DISCUSSION PAPER SERIES

No. 3670

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INDUSTRIAL ORGANIZATION

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ISSN 0265-8003

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Discussion Paper No. 3670
December 2002

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CEPR Discussion Paper No. 3670

December 2002

ABSTRACT

The Variable Value Environment: Auctions and Actions*

This Paper introduces and formally models the variable value environment, where bidders' private values may change over time as a result of both private actions and exogenous shocks. Examples of private actions and exogenous shocks are complementary investments and exogenous changes in bidder's business, respectively. We study mechanisms that lead to efficient allocations, i.e. those in which the final value of the object to the winning bidder, net of the total cost of private actions undertaken by all agents, is maximized. We characterize the first best allocation, and propose a mechanism that yields the first best allocation in equilibrium. This mechanism has an inefficient pooling equilibrium along with an efficient separating equilibrium. To rule out the pooling equilibrium, we introduce a class of almost efficient mechanisms that force players to coordinate on the separating equilibrium.

JEL Classification: D44 and D80
Keywords: auctions and variable value environment

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*The authors are grateful to Rick Ericson, Drew Fudenberg, Paul Milgrom, Anna Mikusheva, Eric Rasmusen, Mike Riordan, Ilya Segal, Eric Stout, Robert Townsend, seminar participants at Harvard/MIT, Columbia, and New Economic School in Moscow, and participants of the Summer Meeting of the Econometric Society 2001 and the European Economic Association Congress 2001 for various helpful comments.

Submitted 15 October 2002

1 Introduction

If an object is available for use at some future date, what is an efficient mechanism for selling it? For example, consider a sale of a hypothetical military base that is scheduled to close in twenty years. Waiting with the sale for twenty years and auctioning off the base immediately before it becomes available probably creates an inefficiency, because the winning bidder might have missed opportunities to invest in assets complementary to the base ownership. In other words, *private actions* that a bidder chooses prior to the actual sale may influence her private value. If a bidder thinks that her likelihood of winning the object is sufficiently low, she would choose not to take costly actions that increase her valuation of the object.[1] On the other extreme, selling the base twenty years before it becomes available seems absurd, because the expected value of the object for each bidder is likely to change over time. Thus, efficiently allocating an object a long time before it becomes available for use is an unlikely possibility. In other words, as long as there are privately observed *exogenous shocks* to private values that are revealed over time, auctioning off the object well in advance (before the exogenous shocks are observed) may be inefficient.

Changes in private values due to *private actions* and *exogenous shocks* are ubiquitous. We will refer to an auction environment, where individual values may change over time as a result of both private actions and exogenous shocks as the *variable value environment.* Private actions, for instance complementary investments, change the private value of an object; and so do exogenous shocks ranging from changes in demand and input prices to changes in tax laws and regulatory environment. A sale of almost any object or service available for use at some known future date is an example of the variable value environment, ranging from leasing a building or a military base to renting a dance club for New Year's Eve.

A sale of an object in a market where search is important inevitably has elements of the variable value environment. Consider the sale of house A that is scheduled to be auctioned off in 20 days. Before house A becomes available bidders may have opportunities to buy other

[1] Alternatively, a bidder may be able to take a free action that increases her value for the object, but decreases her utility from not getting the object. For example, a bidder may sign a contract that is very profitable if the base belongs to the firm, but unprofitable otherwise. Formally, a costly action increasing the value of the asset is equivalent to a free action that lower the reservation utility conditional on not getting an object.

houses, essentially removing themselves from the market. Thus, we can consider an action consisting of "not buying some other reasonably priced house" as an action that boosts the value of the house. A second price auction (or any other single-round auction mechanism) is bound to be inefficient in a market with search because in the variable environment information revelation is necessary for achieving efficiency. The model of the variable value environment not only explains why auctions are rare in markets where search matters, the theory of the variable value environment developed herein also offer insight into auction design for these markets.[2]

One may argue that as long as there is resale opportunity the object should be auctioned off as early as possible and then "the market will allocate it efficiently". However, this argument is flawed on two counts: First, in a generic case, there exists no efficient resale mechanism (Myerson and Satterthwaite, 1983). Second, and most importantly for the current analysis, selling the object early counting on the original buyer to re-allocate the object simply moves the burden of designing an allocation mechanism from one party to the other. To the best of our knowledge, a variable value environment has never been introduced or investigated in the economic literature. The time dimension central for the variable value environment is essentially absent from the auction literature. Although in many auction models bidders use bid history for updating their beliefs, this does not introduce time dimension into the auction environment. Indeed, the multi-period updating is due to a mechanism selected for the auction, and it is the process of the auction that influences the bidders values in these models and not the passage of time. An auction where participants bid for packages of goods in multiple rounds is another example of an auction where a time line is a part of the mechanism but not a part of the auction environment (for a recent example of such a mechanism, see, e.g., Milgrom, 2000, Perry and Reny, 1998).

[2]Even a sale of consumer items via Internet auction houses may have a variable value component. A consumer who considers bidding for an object on an auction that ends in five days may take private actions that influence her private value of the object. For example, a consumer considering bidding for some item may forego opportunities to bid for other similar or complimentary items. Possibly, Internet auction houses such as e-Bay and Amazon incorporated features that allow bidders to buy an object instantly at a sufficiently high price (set by the seller) in order to avoid some of the inefficiencies of using second price auction in the variable value environment. Of course, the use of Buy-It-Now feature in Internet auctions might, perhaps, be explained by factors other than variable value features of the environment. Still, this seems to be a natural explanation.

Auctions with entry costs are related to the variable value environments: Mathematically, decision to pay for costly entry into an auction is equivalent to an action that boosts the private value of the agent. However, there is a number of significant differences. Most importantly, an entry fee is a feature of the mechanism, rather than of the environment. In contrast, ability of agents to take actions influencing the value of the object is a feature of the environment. Entry costs are investigated in various contexts by Milgrom (1981), McAfee and McMillan (1987), Riordan and Sappington (1987), Levin and Smith (1994), Fullerton and McAfee (1999), and Lixin Ye (2000).

The present work attempts to offer insight into auction design for the variable value environment. We start by proposing a formal model of the variable value environment with three periods. In the first period, each party receives a private signal s about its private value for the object. In the second period, a party can take a private, unobservable action at cost c (cost) that increases the value of the object by b (benefit). In the third period, bidders receive independent exogenous shocks t to their private values of the object. For ith bidder, the final reservation price for the object is a function of signals (s_i and t_i) regarding the value obtained in the first and third periods plus the benefit b from taking an action if the bidder took the action ($V(s_i, t_i) + b - c$). Otherwise, the private value of the object is $V(s_i, t_i)$. The identity of the bidder with the highest value can only be established after the third period signals t_i are observed. Thus, an efficient allocation mechanism requires that the ownership of the object is assigned after the third period. Conducting a second price auction after the third period is not efficient (in a world where first period signals are privately observed), since it forces agents to take decisions regarding second period actions in ignorance of the expected private values of other agents. The following example illustrates this simple but essential point.

Example. Let the number of participants be $N = 2$, assume that s_i are privately observed signals independently drawn from the uniform distribution on $[0, 1]$. For simplicity, assume that there is no third period signal, $t_i \equiv 0$.[3] In a symmetric equilibrium with no revelation of

[3] In the special case of all t_i's equal to zero, an efficient allocation rule can be implemented by assigning the ownership of the object by conducting a Vickrey auction at the end of the first period after s_i's are privately learned. Also note that for any non-degenerate distribution of third-period signals, assigning the ownership of the object at the end of the first period is no longer efficient. Of course, the inefficiency of allocating the object at the end of the third period demonstrated by the example does not go away when t_i's are not equal to zero.

the first-period signals, each agent acts if her probability of winning conditional on her own type is higher than $\frac{c}{b}$. That is, agent i acts if $s_i \geq s^* = \frac{c}{b}$. If $s^* = \frac{3}{4}$, then with probability $\frac{1}{16}$ both agents act (which is inefficient), and with probability $\frac{9}{16}$ no agent acts (which is inefficient as well). Therefore, on average there are too few actions ($\frac{1}{2}$ instead of 1). If $s^* = \frac{1}{4}$, the situation is reverse: with probability $\frac{9}{16}$ both agents act, and with probability $\frac{1}{16}$ no agent acts. On average, there are too many actions ($\frac{3}{2}$ instead of 1). This is hardly surprising: without signaling, there are too few actions, when actions are relatively costly ($\frac{c}{b} = \frac{3}{4}$), and there are too many actions, when actions are relatively cheap ($\frac{c}{b} = \frac{1}{4}$).

We consider a problem of designing an efficient mechanism for allocating the object in the environment where signals about bidder's private values (s_i, and t_i) are private signals. First, the concept of allocation needs to be generalized for the variable value environment. For the variable value environment an allocation is defined as the identity of the bidder who receives the object and the list of private actions taken by bidders. The objective of the social planner is to implement an efficient allocation, i.e. to maximize the social surplus, which equals to the expected sum of all bidder's surpluses net of the cost of actions. After the social planner observes the first-period signals s_i obtained by bidders, she has to decide which bidders should act in the second period and which should abstain from actions. Since the exogenous shock of the third period is not known in the second period (when decisions to take actions are made), it may be efficient to have more than one bidder taking an action or to have no bidders at all taking actions.[4] Theorem 1 establishes that if the social planner orders an agent with the first-period value s_i to act, then she also orders all agents with value greater than s_i to act.

Of course, an all-knowing and well-intentioned social planner is rarely available in the real world. What happens if there is no social planner but all the information is common knowledge, i.e. signals obtained by a bidder about her private value are observed by all players? Theorem 2 establishes that the efficient allocation can be achieved in a decentralized case. (This is the same first-best allocation that can be achieved by the social planner.)

The above mentioned results rely on bidders' private values being common knowledge. A more realistic case, where bidders privately observe their valuations, is of primary interest.

[4] Indeed, for a given distribution of the third-period exogenous shock, it becomes inefficient for anybody to undertake an action as the cost of action approaches the benefit. On the other extreme, if the cost of action approaches zero it becomes efficient for more than one agent to undertake an action.

Can an efficient allocation be achieved in that case? It is straightforward that an efficient allocation can not be attained without revelation of bidders' private signals (s_i) prior to the second period. If the object is allocated to the bidder with the highest value following the third period (using, say, a second price sealed bid auction) adding a cheap talk stage following the first period will not result in any information revelation and thus would lead to an inefficient outcome (allocation).[5] In the cheap talk stage each bidder would claim to be 'the high type' because the higher is the perceived type of a bidder, the less likely are the other bidders to undertake actions and thus the lower are the subsequent bids for the object by other players. Theorem 3 and Theorem 5 show that there exists an efficient mechanism, where private information is revealed in the first round and the object is assigned in the second round. The first round takes place after private signals s_i are received by agents. In the first round bidders reveal their private signals s_i by making payments (we show that the higher is the private signal s_i, the higher is the agent's willingness to pay for reporting to other agents that the value of her private signal s_i is high). The second round consists of a second price sealed bid auction conducted after signals t_i are received.

As long as private signals s_i are truthfully revealed in the first round, the subgame corresponding to the second round is identical to the complete information game. Theorem 3 establishes that the mechanism described above has an efficient separating equilibrium. Unfortunately, this mechanism also has an inefficient pooling equilibrium. To rule out the pooling equilibrium, we propose a class of mechanisms that force players to coordinate on the separating equilibrium. We refer to mechanisms from this class as "$\varepsilon-$efficient mechanisms." We prove that one can always choose an $\varepsilon-$efficient mechanism which yields an efficient allocation with probability arbitrarily close to one. An $\varepsilon-$efficient mechanism consists of two rounds. The first round takes place after private signals s_i are received by agents: a non-transferable discount for amount ε is sold via a sealed-bid all-pay auction. After the all-pay auction all bids are made public. The ε discount can only be used in the second round auction. In the second round the object is sold using a Vickrey auction (if the winner of the Vickrey auction is a holder of the ε discount, she pays the second highest bid minus

[5]The condition that the object is allocated to the bidder with the highest value following the third period is a necessary, but not sufficient condition for efficient allocation in the variable value environment. This is because efficiency of an allocation depends on the set of players that take actions in the second period. As we mentioned before, the winning bidder might have forgone investment opportunities enhancing the value of the object.

ε). Note that for $\varepsilon = 0$ this mechanism is identical to the efficient mechanism described above. Theorem 6 shows that an arbitrarily small positive ε forces agents to coordinate on a separating equilibrium that yields an efficient allocation with probability converging to one as ε converges to zero. In spirit, this mechanism is very close to virtual implementation, a pure theoretical concept in mechanism design (e.g., Maskin and Sjostrom, 1999). The proposed mechanism could easily be used in real large-stake auctions, so it gives a first example of a real-world implementable virtual implementation mechanism.

The rest of the paper is organized as follows. In Section 2, we introduce the formal model of the variable value environment. In Section 3, an efficient mechanism that has a fully separating Bayesian-Nash equilibrium is described. Section 4 introduces the $\varepsilon-$efficient mechanism and establishes that it has a unique robust equilibrium. Section 5 concludes. The Appendix contains mathematical details and proofs.

2 The Environment

The variable value environment is an environment, where bidders' expected value of the object changes stochastically over time. This paper focuses on a three period model: it is the simplest possible model that exhibits essential features of the variable value environment, and offers insights into auction design in such an environment. There are N identical agents. In the first and the third periods, agents receive independent signals about their private values of the object. In the second period, each agent has an opportunity to take a costly action that increase her private value of the object.

Timing

Period 1. Each agent receives a signal $s_i \geq 0$ about her private values, drawn independently from the same atomless distribution.

Period 2. Each agent i has an opportunity to take an unobservable action, i.e. choose $a_i \in \{0, 1\}$, which increases the agent's private value by ba_i and costs $ca_i \geq 0$ (obviously, only the case of $b > c$ is of interest). When $a_i = 1$ we say that the agent i undertakes the action or simply 'acts'; if $a_i = 0$ we say the agent i abstains from acting or skips the action.

Period 3. Agents receive independent signals $t_i \geq 0$ about their private values. We assume that a higher first-period signal s_i makes a higher second-period t_i more likely. Formally, if

$s_i > s_i'$, then the distribution of t_i conditional on s_i stochastically dominates the distribution of t_i conditional on s_i'.[6]

Agent's i private value of the object equals $V_i = V(s_i, t_i)$, which depends on her first and third period signals plus the benefit from taking an action. Thus the utility of the agent is given by:

$$U_i = \begin{cases} V(s_i, t_i) + (b-c)a_i - p_i, & \text{if the agent } i \text{ wins the object} \\ -ca_i - p_i, & \text{otherwise,} \end{cases}$$

where p_i denotes the total amount of payments made by the agent i within a mechanism (i.e. not including c).[7] Note that p_i need not be equal to zero for loosing bidders. We assume that $V(s_i, t_i)$ is continuous and increases in both arguments.

To explore possibilities of efficiently allocating the object within the model, we need to extend the concepts of allocation, efficiency, and social surplus to the variable value environment. *Social surplus* is the value of the object to the agent that gets the object minus the cost of actions taken by all agents: $S = V(s_j, t_j) + ba_j - \sum_{i=1}^{N} ca_i$, where j is the identity of the agent that receives the object. An *allocation* is a vector consisting of the list of agents who took actions and the identity of the agent who received the object. An allocation needs to specify the identities of agents who took actions because actions affect the social surplus. An equilibrium strategy profile of a mechanism (e.g., an auction) is referred to as an *allocation rule.* If a mechanism has multiple equilibria, each equilibrium strategy profile defines an allocation rule.

Any allocation rule induces a probability distribution over values of social surplus induced by a mechanism or by a social choice rule adopted by the social planner. Allocation rules can be ranked in terms of efficiency by comparing corresponding expected values of the social surplus. An allocation rule is *efficient* (first-best), if it yields the same expected social surplus as the maximum expected social surplus that can be achieved by the social planner, who observes all signals received by agents, orders agents to take or not to take actions, and, finally, assigns the object.

[6] For instance, this condition holds if random variables s_i and t_i are affiliated (Milgrom and Weber, 1982), which includes independent variables as a particular case.

[7] It is possible to extend our model to the case when the utility function takes the form $U_i(s_i, t_i, a_i)$, where a_i is continuous, and higher values of a_i makes a higher third-period signal t_i more likely. However, it would make exposition much more complex, while providing no new insight.

3 Efficient Mechanism in the Variable Value Environment

In this section, we study mechanism design in the variable value environment. We start with considering a benchmark case of the efficient mechanism for allocating the object that can be achieved by a social planner who knows all the private information available to bidders. Then we consider a mechanism that allocates the object efficiently in the incomplete information case.

3.1 The Social Planner's Problem

Let us start by characterizing the solution to the social planner problem. After observing the first period signals, the social planner decides which agents should act in the second period. Formally, there is a mapping of a vector of the first period signals into a vector of the second period actions $\mathbf{a}^* = \mathbf{a}^*(\mathbf{s})$.[8] At the end of the third period the social planner assigns the object, thus mapping a triplet of vectors $(\mathbf{s}, \mathbf{a}, \mathbf{t})$ into a number between 1 and N. The final assignment of the object is easily characterized. The social surplus maximization calls for assigning the object to the agent with the highest ex-post private value: if the efficient allocation assigns the object to the agent j, then for any $i \neq j$, we have $V(s_j, t_j) + ba_j \geq V(s_i, t_i) + ba_i$. Thus, assigning the object before agents have learned their final values of the object is likely to be inefficient. Obviously, in the variable value environment, giving the object to the agent with the highest ex-post value is necessary, but not sufficient for efficiency. It remains to characterize the function $\mathbf{a}^*(\mathbf{s})$ that describes the second period actions needed to maximize the expected social surplus, given $\mathbf{s}$. So, the social planner's problem might be written as follows:

$$\max_{\mathbf{a}} E_{\mathbf{t}}[S|\mathbf{s}, \mathbf{a}] = \max_{\mathbf{a}} \left\{ E_{\mathbf{v}} \max_i \{V(s_i, t_i) + ba_i\} - c \sum_{j=1}^{N} a_j \right\}.$$

Before proceeding to general results, let us illustrate this problem with a simple example.

Example. This is essentially a continuation of the Example from the introduction. Suppose that $N \geq 2$. If there is no third-period uncertainty ($t_i \equiv 0$), then the social planner chooses

[8] In the most general case, the social planner may assign mixed strategies to the agents. We show later that almost surely, the social planner problem has a unique pure strategy solution. Consequently, we focus on pure strategies of the social planner.

exactly one agent to act – the one with the highest first-period signal. On the other extreme, if there is no first-period signal ($s_i \equiv 0$), and the cost of action is sufficiently cheap, then the social planner would assign all agents to act.

It is useful to introduce a function $G_i(\mathbf{s}, \mathbf{a}_{-i})$ representing the difference in the expected social surplus that results from the agent i acting and not acting (keeping the actions of other agents unchanged):

$$G_i(\mathbf{s}, \mathbf{a}_{-i}) = E_{\mathbf{t}}[S|\mathbf{s}, \mathbf{a}_{-i}, a_i = 1] - E_{\mathbf{t}}[S|\mathbf{s}, \mathbf{a}_{-i}, a_i = 0]. \tag{1}$$

Since the social planner maximizes social surplus, the expected surplus in the above formula should be computed under assumption that after the third period, the social planner allocates the object to the agent with the highest value. The social planner faces the following trade-off: each additional agent's act increases the expected private value of the agent who receives the object, but is associated with the cost of c. Let $\mathbf{a}(m) = \mathbf{a}(m, \mathbf{s})$ denote the vector of actions, where the agents with the highest m first-period signals act, while the other $N - m$ agents skip action.

Theorem 1 *For a given vector of the first-period private signals* $\mathbf{s}$*, there exists a threshold* $r^* = r^*(\mathbf{s})$ *such that the social planner assigns agents with the highest* r^* *first-period signals to act.*[9]

Proof. To prove Theorem 1, we need to establish the following Lemma (a proof is relegated to the Appendix).

Lemma 1. *Consider vectors of actions* a *and* a' *such that* $\sum_i a_i = \sum_i a'_i$, $a_i = a'_i$ *for all* $i \neq j, k$, *and let* $a_j = 1$, $a_k = 0$, $a'_j = 0$, *and* $a'_k = 1$. *If* $s_j \geq s_k$, *then the expected social surplus from* a *is greater than that from* a'.

This Lemma shows that a vector of actions maximizing the expected social surplus must be of the form $\mathbf{a}(m)$ for some m, $0 \leq m \leq N$. Since there is a finite number of possible m's, there exists some r^* such that $\mathbf{a}(r^*)$ is the global maximizer of the expected social surplus. This completes the proof of Theorem 1.■

Definition of $G_i(\mathbf{s}, \mathbf{a}_{-i})$ implies that an action vector maximizing the social surplus must satisfy $G_i(\mathbf{s}, \mathbf{a}_{-i}) \geq 0$ when $a_i = 1$ and $G_i(\mathbf{s}, \mathbf{a}_{-i}) \leq 0$ when $a_i = 0$.

[9] r^* is determined almost uniquely: The event that the expected social surplus is maximized by more than one action vector of the form $\mathbf{a}(r^*)$ and $\mathbf{a}(r^{**})$ where $r^* \neq r^{**}$ has zero probability.

3.2 Complete Information

Now we turn to a world without an all-knowing and well-intentioned social planner. We consider the case, where agents act non-cooperatively, given that the first-period signals are common knowledge. This is an essential step towards mechanism design for the incomplete information case.

One might expect that in the decentralized case too many or too few players may take actions, since they may not fully internalize the effect of their private actions on other players. We show that an efficient allocation can be achieved in a decentralized case, when bidders know each other's first-period signals. Theorem 2 states that in this case there exists an equilibrium outcome of a second price auction conducted at the end of the third period that yields an efficient allocation, the same allocation as the first best obtained by the social planner.

Theorem 2 *If first-period signals* $\mathbf{s}$ *are public knowledge, there exists a socially efficient Subgame Perfect Bayesian-Nash equilibrium of a second price sealed bid auction conducted at the end of the third period. In this equilibrium, agents take unobservable actions as if they were assigned by the social planner resulting in the allocation rule characterized in Theorem 1.*

The basic intuition is as follows: the expected increase in an agent's utility from taking an action is exactly equal to the change in the expected social surplus due to her action.[10] Then the fact that $\mathbf{a}(r^*)$ is the social planner's optimal choice ensures that $\mathbf{a}(r^*)$ is an equilibrium vector of actions in the non-cooperative game.

Proof of Theorem 2. We introduce a function $g_i(\mathbf{s}, \mathbf{a}_{-i})$ defined as the change in the expected utility of the agent i as a result of taking an action instead of skipping it, and prove the following assertion (a proof is in the Appendix).

Lemma 2. $g_i(\mathbf{s}, \mathbf{a}_{-i}) = G_i(\mathbf{s}, \mathbf{a}_{-i})$.

Now observe that if $\mathbf{a}$ is a solution to the social planner's problem, then $G_i(\mathbf{s}, \mathbf{a}_{-i}) \geq 0$ when $a_i = 1$ and $G_i(\mathbf{s}, \mathbf{a}_{-i}) \leq 0$ when $a_i = 0$. Indeed, if $G_i(\mathbf{s}, \mathbf{a}_{-i}) < 0$ when $a_i = 1$, the agent's i switch from acting to non-acting would strictly increase the expected social surplus, contradicting the choice of $\mathbf{a}$. Similarly, $G_i(\mathbf{s}, \mathbf{a}_{-i}) \leq 0$ when $a_i = 0$. Then Lemma

[10] The logic behind the result is similar to the one that insures efficient entry in McAfee and McMillan (1987) and Levin and Smith (1994).

2 asserts that for the change in private benefits we have $g_i(\mathbf{s}, \mathbf{a}_{-i}) \geq 0$ for agents that act, and $g_i(\mathbf{s}, \mathbf{a}_{-i}) \leq 0$ for others. Thus, no agent has incentives to deviate, and Theorem 2 is proven.■

Here and in the rest of the paper the term 'equilibrium' is reserved for a subgame-perfect Bayesian-Nash equilibrium. The efficient equilibrium described in Theorem 2 seems to be a natural focal point. However, the game has a coordination component: there are other Bayesian equilibria that are not efficient. For example, if there are only two players, there might be two equilibria: one with the highest-ranked agent acting and the other abstaining, and another one with the second-ranked agent acting and the highest-ranked abstaining.

3.3 Incomplete Information: Ex-post Efficient Equilibrium

Now we are ready to investigate the incomplete information case. Here we consider a three-period model of the variable value environment similar to the one considered in the preceding section. The only difference is that here bidders' signals regarding their private values ($\mathbf{s}$ and $\mathbf{t}$) are observed privately. Now a mechanism consisting of an auction conducted after the third period no longer leads to an efficient allocation, since under such mechanism agents take second-period actions without knowledge of the private signals obtained by other players.[11] Obviously, an efficient allocation rule can not always assign the final ownership of the object prior the end of the third period.

Is it possible to allocate an object efficiently in the variable value environment of incomplete information? This question is answered affirmatively by Theorem 3. We explicitly construct an efficient allocation mechanism, which consists of two rounds: The private information is revealed by announcing it in the first round that takes place after the first-period-private-signals are observed; the ownership of the object is assigned in the second round that

[11] For the sake of completeness, one can consider the no-signalling case, where an auction is conducted after the third period and no signaling takes place before the second period. (Note that cheap talk communication following the first stage is not credible because everybody has an incentive to exaggerate his signal.) To describe the symmetric equilibria of this game, one can show that there exists a unique constant s^* such that any agent acts if her first-stage value s_i is higher or equal to s^*, and abstains from acting otherwise. In the equilibrium, the expected number of actions is $N(1 - F_s(s^*))$. So, in some cases, there are too few actions, while in others there are too many. This is a generalization of the Example from the Introduction. Also, there are a number of asymmetric equilibria. Of course, an asymmetric equilibrium can not lead to an efficient allocation rule.

takes place after the third-period private signals are observed by bidders.

Rounds of the Efficient Mechanism:

1.a Part a of the first round takes place at the end of the first period (after the private signals $\mathbf{s}$ have been received by agents). In part a of the first round, all agents make simultaneous public announcements $\widehat{s}_i$ about their private values s_i.

1.b Part b of the first round immediately follows part a. Each agent voluntarily selects a payment amount, $h_i \geq 0$, that depends on the announcements of other agents, as well as on her own announcement. (These payments $h_i(\widehat{\mathbf{s}})$ are necessary to make announcements credible.)

2. The second round takes place at the end of the third period, after agents observe their private signals t_i. In the second round, the ownership of the object is assigned using a second-price sealed-bid auction.

Theorem 3 *There exists a subgame perfect Bayesian-Nash equilibrium of the Signaling Mechanism that yields an efficient allocation rule.*

Proofs of Theorem 3 and all subsequent results are relegated to the Appendix. Here, let us discuss the logic behind the result. First, note that if the first period signals are revealed truthfully, the remaining subgame is identical to the game where first-period signals $\mathbf{s}$ are common knowledge. Theorem 2 established that an efficient allocation is an equilibrium of that game. Consequently, in order to establish existence of an efficient allocation mechanism, it suffices to show that for some payment schedule $h_i(\widehat{\mathbf{s}})$, truthful reporting is an equilibrium, when agents anticipate that the equilibrium characterized in Theorem 2 will be played in the remaining subgame. The intuition behind the possibility of truthful revelation is as follows. The higher is the first period signal s_i received by an agent i, the higher is that agent's relative willingness to pay in order to signal that her value of s_i is high. Agents are willing to pay in order to reveal their first period signals, because this information discourages other agents from taking actions, thus increasing the probability of winning for the agent i and decreasing the expected price that she will pay for the object (in the subsequent second price auction) conditional on winning. The expected price decrease affects agents with different private values differently. For instance, someone with a very low first-period signal is unlikely to win the object, thus her willingness to pay for sending a signal that depress the price of the object is lower than that of an agent with a relatively high first-period signal about her private value. This observation, which is critical to the existence of a separating signaling equilibrium, is formalized in Lemma 3. This Lemma establishes an appropriate analog of the increasing-differences property (Milgrom and Shannon, 1994) for the pay-offs in the subgame.

Lemma 3. *Let $E\pi_i(s_i, \hat{s}_i, s_{-i})$ be bidder i's expected pay-off gross of $h_i(\widehat{\mathbf{s}})$, when her true private signal is s_i, while other agents believe that the vector of first-period private signals is $(\hat{s}_i, s_{-i})$. For any s_{-i} and any $\hat{s}_i' > \hat{s}_i$, and any $s_i' > s_i$,*

$$E\pi_i(s_i', \hat{s}_i', \mathbf{s}_{-i}) - E\pi_i(s_i', \hat{s}_i, \mathbf{s}_{-i}) \geq E\pi_i(s_i, \hat{s}_i', \mathbf{s}_{-i}) - E\pi_i(s_i, \hat{s}_i, \mathbf{s}_{-i}). \quad (2)$$

In the above Lemma, $E\pi_i(s_i, \hat{s}_i, \mathbf{s}_{-i})$ is the expected pay-off of agent i in the mechanism described in Section 3.2, when the first period private signals are given by $(s_i, \mathbf{s}_{-i})$ and player i plays the best response to the action profile of players $-i$ given by $\mathbf{a}(r^*(\hat{s}_i, \mathbf{s}_{-i}))$. (The action profile $\mathbf{a}(r^*(\hat{s}_i, \mathbf{s}_{-i}))$ is characterized in Theorem 1.) Essentially, $E\pi_i(s_i, \hat{s}_i, \mathbf{s}_{-i})$ is the pay-off received by agent i in the subgame computed under an assumption that all first round announcements are believed to be truthful, and that agent i reported $\hat{s}_i$, while her true private value is s_i.

Lemma 3 states that the same change in announcement (from $\hat{s}_i$ to $\hat{s}_i'$) brings more in expected surplus to the agent with relatively high true signal, s_i'. Note that $E\pi_i(s_i, \hat{s}_i, \mathbf{s}_{-i})$ is not the same as the expected utility of agent i, because it does not include the payments h_i made in the first round of the mechanism. The agent's utility is given by $E\pi_i(s_i, \hat{s}_i, \mathbf{s}_{-i}) - h_i$. Thus, truthful reporting s_i is consistent with an equilibrium, if there exists a payment schedule $h(\hat{s}_i, \mathbf{s}_{-i})$ such that incentive compatibility and individual rationality constraints are satisfied. Namely, for any agent i and all $(s_i, \hat{s}_i, \mathbf{s}_{-i})$ the payments should satisfy the following conditions:

$$E\pi_i(s_i, s_i, \mathbf{s}_{-i}) - h(s_i, \mathbf{s}_{-i}) \geqslant E\pi_i(s_i, \hat{s}_i, \mathbf{s}_{-i}) - h(\hat{s}_i, \mathbf{s}_{-i}) \quad \text{(IC)}$$

$$E\pi_i(s_i, s_i, \mathbf{s}_{-i}) - h(s_i, \mathbf{s}_{-i}) \geqslant E\pi_i(s_i, \hat{s}_i = 0, \mathbf{s}_{-i}) \quad \text{(IR)}$$

Note that finding $h(\hat{s}_i, \mathbf{s}_{-i})$ that satisfies the above constraints is sufficient for proving the claim of Theorem 3. Such payment schedule $h_i(\hat{s}_i, \mathbf{s}_{-i})$ is characterized in Theorem 4. Before proceeding to Theorem 4, we need to introduce one more definition.

Consider the efficient allocation rule characterized in Theorem 1. It implies that for any vector of the first period private signals $\mathbf{s}_{-i}$, there exists a sequence $0 = \bar{s}_i(k_i^*) \leq \bar{s}_i(k_i^* - 1) \leq \ldots \leq \bar{s}_i(1) \leq \bar{s}_i(0) < \infty$, where $\bar{s}_i(k)$ is defined to be the minimal type of i such that exactly

k highest-ranked agents (different from the agent i herself) act in the subgame equilibrium described in Theorem 2. Let $k_i^* = k_i^*(0, \mathbf{s}_{-i})$ be the number of agents acting, when i has the lowest possible type (zero). Within each segment described above, an agent's i report is irrelevant to the other agents' decisions on whether or not to act.

As above, let $\mathbf{a}(m)$ denote the vector of actions, where the agents with the highest m first-period signals act, while the other $N-m$ agents skip action. Note that $\mathbf{a}(m)$ is a function of the vector of first-period signals $\mathbf{s}$.

Theorem 4 *The following payments are consistent with an efficient equilibrium of the Efficient Mechanism. For any i,*

$$\begin{aligned} h_i(\hat{s}_i, \hat{\mathbf{s}}_{-i}) &= 0, \text{ whenever } \bar{s}_i(k_i^*) \le \hat{s}_i \le \bar{s}_i(k_i^* - 1), \\ h_i(\hat{s}_i, \hat{\mathbf{s}}_{-i}) &= h_i(\bar{s}_i(k), \hat{\mathbf{s}}_{-i}) + E\pi_i(\bar{s}_i(k), \mathbf{a}(k)) - E\pi_i(\bar{s}_i(k), \mathbf{a}(k+1)), \end{aligned} \tag{3}$$

whenever $\bar{s}_i(k) < \hat{s}_i \le \bar{s}_i(k-1)$, $k < k_i^*$.

Theorem 4 shows that for any agent i, the payment schedule satisfies incentive compatibility and individual rationality (IC and IR, respectively) constraints. Then, if agents in the set $-i$ report their type truthfully, $\hat{\mathbf{s}}_{-i} = \mathbf{s}_{-i}$, the payment scheme for the agent i given by (3) induces her to report her type truthfully, $\hat{s}_i = s_i$. The proof of Theorem 3 is based on combining Theorem 2 and Theorem 4.

Proof of Theorem 3. Lemma 3 proves that the above payment schedule induces truthful reporting by agent i, provided that all other agents' reports are truthful. The beliefs supporting the equilibrium in the signaling stage are straightforward: if a payment by an agent i is defined by (3), then the agents first-period signal is perceived to lie within the respective range. In the subgame that starts after the first-period signals are revealed, agents play according to the strategies described in Theorem 2.■

In the above equilibrium, each agent reports her type truthfully regardless of the other agents' types given that these types are reported truthfully.[12] This is a kind of an ex-post equilibrium (Perry and Reny,1999), where no agent regrets her announcement after learning the other agents' types; thus, this mechanism is similar in spirit to the well-known Vickrey-Clarke-Groves mechanism (e.g., Vickrey, 1961, Krishna and Perry, 1998). However, unlike

[12] As usual, the revelation principle (Myerson, 1979) allows us to assume that agents report their types directly, rather than conveying information via a special set of signals.

the Vickrey-Clarke-Groves mechanism, this is a two-round mechanism, where payments made in the signaling round of the mechanism have no direct impact on allocating of the object — these payments influence the allocation of object indirectly by shaping beliefs about first period signals.

3.4 Ex-ante Efficient Equilibrium

The mechanism described above provides an ex-post efficient ex-post equilibrium. In such an equilibrium, agents' payments may depend on the other agents' announcements. Below we show that the Efficient Mechanism described in the previous section also has an ex-ante efficient separating equilibrium. In this equilibrium, agents make no announcements (or make uninformative announcements) in the cheap talk stage of the mechanism. In the stage 1b they simultaneously make publicly observable payments H_i; an agent decides on the payment size without knowing the private signals of other agents. We show that there exists a fully separating equilibrium where there is a unique payment corresponding to each private signal s_i. Consequently, agents no longer need to make announcements, because the announcements of their private signals are revealed in the size of payments they make.

Theorem 5 *There exists an efficient equilibrium in the Efficient Mechanism, where agents simultaneously make payments $H(s_i)$ that depend only on their private information s_i. Equilibrium payments are given by $H_i(\hat{s}_i) = E_{\mathbf{s}_{-i}} h_i(\hat{s}_i, \mathbf{s}_{-i})$, where $h_i(\hat{s}_i, \mathbf{s}_{-i})$ are equilibrium payments defined in Lemma 3.*

Let us discuss the intuition behind the proof of Theorem 5. According to Theorem 2, an efficient allocation can be obtained, when the first period signals s_i are common knowledge. It remains to show that the signaling mechanism proposed above is incentive compatible when an efficient equilibrium is chosen in the subgame following the signaling stage. More formally we need to show that for any $\mathbf{s}_{-i}$,

$$s_i \in \arg\max_{\hat{s}_i} \left\{ E_{\mathbf{s}_{-i},\mathbf{v}} \pi_i(s_i, \hat{s}_i, \mathbf{s}_{-i}) - H_i(\hat{s}_i) \right\}$$

(note that here expectation is taken with respect to $\mathbf{s}_{-i}$ and $\mathbf{v}$). This result is a straightforward corollary to the existence of an ex-post equilibrium established in Lemma 3. Existence of this 'ex-ante' separating equilibrium essentially follows from the fact that if the agent's i

truth-telling is a best reply to *any* vector $\mathbf{s}_{-i}$ of other agents' signals, than it is a best reply on the average as well.

Proof of Theorem 5. It suffices to observe that

$$s_i \in \arg\max_{\hat{s}_i} \{E_{\mathbf{t}}\pi_i(\hat{s}_i, \mathbf{s}_{-i}) - h_i(\hat{s}_i, \mathbf{s}_{-i})\}$$

for any $\mathbf{s}_{-i}$ and any s_i, and take sum over all $\mathbf{s}_{-i}$.

Then note that all $h_i(\hat{s}_i, \mathbf{s}_{-i})$ and thus the function $H(s_i)$ increase in the bidder's i first-period signal s_i. This allows to use $H(s_i)$ to report the true value of s_i. Beliefs are straightforward.■

Although existence of ex-ante equilibrium follows from existence of an ex-post equilibrium, it is a useful result. It shades some light on the maneuvering that bidders often make prior to an auction: for example, firms preparing to participate in a large-scale privatization auction or competing for a procurement contract might engage in costly signaling in order to discourage potential rivals. For example, let us consider the history of bidding for Los Angeles license in 1995 broadband auction for mobile-phone licenses.[13] One bidder, Pacific Telephone, possibly started with a higher private value than other bidders due to experience in California market and possible synergies between its wireline and wireless businesses. There were a number of important decisions (actions) that each bidder had to make before the auction for Los Angeles license, these included forming alliances, making investments and formulating strategies for other markets. It appears that Pacific Telephone signaled to other bidders (and would-be bidders) that it anticipates winning California. Pacific Telephone made public statements like 'If somebody takes California away from us, they'll never make any money'.[14] To make these statement credible, Pacific Telephone made investments that were of little value without winning Los Angeles license[15]. As a result, some potential bidders (including the industry giants such as Bell Atlantic, GTE, and MCI) were discouraged from participating in the auction. (Thus failing to undertake an action, in our interpretation). In fact, GTE and Bell Atlantic took actions that made them ineligible for the auction.

[13] We thank Paul Milgrom for suggesting this example of the variable value environment.

[14] *Wall Street Journal,* October 31, 1994.

[15] Some of the investments made by Pacific Telephone might be interpreted as actions and others as signals. Essentially, running a PR campaign aimed at signaling that Pacific Telephone is determined to win Los Angeles license can be interpreted as signaling. In contrast, making unobservable arrangements made to expedite creation of the wireless service in Southern California can be interpreted as an action.

As a result, revenues were quite low compared to initial estimates.[16] Applying the logic of our model highlights the importance of signaling that discourages competitors from taking actions that increase the value of the prize for them.

In the above example the costs of making reports credible have not been captured by the auctioneer. There is no reason why bidders would opt to announce their types by writing checks to the auctioneer, and not by burning money in some other way. Another problem is that the game considered above has an inefficient pooling equilibrium along with an efficient separating one. Thus, there is no guarantee that an efficient equilibrium is selected. In the following section we introduce an $\varepsilon-$efficient mechanism, which is similar to the ex-ante equilibrium considered here, but free of its main disadvantages. First, an $\varepsilon-$efficient mechanism insures coordination on the efficient equilibrium in the subgame. Second, it allows seller to capture the signaling costs of bidders. The sacrifice that must be made in order to gain robustness and capture signaling costs is an arbitrarily small loss in efficiency.

4 Robust ε-Efficient Mechanism

The efficient mechanism described in the previous section can be viewed as a two-stage auction. The reporting stage, where agents simultaneously make payments that reveal their types, can be replaced with a sealed-bid all-pay auction, where the object being sold is worth nothing (zero). Theorem 5 established existence of an efficient Bayesian-Nash equilibrium of this two stage auction. Unfortunately, this is not a unique equilibrium: a pooling equilibrium, where everybody bids zero in the signaling stage, is a natural focal point. Nevertheless, introducing an arbitrarily small inefficiency into the auction design can force bidders to coordinate on an efficient separating equilibrium. We will refer to such a mechanism as an $\varepsilon-$*efficient mechanism.* We start with describing an $\varepsilon-$efficient mechanism and then proceed to establish efficiency properties of this mechanism in Theorem 6.

[16] Granted, this is not the only possible interpretation of the 1995 auction for Los Angeles licence. Klemperer (2000) considers the history of this auction and suggests that the winner's curse played an important role because the winner's curse is particularly powerful in auctions where one bidder has an advantage. For a theoretical argument that uses this logic, see also Bulow, Huang, and Klemperer (1999). The outcome of that auction was probably determined by a constellation of a large number of factors. Revenue in the auction for Los Angeles licence were low in comparison with spectrum auction in Chicago; however, it is not clear if asymmetry among bidders and the winner curse were more severe in California.

Rounds of $\varepsilon-$Efficient Mechanism

1. The first (reporting) round takes place at the end of the first period (after the private signals $\mathbf{s}$ have been received by agents, but before agents take actions). In this round one coupon is sold via all pay sealed bid auction.[17] All bids are announced at the end of the round. The coupon sold in the signaling round entitles its owner to a discount of size ε for the price in the final auction (the discount coupon is not-transferable, only the winner of the final auction can benefit from having the coupon).

2. The second round (final) auction takes place at the end of the third period, after agents observe private signals $\mathbf{v}$. In the second round the ownership of the object is assigned using a second price sealed bid auction. (If the highest bidder in the final round is the owner of the $\varepsilon-$coupon, then she pays the second highest bid minus ε.)

There are two rounds and three decision nodes in an ε-efficient mechanism. At the first decision node, agents make bids in an all-pay auction, i.e. the i's actions space is $\{H_i|H_i \geq 0\}$. The information set of agent i at the first decision node is given by s_i. The first round strategy is described by the probability distribution $\rho_i(\cdot; s_i)$ over the set of pure strategies $\{H_i|H_i \geq 0\}$. At the second decision node, agents make a decision to act or not to act. The information set of agent i at the second decision node is given by $(s_i, H_i, \mathbf{H}_{-i}, \mathbf{w})$, where $\mathbf{w}$ is an N-dimensional vector with $w_k = 1$ if the agent k won the coupon in the all pay auction, and $w_k = 0$ otherwise. (There is a unique vector $\mathbf{w}$ consistent with vector of payments $\mathbf{H}$, unless there is a tie). The probability that agent i acts ($a_i = 1$) is denoted by $\lambda_i = \lambda_i(s_i, H_i, \mathbf{H}_{-i}, \mathbf{w})$. At the third decision node, agents submit bids in the second price sealed-bid auction. At this moment, the information sets are $(s_i, H_i, \mathbf{H}_{-i}, \mathbf{w}, a_i, t_i)$. It is well known that in an equilibrium in weakly dominant strategies of a private value Vickrey auction bidders bid their true values. Thus, equilibrium bids are given by $V(s_i, t_i)+ba_i+\varepsilon w_i$.

Clearly, an ε-efficient mechanism has multiple equilibria. Some of these equilibria are highly implausible. In order to rule out such equilibria we introduce a restriction on strategies in the spirit of 'intuitive' criteria such as D1 of Cho and Kreps (1987) or stability of Kohlberg and Mertens (1986).

Definition. A strategy of an agent j is *monotonic*, if two vectors $\mathbf{H}_{-j}$ and $\mathbf{H}'_{-j}$ differ only

[17] In an all pay sealed bid auction every agent submits a sealed bid. All agents have to pay the amount of their bids regardless of whether or not they won the object. The agent with the highest bid receives the object. (In case of a tie the winner is randomly chosen from the set of highest bidders.) Fullerton and McAfee (1999) use an all-pay auction in their 'contestant selection auction'.

in component i so that $H_i > H'_i$, then $p_j(s_j, H_j, \mathbf{H}_{-j}, \mathbf{w}) \leq p_j(s_j, H_j, \mathbf{H}'_{-j}, \mathbf{w}')$.

In words, a monotonic strategy of an agent j assumes that for any history, the probability that the agent j takes an action is non-increasing in the size of the payment that some agent i, $i \neq j$ makes in the signaling stage.

As we will see, the requirement that the strategies are monotonic rules out the 'bizarre' equilibrium, where all agents bid zero in the signaling stage and an agent who bids a positive amount is perceived to be of the lowest type. Basically, there are two reasons why an equilibrium strategy may not be monotonic: First, perverse beliefs may sustain an equilibrium in strategies that are not monotonic. An example of such 'unnatural' beliefs is as follows: The more an agent bids for a discount coupon, the lower is her perceived s_i. Obviously, this is counter-intuitive: the higher is an agent's s_i, the more she values the discount coupon. The second possibility stems from coordination aspect of the game. If bids in the signaling stage are used as coordination devices for selecting a Bayesian-Nash equilibrium in the remaining subgame, an equilibrium resulting from these beliefs may include strategies that are not monotonic.

Definition. A *robust equilibrium* of an $\varepsilon-$efficient mechanism is any symmetric subgame-perfect Bayesian-Nash equilibrium in monotonic strategies.

Theorem 6 *For an ε-efficient mechanism, the following is true:*

(i) There exists a robust equilibrium.

(ii) The robust equilibrium is unique.

(iii) The probability that the robust equilibrium yields an efficient allocation converges to one as $\varepsilon \to 0$.

Before proceeding to a formal proof, let us sketch the intuition behind this result. A pooling equilibrium where everybody bids zero for the coupon is not robust. Indeed, if everybody bids zero for the discount, it can be purchased for an arbitrarily small amount. Thus, the pooling equilibrium is sustainable only if bidders are discouraged from bidding a positive amount by a belief that a positive bid would encourage other bidders to act more aggressively in the action stage. However, this belief is inconsistent with strategies being monotonic. The same argument applies to any partially pooling equilibrium. We show that there are no equilibria in mixed strategies, because the willingness to pay for the discount is an increasing function of the bidder's signal. Efficiency of a robust equilibrium follows from

Theorem 5 that establishes that for $\varepsilon = 0$, there exists an efficient symmetric equilibrium. To prove asymptotic efficiency of a robust equilibrium, we show that when ε approaches 0, the robust equilibrium converges to the equilibrium described in Theorem 5.[18]

Proof of Theorem 6.

(i) The proof of existence follows the pattern of the proof of Theorem 3. Construction of an ex-ante equilibrium in the previous section used existence of an ex-post equilibrium in a mechanism, where credibility payments are allowed to be functions of announcements. Here, we use the same idea. As an intermediate step, consider a mechanism, where the discount is not auctioned off using an all-pay auction. Instead, bidders announce their types in the reporting stage (much like in the mechanism described in Section 3). After the announcement, bidders make payments $h_i(\widehat{s}_i, \widehat{\mathbf{s}}_{-i})$ to make the announcement credible, and the bidder with the highest announced s_i receives the $\varepsilon-$discount.

Rounds of the "intermediate" mechanism:

1. After each agent privately learns s_i, all agents simultaneously announce their types in the cheap talk stage. Afterwards, each agent must make a payment of $h_i(\widehat{s}_i, \widehat{\mathbf{s}}_{-i})$. The agent with the highest first period announcement $\widehat{s}$ receives the discount coupon (ties are broken using a lottery). Agents take action after observing announcements $\widehat{\mathbf{s}}$.

2. After the third period signals $\mathbf{v}$ are revealed, the object is sold via a second-price sealed-bid auction.

We shall show that there exists a payment schedule $h_i(\widehat{s}_i, \widehat{\mathbf{s}}_{-i})$ such that truthful reporting supported by paying $h_i(\widehat{s}_i, \widehat{\mathbf{s}}_{-i})$ is an ex-post equilibrium. The private value of the bidder with the highest first-period signal is essentially boosted by the amount equal to the discount ε. Let $\widetilde{\mathbf{s}}(\mathbf{s},\widehat{\mathbf{s}})$ be a vector of 'adjusted' private value signals, where $\widetilde{s}_i = s_i + \varepsilon$ if $\widehat{s}_i > \widehat{s}_j = s_j$ for all $j \neq i$, and $\widetilde{s}_i = s_i$ otherwise. Assuming $\widehat{\mathbf{s}}_{-i} = \mathbf{s}_{-i}$, we study the ith agent incentives to misreport the true signal s_i. If all the equilibrium reports $\widehat{s}_i$ are truthful, then the subgame after ε discount is assigned is identical to the game considered in Section 3.1. The equilibrium expected pay-off of agent i of the subgame, which does not include h_i, is denoted by $E\widetilde{\pi}_i(s_i, \hat{s}_i, \mathbf{s}_{-i})$. One can express $E\widetilde{\pi}_i(s_i, \hat{s}_i, \mathbf{s}_{-i})$ in terms of $E\pi_i(s_i, \hat{s}_i, \mathbf{s}_{-i})$ (defined in Lemma 3) using 'adjusted' private signals. Let $\widehat{\widetilde{\mathbf{s}}}$ denote a vector of perceived 'adjusted' signals of agents; the ith component of $\widehat{\widetilde{\mathbf{s}}}$ is $\widehat{\widetilde{s}}_i = \widehat{\widetilde{s}}_i(s_i, \widehat{s}_i, \widehat{\mathbf{s}}_{-i}) = \widetilde{s}_i + (s_i - \widehat{s}_i)$. That is, $\widetilde{\mathbf{s}}$ is a vector of 'adjusted' private value signals and $\widehat{\widetilde{\mathbf{s}}}$ is public perception about $\widetilde{\mathbf{s}}$. Now

[18] Also, if the ε-efficient mechanism yields an inefficient outcome, efficiency losses are of magnitude ε.

we can write $E\widetilde{\pi}_i(s_i, \hat{s}_i, \mathbf{s}_{-i}) = E\pi_i(\widetilde{s}_i, \widehat{\widetilde{s}}_i, \widetilde{\mathbf{s}}_{-i})$.

To prove that a separating equilibrium exists, we need to formulate an increasing-differences condition similar to (2).

Claim. *For any $N-1$-tuple of truthful reports $\mathbf{s}_{-i}$, and any $\hat{s}'_i \geq \hat{s}_i$, $s'_i \geq s_i$,*

$$E\widetilde{\pi}_i(s'_i, \hat{s}'_i, \mathbf{s}_{-i}) - E\widetilde{\pi}_i(s'_i, \hat{s}_i, \mathbf{s}_{-i}) \geq E\widetilde{\pi}_i(s_i, \hat{s}'_i, \mathbf{s}_{-i}) - E\widetilde{\pi}_i(s_i, \hat{s}_i, \mathbf{s}_{-i}) \tag{4}$$

To prove the claim, we need to consider three cases: (a) the agent wins the ε discount if she makes announcement $\hat{s}'_i$ but not $\hat{s}_i$; (b) an agent wins the discount for either announcement $\hat{s}'_i$ or $\hat{s}_i$; (c) neither $\hat{s}'_i$, nor $\hat{s}_i$ are high enough to win the discount.

For (b) and (c), (4) follows immediately from Lemma 3. It remains to show that it also holds for the case (a). Denote $\mathbf{s}_{-i} = (s^m_{-i}, \mathbf{s}^{-m}_{-i})$, where s^m_{-i} is the largest component of the vector $\mathbf{s}_{-i}$ and $\mathbf{s}^{-m}_{-i}$ is an $N-2$-dimensional vector that consists of all components of vector $\mathbf{s}_{-i}$ other than its largest component s^m_{-i}. Applying the new notation, one gets $E\widetilde{\pi}_i(s_i, \hat{s}_i, \mathbf{s}_{-i}) = E\widetilde{\pi}_i(s_i, \hat{s}_i, s^m_{-i}, \mathbf{s}^{-m}_{-i})$. In case (a), we have $\widetilde{\mathbf{s}}^{-m}_{-i} = \mathbf{s}^{-m}_{-i}$. Therefore, one can re-write (4) as follows:

$$E\pi_i(s'_i + \varepsilon, \hat{s}'_i, s^m_{-i}) - E\pi_i(s'_i, \hat{s}_i, s^m_{-i} + \varepsilon) \geq E\pi_i(s_i + \varepsilon, \hat{s}'_i, s^m_{-i}) - E\pi_i(s_i, \hat{s}_i, s^m_{-i} + \varepsilon). \tag{5}$$

Let

$$\begin{aligned}
X &= V(s_i, t_i) + \varepsilon - \max_{j \neq i} \left\{ V(s_j, t_j) + b a^*_j(\hat{s}'_i, s^m_{-i}, \mathbf{s}^{-m}_{-i}) \right\}, \\
X' &= V(s_i, t_i) + \varepsilon - \max_{j \neq i} \left\{ V(s_j, t_j) + b a^*_j(\hat{s}_i, s^m_{-i} + \varepsilon, \mathbf{s}^{-m}_{-i}) \right\}, \\
Y &= V(s'_i, t_i) + \varepsilon - \max_{j \neq i} \left\{ V(s_j, t_j) + b a^*_j(\hat{s}'_i, s^m_{-i}, \mathbf{s}^{-m}_{-i}) \right\}, \\
Y' &= V(s'_i, t_i) + \varepsilon - \max_{j \neq i} \left\{ V(s_j, t_j) + b a^*_j(\hat{s}_i, s^m_{-i} + \varepsilon, \mathbf{s}^{-m}_{-i}) \right\}
\end{aligned}$$

We know that $X' \succeq X$, $Y' \succeq Y$. Then

$$\begin{aligned}
E\pi_i(s'_i + \varepsilon, \hat{s}'_i, s^m_{-i}) - E\pi_i(s_i + \varepsilon, \hat{s}'_i, s^m_{-i}) &= EY^+ - EX^+, \\
E\pi_i(s'_i, \hat{s}_i, s^m_{-i} + \varepsilon) - E\pi_i(s_i, \hat{s}_i, s^m_{-i} + \varepsilon) &= EY'^+ - EX'^+.
\end{aligned}$$

Using Lemma A3 (from the Appendix) completes the proof of (5).

Since (5) holds, there exists an ex-post separating equilibrium in the "intermediate mechanism". Using existence of an ex-post equilibrium, we can apply the same argument as in the proof of Theorem 5 to establish existence of ex-ante separating signaling mechanism, where agents make signaling payments that are strictly increasing in their signals. This completes the proof of existence.

Now we shall prove that any robust equilibrium is unique, fully separating, and 'almost efficient'.

In an equilibrium, the probability of any particular bid value H in the signaling stage is zero. Indeed, if there is a positive mass of agents that plays some H_{mass} with positive probability, then there is a positive probability of a tie. Then an agent playing H_{mass} can increase the likelihood of winning the discount $\varepsilon > 0$ by increasing her bid by an infinitesimal amount. Since the strategies are monotonic, none of the agents would increase their likelihood of taking actions. Thus, such a deviation would be profitable.

Probability that players in the set $-i$ take actions is denoted here as $\mathbf{p}_{-i}$. Let $\Pi(s_i, \mathbf{p}_{-i}, \mathbf{s}_{-i})$ denote the pay-off of player i in the subgame after signaling payments H's are sunk. We want to show that if $\mathbf{p}_{-i} \geq \mathbf{p}'_{-i}$ then for every $s'_i > s_i$ we have

$$\Pi(s'_i, \mathbf{p}_{-i}, \mathbf{s}_{-i}) - \Pi(s_i, \mathbf{p}'_{-i}, \mathbf{s}_{-i}) \leq \Pi(s'_i, \mathbf{p}_{-i}, \mathbf{s}_{-i}) - \Pi(s_i, \mathbf{p}^{'}_{-i}, \mathbf{s}_{-i}). \tag{6}$$

Essentially this condition says that any decrease in "final" private values of player in the set $-i$ is more valuable for player i with a larger first period private signal. Inequality (6) follows from the proof of Lemma 3.

Let us show that all robust equilibria are separating. In a robust equilibrium, actions taken by players depend on their private signals and the announcements of other players. Thus, we can write $\mathbf{p}_{-i} = \mathbf{p}_{-i}(\mathbf{s}_{-i}, \mathbf{H}_{-i}, H_i)$ and $\mathbf{p}'_{-i} = \mathbf{p}_{-i}(\mathbf{s}_{-i}, \mathbf{H}_{-i}, H'_i)$. (From above, we can conclude that a tie is a measure zero event; and thus have no impact on expected payoffs.) For monotonic strategies $\mathbf{p}_{-i} \geq \mathbf{p}'_{-i}$ for $H'_i > H_i$ (the inequality holds for all components). Inequality (6) implies that $H(s)$ is weakly increasing in s. Now we can conclude that $H(s)$ is strictly increasing in s (expect perhaps for a measure-zero set).

Let us show that in equilibrium, $p_i(\mathbf{H}_{-i}, H_i(s_i), s_i)$ is non decreasing in s_i. Indeed, $\mathbf{p}_{-i} = \mathbf{p}_{-i}(\mathbf{s}_{-i}, \mathbf{H}_{-i}, H_i)$ is weakly decreasing in H_i. Thus, according to single crossing condition, if agent with a first period signal s_i acts with positive probability $p_i(\mathbf{H}_{-i}, H_i(s_i), s_i) > 0$, then any agent with a signal $s'_i > s_i$ strictly prefers to act, and $p_i(\mathbf{H}_{-i}, H_i(s'_i), s'_i) = 1$. Therefore, there exists a unique equilibrium in the subgame that is consistent with a robust equilibrium strategy profile. In this equilibrium, all agents with private values exceeding some critical value $s^*(\mathbf{H})$ act.

From the previous paragraph and Theorem 2, it follows that $\varepsilon-$efficient mechanism yields an efficient allocation with probability converging to one as ε converges to zero.

To establish uniqueness of the robust equilibrium, we use a standard argument (e.g., Klemperer, 1999). Condition (6) implies that $\frac{dH(s)}{ds}$ is the same in any robust equilibrium. In Step 5, we showed that there is a unique robust equilibrium in the subgame following the all-pay auction. It remains to show that $H(0) = 0$. Suppose otherwise, say $H(0) = H_0 > 0$. For a player with $s_i = 0$, $H(0) = 0$ is a profitable deviation: Indeed, after this she does not change the perception of her type (she is correctly perceived to have $s_i = 0$). It was demonstrated that in a robust equilibrium each player either acts with probability one or zero (except perhaps for a set of measure zero). Thus, the deviation can only cause other players to increase the probability with which they act; however, given the set of players that act, non of the players that do not act in a robust equilibrium would choose to act. ■

Let us now consider an example illustrating that the all pay auction part of the ε-efficient mechanism is crucial for ensuring that any robust equilibrium is separating and nearly efficient.

Example. Suppose the all-pay auction is replaced with a second-price sealed-bid auction. When a sufficiently small discount is auctioned off via a second price auction, the following inefficient pooling equilibrium is robust: all agents bid ε for the discount of size ε. Indeed, we need to specify beliefs that support this equilibrium. If an agent deviates by bidding less than ε, she is perceived to have the lowest possible signal s_i. Thus, there are no incentives to bid less than ε, provided that ε is sufficiently small. If an agent bids more than ε, the beliefs of other agents about her type are the same as if she bids ε. Thus, bidding more than ε is a bad strategy: If there are N agents bidding ε each in a second-price auction, each of them has a $\frac{1}{N}$ chance of getting the discount. The winner of the discount "envy" the bidders who did not win the discount, and thus do not have to pay anything in the signaling stage. By bidding more than ε, an agent insures that she wins the discount and will have to pay for it, thus, making herself worse off. In contrast, there are no robust pooling equilibrium of the ε-efficient mechanism (by Theorem 6). For instance, if all agents bid ε for the discount, bidding slightly more than ε is a profitable deviation.

5 Conclusion

Our model captures essential features of the variable value environment. The model describes the sale of an object available for use at some future date (such as a military base scheduled to close in twenty years). Suppose several firms from different industries are considering buying the base in order to convert it into a manufacturing facility. In this case, investments complementary with ownership of the base are modeled as actions and demand shocks, and unexpected developments in bidder's business are modeled as exogenous shocks. We put forth an efficient mechanism for allocating an object in such environment (Theorems 3 and 6). At least in theory, the ε-efficient auction proposed herein has several important advantages under this environment. First, it has a unique robust equilibrium. Second, this equilibrium yields efficient allocation with near certainty (see Theorem 6). An ε-efficient auction seems simple and intuitive enough to have viable practical applications; it might be a first real-world example of 'virtual implementation'. Albeit, no amount of theorizing can guarantee that it performs well with human decision makers. Thus, comparison of ε-efficient auction and other types of auctions may be a high pay-off project for an experimental economist. Note that ε-efficient auction is preferable to a Vickrey auction even if it is not certain whether or not the environment has a variable value component. Indeed, Vickrey auction is a subgame of ε-efficient auction. In general, a fully separating equilibrium collapses, if the first period signals have a large common value component.[19] In the extreme case where no information revelation takes place in the signaling round an ε-efficient auction yields a negligibly small loss in efficiency relative to one round Vickrey auction. However, as long as information revelation occurs in the signaling round of the ε-efficient auction, the additional information is likely to improve performance of the Vickrey auction conducted in the second round relative to a standard Vickrey auction. It is straightforward to show that ε-efficient auction is efficient in independent private value environment. Obviously, in an independent private value case using a simpler efficient mechanism is more practical. However, it is reassuring that using ε-efficient auction does no harm even if the environment has no variable value

[19] Note that in a variable value setting each agent would like to convince other bidders that her signal is high in order to discourage other bidders from taking actions; in contrast, if first period signals have a large common value component agents would like to convince other bidder's that their signals are low in order to depress the price in the final auction. Thus, if the common value component is large the information revelation in the signaling round can be limited or even non-existent.

features. In short, ε-efficient auction seem to offer substantial benefits with a minimum downside.

The model considered in this paper is sufficiently rich to offer insight into understanding of the variable value environment. However, many important examples of the variable environment may be better captured by variations of our models. Markets where search matters are a particularly significant examples of the variable value environment. The theory of the variable value environment developed herein explains why sellers in such an environment are reluctant to use auctions. Indeed, we established that in the variable value environment (e.g. the housing market) standard auction mechanisms, such as Vickrey auction or first price auction, are inefficient. The signaling mechanism and ε-efficient auction proposed in this paper inform our intuition for mechanism design in the variable value environments ranging from market for capital equipment to the housing market.

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APPENDIX

To prove propositions in the body text, we need some auxiliary notation and lemmas. For any number (function) x, let $x^+ = \max\{x, 0\}$. A random variable X (first-order) *stochastically dominates* a random variable Y (denoted $X \succeq Y$) if and only if for cumulative density functions, one has $F_X(z) \leq F_Y(z)$ for any $z \in \mathbf{R}$. An equivalent condition is that $Eh(X) \geq Eh(Y)$ for any increasing function h (e.g., Levy, 1992).

Lemma A1. *Suppose that X, Z and Y, W are random variables, and in both pairs variables are independent of each other. Suppose that $X \succeq Y$, $W \succeq Z$. Then $X - Z \succeq Y - W$.*

Proof. We need to prove that for any t, $F_{X-Z}(t) \leq F_{Y-W}(t)$. One has

$$\begin{aligned}
F_{X-Z}(t) &= \int_{-\infty}^{\infty}\left[\int_{x-t}^{\infty} dF_Z(z)\right] dF_X(x) = \int_{-\infty}^{\infty}(1 - F_Z(x-t))dF_X(x) \\
&\leq \int_{-\infty}^{\infty}(1 - F_W(x-t))dF_X(x) = \int_{-\infty}^{\infty}\left[\int_{x-t}^{\infty} dF_W(w)\right] dF_X(x) \\
&= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{z+t} dF_X(x)\right] dF_W(w) = \int_{-\infty}^{\infty} F_X(w+t)dF_W(z) \\
&\leq \int_{-\infty}^{\infty} F_Y(w+t)dF_W(z) = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{w+t} dF_Y(y)\right] dF_W(z) = F_{Y-W}(t).
\end{aligned}$$

■

Lemma A2. *For any random variables X and Y such that $X \succeq Y$, and a random variable Z, which is independent of X, Y,*

$$\max\{X, Z\} \succeq \max\{Y, Z\}.$$

Proof. Straightforward.■

Lemma A3. *For any random variables X and Y such that X stochastically dominates Y, and any constant $z \geq 0$,*

$$E(X+z)^+ - EX^+ \geq E(Y+z)^+ - EY^+.$$

Proof. For any $z \geq 0$, the function $h_z(x) = (x+z)^+ - x^+$ is a bounded increasing function of x. Therefore, the definition of stochastic dominance yields that $Eh_z(X) \geq Eh_z(Y)$.■

Lemma A4. *For any independent random variables X, Y, Z such that $X \succeq Y$, and any constant $t \geq 0$,*

$$E\max\{X+t, Y, Z\} \succeq E\max\{X, Y+t, Z\}.$$

Proof. For any numbers x, y, and z, $\max\{x,y\} = (x-y)^+ + y$. We start with the following identites

$$\begin{aligned} \max\{X+t, Y, Z\} &= (X+t-\max\{Y,Z\})^+ + \max\{Y,Z\}, \\ \max\{X, Y, Z\} &= (X-\max\{Y,Z\})^+ + \max\{Y,Z\}. \end{aligned}$$

Then

$$\begin{aligned} \max\{X+t, Y, Z\} - \max\{X,Y,Z\} &= (X+t-\max\{Y,Z\})^+ - (X-\max\{Y,Z\})^+, \\ \max\{X, Y+t, Z\} - \max\{X,Y,Z\} &= (Y+t-\max\{X,Z\})^+ - (Y-\max\{X,Z\})^+. \end{aligned}$$

From Lemma A2, we know that $\max\{X,Z\} \succeq \max\{Y,Z\}$. Lemma A1 implies that $X - \max\{Y,Z\} \succeq Y - \max\{X,Z\}$. Using Lemma A3 completes the proof. ■

Lemma A5. *Let $q(x,y)$ be a continuous function increasing in both arguments, and let X, Y be two random variables. For any realizations $x_1 > x_2$ of the random variable X, the distribution of Y conditional on x_1 (first-order) stochastically dominates the distribution of Y conditional on x_2. Then $q(x_1, Y) \succeq q(x_2, Y)$.*

Proof. Define $\tau(x,z)$ to satisfy $q(x, \tau(x,z)) = z$. Clearly, $\tau(x,z)$ is increasing in z. Now $F_{q(x_1,Y)}(z) = F_{Y|x_1}(\tau(x_1,z)) \leq F_{Y|x_1}(\tau(x_2,z))$ and $F_{q(x_2,Y)}(z) = F_{Y|x_2}(\tau(x_2,z)) \geq F_{Y|x_1}(\tau(x_2,z))$, the latter inequality following from the fact that $Y|x_1 \succeq Y|x_2$. Therefore, for any z, $F_{q(x_1,Y)}(z) \leq F_{q(x_2,Y)}(z)$. ■

Proof of Lemma 1. Let $\widetilde{\mathbf{a}}$ be a vector of actions with $\widetilde{a}_j = \widetilde{a}_k = 0$ and $\widetilde{a}_i = a_i = a'_i$ for all $i \neq j,k$. Lemma A4 yields that $V(s_j, T_j) \succeq V(s_k, T_k)$ whenever $s_j \geq s_k$. Now one can use Lemma A3 with the constant $b\widetilde{a}_j = b\widetilde{a}_k$.■

Proof of Lemma 2. Let $Z = \max_{j \neq i}\{V(s_j,t_j) + ba_j\})$, and $X = V(s_i, t_i)$. By definition,

$$g_i(\mathbf{s}, \mathbf{a}_{-i}) = E(X+b-Z)^+ - E(X-Z)^+.$$

Using the formula $\max\{x,y\} = (x-y)^+ + y$, we get

$$\begin{aligned} G_i(\mathbf{s}, \mathbf{a}_{-i}) &= E\max\{X+b, Z\} - E\max\{X, Z\} \\ &= E(X+b-Z)^+ + EZ - (E(X-Z)^+ + EZ) \\ &= E(X+b-Z)^+ - E(X-Z)^+ = g_i(\mathbf{s}, \mathbf{a}_{-i}), \end{aligned}$$

as claimed.■

Proof of Lemma 3. First, we claim that for any i, and for any $\mathbf{s}$ and $\tilde{\mathbf{s}}$ such that $\mathbf{s}_{-i} \leq \tilde{\mathbf{s}}_{-i}$ and $s_i = \tilde{s}_i$, $a_i^*(\tilde{\mathbf{s}}) \leq a_i^*(\mathbf{s})$. Indeed, let $X_i(\mathbf{s}) = V(s_i, t_i)$ and $Z_i(\mathbf{s}) = \max_{j \neq i} \left\{ V(s_j, t_j) + ba_j^* \right\}$.

Suppose that $a_j^*(s_i, \mathbf{s}_{-i}) \leq a_j^*(s_i, \tilde{\mathbf{s}}_{-i})$ for all $j \neq i$. Then

$$X_i(s_i, \mathbf{s}_{-i}) - Z_i(s_i, \mathbf{s}_{-i}) \succeq X_i(s_i, \widetilde{\mathbf{s}}_{-i}) - Z_i(s_i, \widetilde{\mathbf{s}}_{-i})$$

by Lemmas A1 and A2. To prove that $g_i(\mathbf{s}) \geq g_i(\tilde{\mathbf{s}})$, recall that

$$g_i(\mathbf{s}) = E\left(X_i(\mathbf{s}) + b - Z_i(\mathbf{s})\right)^+ - E\left(X_i(\mathbf{s}) - Z_i(\mathbf{s})\right)^+,$$

and then apply Lemma A3 to prove the claim. By definition, $g_i(\mathbf{s}) \geq g_i(\tilde{\mathbf{s}})$ implies that $a_i^*(\tilde{\mathbf{s}}) \leq a_i^*(\mathbf{s})$.

It is enough to consider the case of $a_j^*(s_i, \mathbf{s}_{-i}) \leq a_j^*(s_i, \tilde{\mathbf{s}}_{-i})$ for all $j \neq i$. Indeed, if a switch from 1 to 0 occurred with an agent that ends up higher than i as a result of increase from $\mathbf{s}_{-i}$ to $\tilde{\mathbf{s}}_{-i}$, then it is definite that $a_i^*(\tilde{\mathbf{s}}) = 0$, and thus $a_i^*(\tilde{\mathbf{s}}) \leq a_i^*(\mathbf{s})$ for any $a_i^*(\mathbf{s})$. Otherwise (if a change have occurred with an agent ranked lower than the agent i), $a_i^*(\mathbf{s}) = 1$.

Second, we claim that the function g_i increases with s_i. The first claim shows, in particular, that if s_i increases, while $\mathbf{s}_{-i}$ is constant, the number of agents acting (weakly) decreases. Thus, the random variable $X_i(s_i, \mathbf{s}_{-i}) - Z_i(s_i, \mathbf{s}_{-i})$ raises in terms of stochastic dominance, and Lemma A3 applies.

Now we shall prove that

$$E\pi_i(s_i', \hat{s}_i') - E\pi_i(s_i, \hat{s}_i') \geq E\pi_i(s_i', \hat{s}_i) - E\pi_i(s_i, \hat{s}_i),$$

which is equivalent to (2).

Define $X_i = V(s_i, t_i)$, $X_i' = V(s_i', t_i)$, $Y_i = \max_{j \neq i} \left\{ V(s_j, t_j) + ba_j^*(\hat{s}_i, \mathbf{s}_{-i}) \right\}$, and $Y_i' = \max_{j \neq i} \left\{ V(s_j, t_j) + ba_j^*(\hat{s}_i', \mathbf{s}_{-i}) \right\}$.

$$\begin{aligned} E\pi_i(s_i', \hat{s}_i) - E\pi_i(s_i, \hat{s}_i) &= E(X_i' - Y_i)^+ - E(X_i - Y_i)^+, \\ E\pi_i(s_i', \hat{s}_i') - E\pi_i(s_i, \hat{s}_i') &= E(X_i' - Y_i')^+ - E(X_i - Y_i')^+, \end{aligned}$$

and so it remains to prove that

$$E(X_i' - Y_i')^+ - E(X_i - Y_i')^+ \geq E(X_i' - Y_i)^+ - E(X_i - Y_i)^+.$$

The two claims proved above yield that $X_i' \succeq X_i$ and $Y_i \succeq Y_i'$. Using Lemma A3 (for each non-negative constant) completes the proof.■

Proof of Theorem 4. Let s_i be the true agent's i first-period signal, and consider k such that $\bar{s}_i(k) < \hat{s}_i \leq \bar{s}_i(k-1)$. Since $\hat{\mathbf{s}}_{-i}$ is fixed throughout the argument, we suppress the notation. Truthful reporting brings the expected utility of

$$E\pi_i(s_i, \mathbf{a}(k)) - h_i(s_i) = E\pi_i(s_i, \mathbf{a}(k)) - h_i(\bar{s}_i(k)) - E\pi_i(\bar{s}_i(k), \mathbf{a}(k)) + E\pi_i(\bar{s}_i(k), \mathbf{a}(k+1)).$$

First, we prove that the agent i has no incentives to under-report her first-period signal, i.e. to report $\hat{s}_i < s_i$. Consider incentives the agent i with the first-period signal $\bar{s}_i(k)$ faces. For any ε such that $\bar{s}_i(k) - \bar{s}_i(k+1) > \varepsilon > 0$, she is indifferent between reporting $\bar{s}_i(k)$ and reporting $\bar{s}_i(k) - \varepsilon$. Indeed, the 'credibility payment' is the same and the number of acting rivals is the same $(k+1)$. The condition (2) assures that if the agent with $\bar{s}_i(k)$ is indifferent between reporting $\bar{s}_i(k)$ to reporting $\bar{s}_i(k) - \varepsilon$, then the agent with $s_i > \bar{s}_i(k)$ (weakly) prefers reporting $\bar{s}_i(k)$ to reporting $\bar{s}_i(k) - \varepsilon$. Thus, $\hat{s}_i$ can not be less than $\bar{s}_i(k)$. (To rule out reports below $\bar{s}_i(k+1)$, one can consider incentives the $\bar{s}_i(k+1)$-agent faces.) It remains to show that $\hat{s}_i$ (weakly) exceeds $\bar{s}_i(k)$. So, we need to prove that

$$E\pi_i(s_i, \mathbf{a}(k)) - h_i(\bar{s}_i(k)) - E\pi_i(\bar{s}_i(k), \mathbf{a}(k)) + E\pi_i(\bar{s}_i(k), \mathbf{a}(k+1)) \geq E\pi_i(s_i, \mathbf{a}(k+1)) - h_i(\bar{s}_i(k)),$$

or equivalently,

$$E\pi_i(s_i, \mathbf{a}(k)) - E\pi_i(s_i, \mathbf{a}(k+1)) \geq E\pi_i(\bar{s}_i(k), \mathbf{a}(k)) - E\pi_i(\bar{s}_i(k), \mathbf{a}(k+1)),$$

but this is true by (2). Since the agent i having the signal s_i is indifferent between reporting s_i and reporting any signal that is larger than $\bar{s}_i(k)$ and does not exceed s_i, the proof that the agent i has no incentives to under-report her signal is complete.

The proof that there is no incentives to over-report the signal is somewhat symmetric. The s_i-agent is indifferent between reporting the true signal and reporting $\bar{s}_i(k-1)$. Indeed, the mechanism assumes that the agents with reports s_i and $\bar{s}_i(k-1)$ pay the same amount. Now, for any ε such that $\bar{s}_i(k-2) - \bar{s}_i(k-1) > \varepsilon > 0$, the agent with $\bar{s}_i(k-1)$ is indifferent between reporting the true signal and reporting $\bar{s}_i(k-1) + \varepsilon$. To see this, note that

$$\begin{aligned} E\pi_i(\bar{s}_i(k-1), \mathbf{a}(k)) - h_i(\bar{s}_i(k-1)) &= E\pi_i(\bar{s}_i(k-1), \mathbf{a}(k-1)) - h_i(\bar{s}_i(k-1)) \\ &\quad - E\pi_i(\bar{s}_i(k-1), \mathbf{a}(k-1)) + E\pi_i(\bar{s}_i(k-1), \mathbf{a}(k)). \end{aligned}$$

By (2),

$$E\pi_i(\bar{s}_i(k-1), \mathbf{a}(k-1)) - E\pi_i(\bar{s}_i(k-1), \mathbf{a}(k)) \geq E\pi_i(s_i, \mathbf{a}(k-1)) - E\pi_i(s_i, \mathbf{a}(k)).$$

Thus, if the $\bar{s}_i(k-1)$ is indifferent between reporting the truth and reporting $\bar{s}_i(k-1)+\varepsilon$, the s_i-agent (weakly) prefers to report $\bar{s}_i(k-1)$ (which is pay-off equivalent to reporting the truth), than to report $\bar{s}_i(k-1)+\varepsilon$. To show, that $\hat{s}_i$ would not exceed $\bar{s}_i(k-2)$, one should consider the incentives the $\bar{s}_i(k-2)$-agent faces, etc. Therefore, the agent i has no incentives to over-report her first-period signal.$\blacksquare$