DISCUSSION PAPER SERIES

No. 4070

MICROECONOMIC SOURCES OF EQUITY RISK

Michael R Wickens

FINANCIAL ECONOMICS and INTERNATIONAL MACROECONOMICS

Centre for Economic Policy Research

www.cepr.org

Available online at: **www.cepr.org/pubs/dps/DP4070.asp**

ISSN 0265-8003

MICROECONOMIC SOURCES OF EQUITY RISK

Michael R Wickens, University of York and CEPR

Discussion Paper No. 4070
September 2003

Centre for Economic Policy Research
90–98 Goswell Rd, London EC1V 7RR, UK
Tel: (44 20) 7878 2900, Fax: (44 20) 7878 2999
Email: cepr@cepr.org, Website: www.cepr.org

CEPR Discussion Paper No. 4070

September 2003

ABSTRACT

Microeconomic Sources of Equity Risk

Surprisingly there are very few estimates of the equity risk premium period-by-period that satisfy a no-arbitrage condition, despite the vast literature on the subject. This is mainly due to the difficulties of estimation. Using the stochastic discount factor (SDF) model based on observable macroeconomic factors – as opposed to unobservable (latent) affine factors – and a new econometric methodology, we provide new estimates of the equity risk premium for the US and the UK based on monthly data 1975-2001. We obtain estimates of the risk premium for many of the best-known versions of consumption CAPM including time-separable power utility and time-nonseparable Epstein-Zin utility. We also show why many of the formulations of these models are unable to provide estimates of the risk premium. A related, and rapidly growing, literature that adopts a more statistical approach focuses on the empirical relation between the return on equity (or the Sharpe ratio) and return volatility. We argue that SDF theory implies that this relation is misconceived.

JEL Classification: C32, C51, E44 and G12
Keywords: consumption CAPM, Epstein-Zin model, equity risk premium, multivariate GARCH with no-arbitrage and stochastic discount factor model

Michael R Wickens
Department of Economics and
Related Studies
University of York
Heslington
York
YO1 5DD
Tel: (44 1904) 433 764
Fax: (44 1904) 433 575
Email: mike.wickens@york.ac.uk

Submitted 23 September 2003

"One definition of an economist is somebody who sees something happen in practice and wonders if it will work in theory." Ronald Reagan.

1 Introduction

The existence of an equity risk premium is easy to establish. The problem has been to find theoretical models that can explain it and to estimate the implied time-varying risk premia. Surprisingly, given the large literatures on modelling equity returns and on the equity premium puzzle, there are no estimates of a time-varying equity risk premium based on models with observable factors, such as general equilibrium models, that satisfy a no-arbitrage condition. Nearly all of the evidence on general equilibrium models has been obtained from either GMM estimates of the associated Euler equation, or by the use of calibration, neither of which provide estimates of the risk premium itself or show how it varies over time. The only estimates of a time-varying equity risk premium available are based on the use of unobservable affine factor models.

In this paper we propose a new econometric methodology which provides estimates of both the parameters of asset pricing model that are based on observable macroeconomic factors, and of the implied time-varying equity risk premium. Most of the best-known general equilibrium theories of asset pricing are of this type. These include consumption-based CAPM (C-CAPM) with power utility and the Epstein-Zin general equilibrium model with time non-separable preferences. These two theories are special cases of the stochastic discount factor (SDF) model of asset pricing, as is CAPM, a partial equilibrium model not involving macroeconomic factors - see for example Cochrane (2001). Our methodology can be applied more generally to the SDF model. For a survey of these problems and the methodology see Smith and Wickens (2002) and for an implementation of the methodology to FOREX see Wickens and Smith (2001).

Hansen and Singleton (1983) were one of the first to estimate a general equilibrium model of equity returns. Their model was based on power utility and they estimated the Euler equation by GMM. This was also the method employed by Epstein and Zin (1991) who used a constant elasticity version of Kreps and Porteus (1978) time non-separable utility. The main alternative has been to use calibration methods. The best-known example of this approach is that of Mehra and Prescott (1985) who used power utility. Campbell and Cochrane (1999) calibrated the Epstein-Zin model. The main finding for these models is that the coefficient of relative risk aversion must be implausibly large in order to match the observed (ex-post) risk premium. This is the equity premium puzzle. The response has been to try to modify the theory so that it possesses other means of capturing the risk premium, and doesn't have to rely solely on the coefficient of relative risk aversion to do so. One route is to assume that the utility function is time non-separable. This permits the coefficient of relative risk aversion to be different from the elasticity of inter-temporal substitution, and so introduces additional variables into the expression for the equity risk premium. Again the resulting risk premia are not estimated directly. GMM estimates of the Euler equation are obtained by Epstein and Zin (1991)), more ad hoc estimation methods are used by Campbell (1996), and calibration of the risk premium is undertaken by Campbell (2002).

A widely used alternative time non-separable model to that of Epstein-Zin model is the habit persistence model of Abel (1990) and Constantinides (1990). This has the effect of causing the inter-temporal marginal rate of substitution to become more variable and so be better able to match the observed volatility of equity returns. This approach has been implemented empirically by Abel (1990) using GMM, and by Campbell and Cochrane (1999) using calibration. Unfortunately both implementations were based on unconditional moments, and not conditional moments, which obscured the fact that the particular models of habitual consumption used are unable to deliver a non-zero time-varying risk premium.

The greater generality of the SDF model over with general equilibrium models allows us to consider whether there are other priced sources of equity risk. General equilibrium models imply that investors are concerned with future consumption, and in particular, consumption next period. The main holders of equity are financial institutions, especially pension funds. They act on behalf of investors' consumption at a much more distant point in the future. In assessing risk, financial institutions focus largely on short-term performance, and on the value of the portfolio. This suggests that the market equity risk premium may be more influenced by short-term price risk than longer-term considerations of consumption. The factors that affect the price of equity in the short term are associated with the business cycle and inflation. We therefore examine whether output and inflation are additional priced sources of equity risk. Similar considerations led to the finding in Wickens and Smith (2001) that the FOREX risk premium is better explained by short-term exchnage rate risk as captured by output and money growth rates, than consumption

In most existing empirical implementations of the SDF model, the "factors" are latent variables. Equations for the latent factors must be specified. Since data only exist for the returns, estimates of the parameters of these equations must be obtained from the likelihood function for returns. Estimates of the factors and a time-varying risk premium can then be backed out of the returns data. This is the approach commonly followed for bond pricing and has been used for pricing equity by Bekaert and Grenadier (1999). It is not, however, suitable for CAPM and C-CAPM because here the factors are observable macroeconomic variables, namely, consumption growth (among others) for C-CAPM, and the market interest rate in the case of CAPM. Ang and Piazzesi (2000) combine observed and unobserved factors, but are unable to measure the risk premium. The reason we propose our new econometric methodology is to obtain direct estimates of a time-varying risk premium for the SDF model when the factors are observable.

In the SDF model the risk premium is represented by the conditional covariances of excess equity returns with the factors. Our econometric methodology involves modelling the joint distribution of the excess return on equity and the observable factors using a multivariate $t-$distribution in which the covariance matrix is assumed to be generated by multivariate GARCH and the conditional mean of the distribution of the excess return is constrained to satisfy the no-arbitrage condition.

There is another, rapidly increasing, literature that focusses on the relation between equity returns and their volatility, see for example Campbell (1987), Baillie and DeGennero (1988), Campbell and Hentschel (1992), Glosten, Jagannathan and Runkle (1993), Scruggs (1998) and Lettau and Ludvigson (2002). This is motivated largely by statistical considerations than theoretical considerations such as satisfying a no-arbitrage condition. As a result, in principle, these statistical models admit unlimited arbitrage opportunities. One of the main findings is that average equity returns are positively related to their volatility. The usual interpretation, informally offered, is that higher volatility implies greater risk, and larger returns are required to compensate for this. In other words, the volatility is capturing a risk premium. Strictly, however, this explanation cannot be correct as risk should be expressed in terms of the covariance of returns with other factors, and not the volatility of returns. It would seem, therefore, that these statistically-based models are simply picking up the fact that one component of all of these covariances is the volatility of returns.

Using monthly data for the US and UK from 1975 to 2001, we confirm previous findings that the estimates of the coefficients for the power utility and Epstein-Zin models are implausibly large. We show that CAPM can be rejected as it ignores significanty priced sources of risk. A two-factor SDF model with consumption growth and inflation as the observable factors is preferred to the general equilibrium models (largely on the grounds that the coefficient estimates are not inconsistent with an SDF model), to CAPM whose restrictions are rejected, and to purely statistical models that relate equity returns to their volatility. We find that when included together with consumption, output is not a priced source of risk. The estimated equity risk premia display considerable time variation, tending to increase sharply during periods of negative excess returns before slowly declining, but no corresponding jump is discernable when excess returns are high.

2 Theoretical models of the equity risk premium

2.1 The SDF model of asset pricing for equity returns

The SDF model is based on the simple idea that P_t, the price of an asset at the beginning of period t, is determined by the expected discounted value of its pay-off at the start of period $t+1$, namely, X_{t+1}:

$$P_t = E_t[M_{t+1}X_{t+1}] \tag{1}$$

where M_{t+1} is the stochastic discount factor, or pricing kernel (see Cochrane (2001) for a survey of SDF theory). For equity, the payoff is $X_{t+1} = P_{t+1} + D_{t+1}$, where D_{t+1} are dividend payments assumed to be made at the start of period $t+1$. The pricing equation can also be written

$$1 = E_t[M_{t+1}\frac{X_{t+1}}{P_t}] = E_t[M_{t+1}R_{t+1}], \tag{2}$$

where $R_{t+1} = 1 + r_{t+1} = X_{t+1}/P_t = (P_{t+1} + D_{t+1})/P_t$ is the gross return and r_{t+1} is the return.

Taking logarithms and assuming log-normality - and noting that if $\ln x$ is $N(\mu, \sigma^2)$ then $\ln E(x) = \mu + \frac{\sigma^2}{2}$ - we obtain

$$\begin{aligned} 0 &= \ln E_t[M_{t+1}R_{t+1}] \\ &= E_t[\ln(M_{t+1}R_{t+1})] + V_t[\ln(M_{t+1}R_{t+1})]/2 \\ &= E_t(m_{t+1}) + E_t(r_{t+1}) + V_t(m_{t+1})/2 + V_t(r_{t+1})/2 + cov_t(m_{t+1}, r_{t+1}) \end{aligned}$$

where $m_{t+1} = \ln M_{t+1}$. Hence the pricing equation can be written

$$E_t(r_{t+1}) + E_t(m_{t+1}) + V_t(m_{t+1})/2 + V_t(r_{t+1})/2 = -cov_t(m_{t+1}, r_{t+1})$$

If the asset is risk-free then its return is known at the start of period t implying $r_{t+1} \equiv r_t^f$, $E_t(r_{t+1}) = r_t^f$ and $V_t(r_{t+1}) = 0$. The pricing equation for the risk-free asset can therefore be written

$$E_t(m_{t+1}) + r_t^f + \frac{1}{2}V_t(m_{t+1}) = 0.$$

Subtracting the two pricing equations gives the expected excess return on the risky asset

$$E_t(r_{t+1} - r_t^f) + \frac{1}{2}V_t(r_{t+1}) = -Cov_t(m_{t+1}, r_{t+1}). \tag{3}$$

This is the key no-arbitrage condition that all correctly priced assets must satisfy *when their returns are lognormally distributed*. The right-hand side is the risk premium and $\frac{1}{2}V_t(r_{t+1})$ is the Jensen effect. We note that $V_t(r_{t+1}) = V_t(r_{t+1} - r_t^f)$ and $Cov_t(m_{t+1}, r_{t+1}) = Cov_t(m_{t+1}, r_{t+1} - r_t^f)$ as r_t^f is known at time t.

2.1.1 Real versus nominal returns

The pricing equation (1) and the no-arbitrage condition (3) hold whether the variables, including the discount factor, are expressed in nominal or real terms. Although it is common to specify the discount factor in real terms, as a real risk-free rate does not exist in practice, we shall specify returns in nominal terms. Assuming no default risk, the nominal risk-free rate is a Treasury bill rate. Accordingly, we need to re-express the no-arbitrage condition.

We assume that equations (1) and (3) are expressed in real terms with M_{t+1} as the real ex-post discount factor and r_{t+1} and r_t^f as real ex-post rates of return. We now let i_{t+1} and i_t^f be the respective nominal rates of return and we let P_t^c be the consumer price index at the start of period t. Equation (2) can therefore be written

$$\begin{aligned} 1 &= E_t[M_{t+1}.(1+r_{t+1})] \\ &= E_t[M_{t+1}.\frac{1+i_{t+1}}{1+\pi_{t+1}}] \\ &= E_t[\frac{M_{t+1}}{1+\pi_{t+1}}.(1+i_{t+1})] \end{aligned}$$

where $1+r_{t+1} = \frac{1+i_{t+1}}{1+\pi_{t+1}}$ and $1+\pi_{t+1} = \frac{P_{t+1}^c}{P_t^c}$. It can be shown that the no-arbitrage condition for nominal returns and a real discount factor is

$$E_t(i_{t+1} - i_t^f) + \frac{1}{2}V_t(i_{t+1}) = -Cov_t(m_{t+1}, i_{t+1}) + Cov_t(\pi_{t+1}, i_{t+1}). \quad (4)$$

Thus there is an extra term in the conditional covariance of inflation with the nominal (excess) return. We now consider a number of models of the real discount factor.

2.2 Consumption-based models

2.2.1 C-CAPM with power utility

The canonical model of the discount factor is the consumption-based CAPM. We consider this for nominal asset returns. Asset prices derive their value from the expected consumption streams of investors who choose to

$$\max_{C_t} \mathcal{U}_t = U(C_t) + \beta E_t(\mathcal{U}_{t+1})$$

subject to the nominal budget constraint

$$P_t^c C_t + W_{t+1} = P_t^c Y_t + W_t(1+i_t)$$

where C_t is real consumption, Y_t is real non-asset income and W_t is nominal financial wealth at the start of period t. The solution is the Euler equation

$$E_t[\frac{\beta U'(C_{t+1})}{U'(C_t)} \frac{P_t^c}{P_{t+1}^c}(1+i_{t+1})] = 1.$$

This implies that the C-CAPM has implicitly defined the real SDF as

$$M_{t+1} = \frac{\beta U'(C_{t+1})}{U'(C_t)}$$

For the power utility function $U = \frac{C_t^{1-\sigma}-1}{1-\sigma}$ with constant coefficient of relative risk aversion σ, the real discount factor, or marginal rate of substitution between periods t and $t+1$, is

$$M_{t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\sigma}$$

Since real consumption is usually defined in ex-post terms, the discount factor will also be in real ex-post terms. Taking logarithms, and ignoring all constants, we obtain

$$m_{t+1} = -\sigma \Delta c_{t+1}$$

where $c_t = \ln C_t$. The no-arbitrage condition in real terms becomes

$$E_t(r_{t+1} - r_t^f) + \frac{1}{2} V_t(r_{t+1}) = \sigma Cov_t(\Delta c_{t+1}, r_{t+1})$$

The interpretation of the real equity risk premium is that investors lose utility today by not consuming. To compensate investors who defer the utility from an extra unit of consumption today they need additional marginal utility from future consumption. Because marginal utility declines as consumption increases, a higher level of consumption is needed in the future. The return on the investment must be large enough to generate the required consumption in the future. The greater the consumption needed, the larger the return must be, hence the risk premium is larger the greater the predicted covariance between consumption and returns. Another way to think about the equity risk premium is to note that over the business cycle equity returns tend to be positively correlated with consumption growth. In the downturn, when consumption growth tends to be low, equity returns are also low, hence risk is associated with the predicted covariance of returns with consumption.

In nominal terms the no-arbitrage condition can be written

$$E_t(i_{t+1} - i_t^f) + \frac{1}{2} V_t(i_{t+1}) = \sigma Cov_t(\Delta c_{t+1}, i_{t+1}) + Cov_t(\pi_{t+1}, i_{t+1}) \quad (5)$$

Thus the nominal risk premium involves the nominal return and has a second covariance term associated with consumer price inflation. The greater the covariance between nominal returns and inflation, the larger the risk premium. We have argued that the larger the future consumption needed, the higher real returns must be. This is also true for nominal returns. The extra risk is that nominal returns will need to be larger solely due to inflation.

2.2.2 Time non-separable preferences

Epstein-Zin utility C-CAPM with power utility is a von-Neumann-Morgenstern (VNM) type of utility function. It restricts the coefficient of relative risk aversion to be equal to the elasticity of inter-temporal substitution. This restriction can be relaxed using a non-VNM type of utility function which is time non-separable. A general formulation of time non-separable utility proposed by Kreps and Porteus (1987) is

$$\mathcal{U}_t = \mathcal{U}[C_t,\ E_t(\mathcal{U}_{t+1})]$$

Giovannini and Weil (1989) show that maximising $\mathcal{U}_t$ subject to the budget constraint

$$\begin{aligned} W^R_{t+1} &= R^m_{t+1}(W^R_t - C_t) \\ R^m_{t+1} &= \textstyle\sum_{j=1}^n \alpha_{kt} R_{j,t+1}, \qquad \sum_{j=1}^n \alpha_{kt} = 1 \end{aligned}$$

where R^m_{t+1} is the real return on the market portfolio of all invested real wealth W^R_t, $R_{k,t+1}$ is the return on the k^{th} asset $\{k = 1,..., n\}$ and α_{kt} is its portfolio share, gives the following Euler equations for the market and for any two assets k_1 and k_2:

$$\begin{aligned} E_t[\mathcal{U}_{2,t}\frac{\mathcal{U}_{1,t+1}}{\mathcal{U}_{1,t}}R^m_{t+1}] &= 1 \\ E_t[\mathcal{U}_{2,t}\frac{\mathcal{U}_{1,t+1}}{\mathcal{U}_{1,t}}R_{k_1,t+1}] &= E_t[\mathcal{U}_{2,t}\frac{\mathcal{U}_{1,t+1}}{\mathcal{U}_{1,t}}R_{k_2,t+1}] \end{aligned}$$

where $\mathcal{U}_{1,t} = \frac{\partial \mathcal{U}}{\partial C_t}$ and $\mathcal{U}_{2,t} = \frac{\partial \mathcal{U}}{\partial E_t(\mathcal{U}_{t+1})}$. Thus the real discount factor is

$$M_{t+1} = \mathcal{U}_{2,t}\frac{\mathcal{U}_{1,t+1}}{\mathcal{U}_{1,t}}$$

If r_{t+1} is the real return to equity and r^f_t is a real risk-free rate then it follows that

$$E_t[\mathcal{U}_{2,t}\frac{\mathcal{U}_{1,t+1}}{\mathcal{U}_{1,t}}(r_{t+1} - r^f_t)] = 0.$$

Epstein and Zin (1989, 1990, 1991) have implemented a special case of this based on the constant elasticity of substitution (CES) function:

$$\mathcal{U}_t = \left[(1-\beta)C_t^{1-\frac{1}{\gamma}} + \beta[E_t(\mathcal{U}_{t+1})]^{\frac{1-\frac{1}{\gamma}}{1-\sigma}}\right]^{\frac{1}{1-\frac{1}{\gamma}}}$$

where β is the discount rate, σ is the coefficient of relative risk aversion and γ is the elasticity of inter-temporal substitution. In the separable power utility case $\sigma = 1/\gamma$. The Euler equation derived by Epstein and Zin is

$$E_t\left\{\left[\beta(\frac{C_{t+1}}{C_t})^{-\frac{1}{\gamma}}R^m_{t+1}\right]^{\frac{1-\sigma}{1-\frac{1}{\gamma}}}\right\}=1$$

implying that the real stochastic discount factor is

$$M_{t+1}=\left[\beta(\frac{C_{t+1}}{C_t})^{-\frac{1}{\gamma}}\right]^{\frac{1-\sigma}{1-\frac{1}{\gamma}}}(R^m_{t+1})^{\frac{1-\sigma}{1-\frac{1}{\gamma}}-1}$$

and its logarithm is

$$m_{t+1}=\frac{1-\sigma}{1-\gamma}\Delta c_{t+1}-\frac{1-\gamma\sigma}{1-\gamma}r^m_{t+1}$$

It follows that for the excess return on equity,

$$E_t\left\{\left[\beta(\frac{C_{t+1}}{C_t})^{-\frac{1}{\gamma}}\right]^{\frac{1-\sigma}{1-\frac{1}{\gamma}}}(R^m_{t+1})^{\frac{1-\sigma}{1-\frac{1}{\gamma}}-1}(r_{t+1}-r^f_t)\right\}=0$$

and, assuming log-normality, the no-arbitrage condition is

$$E_t(r_{t+1}-r^f_t)+\frac{1}{2}V_t(r_{t+1})=-\frac{1-\sigma}{1-\gamma}Cov_t(\Delta c_{t+1},r_{t+1})+\frac{1-\gamma\sigma}{1-\gamma}Cov_t(r^m_{t+1},r_{t+1}),$$

a result first derived by Campbell, Lo and MacKinlay (1997).

The no-arbitrage condition for nominal returns requires the nominal budget constraint which becomes

$$W_{t+1}=(1+i^m_{t+1})(W_t-P^c_tC_t)$$

where i^m_{t+1} is the nominal market return and W_t is nominal wealth. This can be re-written as

$$\begin{aligned}P^c_{t+1}W^R_{t+1} &= (1+i^m_{t+1})P^c_t(W^R_t-C_t)\\ W^R_{t+1} &= \frac{1+i^m_{t+1}}{1+\pi_{t+1}}(W^R_t-C_t)\end{aligned}$$

Hence, in the Euler equation we simply replace R^m_{t+1} by $\frac{1+i^m_{t+1}}{1+\pi_{t+1}}$ to obtain

$$E_t\left\{\left[\beta(\frac{C_{t+1}}{C_t})^{-\frac{1}{\gamma}}\right]^{\frac{1-\sigma}{1-\frac{1}{\gamma}}}(\frac{1+i^m_{t+1}}{1+\pi_{t+1}})^{\frac{1-\sigma}{1-\frac{1}{\gamma}}-1}\frac{1}{1+\pi_{t+1}}(i_{t+1}-i^f_t)\right\}=0$$

The no-arbitrage condition for the nominal return on an individual asset with Epstein-Zin preferences satisfying this version of the budget constraint is therefore

$$E_t(i_{t+1}-i^f_t)+\frac{1}{2}V_t(i_{t+1})=-\frac{1-\sigma}{1-\gamma}Cov_t(\Delta c_{t+1},i_{t+1})-\frac{\gamma(1-\sigma)}{1-\gamma}Cov_t(\pi_{t+1},i_{t+1})+\frac{1-\gamma\sigma}{1-\gamma}Cov_t(i^m_{t+1},i_{t+1})$$

A difficulty in implementing this in practice is choosing a market rate of return. If it is assumed that the market portfolio consists of equity and a risk-free asset, the market return would then be a weighted average of the two:

$$i^m_{t+1}=\theta_ti_{t+1}+(1-\theta_t)i^f_t$$

If in addition the portfolio weight is constant with $\theta_t = \theta$, then the Epstein-Zin no-arbitrage condition becomes the fixed parameter model

$$E_t(i_{t+1} - i_t^f) + (\frac{1}{2} - \frac{\theta(1-\gamma\sigma)}{1-\gamma})V_t(i_{t+1}) = -\frac{1-\sigma}{1-\gamma}Cov_t(\Delta c_{t+1}, i_{t+1}) - \frac{\gamma(1-\sigma)}{1-\gamma}Cov_t(\pi_{t+1}, i_{t+1}) \tag{6}$$

The interpretation of the coefficients is now changed compared with the no-arbitrage condition for power utility. The coefficient on $Cov_t(\Delta c_{t+1} i_{t+1})$ is no longer the coefficient of relative risk aversion, and the coefficients on $V_t(i_{t+1})$ and $Cov_t(i_{t+1}, \pi_{t+1})$ are no longer restricted. For the model

$$E_t(i_{t+1} - i_t^f) = \beta_1 V_t(i_{t+1}) + \beta_2 Cov_t(\Delta c_{t+1}, i_{t+1}) + \beta_3 Cov_t(\pi_{t+1}, i_{t+1}) \tag{7}$$

we note that $\sigma = \beta_2 - \beta_3 + 1$ and $\gamma = \frac{\beta_3}{\beta_2}$.

Habit persistence An alternative example of time non-separable utility is the habit persistence model. This assumes that

$$U_t = U(C_t, X_t)$$

where X_t is the habitual level of consumption. Again the aim is to introduce additional terms in the risk premium. Various specific functional forms for the utility function have been suggested. Constantinides (1990) proposed the form

$$U_t = \frac{(C_t - \lambda X_t)^{1-\sigma} - 1}{1-\sigma}$$

The stochastic discount factor implied by this is

$$M_{t+1} = \beta \left(\frac{C_{t+1} - \lambda X_{t+1}}{C_t - \lambda X_t} \right)^{-\sigma}$$

By a suitable choice of λ and X_t it is possible to produce a discount factor that displays greater volatility and hence a larger risk premium.

Campbell and Cochrane (1999) implemented this with the restriction that $\lambda = 1$. They introduced the concept of surplus consumption, defined as $S_t = \frac{C_t - X_t}{C_t}$. The discount factor can then be written

$$M_{t+1} = \beta \left(\frac{C_t}{C_{t+1}} \frac{S_{t+1}}{S_t} \right)^{-\sigma}$$

Assuming log normality, the discount factor is

$$m_{t+1} = \sigma \Delta c_{t+1} - \sigma \Delta s_{t+1}$$

no-arbitrage condition is

$$E_t(r_{t+1} - r_t^f) + \frac{1}{2} V_t(r_{t+1}) = -\sigma Cov_t\left(\Delta c_{t+1}, r_{t+1}\right) + \sigma Cov_t\left(s_{t+1}, r_{t+1}\right)$$

where $s_{t+1} = \ln S_{t+1}$. The attraction of this approach is the inclusion of the extra term to account for the risk premium; the problem lies in the way it has been implemented.

Campbell and Cochrane assume that s_{t+1} is generated by an AR(1) process with a disturbance term whose variance depends on s_t. They do not estimate the resulting model but, like Constantinides, use calibration methods. By calibrating the *unconditional* variance of the error term of the AR process suitably, it is possible to force the covariance between r_{t+1} and s_{t+1} to be of the necessary size. By this means success is virtually guaranteed. Unfortunately, if s_{t+1} is an independent AR(1) process, the implied *conditional* covariance r_{t+1} and s_{t+1} is necessarily zero As a result, this version of the habit persistence hypothesis would be incapable of providing any additional explanation of the risk premium.

A similar analysis applies to the habit persistence utility function proposed by Abel (1990)

$$U_t = \frac{\left(\frac{C_t}{X_t}\right)^{1-\sigma} - 1}{1-\sigma}$$

where X_t is a function of past consumption, for example $X_t = C_{t-1}^{\delta}$. The stochastic discount factor becomes $M_{t+1} = \beta \left(\frac{\frac{C_{t+1}}{X_{t+1}}}{\frac{C_t}{X_t}} \right)^{-\sigma}$ and so

$$m_{t+1} = -\sigma \Delta c_{t+1} + \sigma \Delta x_{t+1}$$

where $x_t = \ln X_t$. Assuming log normality, the no-arbitrage condition is now

$$E_t(r_{t+1} - r_t^f) + \frac{1}{2} V_t(r_{t+1}) = \sigma Cov_t\left(\Delta c_{t+1}, r_{t+1}\right) - \sigma Cov_t(\Delta x_{t+1}, r_{t+1})$$

If, however, it is assumed that $x_{t+1} = \delta c_t$ then

$$Cov_t(\Delta x_{t+1}, r_{t+1}) = \delta Cov_t(\Delta c_t, r_{t+1}) = 0$$

and so, once again, the habit persistence model is of no help in providing additional terms in the risk premium.

Thus, the problem is that if habit is solely related to past consumption - which is its natural meaning - then the *conditional* covariance of returns with the discount factor are unaffected. As unconditional moments are used in GMM estimation of the Euler equation, measuring habit by past consumption does not prevent the parameters of the habit persistence model from being estimated. It is only when trying to estimate the risk premium itself that the problem with the habit persistence model becomes evident.

A similar problem arises in the model of Bekaert and Grenadier (1999). Starting with C-CAPM, they argue that in equilibrium the consumption process C_t must equal the exogenous aggregate real dividend process D_t. The log stochastic discount factor is then

$$m_{t+1} = \ln\beta - \sigma\Delta d_{t+1}$$

where $\ln D = d$. They assume that the dividend process is driven by real productivity shocks x_t so that

$$\Delta d_{t+1} = \frac{\sigma}{2}\sigma_d^2 + \frac{\ln\beta}{\sigma} + \frac{1}{\sigma}x_t + \sigma_d\xi_{t+1}$$

and they assume that x_t has the CIR process

$$x_{t+1} - \mu = \theta(x_t - \mu) + \sigma_x\sqrt{x_t}\varepsilon_{t+1}$$

where ξ_t and ε_t are assumed to be $iid(0,1)$ processes.

Had they derived the no-arbitrage equation for pricing equity, they would have found that it is

$$E_t(r_{t+1} - r_t^f) + \frac{1}{2}V_t(r_{t+1}) = \sigma Cov_t(\Delta d_{t+1}, r_{t+1}) = 0,$$

implying that the equity risk premium is zero. If the dividend process had been written instead as

$$\Delta d_{t+1} = \frac{\sigma}{2}\sigma_d^2 + \frac{\ln\beta}{\sigma} + \frac{1}{\sigma}x_{t+1} + \sigma_d\xi_{t+1}$$

then the no-arbitrage condition would become

$$E_t(r_{t+1} - r_t^f) + \frac{1}{2}V_t(r_{t+1}) = \sigma_x\sqrt{x_t}Cov_t(\varepsilon_{t+1}, r_{t+1}).$$

This would be a more useful formulation, though probably still too restrictive to adequately capture the equity risk premium.

2.3 CAPM

CAPM relates the expected excess return on equity to the excess return of the market portfolio through

$$E_t(r_{t+1} - r_t^f) = \beta_t E_t(r_{t+1}^m - r_t^f)$$

where

$$\beta_t = \frac{Cov_t(r_{t+1}^m, r_{t+1})}{V_t(r_{t+1}^m)}$$

and

$$E_t(r_{t+1}^m - r_t^f) = \sigma_t V_t(r_{t+1}^m).$$

Thus beta is in general time-varying. It follows that we can re-write CAPM as

$$E_t(r_{t+1} - r_t^f) = \sigma_t Cov_t(r_{t+1}^m, r_{t+1})$$

We note that $(1+r^m_{t+1}) = \frac{W^R_{t+1}}{W^R_t}$, where as before W^R_t is real wealth, and if consumption is proportional to wealth, as in the life cycle model, we obtain

$$E_t(r_{t+1} - r^f_t) = \sigma_t Cov_t(\Delta c_{t+1}, r_{t+1})$$

Thus we can interpret CAPM as an SDF model in which the discount factor is $\sigma_t(1+r^m_{t+1})$ - or $\sigma_t \frac{\Delta W_{t+1}}{W_t}$. There is no Jensen effect because the assumption of log-normality is not made.

The corresponding expression for nominal returns is

$$E_t(i_{t+1} - i^f_t) = \sigma_t Cov_t(i^m_{t+1}, i_{t+1})$$

If $\sigma_t = \sigma$ and the market consists of equity and the risk free in constant proportions so that $i^m_{t+1} = \theta i_{t+1} + (1-\theta) i^f_t$ then this becomes

$$E_t(i_{t+1} - i^f_t) = \theta\sigma V_t(i_{t+1})$$

This is another special case of the SDF model; one involving a relation between the excess return and its volatility, in which there are no covariance terms.

2.3.1 The relation between equity returns and volatility

Apart from this result based on CAPM, the finding that the return on equity is related to its own variance does not in general satisfy a no-arbitrage condition. Much of the current view of this relation between mean and variance dates from Campbell (1987) who finds that the relation is generally insignificantly positive when the conditional variance term is modelled as a GARCH-in-mean term. This result on the monthly CRSP market-wide index has been further reinforced by Baillie and DeGennero (1990) using a similar methodology. Glosten at al (1993) have subsequently found evidence of a significant negative relationship between mean and conditional variance. They found that including seasonal effects in the GARCH-M model appears to be of importance. Campbell and Hentschel (1992) and Scruggs (1998), using additional factors, find more support for a positive (partial) relation between mean and variance.

A closely related approach is that of Lettau and Ludvigson (2003) who examine the behaviour of the Sharpe ratio using an ad hoc model that relates the Sharpe ratio to the conditional standard deviation of returns. It is not necessary to use an ad hoc model, however, as it is possible to infer the behaviour of the Sharpe ratio from the SDF model by re-writing the SDF no-arbitrage condition as

$$\frac{E_t(r_{t+1} - r^f_t)}{SD_t(r_{t+1})} = -\frac{1}{2} SD_t(r_{t+1}) - \rho_t(m_{t+1}, r_{t+1}) SD_t(m_{t+1}) \tag{8}$$

where $\rho_t(m_{t+1}, r_{t+1})$ is the conditional correlation and SD_t denotes a conditional standard deviation. These models can easily be tested within the SDF framework since they imply that conditional covariances can be excluded from the conditional mean and the coefficient on the own conditional variance should not be restricted to satisfy the Jensen effect.

2.4 Other SDF models

The SDF model shows how any source of risk can be incorporated into an explanation of the risk premium in a way that satisfies the no-arbitrage conditiond. If z_{it} $(i = 1, ..., n)$ are n factors which are jointly log normally distributed with equity returns then the discount factor can be written

$$m_{t+1} = -\sum\nolimits_{i=1}^{n} \beta_i z_{i,t+1}$$

This implies the no-arbitrage condition

$$\begin{aligned} E_t(r_{t+1} - r_t^f) + \frac{1}{2}V_t(r_{t+1}) &= \sum_i \beta_i Cov_t(z_{i,t+1}, r_{t+1}) \\ &= \sum_i \beta_i \ f_{it}, \end{aligned}$$

where the f_{it} are known as *common factors.*

In the absence of the sort of clear theoretical foundations provided by general equilibrium theories of asset pricing, the problem is to identify potential souces of risk to include in the SDF model. The latent factor literature simply assumes that unobserved processes can be specified for the factors. As noted above, general equilibrium models imply that investors are concerned with future consumption, and in particular, consumption next period. The main holders of equity are financial institutions, especially pension funds. They act on behalf of investors' consumption at a much more distant point in the future. In assessing risk, financial institutions focus largely on short-term performance, and on the value of the portfolio. This suggests that the market equity risk premium may be more influenced by short-term price risk than longer-term considerations of consumption. The sort of factors that are likely to affect the price of equity in the short term are associated with the business cycle and inflation. We therefore examine whether output is an additional source of equity risk to consumption and inflation.

2.5 A taxonomy of SDF models

The following is a summary of the models above.

1. C-CAPM

(a) based on time-separable power utility model

(i) with real returns

(ii) with nominal returns

(b) based on the Epstein-Zin time non-separable utility model

(i) with nominal returns and an explicit market return

(ii) with nominal returns but no explicit market return

2. SDF model

(a) based on two macroeconomic factors: consumption and inflation

(b) based on three macroeconomic factors: consumption, inflation and industrial production

(c) based on four factors: three macroeconomic factors and the market return

These can all be represented as restricted versions of the SDF model

$$
\begin{aligned}
E_t(i_{t+1} - i_t^f) &= \beta_1 V_t(i_{t+1}) + \beta_2 Cov_t(\Delta c_{t+1}, i_{t+1}) + \beta_3 Cov_t(\pi_{t+1}, i_{t+1}) \\
&\quad + \beta_4 Cov_t(\Delta q_{t+1}, i_{t+1}) + \beta_5 Cov_t(i_{t+1}^m, i_{t+1})
\end{aligned} \tag{9}
$$

where q_t is the logarithm of a measure of output.

The various models are summarised in Table 1 below. All of the models except the first assume returns are nominal. The first model is C-CAPM with *ex-post* real returns. The different SDF models - including the C-CAPM - can be distinguished by the restrictions they impose on β_i. All of these models restrict the coefficient on the conditional variance of equity returns to be $\beta_1 = -1/2$ (the Jensen effect). There are two exceptions where the SDF model includes the market return as a factor. As the market portfolio includes equity, we replace the covariance involving the market return with the variance of equity and remove the restriction on β_1. This gives an alternative way of expressing the Epstein-Zin (model 4), and the four variable SDF model which includes the market return as a factor (model 7). Similarly, CAPM which involves only the covariance of the asset with the market, can be re-expressed in terms of the conditional variance on equity. The statistical approach relating the conditional mean to its conditional variance also implies that β_1 is unrestricted and all other $\beta_i = 0$ $\{i = 2, ...\}$. The general model in Table 1 does not satisfy the no-arbitrage condition as it does not restrict $\beta_1 = -1/2$; it can therefore be interpreted as another example of the statistical approach.

Model	β_1	β_2	β_3	β_4	β_5
C-CAPM					
1. PU real	$-\frac{1}{2}$	σ	0	0	0
2. PU nominal	$-\frac{1}{2}$	σ	1	0	0
3. EZ: nominal with market return	$-\frac{1}{2}$	$-\frac{1-\sigma}{1-\gamma}$	$-\frac{\gamma(1-\sigma)}{1-\gamma}$	0	$\frac{1-\gamma\sigma}{1-\gamma}$
4. EZ: nominal without market return	$-\frac{1}{2}+\frac{\theta(1-\gamma\sigma)}{1-\gamma}$	$-\frac{1-\sigma}{1-\gamma}$	$-\frac{\gamma(1-\sigma)}{1-\gamma}$	0	0
SDF					
5. Two factors	$-\frac{1}{2}$	β_2	β_3	0	0
6. Three factors	$-\frac{1}{2}$	β_2	β_3	β_4	0
7. Four factors	β_1	β_2	β_3	β_4	0
Other models					
8. CAPM	$\theta\sigma$	0	0	0	0
9. General unrestricted	β_1	β_2	β_3	β_4	β_5

Table 1: Restrictions on the SDF model

3 The Econometric Framework

3.1 Multivariate conditional heteroskedasticity models

We need to model the distribution of the excess return on equity jointly with the macroeconomic factors in such a way that the mean of the conditional distribution of the excess return in period $t+1$, given information available at time t, satisfies the no-arbitrage condition. The conditional mean of the excess return involves selected time-varying second moments of the joint distribution. We therefore require a specification of the joint distribution that admits a time-varying variance-covariance matrix. A convenient choice is the multi-variate GARCH-in-mean (MGM) model.

Let $\mathbf{x}_{t+1} = (r_{t+1} - r_t^f,\ z_{1,t+1},\ z_{2,t+1},...)'$, where $z_{1,t+1},\ z_{2,t+1},...$ include the macroeconomic variables that give rise to the factors in the SDF through their conditional covariances with the excess return. In principle, they may also include additional variables that are jointly distributed with these macroeconomic variables as this may improve the estimate of the joint distribution. The MGM model can then be written

$$\mathbf{x}_{t+1} = \boldsymbol{\alpha} + \boldsymbol{\Gamma}\mathbf{x}_t + \mathbf{B}\mathbf{g}_t + \varepsilon_{t+1}$$

where

$$\boldsymbol{\varepsilon}_{t+1} \mid I_t \sim D[0, \mathbf{H}_{t+1}]$$

$$\mathbf{g}_t = vech\{\mathbf{H}_{t+1}\}$$

The *vech* operator converts the lower triangle of a symmetric matrix into a vector. The distribution is the multivariate $t-$distribution. The first equation of the model is restricted to satisfy the no-arbitrage condition. Thus, in general, the first row of $\boldsymbol{\Gamma}$ is zero and the first row of $\mathbf{B}$ is ($-\frac{1}{2}$, $-\beta_{11}$, $-\beta_{12}$, $-\beta_{13}$,...).

It will be noted that the theory requires that the macroeconomic variables display conditional heteroskedasticity. This is not something traditionally assumed in macro-econometrics, but seems to be present in our data. Ideally, we would like to use high frequency data for asset returns, but very little macroeconomic data are published for frequencies higher than one month, and then only a few variables are available. Although more macroeconomic variables are published at lower frequencies, they tend not to display conditional heteroskedasticity.

Whilst the MGM model is convenient, it is not ideal. First, it is heavily parameterised which can create numerical problems in finding the maximum of the likelihood function due to the likelihood being surface being relatively flat, and hence uninformative. Second, asset returns tend to be excessively volatile. Assuming a non-normal distribution such as a $t-$distribution can sometimes help in this regard by dealing with thick tails. The main problem, however, is not thick tails, but a small number of extreme values. The coefficients of the variance process of the MGM model have a tendency to produce a near unstable variance process in their attempt to fit these extreme values. In principle, a stochastic volatility model, which includes an extra random term in the variance, could capture these extreme values. Unfortunately, as far as we are aware, no multivariate stochastic model with in mean effects in the conditional covariances has been proposed in the literature.

In view of the need to restrict the number of coefficients to estimate, a commonly used specification of $\mathbf{H}_{t+1}$ is the Constant Conditional Correlation model of Ding and Engle (1994) where the dynamics of the conditional covariances are driven by individual GARCH processes for the variances of each variable. Given that the SDF approach focusses on the importance of the contribution of covariances, restricting their dynamics in this way, and not allowing the correlations to be time-varying, seems too restrictive.[1]

As a result, we specify $\mathbf{H}_{t+1}$ using the BEKK model originally proposed by Engle and Kroner (1995). This takes the form:

$$vech(\mathbf{H}_{t+1}) = \mathbf{\Lambda} + \Sigma_{i=0}^{p-1}\mathbf{\Phi}_i vech(\mathbf{H}_{t-i}) + \Sigma_{j=0}^{q-1}\mathbf{\Theta}_j vech(\varepsilon_{t-j}\varepsilon'_{t-j})$$

where $\mathbf{\Lambda}, \mathbf{\Phi}$ and $\mathbf{\Theta}$ may be unrestricted. With $n-1$ factors z_{it} then $\mathbf{\Phi}$ and $\mathbf{\Theta}$ are both square matrices of size $n(n+1)/2$ and $\mathbf{\Lambda}$ is a size $n(n+1)/2$ vector. A formulation of this model which can make implementation easier is the error-correction formulation or VECM BEKK:

$$\mathbf{H}_{t+1} = \mathbf{V}'\mathbf{V} + \mathbf{A}'(\mathbf{H}_t - \mathbf{V}'\mathbf{V})\mathbf{A} + \mathbf{B}'(\varepsilon_t\varepsilon'_t - \mathbf{V}'\mathbf{V})\mathbf{B}.$$

where the first term on the RHS is the long-run or unconditional covariance matrix. This can be initialised with starting values from sample averages. The remaining terms capture short-run deviations from this long run. A restricted version of this formulation is to specify $\mathbf{V}$ to be lower triangular and $\mathbf{A}$ and $\mathbf{B}$ to be symmetric matrices which further reduces parameter numbers.

A comparison of the number of parameters required for $n = 3$ and $p = q = 1$ is: BEKK= $n(n+1)/2 + (p+q)n^2(n+1)^2/4 = 78$;VECM BEKK= $3n^2 = 27$; Restricted VECM BEKK= $3n(n+1)/2 = 18$; Constant Correlation= $3n + n(n-1)/2 = 12$. We require that the covariance function is stationary. This is satisfied if the absolute value of the eigenvalues of $(A \otimes A) + (B \otimes B)$ lie inside the unit circle where $\otimes$ is the Kronecker product.

The structure of the VECM BEKK model that we employ is common to all of the models that we estimate. In each case we condition on the same set of variables in the macroeconomic environment even though the terms in the no-arbitrage condition for the excess return differ between models. Thus the vector $\mathbf{x}_{t+1}$ for the models of the nominal equity return is $\mathbf{x}_{t+1} = (i_{t+1} - i_t^f, \pi_{t+1}, \Delta c_{t+1}, \Delta q_{t+1})'$ whilst that for the real return is: $\mathbf{x}_{t+1} = (r_{t+1} - r_t^f, \pi_{t+1}, \Delta c_{t+1}, \Delta q_{t+1})'$. A first order vector autoregression for the macroeconomic variables is found to be sufficient to capture the serial dependence in their means; a VECM BEKK(1,1) model is found to be adequate for the multivariate variance-covariance process.

For greater generality, instead of assuming that ε_t has a multivariate Normal distribution, we assume that it has a multivariate $t-$distribution. This introduces a technical problem: unlike the

[1] The attraction of the reduction in parameterisation offered by the CCC model has led to an extension to the dynamics in the DCC proposed recently by Engle(2000).

Normal distribution, the moment generating function of the $t-$distribution does not exist and hence, strictly, the logarithm of the Euler equation does not exist.

3.2 GMM Estimation

It is informative to contrast the above approach with the GMM estimation of the Euler equation of the general equilibrium condition. This may be written

$$E_t[(M_{t+1}(\theta)(r_{t+1} - r_t^f)] = 0$$

where we denote that the discount factor depends on the parameters θ. GMM estimation exploits the lack of correlation between the discounted pay-off and the information set used in conditioning. The null hypothesis is that

$$E[(M_{t+1}(\theta)(r_{t+1} - r_t^f)I_t] = 0$$

I_t is the information set which by implication contains no information about $M_{t+1}(\theta)(r_{t+1} - r_t^f)$, the discounted excess return in period $t + 1$.

Two things may be noted. First, unless the information set has time-varying volatility, it will be unlikely to prove a suitable basis for a time-varying risk premium. Surprisingly perhaps, in practice, this has not usually been a consideration. Second, the risk premium itself cannot be obtained even if we knew θ.

4 Results and Risk Premia

4.1 The Data

The data are monthly for the US (1975.6-2001.12) and UK: (1975.6-2001.12). The US data consists of the excess return on equity of Fama and French, real non-durable growth consumption from FRED, CPI inflation from Datastream and the volume index of industrial production volume index from Datastream. The UK data are the MSCI total equity return index, from Datastream, total real non-durable consumption growth specially provided by the NIESR, RPI inflation and the volume index of industrial production both from Datastream. All data are express in equivalent annual percentages.

Table 2 reports various descriptive statistics for the data, including skewness, kurtosis and autocorrelations of levels and squares.

Table 2. Descriptive statistics for the raw data: US and UK

	$i^{us}_{t+1} - i^{us}_{t}$	$i^{uk}_{t+1} - i^{uk}_{t}$	π^{us}_{t+1}	π^{uk}_{t+1}	Δc^{us}_{t+1}	Δc^{uk}_{t+1}	Δq^{us}_{t+1}	Δq^{uk}_{t+1}
Mean	6.2508	5.3242	4.5212	6.0657	2.6784	2.9605	2.7923	1.3879
Std. Dev	54.2440	60.4400	3.6252	7.3884	7.7575	9.5286	8.2967	15.644
Skewness	-1.0174	-0.9398	0.8340	1.8082	-0.0308	-1.4379	0.0055	-0.1004
Excess Kurtosis	3.9780	4.5802	1.0578	6.5753	0.4453	14.4820	1.7726	7.0462
Normality	50.4180	65.9750	37.0090	117.0600	3.6367	273.2900	32.6320	232.6600
$\rho(x_t, x_{t-1})$	0.0339	0.0199	0.7233	0.3903	-0.3686	-0.1959	0.2884	-0.2978
$\rho(x_t, x_{t-2})$	-0.0874	-0.1147	0.5993	0.2793	-0.0178	0.1464	0.2483	0.0082
$\rho(x_t, x_{t-3})$	-0.0556	-0.0948	0.5921	0.2177	0.1587	-0.0759	0.2116	0.1176
$\rho(x_t, x_{t-4})$	-0.0586	0.0229	0.5526	0.2123	-0.1305	-0.1168	0.1225	-0.0109
$\rho(x_t, x_{t-5})$	0.0751	-0.0030	0.5211	0.2749	0.0551	0.0496	0.0941	0.0495
$\rho(x_t, x_{t-6})$	0.0112	-0.0967	0.5144	0.3893	0.0228	-0.0108	0.0798	-0.0989
$\rho(x^2_t, x^2_{t-1})$	0.0225	0.0861	0.7879	0.1669	0.1011	0.2516	0.1529	0.4476
$\rho(x^2_t, x^2_{t-2})$	0.0288	0.0731	0.7108	0.0972	0.0164	0.0301	0.0940	0.0284
$\rho(x^2_t, x^2_{t-3})$	0.0150	-0.0189	0.6399	0.2107	0.0358	0.1624	0.0512	0.0125
$\rho(x^2_t, x^2_{t-4})$	0.0061	-0.0252	0.5963	0.0460	-0.0024	0.0638	0.0698	0.0303
$\rho(x^2_t, x^2_{t-5})$	-0.0054	0.0096	0.5897	0.0645	-0.0175	0.0115	0.0083	0.0219
$\rho(x^2_t, x^2_{t-6})$	0.0070	-0.0491	0.5699	0.2097	-0.1152	0.0215	0.0876	0.0900

Thus the excess returns and most of the macroeconomic variables display excess kurtosis and non-normality, and the inflation rates show volatility persistence. The main issue, however, is whether all of the variables have significant GARCH effects in the model.

4.2 Estimates

4.2.1 A complete set of estimates for one model

To illustrate, a full set of model estimates with their restrictions is reported for the variables $x(t+1)' = \{i_{t+1} - i^f_t, \pi_{t+1}, \Delta c_{t+1}, \Delta q_{t+1}\}$ for the US. The model is C-CAPM with power utility and nominal (ex-post real) returns. An additional variable, output growth, is included in the joint distribution. A dummy variable for October 1987 is also included on the grounds that the excess return for this observation is drawn from a different distribution. This is to prevent an extreme observation from contaminating the estimates of the GARCH parameters for the whole sample. We note that $\mathbf{g}_t = (V_t(i_{t+1}), Cov_t(\pi^{us}_{t+1}, i^{us}_{t+1}), Cov_t(\Delta c^{us}_{t+1}, i^{us}_{t+1}), Cov_t(\Delta q^{us}_{t+1}, i^{us}_{t+1}))'$ and that $\mathbf{A}$ and $\mathbf{B}$ are symmetric matrices.

The main points to note at this stage are that the multivariate GARCH process is well determined with the macroeconomic variables showing significant conditional heteroskedasticity. The conditional covariance of returns with consumption is highly significant, but the size of the coefficient implies an implausibly large coefficient of relative risk aversion, and hence displays the equity premium puzzle.

Estimates of C-CAPM with power utility and real returns for the US 1975.1-2001.11

$$
\begin{aligned}
\mathbf{x}_{t+1} &= \boldsymbol{\alpha} + \boldsymbol{\Gamma}\mathbf{x}_t + \mathbf{B}\mathbf{g}_t + \delta d_t + \varepsilon_{t+1} \\
\boldsymbol{\varepsilon}_{t+1} &\mid I_t \sim D[0, \mathbf{H}_{t+1}] \\
\mathbf{H}_{t+1} &= \mathbf{V}'\mathbf{V} + \mathbf{A}'(\mathbf{H}_t - \mathbf{V}'\mathbf{V})\mathbf{A} + \mathbf{B}'(\varepsilon_t\varepsilon_t' - \mathbf{V}'\mathbf{V})\mathbf{B} \\
\mathbf{g}_t &= vech\{\mathbf{H}_{t+1}\} \\
\mathbf{x}_{t+1} &= \{r_{t+1} - r_t^f, \pi_{t+1}, \Delta c_{t+1}, \Delta q_{t+1}\}'
\end{aligned}
$$

$$
\boldsymbol{\alpha} = \begin{pmatrix} \underset{(-)}{0.0000} \\ \underset{(4.63)}{0.0009} \\ \underset{(7.02)}{0.0040} \\ \underset{(4.99)}{0.0029} \end{pmatrix}, \boldsymbol{\Gamma} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \underset{(0.90)}{0.002} & \underset{(15.02)}{0.645} & \underset{(2.14)}{0.040} & \underset{(0.32)}{0.006} \\ \underset{(1.83)}{0.015} & \underset{(2.57)}{-0.307} & \underset{(7.98)}{-0.402} & \underset{(0.80)}{-0.043} \\ \underset{(0.95)}{0.007} & \underset{(1.89)}{-0.224} & \underset{(141)}{-0.080} & \underset{(5.08)}{0.297} \end{pmatrix}, \mathbf{B} = \begin{pmatrix} -\frac{1}{2} & 1 & \underset{(3.10)}{328.7} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
$$

$$
\mathbf{A} = \begin{pmatrix} \underset{(24.34)}{0.913} & - & - & - \\ \underset{(0.27)}{0.003} & \underset{(25.73)}{0.909} & - & - \\ \underset{(0.07)}{-0.002} & \underset{(1.69)}{0.039} & \underset{(16.07)}{0.901} & - \\ \underset{(2.38)}{0.207} & \underset{(0.47)}{0.035} & \underset{(1.44)}{-0.219} & \underset{(3.47)}{-0.478} \end{pmatrix}, \mathbf{B} = \begin{pmatrix} \underset{(3.46)}{0.110} & - & - & - \\ \underset{(0.07)}{-0.000} & \underset{(4.90)}{0.304} & - & - \\ \underset{(3.71)}{0.032} & \underset{(1.60)}{0.036} & \underset{(0.77)}{-0.043} & - \\ \underset{(3.01)}{0.034} & \underset{(1.07)}{0.031} & \underset{(0.90)}{0.478} & \underset{(6.36)}{0.571} \end{pmatrix}
$$

$$
V'V = \begin{pmatrix} 2415 & - & - & - \\ -0.8 & 8.3 & - & - \\ 43.8 & -0.9 & 46.4 & - \\ -22.0 & 2.5 & 9.6 & 63.6 \end{pmatrix}, \text{correlation matrix=} \begin{pmatrix} 1.00 & - & - & - \\ -0.01 & 1.00 & - & - \\ 0.13 & -0.05 & 1.00 & - \\ -0.06 & 0.11 & 0.18 & 1.00 \end{pmatrix}
$$

The estimate of the 1987.10 dummy d_t is $\delta = -0.250$ (1.80). The three largest eigenvalues for GARCH process are $0.961, 0.949, 0.929$ and the smallest is -0.544. The annualised mean excess return is 6.25% and mean risk premium is 11.31%. The percentage of the variance of the excess return explained by variations in the risk premium is 2.2%. Absolute $t-$statistics are in parentheses.

4.2.2 Estimates of the no-arbitrage equation

US Estimates of the models in Table 1 are presented in Table 3 for US data. Estimates are not provided for models 3 and 9 as they involve using a observable market return, or of model 8, CAPM. Thus all of the models estimated satisfy the no-arbitrage condition. The estimates of the dummy variable are all significant at the one-sided 5% level. The estimates of the degrees of freedom refer to the $t-$distribution which we have assumed throughout due to its greater generality. Absolute $t-$statistics are in parentheses

US	PU real	PU	EZ	SDF2	SDF3	SDF4
Model	1	2	4	5	6	7
$\mathrm{V}_t(i^{us}_{t+1})$	-0.5	-0.5	-3.615 (0.75)	-0.5	-0.5	-3.544 (0.72)
$\mathrm{C}_t(\Delta c^{us}_{t+1}, i^{us}_{t+1})$	328.7 (3.10)	328.5 (3.10)	358.79 (2.38)	296.56 (2.94)	294.33 (2.89)	356.66 (2.34)
$\mathrm{C}_t(\pi^{us}_{t+1}, i^{us}_{t+1})$	0	-1	-423.29 (1.65)	-421.47 (1.69)	-460.65 (1.68)	-434.52 (1.58)
$\mathrm{C}_t(\Delta q^{us}_{t+1}, i^{us}_{t+1})$	0	0	0	0	28.979 (0.29)	8.540 (0.08)
Dummy, d_t	-0.250 (1.80)	-0.250 (1.80)	-0.246 (1.95)	-0.247 (1.90)	-0.246 (2.02)	-0.246 (1.98)
Deg of Freedom	13.530 (3.25)	13.530 (3.25)	13.793 (3.13)	13.744 (3.21)	13.877 (3.15)	13.829 (3.10)
Log-likelihood	4439.37	4439.37	4441.27	4440.84	4440.89	4441.27
Mean risk premium	12.01%	12.00%	8.85%	9.99%	9.75%	8.83%
$\lvert\lambda_{\max}\rvert$	0.961	0.961	0.966	0.966	0.966	0.966
$\bar{\varepsilon}_{t+1}$	-2.853	-2.854	-1.828	-2.812	-2.753	-1.843
$\mathrm{V}(\phi_{t+1})$	58.882	58.881	94.076	76.868	77.885	93.920
$\frac{\mathrm{V}(\phi_{t+1})}{\mathrm{V}(i^{us}_{t+1}+\frac{1}{2}\mathrm{V}_t(i^{us}_{t+1})-\widehat{\alpha}D)}$	0.0225	0.0225	0.0359	0.0293	0.0297	0.0358

Table 3. Estimates of various no-arbitrage equations for the US

There is clearly little difference between the estimates of C-CAPM with power utility and real and nominal returns. The conclusions reached earlier with the respect to nominal returns apply again. Estimates of the Epstein-Zin model in which the market return is assumed to consist of equity and the risk-free are not significantly different from those for power utility. A likelihood ratio test of the two models has a $\chi^2(2)$ test statistic of 0.62, which is not significant. Although the coefficient of the covariance term with inflation is bordering on significance at the one-sided 5% level, we conclude that, in empirical terms, the Epstein-Zin model offers only a minor generalisation of power utility. The implied estimate of the CRRA is $\sigma = \beta_2 - \beta_3 + 1 = 783$ and of the elasticity of inter-temporal substitution is $\gamma = \frac{\beta_3}{\beta_2} = -1.18$. Clearly, neither of these are acceptable.

Unlike the estimates of the general equilibrium models C-CAPM, those of the SDF models do not have any particular theoretical overtones; they can take on any value provided the model satisfies a no-arbitrage condition and the risk premium is non-negative. This is one of the main attractions of the SDF model. Our interest is more in which coefficients are significant and hence which assets are priced sources of risk. The most successful model in these respects is the two-factor SDF model based on consumption and inflation. The coefficients on the consumption and inflation covariances are both siginficant at the 5% level, whereas those for production and the market return (as embodied in the conditional variance) are not.

We do not provide explicit estimates of CAPM as it clear from the estimates of the multi-factor SDF models that it is dominated by the SDF model. In fact, the estimate of the conditional variance is not significant when included on its own. The only version of CAPM that is consistent with our results would be one in which beta is a function of the SDF factors, see equation (8).

For the US, therefore, we find that a two factor SDF model with consumption and inflation as the factors provides the best model.

UK Estimates for the UK data are presented in Table 4.

UK Model	PU real 1	PU 2	EZ 4	SDF2 5	SDF3 6	SDF4 7
$\mathrm{V}_t(i^{uk}_{t+1})$	-0.5	-0.5	2.213 (0.45)	-0.5	-0.5	3.359 (0.74)
$\mathrm{C}_t(\Delta c^{uk}_{t+1}, i^{uk}_{t+1})$	202.99 (2.54)	203.41 (2.54)	153.53 (1.38)	169.51 (1.57)	230.29 (1.85)	257.65 (1.95)
$\mathrm{C}_t(\pi^{uk}_{t+1}, i^{uk}_{t+1})$	0	-1	-690.27 (2.73)	-701.88 (2.75)	-592.05 (2.06)	-490.19 (1.72)
$\mathrm{C}_t(\Delta q^{uk}_{t+1}, i^{uk}_{t+1})$	0	0	0	0	-56.30 (0.60)	-114.12 (1.07)
Dummy, d_t	-0.314 (1.53)	-0.314 (1.53)	-0.321 (1.65)	-0.321 (1.70)	-0.319 (1.80)	-0.318 (1.86)
Deg of Freedom	12.535 (3.16)	12.536 (3.16)	11.469 (3.42)	11.284 (3.43)	11.365 (3.37)	11.628 (3.32)
Log-likelihood	3907.85	3907.84	3912.07	3911.97	3912.04	3912.27
Mean risk premium	7.21%	7.23%	10.81%	10.65%	10.41%	10.54%
$\lvert\lambda_{\max}\rvert$	0.972	0.972	0.979	0.979	0.977	0.976
$\overline{\varepsilon}_{t+1}$	0.147	0.145	-1.486	-1.170	-0.993	-1.433
$\mathrm{V}(\phi_{t+1})$	68.148	68.189	140.082	146.086	139.993	127.526
$\frac{\mathrm{V}(\phi_{t+1})}{\mathrm{V}(i^{uk}_{t+1}+\frac{1}{2}\mathrm{V}_t(i^{uk}_{t+1})-\widehat{\alpha}D)}$	0.0211	0.0211	0.0434	0.0452	0.0433	0.0395

Table 4. Estimates of various no-arbitrage equations for the UK

The estimates for the UK are similar to those for the US. Although the estimates of the CRRA are smaller, they are still far too large to be plausible. In the Epstein-Zin model the covariance of inflation with the return on equity is now significant, but the unrestricted own variance is not. A likelihood ratio test comparing the two models, which is distributed $\chi^2(2)$, is 2.67. This is not significant even at the 10% level. The implied estimate of σ in the Epstein-Zin model is 844.8 and that of γ is -4.50. We conclude once more therefore that in empirical terms the Epstein-Zin model is a minor generalisation of power utility model and gives unreasonable estimates in terms of general equilibrium theory.

Again, the best peforming SDF model is the two factor model with consumption and inflation, with inflation seemingly more important than consumption. Output has more explanatory power than for the US, but is still not significant. CAPM is rejected once more.

4.2.3 Estimates of the time-varying risk premium

Tables 3 and 4 report the estimated mean risk premia and the proportion of the variance of excess returns explained by each model. There are small differences across models and countries, but broadly the estimated mean risk premium is about 10% and the proportion of the excess return explained is about 3% for the US, and 4.5% for the UK. This is, of course, only part of the picture as these figures do not reveal the time variation in the risk premia.

Estimates of the US and UK risk premia for three models: power utility, Epstein-Zin and the two factor SDF model are plotted in Figures 1 and 2. The risk premia are expressed as annualised percentages. The risk premia show considerable variation over time, but are similar for the three models. For both countries, in the main the risk premia are positive, but in the period 1975-1977 there is a tendency for the risk premia to be negative. For the US the Epstein-Zin and SDF risk premia are also negative in 2001.

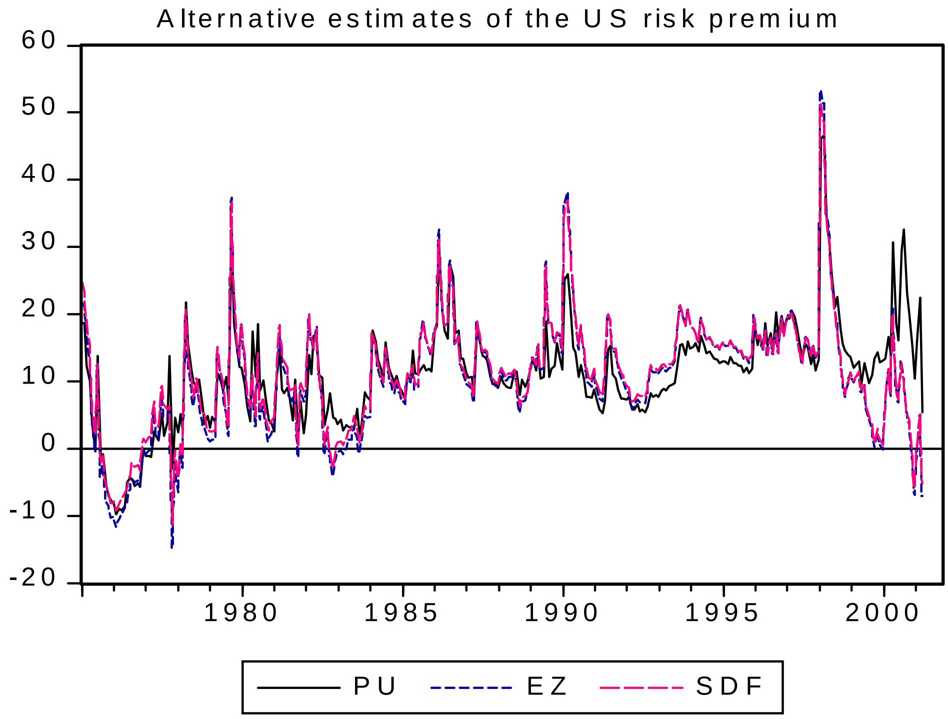

Figure 1.

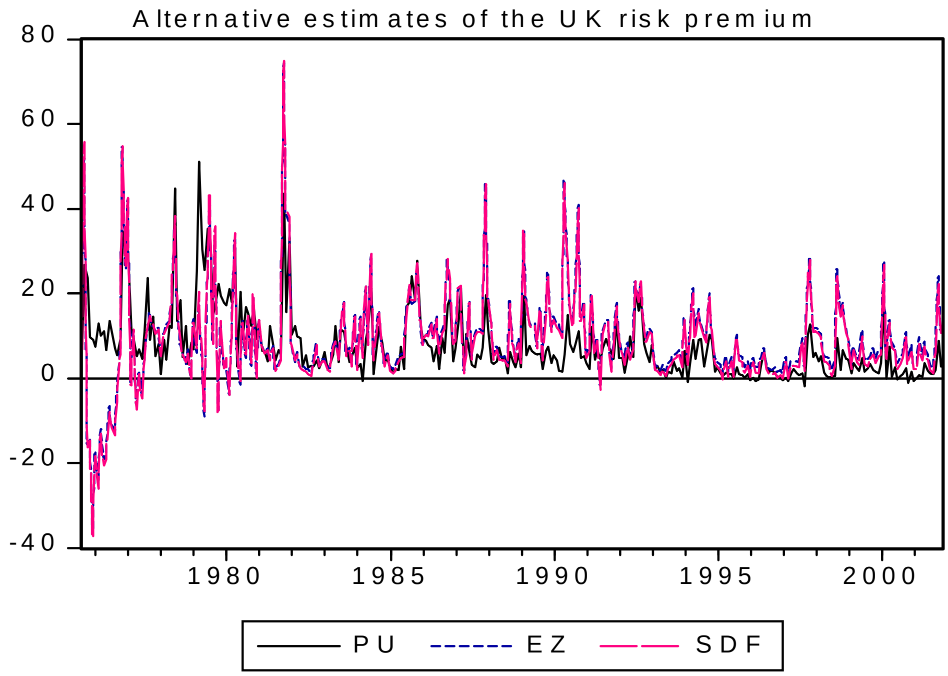

Figure 2.

Comparisons of the risk premia with excess returns (adjusted for the Jensen effect and the October 1987 dummy) can be drawn from Figures 3 and 4. We take as an example the SDF risk premium. It is very interesting to observe that the risk premium tends to jump upwards when excess returns fall sharply and become strongly negative. Although the risk premia fall afterwards as excess returns recover, this occurs more slowly. In contrast, large excess returns are not associated with sharp drops in the risk premium, but the risk premia do fall.

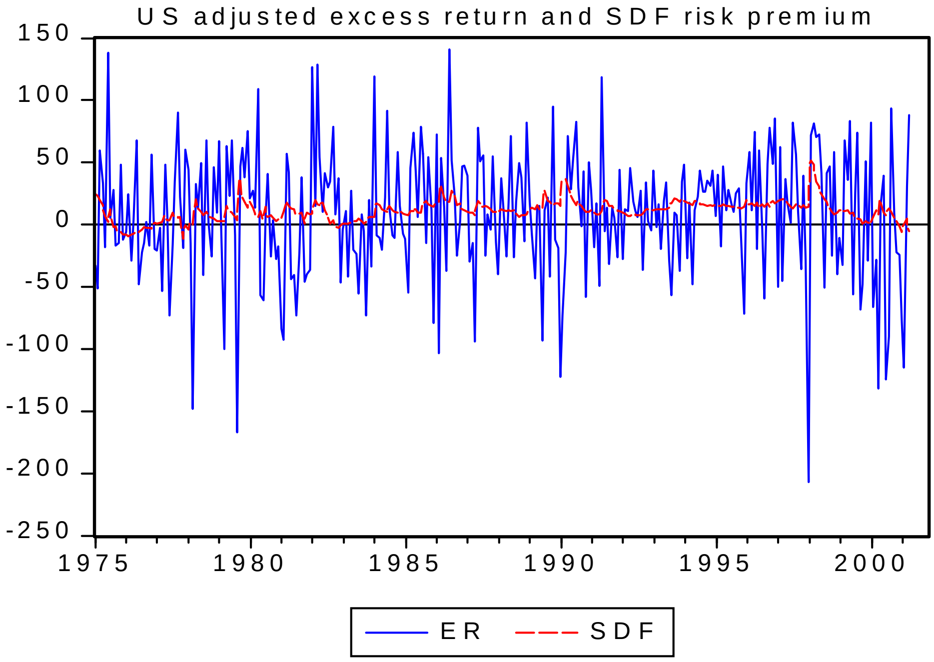

Figure 3.

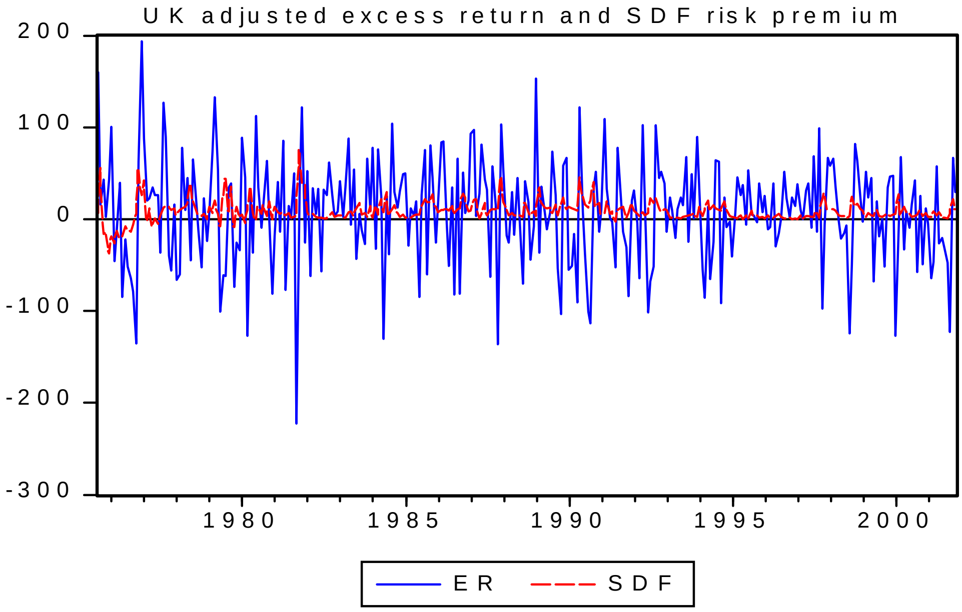

Figure 4.

What is the individual contribution to the risk premium of the different macroeconomic factors? We examine the two-factor SDF risk premium where the factors are consumption and inflation. Figures 5 and 6 plot the SDF risk premium and the conditional covariances of consumption growth and inflation with the excess return on equity. The most obvious point is that the consumption covariance tends to be almost entirely positive, while the inflation covariance tends to be largely negative. This is why the coefficient on the conditional covariance with inflation in the estimated no-arbitrage models are negative. For the US, the inflation covariance increases from 1998 and becomes positive from 1999. This associated with falling inflation over this period. For the UK, both covariances have converged towards zero in the 1990's. As a result, the risk premium has generally been lower over this period than before.

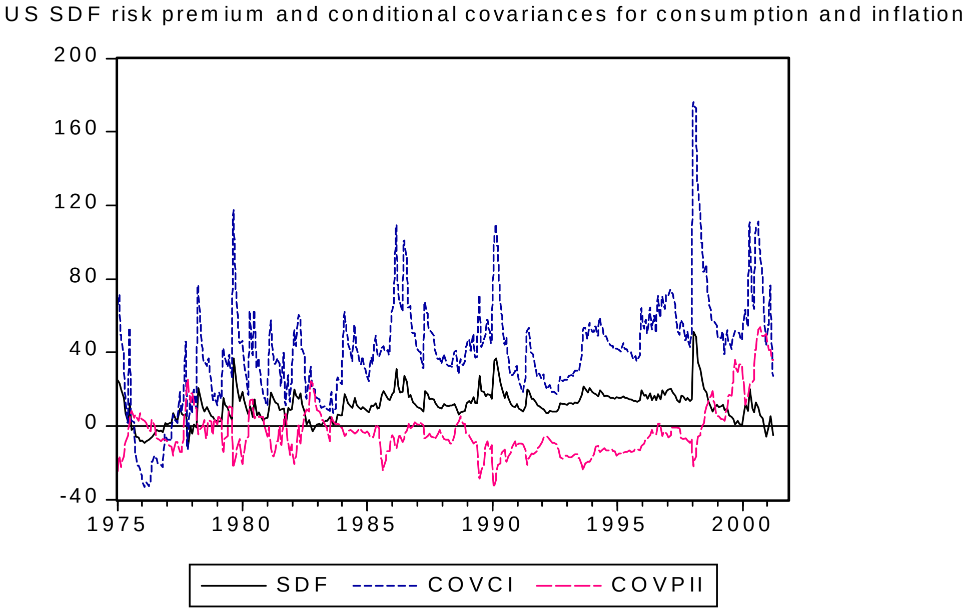

Figure 5.

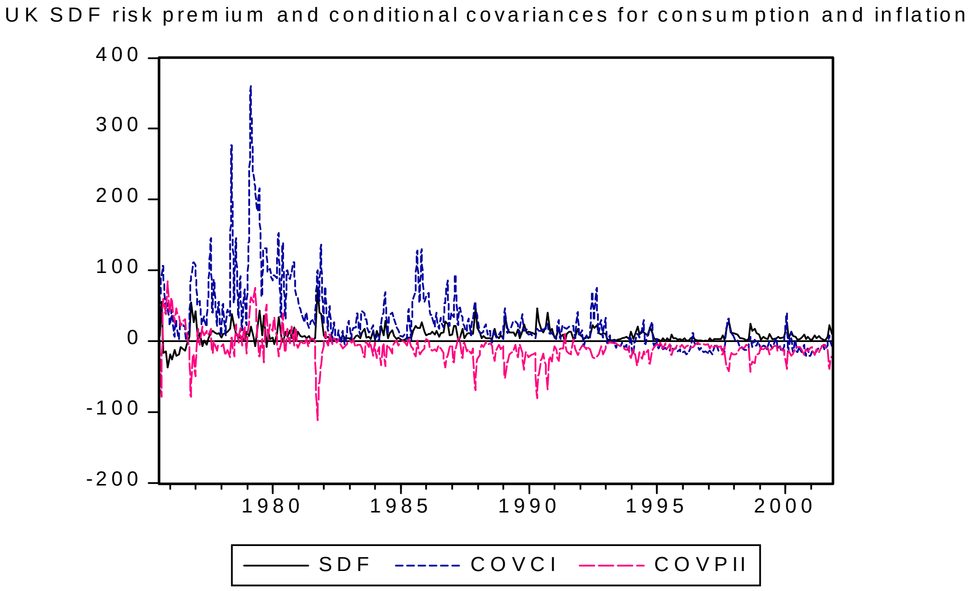

Figure 6.

5 Conclusions

This paper fills a surprising gap in the literature. It provides the first estimates of the time-varying risk premia implied by two key theoretical general equilibrium models, consumption CAPM with time separable power utility and with time non-separable Epstein-Zin preferences. We confirm the findings of other empirical approaches, which do not enable a time-varying risk premium to be extracted, that the coefficient of relative risk aversion is too large for these models to be plausible.

The stochastic discount factor model provides a general framework for pricing assets. Power utility and the Epstein-Zin model are special cases, as is CAPM and other models of time non-separable utility such as the habit persistence model. By focussing on the risk premia implied by these models, it becomes apparent that previous formulations of the habit persistence model do not provide a non-zero time-varying risk premium, and hence can be discarded. An attractive feature of the SDF model, especially given the failure of general equilibrium models, is that shows how assets can be priced in such a way that they satisfy the no-arbitrage condition for any choice of factors, and not just those derived from general equilibrium models. It is also enables some of the restrictions implied by general equilibrium models to be relaxed, and removes the need for the coefficients to be given a specific theoretical interpretation.

Econometric studies of equity returns have tended either to estimate the general equilibrium Euler equation by GMM or use calibration methods. Whilst the former provides estimates of the coefficients, and the latter gives estimates of the mean risk premium, neither provide estimates of a time-varying risk premium. General equilibrium models are examples of observed factor models. The only estimates of a time-varying equity risk premium available are based on the use of unobservable affine factor models. The key distinguishing feature of this paper is a new econometric methodology which permits the estimation of the no-arbitrage condition arising from SDF models with observable factors. We model the joint distribution of the excess return on equity and the observable factors using a multivariate $t-$distribution in which the covariance matrix is assumed to be generated by multivariate GARCH and the conditional mean of the distribution of the excess return is constrained to satisfy the no-arbitrage condition. This entails the presence of conditional covariances in the conditional mean of excess returns. This econometric methodology allows us to estimate the two general equilibrium models, CAPM and various multi-factor SDF models, and to derive estimates their associated time-varying risk premia.

Using monthly data for the US and UK from 1975 to 2001, we find that the best model, both for the US and the UK, is a two-factor SDF model with consumption growth and inflation as the observable factors. As already noted, the resulting estimates of the coefficients for the power utility and Epstein-Zin models are implausible. CAPM can be rejected as it ignores significantly priced sources of risk. The results also imply that the two factor SDF model is to be preferred to purely statistical models that relate equity returns to their volatility.

The estimated equity risk premia that emerge show considerable time variation. They are predominately positive; the exceptions being at the start of our data period in the mid 1970's, and for the US since 1999. There is a case for modifying the model to restrict risk premia to be strictly non-negative. An interesting feature of the estimated risk premia is that they tend to increase sharply during periods of negative excess returns before slowly declining. No corresponding jump is discernable when excess returns are high.

Our conclusion is that this methodology offers a new way to test asset pricing theories, and new insights into the sources of equity risk and new estimates of the behaviour of the equity risk premium. We have obtained similar insights into the pricing of FOREX and are currently examining the use of this methodology for pricing bonds.

6 References

Abel, A.B. (1990) Asset prices under habit formation and catching up with the Joneses, *American Economic Review*, 80, 38-42.

Ang, A. and M. Piazzesi (2000) A no-arbitrage vector autoregression of term strcuture dynamics with macroeconomic and latent variables, Columbia University Working Paper.

Baillie, R.T. and DeGennero (1988), Stock returns and volatility, Michigan State Econometrics and Economic Theory Workshop Paper: 8803.

Bekaert, G. and S.R. Grenadier (1999) Stock and bond pricing in an affine economy, NBER Working Paper 7346.

Campbell, J.Y. (1987) Stock returns and the term structure, *Journal of Financial Economics*, 18, 373-399.

Campbell, J.Y. (1996) Understanding risk and return, *Journal of Political Economy*, 104, 298-345.

Campbell, J.Y. (2002), Consumption-based asset pricing prepared for the *Handbook of the Economics of Finance*, eds G. Constantinides, M. Harris and R. Stulz, Elsevier.

Campbell, J.Y. and J.H.Cochrane (1999) By force of habit: a consumption-based explanation of aggregate stock market behaviour, *Journal of Political Economy*, 107, 205-251.

Campbell, J.Y. and L. Hentschel (1992) No news is good news, *Journal of Financial Economics*, 281-318.

Campbell, J.Y., A. Lo and A. MacKinlay (1997) *The Econometrics of Financial Markets*, Princeton University Press, Princeton, New Jersey.

Cochrane, J. (2001) *Asset Pricing*, Princeton University Press, Princeton, New Jersey.

Connor, G. and R.A.Korajczyk (1995) The arbitrage pricing theory and multifactor models of asset returns, in *Finance, Handbooks in Operational Research and Management Science*, vol 9, eds R.A. Jarrow, V. Maksimovic and W.T.Ziemba, North-Holland, Amsterdam.

Constantinides, G.M. (1990) Habit formation: a resolution of the equity premium puzzle, *Journal of Political Economy*, 98, 519-543.

Ding, Z. and R.F. Engle (1994) Large scale conditional covariaqnce matrix modelling, estimation and testing, mimeo, University of California, San Diego.

Engel, C. (2000) Dynamic conditional correlation - a simple class of multivariate GARCH model, *mimeo University of California, San Diego.*

Engle, R.F. and K.K.Kroner (1995) Multivariate simultaneous generalised GARCH, *Econometric Theory*, 11, 122-150.

Epstein, L.G. and S.E. Zin (1989) Substitution, risk aversion and the temporal behaviour of consumption and asset returns: a theoretical framework, *Econometrica*, 57, 937-968.

Epstein, L.G. and S.E. Zin (1990) First order risk aversion and the equity premium puzzle, *Journal of Monetary Economics*, 26, 387-407.

Epstein, L.G. and S.E. Zin (1991) Substitution, risk aversion and the temporal behaviour of consumption and asset returns: an empirical investigation, *Journal of Political Economy*, 99, 263-286.

Giovannini, R. and P. Weil (1989) Risk aversion and intertemporal substitution in the Capital Asset Pricing Model, National Bureau of Economic Research Working Paper: 2824.

Glosten, L., R. Jagannathan and D Runkle (1993) On the relation between the expected value and the volatility of the nominal excess return on stocks, *Journal of Finance*, 1779-1801.

Hansen, L.P. and K. Singleton (1983) Stochastic consumption, risk aversion and the temporal behaviour of asset returns, *Journal of Political Economy*, 249-265.

Kreps, D and E. Porteus (1978) Temporal resolution of uncertainty and dynamic choice theory, *Econometrica*, 46, 185-200.

Lettau, M. and S. Ludvigson (2002) Measuring and modelling variation in the risk-return tradeoff, prepared for *Handbook of the Economics of Finance*, eds G. Constantinides, M. Harris and R. Stulz, Elsevier.

Mehra, R. and E. Prescott (1985) The equity premium puzzle, *Journal of Monetary Economics*, 15, 145-161.

Scruggs, J. (1998) Resolving the puzzling intertemporal relation between the market riskpremium and conditional market variance: a two-factor approach, *Journal of Finance*, 575-603.

Smith, P.N. and M.R. Wickens (2002) Asset pricing with observable stochastic discount factors, *Journal of Economic Surveys*, 16, 3,397-446.

Wickens, M.R. and P.N. Smith (2001) Macroeconomic sources of FOREX risk, mimeo University of York; http://www-users.york.ac.uk/~pns2/forex.pdf.