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ABSTRACT

Pricing Credit Derivatives with Rating Transitions*

We develop a model for pricing risky debt and valuing credit derivatives that is easily calibrated to existing variables. Our approach is based on expanding the Heath-Jarrow-Morton (1990) term-structure model and its extension, the Das-Sundaram (2000) model to allow for defaultable debt with rating transitions. The framework has two salient features, comprising extensions over the earlier work: (i) it employs a rating transition matrix as the driver for the default process, and (ii) the entire set of rating categories is calibrated jointly, allowing, with minimal assumptions, arbitrage-free restrictions across rating classes, as a bond migrates amongst them. We provide an illustration of the approach by applying it to price credit sensitive notes that have coupon payments that are linked to the rating of the underlying credit.

JEL Classification: G12 and G13
Keywords: credit derivatives, credit sensitive note, HJM model, rating transitions and risky debt

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* The program code presented in this Paper is intended only as pseudo-code and implemented versions may be undertaken at the user's risk. We would like to thank an anonymous referee whose suggestions have improved the exposition in the Paper. The Paper was written while Viral V Acharya was a doctoral student at the Stern School of Business, New York University.

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NON-TECHNICAL SUMMARY

The pricing of credit derivatives is reaching some level of modeling maturity. In particular, ‘reduced form’ models that directly specify the default process or the credit spread have resulted in successful conjoint implementations of term structure models with default models. We contribute to this literature by presenting a discrete-time reduced-form model for valuing risky debt based on the term-structure model of Heath, Jarrow, and Morton (1990).

We extend the HJM model to include risky debt by adding a ‘forward spread’ process to the forward rate process for default risk-free bonds as in Das and Sundaram (2000). Instead of modeling the movement of the spread itself, the engineering of our model focuses on the stochastic process for inter-rating spreads. Working with inter-rating spreads provides any credit spread as the sum of higher rated inter-rating spreads. This approach offers analytical tractability. No restrictions are placed on the correlation between these stochastic processes. The probability of default at any point in time is allowed to depend on the entire history of the process to that point, and is determined from rating transition matrices, exogenously supplied. The model is exible to incorporate any specification for the recovery process that is consistent with the default process and the spread processes.

In Das and Sundaram (2000), the pricing lattice was developed by computing a no-arbitrage tree embedding the riskless term structure and the term structure of credit spreads. While this tree considered the modeling of only a single rating category at a time, this Paper extends that model by calibrating all rating classes jointly on the same pricing lattice. Embedding all rating categories on one pricing lattice requires a set of conditions ensuring consistency across all classes of debt. The additional information required to engineer this comes from the introduction of the rating transition matrix. Thus, in our model, we are now able to price credit derivatives based on multiple classes of debt, which was not possible using simpler models.

To understand the consistency conditions across rating classes, note that the credit rating of a corporate borrower can improve or deteriorate during the life of its issued debt. Thus, the credit spread on its debt contains valuable information about the future credit spreads on debt of all possible rating classes that the borrower could migrate to. This is true for a corporate borrower with any given rating at a point of time. This interdependence of spreads across rating classes immediately implies that calibration of the forward spread process for a given rating class must be undertaken simultaneously with the calibration of the forward spread processes for all other rating classes. Formalizing this interdependence and characterizing the joint calibration process (Proposition 3.2) is the primary contribution of this Paper.

Our model requires as input the government yield curve. In addition, it also uses the term structures of credit spreads for each rating class, available from providers such as Bloomberg. The same source delivers required interest rate and spread volatilities. The model can be efficiently implemented and lends itself most appropriately to pricing of credit derivatives such as credit sensitive notes where the coupon payments are linked to credit quality of the underlying corporate borrower. We provide a numerical example to illustrate the calibration of the model and its use to price credit sensitive notes.

1 Introduction

The pricing of credit derivatives is reaching some level of modeling maturity. In particular, "reduced form" models that directly specify the default process or the credit spread have resulted in successful conjoint implementations of term structure models with default models.[1]

The modeling of default allows a wide choice from amongst a variety of alternatives. In some models, hazard rate functions are specified theoretically (for example Madan and Unal [36], [37], Duffie-Singleton [20], Schönbucher [42]) or calibrated empirically (Wilson [44], Das and Sundaram [11]).

In other models, rating transition matrices are used as the drivers of the stochastic process for default (see, e.g. Arvantis, Gregory and Laurent [3], Bielecki and Rutkowski [4], Das and Tufano [12], Jarrow, Lando and Turnbull [26], Kijima [28], Kijima and Komoribayashi [29], and Lando [33]).[2]

The other modeling choice is that of the recovery rate. Duffie and Singleton [20] suggested the use of "Recovery of Market Value" (RMV), an idea that brings remarkable analytical benefits as well as economic consistency. RMV implies that upon default a zero-coupon risky bond trades for a fraction a of its market value.

We present a discrete-time reduced-form model for valuing risky debt based on the term-structure model of Heath, Jarrow, and Morton [24] (hereafter HJM). We extend the HJM model to include risky debt by adding a "forward spread" process to the forward rate process for default risk-free bonds (see Das and Sundaram [11]). No restrictions are placed on the correlation between the two processes. The probability of default at any point in time is allowed to depend on the entire history of the process to that point, and is determined from rating transition matrices, exogenously supplied. While the model is flexible to incorporate any specification for the recovery process (that is consistent with the default process and the spread processes), we employ the RMV condition for ease of exposition.

In Das and Sundaram [11], the pricing lattice was developed by computing a no-arbitrage tree embedding the riskless term structure and the term structure of credit spreads. While this tree considered the modeling of only a single rating category at a time, this paper offers an extension to modeling all rating classes jointly on the same pricing lattice. Embedding all rating categories on one pricing lattice requires a set of conditions ensuring consistency across all classes of debt. The additional information required to engineer this model comes from the

[1]In contrast, "structural" models starting with Merton [38] explicitly specify the underlying firm-value process. These models either endogenize default as a failure of the equityholders to meet the liabilities of the firm or model default as the event when firm-value reaches some threshold barrier.

[2]Other models are presented in Duffee [15] , Duffie and Huang [18], Duffie, et al [19], Jarrow and Turnbull [27], Kijima and Komoribayashi [29], Kijima [28], and Ramaswamy and Sundaresan [40].

introduction of the rating transition matrix. In our model, we are now able to price credit derivatives based on multiple classes of debt, which was not possible using simpler models.

The model requires as input the government yield curve. In addition, it also uses the term structures of credit spreads for each rating class, available from providers such as Bloomberg. The same source delivers required interest rate and spread volatilities. Instead of modeling the movement of the spread itself, the engineering of our model focuses on the stochastic process for inter-rating spreads. Working with inter-rating spreads provides any credit spread as the sum of higher rated inter-rating spreads. This approach offers analytical tractability. More importantly, it ensures that if the inter-rating spread processes are specified such that they always remain non-negative, then the risky yield curve for a poorer rated security never lies below that of a higher rated security. This restriction on the inter-rating spread processes is easier to model than the latter restriction that credit spreads for same maturity be monotonically decreasing in credit quality.[3]

A more detailed discussion about the implication of allowing for rating transitions is in order before we proceed. Since the credit quality or the rating of a corporate borrower can improve or deteriorate during the life of its issued debt, the credit spread on its debt contains valuable information about the future credit spreads on debt of all possible rating classes that the borrower could migrate to. This is true for a corporate borrower with any given rating at a point of time. This interdependence of spreads across rating classes immediately implies that calibration of the forward spread process for a given rating class must be undertaken simultaneously with the calibration of the forward spread processes for all other rating classes. Formalizing this interdependence and characterizing the joint calibration process (Proposition 3.2) is the primary contribution of this paper. As can be visualized, our model lends itself most appropriately to pricing of credit derivatives such as credit sensitive notes where the coupon payments are linked to credit quality of the underlying corporate borrower.

The remainder of this paper is organized as follows. Section 2 describes the model and underlying assumptions. Section 3 describes the derivation of the recursive representation for the risk-neutral drifts, while Section 4 describes a recursive representation of risky bond prices in our model. Section 5 develops the implementation approach. Sections 6 and 7 consider an actual implementation example of a credit sensitive note that has coupon payments that are linked to corporate rating. Section 8 concludes.

[3] A model which is very close in spirit to that of ours in continuous time is developed by Bielecki and Rutkowski [4]. This model is also based on HJM and uses information about credit spreads coupled with that of transition probabilities and recovery rates to develop a conditionally Markovian model of credit risk. In their paper, they model spreads directly, not inter-rating spreads as we do in this model.

2 The Model

The model is developed in discrete time, since a computer implementation for options with American features and path-dependence is envisaged. We consider an economy on a finite time interval $[0, T^*]$. Periods are taken to be of length $h > 0$; thus, a typical time-point t has the form kh for some integer k. It is assumed that at all times t, a full range of default-free zero-coupon bonds trades, as does a full range of risky zero-coupon bonds, across all rating categories. It is also assumed that markets are free of arbitrage, so there exists an equivalent martingale measure Q for this economy;[4] all references to randomness below and all expectations are with respect to this measure.

For any given pair of time-points (t, T) with $0 \leq t \leq T \leq T^* - h$, let $f(t, T)$ denote the forward rate on the default-free bonds applicable to the period $(T, T + h)$; in words, $f(t, T)$ is the rate as viewed from time t for a default-free lending/investment transaction over the interval $(T, T + h)$. (All interest rates in the model are expressed in continuously-compounded terms.) When $t = T$, the rate $f(t, t)$ will be called the "short rate" and denoted by $r(t)$. The forward rate curve is assumed to evolve according to the process

$$f(t + h, T) \;=\; f(t, T) + \alpha(t, T)h + \sigma(t, T)X_0\sqrt{h}, \tag{2.1}$$

where α is the drift of the process and σ its volatility; and X_0 is a random variable. Both α and σ may depend on other information available at t, such as the time–t forward rates. To keep notation simple, we have suppressed this possible dependence.

We assume a set of risky bonds from $K+1$ rating classes, indexed by $k = 1, \ldots, K+1$. Class $K + 1$ is the default state and it is natural to assume that credit quality deteriorates from Class 1 down to Class $K+1$. It will be assumed in all that follows that once a bond is in default state, i.e. in Class $K + 1$, it does not trade, and its price, net of any recovery upon default, is zero. For $0 \leq t \leq T \leq T^* - h$, let $\varphi(t, T) = [\varphi_1(t, T), ..., \varphi_k(t, T), ..., \varphi_K(t, T)]'$ denote the "forward rate" on the risky bonds implied from the spot yield curve. To be precise, $\varphi_k(t, T)$ is the rate as viewed from time t on a risky lending transaction over the interval $(T, T + h)$ where the risky loan has a credit quality that corresponds to rating class k. The *forward inter-rating spreads* are defined as the spreads between successive rating categories. These comprise a vector:

$$\mathbf{s}(t, T) = [s_1(t, T), ..., s_k(t, T), ..., s_K(t, T)]'$$

where s_1 is the spread between the best quality corporate bond and riskless bonds, s_2 is the spread between the second risk level and first risk level of risky bonds, etc. In general s_k is the

[4]Specifically, we assume that Q is an equivalent martingale measure with respect to the money-market account $B(t)$ defined in (3.1) below. See Harrison and Kreps [22] or Harrison and Pliska [23] for the role of equivalent martingale measures in securities modeling.

spread between the kth risk level and the $(k-1)$st risk level. Therefore, the forward rates on risky debt are related to the inter-rating spreads as

$$\varphi_k(t,T) = f(t,T) + s_1(t,T) + \ldots + s_k(t,T), \quad \forall k. \tag{2.2}$$

As long as $s_k(t,T) > 0$ for all k, we are assured that credit spreads are increasing as the quality level decreases.

Next, we make assumptions concerning the evolution of these forward inter-rating spreads (and, thus, of the forward rates on the risky bonds). We take these to follow the process

$$s_k(t+h,T) \;=\; s_k(t,T) + \beta_k(t,T)h + \eta_k(t,T)'\mathbf{X}\sqrt{h}, \quad \forall k, \tag{2.3}$$

where

$$\begin{aligned}
\beta(t,T) &= [\beta_1(t,T), \ldots, \beta_k(t,T), \ldots, \beta_K(t,T)]' \in R^K \\
\eta(t,T) &= [\eta_1(t,T), \ldots, \eta_k(t,T), \ldots, \eta_K(t,T)] \in R^{L\times K}, \eta_k(t,T) \in R^L
\end{aligned}$$

are the drift and volatility coefficients, respectively, and $\mathbf{X} \in R^L$ are possibly correlated random variables. Note that L is the dimension of the space of diffusion factors that affect the spread processes, and in general, can be smaller than, greater than, or equal to K. Both β and η may depend on other information available at t. At this point, we place no restrictions on the joint distribution of X_0 and $\mathbf{X}$. When illustrating implementation of the model in a later section, we will assume that the random variables $(X_0, \mathbf{X})$ take values on a discrete state-space.

We will denote by $P(t,T)$ the time–t price of a default-free zero-coupon bond of maturity $T \geq t$, and by $\Pi_k(t,T)$ its risky counterpart in the k^{th} rating class. Note that, by definition, we have

$$P(t,T) \;=\; \exp\left\{-\sum_{i=t/h}^{T/h-1} f(t,ih)\cdot h\right\} \tag{2.4}$$

$$\Pi_k(t,T) \;=\; \exp\left\{-\sum_{i=t/h}^{T/h-1} \varphi_k(t,ih)\cdot h\right\}, \quad \forall k \tag{2.5}$$

Default is modeled using a Markov chain that governs the transitions of each bond from one rating level k to another, in a time period of length h. We denote this markov chain as:

$$D = \begin{bmatrix} q_{1,1} & \cdots & \cdots & q_{1,K+1} \\ \vdots & & & \vdots \\ q_{K,1} & & & q_{K,K+1} \\ q_{K+1,1} & \cdots & \cdots & q_{K+1,K+1} \end{bmatrix}$$

Since default is modeled as an absorbing state, we can rewrite

$$D = \begin{bmatrix} q_{11} & \cdots & \cdots & q_{1,K+1} \\ \vdots & & & \vdots \\ q_{K,1} & & & q_{K,K+1} \\ 0 & 0 & \ldots & 1 \end{bmatrix}$$

where the time step h determines the probability of transition. This approach was first employed in the work of Jarrow, Lando and Turnbull [26].[5] This matrix may be obtained empirically from past data on rating transitions, and is available from Moody's and Standard & Poor's. We continue to make the assumption that Jarrow, Lando and Turnbull [26] make which is that the rating transition process is independent of the diffusion processes. The matrix elements need not be constants, and could be functions of the information set as well as time. To reduce the notational burden, we shall often denote the elements of D without the time-dependence. The dependence will be clear from the context.

The spreads on the risky bonds represent the cost of default, and as such, depend on both the probability of default (which in turn depends upon the sequence of rating transitions till maturity of the bond) as well as the amount that bond holders expect to recover in the event of default. Note that we have assumed $\Pi_{K+1}(t,T) \equiv 0,\ \forall t$, the price of a risky bond once in default.

Given that default has not occurred up to t, we denote by $\lambda_k(t) \equiv q_{k,K+1}(t)$, the probability of default by time $t+h$ from state k. Concerning the recovery rate, we will use the "Recovery of Market Value" or RMV condition of Duffie and Singleton [20]. Let Φ^t denote the recovery amount in the event of default at t. The RMV condition then states that conditional on default occurring at time $t+h$, the time–t expectation $E^t[\Phi^{t+h}]$ of the amount bondholders will receive is given by

$$E^t[\Phi^{t+h}] \;=\; \phi_k(t) E^t[\Pi(t+h,T) \mid \text{No Default}], \tag{2.6}$$

where $\phi_k(t)$ denotes the time–t "recovery rate," if the state from which default occurs is k. Recovery rates may be chosen so as to be different depending on the rating class from which the bond has moved to the default state. As with λ, $\phi_k(t)$ may also depend on all information in the model up to and including period t. It may also depend on the subordination level of the bond. The recovery rate may not be specific to the initial rating class, since eventually all defaulting bonds are in the default category.

[5] Several extensions of this model have been undertaken since then, such as the papers by Arvanitis, Gregory and Laurent [3], Das and Tufano [12], Kijima [28], Kijima and Komoribayashi [29], and Lando [33].

The following preliminary result relating short spreads to the default probabilities and recovery rates under Q is fundamental to our analysis and will come in handy in the rest of the paper:

$$\sum_{j=1}^{k} s_j(t,t) \;=\; -\frac{1}{h}\ln[1-\lambda_k(t)+\lambda_k(t)\phi_k(t)], \quad \forall k. \tag{2.7}$$

To see (2.7), consider a risky bond at t that matures at $(t+h)$. By definition, its time–t price is given by

$$\Pi_k(t,t+h) \;=\; \exp\{-(f(t,t)+\sum_{j=1}^{k} s_j(t,t))\cdot h\}, \quad \forall k. \tag{2.8}$$

Now, a one period investment in this bond fetches a cash flow of \$1 at time $(t+h)$ if there is no default at $t+h$, and a cash flow of $\phi_k(t)$ if there is a default. When discounted at the short rate, the expected cash flow (in the risk-neutral world) must equal the initial price of the bond, so we obtain

$$\Pi_k(t,t+h) \;=\; \exp\{-f(t,t)h\}[1-\lambda_k(t)+\lambda_k(t)\phi_k(t)], \quad \forall k. \tag{2.9}$$

Expression (2.7) is an immediate consequence of (2.8) and (2.9).

The model's objective is to develop a risk-neutral lattice for pricing risky debt. This is undertaken in several steps. First, the lattice of default-free interest rates is generated by solving for the risk-neutral drifts so that all discounted default-free securities are martingales. Then, a lattice for credit spreads is superimposed on the first lattice, and risk-neutral drifts are computed for the forward spread process so that the discounted prices of risky debt are martingales. Finally, the recursive structure of the model is used together with a specific assumption concerning the default process to illustrate implementation of the model. We begin with identification of the risk-neutral drifts.

3 Identifying the Risk-Neutral Drifts

In this section, we derive recursive expressions for the drifts α and β of the forward-rate and spread processes, respectively, in terms of the volatilities σ and η. To this end, we define $B(t)$ to be the time–t value of a "money-market account" that uses an initial investment of \$1, and rolls the proceeds over at the default-free short rate:

$$B(t) \;=\; \exp\left\{\sum_{i=0}^{t/h-1} r(ih)\cdot h\right\}. \tag{3.1}$$

We assume without loss of generality that the equivalent martingale measure Q was defined with respect to $B(t)$ as numeraire; thus, under Q all asset prices in the economy discounted by $B(t)$ will be martingales.

Following the extensive literature in the Heath-Jarrow-Morton (HJM) [24] framework, we will first identify the risk-neutral drifts α of the default-free forward rates in terms of the volatilities σ of these rates. As is standard in HJM-type models of interest rates, these risk-neutral drifts can be expressed entirely in terms of the forward-rate volatilities. To be precise, we obtain the following result:

Proposition 3.1 (Drift Terms of Forward Interest Rate Process) *There exists a recursive relationship between the risk-neutral drifts α and the volatilities σ at each t:*

$$\sum_{i=t/h+1}^{T/h-1} \alpha(t, ih) \;=\; \frac{1}{h^2} \ln \left(E^t \left[\exp \left\{ - \sum_{i=t/h+1}^{T/h-1} \sigma(t, ih) X_0 h^{3/2} \right\} \right] \right). \tag{3.2}$$

Next, we turn to the drifts $\beta(t, T)$. Note that these are drift terms for forward inter-rating spread processes for different rating classes. The calculation of these drifts is non-standard and involved due to a simple reason that sits at the heart of this paper: a risky bond with a current rating may move to a different rating class tomorrow. This implies that the price of such a bond today (and hence the spread process for its rating class) implicitly captures information about future risk-neutral drifts of the spread processes for other rating classes. This implies no-arbitrage restrictions on how these drifts evolve with respect to each other. It turns out however that the drifts can be calculated in a simple bootstrap manner as discussed below:

Proposition 3.2 (Drift Terms of Forward Inter-rating Spread Processes) *Assume independence of the rating transition process from the diffusion processes $(X_0, \mathbf{X})$. Further, denote $\theta_j(t, ih) = \sum_{l=1}^{j} \beta_l(t, ih)$. Then, $\forall t$, $\forall k$ (at each state), the drift terms $\{\beta_j(t, ih)\}$ are given indirectly by the solution $\{\theta_j(t, ih)\}$ to the following system of K linear equations in K unknowns $(x_j, j = 1, \ldots, K)$:*

$$\sum_{j=1}^{K} a_{k,j} \cdot b_{k,j} \cdot x_j = 1 \tag{3.3}$$

where

$$a_{k,j} \;=\; \frac{q_{k,j}(t)}{1 - \lambda_k(t)} \cdot \exp \left\{ - \sum_{i=\frac{t}{h}+1}^{\frac{T}{h}-1} \alpha(t, ih) \cdot h^2 \right\} \cdot \exp \left\{ - \sum_{i=\frac{t}{h}+1}^{\frac{T}{h}-1} (\varphi_j(t, ih) - \varphi_k(t, ih)) \cdot h \right\},$$

$$
\tag{3.4}
$$

$$
b_{k,j} = E^t\left[\exp\left\{-\sum_{i=\frac{t}{h}+1}^{\frac{T}{h}-1}\left(\sigma(t,ih)X_0+\sum_{l=1}^{j}\eta_l(t,ih)'\mathbf{X}\right)\cdot h^{\frac{3}{2}}\right\}\right], \text{ and} \tag{3.5}
$$

$$
x_j = \exp\left\{-\sum_{i=\frac{t}{h}+1}^{\frac{T}{h}-1}\theta_j(t,ih)\cdot h^2\right\}. \tag{3.6}
$$

Note that at each state at time t in rating class k, the terms $a_{k,j}$ can be computed knowing the transition probabilities, the α drift terms, and the spread levels at that state. Similarly, the terms $b_{k,j}$ can be computed by taking the expectation over the diffusion processes $(X_0, \mathbf{X})$, as is illustrated in pseudo code later, knowing the term structure of forward interest rate volatilities and forward inter-rating spread volatilities. Thus solving the above system of linear equations (using standard algorithms such as Gauss-Seidel for example), we obtain x_j terms, which in turn yield θ_j terms. Since the system is to be solved in a bootstrap manner starting with $T-1$, the drift terms $\beta_j(t,\cdot)$ can then be backed out from the knowledge of $\theta_j(\cdot,\cdot)$.

Note that the possibility of rating transitions prevents this representation from providing an analytical expression for drift terms $\beta(t,T)$, as is obtained in the single-rating model of Das and Sundaram [11]. Note however that the expectation in relation (3.5) over all possible sample-paths of the state-space for X_0 and $\mathbf{X}$, can be computed numerically using a lattice as we illustrate in this paper. This computes the derivation of the risk-neutral drifts in terms of the volatilities.

4 A Recursive Representation of the Risky Bond Prices

The prices of a risky bond in our model, as in Das and Sundaram [11], have a recursive representation, which leads, in turn, to a representation in terms of bond prices of short maturities, i.e. of the form $\Pi_k(\tau, \tau+h)$. We describe this representation here. While in Das and Sundaram, the recursive representation entails one level of recursion at each time step, with possible rating transitions, our representation forks into K levels of recursion at each time step.

It is straightforward to show (see equation A.10 in Appendix A) that

$$
\exp\{-\varphi_k(t,t)\cdot h\}\cdot E_k^t\left[\Pi(t+h,T)\mid \text{No Default}\right] = \Pi_k(t,T). \tag{4.1}
$$

Rearranging terms and using the fact that $\exp\{-\varphi_k(t,t)\cdot h\} = \Pi_k(t,t+h)$, we now obtain

$$
\Pi_k(t,T) = \Pi_k(t,t+h)\cdot E_k^t\left[\Pi(t+h,T)\mid \text{No Default}\right] \tag{4.2}
$$

$$= \quad \Pi_k(t, t+h) \cdot \sum_{j=1}^{K} \frac{q_{k,j}(t)}{1-\lambda_k(t)} E^t \left[\Pi_j(t+h, T)\right]. \tag{4.3}$$

We can now iterate on the expression for $\Pi_j(t+h, T)$ in terms of the transition probabilities $q_{j,l}(t+h)$ and $E^{t+h}\left[\Pi_l(t+h, T) \mid \text{No Default}\right], \ l = 1, \ldots, K$.

The recursive structure of the prices of risky bonds as described in (4.3) facilitates computation of these prices. Note that since all terms on the right hand side have the form $F(\tau, \tau+h)$, we can make use of relation (2.9) to employ the forward spread components (i.e., the default and recovery rates) in this process.

5 Implementation of the Model

To be able to implement the model, we must be more precise about quantities that have so far been left unspecified, viz., the random variables X_0 and $\mathbf{X}$. In this section, we describe the assumptions that we will use in the rest of this paper. These assumptions were chosen with an eye towards simplicity both in exposition and in implementation, but they are primarily meant to be illustrative; alternative assumptions may, of course, be similarly handled.

We first assume that $K = 2$, so that the three possible states of the corporate bond are *Investment Grade* ($k = 1$), *Speculative Grade* ($k = 2$), and *Default State* ($k = 3$). We make the discrete-time assumption that X_0 and $\mathbf{X}$, i.e. X_0, X_1, and X_2 are binomial random variables, specifically, that each takes on the values ± 1 with probability $1/2$. We assume that the pairwise correlation between X_0 and X_1 is ρ_1, between X_0 and X_2 is ρ_2 and between X_1 and X_2 is ρ_3. So, the assumed joint distribution of (X_0, X_1, X_2) is

$$(X_0, X_1, X_2) \;=\; \begin{cases} (+1,+1,+1), & \text{w.p. } q_{uuu} = (1+\rho_1+\rho_2+\rho_3)/8 \\ (+1,+1,-1), & \text{w.p. } q_{uud} = (1+\rho_1-\rho_2-\rho_3)/8 \\ (+1,-1,+1), & \text{w.p. } q_{udu} = (1-\rho_1+\rho_2-\rho_3)/8 \\ (+1,-1,-1), & \text{w.p. } q_{udd} = (1-\rho_1-\rho_2+\rho_3)/8 \\ (-1,+1,+1), & \text{w.p. } q_{duu} = (1-\rho_1-\rho_2+\rho_3)/8 \\ (-1,+1,-1), & \text{w.p. } q_{dud} = (1-\rho_1+\rho_2-\rho_3)/8 \\ (-1,-1,+1), & \text{w.p. } q_{ddu} = (1+\rho_1-\rho_2-\rho_3)/8 \\ (-1,-1,-1), & \text{w.p. } q_{ddd} = (1+\rho_1+\rho_2+\rho_3)/8 \end{cases} \tag{5.1}$$

We note that, in general, the correlation coefficients may not be equal to zero or even constant over the tree. For some numerical estimates of the correlation coefficient between corporate spreads and interest rates in general, see Das and Sundaram [11], and Das and Tufano [12].

Next, we look at the components of the forward rates, namely the default rate $\lambda_k(t)$, and the recovery rate $\phi_k(t)$. Using equation (2.7), it is clear that knowing the forward spreads $s_k(t,t)$

and either of $\lambda_k(t)$ or $\phi_k(t,t)$ for all k, implies the other. Unlike Das and Sundaram [11], where an additional specification is required linking the default rate $\lambda(t)$ to the interest rate and the spread variables, in our model, $\lambda_k(t)$'s are to be used based on the rating transition matrix. In particular, $\lambda_k(t) \equiv q_{k,K+1}(t)$.

One last, and non-trivial, issue remains before we can discuss the engineering details of model implementation. Estimates of the probabilities provided in standard rating transition matrices (e.g. of Moody's and Standard and Poor's) based on historical data cannot be directly used in our model, since our model (including the probability of default $\lambda_k(t)$) is set in the risk-neutral world. Thus, a translation from the actual to the risk-neutral measure is required. To this end, suppose that $\lambda_k^P(t)$ denotes the actual probability of default. We will make the natural assumption that the recovery rates are the same in the risk-neutral and actual worlds, so realized cash flows coincide in the two cases. Letting $\xi_k(t)$ be the time–t premium for bearing default risk corresponding to rating state k, the analog of (2.7) under the actual probabilities is easily derived:

$$\exp\{-\sum_{j=1}^{k} s_j(t,t)h\} \;=\; \exp\{-\xi_k(t)h\}[1 - \lambda_k^P(t) + \phi_k(t)\lambda_k^P(t)]. \tag{5.2}$$

The difference between (2.7) and (5.2) is simply that the relationship (2.7) is developed in the risk-neutral world, where—by definition—there is no premium for bearing risk. Expression (5.2) follows the same derivation but is set in the actual world, where we would expect that the risk-premium term $\xi_k(t)$ would be positive.

Comparing (2.7) and (5.2), we may express $\lambda_k(t)$ in terms of $\lambda_k^P(t)$ and the risk-premium $\xi_k(t)$:

$$\lambda_k(t) \;=\; \lambda_k^P(t)\left[\frac{1 - \exp\{-\sum_{j=1}^{k} s_j(t,t)h\}}{1 - \exp\{-(\sum_{j=1}^{k} s_j(t,t) - \xi_k(t))h\}}\right]. \tag{5.3}$$

Expression (5.3) implies the intuitive condition that $\lambda_k > \lambda_k^P$ whenever the risk-premium ξ_k is positive.

These expressions may be used in conjunction with equation (2.7) to estimate the risk-premium terms $\xi_k(t)$. Specifically, we get

$$\phi_k(t) \;=\; \frac{1}{\lambda_k^P(t)}[\exp\{-(\sum_{j=1}^{k} s_j(t,t) - \xi_k(t))h\} - 1 + \lambda_k^P(t)]. \tag{5.4}$$

In estimation, we can use $\phi_k(t)$ to be the average recovery rate observed historically for the rating class k, $\bar{\phi}_k(t)$. Thus, knowing the actual recovery rate $\bar{\phi}_k(t)$, the actual default rate $\lambda_k^P(t)$

and the actual spot inter–rating spreads $s_j(t,t)$, $j = 1, \ldots, K$, the risk-premium terms $\xi_k(t)$ can be backed out using equation (5.4). Or, as in Das and Sundaram [11], we can assume that the risk-premium terms are given by $\xi_k(t,t) = \nu_k \sum_{j=1}^{k} s_j(t,t)$ for scalar ν_k, and use equation (5.4) to back out implied recovery rate functions $\phi_k(t)$.

An additional complication remains, which is to adjust the remaining elements of the historical transition matrix to obtain the risk-neutral transition matrix. We make an assumption similar to that in Jarrow, Lando and Turnbull [26], and assume that

$$q_{k,l}(t) = \delta_k(t) \cdot q^P_{k,l}(t), \ \forall l \neq k; \tag{5.5}$$

$$q_{k,k}(t) = 1 + \delta_k(t) \cdot [q^P_{k,k}(t) - 1], \text{ where} \tag{5.6}$$

$$\delta_k(t) = \frac{\lambda_k(t)}{\lambda^P_k(t)}. \tag{5.7}$$

Note that $q_{k,l}(t)$ refers to transition probabilities in the risk-neutral matrix whereas $q^P_{k,l}(t)$ refers to transition probabilities in the historical matrix. Also, $q^P_{k,K+1}(t) = \lambda^P_k(t)$ so that $q_{k,K+1}(t) = \lambda_k(t)$. Such a "spread" transformation, where the mass spreads from the diagonal term of the transition matrix to the off-diagonal terms, has also been employed by Wilson [44]. Note that more sophisticated techniques for estimation of $\delta_k(t)$ would try to minimize error over the entire transition matrix data rather than just the default transition probability, $\lambda_k(t)$, as we did in the possible estimation techniques described in the previous paragraph.

6 Lattice Implementation

We describe in this section an implementation of our model using a lattice. The lattice has a multidimensional structure, since it combines the evolution of interest rates and inter-rating spreads for different rating classes. At the same time, the rating transition process is superimposed on top of this lattice. This superimposition is straightforward since we have assumed that the rating transition process is independent of the diffusion processes.

First, let us look only at the multidimensional structure for the interest rate and the spread processes. We assume as before that there are only three possible rating classes: *Investment Grade* (denoted as I), *Speculative or Junk Grade* (denoted as J), and *Default State* (denoted as D). Thus $K = 2$, and we have two inter-rating spread processes s_I and s_J. Thus, along with the interest rate process, we obtain a triple-binomial structure with eight branches emanating from each node of the lattice. This part of the lattice looks similar to that in Das and Sundaram ([11]). Specifically, once the risk-neutral drifts $\alpha(\cdot), \beta(\cdot)$ have been computed at any t, the possible values of the forward rates and forward spreads one period out are readily obtained using (2.1) and (2.3). At each state, the current rating class is known as well. Thus, if further given the forward and spread curves $F(\tau) = (f(\tau, \cdot))$, $S_I(\tau) = (s_I(\tau, \cdot))$, and $S_J(\tau) = (s_J(\tau, \cdot))$ at any τ, and knowing the one-period default probability $\lambda(\tau)$ as the default probability in one-period for the current rating class, the recovery rate $\phi(\tau)$ can be computed as described in the previous section. So far, at each node on the lattice we have information related to all three risks involved in the valuation of risky debt (interest rates, default probabilities, and recovery rates). However, in order to obtain the possible one-period ahead values of risky debt, we need to superimpose the rating transition process on the lattice. We describe this the next.

From each of the eight nodes of the triple-binomial spread lattice, three rating transitions emanate. Thus, if the current rating class at the source node was k, then three transitions $k \to I$, $k \to J$, and $k \to D$ are possible, with probabilities $q_{k,I}$, $q_{k,J}$, and $q_{k,D}$ respectively. From each of the sixteen non-default states so obtained (note that the default state D is an absorbing state), another superimposition of triple-binomial lattice and rating transition matrix evolves.

Thus, at each node, we carry the information set $(F, S_I, S_J, \lambda_k, \phi_k)$ where k is the current rating. As in Das and Sundaram ([11]), we also carry at each node the state-price of the node and cumulative default probability till the node. This provides all the information that is necessary to price a wide range of standard credit instruments and derivatives. Figure 1 illustrates the rating migration process superimposed on the triple-binomial lattice at one of the nodes. The up and down states for the interest rate process, F_u and F_d, correspond to $X_0 = +1$ and $X_0 = -1$, respectively, with similar notation used for S_I and S_J as well.

The code for calibrating the tree is described in Appendix B. For simplicity, we have assumed in the code that $\sigma(t,T)$, $\eta_I(t,T)$, and $\eta_J(t,T)$, depend only on T. Also, as assumed all through the text, the correlation coefficients between X_0, X_1, X_2 and the rating transition matrix D are assumed constant. To consider a numerical example, we consider the calibration exercise for a tree of three periods with the following parameter specifications.

X_1 and X_2 are assumed to be perfectly correlated ($\rho_3 = 1$). The correlations between X_0 and X_1, and between X_0 and X_2, are assumed to be identical, $\rho_1 = \rho_2 = 0.25$. The time-step in the tree is $h = 0.5$ (half a year). The initial values for forward risk-free rate and inter-rating spreads, and the volatility terms of forward risk-free rate and inter-rating spread processes, are as described in the table below:

i	$f(t, t+ih)$	$s_I(t, t+ih)$	$s_J(t, t+ih)$	$\sigma(t, t+ih)$	$\eta_I(t, t+ih)$	$\eta_J(t, t+ih)$
1	0.06	0.02	0.04	0.010	0.005	0.005
2	0.07	0.02	0.04	0.011	0.006	0.006
3	0.08	0.03	0.05	0.012	0.006	0.007

The rating transition process under the risk-neutral measure is taken to be:

$$D = \begin{bmatrix} 0.70 & 0.20 & 0.10 \\ 0.10 & 0.75 & 0.15 \\ 0 & 0 & 1 \end{bmatrix}$$

Under this parameter specification, the tree for the evolution of the risk-free forward rate and the inter-rating forward spreads is shown in Table 1. Note that unlike Figure 1, the rating transitions are not shown as superimposed in this tree even though the probabilities of these transitions are required for accurate no-arbitrage calibration of the risk-neutral drifts (Proposition 3.2). In addition, the number of branches is reduced since the two inter-rating spread processes are assumed to be perfectly correlated. Thus at first period, there are four nodes possible, viz. uu, ud, du, and dd, with probabilities, $\frac{1}{4}(1+\rho)$, $\frac{1}{4}(1-\rho)$, $\frac{1}{4}(1-\rho)$, and $\frac{1}{4}(1+\rho)$, i.e., 0.3125, 0.1875, 0.1875, and 0.3125, respectively. From each of these nodes, four nodes emanate again.

At each node in the tree at time ih (initial node being $i = 0$), the three columns indicate the forward risk-free rate, the forward inter-rating spread between risk-free and investment grade rating (I), and the forward inter-rating spread between investment grade rating (I) and speculative grade rating (J), respectively, for maturities $(i+1)h, \ldots, T = 1.5$ years. Using these forward rates, the tree for zero bond prices for risk-free bond, investment grade bond, and speculative grade bond, can be readily constructed using equations (2.4) and (2.5). This tree is shown in Table 2. The zero bond prices constitute the fundamental prices using which other instruments can be priced.

We next illustrate this by pricing a credit-related instrument in our framework, taking the example of *Credit Sensitive Note*, whose valuation requires modeling both default risk as well as rating migrations. Other instruments can be priced analogously.

7 Example: Credit Sensitive Note

A Credit Sensitive Note (CSN) is a corporate coupon bond whose coupon is linked to the rating of the corporate. For example, in June 1989, Enron Corp. issued \$100 ml. in non-callable 9.5% Credit Sensitive Notes, to mature on June 15, 2001. The coupon on these notes was linked to Enron's credit rating, as measured by either Standard & Poor's or Moody's. The coupon on the notes was structured such that, when Enron's credit rating changed (at the time of issuance, its outstanding senior debt had ratings BBB- and Baa3, respectively under the two rating agencies), the coupon rate changed as well. To be specific, the coupon rate was set to drop incrementally for improvements in Enron's ratings, and it would climb steeply if the rating deteriorated. The exact schedule of coupon is tabulated below.[6]

Moody's Rating	S&P rating	Coupon Rate
Aaa	AAA	9.20 %
Aa1-Aa3	AA+ - AA−	9.30 %
A1-A3	A+ - A−	9.40 %
Baa1-Baa3	BBB+ - BBB−	9.50 %
Ba1	BB+	12.00 %
Ba2	BB	12.50 %
Ba3	BB−	13.00 %
B1 or lower	B+ or lower	14.00 %

In our setup, we will assume that the coupon amount on a coupon payment date is linked to the corporate rating prevailing at the previous coupon payment date. In our three rating classes model, the CSN has a coupon of c_I for investment grade rating and c_J for junk grade rating. Such a note cannot be priced using a pure spread-based model of credit or a pure intensity-based model of credit. The model described in this paper, however, lends itself appropriately to the valuation of a CSN.

The valuation of CSN along the lattice in our model is straightforward. At each node of the lattice, the current rating class is available in the information set at the node. This determines the coupon payment scheduled for next coupon payment date. The "up-grading" and "down-grading" along the lattice produce the resetting of the coupon during the life of the schedule, as per the coupon vs. rating schedule. Thus, discounting the cashflows in default and non-default states, and moving backwards along the tree yields the price of the CSN.

To be more precise, using the recursive implementation discussed in Section 4, the price of the credit sensitive note, $CSN_k(t,T)$, is given as:

$$CSN_k(t,T) \quad = \quad \Pi_k(t,t+h) \cdot \left[c_k + \sum_{j=1}^{K} \frac{q_{k,j}(t)}{1-\lambda_k(t)} \cdot E^t \left[CSN_j(t+h,T) \right] \right]. \tag{7.1}$$

Note that $\Pi_k(t,t+h)$ is already available from the zero bond price tree, c_k represents the coupon next period which is "set" today given the current rating 'k', and the second term inside $[\ \cdot\ ]$ represents the value of the credit sensitive note at the nodes tomorrow after possible rating

[6]A more recent example of a CSN comes from an issue by Olivetti, which announced on June 7th, 2000, that it plans to link the coupon on 18 billion euros ($17 billion) of bonds sold by itself and its Tecnost SpA unit to their credit rating. Investors will be paid off if the rating worsens, less if the grade recovers. As stated by Olivetti's chief financial officer, Luciano La Noce - "The coupon adjustment will be applicable to all of the outstanding issues. Going forward, we think having these sort of volatility protection measures associated with our bonds should result in a lower capital cost." (source: Bloomberg).

migrations. The code for this procedure is described in Appendix B.

As an illustration, we consider a variant of the Enron CSN that has 1.5 years to maturity (three period note with $h = 0.5$ year). The coupons for different ratings are: $c_I = 0.04675$, and $c_J = 0.06375$, which correspond to semi-annual coupons of 9.35% and 12.75%, respectively. Using the recursive scheme described above (or simply by reducing the scheme to a backwards induction procedure), the CSN can be priced off the zero bond price tree. The tree for CSN prices is described in Table 3. At each node, the two columns represent the CSN price for investment grade and speculative grade ratings, respectively. For example, at $t = 0$, the CSN price is 0.994146 if the underlying credit has investment grade rating, but the price would be lower at 0.984822 if the underlying credit had speculative grade rating.

At $t = 1.0$ year, the price of the note is easy to determine since its coupon payment is "set" for maturity at $T = 1.5$ years. The price is thus simply equal to

$$(1.0 + c_k) \cdot \Pi_k(t, t+h),$$

where k is the current rating of the underlying credit. Consider now the state of the world uu at $t = 0.5$ year when the underlying credit has speculative grade rating. Its price can be computed using equation (7.1) as:

$$CSN_J(t = 0.5, T = 1.5) \;=\; 0.93169 \cdot [\; 0.06375 +$$

$$\tfrac{0.10}{0.85} \cdot [\; (0.3125)(0.980370) + (0.1875)(0.986270) + (0.1875)(0.986270) + (0.3125)(0.992206) \;] +$$

$$\tfrac{0.70}{0.85} \cdot [\; (0.3125)(0.968530) + (0.1875)(0.977286) + (0.1875)(0.974359) + (0.3125)(0.983168) \;]$$

$$] \;=\; 0.969719.$$

To see how the credit sensitivity of the coupon payments plays a role in the pricing of the CSN, Table 4 shows the tree for prices for a credit *insensitive* note that has a fixed coupon of 0.04675, irrespective of the rating of the underlying credit. Consider the price of this note at $t = 0$ with investment grade rating. It is 0.985483, whereas that of the otherwise identical CSN is 0.994146. The difference in value comes from two parts: (i) At all nodes at $t = 0.5$ year, if the rating were to "fall" to speculative grade, the CSN would have an upward jump in coupon payment from 0.04675 to 0.06375. (ii) At all nodes at $t = 0.5$ year, even if the rating were to "stay" as investment grade, the price of the CSN would be higher due to the increase in future coupon payments whenever there is a downgrading. Both these effects are observed by comparing $t = 0.5$ year prices across the trees for credit sensitive note (Table 3) and credit insensitive note (Table 4).

Instruments other than credit sensitive notes that have embedded optionality that is tied to credit quality of the underlying can be priced analogously in a relatively simple manner using our approach.

8 Concluding Comments

This paper develops a model for the pricing of credit derivatives using observables. The model is (i) arbitrage-free, (ii) accommodates path-dependence, (iii) allows for all rating classes in one consistent lattice framework, and (iv) can handle a range of securities that have a credit related component. The computer implementation uses a recursive scheme that is convenient and seamlessly processes forward induction and backward recursion, needed to compute more complicated derivative securities. While the model is rich and flexible enough to price any credit-related instrument, it is particularly appropriate for pricing credit sensitive notes that have payments linked to rating transitions.

A Proofs

Proof of Proposition 3.1: Let $Z(t,T)$ denote the price of the default-free bond discounted using $B(t)$:

$$Z(t,T) = \frac{P(t,T)}{B(t)}. \tag{A.1}$$

Since Z is a martingale under Q, for any $t < T$ we must have $Z(t,T) = E^t[Z(t+h,T)]$, or, equivalently,

$$E^t\left[\frac{Z(t+h,T)}{Z(t,T)}\right] = 1. \tag{A.2}$$

Now, $Z(t+h,T)/Z(t,T) = (P(t+h,T)/P(t,T)) \cdot (B(t)/B(t+h))$. Using (2.4), some algebra reveals the first term to be

$$\frac{P(t+h,T)}{P(t,T)} = \exp\left\{-\left(\sum_{i=t/h+1}^{T/h-1}[f(t+h,ih)-f(t,ih)]\cdot h\right)+f(t,t)h\right\}. \tag{A.3}$$

The second term $B(t)/B(t+h)$ is evidently just $\exp\{-f(t,t)h\}$. Combining these, we obtain

$$\frac{Z(t+h,T)}{Z(t,T)} = \exp\left\{-\sum_{i=t/h+1}^{T/h-1}[f(t+h,ih)-f(t,ih)]\cdot h\right\}, \tag{A.4}$$

Using (A.4) in (A.2), the martingale condition becomes

$$E^t\left[\exp\left\{-\sum_{i=t/h+1}^{T/h-1}[f(t+h,ih)-f(t,ih)]\cdot h\right\}\right] \;=\; 1. \tag{A.5}$$

Substituting for $(f(t+h,ih)-f(t,ih))$ from (2.1), this is the same as

$$E^t\left[\exp\left\{-\sum_{i=t/h+1}^{T/h-1}[\alpha(t,ih)h^2+\sigma(t,ih)X_0h^{3/2}]\right\}\right] \;=\; 1. \tag{A.6}$$

Since $\alpha(t,\cdot)$ is known at t, it may be pulled out of the expectation. This gives us after some rearranging the promised recursive expression relating the risk-neutral drifts α to the volatilities σ at each t:

$$\sum_{i=t/h+1}^{T/h-1}\alpha(t,ih) \;=\; \frac{1}{h^2}\ln\left(E^t\left[\exp\left\{-\sum_{i=t/h+1}^{T/h-1}\sigma(t,ih)X_0h^{3/2}\right\}\right]\right). \tag{A.7}$$

Proof of Proposition 3.2: Pick any $t<T$ and lets suppose that the time-t rating class of the bond is k. Consider a one-period investment in this bond at t. Then, at time $t+h$, there is a set of possible values $\Pi_j(t+h,T), \forall j=1,\ldots,K+1$, since the bond may remain in its time-t rating class k, or move to any other rating class j. Thus, we have

$$E^t_k[\Pi(t+h,T)\mid \text{No Default}] = E^t\left[\sum_{j=1}^{K}\frac{q_{k,j}(t)}{1-\lambda_k(t)}\cdot\Pi_j(t+h,T)\right].$$

The expectation in RHS above is over the state-space $(X_0,\mathbf{X})$. Note that $\lambda_k(t)\equiv q_{k,K+1}(t)$ and the sum inside the expectation is over all possible rating classes at $t+h$, conditional on no-default at $t+h$.

If the bond has defaulted in the period $(t,t+h]$, there is a cashflow at $t+h$ due to the recovery upon default. By RMV assumption (2.6), the expected cashflow is $\phi_k(t)E^t_k[\Pi(t+h,T)]$. Since the probability of default by $t+h$, given rating class at time t is k, is $\lambda_k(t)$, the undiscounted expected value of the bond is

$$(1-\lambda_k(t))\,E^t_k[\Pi(t+h,T)\mid\text{No Default}]+\lambda_k(t)\phi_k(t)\,E^t_k[\Pi(t+h,T)\mid\text{No Default}],\ \forall k \tag{A.8}$$

which is the same as

$$[1-\lambda_k(t)+\lambda_k(t)\phi_k(t)]\,E^t_k[\Pi(t+h,T)\mid\text{No Default}], \quad \forall k. \tag{A.9}$$

By definition of Q, when discounted at the short rate $r(t)$, this expected cash flow must equal $\Pi_k(t,T)$, so we have

$$E^t\left[\frac{[1-\lambda_k(t)+\lambda_k(t)\phi_k(t)]\, E_k^t[\Pi(t+h,T)\mid \text{No Default}]}{\exp\{r(t)h\}\Pi_k(t,T)}\right] \;=\; 1, \quad \forall k. \tag{A.10}$$

Now, using relations (2.7) and (2.5), and the definitional relation $s(t,t)=\varphi(t,t)-f(t,t)$, we get

$$\begin{aligned}
\Pi_k(t,T)\cdot\exp\{[r(t)+\sum_{j=1}^{k}s_j(t,t)]h\} &= \exp\left\{-\sum_{i=\frac{t}{h}+1}^{\frac{T}{h}-1}\varphi_k(t,ih)\cdot h\right\}, \text{ and} && \text{(A.11)}\\
E_k^t[\Pi(t+h,T)\mid \text{No Default}] &= E^t\left[\sum_{j=1}^{K}\frac{q_{k,j}(t)}{1-\lambda_k(t)}\cdot\exp\left\{-\sum_{i=\frac{t}{h}+1}^{\frac{T}{h}-1}\varphi_j(t+h,ih)\cdot h\right\}\right] && \text{(A.12)}
\end{aligned}$$

Using the above two equations, we get the implicit equation for the drift terms $\beta(t,T)$:

$$E^t\left[\sum_{j=1}^{K}\frac{q_{k,j}(t)}{1-\lambda_k(t)}\exp\left\{-\sum_{i=\frac{t}{h}+1}^{\frac{T}{h}-1}(\varphi_j(t+h,ih)-\varphi_k(t,ih))\cdot h\right\}\right]=1. \tag{A.13}$$

Now, writing $\varphi_j(t+h,ih)-\varphi_k(t,ih)$ as $\varphi_j(t+h,ih)-\varphi_j(t,ih)+\varphi_j(t,ih)-\varphi_k(t,ih)$, and using relations (2.1), (2.2), and (2.3), we can rewrite the above equation as

$$\begin{aligned}
E^t\ [\ \sum_{j=1}^{K}\frac{q_{k,j}(t)}{1-\lambda_k(t)}\cdot\exp&\left\{-\sum_{i=\frac{t}{h}+1}^{\frac{T}{h}-1}\left(\alpha(t,ih)\cdot h^2+\sigma(t,ih)X_0\cdot h^{\frac{3}{2}}\right)\right\}\cdot\\
\exp&\left\{-\sum_{i=\frac{t}{h}+1}^{\frac{T}{h}-1}\sum_{l=1}^{j}\left(\beta_l(t,ih)\cdot h^2+\eta_l(t,ih)'\mathbf{X}\cdot h^{\frac{3}{2}}\right)\right\}\cdot\\
\exp&\left\{-\sum_{i=\frac{t}{h}+1}^{\frac{T}{h}-1}(\varphi_j(t,ih)-\varphi_k(t,ih))\cdot h\right\}\ ]\ =\ 1.
\end{aligned}\tag{A.14}$$

Using the notation $\theta_j(t,ih)=\sum_{l=1}^{j}\beta_l(t,ih)$, and assuming independence of the rating transition process from the diffusion processes $(X_0,\mathbf{X})$, we get $\forall t,\ \forall k$, at each state, a system of K linear equations in K unknowns $(x_j, j=1,\ldots,K)$:

$$\sum_{j=1}^{K}a_{k,j}\cdot b_{k,j}\cdot x_j=1 \tag{A.15}$$

where

$$a_{k,j} = \frac{q_{k,j}(t)}{1-\lambda_k(t)} \cdot \exp\left\{-\sum_{i=\frac{t}{h}+1}^{\frac{T}{h}-1} \alpha(t,ih)\cdot h^2\right\} \cdot \exp\left\{-\sum_{i=\frac{t}{h}+1}^{\frac{T}{h}-1} (\varphi_j(t,ih)-\varphi_k(t,ih))\cdot h\right\}, \tag{A.16}$$

$$b_{k,j} = E^t\left[\exp\left\{-\sum_{i=\frac{t}{h}+1}^{\frac{T}{h}-1}\left(\sigma(t,ih)X_0 + \sum_{l=1}^{j}\eta_l(t,ih)'\mathbf{X}\right)\cdot h^{\frac{3}{2}}\right\}\right], \text{ and} \tag{A.17}$$

$$x_j = \exp\left\{-\sum_{i=\frac{t}{h}+1}^{\frac{T}{h}-1} \theta_j(t,ih)\cdot h^2\right\}. \tag{A.18}$$

B Implementation Code for Pricing a CSN

```
/* Program to generate the HJM tree with rating transitions and default
risk recursively, and then to price a Credit Sensitive Note */

CRD(f0,fsig0,s0[],ssig0[],phi[],rho[],h,coupon[],q[][],current_rating,n) {

    /* Note that in our example, K = 2 */
    /* n : number of periods */
    /* q : assumed to be the risk-neutral transition matrix */
    q_uuu = 0.125 * (1 + rho[1] + rho[2] + rho[3]) ;
    q_uud = 0.125 * (1 + rho[1] - rho[2] - rho[3]) ;
    q_udu = 0.125 * (1 - rho[1] + rho[2] - rho[3]) ;
    q_udd = 0.125 * (1 - rho[1] - rho[2] + rho[3]) ;
    q_duu = 0.125 * (1 - rho[1] - rho[2] + rho[3]) ;
    q_dud = 0.125 * (1 - rho[1] + rho[2] - rho[3]) ;
    q_ddu = 0.125 * (1 + rho[1] - rho[2] - rho[3]) ;
    q_ddd = 0.125 * (1 + rho[1] + rho[2] + rho[3]) ;

    CRVAL(level,f,fsig,s[],ssig[],phi[],rating) {

     /* i, j, k, m  : counters; alpha, beta : arrays for drift terms */
     if (level == n-1) {
         result = exp(-s[rating]*h) * (1.0 + coupon[rating]) ;
     }
```

```
else {                   /* level < n-1 */

    m = n - level ;
    cur_f = Take[f,-m] ; /* Takes m bottom elements of the array */
                         /* for m timesteps upto T */
    cur_s = Take[s,-m] ;
    cur_fsig = Take[fsig,-m] ;
    cur_ssig = Take[ssig,-m] ;
    cur_phi  = Take[phi, -m] ;
    sqh = Sqrt(h) ;

    /* Determine alpha drift terms */
    for (i=m; i>=1; i--) { /* Proceed in a bootstrap manner */
       if (i == m) {       /* last period forward drift     */
           alpha[i] = log (0.5 * (exp(-cur_fsig[i]*h*sqh) +
                           exp( cur_fsig[i]*h*sqh)) / (h*h) ;
       }
       else {              /* i < m */
           alpha[i] = log (0.5 *
             (exp(-Sum[cur_fsig[k],{k,i,m}]*h*sqh) +
              exp( Sum[cur_fsig[k],{k,i,m}]*h*sqh)) / (h*h)
             - Sum[alpha[k],{k,i+1,m}] ;
       }
    }

    /* Determine beta drift terms */
    for (i=m; i>=1 ; i--) {
        for (k=1; k<K+1; k++) {
           for (j=1; j<K+1; j++) {
              a[i,k,j] = q[k][j] * exp(-Sum[alpha[l],{l,i,m}]*h*h) *
                         exp((-Sum[Sum[cur_s[p,l],{p,1,j}],{l,i,m}]
                              +Sum[Sum[cur_s[p,l],{p,1,k}],{l,i,m}])*h) ;
              sum_fsig    = Sum[cur_fsig[l],{l,i,m}] ;
              sum_ssig_uu = Sum[cur_ssig[1,l]+cur_ssig[2,l],{l,i,m}] ;
              sum_ssig_ud = Sum[cur_ssig[1,l]-cur_ssig[2,l],{l,i,m}] ;
              sum_ssig_du = Sum[-cur_ssig[1,l]+cur_ssig[2,l],{l,i,m}] ;
              sum_ssig_dd = Sum[-cur_ssig[1,l]-cur_ssig[2,l],{l,i,m}] ;
              b[i,k,j] =
```

```
                (q_uuu * exp(-h*sqh*(sum_fsig+sum_ssig_uu)) +
                 q_uud * exp(-h*sqh*(sum_fsig+sum_ssig_ud)) +
                 q_udu * exp(-h*sqh*(sum_fsig+sum_ssig_du)) +
                 q_udd * exp(-h*sqh*(sum_fsig+sum_ssig_uu)) +
                 q_duu * exp(-h*sqh*(-sum_fsig+sum_ssig_uu)) +
                 q_dud * exp(-h*sqh*(-sum_fsig+sum_ssig_ud)) +
                 q_ddu * exp(-h*sqh*(-sum_fsig+sum_ssig_du)) +
                 q_ddd * exp(-h*sqh*(-sum_fsig+sum_ssig_uu))) ;
       }
       x[i,k] = Solve_Gauss_Seidel(a[i,k]*b[i,k],One[K]) ;
       if (i == m)              /* Last time period */
           theta[i,k] = - log(x[i,k]) / (h*h) ;
       else                     /* i < m */
           theta[i,k] = - log(x[i,k]/x[i+1,k]) / (h*h) ;
       if (k == 1)
           beta[i,k]  = theta[i,k] ;
       else
           beta[i,k]  = theta[i,k] - theta[i,k-1] ;
    }
}

/* This is the interest rate plus inter-rating spread part of
   the lattice */
fu = cur_f + alpha * h + cur_fsig * sqh ;
fd = cur_f + alpha * h - cur_fsig * sqh ;
su = cur_s + beta  * h + cur_ssig * sqh ; /* Vector operation */
sd = cur_s + beta  * h - cur_ssig * sqh ; /* Vector operation */

/* This is the rating transition part of the lattice */
/* The code below is shown for "uuu" node.
   It must be repeated for "uud"..."ddd" nodes to complete
   the branching. */

/* "uuu" node */
s_uuu[1] = su[1] ;
s_uuu[2] = su[2] ;
result_uuu = 0.0 ;
```

```
        for (i=1; i<K+1; i++) { /* Non-default states */
            result_uuu += q[rating][i] / (1 - q[rating][K+1]) *
                CRVAL(level+1,fu,cur_fsig,s_uuu,cur_ssig,cur_phi,i)) ;
        }

        /* "uud" node */
        s_uud[1] = su[1] ;
        s_uud[2] = sd[1] ;
        ...................
        /* Repeat for "udu", "udd", "duu", "dud", "ddu", "ddd" states */
        ...................

        result = exp(-s[rating][1]*h) *      /* discount at spot short rate */
              (coupon[rating] +              /* Credit Sensitivity of Note  */
               q_uuu * result_uuu +
               q_uud * result_uud +
               q_udu * result_udu +
               q_udd * result_udd +
               q_duu * result_duu +
               q_dud * result_dud +
               q_ddu * result_ddu +
               q_ddd * result_ddd) ;
    }  /* end if level < n - 1 */
    return result ;
   }
   return CRVAL(0,f0,fsig0,s0,ssig0,phi,current_rating) ;
}
```

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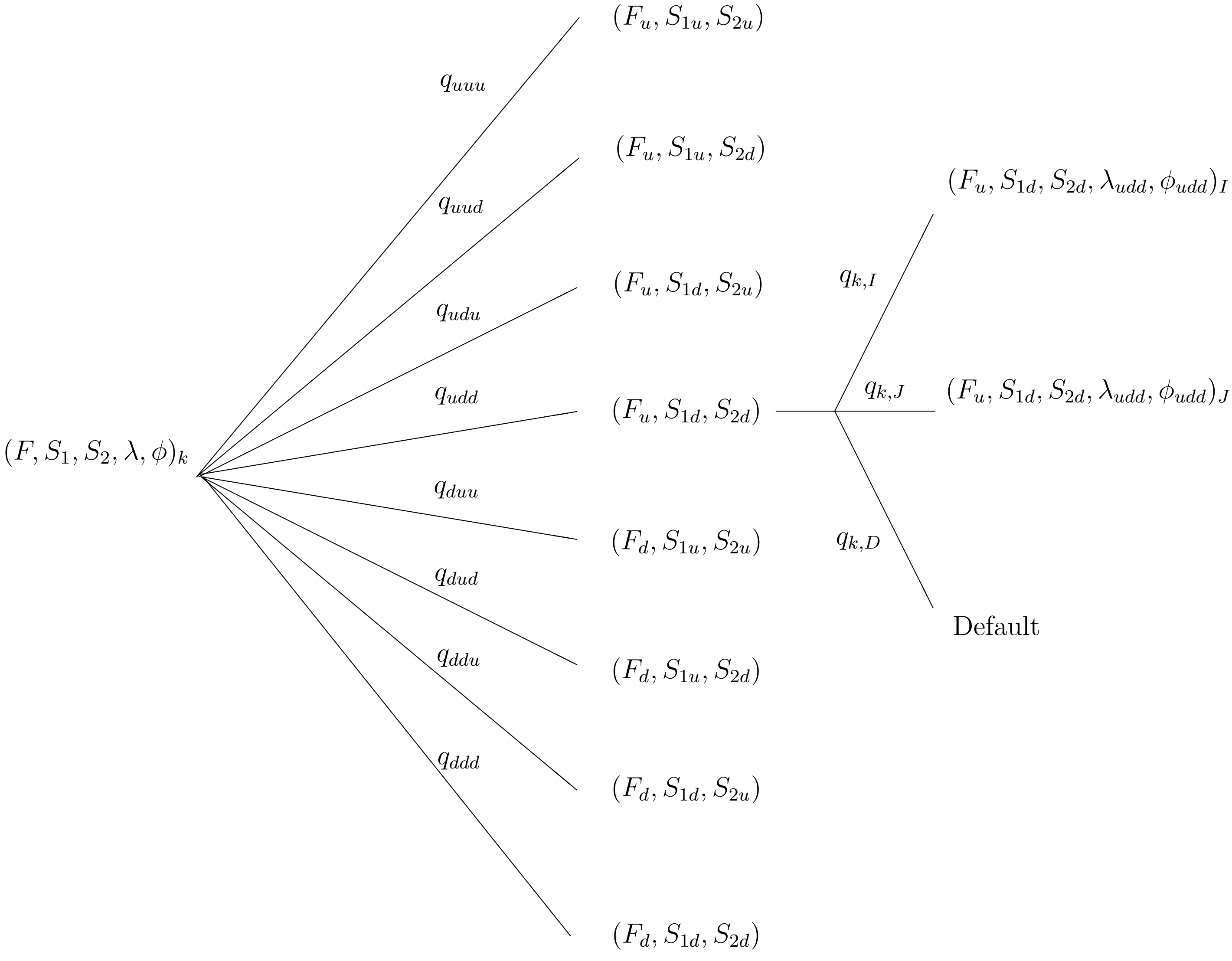
Figure 1: Multi-dimensional Interest Rate plus Spreads Tree and the Rating Transitions



The figure illustrates how the multi-dimensional tree of interest rate and inter-rating spread processes combines with the rating transitions to yield the complete lattice. The left part of the braching shows the eight nodes that emanate from a starting rating class k, depending upon the realizations of the binomial variates (X_0, X_1, X_2). The right part of the branching shows for one of these eight nodes, in particular the up-down-down node, the rating transitions to the three possible states I, J, and D. The probabilities $q_{uuu}, \ldots, q_{ddd}$ for the interest rate plus spreads lattice are given by the equation (5.1), whereas the probabilities for the rating transitions, $q_{k,I}$, $q_{k,J}$, and $q_{k,D}$, are given by the rating transition matrix at the corresponding node. Default is an absorbing state.

Table 1: Tree for Forward Risk-free Rate and Inter-rating Spreads

This table shows a tree of forward risk-free rate and forward inter-rating spreads, calibrated as per the parameter specification in the text. The calibration of the risk-neutral drifts is done using the results in Proposition 3.1 and Proposition 3.2.

t=0			t=0.5				t=1.5			
							uu	0.092017	0.039013	0.056522
			uu	0.075504	0.023003	0.043004	ud	0.092017	0.027013	0.050522
				0.086013	0.033007	0.053515	du	0.080017	0.039013	0.056522
							dd	0.080017	0.027013	0.050522
							uu	0.092017	0.033013	0.049522
			ud	0.075504	0.017002	0.037004	ud	0.092017	0.021013	0.043522
F	S_I	S_J		0.086013	0.027007	0.046515	du	0.080017	0.033013	0.049522
0.06	0.02	0.04					dd	0.080017	0.021013	0.043522
0.07	0.02	0.04								
0.08	0.03	0.05					uu	0.080017	0.039013	0.056522
			du	0.064504	0.023002	0.043004	ud	0.080017	0.027013	0.050522
				0.074013	0.033007	0.053515	du	0.068017	0.039013	0.056522
							dd	0.068017	0.027013	0.050522
							uu	0.080017	0.033013	0.049522
			dd	0.064504	0.017002	0.037004	ud	0.080017	0.021013	0.043522
				0.074013	0.027007	0.046515	du	0.068017	0.033013	0.049522
							dd	0.068017	0.021013	0.043522

Table 2: Tree for Risk-free, Investment Grade, and Speculative Grade Zero Bond Prices

This table shows a tree of zero bond prices for risk-free, investment grade, and speculative grade bonds. The underlying forward rate tree is as in Table 1 and its calibration is done using the results in Proposition 3.1 and Proposition 3.2.

t=0			t=0.5				t=1.5			
							uu	0.955034	0.936585	0.910487
			uu	0.962952	0.951940	0.931690	ud	0.955034	0.942221	0.918718
				0.922416	0.896943	0.854684	du	0.960781	0.942221	0.915966
							dd	0.960781	0.947892	0.924247
							uu	0.955034	0.939399	0.916424
			ud	0.962952	0.954800	0.937297	ud	0.955034	0.945052	0.924709
P	Π_I	Π_J		0.922416	0.902341	0.865435	du	0.960781	0.945052	0.921939
0.970446	0.960789	0.941765					dd	0.960781	0.950740	0.930274
0.937067	0.918512	0.882497								
0.900325	0.869358	0.818731					uu	0.960781	0.942221	0.915966
			du	0.968263	0.957190	0.936829	ud	0.960781	0.947892	0.924247
				0.933085	0.907317	0.864570	du	0.966563	0.947892	0.921478
							dd	0.966563	0.953596	0.929809
							uu	0.960781	0.945052	0.921939
			dd	0.968263	0.960066	0.942466	ud	0.960781	0.950740	0.930274
				0.933085	0.912778	0.875445	du	0.966563	0.950740	0.927487
							dd	0.966563	0.956461	0.935873

Table 3: Tree for Credit Sensitive Note Prices

This table shows a tree of prices for the credit sensitive note that is to mature in 1.5 years and has the coupon schedule: $c_I = 0.04675$ and $c_J = 0.06375$.

t=0		t=0.5			t=1.5		
					uu	0.980370	0.968530
		uu	0.981175	0.969719	ud	0.986270	0.977286
					du	0.986270	0.974359
					dd	0.992206	0.983168
					uu	0.983316	0.974846
		ud	0.987674	0.981145	ud	0.989234	0.983659
CSN_I	CSN_J				du	0.989234	0.980713
					dd	0.995187	0.989579
0.994146	0.984822						
					uu	0.986270	0.974359
		du	0.992254	0.980576	ud	0.992206	0.983168
					du	0.992206	0.980223
					dd	0.998177	0.989084
					uu	0.989234	0.980713
		dd	0.998828	0.992132	ud	0.995187	0.989579
					du	0.995187	0.986615
					dd	1.001176	0.995534

Table 4: Tree for Credit Insensitive Note Prices

This table shows a tree of prices for the credit insensitive note that is to mature in 1.5 years and has a coupon $c = 0.04675$. It is identical to the credit sensitive note considered in Table 3 except that its coupon is not linked to the rating of the underlying credit.

t=0		t=0.5			t=1.5		
					uu	0.980370	0.953052
		uu	0.977876	0.941060	ud	0.986270	0.961668
					du	0.986270	0.958787
					dd	0.992206	0.967455
					uu	0.983316	0.959267
		ud	0.984343	0.952229	ud	0.989234	0.967939
CIN_I	CIN_J				du	0.989234	0.965040
					dd	0.995187	0.973764
0.985483	0.960433						
					uu	0.986270	0.958787
		du	0.988917	0.951681	ud	0.992206	0.967455
					du	0.992206	0.964557
					dd	0.998177	0.973278
					uu	0.989234	0.965040
		dd	0.995459	0.962978	ud	0.995187	0.973764
					du	0.995187	0.970848
					dd	1.001176	0.979625