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# INFORMATIVE ADVERTISING AND PRODUCT DIFFERENTIATION

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# ABSTRACT

## Informative Advertising and Product Differentiation

We study informative advertising within a random-utility, non-localized competition model of product differentiation. In a symmetric equilibrium, advertising is sub-optimal when product differentiation is small, and excessive otherwise. Increasing the number of firms may increase or decrease the market price. We emphasise that quasi-concavity of profits may fail, as firms may prefer a high price deviation, targeting consumers that only become informed about their product (a feature that, while present in earlier models of informative advertising, has not received enough attention). As product differentiation becomes small, a symmetric equilibrium does not exist.

JEL Classification: D83, L13 and L15
Keywords: informative advertising, non-localized competition, product differentiation and random utility

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# 1 Introduction

In many markets, there are different product varieties and firms, while competing in prices, understand that some consumers may have a stronger preference for their own product and some for a rival product. At the same time, consumers typically do not know all the competing product varieties or their prices. Thus, a role exists for advertising, as firms may wish to inform the consumers about their products and prices. Naturally, in such a setting, firms' pricing and advertising decisions are interrelated.

There is an extensive and important literature on product differentiation and (a somewhat less extensive) on advertising.[1] We revisit the issue by studying informative advertising within a product differentiation oligopoly model with symmetric firms and non-localized competition. For the main building blocks of our analysis, we intentionally choose to follow "standard" models, so that we can focus on their interaction and highlight their properties. Regarding product differentiation, we follow Perloff and Salop's (1985) "random utility" model, one that has provided the basis for important subsequent analyses in the field (see e.g. Wolinsky, 1986, and Anderson *et al.,* 1992 for a review). In this setting we introduce informative advertising (as first formalized by Butters, 1977). Some important and interesting studies of informative advertising and oligopoly exist, the most well known of which is probably Grossman and Shapiro (1984) who study the issue in the context of Salop's (1979) "circle" model.[2] However, the literature does not include an analysis of informative advertising based on a random-utility model or more generally one of non-localized competition.[3]

Understanding how informative advertising interacts with a (random-utility) non-localized model of product differentiation is important, as it represents one of the most straightforward ways to examine advertising in oligopoly. The analysis presents some technical challenges but also leads to interesting insights. The source of the challenges is that, while constructing the (symmetric) equilibrium, we find that the profit function may be not quasi-concave in each firm's own price.

[1] See e.g. Bagwell (2002) for a review of research on advertising, as well on a collection of papers on "Advertising and Differentiated Products" by Baye and Nelson (2001) that includes evidence from a number of markets.

[2] Tirole (1988, ch.7) presents a reformulation of the Grossman and Shapiro (1984) model, based on Hotelling's (1929) linear city formulation.

[3] Advertising in our model informs consumers about both the given product's existence and its price. Alternative formulations exist; e.g. in Meurer and Stahl (1994) consumers observe prices while firms decide how to inform them about product characteristics; in Bester and Petrakis (1995) consumers know that two firms exist and the price of the product their region but only learn the price of the other firm once they receive an ad; in Baye and Kovenock (1994) advertising may inform consumers about a commitment to a lower price than the rival's. Anderson and Renault (2002) study specifically the issue of advertising content (i.e. product vs. price information, or both). Of course, besides directly informative, advertising has also been viewed as playing different roles, including signaling (e.g. Kihlstrom and Riordan, 1984), coordination (Bagwell and Ramey, 1994) or persuasion (Bloch and Manceau, 1999) to mention just a few.

Equilibrium existence is, thus, not guaranteed and the solution to the first-order conditions represents an equilibrium only for a certain range of the parameters, even though local concavity may hold. The economic intuition for the possible lack of quasi-concavity is important. Under the standard assumptions about "informative" advertising, consumers that have not received an advertisement from a given firm do not purchase its product. Thus, with positive probability, some consumers have only received advertisement from one and only one firm – this firm then enjoys monopoly power relative to these "captive"consumers and may have an incentive to raise its price to a high level. Of course, whether such a high price strategy is profitable or not depends on how many consumers become informed from advertising, which is also endogenously determined. It follows that, when constructing a symmetric equilibrium we have to examine possible deviations not only to prices in the neighborhood of the candidate equilibrium but also to much higher levels. Essentially, it may be optimal for a firm to "gamble" by setting a high price that would be certainly rejected if a consumer receives an advertisement from some rival firm, but would yield high profit otherwise.[4]

One may wonder, since the "complication" described above and due to the shape of the profit function appears in our setting, why it is not also present in other models of informative advertising. The answer is that, in fact, it *is* present but, by focusing on local deviations around the symmetric equilibrium, these analyses have implicitly assumed that a deviation to a high price is not profitable. Therefore, we have also included in the present paper an analysis illustrating how this potential complication manifests itself in the context of the well-known Grossman and Shapiro (1984) model, pointing out that earlier results hold for a restricted parameters' space – for other cases, the intuition from these results is not valid.

The main features of our model and results are as follows. A number of firms compete in differentiated products and are able to inform the consumers via costly advertising about the existence of their products and their prices. Firms choose prices and advertising intensities simultaneously. Consumers choose to purchase a unit of the product that leaves them with the highest net surplus (value minus price). We examine the existence of a symmetric equilibrium and explore its comparative statics and other properties. Specifically, our main focus is on symmetric equilibria where all consumers would purchase one unit when they receive an ad from at least one firm. We show that the profit functions are not, in general, quasi-concave in price, a feature that jeopardizes the existence of equilibrium. However, a symmetric equilibrium does exist for a large space of parameters and we prove that, if exists, it is unique.

When we examine the comparative statics of the equilibrium, we find among other results that an increase in the number of firms does not necessarily reduce the equilibrium price. Still, increasing the number of firms decreases the per-firm profit, implying that if entry is endogenized there is a unique zero profit equilibrium number of firms. We also explore the role of the cost of advertising: an increase in the advertising cost reduces the equilibrium advertising levels and increases prices; it may either decrease or increase the equilibrium profit. We then turn to the welfare properties of the equilibrium. Because of the mathematical complexity that characterizes the model, we employ

[4]The term "gamble" here refers only to any given individual consumer. Aggregate demand is deterministic.

numerical examples to show that the market over or underprovides informative advertising depending on the degree of product differentiation and the number of firms of the market. Further, when allowing for endogenous entry, our numerical examples indicate that the market tends to provide a number of products not very different from the optimal (as also happens under perfect information). We also extend our analysis in two directions: we examine conditions for symmetric equilibria in which not all informed consumers purchase some product; we also modify our model to allow for sequential choices of advertising intensity and prices and discuss the strategic effect of commitment to some advertising levels on price setting.

The remainder of the paper is organized as follows. Section 2 sets up the basic model. Section 3 is the core of the analysis; it examines the structure of the profit functions, constructs the symmetric equilibrium, and examines its comparative statics properties. Section 4 endogenizes the number of products/firms by deriving the equilibrium where firms could enter upon paying an entry cost. Section 5 compares the equilibrium to the social optimum. Section 6 revisits the equilibrium construction in Grossman and Shapiro (1984) focusing on the (lack, in general, of) quasi-concavity of the profit functions. Section 7 extends the analysis in two directions: symmetric equilibria where not all informed consumers buy and sequential choices of advertising and prices. Section 8 concludes. Some technical derivations are relegated to the Appendix.

## 2 The model

There are $n \geq 2$ firms, each selling a single product. The population of consumers is normalized to have measure one and each consumer has unit demand. A given consumer $\kappa$ that consumes one unit of the product of firm $i$ has value $v_i^\kappa$. The values $v_i^\kappa$, for each product $i$ and each consumer $\kappa$, are random draws from a distribution $F(v)$, with corresponding density $f(v)$. To simplify the analysis, we further assume that the values $v_i^\kappa$ are i.i.d. uniform random variables in $[a, b]$. Consumers learn their values before they purchase, so their choices are not made under uncertainty. Each firm has constant per unit production cost $c$, with $0 \leq c < a$, and sets a per unit price for its product.

Consumers are not aware of the existence of the products and their prices and, as a result, there is a role for informative advertising. Firm $i$ chooses advertising intensity $\phi_i \in [0, 1]$, where $\phi_i$ is the proportion of consumers that receive the advertisement ("ad", for short) of firm $i$. All consumers are equally likely to receive a particular ad. A consumer that has received an ad becomes (perfectly) informed about that firm's product and its price. For each firm, the cost of advertising is given by an advertising expenditure function $A(\phi_i)$. Following Grossman and Shapiro (1984), we assume that the advertising technology is such that $A : [0, 1] \rightarrow R_+$ has continuous second derivative, $A' > 0$, and $A'' > 0$ thus, it is increasingly more expensive to reach more consumers.[5] We also assume $A'(0)+c < a$ and, thus, the total (production plus advertising) cost of supplying one unit to at least one consumer is certainly lower than his value for the unit. Advertising is the only way a consumer may learn about a product and its price. Thus, if a consumer has not received an ad from some firm, this consumer has zero demand for this firm's product.

[5] A formulation along these lines was first proposed by Butters (1977).

Our basic model corresponds to the following game. First, all the firms choose simultaneously their prices, $p_i$, and advertising intensities, $\phi_i$. Then, choosing from among all the firms from which he has received an ad, each consumer $\kappa$ purchases one unit from the firm that offers the highest net surplus $v_i^\kappa - p_i$ (assuming that purchase leaves the consumer with a nonnegative net surplus; otherwise he buys nothing). We look for a (subgame-perfect) Nash equilibrium of the game where each firm maximizes its profit, and each consumer maximizes his surplus.

# 3 Equilibrium

We proceed to construct a symmetric equilibrium of the game outlined above. Whereas the existence of asymmetric equilibria may not, in general be ruled out, focusing on symmetric equilibrium allows us to make our analysis directly comparable to earlier results while, at the same time, allows us to highlight the main strategic features of the problem.

Assuming all other $(n-1)$ firms choose the same price $p$ and advertising intensity $\phi$, the profit function of firm $i$ is

$$\pi_i(p_i, p, \phi_i, \phi) = (p_i - c)D_i(p_i, p, \phi_i, \phi) - A(\phi_i), \tag{1}$$

where

$$D_i(p_i, p, \phi_i, \phi) = \phi_i \sum_{k=0}^{n-1} \frac{(n-1)!}{k!(n-1-k)!}(1-\phi)^{n-1-k}\phi^k \Pr(p_i, p, k) \tag{2}$$

is firm $i$'s demand. $\Pr(p_i, p, k)$ is defined as the probability that a consumer that sees the ads of firm $i$ *and* of $k$ other firms chooses to buy from firm $i$.

Thus, the demand function can be thought of as constructed in two steps: first we ask, given the firms' advertising strategies, what is the probability any given consumer sees the ads of certain firms; second, we ask, given that consumer's (realization of) values for the products and the prices, what is the probability the consumer will choose to purchase a particular product. There are $(n-1)!/(k!(n-1-k)!)$ different combinations with which a consumer may receive the ads of $k$ firms, and each such combination occurs with probability $(1-\phi)^{n-1-k}\phi^k$, since each consumer is equally likely to receive a firm's ad.[6]

Let us now proceed by assuming that $p < a$, and thus we are after constructing an equilibrium where all consumers purchase one unit of the product if they have received at least one ad (of course, that $p < a$ holds will have to be verified in equilibrium for each parameters' configuration).[7] We need

[6] Recall that the probability each consumer sees the ad of a given firm is independent from that of seeing the ad of another firm. Note that the first "probability" in the definition of the demand functions reflects the stochastic nature of the adverising technology (it is not known whether a given consumer will receive an ad that has been sent or not). The second "probability" captures the consumer heterogeneity: each consumer's decision is deterministic, but the values are distributed in the population.

[7] Our construction here corresponds to the case where all informed consumers buy some product and is, thus, exactly consistent with other studies of product differentiation, where consumers always purchase the best of the available products. The underlying assumption is that a given a consumer gets a basic reservation value, say $R$, high enough,

to calculate the probabilities $\Pr(p_i, p, k)$. These are as follows (see Appendix A1 for the derivation and for the relevant expressions when the density $f(v)$ can take more general forms than the uniform). When $k \geq 1$, each consumer has to compare the price-value combinations of $k+1$ products and it follows that

$$\Pr(p_i, p, k) = \begin{cases} 1 & if \quad p_i < p - b + a \\ \frac{(b-a) + \left( \left( \frac{p-p_i}{b-a} \right)^k - (k+1) \right)(p-p_i)}{(b-a)(k+1)} & if \quad p_i \in [p-b+a, p] \\ \frac{1}{k+1} \left( \frac{b-a-p_i+p}{b-a} \right)^{k+1} & if \quad p_i \in [p, p+b-a] \\ 0 & if \quad p_i > p+b-a. \end{cases} \tag{3}$$

When $k = 0$, each consumer has received only the advertisement of firm $i$ and thus he purchases the product of firm $i$ if and only if his value for that product exceeds the price. Therefore

$$\Pr(p_i, p, 0) = \begin{cases} 1 & if \quad p_i \leq a \\ \frac{b-p_i}{b-a} & if \quad p_i \in [a, b] \\ 0 & if \quad p_i \geq b. \end{cases} \tag{4}$$

If there is a symmetric equilibrium with $p_i = p$, $c \leq p < a$ and $\phi_i = \phi \in (0,1)$ for all $i$, then the first-order conditions $\partial \pi_i(p, p, \phi, \phi)/\partial p_i = 0$ and $\partial \pi_i(p, p, \phi, \phi)/\partial \phi_i = 0$ should hold. To characterize the equilibrium conditions, we proceed as follows.

### 3.1 Necessary conditions for a symmetric equilibrium

Regarding the first-order conditions with respect to price, we set all prices other than $p_i$ equal to $p$, and all advertising levels equal to $\phi$. From (1) we obtain

$$\frac{\partial \pi_i(p_i, p, \phi, \phi)}{\partial p_i} = D_i(p_i, p, \phi, \phi) - (p_i - c)\frac{\partial D_i(p_i, p, \phi, \phi)}{\partial p_i}, \tag{5}$$

where, from (2),

$$\frac{\partial D_i(p_i,p,\phi,\phi)}{\partial p_i} = \phi \sum_{k=0}^{n-1} \frac{(n-1)!}{k!(n-1-k)!}(1-\phi)^{n-1-k}\phi^k \frac{\partial \Pr(p_i,p,k)}{\partial p_i}. \tag{6}$$

Further, from (3), for $k \geq 1$ we have

$$\frac{\partial \Pr(p_i, p, k)}{\partial p_i} = \begin{cases} 0 & if \quad p_i < p - b + a \\ -\frac{1 - \left( \frac{p-p_i}{b-a} \right)^k}{b-a} & if \quad p_i \in [p-b+a, p] \\ -\frac{1}{b-a} \left( \frac{b-a-p_i+p}{b-a} \right)^k & if \quad p_i \in [p, p+b-a] \\ 0 & if \quad p_i > p+b-a, \end{cases}$$

when consuming one of the products, in addition to the idiosyncratic value of each product. Our formulation of values distributed on $[a, b]$ is equivalent to one where the idiosyncratic value of each product is distributed on $[0, b-a]$ but there is, in addition, a reservation value $R = a$ such that in equilibrium $p \leq R$.

and, from (4), for $k = 0$

$$\frac{\partial \Pr(p_i, p, 0)}{\partial p_i} = 0.$$

Note that, since we are evaluating necessary conditions for a symmetric equilibrium with $p < a$, the relevant branch of $\Pr(p_i, p, 0)$ has slope zero. Also observe that $\partial \Pr(p_i, p, k)/\partial p_i$ is continuous in $p_i$ at $p_i = p - b + a$ and that for $p_i = p$ we have $\partial \Pr(p_i, p, k)/\partial p_i = -1/(b-a)$. Thus, by substituting the symmetry condition $p_i = p$ into (6) we obtain

$$\begin{aligned} \frac{\partial D_i(p,p,\phi,\phi)}{\partial p_i} &= \phi \frac{-1}{b-a} \sum_{k=1}^{n-1} \frac{(n-1)!}{k!(n-1-k)!}(1-\phi)^{n-1-k}\phi^k \\ &= \phi \frac{-1}{b-a}\left(1-(1-\phi)^{n-1}\right). \end{aligned} \tag{7}$$

In equilibrium, all firms share the market equally. Given that $\Pr(p, p, k) = 1/(k+1)$, from (2) we have that, given $\phi$, the equilibrium quantity sold by each firm is equal to

$$D_i(p, p, \phi, \phi) = \phi \sum_{k=0}^{n-1} \frac{(n-1)!}{k!(n-1-k)!}(1-\phi)^{n-1-k}\phi^k \frac{1}{k+1} = \frac{1-(1-\phi)^n}{n}. \tag{8}$$

Recall that, since $p < a$, consumers buy if and only if have received some ad. The probability that a given consumer sees no ad is $(1-\phi)^n$ and thus the numerator in the last expression of (8) measures all the consumers that purchase.

Substituting expressions (7) and (8) in (5), we obtain

$$\frac{\partial \pi_i(p,p,\phi,\phi)}{\partial p_i} = \frac{1-(1-\phi)^n}{n} - (p-c)\frac{\phi}{b-a}\left(1-(1-\phi)^{n-1}\right),$$

and by setting $\partial \pi_i(p, p, \phi, \phi)/\partial p_i$ equal to zero we have the candidate equilibrium price as a function of $\phi$ :

$$p = \frac{(b-a)}{n\phi}\frac{1-(1-\phi)^n}{1-(1-\phi)^{n-1}} + c. \tag{9}$$

In Appendix 2 we show that the right-hand side of (9) is strictly decreasing in $\phi$. This implies that at an equilibrium with $\phi < 1$, the equilibrium price is greater than $(b-a)/n+c$, the equilibrium price in the perfect information case (i.e. under $\phi = 1$).[8] This shows that imperfect information increases the market power of each firm, by lowering each firm's demand elasticity and increases the equilibrium price.

We now turn to the calculation of the equilibrium advertisement level. By setting $p_i = p$ in (2), we obtain $D_i(p, p, \phi_i, \phi) = \phi_i(1 - (1-\phi)^n)/(n\phi)$, and from (1) we obtain:

$$\frac{\partial \pi_i(p,p,\phi_i,\phi)}{\partial \phi_i} = (p-c)\frac{1-(1-\phi)^n}{n\phi} - A'(\phi_i).$$

In a symmetric equilibrium with $\phi \in (0,1)$, $\partial \pi_i(p, p, \phi, \phi)/\partial \phi_i$ must be equal to zero, or, equivalently,

$$\frac{1-(1-\phi)^n}{n\phi}(p-c) = A'(\phi). \tag{10}$$

[8]Equation (9) for $\phi = 1$ (all consumers are informed of all products) yields the equilibrium price $p = (b-a)/n + c$ of the underlying Perloff and Salop (1985) model.

We conclude that, if an interior equilibrium exists, conditions (9) and (10) must hold and the system of these two equations can be solved to determine the equilibrium values $(p, \phi)$. Note that if our maintained assumption $(a - c) > A'(0)$ does not hold, (10) cannot be satisfied for any positive advertising level: since $[1 - (1 - \phi)^n]/n\phi < 1$ and $\phi \in (0, 1)$, the left-hand side of (10) would always be less than the right-hand side in such a case (recall that $A'' > 0$).

The next step in the analysis is to establish that, for each set of parameter values, there is a *unique* pair $(p, \phi)$ that may constitute a symmetric equilibrium. We solve equation (9) for $(p - c)$ and substitute into (10) to obtain:

$$\frac{(b-a)}{n^2\phi^2}\frac{(1-(1-\phi)^n)^2}{1-(1-\phi)^{n-1}} = A'(\phi). \tag{11}$$

Let us define $L(\phi)$ as the left-hand side of equation (11). For $\phi \in (0, 1)$, this function satisfies the following properties: (i) $L(\phi)$ is continuous. (ii) $L(\phi)$ is strictly decreasing in $\phi$ (see Appendix A2). (iii) $\lim_{\phi \to 0} L(\phi) = +\infty$ (apply l' Hospital rule), and $\lim_{\phi \to 1} L(\phi) = (b - a)/n^2 > 0$. It follows that, since $A''(\phi) > 0$, if there is a value of $\phi$ that solves (11) that value is unique. We then turn to the first-order condition with respect to price: we obtain that the left-hand side of (9) is strictly decreasing in $\phi$ (see Appendix A2) and thus uniqueness of $\phi$ also implies uniqueness of a $p$ that solves the first-order conditions. We thus obtain the following result.

**Proposition 1** *If a symmetric equilibrium exists with $p < a$ and $\phi \in (0, 1)$, it is unique and characterized by equations (9) and (10).*

As noted above, the left-hand side of (11) tends to $(b - a)/n^2$ as $\phi$ tends to 1. However, if $A'(1) \leq (b-a)/n^2$, there is no pair $(p, \phi)$ that satisfies both (9) and (10) at the same time. Condition $A'(1) \leq (b - a)/n^2$ means that the marginal cost of advertising is less than its marginal benefit for every symmetric price schedule - recall that the right-hand side of (9) is decreasing in $\phi$. This would imply that, in any symmetric equilibrium, $\phi = 1$. Thus we have:

**Proposition 2** *If $A'(1) > (b - a)/n^2$ there is a unique pair $(p, \phi)$ that satisfies the first-order conditions (9) and (10). If $A'(1) \leq (b - a)/n^2$ then there is no pair $(p, \phi)$ that satisfies (9) and (10). In such a case, if a symmetric equilibrium exists with $p < a$, it involves $\phi = 1$ and $p = (b - a)/n + c$.*

## 3.2 Sufficient conditions

What may prevent the unique pair $(p, \phi)$ that solves (9) and (10) from constituting an equilibrium is that some firm $i$ may have a profitable deviation when all other firms choose $(p, \phi)$. This possibility comes about because the profit function of firm $i$ is not, in general, quasi-concave in $p_i$. So, even when the necessary conditions are satisfied and sufficiency conditions may hold locally, a symmetric equilibrium may fail to exist. The following result describes the second-order conditions of the firms' maximization problem.

**Lemma 1** *(i) $\pi_i(p, p, \phi_i, \phi)$ is strictly concave in $\phi_i$. (ii) $\pi_i(p_i, p, \phi, \phi)$ is not always quasi-concave in $p_i$.*

**Proof.** Regarding the first part of the Lemma, we have

$$\pi_i(p, p, \phi_i, \phi) = \phi_i \frac{1-(1-\phi)^n}{n\phi}(p-c) - A(\phi_i),$$

which is strictly concave in $\phi_i$ (recall that $A''(\phi_i) > 0$).

To prove the second part of the Lemma it is enough to establish the statement via numerical counter-examples (see below). ■

Let us now offer some remarks on the crucial property of the lack, in general, of quasi-concavity of the profit function. The profit function given in (1) is the sum of the functions $(p-c)\Pr(p_i, p, k)$, $k = 0, 1, ..., n-1$, each weighted by certain constants. Each function $(p-c)\Pr(p_i, p, k)$, $k = 1, ..., n-1$, is quasi-concave in $p_i$.[9]. However, each of these functions is maximized at a different price, $p = \frac{b-a}{k}$, which implies the possible non quasi-concavity of $\pi_i(p_i, p, \phi, \phi)$. What appears to play a key role is the function $(p-c)\Pr(p_i, p, 0)$. In particular, assume the case $p+b-a < a$ and consider $p_i \in [p+b-a, a]$.[10] Then, firm $i$ sells only to the consumers that receive its ad. It follows that in this region, the profit function is equal to $\pi_i(p_i, p, \phi, \phi) = (p_i - c)\phi(1-\phi)^{n-1} - A(\phi)$, which is strictly increasing in $p_i$.[11] Thus, in such a case, there is an interval of prices greater than $p$ in which $\pi_i(p_i, p, \phi, \phi)$ is strictly increasing, a property that in turn implies that $\pi_i(p_i, p, \phi, \phi)$ is not quasi–concave in price.

Two examples help illustrate the point (see Figure 1). We employ the parameter values $n = 7, b = 10, c = 0$, and the advertising cost function $A(\phi_i) = er\ln(1-\phi_i)/\ln(1-r)$ with $e = 0.5$ and $r = 0.1$.[12]

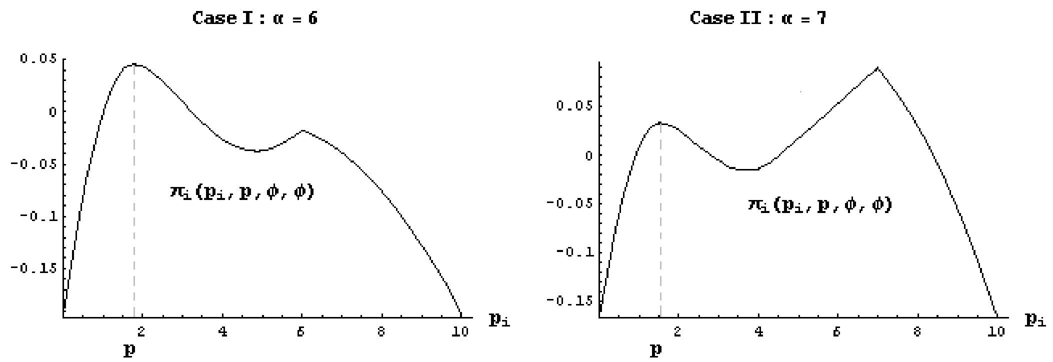


Figure 1.

[9]See Caplin and Nalebuff (1991).

[10]There is a continuum of parameter values that have the property $p+b-a < a$. For example, for $b = 10$, $a = 7$, $n = 5$, $A(\phi_i) = er\ln(1-\phi_i)/\ln(1-r)$, $e = 1$, and $r = 0.1$, the solution to the system of the equations (9) and (10) gives $p \approx 2.32$, where clearly $p + b - a < a$.

[11]This is because even the consumer with the highest valuation for the product of firm $i$, would prefer to buy any other product even if he had the lowest possible valuation for that product: $b - p_i \leq a - p$, or $p_i \geq p + b - a$.

[12]This advertising cost function is borrowed from Grossman and Shapiro (1984). The intuition behind the function is the following: suppose that each ad informs $r$ percentage of the consumer population. Assuming that each consumer is equally likely to receive any ad, if a firm makes $w$ ads, it will inform a percentage $\phi = 1-(1-r)^w$. This means that $w = \ln(1-\phi)/\ln(1-r)$ number of ads are required to reach $\phi$ consumers (since the measure of the population has been normalized to unity). If each ad costs $er$ money units, that is, $e$ per "person" informed by the ad, the cost of informing $\phi$ consumers, is $A(\phi) = er\ln(1-\phi)/\ln(1-r)$.

In both cases, the profit function $\pi_i(p_i, p, \phi, \phi)$ is not quasi-concave. In Case I, we assume $a = 6$, and, then, the corresponding candidate equilibrium pair $(p, \phi) \approx (1.767, 0.334)$, as derived by equations (9) and (10), appears to be consistent with a symmetric equilibrium.[13] In the second case, we assume $a = 7$, and, then, the candidate pair $(p, \phi) \approx (1.517, 0.294)$ is clearly not a symmetric equilibrium strategy profile. In this second case, there is clearly a profitable deviation to a higher price, even though the first-order conditions are satisfied and even though the sufficiency conditions hold locally.

Importantly, there are also cases where the profit function appears to be quasi-concave and, thus, have a unique local maximum.[14] For example, Figure 2 shows the profit function when all parameters are as in the case calculated above, but now we set $a = 4$.

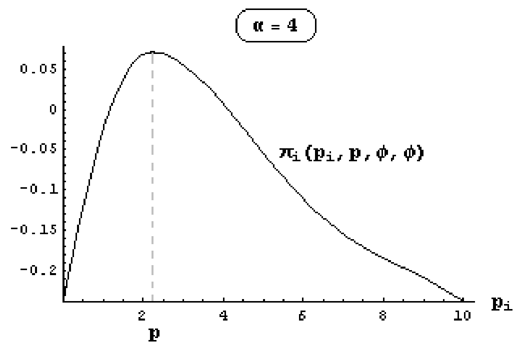


Figure 2.

That the relevant profit function when constructing a symmetric equilibrium is not in general quasi-concave and that an equilibrium may fail to exist even when the first-order and the *local* sufficiency conditions hold is clearly a problematic feature in this analysis. Two remarks are in order. First, that similar complications exist in other related and well-known models of informative advertising (see e.g. Section 6) – our analysis here simply sheds light to this property and emphasizes that the symmetric necessary conditions indeed give us an equilibrium only for some parameter ranges. Second, for each application of the model, a numerical analysis has to be carried out to confirm that the first-order conditions indeed imply an equilibrium. Figure 3 illustrates, numerically calculated, $\pi_i(p_i, p, \phi_i, \phi)$ as a function of $p_i$ and $\phi_i$ for the same set of parameters other than $a$ used above for Figures 1 and 2 (for ease of presentation, the figures have been truncated to report only positive profit levels). In Case I the first order conditions indeed give us the unique symmetric equilibrium $(p, \phi) \approx (1.67, 0.32)$, whereas in Case II they do not as there is a profitable deviation from the candidate point $(p, \phi) \approx (1.57, 0.30)$ to a higher price. For the following steps in our analysis, we proceed by implicitly restricting the parameters so that our conditions indeed characterize an equilibrium.[15]

[13] Still, we are not able to conclude from the diagram that the given $(p, \phi)$ is indeed the equilibrium strategy profile, because a firm may be considering a coordinated deviation in both its price and advertising. For such a conclusion, a maximization over both $p$ and $\phi$ is necessary (see e.g. the numerical examples corresponding to Figure 3 below).

[14] Still, establishing these properties analytically does not appear possible.

[15] Ideally, one would like to partition the parameters space, specifying exactly the regions where a symmetric equilibrium exists. However, the problem is too involved to allow us to do this. For instance, one may think that starting from the candidate equilibrium, a high price deviation would be more desirable for a firm when parameters imply an

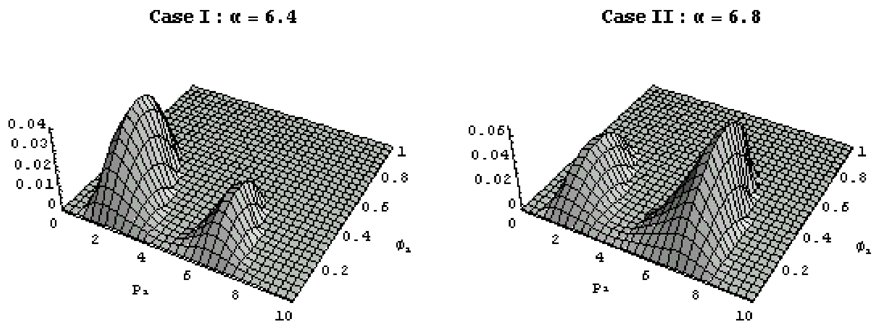


Figure 3.

### 3.3 Vanishing product differentiation

Our set-up allows us to examine the limiting case where product differentiation becomes small ($a \to b$). We show that, in that case, there is no symmetric equilibrium.

First, we show that for $a = b$ there is no symmetric equilibrium. The proof is as follows. If the candidate equilibrium involved $\phi > 0$ and $p > c$, a firm would have a profitable deviation to slightly lowering its price from $p$ and capturing all consumers informed by that firm. If $\phi > 0$ and $p = c$, then all firms would make losses, and any one of them would have an obvious profitable deviation to $\phi = 0$. Finally, if $\phi = 0$ then there are no informed consumers and firms make zero profit. Then, any given firm has a profitable deviation since the level of $\phi$ that maximizes $\phi(b - c) - A(\phi)$ results in a positive (and attainable) profit level – this follows from our assumption $A'(0) < b - c$, that the market is viable.

Next, we turn to the limiting case where the difference $(b - a)$ tends to zero. We show, essentially by exploring the continuity of the relevant conditions, that for $(b - a)$ small a symmetric equilibrium as characterized above fails to exist (see Appendix A3 for the formal proof). Thus, we obtain the following result.

**Proposition 3** *There exists no symmetric equilibrium when $a$ is close enough to $b$.*

### 3.4 Comparative statics

Now we turn to the comparative statics properties of the equilibrium we have derived.

**(i)** $\frac{d\phi}{dn} < 0$

Equation (11) determines implicitly the equilibrium level of advertisements as a function of $n$. Since the derivative of the left-hand side of (11) is decreasing in $\phi$ and in $n$ (see Appendix A2), and $A'' > 0$, it follows by implicit differentiation of (11) that $d\phi/dn < 0$. Thus, a more concentrated market implies more advertising by each firm. The reason behind this result is that informative advertising increases price competition. In order to counterbalance the increase in price competition

increased probability that consumers only receive an ad from that firm, which in turn depends on $\phi$. However, the possible profitability of a deviation also depends on the level of the candidate equilibrium price, which is, of course, jointly endogenous with $\phi$.

by the introduction of an additional firm the firms find it profitable to reduce their advertising. The marginal profitability of the ads also falls as the number of firm increases, because the average number of ads that each consumer receives increases.

**(ii)** $\frac{d\phi}{d(b-a)} > 0$

The result is obtained by implicit differentiation of (11) since the two ratios at the left-hand side of (11) are decreasing in $\phi$ and $A'' > 0$. Thus, in markets with more heterogenous products, firms advertise more. Underlying this result is that a larger $(b-a)$ implies lower price competition, higher profit margins, and thus, increases the marginal profitability of informing the consumers.

**(iii)** $\frac{dp}{dn} \gtrless 0$

We find that an increase of $n$ may, depending on the parameter values and the form of the advertising cost function, increase or decrease $p$. The equilibrium price is given by (9). On the one hand, an increase in $n$ has a direct negative impact on $p$ (by (9) and Appendix 2, Result 2). On the other hand, an increase in $n$ causes a decrease in $\phi$ (see comparative statics (i) above), that tends to increase $p$. Thus, the indirect effect is positive. The final impact is ambiguous. In Appendix A4 we compute the derivative $dp/dn$ analytically. We show that the sign of $dp/dn$ depends on the parameters of the model and the qualitative features of the function $A(\phi)$.

To gain some additional insights about this result note that if the advertising technology is relatively efficient, an increase in $n$ will result in a relatively large decrease in advertising, which, in turn, leads to lower prices (since then consumers are less informed). To illustrate, let us use a parametric example where we employ a relatively efficient advertising technology. Suppose $b = 200$, $a = 100$, $c = 0$, and $A(\phi) = 8\phi^{1.01}$. Then for $n$ taking values 8, 9, and 10 the market prices are 28.553, 28.561, and 28.564, respectively, that is, increasing in the number of firms.

**(iv)** $\frac{dp}{d(b-a)} > 0$

The equilibrium price is given by (9). Since the level of ads, $\phi$, is also endogenous, a change in $(b-a)$ affects the equilibrium price directly (consider a change of $(b-a)$ with $\phi$ kept constant), as well as indirectly, through the impact of $(b-a)$ in $\phi$. On the one hand, as consumer preferences become more diverse, the demand function of each firm becomes less elastic, tending to increase the equilibrium price. On the other hand, an increase in $\phi$ tends to reduce the price. In Appendix A5 we show that the net effect of an increase of $(b-a)$ on $p$ is positive.

For our next three results, we assume that the advertising cost function can be written as $\tilde{A}(\phi, e) = eA(\phi)$, where $e$ is, in the terminology of Grossman and Shapiro (1984), the per-reader cost (of advertising). The parameter $e$ is, essentially, the price of advertising: if, for example, one needs a measure of $A(\phi)$ advertisements in order to reach a measure of $\phi$ consumers, and the cost of *each* advertisement is $e$, the total cost of this advertising campaign is $eA(\phi)$.

**(v)** $\frac{d\phi}{de} < 0$

By implicit differentiation of (11), we obtain

$$\frac{\partial}{\partial \phi}\left(\frac{(b-a)}{n^2\phi^2}\frac{(1-(1-\phi)^n)^2}{1-(1-\phi)^{n-1}}\right)d\phi = \tilde{A}_{\phi\phi}d\phi + \tilde{A}_{\phi e}de. \tag{12}$$

Since the left-hand side of (11) is decreasing in $\phi$ (see Appendix A2), $\tilde{A}_{\phi\phi} > 0$, and $\tilde{A}_{\phi e} = A' > 0$, it follows directly that $d\phi/de$. Thus, more costly advertising leads firms to advertise less.

**(vi)** $\frac{dp}{de} > 0$

By (9) we have that $\frac{dp}{de} = \frac{dp}{d\phi}\frac{d\phi}{de}$, as $p$ depends on $e$ only indirectly, through $\phi$. Since $\frac{d\phi}{de} < 0$ and $\frac{dp}{d\phi} < 0$, the result follows immediately.

**(vii)** $\frac{d\pi}{de} \gtrless 0$

The equilibrium profit as a function of $e$ takes the form

$$\pi_i(p,p,\phi,\phi,e) = \phi\frac{1-(1-\phi)^n}{n\phi}(p-c) - \tilde{A}(\phi,e).$$

From (10) and substituting $\tilde{A}_\phi \equiv \partial A/\partial\phi$ for $A'$, the per-firm profit in the symmetric equilibrium is

$$\pi_i(\phi(e),e) = \phi(e)\tilde{A}_\phi(\phi(e),e) - \tilde{A}(\phi(e),e).$$

It follows that

$$\begin{aligned}
\tfrac{d\pi_i(\phi(e);e)}{de} &= \tilde{A}_\phi(\phi(e);e)\tfrac{d\phi(e)}{de} + \phi(e)\tfrac{d\tilde{A}_\phi(\phi(e);e)}{de} - \tfrac{d\tilde{A}(\phi(e);e)}{de} \\
&= \tilde{A}_\phi(\phi(e);e)\tfrac{d\phi(e)}{de} + \phi(e)\tfrac{d\tilde{A}_\phi(\phi(e);e)}{de} - \left[\tfrac{\partial\tilde{A}(\phi(e);e)}{\partial e} + \tilde{A}_\phi(\phi(e);e)\tfrac{d\phi(e)}{de}\right] \\
&= \phi(e)\tfrac{d\tilde{A}_\phi(\phi(e);e)}{de} - \tfrac{\partial\tilde{A}(\phi(e);e)}{\partial e} \\
&= \phi(e)\left\{\tilde{A}_{\phi\phi}(\phi(e);e)\tfrac{d\phi(e)}{de} + \tilde{A}_{\phi e}(\phi(e);e)\right\} - A(\phi) \\
&= \phi(e)\left\{\tilde{A}_{\phi\phi}(\phi(e);e)\tfrac{d\phi(e)}{de} + A'(\phi)\right\} - A(\phi),
\end{aligned}$$

the sign of which can be either positive or negative. An increase in $e$ affects profit both directly and indirectly. The direct effect is negative: an increase in $e$ increases the cost of reaching any given measure of $\phi$ consumers, $\tilde{A}(\phi,e)$. There are two indirect effects: through $\phi$ and through $p$. Specifically, $\frac{d\phi}{de} < 0$, and $\frac{dp}{de} > 0$. The net impact on profit depends on the parameters and the form of the advertising cost function. To illustrate let us use an example: Suppose $b = 200$, $a = 100$, $n = 5$, $c = 0$. If $A(\phi) = e\phi^{1.2}$, for $e$ taking values 4, 5, and 6, the per-firm equilibrium profit is 0.72, 0.8, and 0.87, respectively; thus, higher advertising cost implies higher per-firm profit. If $A(\phi) = e\frac{r\ln(1-\phi)}{\ln(1-r)}$, and $r = 0.1$, for the same values of $e$, an inverse relationship can be traced.

# 4 Endogenizing the number of firms

In this section, we expand the analysis by endogenizing the number of firms. This extension also serves as a first-step for our welfare analysis below. We consider a "free entry" equilibrium, more precisely one where any firm can enter the market by paying an entry cost $K > 0$. Formally, we add to the initial game a stage, in which firms decide whether to enter the market or not. We examine the subgame-perfect Nash equilibrium of this enlarged game. Given the number of firms, and provided that a symmetric equilibrium exists, our analysis above implies that

$$\begin{aligned}\pi^*(n) &= \tfrac{1-(1-\phi)^n}{n}(p-c) - A(\phi) - K \\ &= \phi\tfrac{1-(1-\phi)^n}{n\phi}(p-c) - A(\phi) - K,\end{aligned}$$

where $\pi^*(n)$ denotes the equilibrium profit of each firm, and $\phi \equiv \phi(n)$ and $p \equiv p(n)$ are the equilibrium advertising level and equilibrium price, respectively, as functions of $n$.

The following proposition ensures that, when entry is endogenized, the equilibrium number of firms is unique.

**Proposition 4** *In the symmetric equilibrium, the profit of each firm is strictly decreasing in the number of firms in the market.*

**Proof.**

$$\begin{aligned}\tfrac{d\pi^*(n)}{dn} &= \tfrac{d}{dn}\left(\phi\tfrac{1-(1-\phi)^n}{n\phi}(p-c)\right) - A'(\phi)\tfrac{d\phi}{dn} \\ &= \phi\tfrac{d}{dn}\left(\tfrac{1-(1-\phi)^n}{n\phi}(p-c)\right) + \left(\tfrac{1-(1-\phi)^n}{n\phi}(p-c)\right)\tfrac{d\phi}{dn} - A'(\phi)\tfrac{d\phi}{dn} \\ &= \phi\tfrac{dA'(\phi)}{dn} + A'(\phi)\tfrac{d\phi}{dn} - A'(\phi)\tfrac{d\phi}{dn}, \quad \text{by (10)} \\ &= \phi A''(\phi)\tfrac{d\phi}{dn} < 0,\end{aligned}$$

since $\frac{d\phi}{dn} < 0$. ■

Note that we have substituted $\frac{dA'(\phi)}{dn}$ for $\frac{d}{dn}\left(\frac{1-(1-\phi)^n}{n\phi}(p-c)\right)$. By totally differentiating (11), we have shown that $d\phi/dn < 0$. The equilibrium condition (10) holds when moving from $\phi$ to $\phi + d\phi$. Thus, the change in $\left(\frac{1-(1-\phi)^n}{n\phi}(p-c)\right)$ is equal to the change in $A'(\phi)$. But the change in $A'(\phi)$ is $\frac{dA'(\phi)}{dn} = A''(\phi)\frac{d\phi}{dn}$.

Since equilibrium per firm profit is strictly decreasing in the number of firms, then, assuming that a symmetric equilibrium with $p < a$ is played after entry, we immediately obtain the following result:

**Corollary** *The "free entry" equilibrium, if it exists, is unique.*

## 5 Welfare

We now turn to the comparison between the market equilibrium and the social optimum. Suppose a social planner chooses the level of advertising $\phi$, taking as given the $n$ firms in the market. Total surplus is then given by

$$W(n,\phi) = \sum_{k=1}^{n} \frac{n!}{k!(n-k)!}(1-\phi)^{n-k}\phi^k M(k) + \left(\frac{(1-(1-\phi)^n)}{n}(p-c) - A(\phi) - K\right)n, \tag{13}$$

where

$$M(k) \equiv \int_a^b v\left(k\,(F(v))^{k-1}\right)f(v)dv - p = \frac{kb+a}{k+1} - p, \tag{14}$$

for $k \geq 1$. $M(k)$ represents the expected value of the maximum surplus of the consumer that receives $k$ ads. The factor in parenthesis at the last term of (13) is the net profit of each of the $n$ firms.

Using (14) and after some manipulations, (13) can be rewritten as

$$W(n,\phi) = z(n,\phi) - (A(\phi) + K)\, n, \tag{15}$$

where

$$\begin{aligned} z(n,\phi) &\equiv \textstyle\sum_{k=1}^{n} \frac{n!}{k!(n-k)!}(1-\phi)^{n-k}\phi^{k}\left(b - c - \frac{b-a}{k+1}\right) \\ &= (b-c)\left(1-(1-\phi)^{n}\right) - (b-a)\left(\frac{1}{(n+1)\phi}\left(1-(1-\phi)^{n+1}\right) - (1-\phi)^{n}\right) \\ &= (a-c)\left(1-(1-\phi)^{n}\right) + (b-a)\left(1 - \frac{1}{(n+1)\phi}\left(1-(1-\phi)^{n+1}\right)\right). \end{aligned} \tag{16}$$

The function $z(n,\phi)$ represents the expectation of the maximum value of each consumer, gross of price and net of production cost, $c$, and $(A(\phi) + K)$ is the advertising plus entry cost per firm. Note that the price variable does not enter the final expression for the total surplus, because for $p < a$ every consumer that receives an ad purchases exactly one unit.

The first issue to examine is whether the market over or underprovides ads for a given number of firms.[16] At a second stage, we examine the socially optimal number of firms and compare it to the "free entry" number. Due to the complexity of the mathematical formulae involved, we have to rely upon numerical techniques. We have tested a wide variety of parameter configurations, and, while we cannot state a formal Proposition, we report:

**Summary of numerical results 1** *For a given number of firms, the market tends to overprovide or underprovide informative advertisements (as compared to the socially optimal level), depending on the degree of product differentiation, as captured by the difference $(b - a)$. Specifically, for small (large) values of $(b - a)$ the market underprovides (overprovides) advertisements as compared to the socially optimal level of advertisements.*

Table 1 presents results that help illustrate the above finding.

The intuition is as follows: As $(b - a)$ shrinks and the products become less differentiated, the market equilibrium price falls (as per comparative statics result (iv)). This decrease in profit margins implies a weakening of the firms' incentives to provide information and results in underprovision of ads. The opposite happens when the market is characterized by relatively high product differentiation and, thus, high profit margins. In that case, each firm has a strong unilateral incentive to advertise and expand its market share, that leads to an equilibrium advertising level that is too high from a social welfare point of view (since advertising is costly). This is an important finding and one that does not follow from the Grossman and Shapiro (1984) formulation. In fact, their work suggests a

[16] In the classic Butters (1977) paper on informative advertising, equilibrium advertising by firms producing a homogeneous product is socially optimal. In important extensions, Stegeman (1991) shows that advertising is inadequate when valuations are heterogeneous and Stahl (1994) finds a similar result with downward sloping individual consumer demands. See also Anderson *et al.* (1992) for a discussion of this issue.

| Numerical examples: Comparison of $\phi^*$ and $\phi^o$ | | |
|---|---|---|
| $c = 0, b = 200, A(\phi) = erln(1-\phi)/ln(1-r), r = 0.1, e = 2$ | | |
| $n$ | **Range of product diversity** | **Result** |
| 5 | $a < (>)125$ | $\phi^* > (<)\phi^o$ |
| 10 | $a < (>)118$ | $\phi^* > (<)\phi^o$ |
| 15 | $a < (>)112$ | $\phi^* > (<)\phi^o$ |
| 20 | $a < (>)107$ | $\phi^* > (<)\phi^o$ |

Table 1: Comparison of the equilibrium ($\phi^*$) and the optimal ($\phi^o$) advertising.

result in sharp contrast to ours, namely, that, for given number of firms, the market overprovides information for all degrees of product differentiation.[17]

We now turn to the issue of the socially optimal number of firms. Again, the mathematical formulae are too complex to allow for formal results, and we have to resort to numerical calculations. In our model, firms decide on price, advertising level, and entry. Since consumers have unit demands and, in equilibrium, all informed consumers purchase, the price decisions of firms do not affect the welfare calculations. However, the decisions for the level of advertisement do affect social welfare, because they determine the number of informed consumers and, thus, dictate the number of trades that take place. Therefore, we have to distinguish between the constrained and unconstrained social optimal number of firms. The former refers to the case where firms compete in advertising (we call this "second-best"), and the latter to the case where the social planner selects both the number of firms and the advertisement levels ("first-best").

**Summary of numerical results 2** *The market, in general, tends to overprovide product variety. However, the difference between the socially optimal (either first-best or second-best number of firms) and the market free entry equilibrium number of firms tends to be small. Greater differences are associated with more costly advertisement technologies.*

To derive the constrained socially optimal number of firms, we maximize (15) with respect to $n$, given that the advertising level associated with each $n$ is the one chosen in equilibrium by the firms, that is, given by (11). To derive the unconstrained socially optimal number of firms we maximize (15) with respect to both $n$ and $\phi$. Table 2 compares the market free-entry equilibrium number of firms with the first-best and the second-best. Due to the nature of the problem and the discreteness of variable $n$, the first-best values calculated in the examples often do not differ from the second-best values. We, thus, indicate in Table 2 separately only the cases where the two values differ.

[17] Although their formal analysis is conducted under the assumption of large $n$, Grossman and Shapiro (1984) report that based on numerical simulations, the conclusion holds true even for small $n$; in an attempt to better relate their analysis to ours, we have replicated such calculations and indeed the above statement appears correct.

| **Numerical examples: Comparison of $n^*$, $n^o$, and $n^{oo}$** | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $Parameters: c=0, b=200, r=0.1, K=0.3, A(\phi)=erln(1-\phi)/ln(1-r)$ | | | | | | | | | | | | | | | | |
| | $e=0.1$ | | $e=0.5$ | | $e=1$ | | $e=1.5$ | | $e=2$ | | $e=2.5$ | | | $e=3$ | | |
| $a$ | $n^*$ | $n^o$ | $n^*$ | $n^o$ | $n^*$ | $n^o$ | $n^*$ | $n^o$ | $n^*$ | $n^o$ | $n^*$ | $n^o$ | $n^{oo}$ | $n^*$ | $n^o$ | $n^{oo}$ |
| 40 | 20 | 19 | 18 | 17 | 18 | 16 | 17 | 16 | 17 | 15 | 17 | 15 | | 17 | 14 | 15 |
| 60 | 18 | 18 | 17 | 16 | 16 | 15 | 16 | 14 | 16 | 14 | 16 | 14 | | 16 | 13 | |
| 80 | 17 | 16 | 16 | 15 | 15 | 14 | 15 | 13 | 15 | 13 | 15 | 12 | | 15 | 12 | |
| 100 | 16 | 15 | 14 | 13 | 14 | 12 | 14 | 12 | 13 | 11 | 13 | 11 | | 13 | 11 | 10 |
| 120 | 14 | 13 | 13 | 12 | 12 | 11 | 12 | 10 | 12 | 10 | 12 | 10 | | 12 | 10 | |
| 140 | 12 | 11 | 11 | 10 | 11 | 9 | 10 | 9 | 10 | 8 | 10 | 8 | | 10 | 8 | |
| 160 | 10 | 9 | 9 | 8 | 9 | 7 | 8 | 7 | 8 | 7 | 8 | 7 | 6 | 8 | 7 | 6 |

Table 2: Comparison of the equilibrium ($n^*$) to the first-best ($n^o$) and the second-best ($n^{oo}$).

Straightforward calculations imply that, under perfect information ($\phi = 1$) the market equilibrium number of firms exceeds the optimum number by one.[18] The results presented in Table 2, suggest that the presence of imperfect competition may further increase the gap between the market free-entry equilibrium number of firms and the socially optimum number of firms, but only by one or, less likely, by two firms. This is because imperfect information reduces the intensity of price competition, leading to a greater number of firms in the market.

## 6 Informative advertising with localized competition

Thus far, we have analyzed the properties of a symmetric equilibrium with informative advertising in a random utility (non localized competition) model of product differentiation. In this section, we revisit earlier work on informative advertising in oligopoly, where the underlying model in one of localized competition. In particular, and as discussed in the Introduction, we analyze aspects of the well-known model of Grossman and Shapiro (1984), which builds on Salop's (1979) circle model. This is an important model, the first that introduced advertising in a differentiated product oligopoly. As such, it has been the basis for extensions and, in general the subject of considerable attention. The reason for returning to this model here is to draw some analogies and illustrate some differences for informative advertising when the underlying model of oligopolistic competition is one of the localized or the non-localized type.

The emphasis is on the properties of the profit function and, in particular, on the lack, in general, of quasi-concavity. As we show, this is a property shared by both models, even though Grossman and Shapiro (1984) have not examined this possibility, implicitly restricting the analysis to parameter ranges where a deviation to a high price is not profitable.[19] We believe this is an interesting, more

[18] With perfect information the equilibrium price is $p = (b-a)/n$, and the profit as a function of $n$ equals $(b-a)/n^2 - K$. It follows that the zero profit equilibrium implies $n^* = \sqrt{(b-a)/K}$. Regarding welfare, we have, by (15) and (16), that $W(n,1) = b - c - \frac{b-a}{n+1} - Kn$, which is maximized at $n^o = \sqrt{(b-a)/K} + 1$.

[19] Thus, Grossman and Shapiro (1984) refer to second-order conditions in a neighborhood of a candidate symmetric

general, feature of oligopoly models of informative advertising and we, thus, attempt to analyze it in the present paper. Before we turn to the formal analysis, let us make two observations. First, that the equilibrium conditions of the underlying models in the absence of uncertainty (and advertising) exhibit similarities. In particular, the equilibrium price in the Salop (1979) "circle" model is $p = c + \frac{t}{n}$, where $t$ is the per-unit "transportation" cost, while in the "random utility" model of Perloff-Salop (1985) with valuations *uniformly* distributed on $[a, b]$ it is easy to show that the equilibrium price is $p = c + \frac{b-a}{n}$.[20] As the former model is the basis for the Grossman-Shapiro (1984) model and the latter for our analysis in the paper, it is possible that, when inserting informative advertising, the models will react in similar ways. Second, note that, on the basis of our analysis, we can conclude that the possible non-existence of a symmetric equilibrium we encounter in our model is a robust feature of informative advertising models of product differentiation, whether the underlying mode of competition is localized or not.[21] The intuition remains that, for certain parameter values, a given firm would prefer to charge a high price and sell to the consumers that have been informed about its product rather than charging the lower (candidate equilibrium) price that would allow the firm to also capture some of the consumers that have been informed about rival products as well. An important observation is that, in the context of a localized competition model (such as the circle model), incorporating uncertainty and informative advertising tends to make competition with firms located at a greater distance more intense. Under certainty, the price equilibrium conditions are determined relative to the closest two rivals and a deviation to a higher price never pays, as consumers located far away would never purchase. On the contrary, when there is a chance consumers are not informed about any rival products, charging a high price may be the optimal strategy, as some consumers may purchase even if they are located "far away".[22]

We now proceed to show that the results by Grossman and Shapiro (1984), regarding a symmetric equilibrium are valid but only for certain regions of the parameters involved. To do this, the key is to determine the complete form of the corresponding demand function of firm $i$, $D_i(p_i, p, \phi, \phi)$, that is, defined over the entire range of prices. We will then be able to observe that the demand function is discontinuous, rendering the profit function discontinuous and non-quasi-concave.

Let us derive the demand function. Following Grossman and Shapiro (1984, pp. 67-70) we proceed by partitioning the consumer population into $n$ groups, where the $kth$ group is the set of consumers to whom the representative firm offers the $kth$ highest surplus among the $n$ firms, assuming all consumers are informed. The demand function is determined by the sizes of these $n$ groups. Suppose, without

equilibrium, which is only part of the story for some parameter values. Tirole (1984, section 7.3.2.2), presents a reformulation of their model based on Hotelling's (1929) linear city and focuses explicitly on the "competitive" case in which firms compete for a common demand (this would roughly correspond, in our model, to the case where firms prefer to compete via a relatively low price for more consumers rather than find a high price deviation more profitable).

[20] Thus, $(b - a)$ plays a role similar to that of $t$, both representing product differentiation. While the equilibrium prices appear be similar, note that the demand and profit functions in the models are different.

[21] Deneckere and Rothschild (1992) study the relation between models of localized and non-localized competition.

[22] Indeed, along these lines, Grossman and Shapiro (1984, p. 67) note that under imperfect information "competition is no longer localized".

loss of generality, that the representative firm $i$ is located at point 0 and firm $i+k$ is located at point $k/n$. Further, suppose that all firms but $i$ set the same price, $p$. We denote by $N_k$ the size of the $kth$ group of consumers and by $z_k$ the consumer that is indifferent between buying from firm $i$ and firm from $i+k$, where $z_k = k/2n + (p - p_i)/2t$. For $p_i$ close enough to $p$ we have $N_1 = 2z_1$, $N_2 = 2(z_2 - z_1) = \frac{1}{n}$, $N_3 = 2(z_3 - z_2) = \frac{1}{n}$, ..., $N_k = 2(z_k - z_{k-1}) = \frac{1}{n}$, ..., $N_n = 1 - 2z_{n-1}$. In order for the consumers of group $N_k$ to purchase from firm $i$, it must be that the consumers of that group receive the ad of firm $i$, but they do not receive the ad of any of the firms that give them surplus greater than the $kth$ highest surplus, that firm $i$ gives to them. This occurs with probability $\phi(1-\phi)^{k-1}$. It follows that

$$D(p_i, p, \phi) = \sum_{k=1}^{n} \phi(1-\phi)^{k-1} N_k. \tag{17}$$

However, as noted above, function (17) represents the demand of firm $i$ only for $p_i$ close enough to $p$. The demand function should be modified when written over prices not close enough to $p$. In particular, consider $p_i$ greater than $p$. The price $p_i$ that makes the consumer located at point 0 indifferent between buying from firm $i$ or from firm $i+1$, is $p_i = p + \frac{t}{n}$.[23] For $p_i > p + \frac{t}{n}$, there are less than $n$ groups of consumers who may demand the product of firm $i$. Specifically, for $p_i \in [p + \frac{t}{n}, p + \frac{2t}{n}]$, there are $n-2$ such groups.[24] This is because prices are such that the product of firm $i$ is dominated by the products of firm $i+1$ and firm $i-1$, that is, all consumers prefer either product $i+1$ or $i-1$ to $i$. It follows that the ranking for the product of firm $i$ starts from position 3. The relevant groups are now $N_1 = 2z_2$, $N_2 = 2(z_3 - z_2) = \frac{1}{n}$, ..., $N_k = 2(z_{k+1} - z_k) = \frac{1}{n}$, $N_{n-2} = 1 - 2z_{n-2}$, and the demand is

$$D(p_i, p, \phi) = \sum_{k=1}^{n-2} \phi(1-\phi)^{k+1} N_k.$$

By an analogous reasoning, for $p_i \in [p + \frac{2t}{n}\ ,\ p + \frac{3t}{n}]$, firm $i$ offers a product that is dominated by these of firms $i \pm 1$ and $i \pm 2$. The $n-4$ groups are now $N_1 = 2z_3$, $N_2 = 2(z_4 - z_3) = \frac{1}{n}$, ..., $N_k = 2(z_{k+2} - z_{k+1}) = \frac{1}{n}$, ..., $N_{n-4} = 1 - 2z_{n-3}$, and the demand is

$$D(p_i, p, \phi) = \sum_{k=1}^{n-4} \phi(1-\phi)^{k+3} N_k.$$

By applying this procedure, we find that, for $p_i \in [p + \frac{(r-1)t}{n}\ ,\ p + \frac{rt}{n}]$ the respective groups are $N_1 = 2z_r$, $N_2 = 2(z_{r+1} - z_r) = \frac{1}{n}$, ..., $N_k = 2(z_{k+(r-1)} - z_{k+(r-2)}) = \frac{1}{n}$, ..., $N_{n-2(r-1)} = 1 - 2z_{n-r}$, and the demand is

$$D(p_i, p, \phi) = \sum_{k=1}^{n-2(r-1)} \phi(1-\phi)^{k+2(r-1)-1} N_k.$$

As $p_i$ increases, we finally reach a point where product $i$ is dominated by all other products. The critical price is $p + \frac{t}{2}\left(\frac{n-1}{n}\right)$ if the number of firms, $n$, is odd and $p + \frac{t}{2}$ if $n$ is even. Thus, if $n$ is odd

[23] This price is the solution to $u - p_i = u - t(\frac{1}{n}) - p$.

[24] The price $p_i = p + \frac{2t}{n}$ makes the consumer located at point 0 indifferent between buying from firm $i$ or firm $i+2$, that is, solves $u - p_i = u - t(\frac{2}{n}) - p$.

we have $D(p_i, p, \phi) = \phi(1-\phi)^{n-1}$ for $p_i \in [p - \frac{t}{2}\left(\frac{n-1}{n}\right), R - t/2]$ and if the number of firms is even we have $D(p_i, p, \phi) = \phi(1-\phi)^{n-1}$ for $p_i \in [p - \frac{t}{2}, R - t/2]$. We use the value $R - t/2$ as the upper bound of the previous interval because the valuations for firm $i's$ product are uniformly distributed in the interval $[R - t/2, R]$. For $p_i \in [R - t/2, R]$ demand is given by $\phi(1-\phi)^{n-1}\left(\frac{R-p_i}{t/2}\right)$ and, for $p_i > R$, demand is zero.

It remains to consider the case of prices $p_i$ lower than $p$. Once again, though now for different reasons, a continuous decrease in $p_i$ gradually reduces the number of groups of consumers to which firm $i$ may sell. Specifically, for $p_i \in [p - \frac{2t}{n}\ , p - \frac{t}{n}]$, the product of firm $i$ dominates the products of firms $i+1$ and $i-1$, and, thus, $N_1 = 2z_2$, $N_2 = 2(z_3 - z_2) = \frac{1}{n}$, ..., $N_k = 2(z_{k+1} - z_k) = \frac{1}{n}$, ..., $N_{n-2} = 1 - 2z_{n-2}$. The demand for the product of firm $i$ is

$$D(p_i, p, \phi) = \sum_{k=1}^{n-2} \phi(1-\phi)^{k-1} N_k.$$

By the same reasoning, when $p_i \in [p - \frac{rt}{n}\ , p - \frac{(r-1)t}{n}]$, the product of firm $i$ dominates the products of firms $i \pm 1, ..., i \pm (r-1)$. The groups are: $N_1 = 2z_r$, $N_2 = 2(z_{r+1} - z_r) = \frac{1}{n}$, ..., $N_k = 2(z_{k+(r-1)} - z_{k+(r-2)}) = \frac{1}{n}$, ..., $N_{n-2(r-1)} = 1 - 2z_{n-r}$, and the demand is

$$D(p_i, p, \phi) = \sum_{k=1}^{n-2(r-1)} \phi(1-\phi)^{k-1} N_k.$$

Finally, we reach a point where $p_i$ is so small that all consumers rank the product of firm $i$ as the most preferred. This threshold is the price $p - \frac{t}{2}\left(\frac{n-1}{n}\right)$ if $n$ is odd and $p - \frac{t}{2}$ if $n$ is even. Thus, if the number of firms is odd, we have $D(p_i, p, \phi) = \phi$ for $p_i < p - \frac{t}{2}\left(\frac{n-1}{n}\right)$ and, if the number of firms is even, we have $D(p_i, p, \phi) = \phi$ for $p_i < p - \frac{t}{2}$.

We observe that the profit function of the Grossman and Shapiro (1984) model is not quasi-concave in price. Note, for example, that at $p_i = p - t/n$, the demand of firm $i$ exhibits a discontinuous jump (reminiscent of the case of perfect information with transportation cost linear in distance). This lack of quasi-concavity may result to non-existence of equilibrium, as shown in Figure 4. In Case I, we use the parameter values $R = 2.5$, $n = 6$, $t = 1$, $c = 0$ and the advertising cost function $A(\phi_i) = er\ln(1-\phi_i)/\ln(1-r)$, with $e = 0.5$ and $r = 0.1$. We observe, that the candidate equilibrium pair $(p, \phi) \approx (0.9445, 0.194)$ is not a symmetric equilibrium profile: with $\phi_i = \phi$ there is, obviously, a profitable price deviation to $p_i = 2$. On the other hand, in Case II the candidate equilibrium pair $(p, \phi) \approx (96.55, 0.67)$ is indeed a symmetric equilibrium profile. The set of parameters we use in Case II is $R = 250$, $n = 8$, $t = 250$, $c = 50$; $e = 3$, $r = 0.1$, with the same advertising cost function as above (this last set of parameters is borrowed from Grossman and Shapiro, 1984).

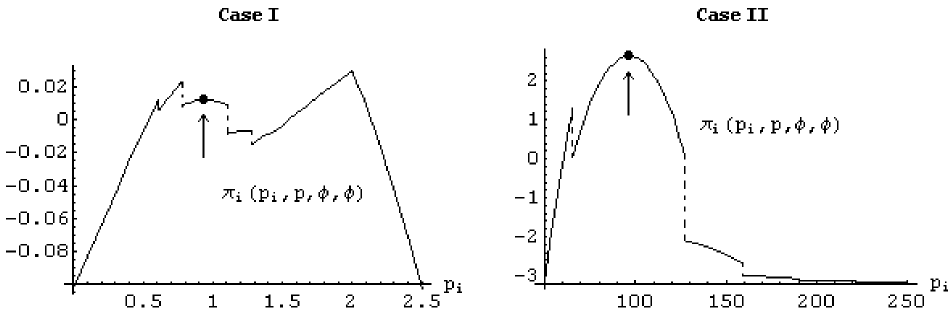


Figure 4.

As a final note, the candidate symmetric equilibrium pair $(p, \phi)$ in Grossman and Shapiro (1984) is characterized by the two first-order conditions:

$$p = \frac{t}{n\phi} \frac{1-(1-\phi)^n}{1-(1-\phi)^{n-1}} + c \tag{18}$$

and

$$\frac{1-(1-\phi)^n}{n\phi}(p-c) = A'(\phi). \tag{19}$$

There is a striking similarity between equations (18) and (19) and the corresponding equations (9) and (10) in our analysis. The only difference is that the per-unit transportation cost $t$ now plays the role of the distance $(b-a)$, that is, the maximum difference in consumers' valuations. As the profit functions $\pi_i$'s in both models are also, similar, for $p_i$ locally around $p$ (with $t$ playing the role of $(b-a)$), it follows that as $t \to 0$ the symmetric equilibrium fails to exist, exactly as we have shown for $(b-a) \to 0$ earlier in the paper.

# 7 Extensions

## 7.1 Symmetric equilibria where not all informed consumers buy.

The main focus of the paper is on the existence of symmetric equilibria with $p < a$, that is, equilibria where all consumers purchase one product provided they have received at least one ad. This practice is consistent with the product differentiation literature, where typically consumer preferences are such that, in equilibrium, they always buy one unit from one of the firms. However, our model also allows us to derive conditions for the complementary case, that of a symmetric equilibrium with $p \in (a, b)$.[25] In such a case, not all informed consumers would purchase: specifically, a consumer $k$ that happens to have valuations $v_i^k$ lower than $p$ for $i = 1, ..., n$, would not purchase any of the products.[26]

[25] Such equilibria could, in principle, be derived in other related models (such as in Grossman and Shapiro, 1984, although they focus exclusively on equilibria where all informed consumers purchase and extending their analysis to other equilibria is not trivial).

[26] For a complete analysis, one should also examine equilibria with $p = a$. In such equilibria all informed consumers purchase but the analysis needs to be modified relative to the $p < a$ case, since now the relevant profit function is not differentiable with respect to price at the candidate equilibrium point. Details are available by the authors upon request.

In the case of perfect information about products and prices (this would correspond to $\phi = 1$ in our model) it can be shown that a symmetric equilibrium with $p \in (a, b)$ may exist. However, for a given set of parameter values, only one symmetric equilibrium may exist, that is, either with $p > a$ or $p < a$ but not both.[27] We cannot formally prove such a result in our model due to its complexity (note that here we have two strategic variables). Still, we can verify for every set of parameters whether the candidate equilibrium involving $p \in (a, b)$ can coexist with the one that involves $p < a$. We have examined the matter numerically and our results indicate that it is not possible to have both equilibria for the same parameter values (this approach, of course, does not constitute a proof). An example is presented after the derivation of the optimality conditions.

Let us now proceed to the formal derivation of the conditions for an equilibrium with $p \in (a, b)$. The profit function of firm $i$ given that all other firms set price $p$ and advertising level $\phi$, is again given by (1) and the demand function by (2). The probability $\Pr(p_i, p, k)$ that a consumer that sees the ads of firm $i$ and of $k$ other firms when firm $i$ charges price $p_i$ and all other firms charge price $p$, is now given by (for a general probability distribution function, $F$):

$$\Pr(p_i, p, k) = \begin{cases} \int_{p_i}^{b} F^k\left(v_i - p_i + p\right) f(v_i) dv_i, & if \quad p_i \geq p \\ \int_{\max\{a, p_i\}}^{b+p_i-p} F^k\left(v_i - p_i + p\right) f(v_i) dv_i + \int_{b+p_i-p}^{b} f(v_i) dv_i, & if \quad p_i \leq p. \end{cases}$$

If the probability distribution function is uniform, we get for $k \geq 1$:

$$\Pr(p_i, p, k) = \begin{cases} 1 & if \quad p_i < p - b + a \\ \frac{(b-a)+(p-p_i)\left(k+1-\left(\frac{p-p_i}{b-a}\right)^k\right)}{(b-a)(k+1)} & if \quad p_i \in [p - b + a, a] \\ \frac{(b-a)+(p-p_i)(k+1)-\left(\frac{p-a}{b-a}\right)^k (p-a)}{(b-a)(k+1)} & if \quad p_i \in [a, p] \\ \frac{(b-a-p_i+p)^{k+1}-(p-a)^{k+1}}{(k+1)(b-a)^{k+1}} & if \quad p_i \in [p, b] \\ 0 & if \quad p_i > b, \end{cases}$$

and for $k = 0$:

$$\Pr(p_i, p, 0) = \begin{cases} 1 & if \quad p_i \leq a \\ \frac{b-p_i}{b-a} & if \quad p_i \in [a, b] \\ 0 & if \quad p_i \geq b. \end{cases} \tag{20}$$

The derivative of the profit function with respect to $p_i$ is

$$\frac{\partial \pi_i(p_i, p, \phi, \phi)}{\partial p_i} = D_i(p_i, p, \phi, \phi) - (p_i - c)\frac{\partial D_i(p_i, p, \phi, \phi)}{\partial p_i}, \tag{21}$$

Note that, for all $k$,

$$\frac{\partial \Pr(p, p, k)}{\partial p_i} = -\frac{1}{b - a}.$$

[27] Details are available by the authors upon request. See also Konishi (1999) who presents related results in the context of a search model.

By substituting the symmetry condition $p_i = p$ we obtain

$$\tfrac{\partial D_i(p,p,\phi,\phi)}{\partial p_i} = \phi\tfrac{-1}{b-a}\textstyle\sum_{k=0}^{n-1}\tfrac{(n-1)!}{k!(n-1-k)!}(1-\phi)^{n-1-k}\phi^k = \phi\tfrac{-1}{b-a}. \tag{22}$$

Given $\phi$, the equilibrium quantity sold by each firm is equal to

$$D_i(p,p,\phi,\phi) = \phi\sum_{k=0}^{n-1}\frac{(n-1)!}{k!(n-1-k)!}(1-\phi)^{n-1-k}\phi^k \Pr(p,p,k) = \frac{1-\left(1-\phi\frac{b-p}{b-a}\right)^n}{n}. \tag{23}$$

Substituting expressions (22) and (23) in (21), we obtain

$$\frac{\partial \pi_i(p,p,\phi,\phi)}{\partial p_i} = \frac{1-\left(1-\phi\frac{b-p}{b-a}\right)^n}{n} - (p-c)\frac{\phi}{b-a},$$

and, by setting $\partial \pi_i(p,p,\phi,\phi)/\partial p_i$ equal to zero, we find that the candidate equilibrium price satisfies

$$p = \frac{(b-a)}{n\phi}\left(1-\left(1-\phi\frac{b-p}{b-a}\right)^n\right) + c. \tag{24}$$

We now turn to the calculation of the equilibrium advertisement level. By setting $p_i = p$ in (2), we obtain $D_i(p,p,\phi_i,\phi) = \phi_i\left(1-\left(1-\phi\frac{b-p}{b-a}\right)^n\right)/(n\phi)$, and from (1) we have

$$\frac{\partial \pi_i(p,p,\phi_i,\phi)}{\partial \phi_i} = (p-c)\frac{\left(1-\left(1-\phi\frac{b-p}{b-a}\right)^n\right)}{n\phi} - A'(\phi_i),$$

which implies that, in equilibrium,

$$(p-c)\frac{\left(1-\left(1-\phi\frac{b-p}{b-a}\right)^n\right)}{n\phi} = A'(\phi). \tag{25}$$

We conclude that a symmetric equilibrium with $p \in (a,b)$ should solve the system of the non-linear first-order conditions (24) and (25).

A note about the second-order conditions is in order here. With respect to the price variable, we note that the profit function is still not quasi-concave. Again, the profit function, as described by (1) and (2), is a weighted sum of the elements of the set $\{(p_i - c)\Pr(p_i,p,k),\ k = 0,...,n-1\}$, each of which is a quasi-concave function of $p_i$; but the sum of quasi-concave functions need not be quasi-concave. As far as the second-order conditions with respect to $\phi$ are concerned, we note that $\pi_i(p,p,\phi_i,\phi)$ is strictly concave in $\phi_i$. Given that the second-order conditions do not always guarantee that an equilibrium obtains where the first-order conditions hold, we have to resort to a numerical solution for each case.

Of course, it is not *a priori* guaranteed that a price that satisfies (24) and (25) also satisfies $p \in (a,b)$. For parameters for which this last condition does not hold, clearly, symmetric equilibrium with $p \in (a,b)$ does not exist. Despite the complications, due to the fact that the first-order conditions do not yield a closed form solution in $(p,\phi)$ and also that the second-order conditions do not guarantee existence, the information presented here is useful. Employing (24) and (25), we can now calculate for a given parametric model the possible symmetric equilibria. To illustrate, consider Table 3: For a given set of parameters we have calculated the candidate equilibrium prices with $p < a$ and $p \in (a, b)$. We have then solved the problem (numerically) and marked with an asterisk the prices that indeed correspond to an equilibrium. Note that for low values of $a$ there is an equilibrium with $p \in (a, b)$. For intermediate values of $a$ there is an equilibrium with $p < a$. For high values of $a$ there is no symmetric equilibrium because the candidate price is so low that firms prefer a high price deviation.

| **Numerical examples** | | |
|---|---|---|
| $n = 5, b = 200, c = 0, r = 0.1, e = 3$ | | |
| $p < a$ | $a$ | $p \in (a, b)$ |
| 47.0 | 20 | 46.8* |
| 42.8 | 40 | 42.6* |
| 38.5* | 60 | 38.4 |
| 34.2* | 80 | 34.0 |
| 29.8* | 100 | 29.6 |
| 25.3* | 120 | 25.2 |
| 20.7* | 140 | 55.5 |
| 15.8 | 160 | 93.9 |

Table 3: Candidate equilibrium prices.

### 7.2 Sequential (advertising-then-price) choices

Our analysis, thus far, deals with the case where firms choose prices and advertising levels at the same time. Here we make some steps towards understanding the case where firms select their ads before setting prices (hence, we refer to this game as the sequential-move game, as opposed to the simultaneous-move game we have presented). Due to the complexity of the problem, we study the case of a duopoly. By substituting $n = 2$ in equations (1), (2), and (3), we obtain the profit function of firm $i$ as follows:

$$\pi_i(p_i, p_j, \phi_i, \phi_j) = p_i \phi_i \left[(1 - \phi_j) + \phi_j \Pr(p_i, p_j, 1)\right] - A(\phi_i).$$

To solve the sequential-move game, the normal way to proceed would be to first derive the equilibrium prices $p_1^*$ and $p_2^*$ as functions of the ad intensities $(\phi_1, \phi_2)$, and then to move these functions up to the first stage and derive the subgame-perfect equilibrium. Following this approach, the first-order condition of the profit function of firm 1 with respect to $\phi_1$ at an interior equilibrium would be

$$\begin{aligned} \frac{d\pi_1(p_1^*(\phi_1,\phi_2), p_2^*(\phi_1,\phi_2), \phi_1, \phi_2)}{d\phi_1} &= \frac{\partial \pi_1}{\partial p_1}\frac{dp_1^*}{d\phi_1} + \frac{\partial \pi_1}{\partial p_2}\frac{dp_2^*}{d\phi_1} + \frac{\partial \pi_1}{\partial \phi_1} \\ &= \frac{\partial \pi_1}{\partial p_2}\frac{dp_2^*}{d\phi_1} + \frac{\partial \pi_1}{\partial \phi_1} = 0, \end{aligned} \tag{26}$$

since $\frac{\partial \pi_1}{\partial p_1} = 0$ by the "envelope condition". However, in our case we cannot obtain closed form solutions for $p_1^*(\phi_1, \phi_2)$ and $p_2^*(\phi_1, \phi_2)$. Nonetheless, we can partly avoid this problem by focusing on symmetric equilibria, that is, equilibria where $\phi_1 = \phi_2 = \phi$ and $p_1^* = p_2^*$ and study their properties, assuming they exist.[28] This will allow us to make a straightforward comparison between the corresponding symmetric equilibria of the two games.

Note first that an interior equilibrium of the simultaneous-move game involves $\frac{\partial \pi_1}{\partial \phi_1} = 0$. In the sequential-move game whether $\frac{\partial \pi_1}{\partial \phi_1}$ is positive or negative, in equilibrium, depends on the sign of the term capturing the strategic effect, $\frac{\partial \pi_1}{\partial p_2}\frac{dp_2^*}{d\phi_1}$. Two questions arise: what is the sign of the strategic effect of the sequential-move game, and whether knowing this sign allows us to compare the two equilibria. The answer to the first question is that, in general, $\frac{\partial \pi_1}{\partial p_2}\frac{dp_2^*}{d\phi_1} < 0$.[29] The answer to the second is that we can compare the two equilibria provided there is a one-to-one relation between $\frac{\partial \pi_1}{\partial \phi_1}$ and $\phi$ (with $\frac{\partial \pi_1}{\partial \phi_1}$ computed at the symmetric equilibrium). Specifically, the first-order condition with respect to $p_i$ is identical in both the simultaneous-move and the sequential-move game and, hence, the price that would prevail at a symmetric equilibrium is given as a function of $\phi$ by (9) in either game. Let us denote this price by $p^*$.[30] Evaluating $\frac{\partial \pi_1}{\partial \phi_1}$ when $\phi_1 = \phi_2 = \phi$, and for the corresponding $p^*$, we obtain:

$$\begin{aligned}
\frac{\partial \pi_1}{\partial \phi_1}|_* &= \frac{\partial [p_1\phi_1[(1-\phi_2)+\phi_2 \Pr(p_1,p_2,1)] - A(\phi_1)]}{\partial \phi_1}|_* \\
&= \frac{\partial [p^*\phi_1[(1-\phi_2)+\phi_2/2)] - A(\phi_1)]}{\partial \phi_1}|_* \\
&= p^*\left[(1-\phi)+\frac{\phi}{2})\right] - A'(\phi) \\
&= p^*(1-\phi/2) - A'(\phi).
\end{aligned}$$

Note that the last expression is decreasing in $\phi$ since $p^*$ is decreasing in $\phi$ (by Appendix 2) and $A'' > 0$. Thus, there is a one-to-one relation between $\frac{\partial \pi_1}{\partial \phi_1}|_*$ and $\phi$.

For the simultaneous-move game it must be that $\frac{\partial \pi_1}{\partial \phi_1}|_* = 0$, whereas for the sequential-move game, in general, $\frac{\partial \pi_1}{\partial \phi_1}|_* > 0$.[31] This implies that the equilibrium advertising level of the sequential-move game must be lower compared to the corresponding of the simultaneous-move game. The intuition is as follows: because of the negative impact of advertisements on the pricing decisions – more ads increase price competition – firms have a lower incentive to advertise in the sequential-move game. In equilibrium firms advertise less compared to the case where they choose advertising levels at the same time with their pricing decisions.

# 8 Conclusion

The model analyzed in this paper represents conceptually one of the most straightforward ways to study advertising in oligopoly. Hence, despite the technical complications, due to the possible non

[28] A complete analysis of the second-order conditions in the sequential-move game is much more complex than in the simultaneous-move game and falls out of the scope of the our analysis.

[29] $\frac{\partial \pi_1}{\partial p_2} > 0$ holds because $\frac{\partial \pi_1(p_1,p_2,\phi_1,\phi_2)}{\partial p_2} = p_1\phi_1\phi_2\frac{\partial \Pr(p_1,p_2,1)}{\partial p_2} > 0$. Further, we find that, in general, $\frac{dp_i^*}{d\phi_j} < 0$ – essentially, we implicitly differentiate the system consisting of the first-order conditions $\frac{\partial \pi_1}{\partial p_1} = 0$ and $\frac{\partial \pi_2}{\partial p_2} = 0$ and compute $\frac{dp_i^*}{d\phi_j}$ at the symmetric equilibrium price. That $\frac{dp_i^*}{d\phi_j} < 0$ has a direct economic interpretation: a firm that increases its advertising level tends to attract more customers and induces the rival firm to lower its price.

[30] $p^*$ is the price that for any given $\phi$ set by the two firms, solves the first-order conditions with respect to prices. The terms $p_1^*$ and $p_2^*$ used above are the corresponding prices for any pair $(\phi_1, \phi_2)$. Of course, if $\phi_1 = \phi_2$, $p_1^* = p_2^* = p^*$.

[31] Because $\frac{\partial \pi_1}{\partial p_2}\frac{dp_2^*}{d\phi_1} < 0$ and in the equilibrium of the sequential-move game requires that (26) holds.

quasi-concavity of profits which in turn implies that a symmetric equilibrium does not always exist, it is worth studying its equilibrium properties. Firms compete strategically in both prices and informative advertising and, in equilibrium, a higher level of advertising tends to lower prices (as consumers gain information about more products and demand elasticity increases). Higher cost of advertising leads to higher prices but may either increase or decrease firms' profits. A higher number of firms (or products, equivalently) may lead to either a higher or to a lower price. Compared to the social optimum, the market tends to overprovide advertising, when the degree of possible product differentiation is large, and to underprovide it otherwise.

That a firm may have an incentive to deviate to a higher price than the one prescribed in the candidate symmetric equilibrium, rather than being an anomalous case, appears a robust characteristic of informative advertising models of product differentiation and it appears worth drawing attention to. The underlying intuition is that, for some parameter values, the equilibrium advertising level makes it likely that a large enough share of consumers will be informed about only one product. Thus, a given firm may wish to exploit its monopoly power over these captive consumers and increase its profit by charging a high price. It follows that, when establishing a symmetric price and advertising equilibrium, attention should be given not only to possible local deviations but also to discrete jumps in prices. We point out that this feature is present not only in our analysis of non-localized competition but also, to the same extent, in models based on localized competition. In this sense, our analysis offers some insights complementary to those of the earlier studies.

A few aspects of advertising in oligopoly may fit in the general framework proposed here, even though they have to be left out of the present analysis. Such topics for future work include cases when consumers are informed about the existence of products while advertising informs them only about prices, cases of vertical (quality) product differentiation, and cases where firms could choose their advertising content.

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# Appendix

### Appendix A1: Derivation of the probabilities $\Pr(p_i, p, k)$

We construct functions (3) and (4) representing the probabilities that consumers purchase from one of the firms and employed in the demand function (2), with the aid of a simple diagram (see figure A1).

Let us consider a pair of products (or, equivalently, firms). We measure on the horizontal axis the value that the typical consumer attaches to the product of firm $i$. On the vertical axis we measure the utility that the typical consumer attaches to the product of firm $j \neq i$.

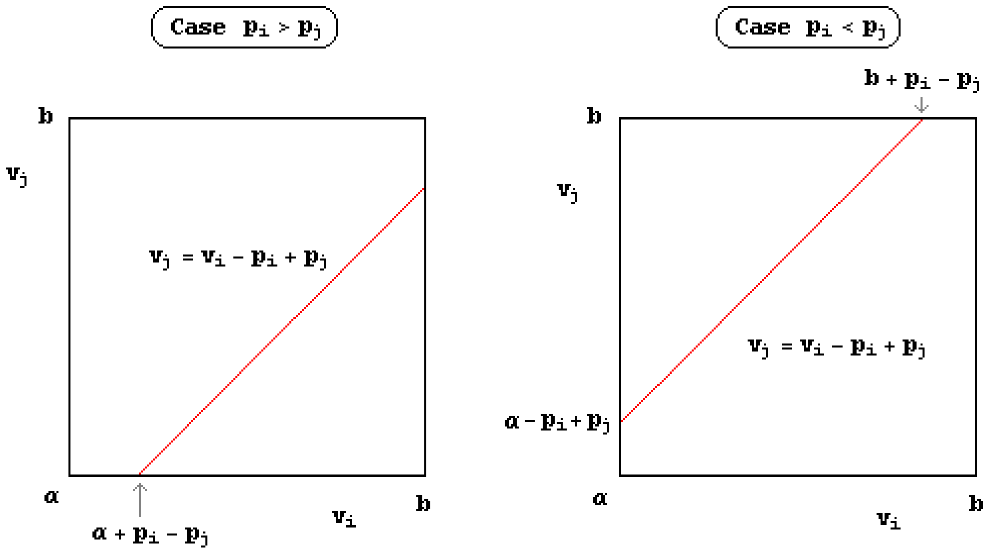


Figure A1.

Regarding the choice between any two products, the typical consumer is characterized by a pair $(v_i, v_j)$, a point in the square. To determine his purchasing decision he also has to consider the prices. The consumer prefers to purchase product $i$ to product $j$ if and only if his values for the two products satisfy $v_j - p_j < v_i - p_i$. This inequality holds in the area of the square that lies below the straight line $v_j = v_i - p_i + p_j$ that represents indifference between the two products. The consumer buys from firm $i$ only if it attaches a pair of values that lies below the line $v_j = v_i - p_i + p_j$, where $v_j - p_j < v_i - p_i$ holds.

Consider, first, the case $p_i > p_j$. The probability that the pair $(v_i, v_j)$ is in the area where $v_j - p_j < v_i - p_i$ is equal to $\int_{a+p_i-p_j}^{b} F(v_i - p_i + p_j) f(v_i) dv_i$, where $F(v_i - p_i + p_j) = \Pr\{v_j < v_i - p_i + p_j$, given $v_i\}$. Note that for $v_i < a + p_i - p_j$ the consumer buys from firm $j$ for every realization of $v_j$ and this threshold defines the lower bound of the integral.

Of course, each product may be competing with more than one other products. To extend the argument to the case where firm $i$ competes with $k$ rival firms, note that since the values $v_i$ are i.i.d., the probability $\Pr\{v_1 < v_i - p_i + p_1, ..., v_k < v_i - p_i + p_k$, given $v_i\}$ equals $(F(v_i - p_i + p))^k =$

$F^k(v_i - p_i + p)$ for $p_j = p$, $j = 1, ..., k$. It follows that the probability that the vector $(v_i, v_1, ..., v_k)$ is in the area where $v_j - p_j < v_i - p_i$, for all $j = 1, ..., k$, is given by

$$\int_{a+p_i-p_j}^{b} F^k(v_i - p_i + p) f(v_i) dv_i.$$

We follow a similar approach to construct the probabilities when $p_i < p_j$. When comparing products $i$ and $j$, note that for $v_i > b + p_i - p_j$, the consumer always prefers product $i$ irrespective of the value $v_i$. Specifically, for $v_i > b + p_i - p_j$ we have $F(v_i - p_i + p_j) = 1$. With $k$ rivals, the probability that the vector $(v_i, v_1, ..., v_k)$ is in the area where $v_j - p_j < v_i - p_i$, for all $j = 1, ..., k$ and $p_j = p$, $j = 1, ..., k$, is given by

$$\int_{a}^{b+p_i-p} F^k(v_i - p_i + p) f(v_i) dv_i + \int_{b+p_i-p}^{b} f(v_i) dv_i.$$

We conclude that when the typical consumer compares the price-value combinations of $k + 1$ products (firm $i$ with price $p_i$ and the other $k$ products with price $p$) he purchases from firm $i$ with probability

$$\Pr(p_i, p, k) = \begin{cases} 1 & if \quad p_i < p - b + a \\ \int_a^{b+p_i-p} F^k(v - p_i + p) f(v) dv + \int_{b+p_i-p}^{b} f(v) dv & if \quad p_i \in [p - b + a, p] \\ \int_{a+p_i-p}^{b} F^k(v - p_i + p) f(v) dv & if \quad p_i \in [p, p + b - a] \\ 0 & if \quad p_i > p + b - a. \end{cases} \tag{A.1}$$

If the consumer has received ads from $k = 0$ other firms, the firms $i$ has no rivals: the consumer purchases product $i$ if and only if his value $v_i$ exceeds the price $p_i$. Thus, we have

$$\Pr(p_i, p, 0) = \begin{cases} 1 & if \quad p_i \leq a \\ 1 - F(p_i) > 0 & if \quad p_i \in [a, b] \\ 0 & if \quad p_i \geq b. \end{cases} \tag{A.2}$$

When $F$ is uniform on $[a, b]$ expressions (A.1) and (A.2) can be rewritten as (3) and (4), respectively.

**Appendix A2: Derivatives of the symmetric equilibrium condition**

We wish to show that the left-hand side of (11) is decreasing in $\phi$. Since

$$\frac{(b-a)}{n^2\phi^2} \frac{(1-(1-\phi)^n)^2}{1-(1-\phi)^{n-1}} = \left(\frac{1-(1-\phi)^n}{n\phi}\right)\left(\frac{(b-a)}{n\phi}\frac{1-(1-\phi)^n}{1-(1-\phi)^{n-1}}\right),$$

it suffices to show that the two components of the product in the corresponding parentheses, are decreasing in $\phi$. The following results ensure these properties. At the same time, we study below the monotonicity of the above expressions with respect to $n$, and report results used in other parts of the paper.

**Result 1** For $\phi \in (0,1)$,

$$\frac{\partial}{\partial \phi}\left(\frac{1-(1-\phi)^n}{n\phi}\right) < 0 \text{ and } \frac{\partial}{\partial n}\left(\frac{1-(1-\phi)^n}{n\phi}\right) < 0, \tag{A.3}$$

**Proof.** By direct calculations we get

$$\frac{\partial}{\partial n}\left(\frac{1-(1-\phi)^n}{n\phi}\right) = \frac{(1-\phi)^n\left(1-\ln(1-\phi)^n\right)-1}{n^2\phi},$$

and

$$\frac{\partial}{\partial \phi}\left(\frac{1-(1-\phi)^n}{n\phi}\right) = \frac{(1-\phi)^{n-1}(1-\phi+n\phi)-1}{\phi^2 n}.$$

The term $(1-\phi)^{n-1}(1-\phi+n\phi)-1$ is negative because it is equal to zero for $\phi = 0$, and $\partial\left[(1-\phi)^{n-1}(1-\phi+n\phi)-1\right]/\partial\phi = -(1-\phi)^{n-2}n(n-1)\phi < 0$ for $\phi \in (0,1)$.

The term $(1-\phi)^n\left(1-\ln(1-\phi)^n\right)-1$ is negative because it is equal to zero for $\phi = 0$, and $\partial\left[(1-\phi)^n\left(1-\ln(1-\phi)^n\right)-1\right]/\partial\phi = (1-\phi)^{n-1}n^2\ln(1-\phi) < 0$ for $\phi \in (0,1)$. ■

**Result 2** For $\phi \in (0,1)$,

$$\frac{\partial}{\partial \phi}\left(\frac{1-(1-\phi)^n}{1-(1-\phi)^{n-1}}\right) < 0 \text{ and } \frac{\partial}{\partial n}\left(\frac{1-(1-\phi)^n}{1-(1-\phi)^{n-1}}\right) < 0,$$

**Proof.** Calculating the derivative, we get

$$\frac{\partial}{\partial \phi}\left(\frac{1-(1-\phi)^n}{1-(1-\phi)^{n-1}}\right) = \frac{(1-\phi)^{n-2}}{(1-(1-\phi)^{n-1})^2}(1-n\phi-(1-\phi)^n), \tag{A.4}$$

which is negative when $\phi \in (0,1)$ because $(1-n\phi-(1-\phi)^n) < 0$ when $\phi \in (0,1]$. The last inequality is true because $\partial(1-n\phi-(1-\phi)^n)/\partial\phi = n\left((1-\phi)^{n-1}-1\right) < 0$ when $\phi \in (0,1)$.

The derivative with respect to $n$ is given by

$$\frac{\partial}{\partial n}\left(\frac{1-(1-\phi)^n}{1-(1-\phi)^{n-1}}\right) = \frac{(1-\phi)^{n-1}\phi\ln(1-\phi)}{(1-(1-\phi)^{n-1})^2}, \tag{A.5}$$

which is negative when $\phi \in (0,1)$. ■

### Appendix A3: Non-existence of equilibrium for $a \to b$

**Proof of Proposition 3.** Suppose that as $a \to b$, we have $\phi \to m$ where $m$ is strictly positive. Then, equation (9) implies that $p \to c$, leaving each firm with strictly negative profit; thus, each firm would have a unilateral incentive to deviate by setting a greater price or by choosing $\phi = 0$. Thus, let us now suppose that $\phi \to 0$.

We calculate the limit of the profit function as $a \to b$ and $\phi \to 0$. We define

$$\pi^*(a) \equiv \tfrac{1-(1-\phi(a))^n}{n}(p(a)-c) - A(\phi(a)),$$

where

$$p(a) - c = \frac{(b-a)}{n\phi(a)}\frac{1-(1-\phi(a))^n}{1-(1-\phi(a))^{n-1}},$$

by (9).

For $\lim_{a\to b}\pi^*(a)$ to be zero it suffices that $\lim_{a\to b}(p(a)-c)$ is finite because we assume here $\phi\to 0$ and $A$ is by assumption continuous with $A(0)=0$. In the following lines, we show that this is the case. We have

$$\lim_{a\to b}(p(a)-c)=\lim_{a\to b}(b-a)\lim_{a\to b}\frac{1}{n\phi(a)}\frac{1-(1-\phi(a))^n}{1-(1-\phi(a))^{n-1}}$$

Assuming $\lim_{a\to b}\phi(a)=0$, we have

$$\lim_{a\to b}\frac{1}{n\phi(a)}\frac{1-(1-\phi(a))^n}{1-(1-\phi(a))^{n-1}}=\lim_{\phi\to 0}\frac{1}{n\phi}\frac{1-(1-\phi)^n}{1-(1-\phi)^{n-1}}=+\infty,$$

and, thus, by l' Hospital rule,

$$\begin{aligned}
\lim_{a\to b}(p(a)-c) &= \lim_{a\to b}\frac{b-a}{\left[\frac{1}{n\phi(a)}\frac{1-(1-\phi(a))^n}{1-(1-\phi(a))^{n-1}}\right]^{-1}}=\lim_{a\to b}\frac{b-a}{\left[\frac{1}{n\phi(a)}\frac{1-(1-\phi(a))^n}{1-(1-\phi(a))^{n-1}}\right]^{-1}}\\
&=\lim_{a\to b}\frac{d(b-a)}{da}\lim_{a\to b}\left[\frac{d}{da}\left\{\frac{n\phi(a)\left[1-(1-\phi(a))^{n-1}\right]}{1-(1-\phi(a))^n}\right\}\right]^{-1}\\
&=-\lim_{a\to b}\left[\frac{d}{da}\left\{\frac{n\phi(a)\left[1-(1-\phi(a))^{n-1}\right]}{1-(1-\phi(a))^n}\right\}\right]^{-1}\\
&=-\lim_{a\to b}\left[\left(\frac{\partial}{\partial\phi}\left\{\frac{n\phi(a)\left[1-(1-\phi(a))^{n-1}\right]}{1-(1-\phi(a))^n}\right\}\right)\frac{d\phi(a)}{da}\right]^{-1}.
\end{aligned}$$

We calculate $d\phi(a)/da$ by the total differential of (11) using the fact that the limit of $\phi$ as $a$ goes to $b$ is zero and after some manipulations we obtain:

$$\lim_{a\to b}(p(a)-c)=(n-1)^2A'(0)<+\infty,$$

since we have assumed $A'(0)<a-c<+\infty$.

Thus, we have established that $\lim_{a\to b}\pi^*(a)=0$. It follows that, for every $p<a$, a given firm's profit as $a\to b$ becomes less than the profit from deviating to $p=a$ and to the strictly positive $\phi$ that maximizes $(a-c)\phi-A(\phi)$. This last expression is the profit function in the limit, when all firms other than $i$ send no ads, and has positive maximum value, since $A'(0)<a-c$. ■

**Appendix A4: The relation between the equilibrium price and the number of firms is not monotonic**

The equilibrium level of ads as a function of $n$ is (implicitly) defined by (11), which we rewrite as

$$\frac{1-(1-\phi)^n}{n\phi}\frac{(b-a)}{n\phi}\frac{1-(1-\phi)^n}{1-(1-\phi)^{n-1}}=A'(\phi), \tag{A.6}$$

while the equilibrium price as a function of $n$ and $\phi$, is given by (9), that is,

$$p-c=\frac{(b-a)}{n\phi}\frac{1-(1-\phi)^n}{1-(1-\phi)^{n-1}}.$$

We define

$$g(n,\phi) \equiv g = \frac{(b-a)}{n\phi}\frac{1-(1-\phi)^n}{1-(1-\phi)^{n-1}},$$

and

$$h(n,\phi) \equiv h = \frac{1-(1-\phi)^n}{n\phi},$$

so that we can re-write (A.6) as

$$h(n,\phi)g(n,\phi) = A'(\phi). \tag{A.7}$$

As for the derivative of equilibrium price, $p$, with respect to $n$, we get

$$\tfrac{dp}{dn} = \tfrac{d(p-c)}{dn} = \tfrac{dg(n,\phi(n))}{dn} = \tfrac{\partial g(n,\phi(n))}{\partial n} + \tfrac{\partial g(n,\phi(n))}{\partial \phi}\tfrac{d\phi(n)}{dn}.$$

All terms are readily available except for $d\phi(n)/dn$. We can compute that term by totally differentiating (A.6), or, equivalently, (A.7) with respect to $n$ and $\phi$. Doing this we get

$$\frac{d\phi(n)}{dn} = \frac{g_n h + g h_n}{A'' - (g_\phi h + g h_\phi)}.$$

By (A.4), (A.5) and (A.3), when $\phi \in (0,1)$ we have

$$\begin{array}{ll} \tfrac{\partial g(n,\phi)}{\partial n} \equiv g_n < 0 & \tfrac{\partial g(n,\phi)}{\partial \phi} \equiv g_\phi < 0 \\ \tfrac{\partial h(n,\phi)}{\partial n} \equiv h_n < 0 & \tfrac{\partial h(n,\phi)}{\partial \phi} \equiv h_\phi < 0 \end{array}$$

It follows that

$$\begin{aligned} \tfrac{dg(n,\phi(n))}{dn} &= \tfrac{\partial g(n,\phi(n))}{\partial n} + \tfrac{\partial g(n,\phi(n))}{\partial \phi}\tfrac{d\phi(n)}{dn} \\ &= g_n + g_\phi \tfrac{g_n h + g h_n}{A'' - (g_\phi h + g h_\phi)} \\ &= \tfrac{1}{A'' - (g_\phi h + g h_\phi)}\left((g_n A'' - g(g_n h_\phi - g_\phi h_n)\right). \end{aligned}$$

Note that $A'' - (g_\phi h + g h_\phi) > 0$, that is, the denominator of the above expression is positive. In Result 3 below, we show that

$$g_n h_\phi - g_\phi h_n < 0.$$

It follows that the sign of $dg(n,\phi(n))/n$ depends on the parameters $n$, $\phi$, and on function $A$.

**Result 3** $g_n h_\phi - g_\phi h_n \leq 0$.

**Proof.** By direct calculation we have that

$$sign\left\{g_n h_\phi - g_\phi h_n\right\} = sign\left\{((-1+(1-\phi)^n)\phi(n-1) - (1+\phi-(1-\phi)^n)n\ln(1-\phi))\right\}. \tag{A.8}$$

We define

$$\Lambda = ((-1+(1-\phi)^n)\phi(n-1) - (1+\phi-(1-\phi)^n)n\ln(1-\phi)).$$

We will show that $\Lambda/(\phi(1-\phi)^n) < 0$ for $\phi \in (0,1)$. Note that

$$\frac{\partial\left[\Lambda/(\phi(1-\phi)^n)\right]}{\partial\phi} = \frac{n}{\phi^2(1-\phi)^{n+1}}\left\{1-(1-\phi)^n-\phi n\right\}\left\{\phi+(1+\phi)\ln(1-\phi)\right\}.$$

The first expression in the curly brackets is positive and the second is negative. Given that $\lim_{\phi\to 0}\Lambda/(\phi(1-\phi)^n) = 0$, it follows that $\Lambda/(\phi(1-\phi)^n) < 0$ for $\phi \in (0,1)$. Since $(\phi(1-\phi)^n) > 0$ for $\phi \in (0,1)$ it follows that $\Lambda > 0$ for $\phi \in (0,1)$. For $\phi = 0$ or 1 we have $\lim_{\phi\to 0}\Lambda = (a-b)/((n-1)12) < 0$ and $\lim_{\phi\to 1}\Lambda = 0$. ■

**Appendix A5: The equilibrium price is increasing in the degree of product differentiation**

The relation between the equilibrium level of ads and the product differentiation parameter $(b-a)$ is given implicitly by equation (11), which we rewrite as

$$\frac{1-(1-\phi)^n}{n\phi}\frac{(b-a)}{n\phi}\frac{1-(1-\phi)^n}{1-(1-\phi)^{n-1}} = A'(\phi). \tag{A.9}$$

The equilibrium price as a function of $(b-a)$ and $\phi$, is given by (9), that is,

$$p-c = \frac{(b-a)}{n\phi}\frac{1-(1-\phi)^n}{1-(1-\phi)^{n-1}}.$$

We define

$$g(b-a,\phi) \equiv g = \frac{b-a}{n\phi}\frac{1-(1-\phi)^n}{1-(1-\phi)^{n-1}},$$

and

$$h(b-a,\phi) \equiv h = \frac{1-(1-\phi)^n}{n\phi},$$

so that we can re-write (A.9) as

$$h(b-a,\phi)g(b-a,\phi) = A'(\phi).$$

It follows that

$$\frac{dp}{d(b-a)} = \frac{d(p-c)}{d(b-a)} = \frac{dg(b-a,\phi(b-a))}{d(b-a)} = \frac{\partial g(b-a,\phi(b-a))}{\partial(b-a)} + \frac{\partial g(b-a,\phi(b-a))}{\partial\phi}\frac{d\phi(b-a)}{db-a}.$$

We calculate $d\phi(b-a))/d(b-a)$ by totally differentiating (A.7). By doing this, we obtain

$$\frac{d\phi(b-a)}{d(b-a)} = \frac{g_{b-a}h + gh_{b-a}}{A'' - (g_\phi h + gh_\phi)}.$$

By (A.4), and (A.3), when $\phi \in (0,1)$ we have

$$\begin{array}{ll} \frac{\partial g(b-a,\phi)}{\partial(b-a)} \equiv g_{b-a} = \frac{b-a}{n\phi}\frac{1-(1-\phi)^n}{1-(1-\phi)^{n-1}} > 0 & \frac{\partial g(b-a,\phi)}{\partial\phi} \equiv g_\phi < 0 \\ \frac{\partial h(b-a,\phi)}{\partial(b-a)} \equiv h_{b-a} = 0 & \frac{\partial h(b-a,\phi)}{\partial\phi} \equiv h_\phi < 0 \end{array}$$

It follows that

$$\begin{aligned} \frac{dg(b-a,\phi(b-a))}{d(b-a)} &= \frac{\partial g(b-a,\phi(b-a))}{\partial(b-a)} + \frac{\partial g(b-a,\phi(b-a))}{\partial\phi}\frac{d\phi(b-a)}{d(b-a)} \\ &= g_{b-a} + g_\phi\frac{g_{b-a}h}{A''-(g_\phi h+gh_\phi)} \\ &= \frac{g_{b-a}}{A''-(g_\phi h+gh_\phi)}(A''-gh_\phi) > 0. \end{aligned}$$

**Appendix A6: The welfare function is concave in $\phi$**

We show that function (15) is concave in $\phi$, when $\phi \in (0,1)$. By (16) we have that

$$z(\phi, n) = (a-c)\left(1-(1-\phi)^n\right) + (b-a)\left(1 - \frac{1}{(n+1)\phi}\left(1-(1-\phi)^{n+1}\right)\right).$$

It is enough that $(1-(1-\phi)^n)$ and $1-(1-\phi)^{n+1}/((n+1)\phi)$ are concave in $\phi$. For the first term, we have

$$\frac{\partial^2}{\partial \phi^2}\left(1-(1-\phi)^n\right) = -(1-\phi)^{n-2}(n-1)n < 0,\ \ \phi \in [0,1).$$

For the second term, we have

$$\frac{\partial^2}{\partial \phi^2}\left(-\frac{1-(1-\phi)^{n+1}}{(n+1)\phi}\right) = \frac{\phi(1-\phi)^{n-1}(2(1-\phi)+\phi n) - \frac{2\left(1-(1-\phi)^{n+1}\right)}{n+1}}{\phi^3}.$$

We define

$$x(\phi) \equiv \phi(1-\phi)^{n-1}(2(1-\phi)+\phi n) - \frac{2\left(1-(1-\phi)^{n+1}\right)}{n+1}.$$

We now show that $x(\phi) < 0$ when $\phi \in (0,1)$. We have that $x(0) = 0$, and

$$\frac{\partial x(\phi)}{\partial \phi} = -(1-\phi)^{n-2}\phi^2 n(n-1) < 0,$$

when $\phi \in (0,1)$. It follows that $x(\phi) < 0$ for all $\phi \in (0,1)$.