vectors and to each other, and of w, w^k and the w^i; that is to say, what is the relationship between market prices and shadow prices and how sensitive are the shadow prices to what is assumed about how tax rates change as public sector production changes. The mathematical characterization of the shadow prices at this level of generality is not suggestive of conclusions, so now I turn to a more specific model in which examples may be computed.

3.0 THE MODEL

The consumption side of the model is described by the utility function

$$u = \prod_{i=1}^{n} (c_i - a_i)^{b_i}$$

where $\sum_{i=1}^{n} b_i = 1$, from which are derived the compensated demand functions

$$c_i(q,u) = a_i + u(b_i/q_i) \prod_{k=1}^{n} (q_k/b_k)^{b_k}$$

so that the elements of the substitution matrix C are

$$\partial c_i(q,u)/\partial q_j = u(b_i/q_i)(b_j/q_j) \prod_{k=1}^{n} (q_k/b_k)^{b_k} \qquad (i \neq j)$$

$$-u(1-b_i)(b_i/q_i^2) \prod_{k=1}^{n} (q_k/b_k)^{b_k} \qquad (i=j)$$

where the a_i and the b_i are constants.

The production side is more complicated. Production of each good is assumed to require only primary factors of production, so that there are no intermediate inputs, and there is no joint production. Output of good j is given by the CES production function

$$x_j = (\sum_{k=1}^{n} \beta_{kj} z_k^{-\alpha_j})^{-1/\alpha_j} \qquad (\alpha_j > -1, \neq 0)$$

from which it is straightforward to derive the cost function and the cost-minimising input coefficients

$$a_{ij}(w) = (w_i/\beta_{ij})^{-1/(1+\alpha_j)} \gamma_j^{1/\alpha_j}$$

where

$$\gamma_j = \sum_{j=1}^{n} \beta_{kj} (w_k/\beta_{kj})^{\alpha_j/(1+\alpha_j)}$$

a_{ij} being the cost-minimising quantity of input i used in the production of one unit of good j. The effect on total demand of input i of a change in the price of input r is given by

$$s_{ir} = \sum_{j=1}^{n} (a_{ij}/ w_r) x_j$$

Now writing the matrix $[a_{ij}]$ of input coefficients as **A**, the equilibrium conditions on the production side of the model are

$$\mathbf{A}\mathbf{x} = \mathbf{v}$$

$$\mathbf{A}'\mathbf{w} = \mathbf{p}$$

The total differential of these equations is

$$\begin{pmatrix} S & A \\ A' & 0 \end{pmatrix} \begin{pmatrix} dw \\ dx \end{pmatrix} = \begin{pmatrix} dv \\ dp \end{pmatrix} \qquad (12)$$

where S is the matrix $[s_{ij}]$, since the envelope property of cost-minimisation implies that $dA'w = 0$. If $m = n$, and if A is nonsingular, the equation system inverts to give

$$\begin{pmatrix} dw \\ dx \end{pmatrix} = \begin{pmatrix} 0 & (A')^{-1} \\ A^{-1} & -A^{-1}S(A')^{-1} \end{pmatrix} \begin{pmatrix} dv \\ dp \end{pmatrix}$$

from which it follows that the matrix X is $-A^{-1}S(A')^{-1}$. We also see that $\partial w(p,v)/\partial v = 0$, which is the well-known local factor-price-equalisation property for the case of equal numbers of goods and factors. Finally, note that the factor shadow prices $w^k = k[\partial x/\partial v] = kA^{-1}$ have the property that $w^k A = k$, so that all private sector production activities have zero profitability at shadow prices as well as at market prices, a property noted by Diamond and Mirrlees (1976).

In the examples reported in the next section, there is the same number of goods and factors, but it is worth noting what are the properties of the system when m, the number of factors, is not equal to n, the number of goods.

In the case where $m>n$, the equation system (12) can be inverted if the matrix $\hat{S} = AA' - S$ is nonsingular (see Diewert and Woodland (1977)) and it is easily confirmed that

$$\begin{pmatrix} S & A \\ A' & 0 \end{pmatrix}^{-1} = \begin{pmatrix} D & Z' \\ Z & X \end{pmatrix}$$

where

$$B = [A'\hat{S}^{-1}A]^{-1}$$

$$D = -\hat{S}^{-1} + \hat{S}^{-1}ABA'\hat{S}^{-1}$$

$$Z = B'A'\hat{S}^{-1}$$

$$X = -I + B$$

so that $\partial w/\partial v$, $\partial w/\partial p$, $\partial x/\partial v$, $\partial x/\partial p$ are respectively given by D, Z', Z, and D. Now we do not have factor price equalisation, and the expression for X is more complex, but it is easily confirmed that the shadow prices have the Diamond-Mirrlees property.

The case of m<n, where there are more goods than factors, is more problematic. It is impossible to derive a supply function $x(p,v^X)$ because the equilibrium equations do not define a unique x for most possible values of p,v^X. The derivation of shadow prices in this case is rather different from the cases above and is discussed in a separate paper (Smith (1984)).

4.0 COMPUTATION OF SHADOW PRICES

In all the examples which we have computed, the initial equilibrium is one in which there is no public production. First, equilibrium prices and quantities were computed as follows. The parameters of the consumption and production systems were set, and exogenous tax rates and factor endowments specified. An initial guess at the factor price vector was made, from which were calculated the input coefficients, the goods prices (set equal to costs of production) and the full employment output levels. Next from the consumer income which such output levels implied were calculated the consumer demands and the implied factor demands. The initial guessed factor prices were then modified according to a fixed adjustment rule in response to the calculated excess demands, and the whole procedure repeated until it converged to equilibrium. Then the computation of shadow prices required simply computation of the equilibrium values of the matrices C, CT, X, and Z, from the equilibrium p, q and A; and then the h^i and k goods shadow prices together with the corresponding factor shadow prices were computed as indicated in section 2.0.

All the examples reported below have three goods and three factors. The "minimum" consumption levels in the demand system are $a_1 = 0$, $a_2 = 0.5$, $a_3 = 1.0$; while the marginal consumption coefficients are $b_1 = 0.5$, $b_2 = 0.3$, $b_3 = 0.2$. The production function elasticity coefficients are $\alpha_1 = -0.3$, $\alpha_2 = -0.2$, and $\alpha_3 = 2.0$, while the matrix $[\beta_{kj}]$ of scale coefficients is

$$\begin{pmatrix} 0.01 & 0.5 & 0.01 \\ 0.1 & 0.5 & 0.001 \\ 1.0 & 0.5 & 0.0001 \end{pmatrix}$$

The supplies of the three factors are $v_1 = 50$, $v_2 = 10$, $v_3 =$

40. Without loss of generality all prices are expressed in terms of the numeraire $p_1 = 1$, and the tax rate on the first good is zero (though of course only in the calculation of the h^1 prices are the tax rates held constant with respect to this numeraire). All shadow prices are also expressed as prices relative to the shadow price of the first good.

Example 1 is typical of equilibria with small distortion levels. The shadow price vectors are close together and in the distorted markets the shadow prices for goods are strictly between the producer and consumer prices.

It is Example 2 which illustrates how shadow prices may easily in this model have properties which contradict the conventional wisdom. First note that h^1 and k are significantly different from each other (and that, therefore, w^1 and w^k diverge significantly also). More important, so far from satisfying a "weighted-average" rule of the type suggested by the partial equilibrium argument of the introduction, each shadow price vector has an entry that lies outside the range of the producer and consumer price: k_3, h_3^2, and h_3^3 all exceed both p_3 and q_3, while h_2^1 is less than both p_2 and q_2.

It should be emphasised that the construction of this example has no peculiar features which would suggest that it is unusual or pathological. The distortion levels are large but not ridiculously large: the effective protection rates afforded to many industrial activities in developing countries and the subsidy rates granted to many agricultural activities in developed countries are in the 25% - 100% range or even larger. The features of Example 2 have been replicated in many other examples which we have computed: both the sensitivity of shadow prices to the definition of the constancy of tax rates, and shadow prices lying outside the range of producer and consumer prices are quite typical of examples with moderate distortion levels. On the other hand, it is worth repeating at this point that not too much weight should be given to small-scale examples in which the parameters are chosen arbitrarily.

Examples 3 and 4 are of a pathological nature; displaying the consequences of very high distortion levels. In Example 3, some factor shadow prices are negative: social welfare would be raised by disposal of the relevant factor. This is the phenomenon known in the international trade literature as "immiserising growth". In Example 4, some goods have negative shadow prices, indicating that welfare would be raised by disposal of the relevant good. (When h^1 satisfies (11) so does $-h^1$, and in Example 4 the shadow price vector quoted is, in each case, the one for which $h^i c_u > 0$.)

At this point, it is natural to turn back to the partial equilibrium argument in the introduction and ask why the answer it suggests turns out to be frequently so misleading.

The reason is simple: the partial equilibrium argument takes the supply curve as representing marginal social cost. In the event that there are distortions elsewhere in the economy, social and private marginal cost will not coincide. To put the point in a concrete example, large agricultural subsidies will lead to inefficient expansion of the agricultural sector and tend to raise the market price of land above the shadow price. But this will then tend to raise the supply curve in other land-using sectors above the social marginal cost curve and make the "weighted-average" argument untenable as it stands.

Example 1: $t_1 = 0$, $t_2 = 0.1$, $t_3 = 0.1$.

	0.0094	0.4281	0.1706
$A =$	0.0254	0.0575	0.0464
	0.8859	0.0722	0.0229

$x =$	36.381	75.102	102.605
$p =$	1.0000	0.2660	0.1302
$q =$	1.0000	0.2926	0.1432
$w =$	0.2622	1.3067	1.0885
$k =$	1.0000	0.2792	0.1375
$h^1 =$	1.0000	0.2786	0.1372
$h^2 =$	1.0000	0.2792	0.1375
$h^3 =$	1.0000	0.2792	0.1375
$w^k =$	0.2819	1.3925	1.0859
$w^1 =$	0.2810	1.3886	1.0860
$w^2 =$	0.2820	1.3918	1.0859
$w^3 =$	0.2819	1.3934	1.0858

Example 2: $t_1 = 0$, $t_2 = 1.0$, $t_3 = -0.25$.

		0.0116	0.4896	0.1788
A	=	0.0224	0.0490	0.0449
		0.8892	0.0691	0.0229
x	=	37.530	44.459	155.469
p	=	1.0000	0.2561	0.1296
q	=	1.0000	0.5123	0.0972
w	=	0.2267	1.4301	1.0856
k	=	1.0000	0.2853	0.1382
h^1	=	1.0000	0.2305	0.1103
h^2	=	1.0000	0.2855	0.1363
h^3	=	1.0000	0.2849	0.1421
w^k	=	0.2935	1.3556	1.0866
w^1	=	0.2099	1.0626	1.0951
w^2	=	0.3011	1.2814	1.0884
w^3	=	0.2784	1.5037	1.0831

Example 3: $t_1 = 0$, $t_2 = 20.0$, $t_3 = -0.95$.

$$A = \begin{matrix} 0.0010 & 0.2737 & 0.1914 \\ 0.0007 & 0.0118 & 0.0385 \\ 0.9585 & 0.3628 & 0.0444 \end{matrix}$$

x =	29.183	1.411	259.092
p =	1.0000	0.9154	0.9046
q =	1.0000	19.2237	0.0452
w =	1.2904	15.9116	1.0301
k =	1.0000	0.4573	0.1001
h^1 =	1.0000	0.3700	0.0299
h^2 =	1.0000	0.4798	0.0358
h^3 =	1.0000	0.4421	0.1434
w^k =	0.2904	-0.0486	1.0430
w^1 =	-0.0162	-0.3478	1.0436
w^2 =	0.4846	-2.6898	1.0448
w^3 =	0.1595	1.7310	1.0418

Example 4: $t_1 = 0$, $t_2 = -0.95$, $t_3 = -0.1$.

$$A = \begin{matrix} 0.0020 & 0.2477 & 0.1542 \\ 0.0083 & 0.0481 & 0.0462 \\ 0.9181 & 0.1663 & 0.0299 \end{matrix}$$

x =	8.119	193.258	13.743
p =	1.0000	0.5054	0.2832
q =	1.0000	0.0253	0.2549
w =	0.7719	2.8633	1.0616
k =	1.0000	0.3345	0.1536
h^1 =	-1.0000	-0.0427	0.0722
h^2 =	1.0000	0.3368	0.1556
h^3 =	-1.0000	0.0920	0.2151
w^k =	0.3375	1.5005	1.0749
w^1 =	-0.3544	-1.0896	1.0998
w^2 =	0.3404	1.5344	1.0746
w^3 =	-0.2416	-4.5770	1.1311

5.0 EXTENSIONS OF THE MODEL

From section 3.0, it should be clear that the computation of shadow prices in larger-scale models (with $m \geq n$) would be straightforward. More significant, however, is the question of whether the methods used in this paper could be extended to richer specifications of second-best constraints.

5.1 Many Consumers, Lump-sum Transfers

The first extension serves to justify the contention on page 1 that the single-consumer case is equivalent to the case in which the government has the ability to undertake lump-sum transfers.

Suppose now that there are H individuals with utility functions $u^h(c^h)$ and that the government's objective is to maximise the social welfare function $W(u^1,\ldots,u^H)$, with, as before, the consumer tax rates t_i fixed. It is convenient now, for the sake of cmparison with the case where there is no lump-sum redistribution, to represent consumer demand by the uncompensated demand functions $c^h(q,m^h)$ so that the equilibrium condition is

$$\sum_{h=1}^{H} c^h(q,m^h) = x(p,v^x) + g \qquad (13)$$

(which, as usual, implies the Walras' law relationship that $\sum_{h=1}^{H} m^h = \tau c + wv + [pg - wv^g])$.

Using Roy's identity gives

$$dW = \sum_{h=1}^{H} \beta^h d\omega^h$$

where $\beta^h = (\partial W/\partial u^h)(\partial V^h/\partial m)$ is the social marginal value of individual h's money income, V^h being the indirect utility function $V^h(q,m^h) = u^h(c^h(q,m^h))$, and where $d\omega^h$ is the money measure of a marginal change in individual h's utility

$$d\omega^h = dm^h - \sum_{j=1}^{n} c_j^h dq_j$$

and, from the Slutsky equation, the change in aggregate consumption becomes

$$dc = Cdq + \sum_{h=1}^{H} (\partial c^h(q,m^h)/\partial m) d\omega^h.$$

where $C = \sum_{h=1}^{H} \partial c^h(q,u^h)/\partial q$ is the aggregate Slutsky matrix.

Now corresponding to Propositions 1 and 2 in section 2.0 we obtain:

<u>Proposition 1A</u>: There does not exist a welfare-improving equilibrium-preserving change in prices and in income distribution if and only if there exists a positive scalar λ and a non-negative vector k such that

$$\lambda\beta^h = k(\partial c^h/\partial m) \qquad \text{for all } h$$

and

$$0 = k(X - CT)$$

Proposition 2A: For given dt, dv, dg and dv^g, there exists a welfare-improving, equilibrium-preserving change in prices and in income distribution if and only if

$$-kCPdt + w^k dv + [kdg - w^k dv^g] > 0$$

Thus, apart from an additional condition that explicitly states that income is optimally distributed, this case is identical to the single-consumer case.

The analysis of (strictly) Pareto-improving changes is very closely related. The criterion for welfare improvement is now that $d\omega^h > 0$ for all h. (The case of non-strict Pareto improvements is not significantly different.) It is straightforward to obtain:

Proposition 1B: There does not exist a strictly Pareto-improving, equilibrium-preserving change in prices and in income distribution if and only if there exist non-negative scalars λ^h, of which at least one is positive, and a non-negative vector k such that

$$\lambda^h = k(\partial c^h/\partial m) \quad \text{for all h}$$

and

$$0 = k(X - CT)$$

Proposition 2B: For given dt, dv, dg and dv^g, there exists a strictly Pareto-improving, equilibrium-preserving change in prices and in income distribution if and only if

$$-kCPdt + w^k dv + [kdg - w^k dv^g] > 0$$

Therefore, apart from making the relative social marginal costs of individual's income endogenous, this case is no different from the previous one.

5.2 Single Consumer, Some Taxes Controlled

The previous subsection serves to emphasise the restrictive nature of the assumptions made in the model presented in this paper. The rest of this section is addressed to the question of whether some of these assumptions could be relaxed.

The first such relaxation to consider is to allow some of the tax rates to be optimally chosen. Let (5) be written as

$$c_u du = (X - CT)dp - (CP)^c dt^c - (CP)^f dt^f + Zdv + (dg - Zdv^g)$$

where t^c are the controlled (endogenous) tax rates while t^f are the exogenously fixed tax rates, and CP is correspondingly partitioned. Hence

Proposition 1C: There does not exist a welfare-improving equilibrium-preserving change in prices and in the controlled tax rates, that is, a du > 0 and a dp and dt^c satisfying

$$-c_u du + (X - CT)dp - (CP)^c dt^c \geq 0$$

if and only if there exists a positive scalar λ and a non-negative vector k such that

$$\lambda = kc_u$$

$$0 = k(X - CT)$$

and

$$0 = k(CP)^c$$

Proposition 2C: For given dt^f, dv, dg and dv^g, there exists a welfare-improving, equilibrium-preserving change in the controlled tax rates if and only if

$$-k(CP)^f dt^f + w^k dv + [kdg - w^k dv^g] > 0$$

All of this looks remarkably similar to the earlier theory, in that the only change is the addition of a set of conditions characterising the optimal choice of the controlled tax rates, these conditions being a simple version of what Drèze and Stern (1983) call "generalized Ramsey-Boiteux" conditions. In practice, however, there are two significant differences. Computation of an set of shadow prices no longer follows directly from the computation of equilibrium prices:

the optimal levels of the controlled taxes must simultaneously be computed, presumably iteratively. More important, perhaps is the problem of reconciling observed levels of tax rates which are being assumed to be optimally chosen with the assumed welfare function. On this type of "inverse optimum" problem in a rather different context see Ahmad and Stern (1983).

5.3 Single Consumer, Some Prices Fixed

It is now relatively straightforward to introduce the possibility that some prices are fixed. In many contexts, this may be a better reflection of the political constraints on government action. For example, while in the short run it may be defensible to model farm subsidies as fixed, in the longer run it seems more plausible that farm prices should be the objective of protectionist agricultural policy.

For simplicity, I confine attention to the empirically more relevant case where it is some producer prices which are fixed and write (5) as

$$c_u du = (X - CT)^c dp^c + (X - CT)^f dp^f - (CP)^c dt^c - (CP)^f dt^f + Zdv + (dg - Zdv^g)$$

where p^c are the controlled (endogenous) prices while p^f are the exogenously fixed prices, and $X - CT$ is correspondingly partitioned. Now, it is better to refer to the tax rates t^c

as "endogenous" rather than "controlled", since in the presence of some fixed producer prices, some other prices will have to adjust endogenously to clear the markets. On this too see Drèze and Stern (1983). Hence

Proposition 1D: There does not exist a welfare-improving equilibrium-preserving change in the endogenous prices and taxes that is, a $du > 0$ and a dp^c and dt^c satisfying

$$-c_u du + (X - CT)^c dp^c - (CP)^c dt^c \geq 0$$

if and only if there exists a positive scalar λ and a non-negative vector k such that

$$\lambda = kc_u$$

$$0 = k(X - CT)^c$$

and

$$0 = k(CP)^c$$

Proposition 2D: For given dp^f, dt^f, dv, dg and dv^g, there exists a welfare-improving, equilibrium-preserving change in the endogenous prices and taxes if and only if

$$k(X - CT)^f dp^f - k(CP)^f dt^f + w^k dv + [kdg - w^k dv^g] > 0$$

Again, the changes in Proposition 1D are, from the viewpoint of actual calculation of shadow prices, more fundamental than they seem at first sight.

The very assumption that market-clearing equilibrium is attained implies that there are a sufficient number of endogenous variables. If not, we should have to introduce yet more complications arising from the non-clearing of markets. But even when markets clear, the determination of k would be considerably more complicated than was the case in the model of section 2.0.

5.4 Single Consumer, Some Quantities Fixed

Yet another way of modelling external constraints on government action is to let some *quantities* be fixed: this would be appropriate, for example, in the case of the "national defence" case for promoting particular activities. For simplicity, I confine attention to the case where the output levels of some goods are fixed.

Let the superscript f denote the subset of goods of which total output is fixed. (5) now has added to it

$$dc^f = X^f dp + Z^f dv + (dg^f - Z^f dv^g) = 0$$

and we obtain

Proposition 1E: There does not exist a welfare-improving equilibrium-preserving change in prices which satisfies the output constraints if and only if there exists a positive scalar λ, a non-negative vector k, and a vector η such that

$$\lambda = kc_u$$

$$0 = k(X - CT) + \eta X$$

where the vector η has zero entries corresponding to the unconstrained outputs.

Proposition 2E: For given dt, dv, dg, dv^g, dc^f, there exists a welfare-improving, equilibrium-preserving change in-prices satisfying the output constraints if and only if

$$-kCPdt + w^\eta dv + [(k+\eta)dg - w^\eta dv] - dc^f > 0$$

where

$$w^\eta = (k+\eta)Z$$

In short, as one would expect, the shadow prices of each of the constrained goods now has an extra element corresponding to the social cost of the constraint.

Note why the shadow price of a good whose output is constrained in this way is calculated differently from the shadow price of a factor. Each is a quantity whose total net output is fixed, but the quantities of factors available do not enter the utility function.

5.5 Many Consumers, No Lump-sum Transfers

Now I turn to consider briefly how the analysis would be changed by the removal of the assumption that lump-sum redistribution is available and used as a government policy.

For the sake of comparability with the model of sections 2.0-4.0, I assume that there are some tax rates fixed outside the control of the government, but that all prices are variable. Rents on the factors are assumed to be taxed at 100%. In the notation of 5.1, the money value of individual h's utility change is

$$d\omega^h = -\sum_{j=1}^{n} c_j^h dq_j$$

and the change in aggregate consumption is, as before,

$$dc = Cdq + \sum_{h=1}^{H} (\partial c^h(q,m^h)/\partial m) d\omega^h$$

The criterion for a welfare improvement is that

$$\sum_{h=1}^{H} \beta^h \left(-\sum_{j=1}^{n} c_j^h dq_j\right) > 0$$

while the equilibrium equation is

$$\sum_{h=1}^{H} (\partial c^h(q,m^h)/\partial m)\left(\sum_{j=1}^{n} c_j^h dq_j\right) + (X - CT)dp - (CP)^c dt^c - (CP)^f dt^f + Zdv + (dg - Zdv^g) = 0$$

from which follow in a now familiar pattern

Proposition 1E: There does not exist a welfare-improving equilibrium-preserving change in the prices and the controlled tax rates, that is a dp, dt^c satisfying

$$\sum_{h=1}^{H} \beta^h \left(-\sum_{j=1}^{n} c_j^h (1+t_j) dp_j - \sum_{j \in C} c_j^h p_j dt_j\right) > 0$$

and

$$\sum_{h=1}^{H} (\partial c^h(q,m^h)/\partial m)\left(\sum_{j=1}^{n} c_j^h (1+t_j) dp_j + \sum_{j \in C} c_j^h p_j dt_j\right) + (X - CT)dp - (CP)^c dt^c \geq 0$$

where C is the set of indices corresponding to the controlled tax rates, if and only if there exists a positive scalar λ and a non-negative vector k such that

$$\sum_{h=1}^{H}[\lambda\beta^h - k(\partial c^h(q,m^h)/\partial m]c_j^h(1+t_j) = k(X - CT)_j$$
$$\sum_{h=1}^{H}[\lambda\beta^h - k(\partial c^h(q,m^h)/\partial m]c_j^h p_j = -k(CP)_j \qquad j\epsilon C$$

Proposition 2F: For given dv, dg and dv^g, there exists a welfare-improving, equilibrium-preserving change in prices and in the controlled tax rates if and only if

$$w^k dv + [kdg - w^k dv^g] > 0$$

Now the shadow pricing rules incorporate what Drèze and Stern call the "distributional characteristics" of the goods and their calculation would be correspondingly harder. (In the statement of Proposition 2F, I have omitted the condition describing welfare-improving changes in the fixed tax rates, as they are more complex than in previous cases, and not relevant to the topic of this paper.)

In the special case where all tax rates are controlled, the Diamond-Mirrlees (1971) case, the above is easily shown to imply that k coincides with p, so there is aggregate production efficiency. In the case where the government has the instrument of a poll tax available to it, one extra optimality condition is added. If there were untaxed rents, there would be yet further complications. Finally, observe that all of this could be translated with no great change into statements about Pareto improvements.

6.0 CONCLUSION

The last section has emphasised the fact that the model presented in the core of this paper is based of rather restrictive assumptions. From one viewpoint that is not something to be apologetic about. To show the fragility and unreliability of partial equilibrium reasoning about general equilibrium situations it is quite appropriate to use the simplest model, in which the partial equilibrium model has the best chance of being applicable.

However, the computation of shadow prices in a simple general equilibrium model clearly raises the issue of whether a similar exercise could be undertaken in a computable general equilibrium model of more realistic scale, calibrated against real data, and including a more realistic range of second-best constraints. The analysis of the previous section suggests that at least the last of these three objectives could be met, in the sense that the general methods used in deriving the results presented above could be extended.

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