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HETEROGENEITY OF INVESTORS AND ASSET PRICING IN A RISK-VALUE WORLD

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ABSTRACT

Heterogeneity of Investors and Asset Pricing in a Risk-Value World*

Portfolio choice and the implied asset pricing are usually derived assuming maximization of expected utility. In this Paper, they are derived from risk-value models that generalize the Markowitz-model. We use a behaviourally-based risk measure with an endogenous or exogenous benchmark. If the risk measure is modelled by a negative HARA-function, then sharing rules are convex or concave relative to each other. A measure of heterogeneity of investors is derived. More heterogeneity (a) raises convexity/concavity of sharing rules and, thus, the need of investors to trade options, (b) increases convexity of the pricing kernel, (c) raises option prices relative to the price of the under-lying asset and (d) raises the variance and kurtosis of the risk-neutral probability distribution of the aggregate pay-off.

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1 Introduction

Recently, traditional theories of asset pricing have come under intensive discussion. Quite a number of market phenomena have been discovered which cannot or at least cannot easily be explained by traditional theories. The equity premium puzzle [*Mehra* and *Prescot* 1985], short- and long term predictability of stock returns [*Jegadeesh* and *Titman* 1993, *De Bondt* and *Thaler* 1985], high volatility of stock prices [*Shiller* 1981, 1989] and the success of strategies based on value and size [*Vuolteenaho* 2002] are just examples for those phenomena. Interestingly enough, these puzzles are not only found at the New York Stock Exchange, but they are also present in markets world wide, see, e.g., *Rouwenhorst* (1998) and *Schiereck*, *De Bondt* and *Weber* (2000) for the persistence of short-term predictability. These results are complemented by the analysis of individual trading behavior [*Odean* 1998] reflecting biases known from psychological research as well as from experimental work which shows that expected utility does not adequately describe human behavior [*Gneezy, Kapteyn* and *Potters* (in press), *Sarin* and *Weber* 1993, *Thaler et al* 1997, *Weber* and *Camerer* 1998].

The question remains how these puzzles can be solved. Apart from market microstructure models, most models are based on assumptions strong enough to imply the existence of a representative investor. These models cannot explain the puzzles mentioned before, nor the trade of derivatives nor the strong trading volume of stocks. Therefore we provide a further attempt to explore the implications of investor heterogeneity on asset pricing. In contrast to the existing literature, we use a risk-value model to derive efficient portfolios. This approach allows us to obtain an explicit measure of investor heterogeneity similar to a variance measure. We show that more heterogeneity implies a stronger need for trading options and higher option prices relative to the price of the underlying. We explicitly relate the measure of investor heterogeneity to the shape of the pricing kernel and to relative asset pricing. These results also yield testable implications. For example, the theory predicts that in times of high trading volume, indicating strong investor heterogeneity, options should be more expensive relative to the underlying. The latter can be proxied by the implied volatility, derived from the Black-Scholes model, divided by the actual volatility. We also present some very preliminary empirical evidence.

The need to trade options derives from nonlinear sharing rules. They relate our model to the literature on portfolio insurance [*Leland* 1980, *Brennan* and *Solanki* 1981, *Benninga* and *Blume* 1985, *Franke, Stapleton* and *Subrahmanyam* 1998, *Grossman* and *Zhou* 1996, *Benninga* and *Mayshar* 2000, *Carr* and *Madan* 2001]. *Leland* defines portfolio insurance as a portfolio policy which leads to a convex sharing rule. The intuition behind his concept is that a convex sharing rule gives the investor a higher payoff when the aggregate payoff is low (= low state) as compared to a linear or concave sharing rule.

Nonlinear sharing rules are obtained if investors differ in their preferences, in their expectations and/or in their environment. *Benninga* and *Mayshar* assume that every investor has constant relative risk aversion, but the level of risk aversion differs across investors. *Carr* and *Madan* consider the case whrere investors have different HARA-utility functions and, perhaps, different lognormal expectations. *Grossman* and *Zhou* analyse an equilibrium with two agents being the same, but one has to make sure that his wealth never falls below a given floor. *Franke, Stapleton* and *Subrahmanyam* show for investors with HARA-utility functions that those with high additive background risk buy portfolio insurance from those with low background risk. A very recent paper by *Cassano* (2002) relates the option trading volume to differences in lognormal expectations.

This paper derives an explicit measure of investor heterogeneity based on different preferences retaining homogeneous expectations and homogeneous environments. The main insights of our analysis are based on certain properties of agents' demand functions for state-contingent claims. In particular, all agents' relative demand functions are either convex, linear or concave. A relative demand function relates the demand for state-contingent claims of one investor to the demand of some other investor. If investor i's demand function is convex relative to that of investor j, then investor i buys portfolio insurance from investor j.

As we know from *Leland*, in an expected utility framework we obtain such a result if in an equilibrium allocation the absolute risk aversion of agent i increases faster in his wealth than that of agent j. In contrast to the previous literature, we do not start from expected utility theory. Instead we start from a risk-value model. Such an approach is quite unusual in finance papers and needs justification.

There are quite a number of recent theories of choice which depart from expected utility (*Camerer* 1995 and *Starmer* 2000). We assume that investors use risk-value models for decision making. In a risk-value model, risk and value are taken as primitives. To evaluate an alternative, the decision maker first separately derives value and risk of this alternative. Then value and risk are combined into an overall preference which may differ across decision makers even though they measure risk in the same manner. We will motivate this separation first and then talk about risk value models in more detail.

The separation of value and risk is quite popular in finance. Investors usually talk about the risk of an investment which then is evaluated against its expected return, a proxy for value. Thus, decisions are made by evaluating risk and value separately and trading off both components. This allows different investors to have different tradeoffs even though their risk measures may be the same. The explicit consideration of the investment risk has become even more important in the light of recent regulations which require broker houses to inform their clients about the riskiness of their investments. Risk judgments have also become quite important in bank regulation and management. Finally, in board meetings risk and return are usually discussed separately allowing board members to use their personal tradeoffs in evaluating investment alternatives.

While the strength of the expected utility principle is its strong axiomatic foundation, a weakness is that the utility function simultaneously determines the risk measure and the tradeoff between risk and return. There is little freedom to vary the tradeoff once the risk measure has been determined. Consider, for example, an investor who measures risk by the variance of his wealth e. Then he uses a quadratic utility function $u(e) = -(a - e)^2$. The tradeoff between risk and return on an iso-utility curve is given by the slope $(1/2)\ d\sigma^2(e)/dE(e) = a - E(e)$. In other words, the tradeoff between risk and return is necessarily strictly proportional to $a - E(e)$ and independent of the risk level. The only degree of freedom is to change the constant a. This appears very restrictive. There is little reason to believe that risk perception and the risk-return tradeoff are so strictly intertwined. Therefore, we start from a more realistic risk-value model.

The necessity (or at least the possibility) of treating risk as an own construct is matched by a well developed body of psychological research showing that risk perception (measuring risk) and risk attitude (evaluating risk) can be consistently assessed. In Section 2 we will demonstrate some of this psychological research in more detail. Results suggest that risk perception can be modelled as the expected value of the transformed deviation of the outcome from a benchmark. The benchmark can be exogenous (any target or aspiration level) or endogenous, e.g. the expected outcome. The transformation is achieved by a risk function. Empirical work suggests that the risk function can be a monotonically decreasing convex function, with positive deviations from the benchmark reducing risk and negative deviations increasing risk. Such a risk function reflects the practitioners' view that negative deviations constitute "risk" and positive deviations "chances". At some point in our analysis, we will restrict the risk function to be of the negative HARA (hyperbolic absolute risk aversion)-type. This measure still captures major empirical findings on risk perception.

An investor's trade-off between the risk and the value of an alternative is usually labelled his attitude towards risk. *Markowitz* and his followers propose (μ, σ)-models which are consistent with expected utility and with risk-value models. *Coombs* (1969) was one of the first who suggested to use risk-value models independently of expected utility theory.[1] There are first experimental studies of risk-value models in the context of financial decision making. *E.Weber* and *Hsee* (1998) find that Chinese students have a higher willingness to pay for risky assets than US-students. Analysing this behavior in the light of a risk-value model they find that the trade-off between risk and return is not significantly different for both groups, however Chinese subjects perceive the investment alternative to be less risky than US-students. Moreover, empirical research has shown that the perception of the value of an alternative does not differ between individuals or groups

[1]Recently, *Jia* and *Dyer* (1996) have shown that an expected utility model can be written as a risk-value model if and only if a strong condition, called risk independence, holds. Basically, this condition requires that lotteries with the same expected payoff can be preference ranked by their risk measures such that this ranking does not change when the expected payoff changes [see *Bell* 1995a, b for further discussion]. This result demonstrates that risk-value models can be quite different from expected utility and, thus, in the spirit of behavioral research allow for a wider and different type of investor behavior.

[*E.Weber*, *Anderson* and *Birnbaum* 1992]. It is therefore that in section 2 we concentrate on the discussion of measurement of perceived risk. A review of the literature on risk-value models can be found in *Sarin* and *Weber* 1993).

Using this risk-value framework we get two classes of results. First, we determine the shape of the investors' sharing rules in equilibrium. Two definitions of sharing rules will be used. The absolute sharing rule is the function which relates the investor's portfolio payoff to the aggregate payoff, i.e. the exogenously given payoff to all investors. The relative sharing rule is the function which relates an investor's payoff to the payoff of another investor. *Cass* and *Stiglitz* (1970) and *Rubinstein* (1974) showed that for expected utility maximizers using a utility function of the HARA-class with the same exponent, relative sharing rules are linear. In risk-value models where risk is measured by a negative HARA function, relative sharing rules are linear if and only if this risk function is quadratic. Otherwise strictly convex or concave relative sharing rules are obtained. More heterogeneity among investors translates into more concavity or convexity of relative sharing rules.

Second, our equilibrium analysis yields a pricing kernel which is declining and convex in aggregate consumption. As *Dybvig* (1988) pointed out, in such a setting there exists a von Neumann-Morgenstern utility function such that a representative investor with this function would imply the same pricing kernel. Hence there would be no need to talk about heterogeneity of investors. The important contribution of this paper is, however, to derive a measure of investor heterogeneity and the impact of investor heterogeneity on this pricing kernel. It will be shown for HARA-based risk functions that more heterogeneity a) increases convexity/concavity of relative sharing rules and, thus, the need of investors to trade options, b) raises convexity of the pricing kernel, c) raises option prices relative to the price of the underlying asset and

d) raises the variance and the kurtosis of the risk-neutral probability distribution of the aggregate payoff. Hence the implied volatility derived from the Black-Scholes model increases with heterogeneity. This might explain the observation that stock index options appear to be more expensive than suggested by the Black/Scholes model [*Christensen* and *Prabhala* 1998]. Also, our results help to explain why the risk-neutral probability distributions estimated from stock index option prices are leptokurtic and negatively skewed [*Longstaff* 1995, *Brenner* and *Eom* 1996, *Jackwerth* 2000].

The paper is organized as follows. In section 2, we will review some of the theoretical and empirical research in decision theory on how to measure risk. Based on this analysis, we will discuss some general properties a risk measure should have. In section 3, risk-value efficient portfolios are derived. Equilibrium is analysed in sections 4 and 5. In section 4, we first investigate individual sharing rules for a rather general class of risk functions and then for HARA functions. In section 5 we present the measure of investor heterogeneity, its relation to the shape of the pricing kernel and the impact on option prices. Also some preliminary empirical evidence is documented. Section 6 summarizes the main results.

2 Risk Measurement

2.1 Background[2]

The separation of value and risk is quite popular in finance. There are two streams of research dealing with the problem of how to measure risk. On the one hand, in financial economics risk is usually defined based on expected utility or some other normative concept. Variance, the inclusion of third moments of the return distribution [*Kraus* and *Litzenberger* 1976] and semi-

[2]See *Sarin* and *Weber* (1993) and *E. Weber* (1997) for reviews on risk measurement.

variance [*Bawa* and *Lindenberg* 1997] are examples of such measures of risk. Recently, value at risk has been added to the list of risk measures. On the other hand, in psychology the aim is to measure risk, similar to measuring happiness or utility [*Kahneman*, *Wakker* and *Sarin* 1997]. As argued before, this descriptive approach will help us to better understand and predict, what people think when they talk about the riskiness of an investment alternative.

Meanwhile, a substantial literature has developed on how to measure risk, see *Brachinger* and *Weber* [1997] or *M. Weber* [2001] for an overview on theoretical aspects and *E. Weber* [2001] for an overview on empirical research. In the following, we will present just a few relevant elements of this literature. One important finding is that risk can be measured as a stable construct, i.e. perceived risk can be assessed in a reliable manner. This is true for the measurement of the perceived riskiness of gambles, e.g. *Keller et al.* [1986], as well as for the assessment in an investment context, e.g. *E. Weber* and *Milliman* [1997]. To model risk perception, the literature on behavioral decision making has proposed and tested different theories [see also *Jia*, *Dyer* and *Butler* 1999 for an overview]. An exponential model [*Sarin* 1984] and a power function model [*Luce* and *E. Weber* 1986] were found to fit the data quite well [*Keller et al.* 1986 and *E. Weber* and *Bottom* 1990]. These models define risk as the expected value of a function of the outcomes or of the deviations of the outcomes from a possibly endogenous benchmark. Thus, the risk of a random variable e can be written as

$$Risk(e) = E[F(e - \bar{e})] \tag{1}$$

F is a function with $F(0) = 0$, e denotes the random payoff of the alternative and $\bar{e}$ the benchmark. This risk measure is general enough to include risk measures which describe people's risk perception, e.g. the exponential risk measure. *Jia* and *Dyer* (1996) define a standard measure of risk as in equation (1) with $(-F)$ being a von Neumann-Morgenstern utility function. The benchmark $\bar{e}$ equals the expected payoff.

There are two fundamental ways for defining a benchmark: exogenously or endogenously. An exogenous benchmark is set by the investor, e.g., it can be the outcome level the investor wants to surpass. This benchmark can have any sign. This benchmark may be affected by portfolio gains or losses in previous periods as suggested by *Barberis*, *Huang* and *Santos* (2001) and by other factors which have been shown to be important in financial behavior.

An endogenous benchmark depends on the characteristics of the payoff distribution. The most prominent endogenous benchmark is the expected value of the payoffs as in variance and other moments. This benchmark implies that the risk measure is location free, i.e. risk does not change if the return distribution is shifted by adding or subtracting a positive number. Hence risk is independent of the expected payoff. This is a desirable property since the expected payoff (value) is already used as a primitive in the preference function.

2.2 Properties of a Risk Measure

We now describe the risk measure in more detail. Risk will be measured according to equation (1) as the expectation of a function of the deviation of a random variable e from a benchmark $\bar{e}$. In the case of an exogenous benchmark $\bar{e}$ is a given number, in the case of an endogenous benchmark $\bar{e}$ is the expected value $E(e)$.

For simplicity we define the deviation $\hat{e} := e - \bar{e}$.

As a next step we postulate three key properties of the risk function F which are based on the psychological studies cited before:

i) Outcomes above the benchmark reduce risk and outcomes below increase risk, $F(\hat{e}_1) > F(0) > F(\hat{e}_2)$ for $\hat{e}_1 < 0 < \hat{e}_2$. In addition, we require monotonicity: A higher payoff will contribute less to risk than a lower payoff, thus $F' < 0$.

ii) Mean preserving spreads increase risk, thus $F'' > 0$.

iii) The sensitivity to a mean preserving spread is larger in the loss domain (relative to the benchmark) than in the gain domain. Requiring monotonicity then implies that the sensitivity decreases if the payoff increases; thus we require $F''' < 0$.[3]

As proposed by *Jia* and *Dyer* (1996), risk functions with $F' < 0$, $F'' > 0$, and $F''' < 0$ correspond to utility functions with $u' > 0$, $u'' < 0$ and $u''' > 0$, i.e. with positive prudence [*Kimball* 1990].

Standard models of risk perception, e.g. the exponential model, have the properties noted above. Note that variance neither fulfills property i) nor property iii). Risk judgments found in a number of empirical studies, see, e.g., *Keller, Sarin* and *Weber* (1986), are consistent with the risk ranking implied by these properties. Property iii) is reflected in a variety of empirical results that show that people judge alternatives with potential catastrophic outcomes as being especially risky, see, e.g., *Slovic* (1987).

In the following sections we will derive results using properties i) - iii). The risk function, in general, varies from investor to investor. It is defined on the range $(\underline{\hat{e}}, \overline{\hat{e}})$. In order to guarantee optimal internal solutions to portfolio choice, we add the assumption that $F'(\hat{e}) \to -\infty$ for $\hat{e} \to \underline{\hat{e}}$ and $F'(\hat{e}) \to$

[3] The necessity of $F''' < 0$ can be seen as follows:
Let y, z be two states with the same payoff and the same probability p; in both states the payoff deviates from the expected value by Δ. Their contribution to the total risk of the portfolio is then given by $2pF(\Delta)$. Now replace the payoff deviation Δ in the states y and z by a mean preserving spread around Δ, that is: the deviation from the expected payoff is $\Delta - \alpha$ in state y and $\Delta + \alpha$ in state z ($\alpha > 0$). Notice that the expected payoff is not changed and hence it does not influence the risk contribution of the other states. The new contribution of the states y, z to the total risk is $pF(\Delta - \alpha) + pF(\Delta + \alpha)$. The risk increase, denoted RI_α is then given by: $RI_\alpha(\Delta) = pF(\Delta - \alpha) + pF(\Delta + \alpha) - 2pF(\Delta)$ and the strict convexity of F is equivalent to $RI_\alpha > 0$ for all Δ and all $\alpha > 0$. We require for an increase in Δ that $RI'_\alpha(\Delta) = pF'(\Delta - \alpha) + pF'(\Delta + \alpha) - 2pF'(\Delta) < 0$. This holds iff $F''' < 0$.

0 for $\hat{e} \rightarrow \overline{\hat{e}}$. At the end, in order to get further results, we need more detailed information about the risk measure. At that point we will assume that the risk function belongs to the set of negative HARA-functions. We consider negative HARA-functions with properties i) - iii). They include the exponential function, an important function to describe people's risk judgments. The negative HARA- class is defined as

$$F(\hat{e}) \;\; = -\frac{1-\gamma}{\gamma}\left(A+\frac{\hat{e}}{1-\gamma}\right)^{\gamma} \tag{2}$$

where $\gamma \in \mathrm{I\!R} \backslash \{ 0, 1 \}$; $A > 0$. In the case $\gamma = 0$ we obtain $F(\hat{e}) = -ln(A + \hat{e})$ and in the case $\gamma = -\infty$ we get $F(\hat{e}) = exp(-B\hat{e})$ with $B > 0$. For $\gamma > -\infty$ the domain of F is constrained by $(A + \hat{e}/(1-\gamma)) > 0$. A has to be sufficiently high. For $\gamma < 1$, we need $inf\ \hat{e} > -A(1-\gamma)$; for $\gamma > 1$, we need $sup\ \hat{e} < -A(1-\gamma)$. Since it makes little sense to constrain $\hat{e}$ from above, we shall mostly assume $\gamma < 1$. We have $F' < 0, F'' > 0$ and

$$F'''\ (\hat{e}) \; = \frac{\gamma-2}{1-\gamma}\left(A+\frac{\hat{e}}{1-\gamma}\right)^{\gamma-3} \tag{3}$$

As we require $F'''(\hat{e}) < 0$, $\gamma < 1$ or $\gamma > 2$ is implied. Therefore, we will only consider functions with $\gamma < 1$ or $\gamma > 2$, mostly $\gamma < 1$.

3 Efficient Portfolios

We assume a two date-economy with a perfect and complete capital market. At date 0 investors choose their portfolios which pay off at date 1. A state of nature at date 1 is defined by the exogenously given aggregate payoff ε; i.e. the sum of payoffs to all investors.[4] ε is a positive variable, $\varepsilon \in (\underline{\varepsilon}, \overline{\varepsilon})$ with

[4]The market is said to be complete if for every $\varepsilon^0 \in \mathrm{I\!R}^+$ there exists a claim which pays off \$ 1 if $\varepsilon > \varepsilon^0$ and zero otherwise [see *Nachman* 1988]. We assume the existence of

the probability density being positive for every $\varepsilon \in (\underline{\varepsilon}, \overline{\varepsilon})$.[5] Since we are not interested in time preferences, the whole analysis is done in forward terms. Equivalently, the risk-free rate can be assumed to be zero. Define:

$p_\varepsilon :=$ probability density of ε,

$e_\varepsilon :=$ number of claims contingent on state ε, purchased by the investor; each claim pays off \$1 if and only if state ε obtains,

$\pi_\varepsilon :=$ forward pricing kernel; for every state ε it denotes the price of a claim contingent on that state divided by the state probability density, $\pi_\varepsilon > 0$; $E(\pi) = 1$.

$W_0 :=$ the investor's initial forward endowment (wealth), $W_0 > 0$. Since the investor is endowed with state-contingent claims, W_0 equals the forward market value of these claims.

$R^* :=$ expected gross portfolio return on the forward endowment required by the investor.

The pricing kernel is assumed to be twice continuously differentiable. To simplify notation, the state index will be dropped unless necessary for clarity of exposition.

In risk-value models the investor, first, derives the set of risk-value efficient portfolios and, second, chooses one of these portfolios according to his tradeoff between risk and value. The advantage of our analysis is that asset pricing in a risk-value equilibrium can be analyzed as in the CAPM-world without making assumptions about this tradeoff.

A risk-value efficient portfolio minimizes the risk subject to the constraint that the expected portfolio payoff $E(e)$ does not fall below some exogenously given value $W_0 R^*$ (payoff constraint). Hence, the expected gross portfolio return has to be equal or higher than R^*. In this section, the prices for state-

$\underline{\varepsilon}, \overline{\varepsilon} \in \mathbb{R}^+ \cup \{\infty\}$such that the state space is identified by $(\underline{\varepsilon}, \overline{\varepsilon})$.

[5] Assuming a continuous state space does not appear to be essential, but allows us to differentiate with respect to ε.

contingent claims are assumed to be exogenously given. Then an efficient portfolio is the solution to the following problem.

Minimize

$$E\ [F\ (\hat{e})] \tag{4}$$

subject to the budget constraint:

$$E\ [e\pi] = W_0 \tag{5}$$

and the payoff constraint:

$$E\ (e) \geq W_0 R^* \tag{6}$$

Varying R^* parametrically allows us to derive all risk-value efficient portfolios. In the following the negative marginal risk function

$$f\ (\hat{e}) := -\frac{\partial\ F\ (\hat{e})}{\partial\ \hat{e}} \tag{7}$$

will be of special importance. From the properties of the risk function $F(F' < 0,\ F'' > 0$, and $F''' < 0)$, we immediately get: $f > 0,\ f' < 0,\ f'' > 0$.

3.1 Risk-Value Models With an Endogenous Benchmark

Using this notation we can write the first order condition for a solution to the minimization problem (4) - (6) with an endogenous benchmark $\bar{e} = E(e)$, $(\eta$

is the Lagrange-multiplier of the budget constraint (5) and λ the Lagrange-multiplier of the payoff constraint (6)) as[6] (after dividing by the probability density)

$$-f(\hat{e}_\varepsilon) + E[f\,(\hat{e})] = \eta\pi_\varepsilon + \lambda;\ \forall\ \varepsilon. \tag{8}$$

As $E(\pi) = 1$, taking expectations yields

$$0 = \eta + \lambda \tag{9}$$

Since raising R^* raises the risk of the efficient portfolio, $\lambda > 0$ so that, by (9), $\eta < 0$. Substituting λ in equation (8) yields

$$-E[f(\hat{e})] + f\,(\hat{e}_\varepsilon) = \eta\,[1 - \pi_\varepsilon]\,;\ \forall\ \varepsilon \tag{10}$$

Defining $\theta_\varepsilon = (1 - \pi_\varepsilon)$ this equation can be rewritten as :

$$-E[f(\hat{e})] + f(\hat{e}_\varepsilon) = \ \eta\ \theta_\varepsilon;\ \forall\ \varepsilon \tag{11}$$

The investor's efficient portfolio is characterized by equation (11) and constraints (5) and (6). θ_ε is the difference between the forward price of the risk-free claim, 1, and the probability deflated forward price of a state ε-contingent claim. θ_ε is negative if the probability deflated forward price of a state ε-contingent claim exceeds the forward price of the risk-free claim. θ has zero expectation. The higher θ_ε, the cheaper are the state ε-contingent

[6] The solution of the minimization problem exists and is unique if a) for every e satisfying the constraints (5) and (6) the value of the objective function is finite and b) for every λ, η and e satisfying the first order condition (8) $E[e\pi]$ is finite [*Back* and *Dybvig* 1993].

claims, and the more state -contingent claims the investor buys because a higher level of these claims raises $-f(\hat{e}_\varepsilon)$.

The risk-value efficient frontier can be derived by varying parametrically the required expected return R^*. If R^* is not greater than 1, then the Lagrange multiplier $\lambda = 0$ and risk is zero. All endowment is invested in the risk-free asset so that e_ε is the same for every state. λ grows with R^* ($R^* > 1$), because the objective function is strictly convex. Since $\lambda = dE[F(\hat{e})]/d(W_0 R^*)$ for efficient portfolios, it follows that the risk-value efficient frontier is strictly convex as shown in figure 1.

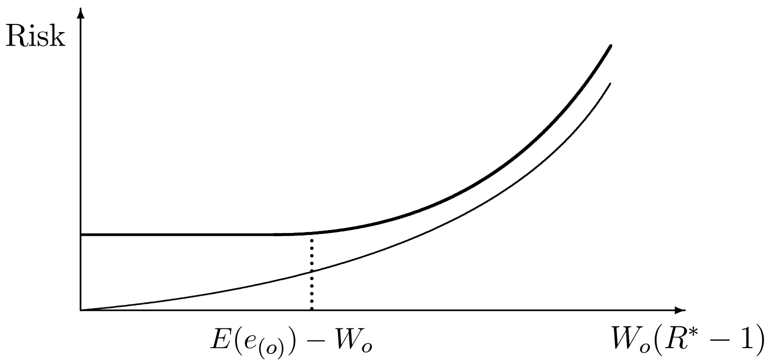


Figure 1: The risk-value efficient frontier. It depicts the minimal portfolio risk as a function of the required expected portfolio return R^*. The thin curve represents the frontier for an endogenous benchmark, the thick curve for an exogenous benchmark.

3.2 Risk-Value Models With an Exogenous Benchmark

Now consider risk functions with an *exogenous* benchmark $\bar{e}$; $\bar{e} \gtreqless 0$. Then in the first order condition (8) for a risk-value efficient portfolio the second term disappears since portfolio choice has no effect on the benchmark. Hence the first order condition reads:

$$-f(\hat{e}_\varepsilon) = \eta\pi_\varepsilon + \lambda; \forall\varepsilon. \tag{12}$$

Again, $\lambda > 0$ so that $\eta < 0$ follows.

Taking expectations yields

$$-E[f(\hat{e})] = \eta + \lambda \tag{13}$$

so that subtraction of (12) from (13) leads to

$$-E[f(\hat{e})] + f(\hat{e}_\varepsilon) = \eta\theta_\varepsilon;\ \forall\ \varepsilon. \tag{14}$$

Compare the solution of the exogenous benchmark model with that of the endogenous benchmark model. From equation (8), for an endogenous benchmark an increase in some payoff lowers [raises] risk if the payoff $\hat{e}_\varepsilon$ is lower [higher] than $f^{-1}(E[f(\hat{e})])$. This follows from the impact of the payoff increase on the benchmark. From equation (12), in the case of an exogenous benchmark, raising the payoff in some state always reduces risk. The risk reduction is higher the lower the payoff. Therefore, the risk reduction induced by raising the payoff in some state minus the expected risk reduction induced by raising the payoff in every state, $-f(\hat{e}_\varepsilon)+E[f(\hat{e})]$, has the same properties as the risk reduction in the endogenous benchmark model. This explains why the first order conditions (11) and (14) look precisely the same although the optimal portfolios are different because of the different benchmarks.

In spite of the formal identity of (11) and (14) the optimal solutions to both problems differ substantially. If the payoff constraint (6) is not binding, then the optimal solution to the endogenous benchmark-problem is to buy only the risk free asset. The exogenous benchmark-problem without the payoff constraint is formally the same as the traditional state preference -EU-choice problem Max $E[u(e)]$ subject to the budget constraint (5). This follows since minimizing risk with the risk function being a negative utility function is formally the same as maximizing expected utility. Hence the investor minimizing risk with an exogenous benchmark chooses a risky portfolio $e_{(0)}$ even if the payoff constraint (6) is not binding. This implies $E(e_{(0)}) > W_0$ or, equivalently, an expected gross portfolio return $E(R) > 1$. The payoff constraint becomes binding only if the required expected payoff is raised above this expectation, i.e. if $W_0 R^* > E(e_{(0)})$; (see figure 1).

We summarize the differences between risk-value models and expected utility models as follows.

1. The risk-value model with an exogenous benchmark is formally the same as the expected utility model if the payoff constraint (6) is not binding. If the payoff constraint is binding, then the investor has to take more risk so as to satisfy the payoff constraint. Hence the payoff constraint adds a new element to the optimization. For risk functions with an endogenous benchmark the investor takes risk if and only if the payoff constraint is binding.

2. If the benchmark is endogenous, then risk measurement depends on this endogenous benchmark which conflicts with the axioms of von Neumann/Morgenstern.[7]

4 Equilibrium: Investors' Sharing Rules

4.1 General Risk Functions

Next equilibrium in the capital market will be investigated. We assume the existence of an equilibrium. If there exist multiple equilibria, we analyse anyone of them. Every investor chooses a risk-value efficient portfolio. All investors are assumed to have homogeneous expectations. First, individual sharing rules will be analysed in equilibrium (Section 4). After considering general risk functions (Section 4.1), we restrict ourselves to HARA-based risk functions (Section 4.2). Second, the equilibrium pricing kernel will be derived and analyzed (Section 5).

[7] A third difference between risk-value models with an endogenous benchmark and EU-models relates to satiation. In the EU-model, marginal utility is always positive, by assumption. In the risk-value model, the investor may be worse off if she receives an additional payoff in some state with a high portfolio return. Consider as an example the preference function $P(\bar{e}, risk)$ with $\alpha > 0$ and $\bar{e} = E(e)$, $P(\bar{e}, risk) = \alpha\bar{e} - risk$, with $risk$ as defined in equation (1). Differentiate the preference function with respect to e_ε . This yields:

$$\frac{\partial\ P}{\partial\ e_\varepsilon} = p_\varepsilon \left[\alpha + f(\hat{e}_\varepsilon) - E[f\ \ (\hat{e})]\right].$$

If the portfolio payoff is random, there must exist a state ε with $f(\hat{e}_\varepsilon) < E[f(\hat{e})]$ (for example, the state with the lowest π_ε). Then, given a sufficiently small α, we have $\partial\ P\ /\partial e_\varepsilon < 0$. Hence an increase in the state ε-portfolio return may reduce the investor's welfare which contradicts the usual assumption of non-satiation. If, however, all prices for state-contingent claims, π_ε, are positive, then the investor always chooses his/her optimal portfolio such that he/she never reaches or crosses satiation. This follows since a risk free-asset exists and the investor can always purchase fewer claims contingent on these critical states, invest the saved money in the risk-free asset and, thereby, increase his/her welfare.

Individual investors are indexed by i. Hence all investor-dependent variables have to be indexed by i. Then condition (11) for an endogenous benchmark reads

$$-E[f_i\,(\hat{e}_i)] + f_i(\hat{e}_{i\varepsilon}) = \eta_i\,\theta_\varepsilon\ ;\quad \forall\ \varepsilon, i. \tag{15}$$

Condition (14) for an exogenous benchmark is the same. Unless stated otherwise, the following results hold for efficient portfolios with an endogenous and with an exogenous benchmark. $\lambda_i > 0$ and $\eta_i < 0$, will be assumed throughout. Then we can derive a proposition which relates investors' portfolio choices to the pricing of state-contingent claims.

Proposition 1 *: For every investor, his/her optimal payoff is decreasing and convex in the probability-deflated price for state-contingent claims.*

Proof. [8]

Differentiate equation (15) with respect to θ. Recall, $f_i(\hat{e}_i) > 0$, $f_i'(\hat{e}_i) < 0$, and $f_i''(\hat{e}_i) > 0$. Hence it follows that e_i is an increasing and convex function in θ. As $\theta = 1 - \pi$, e_i is a decreasing, convex function in π. This proves proposition 1 ■

An absolute sharing rule relates investor i's payoff e_i to the aggregate payoff ε. Proposition 1 states $de_i/d\pi < 0$. Aggregation across investors implies $d\varepsilon/d\pi < 0$. Hence it follows that $de_i/d\varepsilon > 0$. The positive slope of the absolute sharing rule does not come as a surprise. This is still in line with EU-theory. The major difference between the risk-value and the EU-model is reflected in the shapes of the sharing rules. Equation (15) implies for two investors i and j in the risk-value model

[8] By the implicit function theorem, $\hat{e}_i\ (\varepsilon)$ is twice continuously differentiable since f_i $(\hat{e}_i)$ and $\theta_\varepsilon = \theta(\varepsilon)$ are twice continuously differentiable.

$$\frac{-E[f_i\ (\hat{e}_i)] + f_i(\hat{e}_{i\varepsilon})}{-\eta_i} \quad = \quad \frac{-E[f_j\ (\hat{e}_j)] + f_j(\hat{e}_{j\varepsilon})}{-\eta_j}\ ; \quad \forall\ \varepsilon. \tag{16}$$

Define $s_i := E[f_i(\hat{e}_i)]/-\eta_i$ to be investor i 's sharing constant. Equation (16) can be written then as

$$-s_i + \frac{f_i(\hat{e}_{i\varepsilon})}{-\eta_i} \quad = \quad -\ s_j + \frac{f_j(\hat{e}_{j\varepsilon})}{-\eta_j}\ ; \quad \forall\ \varepsilon. \tag{17}$$

The new element in risk-value models as compared to expected utility models are the sharing constants. They are generated in the case of an exogenous benchmark by the payoff constraint and, in the case of an endogenous benchmark, by the impact of a payoff change on the benchmark. In order to grasp the intuition behind the sharing constants, consider the inverse sharing constant - $\eta_i/E[f_i\ (\hat{e}_i)]$. $-\eta_i$ is the efficient increase in risk due to a marginal reduction in the initial endowment available for buying claims, holding the required expected payoff W_0R^* constant. This risk increase depends on the risk function which is determined up to a linear positive transformation. Hence we need to standardize these marginal risk increases to make them comparable across investors. This is done by dividing - η_i through the expected slope of the risk function based on the efficient portfolio, $E[f_i\ (\hat{e}_i)]$. Hence the inverse sharing constant measures the standardized efficient risk increase due to a marginal reduction in the initial endowment.

In order to gain some insight into the mechanics of the risk-value model, we analyse the impact of changes in initial endowment and in the required expected return on an investor's sharing constant. For simplicity of notation, we drop the index i in Lemma 1.

Lemma 1: *Consider risk-value efficient portfolios under the condition* $\lambda > 0$ and $\eta < 0$. *Then, given the prices of state-contingent claims, the sharing constant s declines when*

– the initial endowment W_0 *increases, or*

– the required expected return R^* *increases.*

Proof. See Appendix A.

From Lemma 1 it is apparent that the sharing constants differ across investors. The sharing constants, in fact, prohibit linear sharing rules. This is illustrated by proposition 2 which does not constrain $f''(\hat{e})$ to be positive. It should be noted that each investor has a linear absolute sharing rule e_ε if all relative sharing rules are linear, and vice versa.

Proposition 2 *: Let* $f''(\hat{e})$ *be unconstrained in sign. Then in an equilibrium with risk-value models every investor has a linear (absolute) sharing rule if and only if every investor uses a quadratic function* $F(\hat{e})$.

Proof. See Appendix B.

Proposition 2 provides a strong result about the shape of the sharing rules. In a risk-value world the sharing rules are linear if and only if every investor uses variance or a related quadratic risk measure (which both behaviorally are not appropriate), or, equivalently, if and only if the sharing constant disappears. This is true only for quadratic risk functions. Then $f(\hat{e}) = a + b\hat{e}$ so that $f(\hat{e}) - E(f(\hat{e})) = b(e - E(e)) = b\hat{e}$. In contrast, for EU- models, *Rubinstein* (1974) has shown that linear sharing rules are obtained whenever all investors have a HARA-utility function with the same γ.

In the following, we require again $f''(\hat{e}) > 0$ and analyse the sharing rule of investor i relative to that of investor j, i.e. the relative sharing rule $e_i(e_j)$. In analogy to *Leland* (1980), we say that investor i purchases portfolio insurance from investor j if his sharing rule e_i is strictly convex in e_j. Proposition 3 provides conditions for trading portfolio insurance.

Proposition 3 *: In a risk-value equilibrium the following statements are equivalent:*
– Investor i's sharing rule is strictly convex [linear] [strictly concave] relative to that of investor j.
– The coefficient of absolute prudence of investor i's risk function, $-f_i''(\hat{e}_i)/f_i'(\hat{e}_i)$*, multiplied by* de_i/de_j*, is everywhere greater than [equal to] [smaller than] the coefficient of absolute prudence of investor j's risk function.*
–

$$-\eta_i \frac{f_i''(\hat{e}_i)}{[f_i'(\hat{e}_i)]^2} > [=][<] - \eta_j \frac{f_j''(\hat{e}_j)}{[f_j'(\hat{e}_{jj})]^2}. \tag{18}$$

Proof. See Appendix C.

Proposition 3 provides necessary and sufficient conditions for the shape of an investor's sharing rule relative to that of another one. The shape depends on the investors' coefficients of absolute prudence given efficient portfolios.

4.2 HARA-Based Risk Functions

In this section, we derive more specific results assuming that the risk function F belongs to the negative HARA-class. $F(\hat{e})$ is given by equation (2) with $\gamma > 2$ or $\gamma < 1$. γ is assumed to be the same for all investors.

A necessary condition for the existence of an equilibrium in the case of a non-exponential risk function ($\gamma > -\infty$) is

$$\sum_i (A_i + \frac{\hat{e}_{i\varepsilon}}{1-\gamma}) \equiv A + \frac{\hat{e}_\varepsilon}{1-\gamma} > 0, \quad \forall\, \varepsilon \quad \text{with } A \equiv \sum_i A_i \text{ and } \hat{e}_\varepsilon \equiv \sum_i \hat{e}_{i\varepsilon}.$$

This condition must hold because $A_i + \hat{e}_{i\varepsilon}/(1-\gamma) > 0 \quad \forall i, \varepsilon$ is required by the FOC (11) resp. (14). Let $\bar{e} \equiv \sum_i \bar{e}_i$, then $\hat{e}_\varepsilon = \varepsilon - \bar{e}$. Hence an equilibrium requires $A + (\varepsilon - \bar{e}) \ / \ (1-\gamma) > 0$. For the more important case $\gamma < 1$ this implies $\varepsilon \geq \bar{e} - A(1-\gamma)$, $\forall \ \varepsilon$. If $W_0 \equiv \sum_i W_{0i}$, then in equilibrium W_0 is the forward market value of the aggregate payoff ε, i.e. $W_0 = E(\varepsilon\pi(\varepsilon))$.

We first show that investor i's sharing rule relative to that of investor j is either concave, linear or convex.

Proposition 4 *: Consider two investors i and j who measure risk by a negative HARA-function with γ being the same for both. Then the following statements are equivalent:*

– Investor i's sharing rule is strictly convex [linear] [strictly concave] relative to that of investor j.

– Investor i's sharing constant is smaller than [equal to][greater than] that of investor j.

Proof. See Appendix D.

Proposition 4 illustrates our earlier claim that the sharing constants in the optimality condition (11) resp. (14) generate room for a larger variety of sharing rules. While in the EU-model with HARA-utility all investors have linear sharing rules, in the risk-value model one investor's sharing rule relative to another one's is concave, linear or convex. It is the difference between the sharing constants of both investors which determines convexity, linearity and concavity. If all sharing constants are the same, then linearity obtains as in the EU- model, and a representative investor exists. Conversely, heterogeneity of investors may be measured by differences in the sharing constants.

Corollary: *The convexity of investor i's sharing rule relative to that of investor j grows with the difference in the sharing constants $(s_j - s_i)$.*

Proof. Consider a sequence of investors such that $s_j - s_{j+1} = \Delta$ with Δ being a small positive number; $j = 1, ..., J-1$.. Then the convexity of $\hat{e}_{j+1}(\hat{e}_j)$ is positive. Hence the convexity of $\hat{e}_{j+k}(\hat{e}_j)$ must increase in k, and, hence, in $s_j - s_{j+k}$ ■

More insight can be gained by analysing the investor's sharing rule in terms of the gross return R rather than the payoff e. Since $R := e/W_0$, the risk function can also be written as [$\widehat{R} := R - \overline{R}$ with $\overline{R} = E(R)$ for an endogenous benchmark resp. $\overline{R}$ being the exogenous benchmark]

$$E[F(\widehat{R})] = \frac{1-\gamma}{\gamma} E\left(\frac{A}{W_0} + \frac{\widehat{R}}{1-\gamma}\right)^{\gamma}$$

except for the case of the exponential risk function ($\gamma = -\infty$). In this case $E[F(\widehat{R})] = E[exp(-BW_0\widehat{R})]$.

The investor minimizes his risk subject to the budget contraint $E(R\pi) = 1$ and the return constraint $E(R) \geq R^*$. The solution gives the efficient return sharing rule.

The ratio A/W_0 is crucial for risk measurement if $\gamma > -\infty$. Depending on the investor, the ratio A/W_0 might be independent of W_0 or not. It might also depend on the investor's past portfolio performance as hypothesized by *Barberis* and *Huang* (2001). Define

$$\frac{A^*}{W_0} = \frac{A}{W_0} - \frac{\overline{R}}{1-\gamma}.$$

Then

$$E[F(\widehat{R})] = -\frac{1-\gamma}{\gamma} E\left(\frac{A^*}{W_0} + \frac{R}{1-\gamma}\right)^{\gamma}. \tag{19}$$

Investors differ in terms of $W_0\ /A^*$ for $\gamma > -\infty$ resp. BW_0 for $\gamma = -\infty$ and the required expected return R^*. Hence the sharing constants of two investors differ because of differences in these parameters. Applying Lemma 1 shows that the sharing constant of an investor declines when W_0/A^* resp. BW_0 or the required expected return R^* increases.

This proves

Proposition 5 *: Given the pricing kernel, the difference between the sharing constants of investors i and j, $(s_i - s_j)$, is monotonically increasing in $(W_{0j}/A_j^* - W_{0i}/A_i^*)$ for $\gamma > -\infty$ resp. $(B_j W_{0j} - B_i W_{0i})$ for $\gamma = -\infty$ and $(R_j^* - R_i^*)$.*

Before we discuss the sharing rules of these investors, it is helpful to understand the impact of W_0/A^* resp. BW_0. The curvature of the risk function, $-f'(\widehat{R})/f(\widehat{R})$, similar to absolute risk aversion, increases monotonically in W_0/A^* resp. BW_0. Therefore we denote W_0/A^* resp. BW_0 as the investor's risk sensitivity. With $1 > \gamma > -\infty$, risk sensitivity is higher for a more aggressive investor, i.e. an investor, who uses a higher exogenous benchmark return or, in the case of an endogenous benchmark, demands a higher expected portfolio return.

Proposition 6 characterizes the sharing rules of investors.

Proposition 6 *: Consider two investors i and j who have HARA-risk functions with the same γ.*

a) Suppose that the risk sensitivity, $W_0\ /A^*$ for $\gamma > -\infty$ resp. BW_0 for $\gamma = -\infty$, *is higher for investor i than for investor j and/or she demands a higher expected portfolio return. Then investor i's sharing rule is strictly convex relative to that of investor j.*

b1) Suppose that for both investors the risk sensitivity is the same, but investor i demands a higher expected portfolio return. Then there exists some portfolio return R^1 such that

$$R_i < [=][>] R_j \quad \text{for} \quad R_j < [=][>] R^1.$$

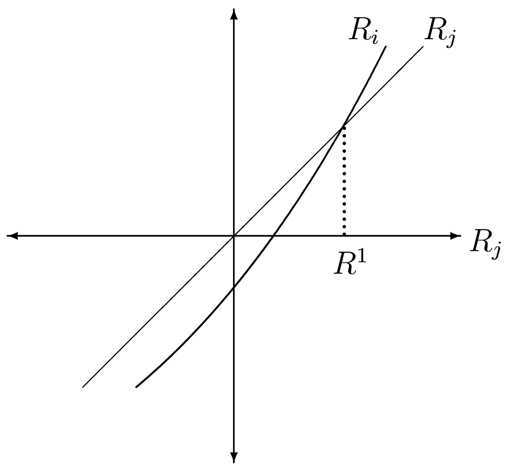
b2) Suppose that the risk sensitivity is higher for investor i but both investors demand the same expected portfolio return. Then there exist R^0 and R^{00} with $R^0 < R^{00}$ such that

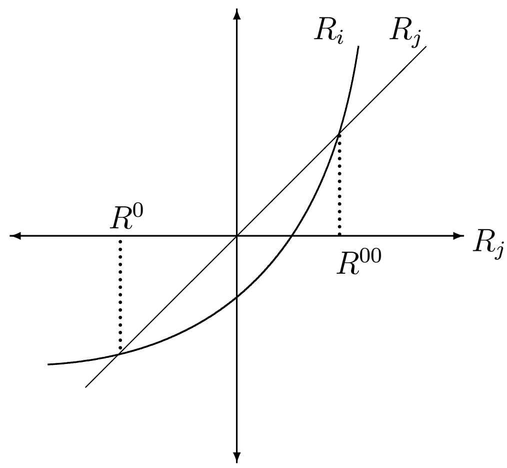
$$R_i > R_j \text{ for } \quad R_j < R^0 \text{ and } R_j > R^{00},$$

$$R_i < R_j \quad \text{ for } R^0 < R_j < R^{00}.$$

Proof. See Appendix E.

Figure 2: Relative sharing rules.





b1) Investor i demands higher expected portfolio return. His sharing rule is convex relative to that of investor j such that R_i and R_j intersect once.

b2) Investor i has a higher risk sensitivity. His sharing rule is convex relative to that of investor j such that both intersect twice.

Proposition 6 is illustrated in Figure 2. If two investors i and j have the same expected portfolio return and the same risk sensitivity, then their portfolio returns are the same in every state. If investor i has a higher risk sensitivity and/or demands a higher expected portfolio return, then her sharing rule is convex relative to that of the other investor. The special cases b1) and b2) will help to understand the intuition behind this result.

Proposition 6 b1) says that an investor i who ceteris paribus demands a higher expected portfolio return than investor j chooses a portfolio such that in the low states ($R_j < R^1$) her portfolio return is lower and in the high states ($R_j > R^1$) it is higher. Hence, a higher expected return forces her to take more risk. Now suppose, she would choose a sharing rule which is linear relative to that of the other investor but has higher slope (such as in EU-portfolio analysis). This would raise her risk dramatically because her return would fall strongly in the low states and the strict concavity of her marginal risk function would reinforce the impact on risk. Therefore, she rebalances her portfolio by buying more claims in the low states yielding a convex sharing rule relative to investor j. This result is consistent with the psychological evidence that risk perception is driven to a large extent by the worst outcomes. Therefore, investors who demand a higher expected return *buy* portfolio insurance.

Proposition 6 b2) states that, given the same expected return, the more risk sensitive investor chooses a portfolio such that her return is higher in the very low and in the very high states, but lower in between. The more risk sensitive investor is concerned, in particular, about large negative deviations of her return from the benchmark return in the very low states and, therefore, reduces these to reduce her risk. If she would buy less claims in all the higher states to pay for the additional claims in the low states, then this would lower her expected portfolio return. In order to avoid this, she buys more claims in the very high states in which claims are cheap.

Nonlinear sharing rules are also obtained in EU - equilibria based on HARA utility with distortions. *Grossman* and *Zhou* (1996) analyze the equilibrium for two investors who maximize expected utility. Suppose that both optimization problems are identical except that the second investor also makes sure that his random wealth never falls below a given floor. Then this investor buys options from the other one. *Benninga* and *Mayshar* (2000) prove a related result for an equilibrium with two investors who maximize their expected utility without a floor. Both have constant relative risk aversion, but at different levels. Then the investor with higher relative risk aversion buys options from the other investor. *Franke, Stapleton* and *Subrahmanyam* (1998) obtain a related result when investors have HARA-utility and face background risk.This risk destroys linearity of the sharing rules.A poor investor suffers more from a given level of background risk than a rich investor and, therefore, tends to buy portfolio insurance.

5 The Pricing Kernel and Investor Heterogeneity

5.1 General Risk Functions

In this section we analyze the pricing kernel in a risk-value equilibrium. We first investigate how in equilibrium the price for state-contingent claims is related to the aggregate payoff. Proposition 7 provides the result. Note that proposition 7 is true also if some investors have endogeneous and the others have exogenous benchmarks.

Proposition 7 *: In a risk-value equilibrium the probability-deflated price for state-contingent claims, π_ε, is decreasing and convex in the aggregate payoff ε.*[9]

Proof: The sum of the payoffs of individual investors is the aggregate payoff. Hence proposition 1 implies that the aggregate payoff is decreasing and convex in π. Therefore π must be decreasing and convex in the aggregate payoff ■

The result of proposition 7 does not come as a surprise. It is also obtained in an EU-equilibrium if every investor has a von Neumann-Morgenstern utility function with positive prudence.

[9] For very high aggregate payoffs the market price for state-contingent claims could be negative. This would violate the assumption of arbitrage-free markets. A negative price for some state-contingent claims requires, however, that every investor is beyond satiation in that state, a possibility that can be safely ignored (see also footnote 7).

5.2 HARA-Based Risk Functions

We now assume that all investors have HARA-based risk functions with the same γ. We will analyse the impact of heterogeneity of investors on the pricing kernel, in particular on the pricing of options. In the following, we, first, present a measure of investor heterogeneity. Second, we discuss the impact of investor heterogeneity on the convexity of the pricing kernel, and, third, on the level of prices for state-contingent claims and of option prices.

Two investors are said to be heterogeneous if their sharing constants differ. The more they differ, the more heterogeneous the investors are. Heterogenity of investors will be measured by the "variance" of their sharing constants.

Let $g_{i\varepsilon}$ denote the state-dependent share of investor i in the aggregate payoff ε, distorted by A_i and A.

$$g_{i\varepsilon} = \left(A_i + \frac{\hat{e}_{i\varepsilon}}{1-\gamma}\right) / \left(A + \frac{\hat{e}_{\varepsilon}}{1-\gamma}\right) \quad \text{for} \quad \gamma > -\infty,$$

$$g_{i\varepsilon} = g_i = \frac{1/B_i}{\sum_j 1/B_j} \quad \text{for} \quad \gamma = -\infty,$$

so that $\sum_i g_{i\varepsilon} = 1, \ \forall\varepsilon$. Moreover, define $V(\varepsilon)$, a hyperbolic "variance" measure of the sharing constants,

$$V(\varepsilon) \equiv \sum_i g_{i\varepsilon} \left(\frac{[\pi_\varepsilon - 1 + s_i]^{-1}}{\sum_j g_{j\varepsilon}[\pi_\varepsilon - 1 + s_j]^{-1}} - 1 \right)^2 \qquad (20)$$

This variance measure is endogenous since all terms are determined by the equilibrium. It basically measures the differences between the sharing constants which are determined by the differences in risk sensitivities and required portfolio returns (Proposition 5). If all sharing constants are the same, then this "variance" is zero. Thus, this "variance" tends to increase with differences among investors in risk sensitivities and required portfolio returns. Therefore we measure investor heterogeneity by $V(\varepsilon)$.

5.2.1 The Convexity of the Pricing Kernel

Next we show that the convexity of the pricing kernel increases monotonically in the measure $V(\varepsilon)$ of investor heterogeneity. Convexity $c(\varepsilon)$ is defined as

$$c(\varepsilon) \equiv -\frac{\pi''(\varepsilon)}{\pi'(\varepsilon)}$$

Proposition 8 *: Assume that every investor uses a risk function belonging to the HARA-class with γ being the same for every investor. Then for $\gamma > -\infty$ the convexity of the pricing kernel is*

$$\begin{aligned} c(\varepsilon) &= \left(A + \frac{\hat{e}_\varepsilon}{1-\gamma}\right)^{-1} \frac{\gamma-2}{\gamma-1}[1+V(\varepsilon)] \\ &= \left(A + \frac{\hat{e}_\varepsilon}{1-\gamma}\right)^{-1} \left[\frac{\gamma-2}{\gamma-1} + \frac{1}{\pi'(\varepsilon)} \sum_i s_i \frac{d^2 e_i}{d\varepsilon^2}\right]; \qquad \forall \varepsilon. \quad (21) \end{aligned}$$

For the exponential risk function ($\gamma = -\infty$) equation (21) holds with $[A + \hat{e}_\varepsilon/(1-\gamma)]^{-1}$ being replaced by $\left[\sum_j 1/B_j\right]^{-1}$.

Proof. See Appendix F.

Proposition 8 reveals the impact of investor heterogeneity on the convexity of the pricing kernel. By equation (21), the heterogeneity is reflected in the variance measure of the sharing constants which depends on ε. If all sharing constants are equal, then $V(\varepsilon) = 0$. Hence the convexity of the pricing kernel is minimal. The higher the variance is, the higher is the convexity of the pricing kernel.

Alternatively, the heterogeneity between investors is reflected in the convexities of their absolute sharing rules. The second part of equation (21) shows that the convexity of the pricing kernel is minimal if all sharing rules are linear. The sum $\sum_i s_i d^2 e_i / d\varepsilon^2$ is a weighted sum of the convexities $d^2 e_i / d\varepsilon^2$. If the weights s_i were the same across investors, then the sum would be zero. Hence it is the difference in weights which makes the sum nonzero. Proposition 5 tells us that the sharing constant s_i is determined by investor i's risk sensitivity and her required portfolio return. The higher these parameters are, the lower is her sharing constant, the more convex is her sharing rule. Consider, for example, an economy with two investors only. Investor i (h) has low (high) risk sensitivity, both demand the same portfolio return R^*. Then investor i (h) buys a concave (convex) sharing rule. Since

$$0 < s_h < s_i \text{ and } -d^2 e_i / d\varepsilon^2 = d^2 e_h / d\varepsilon^2, \quad \sum_j s_j d^2 e_j / d\varepsilon^2 < 0, \ \forall \varepsilon.$$

In general, the inverse relationship between s_i and $d^2 e_i / d\varepsilon^2$ across investors renders the sum $\sum_i s_i d^2 e_i / d\varepsilon^2$ negative. Hence, divided by $\pi'(\varepsilon)$, it is positive.

Interestingly, heterogeneity of investors always raises the convexity of the pricing kernel. This is intuitively appealing since investors being very risk sensitive and/or demanding a high portfolio return have a strong appetite for portfolio insurance, i.e. they have a strong appetite for claims in the very low and in the very high aggregate payoff-states. This makes these claims relatively more expensive.

In order to illustrate this, consider the case $1 > \gamma > -\infty$ and let investor h have the lowest sharing constant. She is very risk sensitive so that her

W_0/A^* is high. Since W_0/A^* is high, $-A^*\ /W_0$ is high. Hence in a state with a very low aggregate payoff, this investor has to buy enough claims so as to assure $R > -\ A^*(1-\gamma)/W_0$ and constrain her risk. Therefore she has to buy a substantial fraction of the available claims driving up the price of these claims. In the very high payoff-states she buys a very high fraction of the available claims. This can be seen by analyzing $g_{h\varepsilon}$, investor h's share of claims on the aggregate payoff.

Proposition 9 *: Assume* $1 > \gamma > -\infty$. *Let investor* h *be the unique investor with the lowest sharing constant. Then* $g_{h\varepsilon}$ *increases monotonically in* ε *and approaches 1 for* $\varepsilon \to \infty$.

Proof. See Appendix G.

Proposition 9 shows that investor h being very risk sensitive and/or demanding a high portfolio return buys a very high fraction of all claims in the very high-payoff states. In these states claims are relatively cheap enabling the investor to earn a high return.

Since $g_{h\varepsilon} \to 1$ for $\varepsilon \to \infty$, $V(\varepsilon) \to 0$ for $\varepsilon \to \infty$. Investor h dominates pricing in the high payoff-states so that the measure of investor heterogeneity converges to zero.

Similarly, in the case of exponential risk functions ($\gamma = -\infty$) it can be shown for investor h having the lowest sharing constant that $e_{h\varepsilon}/\varepsilon$ approaches 1 for $\varepsilon \to \infty$ and that $V(\varepsilon) \to 0$ for $\varepsilon \to \infty$.

$V(\varepsilon) \to 0$ also for very low values of ε for which $\pi_\varepsilon \to \infty$. Hence heterogeneity of investors has a small effect on the convexity of the pricing kernel in the low and in the high states and a large effect in the intermediate states.

It is interesting to compare the convexity in this risk-value equilibrium to that in an EU-equilibrium in which all investors use a HARA-utility function with the same exponent γ. Assume that $\gamma > -\infty$ and the sum $\sum_i A_i$ in the

EU-world equals $\left[\sum_i (A_i - \overline{R_i}/[1-\gamma])\right]$ in the risk-value world. If $\gamma = -\infty$, then asssume $\sum_i (1/B_i)$ to be the same in both worlds. Then the convexity of the EU-pricing kernel is the same as that of the risk-value pricing kernel if all sharing rules are linear in the risk-value equilibrium. Since sharing rules are linear in the EU-equilibrium anyway, this finding reinforces the importance of heterogeneity of investors: If in both worlds there exists a representative investor, i.e. if all sharing rules are linear, then the convexity of the pricing kernel is the same in both worlds. Otherwise the convexity is higher in the risk-value world because a representative investor does not exist.

5.2.2 The Level of Prices for State-Contingent Claims

In the previous section we have argued that the demand for portfolio insurance of some investors makes claims in the high and low aggregate payoff-states relatively more expensive. In this section, we wish to make a statement on the absolute levels of prices for state-contingent claims. This requires a calibration of the economy since, given the parameters A and γ, the measure of heterogeneity, $V(\varepsilon)$, does not uniquely determine the levels of prices.

A natural way of calibrating the economy is to consider the observable forward price of the aggregate payoff distribution, $E[\varepsilon\pi(\varepsilon)]$, as given and to analyze the pricing kernel for different levels of investor heterogeneity. The next proposition shows for the calibrated economy that more heterogeneity raises the prices of claims in low and high aggregate payoff-states.

Proposition 10 *: Assume that the measure of investor heterogeneity, $V(\varepsilon)$, increases everywhere, but the forward price of the aggregate payoff distribution stays the same. Let $\pi_1(\varepsilon)$ and $\pi_2(\varepsilon)$ denote the pricing kernels under the higher and the lower level of investor heterogeneity. Then there exist ε_0 and ε_{00} with $\varepsilon_0 < \varepsilon_{00}$ such that*

$\pi_1(\varepsilon) > \pi_2(\varepsilon) \quad \forall\, \varepsilon < \varepsilon_0$ and $\forall\, \varepsilon > \varepsilon_{00}$, and

$\pi_1(\varepsilon) < \pi_2(\varepsilon) \quad \forall\, \varepsilon \in (\varepsilon_0, \varepsilon_{00})$.

Proof. : Appendix H.

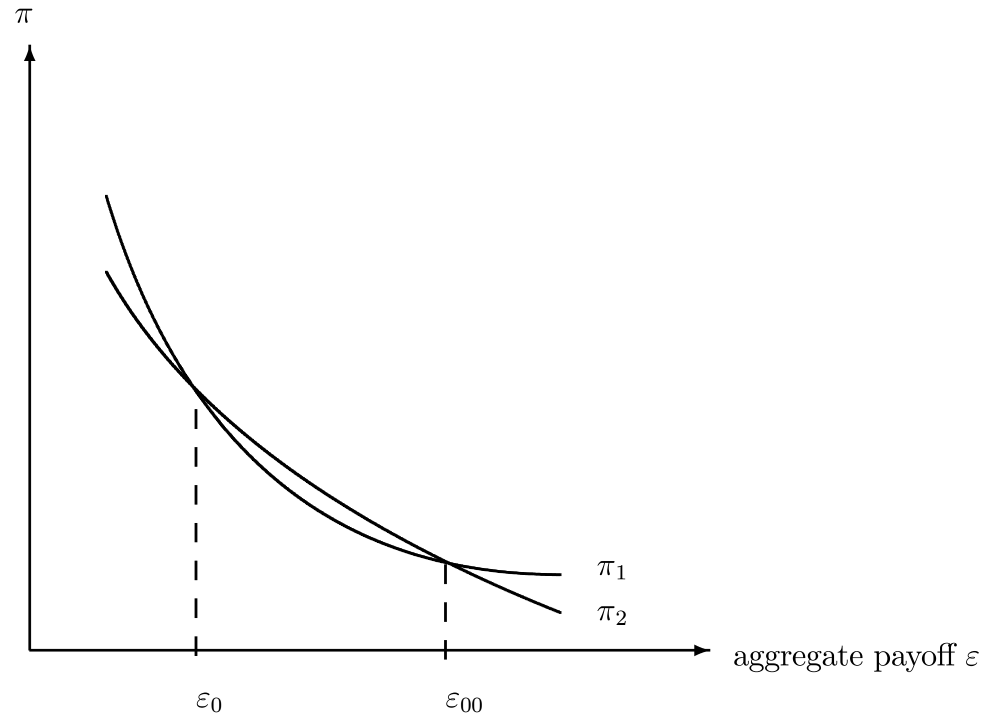


Figure 3: The pricing kernels. The pricing kernel π_1 under higher investor heterogeneity intersects the kernel π_2 under lower investor heterogeneity twice, first from above and then from below.

Proposition 10 is illustrated in Figure 3. It confirms our intuition on the impact of investor heterogeneity on the convexity of the pricing kernel. More heterogeneity implies more demand for portfolio insurance, in particular for claims in the low and in the high ε-states. This raises the prices of these claims. Since the expectation of the forward pricing kernel is always equal to 1, the prices for claims in the medium ε-states must decline. This is consistent with more convexity. The more heterogeneous investors are, the more dramatic are these pricing effects.

From Figure 3 it is apparent that an increase in investor heterogeneity shifts mass of the risk-neutral probability distribution of ε from the middle to the tails. Since, by assumption, the forward price of ε is given, the mean of the risk-neutral probability distribution does not change. Therefore the variance and the kurtosis of this distribution must increase with investor heterogeneity. This proves

Corollary: *Under the conditions of Proposition 10, an increase in investor heterogeneity raises the variance and the kurtosis of the risk-neutral probability distribution of the aggregate payoff.*

Figure 3 also shows that European calls on ε with a strike price above ε_{00} and European puts with a strike price below ε_0 are more expensive under pricing kernel $\pi_1(\varepsilon)$ as compared to $\pi_2(\varepsilon)$. Based on a result of *Franke, Stapleton* and *Subrahmanyam* (1999), the next proposition shows that this holds for all European options.

Proposition 11 *: Assume that the measure of investor heterogeneity, $V(\varepsilon)$, increases everywhere, but the forward price of the aggregate payoff distribution stays the same. Then the forward prices of all European options on the aggregate payoff increase.*

Proof. : Under the assumptions of Proposition 11, Proposition 10 holds. Hence, the proof of Proposition 11 is the same as the proof of Theorem 1 in *Franke/Stapleton/Subrahmanyam* (1999)■

The important result is that all European options are more expensive relative to the underlying asset, the more heterogeneous investors are. Stated differently, the more the sharing rules deviate from linearity, the higher is the investors' need for option trading, the more expensive are European options relative to the underlying asset. Hence investor heterogeneity might provide an explanation for the observation that stock index options appear to be more expensive than suggested by the Black/Scholes model [*Christensen* and *Prabhala* 1998]. Also, our results could help to explain that the risk-neutral probability distributions estimated from stock index option prices are leptokurtic and negatively skewed [*Longstaff* 1995, *Brenner* and *Eom* 1996, *Jackwerth* 2000].

5.3 Preliminary Empirical Evidence

In the following we provide preliminary empirical evidence on investor heterogeneity and option pricing. Investor heterogeneity manifests itself, among other things, in trading activity. In a one period-setting, the need for options should show up in option trading volume. Investors can, however, rebalance their portfolios almost continuously and trade the underlying and risk-free claims to synthesize option payoffs. Therefore it may be more promising to analyse trading activity in the underlying. In any case, more trading activity indicates stronger heterogeneity of investors.

There is no general measure of how expensive options are relative to the underlying. The option price depends on the risk adjusted probability distribution which is the product of the true probability distribution and the standardized pricing kernel. Given the latter the pricing of the option relative to the underlying can be measured correctly only if the true probability distribution is known. We circumvent this problem by referring to the Black-Scholes setting. We proxy for this pricing measure by the implied over the actual volatility, $s \equiv \sigma_{implied}/\sigma_{actual}$. Hence a simple test of the impact of investor heterogeneity on option pricing is a regression of s on trading volume. Since trading volume x grows over time, we include a linear time trend t. This leads to the following linear regression

$$s = a + bt + cx + \varepsilon$$

$\frac{Strike}{S\&P-500}$	Variable	Coefficient	t−Statistik	Prob	Adj. R^2
0.95	t	0,000174	4,917701	0,0000	0,162437
	X	4,73E-10	1,759755	0,0786	
1.00	t	3,31E-05	1,118702	0,2634	0,040166
	X	3,96E-10	1,836543	0,0664	
1.05	t	-4,53E-05	-1,552550	0,1207	0,028372
	X	6,77E-10	3,219382	0,0013	

Table 1: Results from OLS-regressions of implied over actual volatility on calendar time t and NYSE-trading volume x. For options with strike over S&P-500 prices of 0.95, 1.00 and 1.05, the regression coefficient for the time trend and the trading volume, the t-statistics based on Newey-West, the associated probabilities and the adjusted R^2 are displayed.

with ε being a noise term.

The theoretical results relate to the aggregate payoff. Therefore we analyse daily data on S&P-500 options, from March 29, 1988 to the end of 1995.[10] For every day the implied volatility is derived from the options closest to 60 days to maturity. Three options are distinguished, those whose strike over the actual S&P-500 price is closest to 0.95, to 1.00 and to 1.05, respectively, at 9 a.m. Chicago time. The daily implied volatility of such an option is the average calculated over the day. The actual volatility is derived from the returns of the S&P-500 over the next 60 calendar days, based on closing prices.

To proxy for investor heterogeneity, we first use the daily trading volume of all S&P-500 options at the CBOE. But no significant results were found. This is in line with Chan, Chung and Fong (2002) who found only little predictive ability for stock and option quote revisions of option net trading volume (buyer-initiated volume minus seller-initiated volume), but a much better predictive ability of stock net trading volume. Therefore we use,

[10]The data were provided very generously by Jens Jackwerth.

second, the daily NYSE-stock trading volume as a proxy for investor heterogeneity. This measure adds up the number of all traded stocks irrespective of stock prices. The results are presented in table I.

Table I suggests a positive impact of the NYSE-trading volume on the implied over actual volatility. The significance of the regression coefficient is highest for the options with strike over S&P-500 price of 1.05 and weakest for 0.95. This is surprising since many investors display a need for portfolio insurance which primarily shows up in the trade of low moneyness (0.95)-options. Hence we would expect the highest significance of the trade volume coefficient for these options. This argument may, however, be misleading for the observation period which was mostly bullish. The time trend displays a highly significant positive coefficient for 0.95, but otherwise the coefficient is not significant and also varies in sign. The explanatory power of the regression, measured by the adjusted R^2, depends strongly on the strike over S&P-500 price.

Although the results are consistent with our theoretical results, they should be interpreted with much caution. First, the results depend on the observation period. For some other observation periods, we find no significant regression coefficients of trading volume. Second, trading volume is a crude measure of trading activity since it aggregates the number of traded stocks irrespective of stock prices. Third, trading volume is driven by all kinds of investor heterogeneity including heterogeneous expectations, heterogeneous environments, and heterogeneous investment opportunity sets. Fourth, there may exist nonlinearities which are not taken into account. Therefore the evidence presented in table 1 can only be viewed as preliminary. More empirical research is needed.

6 Conclusion

The paper considers portfolio choice and asset pricing in a world where investors' preferences are modelled by a risk-value approach. We consider risk functions with an exogenous and with an endogenous benchmark. Both models yield similar results. In contrast to expected utility, risk-value models do not constrain the tradeoff between value and risk given a risk measure. This approach is consistent with the widely observable separation of value and risk in finance. We have defined properties of the risk measure based on empirical findings on risk judgments.

Looking at efficient individual sharing rules in risk-value models, the first order conditions display an additional term, the sharing constant. This constant generates a larger variety of sharing rules as compared to expected utility models. In the risk-value world with the risk function being a negative HARA-function, these sharing constants determine whether an investor's sharing rule is convex, linear or concave relative to that of another investor. Highly risk sensitive investors buy portfolio insurance from less risk sensitive investors. The more aggressive investors, i.e. those who demand a higher expected portfolio return, take more risk, but also buy portfolio insurance.

In the risk-value world the convexity of the pricing kernel increases with the heterogeneity of investors which is reflected in the non-linearity of their sharing rules. The more heterogeneous investors are, the more demand for option trading exists, the more convex is the pricing kernel, the more expensive are European options relative to the underlying asset, the higher are the variance and the kurtosis of the risk-neutral probability distribution. This is explained by the stronger demand of risk sensitive investors for claims in the low and high aggregate payoff-states which drives up the prices of these claims and lowers the prices of the other claims. Also the more aggressive investors buy portfolio insurance driving up the prices for claims in the low states.

These results suggest a need for further theoretical research on asset pricing and investor heterogeneity. In addition, more empirical research is needed.

Appendix A: Proof of Lemma 1

First, we consider an increase in W_0 to bW_0 holding the required expected return R^* constant; $b > 1$. Then η changes to $a\eta$; $a > 0$. Define $\hat{e}_1 := (e - \bar{e})$ for the initial endowment W_0 and define $\hat{e}_b := (e - \bar{e})$ for the initial endowment bW_0. Then we need to show that

$$\frac{E[f(\hat{e}_1)]}{-\eta} > \frac{E[f(\hat{e}_b)]}{-a\eta}$$

or

$$aE[f(\hat{e}_1)] > E[f(\hat{e}_b)]. \tag{22}$$

From equation (15) it follows that $\forall\ \varepsilon$,

$$-E[f(\hat{e}_b)] + f(\hat{e}_{b\varepsilon}) = a\eta\ \theta_\varepsilon = a(-E[f(\hat{e}_1)] + f(\hat{e}_{1\varepsilon})). \tag{23}$$

As the mean absolute deviation between payoffs across states has to grow with W_0, the monotonicity of f implies that also the mean absolute deviation $\mid E[f(\hat{e})] - f(\hat{e})] \mid$ has to grow. Hence $a > 1$. Now assume, by contradiction, that inequality (22) is not true. Then equation (23) implies

$$f(\hat{e}_{b\varepsilon}) \geq af(\hat{e}_{1\varepsilon}); \quad \forall\ \varepsilon. \tag{24}$$

As $a > 1$ and $f > 0$, this implies

$$f(\hat{e}_{b\varepsilon}) > f(\hat{e}_{1\varepsilon}); \quad \forall\ \varepsilon.$$

Since $f' < 0$, it follows that $\hat{e}_{b\varepsilon} < \hat{e}_{1\varepsilon}$ and hence

$$\hat{e}_{b\varepsilon} < \hat{e}_{1\varepsilon}; \quad \forall \, \varepsilon,$$

which contradicts the budget constraint (5). Therefore inequality (22) must be true.

Second, we consider an increase in R^* so that $\hat{e}_1$ changes to $\hat{e}_{b^o}$. Then η changes to $a^o\eta$. Hence the sharing constant decreases if inequality (22) holds with a and b being replaced by a^o and b^o. Therefore the same method by which the first part of Lemma 1 has been proven can be applied here. ■

Appendix B: Proof of Proposition 2

a) Sufficiency: Suppose that $F(\hat{e})$ is a quadratic function. Then $f(\hat{e}) = a + b\hat{e}$. Hence (17) implies linear relative sharing rules for two investors i and j. Therefore all absolute rules are also linear.

b) Necessity: Differentiate (15) with respect to ε; this yields

$$f_i'(\hat{e}_{i\varepsilon})\frac{de_i}{d\varepsilon} = \eta_i \, \frac{d\theta}{d\varepsilon} \tag{25}$$

Now suppose a linear absolute sharing rule for every investor: $\hat{e}_{i\varepsilon} = \alpha_i + \beta_i \, \varepsilon$, so that $\frac{de_i}{d\varepsilon} = \beta_i$. Then it follows from (25) for any two investors i and j

$$f_i'(\hat{e}_{i\varepsilon})\frac{\beta_i}{\eta_i} = f_j'(\hat{e}_{j\varepsilon})\frac{\beta_j}{\eta_j}; \quad \forall \, \varepsilon. \tag{26}$$

In the traditional state preference-model, the investor maximizes his expected utility $E\left[u\left(e\right)\right]$ subject to the budget constraint (5). Let μ denote the Lagrange-multiplier of the budget constraint. Then the first order conditions imply for two investors i and j

$$u_i'(e_{i\varepsilon})\frac{1}{\mu_i} = u_j'(e_{j\varepsilon})\frac{1}{\mu_j}; \quad \forall\ \varepsilon. \tag{27}$$

Cass and *Stiglitz* (1970) have shown that in the EU-model all investors can have linear sharing rules only if $u(e)$ belongs to the HARA class with the exponent γ being the same for all investors. Since (26) is formally the same as (27), it follows for the risk-value model that all investor can have linear sharing rules only if $f(\hat{e})$ belongs to the HARA class with $(\gamma - 1)$ being the same for all investors. Therefore we can rewrite (17) as

$$-\frac{1}{\eta_i}\left(A_i + \frac{\alpha_i + \beta_i\varepsilon}{1-\gamma}\right)^{\gamma-1} + \frac{1}{\eta_j}\left(A_j + \frac{\alpha_j + \beta_j\varepsilon}{1-\gamma}\right)^{\gamma-1} = s_i - s_j; \quad \forall\ \varepsilon.$$

Suppose that $s_i \neq s_j$. Then, the last equation can hold for every ε only if $\gamma = 2$. Hence $F(\hat{e})$ must be quadratic for every investor. ■

Appendix C: Proof of Proposition 3

Differentiating equation (16) with respect to e_j yields

$$\frac{f_i'(\hat{e}_i)}{-\eta_i}\frac{de_i}{de_j} = \frac{f_j'(\hat{e}_j)}{-\eta_j}. \tag{28}$$

Hence de_i/de_j is a constant if and only if $f_j'(\hat{e}_j)/f_i'(\hat{e}_i)$ is a constant. Then investor i's sharing rule is linear relative to that of investor j. Investor i's sharing rule is strictly convex [concave] relative to that of investor j if $f_j'(\hat{e}_j)/f_i'(\hat{e}_i)$ is strictly increasing [decreasing] in e_j and, hence, in the aggregate payoff ε. This result can be restated using the coefficient of the negative third to the second derivative of the F-function, $-f_i''(\hat{e}_i)/f_i'(\hat{e})$. This coefficient is called the coefficient of absolute prudence of the investor's risk function; it is the analogue to Kimball's coefficient of absolute prudence. Multiplying equation (28) by -1, taking logs and differentiating with respect to e_j yields

$$\frac{d\ ln(de_i/de_j)}{de_j} = -\frac{f_j''(\hat{e}_j)}{-f_j'(\hat{e}_j)} + \frac{f_i''(\hat{e}_i)}{-f_i'(\hat{e}_i)} \frac{de_i}{de_j}. \tag{29}$$

This proves the equivalence of the first two statements in Proposition 3. Substituting de_i/de_j in equation (29) from equation (28) shows that $d\ ln(de_i/de_j)/de_j >$ [=] [<] 0 if and only if (18) holds. ■

Appendix D: Proof of Proposition 4

For any HARA function,

$$\frac{f_i''(\hat{e}_i)}{[f_i'(\hat{e}_i)]^2} = \frac{\gamma - 2}{\gamma - 1}/f_i(\hat{e}_i).$$

Hence, by the last statement of Proposition 3, $e_i(e_j)$ is strictly convex [linear] [strictly concave] if and only if

$$\frac{f_i(\hat{e}_i)}{-\eta_i} < \ [=]\ [>] \frac{f_j(\hat{e}_j)}{-\eta_j}; \quad \forall\ e_j.$$

Hence, by equation (17), $e_i(e_j)$ is strictly convex [linear] [strictly concave] if and only if investor i's sharing constant is smaller than [equal to] [greater than] that of investor j. This proves Proposition 4. ■

Appendix E: Proof of Proposition 6

The proof will be shown for $\gamma > -\infty$. It is the same for $\gamma = -\infty$.

a) First, we prove statement a). Consider the minimisation of the objective function (19) s.t. $E[R\pi] = 1$ and $E(R) \geq R^*$.
Hence for two investors i and j with $A^*_{0i}/W_{0i} = A^*_{0j}/W_{0j}$ and $R^*_i = R^*_j$ it follows that the efficient portfolio returns are the same. By Lemma 1, investor i's sharing constant decreases when her initial endowment increases or when she demands a higher expected portfolio return. Then, by Proposition 4, her sharing rule is strictly convex relative to that of investor j.

b1) Now we prove statement b1). As $R_i(R_j)$ is strictly convex, the curves $R_i(R_j)$ and $R_j(R_j)$ can intersect at most twice. They have to intersect at least once, otherwise the budget constraint $E[R\pi] = 1$ cannot hold for both. Hence we have to show that two intersections are impossible. Equation (28) yields in the HARA-case, starting from objective function (19),

$$\frac{dR_i}{dR_j} = \left(\frac{\frac{A_i^*}{W_{0i}} + \frac{R_i}{1-\gamma}}{\frac{A_j^*}{W_{0j}} + \frac{R_j}{1-\gamma}} \right)^{2-\gamma} \frac{\eta_i}{\eta_j}.$$

At an intersection $R_i = R_j$ so that $W_{0i}/A^*_i = W_{0j}/A^*_j$ implies that the bracketed term equals 1. Hence the slope dR_i/dR_j at an intersection is unique. Therefore convexity of $R_i(R_j)$ rules out two intersections. Given one intersection at $R_j = R^1$, a higher expected portfolio return can be obtained only if in the states $R_j < R^1$ relatively expensive claims are sold and in the states $R_j > R^1$ relatively cheap claims are bought. Thus $dR_i/dR_j > 1$ for $R_j = R^1$.

b2) Finally we prove statement b2). The strict convexity of $R_i(R_j)$ implies that $R_i(R_j)$ and $R_j(R_j)$ can intersect at most twice. $R_i^* = R_j^*$ requires at least one intersection. Suppose there exists one intersection only. As $\pi(\varepsilon)$ is monotonically decreasing, purchasing claims in some low states $\varepsilon < \varepsilon^+$ and selling claims in some high states $\varepsilon > \varepsilon^+$ such that this transaction is self-financing implies a lower expected return. Thus, one intersection contradicts $R_i^* = R_j^*$ so that two intersections are implied. ■

Appendix F: Proof of Proposition 8

Differentiate the first order condition (15) with respect to ε. This yields

$$f_i'(\hat{e}_{i\varepsilon})\frac{de_i}{d\varepsilon} = -\eta_i \pi'(\varepsilon); \quad \forall\ i, \varepsilon. \tag{30}$$

Multiply this equation by -1, take logs and differentiate with respect to ε. This yields

$$\frac{f_i''(\hat{e}_{i\varepsilon})}{f_i'(\hat{e}_{i\varepsilon})}\ \frac{de_i}{d\varepsilon} + \frac{d^2 e_i/d\varepsilon^2}{de_i/d\varepsilon} = -c(\varepsilon); \quad \forall\ i, \varepsilon. \tag{31}$$

First, we prove the first part of equation (21). Multiply equation (31) by $de_i/d\varepsilon$ and aggregate across investors. This yields

$$\sum_i \frac{f_i''(\hat{e}_{i\varepsilon})}{f_i'(\hat{e}_{i\varepsilon})}\ \left(\frac{de_i}{d\varepsilon}\right)^2 = -c(\varepsilon); \quad \forall\ \varepsilon, \tag{32}$$

since $\sum_i d^2 e_i/d\varepsilon^2 = 0$.

In the HARA-case with $\gamma > -\infty$,

$$\frac{f_i''(\hat{e}_{i\varepsilon})}{-f_i'(\hat{e}_{i\varepsilon})} = \frac{\gamma-2}{\gamma-1}\left(A_i + \frac{\hat{e}_{i\varepsilon}}{1-\gamma}\right)^{-1} = \frac{\gamma-2}{\gamma-1}\,\frac{-f_i'(\hat{e}_{i\varepsilon})}{f_i(\hat{e}_{i\varepsilon})};\quad \forall\, i,\varepsilon \tag{33}$$

Hence

$$c(\varepsilon) = \frac{\gamma-2}{\gamma-1}\sum_i \left(A_i + \frac{\hat{e}_{i\varepsilon}}{1-\gamma}\right)\left(\frac{de_i/d\varepsilon}{A_i + \frac{\hat{e}_{i\varepsilon}}{1-\gamma}}\right)^2;\quad \forall\, \varepsilon \tag{34}$$

Divide (30) by $f_i'(\hat{e}_{i\varepsilon})$ and aggregate. This yields

$$1 = \pi'(\varepsilon)\sum_i -\eta_i/f_i'(\hat{e}_{i\varepsilon}) = \pi'(\varepsilon)\sum_i \left(A_i + \frac{\hat{e}_{i\varepsilon}}{1-\gamma}\right)(\eta_i/f_i(\hat{e}_{i\varepsilon})) \tag{35}$$

Now divide (30) by $f_i(\hat{e}_{i\varepsilon})$. This and (35) yield

$$\frac{f_i'(\hat{e}_{i\varepsilon})}{f_i(\hat{e}_{i\varepsilon})}\,\frac{de_i}{d\varepsilon} = \frac{-\eta_i}{f_i(\hat{e}_{i\varepsilon})}\pi'(\varepsilon) = \frac{-\eta_i/f_i(\hat{e}_{i\varepsilon})}{\sum_j \left(\eta_j/f_j(\hat{e}_{j\varepsilon})\right)\left(A_j + \frac{\hat{e}_{j\varepsilon}}{1-\gamma}\right)} \tag{36}$$

The left hand side of this equation, multiplied by -1, equals the term in the squared bracket of (34). Hence it follows that

$$c(\varepsilon) = \frac{\gamma-2}{\gamma-1}\sum_i \frac{A_i + \frac{\hat{e}_{i\varepsilon}}{1-\gamma}}{\left[\sum_j \left(\eta_j/f_j(\hat{e}_{j\varepsilon})\right)\left(A_j + \frac{\hat{e}_{j\varepsilon}}{1-\gamma}\right)\right]^2}\left(\frac{-\eta_i}{f_i(\hat{e}_{i\varepsilon})}\right)^2;\quad \forall\, \varepsilon. \tag{37}$$

Define $g_{i\varepsilon} \equiv \left(A_i + \frac{\hat{e}_{i\varepsilon}}{1-\gamma}\right) / \left(A + \frac{\hat{e}_{\varepsilon}}{1-\gamma}\right)$ $\forall\, i,\varepsilon$ so that $\sum_i g_{i\varepsilon} = 1\ \forall\, \varepsilon$.

Then (37) yields

$$
\begin{aligned}
c(\varepsilon) &= \frac{\gamma-2}{\gamma-1}\left(A+\frac{\hat{e}_\varepsilon}{1-\gamma}\right)^{-1}\sum_i g_{i\varepsilon}\left(\frac{-\eta_i/f_i(\hat{e}_{i\varepsilon})}{\sum_j g_{j\varepsilon}(-\eta_j/f_j(\hat{e}_{j\varepsilon})}\right)^2 \\
&= \frac{\gamma-2}{\gamma-1}\left(A+\frac{\hat{e}_\varepsilon}{1-\gamma}\right)^{-1}[1+V(\varepsilon)]\,; \quad \forall\,\varepsilon, \qquad (38)
\end{aligned}
$$

with $V(\varepsilon)$ being defined in equation (20). The latter follows from the FOC $f_i(\hat{e}_{i\varepsilon})/-\eta_i = \pi_\varepsilon - 1 + s_i;\ \forall i$. This proves the first part of equation (21).

Now we we prove the second part of equation (21). Substitute $f_i''(\hat{e}_{i\varepsilon})/f_i'(\hat{e}_{i\varepsilon})$ in equation (31) from (33) and multiply the equation by $-(A_i + \hat{e}_{i\varepsilon}/(1-\gamma))$. This yields

$$
\frac{\gamma-2}{\gamma-1}\frac{de_i}{d\varepsilon} - \frac{A_i+\hat{e}_{i\varepsilon}/(1-\gamma)}{de_i/d\varepsilon}\,\frac{d^2e_i}{d\varepsilon^2} = c(\varepsilon)\left(A_i+\frac{\hat{e}_{i\varepsilon}}{1-\gamma}\right); \quad \forall\, i,\varepsilon. \qquad (39)
$$

The factor of $d^2e_i/d\varepsilon^2$ equals the inverse first term in equation (36). Hence it follows from equation (36) and (15) that this factor equals

$$
\frac{f_i(\hat{e}_{i\varepsilon})}{-\eta_i}\,\frac{1}{\pi'(\varepsilon)} = \frac{-1+\pi_\varepsilon+s_i}{\pi'(\varepsilon)}. \qquad (40)
$$

Insert (40) in (39) and obtain after aggregation across investors since $\sum_i d^2e_i/d\varepsilon^2 = 0$,

$$
\frac{\gamma-2}{\gamma-1} + \sum_i \frac{s_i}{\pi'(\varepsilon)}\, d^2e_i/d\varepsilon^2 = c(\varepsilon)\ \left(A+\frac{\hat{e}_\varepsilon}{1-\gamma}\right); \quad \forall\,\varepsilon \qquad (41)
$$

This proves the second part of equation (21). The proof of proposition 9 is the same for $\gamma = -\infty$. ■

Appendix G: Proof of Proposition 9

For a HARA-based risk function with $1 > \gamma > -\infty$ the FOC (11) resp. (14) yields

$$\left(A_i + \frac{\hat{e}_{i\varepsilon}}{1-\gamma}\right)^{\gamma-1} = -\eta_i[\pi_\varepsilon - 1 + s_i]; \quad \forall\, i, \varepsilon, \tag{42}$$

or

$$A_i + \frac{\hat{e}_{i\varepsilon}}{1-\gamma} = (-\eta_i[\pi_\varepsilon - 1 + s_i])^{\frac{1}{\gamma-1}}; \quad \forall\, i, \varepsilon. \tag{43}$$

Aggregating across investors yields $A + \hat{e}_\varepsilon/(1-\gamma)$. Dividing the aggregate equation by $A_i + \hat{e}_{i\varepsilon}/(1-\gamma)$ yields $1/g_{i\varepsilon}$ and hence,

$$1/g_{i\varepsilon} = \sum_j \left(\frac{-\eta_i[\pi_\varepsilon - 1 + s_i]}{-\eta_j[\pi_\varepsilon - 1 + s_j]}\right)^{\frac{1}{1-\gamma}} \tag{44}$$

so that

$$\frac{d(1/g_{i\varepsilon})}{d\varepsilon} = \sum_j \frac{1}{1-\gamma}\left(\frac{-\eta_i}{-\eta_j}\right)^{\frac{1}{1-\gamma}} \left(\frac{\pi_\varepsilon - 1 + s_i}{\pi_\varepsilon - 1 + s_j}\right)^{\frac{1}{1-\gamma}-1} \frac{s_j - s_i}{(\pi_\varepsilon - 1 + s_j)^2}\, \pi'(\varepsilon). \tag{45}$$

Assume $\gamma < 1$ and $s_h \leq s_j\ \forall i$. Then $\pi'(\varepsilon) < 0$ implies $d(1/g_{h\varepsilon})/d\varepsilon < 0$ so that $dg_{h\varepsilon}/d\varepsilon > 0$.

It remains to be shown for $\gamma < 1$ that $g_{h\varepsilon} \to 1$ for $\varepsilon \to \infty$. For $\varepsilon \to \infty$, $\hat{e}_{i\varepsilon} \to \infty$ for at least one investor. By equation (43), $\hat{e}_{i\varepsilon} \to \infty$ if $\pi_\varepsilon - 1 + s_i \to 0$. Since $\pi_\varepsilon - 1 + s_i > 0\ \forall i, \varepsilon$ and $s_h < s_j\ \forall j,\ j \neq h, \pi_\varepsilon - 1 + s_i \to 0$ for $\varepsilon \to \infty$ must hold for $i = h$ only. Hence $\hat{e}_{h\varepsilon} \to \infty$ and, therefore, by equation (44), $g_{h\varepsilon} \to 1$ for $\varepsilon \to \infty$. It follows from $\pi_\varepsilon - 1 + s_h \to 0$ for $\varepsilon \to \infty$ and $\pi_\varepsilon \geq 0$ that s_h must not exceed 1. ■

Appendix H: Proof of Proposition 10

Franke/Stapleton/Subrahmanyan (1999) have shown that the pricing kernels $\pi_1(\varepsilon)$ and $\pi_2(\varepsilon)$ must intersect at least twice to produce the same forward price of the underlying asset. More than two intersections will be shown to be impossible. By Proposition 8, an increase in $V(\varepsilon)$ raises convexity $c(\varepsilon)$.

$c_1(\varepsilon) > c_2(\varepsilon)\ \forall\varepsilon$ implies

$$-\frac{dln\left[-\pi_1'(\varepsilon)\right]}{d\varepsilon} > -\frac{dln\left[-\pi_2'(\varepsilon)\right]}{d\varepsilon}$$

$$or \qquad -\frac{dln[-\pi_1'(\varepsilon)/-\pi_2'(\varepsilon)]}{d\varepsilon} > 0$$

so that $-\pi_1'(\varepsilon)/-\pi_2'(\varepsilon)$ must decline in ε everywhere. Suppose that $-\pi_1'(\varepsilon)/-\pi_2'(\varepsilon) > 1$ at the first intersection of $\pi_1(\varepsilon)$ and $\pi_2(\varepsilon)$. Hence it must be less than 1 at the second intersection and greater than 1 at the third intersection. But the latter condition contradicts the condition that $-\pi_1'(\varepsilon)/-\pi_2'(\varepsilon)$ declines. Hence a third intersection cannot exist. Also $\pi_1(\varepsilon) > \pi_2(\varepsilon)$ before the first intersection and after the second intersection and vice versa in between follows. ■

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