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ABSTRACT

Optimal Commitment Policy Under Noisy Information*

This Paper studies an advantage of commitment over discretion when a central bank observes only noisy measures of current inflation and output, in the context of an optimizing model with nominal-price stickiness. Under a commitment regime, if current policy turns out to be too expansionary (contractionary) because of the bank's information problem, subsequent policies should be slightly contractionary (expansionary). By following this approach, the central bank can improve the trade-off between the fluctuations of its goal variables caused by economic shocks and those fluctuations caused by the bank's response to measurement error.

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1 Introduction

This paper studies an optimal monetary policy which requires commitment, compared with discretionary optimization, when a central bank's information is noisy. Monetary policy is inevitably conducted under uncertainty about the state of the economy, and economic data available to a central bank at the time policy decisions are taken are subject to a considerable amount of measurement error. For example, Orphanides (1998a) reports that the standard deviation of the first revision of the US output gap is quite large, 0.66 percent, while that of the output gap itself is 1.78 percent. He also reports that the standard deviation of the first revision of the GDP deflator is 0.23 percent, while that of the GDP deflator itself is 0.68 percent. This implies that more than 30 percent of the fluctuations in the preliminary data may be due to measurement error. This information constraint is particularly important for a central bank which follows a Taylor-type policy rule (Taylor (1993)). When the central bank has only preliminary data, the interest rate chosen by a feedback rule is affected by measurement error. A useful analysis of optimal monetary policy needs to take account of this information problem. Recently, there is a growing literature on the implications of imperfect information about the state of the economy on optimal policy (Smets (1998), Orphanides (1998b), Rudebusch (1999), Aoki (2002), Svensson and Woodford (2000), (2001), Swanson (2000), Dotsey and Hornstein (2000)). This line of research shows that certainty equivalence holds in the sense that the optimal response to the estimate of the state of the economy should be independent of the degree of data uncertainty.[1] However, it is also shown that reduced-form policy in terms of observable variables should exhibit policy cautiousness when these observables are subject to measurement errors.

In the present paper, we draw an additional implication of data uncertainty on optimal policy, focusing on an advantage of commitment over discretion under data uncertainty. We use a forward-looking New Keynesian sticky price model in which optimal policy would be time-consistent if information were perfect, so the analysis below purely focuses on the implication of noisy information for the desirability of commitment. It is shown that the optimal plan under noisy information is in general time-inconsistent, even if the optimal plan under perfect information is time-consistent. This result implies that gains from credibility emerge when the central bank's information set is imperfect. Next we derive how a central bank under commitment should systematically react to its past policy mistakes revealed by data revision. It is shown that, under the commitment regime, if current policy turns out to be too expansionary (contractionary) after data revision, the central bank should set subsequent policy slightly contractionary (expansionary).

[1] In particular, see, Svensson and Woodford (2000), (2001).

By following this approach, the central bank can improve the trade-off between the variability of inflation and the output gap caused by underlying economic shocks and the variability caused by the bank's response to measurement errors.[2] In the dynamic New Keynesian framework, Clarida *et al.* (1999) and Woodford (1999) emphasized that the advantage of commitment may emerge even in the absence of the inflation bias problem of Barro and Gordon (1983). This paper provides a contribution to this line of research – the information problem is a reason for the desirability of commitment.

Furthermore, we consider a feedback policy rule that implements the optimal plan. A feedback rule is a policy rule which is consistent with the optimal equilibrium but depends only on the observed endogenous variables. It is shown that the policy rule should involve persistence which is independent of any persistence of economic shocks, as advocated by Woodford (1999), but for different reasons.

The paper is organized as follows. The next section presents the model of the economy and the problem of optimal monetary policy. Section 3 characterizes the optimal evolution of endogenous variables under commitment, and compares the results with those under discretion. Section 4 characterizes the equilibrium and the optimal filtering problem. Section 5 provides some numerical examples to compare in detail the optimal plan with discretionary optimization, and to consider the effects of noise on the optimal policy. Section 6 discusses implementation of the optimal plan by a feedback rule. Section 7 concludes.

2 Central Bank's Problem

2.1 Structural Equations and the Objective of the Central Bank

The model is a simple variant of the dynamic sticky price models which have often been used in the recent research on monetary policy. The structure of the economy is described by a log-linearized Phillips curve and an expectational IS curve (Kerr and King (1996), Woodford (1996), Bernanke and Woodford (1997), Clarida *et al.* (1999), McCallum and Nelson (1999a), (1999b)). The model used throughout the paper is identical to that used in Aoki (2002) who characterizes the discretionary optimal policy under data uncertainty. In the subsequent sections we compare the commitment equilibrium with the discretionary equilibrium that is analyzed in Aoki (2002). Throughout the paper, we compare our commitment plan with the discretionary plan derived in Aoki (2002).

The expectational IS equation is given by

$$y_t = E_t y_{t+1} - \sigma \left[R_t - E_t \pi_{t+1} - \rho_t\right], \quad \sigma > 0, \tag{1}$$

[2] The trade-off facing the central bank in this model will be discussed in detail in Section 5.3.

where y_t, π_t, R_t are the time t output, inflation rate, and the nominal interest rate, respectively.[3] This IS equation can be obtained from a log-linear approximation to the Euler equation for the optimal timing of expenditure by the representative household under perfect financial markets. The parameter σ can be interpreted as the intertemporal elasticity of substitution of expenditure, and the exogenous disturbance ρ_t represents a demand shock, which may arise from autonomous variations in spending not motivated by intertemporal substitution in response to the real interest rate.

The aggregate supply equation is represented by an expectational Phillips curve of the form

$$\pi_t = \kappa (y_t - y_t^n) + \beta E_t \pi_{t+1}, \quad \kappa > 0, \ 0 < \beta < 1, \tag{2}$$

where y_t^n is an exogenous supply shock at time t. It represents the potential level of output, which would be the equilibrium level of output if prices were fully flexible. This Phillips curve can be derived from a log-linear approximation to the first-order condition for the optimal price-setting decision of a firm in sticky price models, such as Calvo (1983)'s staggered price model.[4] The parameter κ is a measure of the speed of price adjustment, and β can be interpreted as the discount factor of price setters. Equations (1) and (2), with a monetary policy reaction function which sets the nominal interest rate, determine the equilibrium paths of inflation, output, and the nominal interest rate.

For concreteness, we assume the following stochastic processes for the demand and supply shocks:

$$y_t^n = \delta y_{t-1}^n + e_{yt}, \quad 0 < \delta < 1, \tag{3}$$

$$\rho_t = \eta \rho_{t-1} + e_{\rho t}, \quad 0 < \eta < 1, \tag{4}$$

$$e_{yt} \sim N\left(0, s_y^2\right), \ e_{\rho t} \sim N\left(0, s_\rho^2\right), \tag{5}$$

where e_{yt} and $e_{\rho t}$ are independent and serially uncorrelated at all leads and lags.

Expectations of the future (i.e., $E_t[\pi_{t+1}]$ and $E_t[y_{t+1}]$) play an important role in determining equilibrium. We can solve equation (1) forward to obtain

$$y_t = -\sigma E_t \sum_{i=0}^{\infty} [R_{t+i} - \pi_{t+1+i}] + \sigma \rho_t,$$

which implies that aggregate demand depends not only on the current short term nominal interest rate, but also on the long-run real interest rates, which in turn depend on the expected future short term interest

[3] All variables are measured in percentage deviations from their steady state values in a steady state with zero inflation.

[4] Roberts (1995) calls aggregate supply equations of this kind "New Keynesian Phillips curves".

rates. On the other hand, since firms cannot continuously adjust their prices, they set their prices on the basis of expectations of future cost and demand conditions, and not just on the basis of current conditions. This results in the expectation term in equation (2).[5]

It is convenient to express the model in terms of the output gap $y_t - y_t^n$. Define $\tilde{y}_t \equiv y_t - y_t^n$, $\tilde{\pi}_t \equiv \pi_t$, and $\tilde{R}_t \equiv R_t - R_t^*$, where R_t^* is the natural interest rate as a function of the demand and supply shocks (ρ_t, y_t^n), defined by

$$R_t^* = \frac{\delta - 1}{\sigma} y_t^n + \rho_t. \tag{6}$$

Then the structural equations can be written as

$$\begin{aligned} \tilde{y}_t &= E_t \tilde{y}_{t+1} - \sigma \left(\tilde{R}_t - E_t \tilde{\pi}_{t+1} \right), \\ \tilde{\pi}_t &= \kappa \tilde{y}_t + \beta E_t \tilde{\pi}_{t+1}. \end{aligned}$$

The interest rate (6) is a version of the so-called "Wicksellian natural rate of interest."[6] It is an equilibrium real interest rate that would arise under flexible prices. Thus it is the real interest rate that equates output to its natural level. The exogenous disturbances that matter for the determination of inflation and the output gap are all summarized by the natural rate, and they matter only through the "interest-rate gaps," $\tilde{R}_t$. For the rest of the analysis, we express the structural equations in matrix form as

$$B x_t = A E_t x_{t+1} + C \tilde{R}_t, \tag{7}$$

where $x \equiv (\tilde{\pi}_t, \tilde{y}_t)$, and

$$A \equiv \begin{bmatrix} \sigma & 1 \\ \beta & 0 \end{bmatrix}, \quad B \equiv \begin{bmatrix} 0 & 1 \\ 1 & -\kappa \end{bmatrix}, \quad C \equiv \begin{bmatrix} -\sigma \\ 0 \end{bmatrix}.$$

The upper row represents the IS equation, and the lower row represents the Phillips curve. By inverting A, we obtain the reduced form

$$\begin{aligned} E_t x_{t+1} &= M x_t + N \tilde{R}_t, \qquad (8) \\ M &\equiv \beta^{-1} \begin{bmatrix} 1 & -\kappa \\ -\sigma & \sigma\kappa + \beta \end{bmatrix}, \quad N \equiv \begin{bmatrix} 0 \\ \sigma \end{bmatrix}. \end{aligned}$$

[5] Empirical support for this forward-looking aggregate supply equation is found in Roberts (1995), Sbordone (1998), and Gali and Gertler (1999). Moreover, Rotemberg and Woodford (1997) show that an estimated model similar to ours (our model is essentially a simplified version of theirs) provides a good description of the actual behavior of inflation, output, and the quarterly average of the Federal Funds rate in the US from 1979 to 1995.

[6] For the recent discussion about this concept, see Blinder (1998) and Woodford (1998,1999).

We turn to the analysis of monetary policy for the economy described in this model. The welfare loss the central bank seeks to minimize is the expected discounted sum of period loss functions

$$L \equiv E\left[\sum_{t=0}^{\infty} \beta^t L_t\right]. \tag{9}$$

Here unconditional expectation is taken with respect to the initial state. The period loss function is the weighted sum of the squared output gap and squared inflation, given by

$$\begin{aligned} L_t &= \frac{1}{2}\left[\pi_t^2 + a\left(y_t - y_t^n\right)^2\right] \\ &= \frac{1}{2} x_t' W x_t, \end{aligned} \tag{10}$$

where W represents the weighting matrix

$$W \equiv \begin{bmatrix} 1 & 0 \\ 0 & a \end{bmatrix}, \quad a > 0.$$

Rotemberg and Woodford (1997) show that the loss measure (9) with (10) is a quadratic approximation to the expected utility of the representative household in the Calvo model, when $a > 0$ is appropriately chosen.[7]

In our model, stabilizing inflation at zero is consistent with stabilizing the output gap, since in any stationary equilibrium (2) implies

$$\pi_t = \kappa \sum_{i=0}^{\infty} \beta^i E_t\left(y_{t+i} - y_{t+i}^n\right). \tag{11}$$

If the central bank successfully stabilized the output gap in each period, then equilibrium inflation would be zero in each period in the rational expectations equilibrium. Therefore, in our model there is no trade-off between eliminating the output gap and stabilizing inflation at zero. It is shown from equations (1) and (2) that this equilibrium requires nominal interest rates equal to the natural rates at all times. In other words, if the central bank sets nominal interest rates equal to the natural rates at all times, it can completely stabilize inflation and the output gap, and such an equilibrium achieves the theoretical minimum of our loss measure (9).

Furthermore, this equilibrium is shown to be time-consistent, i.e., the optimal commitment plan coincides with the optimal discretion plan where the central bank expects itself to re-optimize at each successive date.[8] One reason for this is that the target level of the output gap is zero. This implies that the natural

[7] See, also, Woodford (1999). Our loss measure corresponds to his loss measure when the central bank does not face the zero nominal interest rate bound.

[8] Formal discussion is provided in section 3.

level of output is assumed to be efficient so that the central bank does not have any incentive to increase output beyond its natural level. When the target output gap is not zero, time inconsistency occurs for the same reason as in Barro and Gordon (1983)'s analysis of systematic inflation bias. Another reason is that we do not assume any other reasons for the first-level of welfare not to be attainable, such as "cost-push shocks" assumed in Clarida *et al.* (1999) or the zero bound on nominal interest rates assumed in Woodford (1999).

2.2 Information

Implementation of optimal policy requires knowledge about the states of the economy. A central bank needs to know the realization of the current demand and supply shocks in order to set the current nominal interest rate equal to the natural rate. However, in the real economy, it is very difficult for the central bank to directly observe these shocks. As in Aoki (2002), we assume that the central bank has accurate measures of all relevant state variables in previous periods, but only noisy measures of output and inflation in the current period, at the time that it must set the nominal interest rate. This assumption captures the fact that the central bank has only inaccurate estimates of current economic variables, but has more accurate measures of past variables.[9]

To be precise, the central bank at time t does not observe the true measures of current inflation, output, and the underlying economic shocks, but observes the following measures of inflation and output:

$$Z_t \equiv \left(\pi_t^o, y_t^o\right)',$$

where

$$\begin{aligned} \pi_t^o &\equiv \pi_t + \varepsilon_{\pi t}, \quad y_t^o \equiv y_t + \varepsilon_{yt}, \\ & \varepsilon_{\pi t} \sim N\left(0, \sigma_\pi^2\right), \quad \varepsilon_{yt} \sim N\left(0, \sigma_y^2\right). \end{aligned} \tag{12}$$

Here random variables $\varepsilon_{\pi t}$ and ε_{yt} represent measurement errors. The relationship between x_t and Z_t is given by

$$\begin{aligned} Z_t &\equiv x_t + F \begin{bmatrix} y_t^n \\ \rho_t \end{bmatrix} + \varepsilon_t, \\ F &\equiv \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \varepsilon_t \equiv \left(\varepsilon_{\pi t}, \varepsilon_{yt}\right)'. \end{aligned} \tag{13}$$

[9] See Orphanides (1998a) for the discussion about data revisions.

Now we define the information set of the central bank. Let $S_t \equiv (e_{yt}, e_{\rho t}, \varepsilon_{\pi t}, \varepsilon_{yt})$ be a vector of realization of the shocks at time t. History at time t is a sequence of realizations S_t, denoted by $h_t \equiv \{S_i\}_{i=0}^{t}$. Define the full information of the economy at time t as $\Omega_t \equiv \left\{ \{\pi_i\}_{i=0}^{t}, \{y_i\}_{i=0}^{t}, \{R_t\}_{i=0}^{t}, h_t \right\}$.[10] Then the central bank's information set at time t is defined by

$$I_t \equiv \{Z_t, R_t, \Omega_{t-1}\}.$$

In other words, the central bank's information set at time t consists of the noisy measures of the current inflation and output, the interest rate, and the complete information about the past states (Ω_{t-1}).

Finally, in order to focus our analysis on the implications of the information constraint facing the central bank, we abstract from any uncertainty facing the private agents, as in Aoki (2002) and Svensson and Woodford (2001). The private sector is assumed to have complete knowledge about the states of the economy, including the realization of current inflation and output. One justification for this assumption is that consumption and production decisions are not as dependent on the availability of aggregate data as the policy decision of the central bank. For example, each firm needs to know its own production cost and the prices of close substitutes for the good it produces, but it does not necessarily need to know those of all industries. Furthermore, it is plausible that a firm has more information about variations in its production capacity (i.e., supply shocks) than the central bank does. Therefore, it is of great interest to consider a situation in which the central bank faces more serious measurement problems than the private agents. Our assumption is one of the simplest among such situations.[11]

3 Optimal Plan under Imperfect Information

3.1 The Central Bank's Minimization Problem

We now turn to the central bank's minimization problem under noisy information. First of all, we consider what pattern of the evolution of the endogenous variables (inflation, the output gap, and the interest rate) should be associated in an optimal equilibrium that minimizes the expected discounted sum of period loss functions

$$L \equiv E\left[\sum_{t=0}^{\infty} \beta^t L_t\right]. \tag{14}$$

[10] Term h_t may be redundant in equilibrium, since the central bank can identify h_t from $\{\pi_i\}_{i=0}^{t}, \{y_i\}_{i=0}^{t}, \{R_t\}_{i=0}^{t}$.

[11] See, also, Svensson and Woodford (2001) and Dotsey and Hornstein (2000). An implication of asymmetric information, compared with symmetric information, is that although certainty equivalence still holds the separation principle does not hold any more. This issue is discussed later.

The unconditional expectation is taken with respect to the initial state. In the case of noisy information, the central bank's information set in each period (including period zero) depends on the way the bank responds to observable variables. In other words, information revealed by the observable variables depends on the bank's policy reactions. In this case, the central bank must consider what pattern of the evolution of the observable variables, including those at period zero, would be desirable. Thus we consider a situation where the central bank commits itself before the realization of any random variables to a state-contingent plan that minimizes (14).

The structural equations (1) and (2) constitute part of the constraints facing the central bank. Another important constraint is the way its policy instrument responds to the observable variables. Since the central bank has to choose the nominal interest rate based on its information set, the interest rate at time t should satisfy

$$R_t = R_{t|I_t}, \tag{15}$$

where $z_{t|I_t}$ denotes the expectation of random variable z_t conditional on the bank's information set I_t. In other words, the interest rate has to be measurable with respect to the bank's information set. As mentioned in the previous section, we assume that while the central bank has complete knowledge about the past states of the economy, it has only noisy indicators Z_t of the current state. Under this assumption, equation (15) implies

$$R_t = R_{t|I_t} \equiv E_{t-1}[R_t] + Cov_{t-1}[R_t, Z_t]' Var_{t-1}[Z_t]^{-1} (Z_t - E_{t-1}[Z_t]), \tag{16}$$

where, $E_{t-1}[\cdot]$, $Cov_{t-1}[\cdot]$, and $Var_{t-1}[\cdot]$ represent the conditional expectation, covariance, and variance, respectively, conditional on the full information set at time $t-1$. Equation (16) represents the information constraint facing the central bank.[12] Finally, when constructing the optimal policy, the central bank must take into account the fact that its pattern of action affects the statistical relation between the observable variables and the underlying exogenous economic shocks. Therefore, the observation equation (13) is also one of the constraints of the central bank.

3.2 Optimal Policy in Terms of State of Economy

The central bank minimizes (14) subject to (7), (13), and (16). This kind of linear-quadratic problem can be solved by Lagrangian method (Svensson and Woodford (2000)). Using the law of iterated expectation,

[12] When all of the disturbances are jointly normally distributed, this is identical to the minimum mean square estimator. Even if they are not normally distributed, considering a class of linear policy of this kind is of great interest for its simplicity.

the Lagrangian associated with the minimization problem is given by

$$E \sum_{t=0}^{\infty} \beta^t \Bigg[L_t + \phi_t' \left\{ -Ax_{t+1} + Bx_t - C\left(R_{t|I_t} - R_t^*\right)\right\} \tag{17}$$

$$+ \mu_t' \left\{ Z_t - x_t - F \begin{pmatrix} y_t^n \\ \rho_t \end{pmatrix} - \varepsilon_t \right\} \Bigg],$$

where $\phi_t \equiv (\phi_{1t}\ \phi_{2t})'$ and $\mu_t \equiv (\mu_{1t}\ \mu_{2t})'$ are the Lagrange multipliers associated with the structural equations and the observation equations respectively. In (17), the nominal interest rate at time t has to be chosen based on the bank's information set at time t, so that $R_t = R_{t|I_t}$, which is given by (16). Note that the Lagrange multipliers are measurable with respect to the full information set at time t, but are not necessarily measurable with respect to I_t. Optimal equilibrium is obtained by taking the first order conditions with respect to all endogenous variables. The first order conditions with respect to x_t, R_t, Z_t are given by

$$Wx_t - \beta^{-1}A'\phi_{t-1} + B'\phi_t - \mu_t = 0, \tag{18}$$

$$C'\phi_{t|I_t} = 0, \tag{19}$$

$$-C'\left(\phi_t - \phi_{t|I_t}\right)\left[Cov_{t-1}\left[R_t, Z_t\right]' Var_{t-1}\left[Z_t\right]^{-1}\right]' + \mu_t = 0. \tag{20}$$

Conditions (18) - (20) must hold for any possible states. Equation (19) is conditioned on I_t, since the interest rate must take the same value in each state of the world in I_t. The derivation of (20) is provided in the Appendix. Furthermore, we stipulate the initial condition

$$\phi_{-1} = \mu_{-1} = 0. \tag{21}$$

These Lagrange multipliers are not associated with any actual constraints of the central bank when it solves the minimization problem (17). Equations (19) and (20) imply that

$$\mu_{t|I_t} = 0. \tag{22}$$

Therefore the observation equations are irrelevant for the determination of the optimal plan conditional on the central bank's information set.[13] Notice that μ_t represents the shadow values of relaxing constraints (13). Equation (22) implies that, conditional on the central bank information set, there should be no gain from relaxing constraints (13) in the optimal equilibrium. This in turn implies that the central bank should have no reason to change its policy decision even if it observes that the realized values of the indicators

[13] See Svensson and Woodford (2000), (2001) for further discussion.

do not equal their values which would be optimal if information were perfect. In other words, the central bank must be self-confirmed in the desirable equilibrium. This is a concept of noisy rational expectations equilibrium.

We show in the Appendix that the optimal evolution of the Lagrange multipliers conditional on the bank's information is given by

$$\begin{aligned}
\phi_{t|I_t} &= \lambda_1 \begin{bmatrix} 0 & 0 \\ b & 1 \end{bmatrix} \phi_{t-1} \equiv \lambda_1 D \phi_{t-1}, \qquad (23)\\
\phi_{t+j|I_t} &= \lambda_1^j \phi_{t|I_t} = \lambda_1^{j+1} D \phi_{t-1} \quad \text{for } j \geq 1,
\end{aligned}$$

where $0 < \lambda_1 < 1$ and b is a constant. Taking the conditional expectation of (18) conditional on the bank's information set and substituting (23), one can solve for $x_{t|I_t}$ as

$$\begin{aligned}
x_{t|I_t} &= \Gamma \phi_{t-1}, \qquad (24)\\
\Gamma &\equiv W^{-1}\left(\beta^{-1} A' - \lambda_1 B' D\right).
\end{aligned}$$

Equation (24) shows that the optimal evolution of the endogenous variables depends on the predetermined variable ϕ_{t-1}. Also, we obtain the optimal evolution of the future endogenous variables conditional on I_t as

$$x_{t+j|I_t} = \Gamma \phi_{t+j-1|I_t} = \lambda_1^j \Gamma D \phi_{t-1} \quad \text{for } j \geq 1. \qquad (25)$$

The interest-rate gap that is consistent with the optimal equilibrium can be calculated from (8), (24), and (25) as

$$\begin{aligned}
\tilde{R}_{t|I_t} &= \Lambda \phi_{t-1}, \qquad (26)\\
\Lambda &\equiv \sigma^{-1}\left[-\left(0\ 1\right)\Gamma + \lambda_1 \left(\sigma\ 1\right)\Gamma D\right].
\end{aligned}$$

Then the expected path of the future interest-rate gaps is given by

$$\tilde{R}_{t+j|I_t} = \lambda_1^j \Lambda D \phi_{t-1} \quad \text{for } j \geq 1. \qquad (27)$$

Equations (26) and (27) represent the optimal policy in terms of the state of the economy.[14] In other words, it is an optimal response to the bank's estimate of the state variables. Note that neither λ_1 and Λ depend on σ_π and σ_y. Certainty equivalence therefore holds, in the sense that optimal policy response to the state of the economy is independent of degree of noise.[15] It is worth emphasizing that equation (26)

[14] In Section 4 we derive optimal policy in terms of observable variables.

[15] See Svensson and Woodford (2001) for further discussion.

does not contradict certainty equivalence even if $\tilde{R}_t = 0$ is the optimal commitment policy under perfect information. In the case of perfect information, it is easily shown that $\phi_{t-1} = 0$ at all times in the optimal equilibrium. Intuitively, since inflation and the output gap are perfectly stabilized, there is no gain from relaxing the constraints imposed by the structural equations. It is in general not equal to zero in the case of imperfect information, since perfect stabilization is impossible. However, certainty equivalence does not require that the Lagrange multipliers should evolve in a particular way.

It is of interest to compare the optimal plan with the discretion plan that is analyzed in Aoki (2002). Using the same model, Aoki (2002) characterizes the discretionary optimal policy under the same information structure. The difference between the first order conditions for the commitment plan and those for the discretion plan is the existence of the lagged Lagrange multiplier in (18). Aoki (2002) shows that a central bank under discretion that solves a corresponding problem at time t chooses the processes for dates $i \geq t$ that satisfy the first order conditions (18) - (20), *and*

$$\phi_{t-1} = \mu_{t-1} = 0$$

for all t. In this case, the optimal equilibrium conditional on the central bank's information set is given by

$$x_{t|I_t} = 0$$

for all t. In other words, the central bank seeks to completely stabilize inflation and the output gap in each period conditional on the bank's information set. The interest-rate gap consistent with this equilibrium is $\tilde{R}_{t|I_t} = 0$ at all times, i.e., the central bank should choose the interest rates equal to its conditional estimate of the natural rates.

The optimal commitment plan is time-consistent only if $\phi_{t-1} = 0$ for all t. It is easy to show that $\phi_t = 0$ for all t can be an equilibrium only if $x_t = 0$ for all t, i.e., inflation and the output gap are perfectly stabilized at all times. However, this may not always be possible when the central bank cannot perfectly observe the current demand and supply shocks. In other words, imperfect information prevents the bank from achieving the first-best equilibrium. Then the optimal plan is no longer time-consistent. Equation (24) also implies that period-by-period stabilization of inflation and the output gap is not optimal under imperfect information even if there is no output-gap-inflation variability trade-off. The optimal policy given by (26) and (27) calls for persistent deviations of the nominal interest rates from the natural rates. Note also that, even though no lagged variables play any roles in the determination of current inflation and the output gap in the IS curve and the Phillips curve, the optimal path of the endogenous variables depend on the lagged variables. In other words, the path of the desirable equilibrium given by (24) and (25) involves persistence which is independent of any persistence of the underlying economic shocks. The persistent path of the interest rate given by (27) agrees with Woodford (1999)'s finding that the optimal policy involves persistence in the nominal interest rate.

3.3 Equilibrium and Optimal Filtering

We now fully characterize the optimal equilibrium under commitment, and derive optimal policy reaction function in terms of observable variables. The equilibrium is of course not independent of the central bank's response to the noisy indicators Z_t, since it depends not only on the demand and supply shocks but also on the policy of the central bank. On the other hand, the optimal response to Z_t depends on the equilibrium statistical relation between Z_t and the underlying shocks. Therefore equilibrium and filtering should be simultaneously determined. We take the following two steps to characterize equilibrium and filtering. Firstly, we characterize inflation and the output gap in the desirable equilibrium taking the policy response to Z_t as given, and next we derive the optimal policy response to Z_t.

Firstly, we use the structural equations and the first order conditions to derive the evolution of inflation and the output gap, x_t, in the optimal equilibrium. It is shown in the Appendix that x_t is given by

$$x_t = \Gamma\phi_{t-1} + \Phi^{-1}N\Xi e_t + \Phi^{-1}Nr\varepsilon_t, \tag{28}$$

where $r \equiv Cov_{t-1}[R_t, Z_t]' Var[Z_t]^{-1}$ represents the optimal response coefficients on noisy indicators Z_t. Similarly, the equilibrium Lagrange multipliers are given by

$$\phi_t = \lambda_1 D\phi_{t-1} + \Psi\Phi^{-1}N\Xi e_t + \Psi\Phi^{-1}Nr\varepsilon_t. \tag{29}$$

Equations (28) and (29) jointly determine the evolution of the endogenous variables. Notice that x_t and ϕ_t depend on the specification of the central bank's information.[16] Also, (28) and (29) depend on the policy coefficient r. As is shown in Svensson and Woodford (2001), the separation principle does not hold under asymmetric information, in the sense that estimation of the state of the economy is not independent of optimal policy and the information structure.

Secondly, we derive the optimal response coefficient r to Z_t. The optimal interest rate is given by (26),

[16] In deriving (28) and (29), we use the assumption that the central bank has perfect information about the past state of the economy, as is shown in Appendix.

which is written as

$$
\begin{aligned}
R_t &= R^*_{t|t-1} + Pe_{t|I_t} + \Lambda\phi_{t-1}, \qquad (30)\\
P &\equiv \begin{bmatrix} \frac{\delta-1}{\sigma} & 1 \end{bmatrix}.
\end{aligned}
$$

Therefore, our filtering problem reduces to the optimal updating of the central bank's belief about the innovations in the demand and supply shocks, e_t. To solve the filtering problem we need to derive the equilibrium relationship between Z_t and the underlying economic shocks, using the first order conditions and the structural equations. From (13) and (28), we obtain a relationship between the innovations in Z_t and those in the underlying shocks, given by

$$
(\Gamma\Psi - M - Nr)\left(Z_t - Z_{t|t-1}\right) = \left[(\Gamma\Psi - M)(F - NP)\right] e_t + (\Gamma\Psi - M)\,\varepsilon_t.
$$

Since e_t and ε_t are assumed to be jointly normal, we can calculate the updated belief about e_t as

$$
\begin{aligned}
e_{t|I_t} &= Cov_{t-1}\left[(\Gamma\Psi - M - Nr)\, Z_t, e_t\right]' Var_{t-1}\left[(\Gamma\Psi - M - Nr)\, Z_t\right]^{-1}\\
&\quad * (\Gamma\Psi - M - Nr)\left(Z_t - Z_{t|t-1}\right)\\
&\equiv K\left(Z_t - Z_{t|t-1}\right).
\end{aligned}
$$

Since (30) implies that

$$
r\left(Z_t - Z_{t|t-1}\right) = Pe_{t|I_t},
$$

the optimal coefficient r solves

$$
r = PK, \qquad (31)
$$

where K itself is a function of r. In other words, r is the fixed point of function (31). Note that (31) is consistent with the information constraint (16). Algebraic solution for r is rather complex, so we provide some numerical examples to characterize the properties of r.

4 Numerical Examples

In this section, we provide some numerical examples to discuss the properties of the optimal commitment plan. Parameter values of the structural equations and the welfare function are cited from Rotemberg and Woodford (1997, 1999), who find that these values result in the best fit between the estimated impulse responses of inflation and output to a monetary policy shock in the US and those predicted by their model (our model is essentially a simplified version of theirs). These values are $\beta = 0.99$, $\sigma = 6.365$, $\kappa = 0.0238$, $a = 0.047$, $\eta = 0.92$, $\delta = 0.15$.[17]

Properties of the optimal policy reported below are not sensitive to variations in the values of the structural parameters assumed above. More important parameters are the standard deviations of the measurement errors in comparison to those of the innovations in the underlying economic shocks. The standard deviations of the innovations in the demand and supply disturbances in the Rotemberg-Woodford model are calculated as 1.022 percent and 1.906 percent, respectively.[18] We choose s_ρ and s_y equal to these values. Standard deviations of the measurement errors are based on Orphanides (1998a). Since we assume in our model that data revision is completed one period later, we calculate the standard deviations of the cumulative revisions of the inflation and output measures from his estimates, and choose σ_π and σ_y equal to these values. They are $\sigma_y = 0.89$ percent and $\sigma_\pi = 0.35$ percent. We set these values as our benchmark case, but later in this section we vary these in order to see effects of the degree of noise on optimal policy. The "small noise" case sets $(\sigma_\pi, \sigma_y) = (0.192, 0.488)$, and the "large noise" case sets $(\sigma_\pi, \sigma_y) = (1.400, 3.560)$. In the small noise case, the standard deviations of the measurement errors are about a half of the estimated noise case, and in the large noise case, these are four times larger.

It is of interest to compare the optimal commitment plan with the discretion plan that is analyzed in Aoki (2002). Comparison with the discretion plan would show how the central bank under commitment can improve economic welfare. For this purpose, we also present numerical results under discretion.

4.1 Property of r and Welfare

Figures (1.1) - (1.2) plot the optimal policy coefficient to the noisy indicators, r, against the standard deviation of the measurement errors. Solid lines and dashed lines represent r under commitment and discretion, respectively. Figure (1.1) plots r_π against σ_π, setting $\sigma_y = 0.890$. Conversely, Figure (1.2) plots r_y against σ_y, setting $\sigma_\pi = 0.350$. These figures respectively show that r_π is decreasing in σ_π, and that r_y is decreasing in σ_y under both regimes. Intuitively, when indicators are very noisy, their fluctuations are more likely to be caused by measurement errors rather than demand and supply shocks which the central

[17] Our definition of the demand disturbances follows Bernanke and Woodford (1997). That is,

$$\rho_t = E_t G_{t+1} - G_t$$

where G_t is Rotemberg-Woodford (1999)'s definition of the demand disturbance. If we assume that G_t follows an AR(1) process with autoregressive root 0.92, it is easily shown that ρ_t also follows an AR(1) process with the same autoregressive root.

[18] I thank Marc Giannoni for providing these values. Specifically, these are the standard deviations of the innovations in the processes $E_{t-2}[G_{t+1} - G_t]$ and $E_{t-2} Y_t^S$ where G_t and Y_t^S are Rotemberg-Woodford's demand and supply disturbances. This is because their structural equations coincide with our simpler model only when conditioned on information available two periods earlier.

bank should respond to. Measurement error is therefore a reason for policy cautiousness, even if optimal policy in terms of the state variables (26) exhibits certainty equivalence.[19]

Figure (1.1) also shows that r_π is smaller under commitment than under discretion for larger values of σ_π, but is larger for smaller values of σ_π. This implies that when the degree of noise is larger, the central bank responds to noisy indicators more cautiously under commitment than under discretion. On the other hand, when the degree of noise is smaller, the central bank responds more strongly under commitment.[20] Notice that, in the absence of any measurement errors, our structural model implies that the central bank can attain the theoretical minimum of its loss measure (14) by responding to inflation very rigorously ($r_\pi = \infty$), as is shown in Bernanke and Woodford (1997). In other words, Figure (1.1) implies that the commitment plan converges to the optimal plan under perfect information more quickly than the discretion plan does.

Figures (2.1) and (2.2) plot the loss measure L, defined by (14), against σ_π and σ_y, respectively. Here independent drawings from the same distribution of shocks (demand shock, supply shock, and measurement errors) are assumed to occur in each period, and infinite-horizon stochastic equilibria are computed under each policy regime. The Figures show that L is increasing in the degree of noise under both regimes. They also show that the commitment plan achieves lower L for all possible values of σ_π and σ_y. Table (1) also gives the comparison of L under the two regimes for each case. It shows that the commitment plan does not significantly improve welfare in the estimated noise case. However, in the large noise case, the benefit of commitment becomes significant, improving welfare by 13.21 percent. The commitment plan also improves welfare by 2.18 percent in the small noise case.

4.2 Impulse Responses

How does the commitment plan improve welfare in our model? The policy response to the lagged variables, represented in equation (26), and the private-sector expectations about future monetary policy play important roles. In order to discuss further the properties of the commitment equilibrium, we present some impulse responses of the endogenous variables – inflation, the output gap, the interest-rate gap, and the nominal interest rate – to the economic shocks and the measurement errors. In Figures (3.1) - (4.2), we present the impulse responses under commitment by solid lines, and those under discretion by dashed

[19] Aoki (2002) analytically derives this property in the case of discretion. See, also, Orphanides (1998b), Rudebusch (1998), Smets (1998), Svensson and Woodford (2000), (2001), Swanson (2000).

[20] Intuition for this result will be discussed in Section 5.3. It may appear that r_y does not have this property, but this depends on the assumed value of σ_π in Figure (1.2). Indeed, Table 1 shows that r_y is larger under commitment than discretion in the small noise case.

lines. Since ϕ_{-1} is equal to zero, (26) implies that the central bank under commitment as well as under discretion should choose the interest-rate gap equal to zero in period zero. Therefore, the interest-rate gap in period zero purely represents policy error caused by the information problem, i.e., the bank's inability to determine the nature of the shock. Note also that the paths of the interest-rate gaps after period one in the Figures represent the private-sector expectations about the future monetary policy. Since information is asymmetric between the central bank and the private agents, their expectations are different from each other.[21] What matters for determination of inflation and the output gap at time 0 is *private* expectations about future monetary policy.

In this section, we focus on the impulse responses to supply shock and inflation noise.[22] To save length of the paper we only discuss the large noise case and the small noise case. Since the optimal coefficient r is smaller under commitment than under discretion in both of the estimated noise case and the large noise case, the qualitative properties of the equilibria are essentially the same in these two cases.

4.2.1 Large Noise Case

Responses to a Supply Shock Figure (3.1) shows the impulse responses to a supply shock in the large noise case. Here a unit positive innovation in e_{yt} is assumed to occur in period zero, which increases the natural level of output by one unit and decreases the natural interest rate by 0.13 according to (6). Equation (26) implies that the central bank should have decreased the nominal interest rate in response to that decrease in the natural rate. However, the central bank cannot tell if an observed increase in the output measure is caused by a supply shock, by a demand shock, or by a measurement error. Given the relative size of the variances of the innovations in the demand and supply shocks, the central bank believes that the observed increase in the output measure is more likely to be caused by a demand shock rather than a supply shock, so it increases the nominal interest rate.[23] Here the increase in the interest rate (the positive interest-rate gaps) results in the negative output gap, leading to deflation.

Since the central bank is assumed to have perfect information about the past states of the economy, it will know in period one that a unit supply shock occurred in period zero. Under discretion, the central bank

[21] The central bank expectations at time 0 are given by (27) as

$$\tilde{R}_{t+j|I_0} = \lambda_1^j \Lambda \phi_{-1} = 0, \quad j \geq 1.$$

Private expectations, on the other hand, are given by

$$E_0\left[\tilde{R}_{t+j}\right] = E_0\left[\tilde{R}_{t+j|I_{t+j}}\right] = \lambda_1^j \Lambda \phi_0 \neq 0.$$

[22] Impulse responses to the other shocks are presented in a longer version of the paper, which is available from the author.

[23] Aoki (2002) analytically derives the condition for r_y to be positive in the case of discretionary optimization.

will choose the interest-rate gaps equal to zero in subsequent periods. Thus the path of the nominal rate does not involve any persistence which is independent of the persistence of the natural rate. The resulting impulse responses therefore die out one period later. In contrast, the central bank under commitment will lower the interest rate in the next period, resulting in a negative interest-rate gap by a small amount. The interest-rate gap becomes positive again in period two, and then converges gradually to zero afterward. The rate of convergence is given by $\lambda_1 = 0.899$ under our parameter values. The resulting fluctuations of inflation and the output gap in period zero are smaller under commitment than under discretion, even though r is smaller. Furthermore, under the commitment regime, inflation and the output gap become positive in the next period, and then gradually converge to zero.[24]

A small decrease in the output gap in period zero implies a large increase in the output level. Since the policy responds to the measures of inflation and output (not to the output gap), the nominal interest rate increases more under commitment than it does under discretion. Figure (3.1) shows that the corresponding initial interest-rate gap is larger under commitment, implying that "equilibrium" policy errors may be larger even if r is smaller. However, even if equilibrium policy errors may be larger under commitment, the variabilities of inflation and the output gap are smaller.

The property of the path of the interest rate described above is common for the other shocks. In period one, the central bank under commitment will adjust the interest-rate gap in the direction opposite to that of the policy which it realizes was mistaken in period zero, but by a small amount. In other words, if the policy turns out to be too contractionary (expansionary), the policy in the next period should be slightly expansionary (contractionary). Then the interest-rate gap gradually converges to zero. Next, we discuss the impulse responses to an inflation noise.

Responses to Inflation Noise The impulse responses to an innovation in the measurement error in inflation measure are shown in Figure (3.2).[25] When a unit increase in the inflation noise occurs, the observed measure of inflation increases by one unit (keeping the other things constant). Since the natural interest rate remains constant, the central bank should not have responded to the observed increase in the inflation measure. However, the central bank cannot avoid responding to the observed increase in

[24] The subsequent inflation and the output gaps remain positive even if the subsequent interest rate gaps are positive. This is because what matters for the determination of the endogenous variables are the *real* interest rate gaps. Anticipating future positive inflation, the real interest rate gaps are negative, leading to positive output gaps. These positive output gaps are consistent with positive inflation.

[25] Since measurement errors do not affect the natural interest rate, the responses of the nominal interest rate to the measurement errors are identical to those of the interest rate gap. Therefore, in figures (3.2) and (4.2), we do not present the responses of the nominal interest rate.

the inflation measure because it cannot distinguish the inflation noise from the other shocks. The bank increases the interest rate in response to the increase in the inflation measure. This results in decreases in inflation and the output gap. However, because r is smaller under commitment, the policy error caused by the bank's response to the inflation noise is smaller. As a result, the commitment plan is very successful in stabilizing inflation and the output gap in period zero. Inflation is almost perfectly stabilized under commitment. The subsequent increase in the output gap in the next period is also very small.

4.2.2 Small Noise Case

Here we discuss the impulse responses in the small noise case. The main properties of the impulse responses are the same as those in the large noise case. That is, if the initial policy turns out to be too contractionary, then the policy in the next period should be slightly expansionary, and vice versa. In the small noise case, the commitment plan successfully stabilizes the fluctuations with *larger* r, contrary to the large noise case.[26]

Responses to a Supply Shock Figure (4.1) shows the impulse responses to a supply shock in the small noise case. In this case, the policy response coefficient r is larger under commitment than under discretion. Large r_y may increase the fluctuations due to the supply shock, because the central bank should have decreased the interest rate when an increase in output is caused by a positive supply shock. However, Figure (4.1) shows that the initial decrease in the output gap is smaller under commitment than it is under discretion, even if r_y is larger. There are two reasons for this result. Firstly, the private agents' expectations about the future interest rate gap are important. The positive interest rate gap in period zero indicates that the initial policy is too contractionary. Under commitment, the interest rate gaps will become negative after period one, implying that the subsequent policy will be expansionary. Since the private agents anticipate this future policy loosening, the resulting initial decrease in the output gap is smaller. Secondly, the positive response to the observed increase in the output measure results in deflation. Positive r_π implies that interest rate should decrease. This effect offsets the positive response to the output measure. Since r_π is larger under commitment, this effect is larger. Consistent with this, the initial interest rate gap is smaller under commitment. Indeed, the nominal interest rate decreases in equilibrium, which is the right direction of the response to a positive supply shock. Inflation fluctuation is larger under commitment, but since the order of magnitude is small, its contribution to the loss measure

[26] See Table 1.

L is small.

Responses to Inflation Noise The impulse responses to an innovation in the measurement error in the inflation measure are shown in Figure (4.2). Since the central bank under commitment responds to Z_t more strongly than it does under discretion, it cannot avoid responding to the measurement error strongly. Therefore the initial interest-rate gap (i.e., the initial policy error) is larger under commitment than it is under discretion. The resulting fluctuations of inflation and the output gap are larger under commitment. This implies that the commitment plan does not dominate the discretion plan for all types of shocks. But the overall loss measure L is always smaller under commitment, as Figure (2) shows.

4.3 Commitment and the Trade-off Between the Variability caused by Economic Shocks and the Variability caused by Noise

Finally, we intuitively discuss how the commitment plan can achieve higher welfare under noisy information. First of all, the fluctuations in the central bank's goal variables in our model (x_t) are caused by the bank's inability to determine the nature of the shock. In such an environment, there is a trade-off between the fluctuations in x_t caused by the structural shocks and those caused by the bank's response to the measurement errors. To understand this trade-off, it is important to notice that the variability of x_t is decreasing in the size of r (in particular, the size of r_π) when policy takes a form of a feedback rule which responds to inflation and output measures (see, for example, Bernanke and Woodford (1997), and Rotemberg and Woodford (1999)). In the optimal equilibrium where inflation fluctuation is small, inflation does not respond strongly to the demand and supply shocks, so it is a poor indicator of the underlying economic shocks to which monetary policy should respond. Because inflation changes by only a small amount in response to these shocks, the central bank must respond to observed inflation variations with the rule that has a large inflation coefficient. On the other hand, when the inflation measure is subject to noise, a central bank with large r_π responds to inflation noise strongly, causing large policy error. Large r may therefore increase the variability of x_t caused by the bank's response to noise.

When degree of measurement error is large, small r is desirable in order to avoid responding to the measurement errors. However, the existence of the trade-off implies that policy with small r may increase the fluctuations in x_t due to the underlying structural shocks. Our numerical examples have shown that the central bank under commitment can stabilize the fluctuations caused by the economic shocks with smaller r than under discretion. Under commitment, if it turned out that the initial policy were too expansionary (contractionary), the subsequent policy would be slightly contractionary (expansionary). If the private agents understand this path of future policy, they expect that future interest rates will be adjusted to offset the current policy error. Then the output gap and inflation do not respond to policy errors very much. Therefore, the central bank under commitment can reduce the fluctuations in x_t due to its response to noise without significantly increasing the fluctuations due to the underlying economic shocks. Similarly, when the degree of the measurement errors is small, the central bank can respond to the noisy indicators with larger r to reduce the fluctuations caused by the economic shocks, while it can limit the increased fluctuations caused by its response to noise. The bank can do so by committing to offset policy errors in the subsequent periods.

It is of interest to compare our result to the literature on discretion vs commitment. A classic example is the analysis of systematic inflation bias that results under discretion (Barro and Gordon (1983)). The benefit of commitment is to eliminate inflationary bias. More recently, several authors have shown that, when the private agents are forward-looking, there may be gains from commitment even if the central bank is not trying to increase real output above its potential level. Clarida *et al.* (1999) use a similar model to ours in which the so called "cost-push" shock causes inflation-output gap variability trade-off, and show that commitment can improve such trade-off. Woodford (1999) considers gains from commitment when it is impossible for a central bank to completely stabilize inflation and the output gap because of the presence of the zero nominal interest rate bound. In his model, a commitment plan can stabilize inflation and the output gap without involving large fluctuations in the nominal interest rate. In our model, it is the information constraint that prevents the central bank from achieving the first-best equilibrium. We have shown that gains from commitment may emerge when the central bank's information set is imperfect, and commitment can improve the trade-off described above.

5 A Feedback Rule that Implements the Optimal Plan

We briefly discuss the description of the optimal policy in terms of an interest-rate feedback rule, giving attention to the way in which the forecastable component of interest rates should vary with lagged endogenous variables. There are several reasons why we focus attention on implementation of optimal policy by following a policy rule in terms of only current and lagged observable variables. The optimal policy we obtained in Section 4 is expressed as

$$R_t = R^*_{t|t-1} + \Lambda\phi_{t-1} + r\left(Z_t - Z_{t|t-1}\right), \tag{32}$$

which depends on the past values of the Lagrange multiplier as well as the forecastable parts of the natural rate and the observable variables. The implementation of the policy reaction function (32) thus requires period-by-period updating of the estimates of the economic shocks and the Lagrange multiplier. However, this may not be easy in practice. It is of great interest to characterize a feedback rule that is consistent with the optimal equilibrium but requires less information for implementation by the central bank. Furthermore, if a feedback policy rule depends only on the current and the lagged observable variables, private agents can easily verify the central bank's behaviour and forecast future monetary policy. Since the optimal policy under commitment relies on the private sector's understanding of future monetary policy, policy implementation by a simple rule deserves a lot of emphasis. There is another reason why a feedback rule is useful for policy implementation. The policy rule (32) may not uniquely determine the optimal plan as the equilibrium outcome. This is because rational expectations equilibrium may be indeterminate under some interest-rate feedback rules that include the optimal plan as one of many possible equilibria.[27]

McCallum (1981) shows that an interest-rate rule that does not have a strong "nominal anchor" may result in indeterminacy of equilibrium. In our model, his result may apply to the policy reaction function (32) when r_π becomes small. In this case, it is of great interest to find a feedback rule that is consistent with the optimal equilibrium and results in determinacy.

First of all, we wish to express the forecastable component of the interest rate in (32), namely, $R^*_{t|t-1}$, $Z_{t|t-1}$, and ϕ_{t-1}, in terms of the lagged endogenous variables. Using (28) and (13), we can rewrite these terms as

$$R^*_{t|t-1} = P \begin{pmatrix} \delta & 0 \\ 0 & \eta \end{pmatrix} \begin{pmatrix} y^n_{t-1} \\ \rho_{t-1} \end{pmatrix}, \tag{33}$$

$$Z_{t|t-1} = \Gamma\phi_{t-1} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \delta & 0 \\ 0 & \eta \end{pmatrix} \begin{pmatrix} y^n_{t-1} \\ \rho_{t-1} \end{pmatrix}. \tag{34}$$

We can also express the lagged Lagrange multiplier as a function of the lagged endogenous variables as[28]

$$\phi_{t-1} = \Gamma^{-1} M x_{t-1} + \Gamma^{-1} N \tilde{R}_{t-1}, \tag{35}$$

[27] Bernanke and Woodford (1997) emphasized this problem in the context of inflation forecast targeting policy.

[28] See equation (C.1) of Appendix.

where

$$x_{t-1} = \begin{pmatrix} \pi_{t-1} \\ y_{t-1} \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y^n_{t-1} \\ \rho_{t-1} \end{pmatrix}, \tag{36}$$

$$\tilde{R}_{t-1} = R_{t-1} - P \begin{pmatrix} y^n_{t-1} \\ \rho_{t-1} \end{pmatrix}. \tag{37}$$

Equations (33), (34), and (35) involve the lagged exogenous shocks $\left(y^n_{t-1}, \rho_{t-1}\right)$. So we wish to identify $\left(y^n_{t-1}, \rho_{t-1}\right)$ in terms of the lagged endogenous variables in order to obtain a feedback rule.

From equations (18), (19), (20), (35), (36), and (37), we can solve for $\left(y^n_{t-1}, \rho_{t-1}\right)$ as

$$\begin{pmatrix} y^n_{t-1} \\ \rho_{t-1} \end{pmatrix} = F_1 \begin{pmatrix} \pi_{t-1} \\ y_{t-1} \end{pmatrix} + F_2 R_{t-1} + F_3 x_{t-2} + F_4 \tilde{R}_{t-2} \tag{38}$$

where matrices F_i for $i = 1, 2, 3, 4$ are properly defined. Substituting (33), (34), (38) into (32), we can write the optimal policy reaction function as

$$R_t = rZ_t + G_1 \begin{pmatrix} \pi_{t-1} \\ y_{t-1} \end{pmatrix} + \eta R_{t-1} + G_3 x_{t-2} + G_4 \tilde{R}_{t-2},$$

where matrices G_i for $i = 1, 3, 4$ are properly defined. This in turn can be written as

$$\begin{aligned} R_t &= rZ_t + G_1 \begin{pmatrix} \pi_{t-1} \\ y_{t-1} \end{pmatrix} + \eta R_{t-1} + G_3 \begin{pmatrix} \pi_{t-2} \\ y_{t-2} \end{pmatrix} \\ &\quad + G_4 R_{t-2} - \left(G_3 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + G_4 P \right) \begin{pmatrix} y^n_{t-2} \\ \rho_{t-2} \end{pmatrix}. \end{aligned} \tag{39}$$

Substituting (38) recursively into (39) for the lagged demand and supply shocks, we can express the interest rate as a function of the lagged inflation, output, and interest rate. Algebraic form of the feedback rule is rather complex, but in the estimated noise case, it is well approximated by

$$R_t = \begin{pmatrix} 3.91 \\ 4.43 \end{pmatrix}' Z_t + \begin{pmatrix} -70.10 \\ 2.99 \end{pmatrix}' \begin{pmatrix} \pi_{t-1} \\ y_{t-1} \end{pmatrix} + 0.92 R_{t-1} - 1.27 \pi_{t-2} - (1 - 0.87L)^{-1} 1.36 \pi_{t-3},$$

where L is the lag operator. Therefore, the optimal feedback rule is expressed as

$$\begin{aligned} & (1 - 0.92L)(1 - 0.87L) R_{t-1} \\ = \; & (1 - 0.87L) \begin{pmatrix} 3.91 \\ 4.43 \end{pmatrix}' Z_t + \begin{pmatrix} 35.93 \\ -0.56 \end{pmatrix}' \begin{pmatrix} \pi_{t-1} \\ y_{t-1} \end{pmatrix} + \begin{pmatrix} -32.57 \\ 0.49 \end{pmatrix}' \begin{pmatrix} \pi_{t-2} \\ y_{t-2} \end{pmatrix} - 0.25 \pi_{t-3} \end{aligned} \tag{40}$$

The system of equations (7) with (40) results in determinacy of rational expectations equilibrium if and only if three of the eigenvalues of the system are outside the unit circle (Blanchard and Kahn (1980).) At least under our assumed parameter values, it can be easily shown that the feedback rule (40) results in determinacy of equilibrium.

It is of interest to compare (40) with the optimal feedback rule under discretion that is obtained by Aoki (2002). In the estimated noise case, it is given by

$$(1-0.92L)\,R_t = \begin{pmatrix} 4.36 \\ 4.92 \end{pmatrix}' Z_t + \begin{pmatrix} 30.66 \\ -0.63 \end{pmatrix}' \begin{pmatrix} \pi_{t-1} \\ y_{t-1} \end{pmatrix}.$$

What is of interest is the roots of the lag polynomials of the interest rate. The optimal feedback rule under discretion involves the lagged interest rate insofar as the underlying economic shocks are persistent.[29] On the other hand, presence of the lag polynomial $(1-0.87L)$ in equation (40) shows that the optimal feedback rule under commitment involves the lagged interest rates even if the underlying economic shocks are completely transitory. This feature of the optimal feedback rule results from the fact that the optimal policy (26) involves persistence that is independent of any persistence of the underlying economic shocks.[30]

6 Conclusion

This paper has characterized the optimal monetary policy which requires commitment when a central bank's measures of current inflation and output are subject to noise. Noise causes a trade-off between the variability of inflation and the output gap due to underlying economic shocks and their variability due to the bank's response to noise. In other words, this trade-off is caused by the bank's inability to determine the nature of the shock. We have shown that, compared with the discretion plan, the commitment plan can improve this trade-off. Under the commitment regime, if current policy turned out to be too expansionary (contractionary) because of the measurement problem, subsequent policy should be slightly contractionary (expansionary). It is also shown that, when the degree of measurement error is larger, the central bank responds to noisy indicators more weakly than it would under discretion, reducing the fluctuations of the endogenous variables due to its response to the measurement errors, without significantly increasing their

[29] Here the coefficient of the lag operator, 0.92, represents the persistence of the demand shock.

[30] Furthermore, when the noise becomes larger, the root of this lag polynomial becomes larger than one. This would imply that the interest rate should be "super inertial" as advocated by Woodford (1999). He shows that optimal monetary policy calls for an inertial behavior of the interest rate, and the policy rule that implements the optimal plan requires a super-inertial path of interest rates. When the noise is large, our feedback rule also has this property.

fluctuations due to the economic shocks. Similarly, when the degree of measurement error is smaller, the central bank responds to noisy indicators more strongly to reduce the variability due to the economic shocks, while limiting the increased variability due to its response to the measurement errors.

The numerical examples provided in section 5 show that the benefit from commitment to the estimated US economy may not be very large, given our assumed values of the standard deviations of the measurement errors. However, our numerical examples also show that the benefit from commitment increases significantly as the degree of noise becomes larger. Since the problem of the reliability of data on economic indicators may be more serious in the Euro area given the initial aggregation problems and the lack of reliable historical data, our results may be more relevant for the European Central Bank. Furthermore, the central banks in reality may face a larger degree of data uncertainty than we assume throughout our analysis. We consider a very simple environment in which data revision is complete one period later. However, in reality, data uncertainty is resolved only slowly and perhaps never completely. Then the advantage of commitment may be larger than we have calibrated. A more realistic characterization of the optimal policy under noisy information would require a more complicated model than ours. Nonetheless, our basic argument for the properties of the commitment plan under noisy information would be likely to extend to the more general setting.

A The First Order Condition with Respect to Observable Variables

In this section of the appendix, we derive the first order condition with respect to the observables Z_t. In particular, we provide the details of the differentiation of the terms involving $R_{t|I_t}$.

First of all, we define the derivative of a function of random variables. Let $S_t \equiv (e_{yt}, e_{\rho t}, \varepsilon_{\pi t}, \varepsilon_{yt})$ be a vector of the random variables at time t. History at time t is defined as a sequence of S_t, denoted by $h_t \equiv \{S_i\}_{i=0}^{t}$. A state of the economy is an infinite sequence of the vector of the random variables $h \equiv \{S_i\}_{i=0}^{\infty}$. Let $f(h)$ be the density function of h, and $f(h_t)$ be the marginal density of h_t. Let H and H_t respectively be the support of h and h_t. The Lagrangian for the optimization problem is given by (17), where expectation is taken over h. The interest rate chosen by the central bank must satisfy

$$R_t = R_{t|I_t} \equiv E_{t-1}[R_t] + Cov_{t-1}[R_t, Z_t]' Var_{t-1}[Z_t]^{-1}(Z_t - E_{t-1}[Z_t]), \tag{A.1}$$

where, $E_{t-1}[\cdot]$, $Cov_{t-1}[\cdot]$, and $Var_{t-1}[\cdot]$ represent the conditional expectation, covariance, and variance, conditional on the full information set at time $t-1$. For example,

$$E_{t-1}[x_t(h)] = \int_{H \backslash H_{t-1}} x_t(h) f(h|h_{t-1}) \, d(h \backslash h_{t-1}),$$

where

$$f(h|h_{t-1}) \equiv \frac{f(h)}{f(h_{t-1})}.$$

Here, $d(h \backslash h_{t-1})$ denotes the integration over a vector of random variables $h \backslash h_{t-1} \equiv (h_t, h_{t+1}, h_{t+2}, ...)$, and $H \backslash H_{t-1}$ denotes the support of $h \backslash h_{t-1}$. Now we define the functional derivative as follows. For any variable Z which is a function of a vector of random variables h, define a derivative of a functional $g(Z)$, normalized by probability density $f(h)$, as

$$\frac{\partial}{\partial Z} E[g(Z(h))] \equiv \frac{\partial}{\partial Z} \int_H g(Z(h)) f(h) \, dh \equiv \frac{\partial g}{\partial Z}.$$

Now we derive the first order condition with respect to Z_t, paying attention to the derivative of the term $E\left[\phi_t' C R_{t|I_t}\right]$ in the Lagrangian. Note that we can rewrite the term $E\left[\phi_t' C R_{t|I_t}\right] = E\left[\sigma \phi_{1t} R_{t|I_t}\right]$. Therefore, for simplicity of calculation, we consider the derivative of $E\left[\phi_{1t} R_{t|I_t}\right]$ with respect to Z_t. Since $R_{t|I_t}$ is given by (A.1), we have

$$E\left[\phi_{1t} R_{t|I_t}\right] = E\left[\phi_{1t} E_{t-1}[R_t]\right] + E\left[\phi_{1t} Cov_{t-1}[R_t, Z_t]' Var_{t-1}[Z_t]^{-1}(Z_t - E_{t-1}[Z_t])\right]. \tag{A.2}$$

The first term is expressed as

$$
\begin{aligned}
& E\left[\phi_{1t} E_{t-1}\left[R_t\right]\right] \\
= & \int_H\left[\phi_{1t}(h) \int_{H \backslash H_{t-1}} R_t(h) \frac{f(h)}{f\left(h_{t-1}\right)} d\left(h \backslash h_{t-1}\right)\right] f(h) \, dh \\
= & \int_{H_{t-1}}\left[\int_{H \backslash H_{t-1}}\left[f(h) \phi_{1t}(h)\right] d\left(h \backslash h_{t-1}\right) \int_{H \backslash H_{t-1}} R_t(h) \frac{f(h)}{f\left(h_{t-1}\right)} d\left(h \backslash h_{t-1}\right)\right] dh_{t-1} \\
= & \int_{H_{t-1}}\left[\int_{H \backslash H_{t-1}}\left[\frac{f(h)}{f\left(h_{t-1}\right)} \phi_{1t}(h)\right] d\left(h \backslash h_{t-1}\right) \int_{H \backslash H_{t-1}} R_t(h) \frac{f(h)}{f\left(h_{t-1}\right)} d\left(h \backslash h_{t-1}\right)\right] f\left(h_{t-1}\right) dh_{t-1} \\
= & \int_{H_{t-1}}\left[E_{t-1}\left[\phi_{1t}(h)\right] E_{t-1}\left[R_t(h)\right]\right] f\left(h_{t-1}\right) dh_{t-1}.
\end{aligned}
$$

Similarly, the second term can be expressed as

$$
\begin{aligned}
& E\left[\phi_{1t} Cov_{t-1}\left[R_t, Z_t\right]^{\prime} Var_{t-1}\left[Z_t\right]^{-1}\left(Z_t-E_{t-1}\left[Z_t\right]\right)\right] \\
= & \int_H\left[\phi_{1t}(h) Cov_{t-1}\left[R_t, Z_t\right]^{\prime} Var_{t-1}\left[Z_t\right]^{-1}\left(Z_t-E_{t-1}\left[Z_t\right]\right)\right] f(h) \, dh \\
= & \int_{H_{t-1}}\left[Cov_{t-1}\left[R_t, Z_t\right]^{\prime} Var_{t-1}\left[Z_t\right]^{-1} Cov_{t-1}\left[\phi_{1t}, Z_t\right]\right] f\left(h_{t-1}\right) dh_{t-1}.
\end{aligned}
$$

Therefore, (A.2) reduces to

$$
\begin{aligned}
& E\left[\phi_{1t} R_{t|I_t}\right] \\
= & \int_{H_{t-1}}\left[E_{t-1}\left[\phi_{1t}(h)\right] E_{t-1}\left[R_t(h)\right]\right. \\
& \left.+Cov_{t-1}\left[R_t, Z_t\right]^{\prime} Var_{t-1}\left[Z_t\right]^{-1} Cov_{t-1}\left[\phi_{1t}, Z_t\right]\right] f\left(h_{t-1}\right) dh_{t-1}.
\end{aligned} \tag{A.3}
$$

We wish to calculate a Jacobian matrix of $E\left[\phi_{1t} R_{t|I_t}\right]$ with respect to Z_t, normalized by probability density $f(h)$, denoted as

$$
\begin{aligned}
& D_{Z_t}\left[E\left[\phi_{1t} R_{t|I_t}\right]\right] \\
\equiv & \frac{\partial \, vec\left(E\left[\phi_{1t} R_{t|I_t}\right]\right)}{\partial\left(vec\left(Z_t\right)\right)^{\prime}} \equiv \frac{\partial E\left[\phi_{1t} R_{t|I_t}\right]}{\partial Z_t^{\prime}} \\
= & \frac{\partial}{\partial Z_t^{\prime}} \int_{H_{t-1}}\left[E_{t-1}\left[\phi_{1t}(h)\right] E_{t-1}\left[R_t(h)\right]\right. \\
& \left.+Cov_{t-1}\left[R_t, Z_t\right]^{\prime} Var_{t-1}\left[Z_t\right]^{-1} Cov_{t-1}\left[\phi_{1t}, Z_t\right]\right] f\left(h_{t-1}\right) dh_{t-1},
\end{aligned} \tag{A.4}
$$

where *vec* represents the vec operator. The last line of (A.4) follows from equation (A.3). The definition of Jacobian follows Magnus and Neudecker (1988). The first order condition (20) is obtained from the transpose of the Jacobian calculated below. We now wish to show that

$$
D_{Z_t}\left[E\left[\phi_{1t} R_{t|I_t}\right]\right]=Cov_{t-1}\left[R_t, Z_t\right]^{\prime} Var_{t-1}\left[Z_t\right]^{-1}\left(\phi_{1t}-\phi_{1t|I_t}\right). \tag{A.5}
$$

We use the identification theorem for matrix functions (Magnus and Neudecker (1988), Theorem 5.11) in order to derive(A.5). [31] We first calculate the differential of the second term of equation (A.4), and vectorize it to obtain

$$
\begin{aligned}
& d\,vec\left(Cov_{t-1}\left[R_t, Z_t\right]' Var_{t-1}\left[Z_t\right]^{-1} Cov_{t-1}\left[\phi_{1t}, Z_t\right]\right) \\
= \;& \left\{Var_{t-1}\left[Z_t\right]^{-1} Cov_{t-1}\left[\phi_{1t}, Z_t\right]\right\}' D_{Z_t} Cov_{t-1}\left[R_t, Z_t\right]' dvecZ_t \\
& + \left\{Cov_{t-1}\left[\phi_{1t}, Z_t\right]' \otimes Cov_{t-1}\left[R_t, Z_t\right]'\right\} D_{Z_t} Var_{t-1}\left[Z_t\right]^{-1} dvecZ_t \\
& + \left\{Cov_{t-1}\left[R_t, Z_t\right]' Var_{t-1}\left[Z_t\right]^{-1}\right\} D_{Z_t} Cov_{t-1}\left[\phi_{1t}, Z_t\right] dvecZ_t.
\end{aligned}
\tag{A.6}
$$

Here, each Jacobian matrix is given by

$$
D_{Z_t} Cov_{t-1}\left(R_t, Z_t\right)' = \left[R_t - E_{t-1}\left(R_t\right)\right] I \frac{f\left(h\right)/f\left(h_{t-1}\right)}{f\left(h\right)}, \tag{A.7}
$$

$$
\begin{aligned}
& D_{Z_t} Var_{t-1}\left[Z_t\right]^{-1} \\
= \;& -\Bigg[\left(Var_{t-1}\left[Z_t\right]^{-1} \otimes Var_{t-1}\left[Z_t\right]^{-1}\right) \\
& * \left(I \otimes \left(Z_t - E_{t-1}\left[Z_t\right]\right) + \left(Z_t - E_{t-1}\left[Z_t\right]\right) \otimes I\right)\Bigg] \frac{f\left(h\right)/f\left(h_{t-1}\right)}{f\left(h\right)},
\end{aligned}
\tag{A.8}
$$

$$
D_{Z_t} Cov_{t-1}\left[\phi_{1t}, Z_t\right] = \left[\phi_{1t} - E_{t-1}\left[\phi_{1t}\right]\right] I \frac{f\left(h\right)/f\left(h_{t-1}\right)}{f\left(h\right)}, \tag{A.9}
$$

where I denotes two-by-two identity matrix. Note that each Jacobian matrix is normalized by the density $f\left(h\right)$, and that the term $f\left(h\right)/f\left(h_{t-1}\right)$ represents the conditional density of $h \backslash h_{t-1}$.

Substituting (A.7), (A.8), and (A.9) into (A.6), and using the identification theorem for matrix functions, we obtain

$$
\begin{aligned}
& D_{Z_t}\left[E\left[\phi_{1t} R_{t|I_t}\right]\right] \\
= \;& \left\{Var_{t-1}\left[Z_t\right]^{-1} Cov_{t-1}\left[\phi_{1t}, Z_t\right]\right\}' \left(R_t - E_{t-1}\left[R_t\right]\right) I \\
& - \left\{Cov_{t-1}\left[\phi_{1t}, Z_t\right]' \otimes Cov_{t-1}\left[R_t, Z_t\right]'\right\} \left(Var_{t-1}\left[Z_t\right]^{-1} \otimes Var_{t-1}\left[Z_t\right]^{-1}\right) \\
& * \left(I \otimes \left(Z_t - E_{t-1}\left[Z_t\right]\right) + \left(Z_t - E_{t-1}\left[Z_t\right]\right) \otimes I\right) \\
& + \left\{Cov_{t-1}\left[R_t, Z_t\right]' Var_{t-1}\left[Z_t\right]^{-1}\right\} \left[\phi_{1t} - E_{t-1}\left[\phi_{1t}\right]\right] I.
\end{aligned}
\tag{A.10}
$$

[31] See, also, Chapter 9.5 of Magnus and Neudecker (1988), "Identification of Jacobian Matrices."

The second term of (A.10) can be written as

$$
\begin{aligned}
& -\left\{Cov_{t-1}\left[\phi_{1t}, Z_t\right]' \otimes Cov_{t-1}\left[R_t, Z_t\right]'\right\}\left(Var_{t-1}\left[Z_t\right]^{-1} \otimes Var_{t-1}\left[Z_t\right]^{-1}\right) \\
& *\left(I \otimes\left(Z_t-E_{t-1}\left[Z_t\right]\right)+\left(Z_t-E_{t-1}\left[Z_t\right]\right) \otimes I\right) \\
= & -\left(Cov_{t-1}\left[\phi_{1t}, Z_t\right]' Var_{t-1}\left[Z_t\right]^{-1}\right) \otimes\left(\left\{Cov_{t-1}\left[R_t, Z_t\right]' Var_{t-1}\left[Z_t\right]^{-1}\right\}\right) \\
& *\left(I \otimes\left(Z_t-E_{t-1}\left[Z_t\right]\right)+\left(Z_t-E_{t-1}\left[Z_t\right]\right) \otimes I\right) \\
= & -Cov_{t-1}\left[\phi_{1t}, Z_t\right]' Var_{t-1}\left[Z_t\right]^{-1} \otimes Cov_{t-1}\left[R_t, Z_t\right]' Var_{t-1}\left[Z_t\right]^{-1}\left(Z_t-E_{t-1}\left[Z_t\right]\right) \\
& -Cov_{t-1}\left[\phi_{1t}, Z_t\right]' Var_{t-1}\left[Z_t\right]^{-1}\left(Z_t-E_{t-1}\left[Z_t\right]\right) \otimes Cov_{t-1}\left[R_t, Z_t\right]' Var_{t-1}\left[Z_t\right]^{-1} . \qquad \text{(A.11)}
\end{aligned}
$$

Substituting (A.11) into (A.10) and arranging terms, we obtain

$$
\begin{aligned}
& D_{Z_t}\left[E\left[\phi_{1t} R_{t \mid I_t}\right]\right] \\
= & Cov_{t-1}\left[\phi_{1t}, Z_t\right]' Var_{t-1}\left[Z_t\right]^{-1}\left(R_t-R_{t \mid I_t}\right)+Cov_{t-1}\left[R_t, Z_t\right]' Var_{t-1}\left[Z_t\right]^{-1}\left(\phi_{1t}-\phi_{1t \mid I_t}\right)
\end{aligned}
$$

Since (16) requires that $R_t = R_{t|I_t}$ for all states, we conclude that (A.5) holds. Therefore, we have shown that equation (20) holds. Note that the first order condition (20) is obtained from the transpose of the Jacobian matrix we have calculated.

B Optimal Evolution of Lagrange Multipliers

In this section, we derive the optimal evolution of the Lagrange multipliers conditional on the central bank's information set (equation (23)). For simplicity of calculation, it is convenient not to express the equations in a matrix form. Equations (18), (19), (20), (22) imply that, conditional on the bank's information set, the optimal evolution of the endogenous variables must solve the following system of equations:

$$\tilde{\pi}_{t|I_t} - \beta^{-1}\sigma\phi_{1t-1} + \phi_{2t|I_t} - \phi_{2t-1} = 0, \tag{B.1}$$

$$a\tilde{y}_{t|I_t} + \phi_{1t|I_t} - \beta^{-1}\phi_{1t-1} - \kappa\phi_{2t|I_t} = 0, \tag{B.2}$$

$$\sigma\phi_{1t|I_t} = 0, \tag{B.3}$$

$$\tilde{y}_{t|I_t} = \tilde{y}_{t+1|I_t} - \sigma\left(\tilde{R}_{t|I_t} - \tilde{\pi}_{t+1|I_t}\right), \tag{B.4}$$

$$\tilde{\pi}_{t|I_t} = \kappa\tilde{y}_{t|I_t} + \beta\tilde{\pi}_{t+1|I_t}. \tag{B.5}$$

Recall that $\tilde{\pi} \equiv \pi$, $\tilde{y} \equiv y - y^n$, and $\tilde{R} \equiv R - R^*$. Equations (B.4) and (B.5) are the conditional expectations of the IS curve and the Phillips curve. Note that, at time t, ϕ_{1t-1} and ϕ_{2t-1} are known to the central bank.

The optimal evolution of $\phi_{1t|I_t}$ is therefore given by (B.3). Now we derive the optimal evolution of $\phi_{2t|I_t}$. Using (B.2), (B.3), and (B.5), we obtain

$$\begin{aligned}\tilde{\pi}_{t|I_t} &= \beta\tilde{\pi}_{t+1|I_t} + \kappa\tilde{y}_{t|I_t} \\ &= \beta\tilde{\pi}_{t+1|I_t} + \frac{\kappa}{a\beta}\phi_{1t-1} + \frac{\kappa^2}{a}\phi_{2t|I_t}.\end{aligned} \tag{B.6}$$

On the other hand, from (B.1), $\pi_{t|I_t}$ is given by

$$\tilde{\pi}_{t|I_t} = \beta^{-1}\sigma\phi_{1t-1} - \phi_{2t|I_t} + \phi_{2t-1},$$

and therefore, we have

$$\beta\tilde{\pi}_{t+1|I_t} = \phi_{1t|I_t} - \beta\phi_{2t+1|I_t} + \beta\phi_{2t|I_t}.$$

Substitute these equations into (B.6), and using $\phi_{1t|I_t} = 0$, we obtain

$$\phi_{2t|I_t} = \frac{\beta}{\kappa^2 a^{-1} + 1 + \beta}\phi_{2t+1|I_t} + \frac{\beta^{-1}\left(\sigma - \kappa a^{-1}\right)}{\kappa^2 a^{-1} + 1 + \beta}\phi_{1t-1} + \frac{1}{\kappa^2 a^{-1} + 1 + \beta}\phi_{2t-1}. \tag{B.7}$$

Furthermore, equations (B.3) and (B.7) imply that the difference equation with respect to $\phi_{2t|I_t}$ is given by

$$\phi_{2j+2|I_t} - \frac{\kappa^2 a^{-1} + 1 + \beta}{\beta}\phi_{2j+1|I_t} + \beta^{-1}\phi_{2j|I_t} = 0, \quad \text{for } j \geq t. \tag{B.8}$$

We wish to solve the difference equation (B.8), given the initial condition (B.7). Again, at time t, ϕ_{1t-1} and ϕ_{2t-1} are given and known to the central bank. We can show that the roots of the characteristic equation associated with (B.8) are real, distinct, and satisfy[32]

$$0 < \lambda_1 < 1 < \lambda_2.$$

Therefore, in order for the sequence of $\phi_{2j|It}$ to be bounded, it must satisfy

$$\phi_{2j+1|I_t} = \lambda_1\phi_{2j|I_t}. \tag{B.9}$$

[32] The characteristic equation is given by

$$f(\lambda) = \lambda^2 - \frac{\kappa^2 a^{-1} + 1 + \beta}{\beta}\lambda + \beta^{-1}.$$

Then its determinant is given by

$$\begin{aligned}D &\equiv \left(\frac{\kappa^2 a^{-1} + 1 + \beta}{\beta}\right)^2 - 4\beta^{-1} \\ &= \left(\frac{\kappa^2 a^{-1} - 1 + \beta}{\beta}\right)^2 + \frac{4\kappa^2 a^{-1}}{\beta^2} > 0,\end{aligned}$$

and the characteristic equation satisfies $f(1) = -\kappa^2 a^{-1} < 0$, $f(0) = \beta^{-1} > 0$.

It must satisfy the initial condition (B.7), too. Then,

$$\begin{aligned}\phi_{2t|I_t} &= \frac{\beta}{\kappa^2 a^{-1}+1+\beta}\phi_{2t+1|I_t} + \frac{\beta^{-1}\left(\sigma-\kappa a^{-1}\right)}{\kappa^2 a^{-1}+1+\beta}\phi_{1t-1} + \frac{1}{\kappa^2 a^{-1}+1+\beta}\phi_{2t-1} \\ &= \frac{\beta}{\kappa^2 a^{-1}+1+\beta}\lambda_1\phi_{2t|I_t} + \frac{\beta^{-1}\left(\sigma-\kappa a^{-1}\right)}{\kappa^2 a^{-1}+1+\beta}\phi_{1t-1} + \frac{1}{\kappa^2 a^{-1}+1+\beta}\phi_{2t-1}.\end{aligned}$$

Therefore, we can solve for $\phi_{2t|I_t}$ as

$$\begin{aligned}\phi_{2t|I_t} &= \left(1-\frac{\beta}{\kappa^2 a^{-1}+1+\beta}\lambda_1\right)^{-1}\left(\frac{\beta^{-1}\left(\sigma-\kappa a^{-1}\right)}{\kappa^2 a^{-1}+1+\beta}\phi_{1t-1} + \frac{1}{\kappa^2 a^{-1}+1+\beta}\phi_{2t-1}\right) \\ &= \left\{\left(\kappa^2 a^{-1}+1+\beta\right)-\beta\lambda_1\right\}^{-1}\left\{\beta^{-1}\left(\sigma-\kappa a^{-1}\right)\phi_{1t-1}+\phi_{2t-1}\right\}.\end{aligned}$$

Furthermore, we notice from the characteristic equation that

$$\left(\kappa^2 a^{-1}+1+\beta\right)-\beta\lambda_1 = \lambda_1^{-1}.$$

Therefore, ϕ_{2t_It} is given by

$$\begin{aligned}\phi_{2t|I_t} &= \lambda_1\left[\beta^{-1}\left(\sigma-\kappa a^{-1}\right)\phi_{1t-1}+\phi_{2t-1}\right] \\ &= \lambda_1\left[b\phi_{1t-1}+\phi_{2t-1}\right], \\ b &\equiv \beta^{-1}\left(\sigma-\kappa a^{-1}\right).\end{aligned} \tag{B.10}$$

Using (B.9) and (B.10), we obtain the solution for $\phi_{2t|I_t}$ as

$$\phi_{2t+j|I_t} = \lambda_1^{j+1}\left[b\phi_{1t-1}+\phi_{2t-1}\right], \quad j \geq 0.$$

To summarize, the evolution of optimal Lagrange multipliers conditional on I_t is given by

$$\phi_{1t+j|I_t} = 0, \tag{B.11}$$

$$\phi_{2t+j|I_t} = \lambda_1^{j+1}\left[b\phi_{1t-1}+\phi_{2t-1}\right], \quad j \geq 0. \tag{B.12}$$

C Evolution of endogenous variables

We derive the evolution of inflation and the output gap, x_t, in the optimal equilibrium. Since we assume that the central bank has complete knowledge about the past states of the economy, we have $I_{t+1} \supset \Omega_t$. Then, by the law of iterated expectations, (24) implies

$$\begin{aligned}E_t x_{t+1} &= E_t\left[x_{t+1|I_{t+1}}\right] \\ &= E_t\left[\Gamma\phi_t\right] = \Gamma\phi_t.\end{aligned}$$

Substituting this into (8), we obtain

$$\Gamma\phi_t = Mx_t + N\tilde{R}_t. \tag{C.1}$$

Equation (C.1) implies that

$$\begin{aligned} \Gamma\left(\phi_t - \phi_{t|t-1}\right) &= M\left(x_t - x_{t|t-1}\right) + N\left(R_t - R_{t|t-1}\right) - NPe_t, \\ P &\equiv \left[\begin{array}{cc} \frac{\delta-1}{\sigma} & 1 \end{array}\right]. \end{aligned} \tag{C.2}$$

Here we denote $E_{t-1}[z_t] \equiv z_{t|t-1}$ for notational convenience. Using the first order conditions (18), (19), and (20), we obtain

$$Wx_t - \beta^{-1}A'\phi_{t-1} + B'\phi_t - C'\phi_t r' = 0. \tag{C.3}$$

Rewriting the term $C'\phi_t r'$ as $r'C'\phi_t$, (C.3) is written as

$$Wx_t - \beta^{-1}A'\phi_{t-1} + \left(B' - r'C'\right)\phi_t = 0. \tag{C.4}$$

This equation gives the relationship between the innovations in the equilibrium endogenous variables and those in the Lagrange multiplier as

$$\begin{aligned} \left(\phi_t - \phi_{t|t-1}\right) &= -\left(B' - r'C'\right)^{-1} W\left(x_t - x_{t|t-1}\right) \\ &\equiv \Psi\left(x_t - x_{t|t-1}\right). \end{aligned} \tag{C.5}$$

Substituting (C.5) into (C.2), we obtain

$$\left(\Gamma\Psi - M\right)\left(x_t - x_{t|t-1}\right) = N\left(R_t - R_{t|t-1}\right) - NPe_t. \tag{C.6}$$

Equation (C.6) expresses the innovations in x_t as a function of the innovations in policy and those in the demand and supply shocks.

Next, equation (16) implies that the optimal policy takes a form of

$$\begin{aligned} R_t - R_{t|t-1} &= Cov_{t-1}\left[R_t, Z_t\right]' Var_{t-1}\left[Z_t\right]^{-1}\left(Z_t - E_{t-1}\left[Z_t\right]\right) \\ &\equiv r\left(Z_t - Z_{t|t-1}\right), \end{aligned} \tag{C.7}$$

where r is the optimal policy coefficient derived in Section 3.3. From equation (13), the relationship between the innovation in x_t and those in Z_t are given by

$$\left(Z_t - Z_{t|t-1}\right) = \left(x_t - x_{t|t-1}\right) + Fe_t + \varepsilon_t. \tag{C.8}$$

Substitution of (C.7) and (C.8) into (C.6) yields the deviations of the equilibrium endogenous variables from their forecastable values as a function of the underlying economic shocks and measurement errors

$$x_t - x_{t|t-1} = \Phi^{-1} N \Xi e_t + \Phi^{-1} N r \varepsilon_t, \tag{C.9}$$

where

$$\begin{aligned} \Phi &\equiv \Gamma\Psi - M - Nr, \\ \Xi &\equiv rF - P. \end{aligned}$$

Note also that, under our assumption about the information structure,

$$x_{t|I_t} = \Gamma\phi_{t-1} = x_{t|t-1}.$$

Therefore the deviation from the central bank's optimal plan is also given by (C.9). The equilibrium endogenous variables are thus given by

$$x_t = \Gamma\phi_{t-1} + \Phi^{-1} N \Xi e_t + \Phi^{-1} N r \varepsilon_t. \tag{C.10}$$

Similarly, the equilibrium Lagrange multiplier is given by

$$\phi_t = \lambda_1 D \phi_{t-1} + \Psi\Phi^{-1} N \Xi e_t + \Psi\Phi^{-1} N r \varepsilon_t. \tag{C.11}$$

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	$r = (r_\pi, r_y)$				Loss L		
	Commitment		Discretion		Commitment	Discretion	Ratio (%)
Estimated Noise	3.906	4.428	4.358	4.916	0.2042	0.2044	0.01
Small Noise	106.4	27.15	21.55	8.399	0.1693	0.1730	2.18
Large Noise	0.015	0.291	0.105	0.497	0.5498	0.6224	13.21

Table 1: Optimal Coefficients and Loss Measure

Figure (1.1)

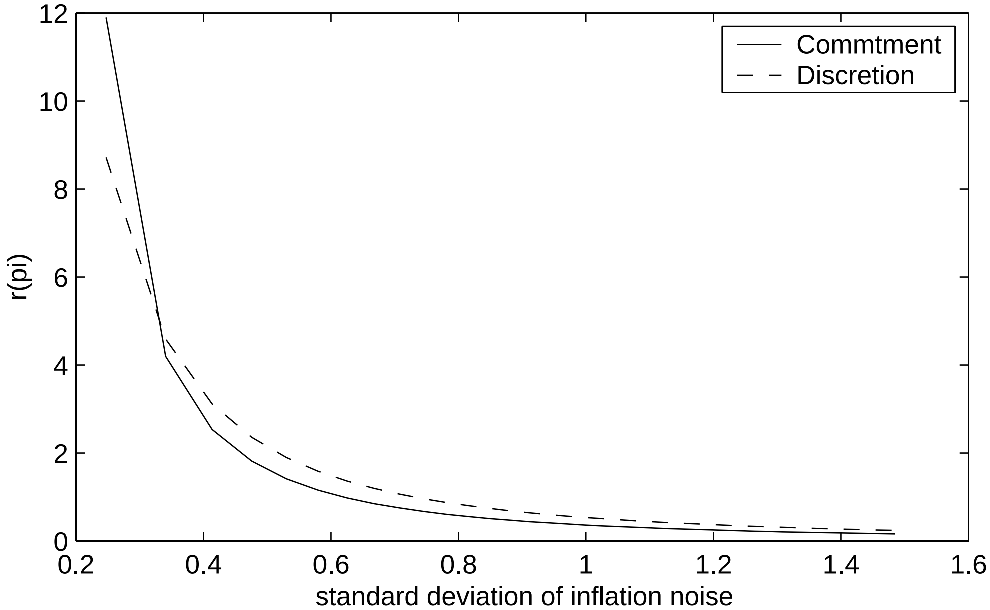
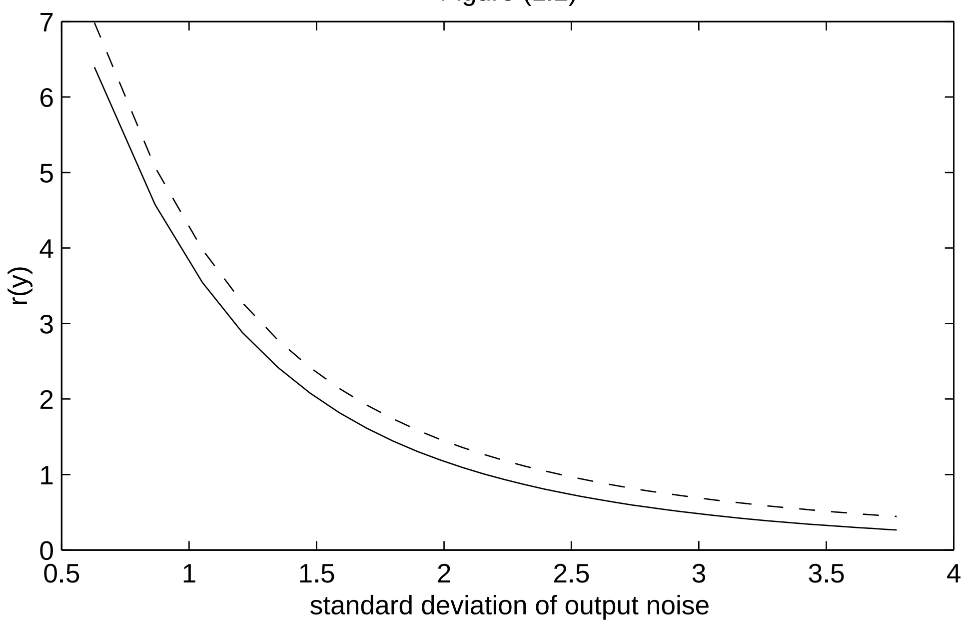


Figure (1.2)



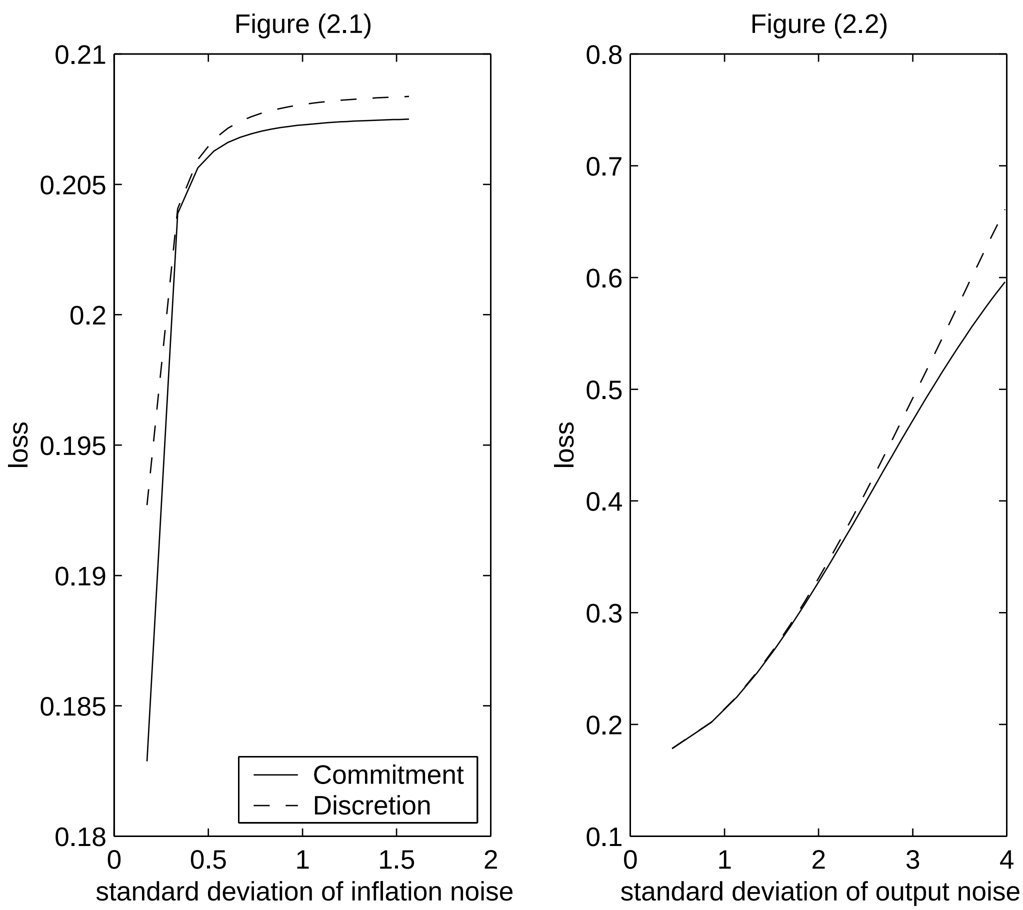
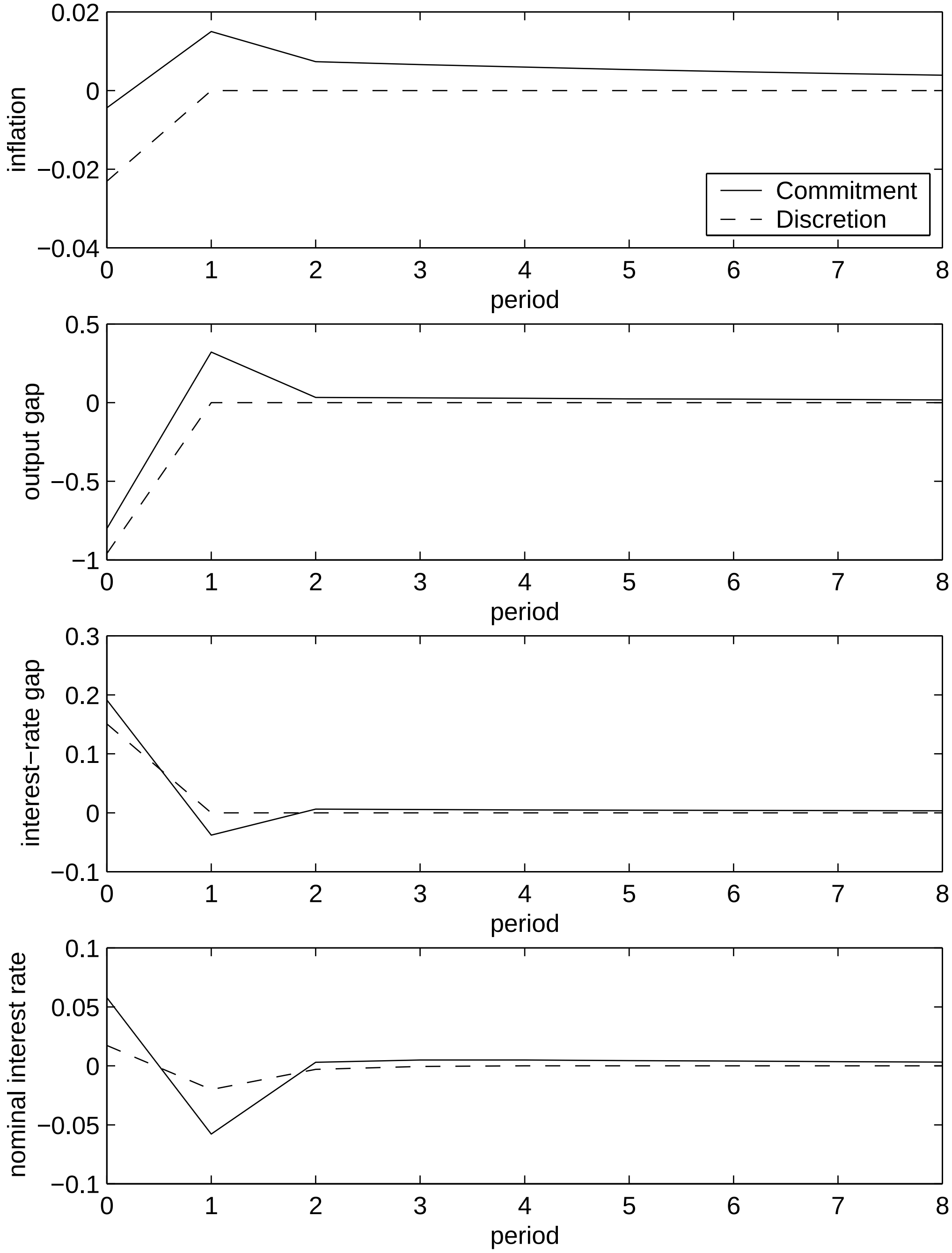
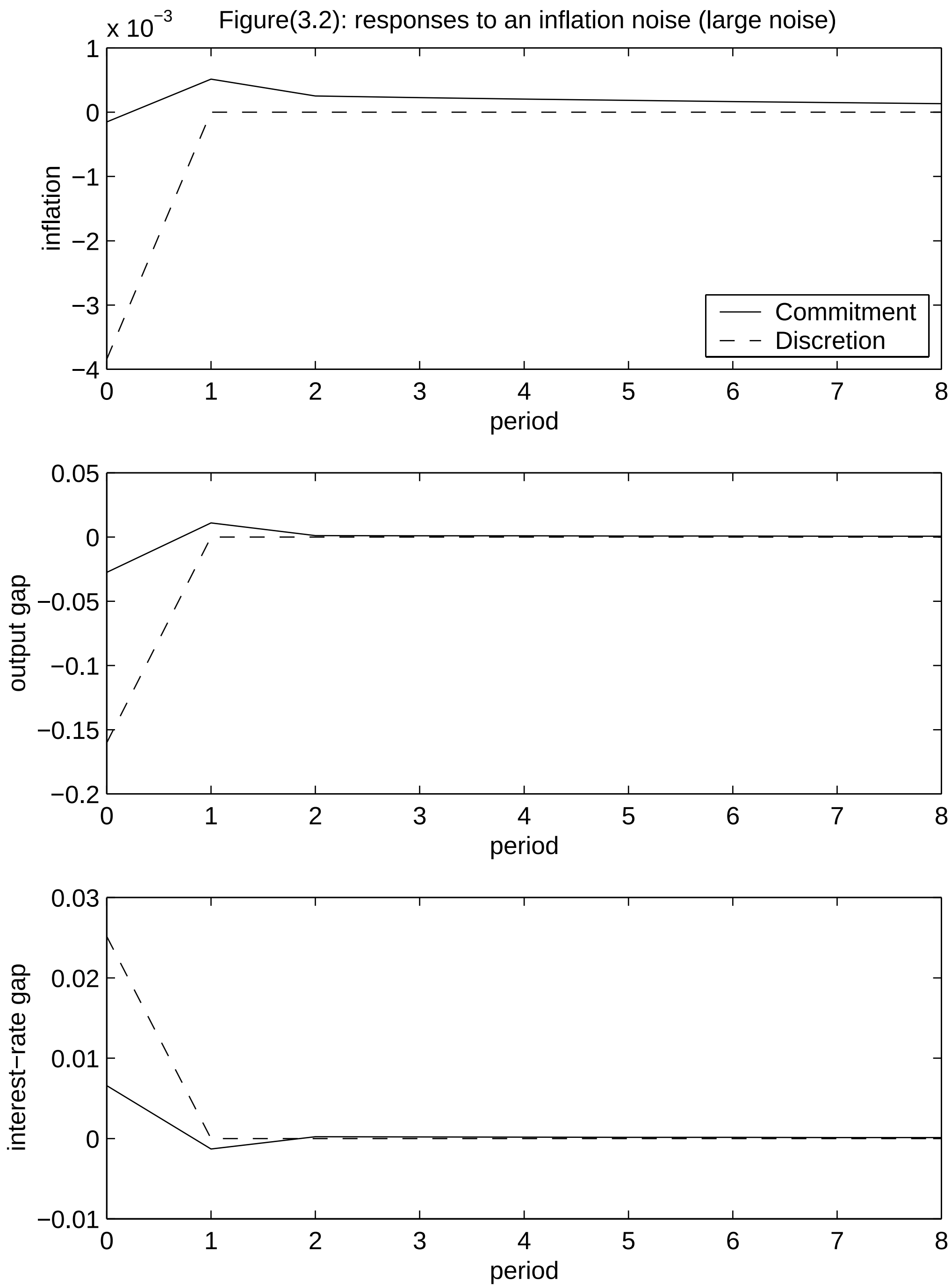


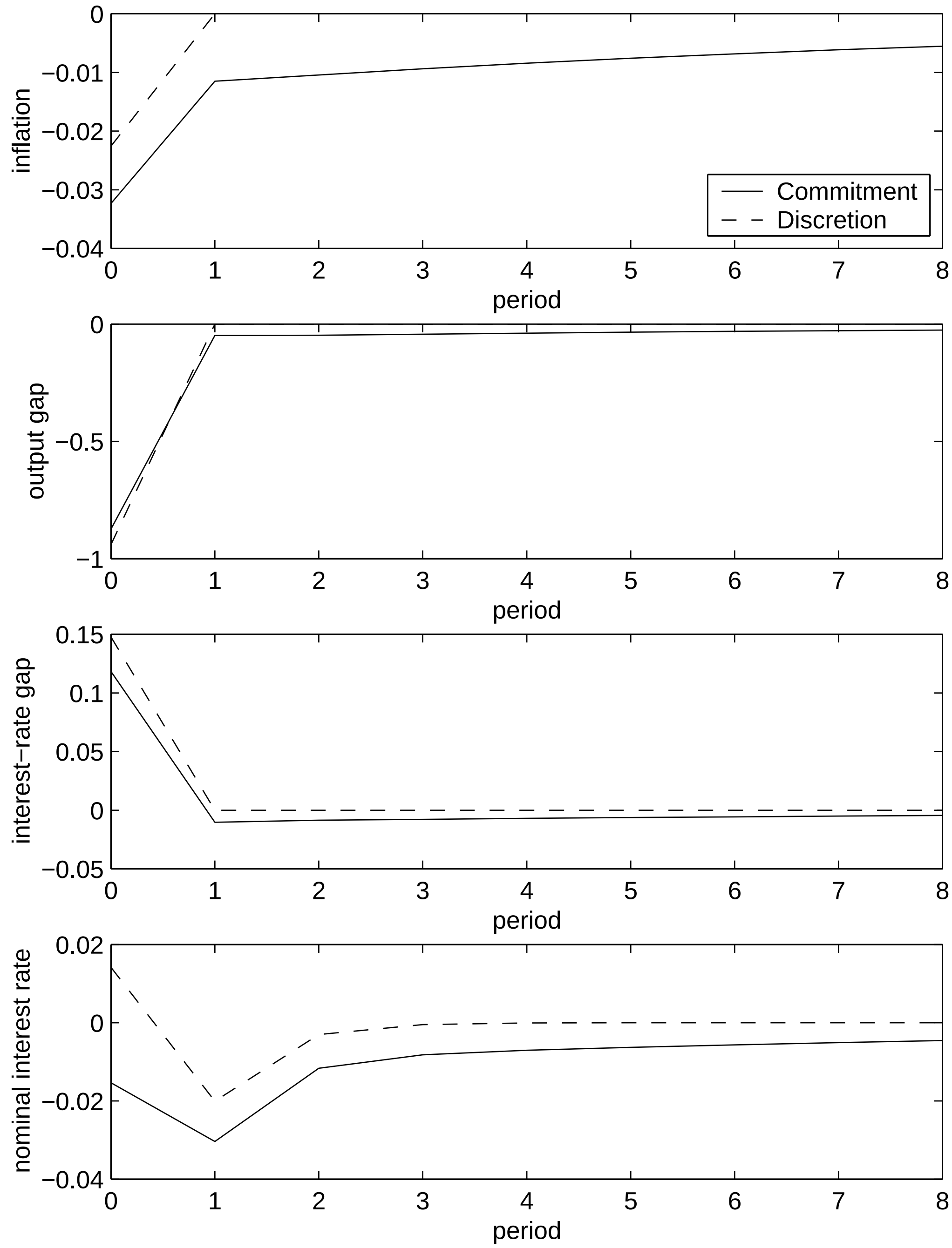
Figure (2.1)
Figure (2.2)
loss
standard deviation of inflation noise
standard deviation of output noise
Commitment
Discretion
0.21
0.205
0.2
0.195
0.19
0.185
0.18
0
0.5
1
1.5
2
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
3
4

Figure(3.1): responses to a supply shock (large noise)




Figure(3.2): responses to an inflation noise (large noise)
x 10^−3
inflation
Commitment
Discretion
period
output gap
period
interest−rate gap
period

Figure(4.1): responses to a supply shock (small noise)



Figure(4.2): responses to an inflation noise (small noise)