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ABSTRACT

Structural Change and the Kaldor Facts of Economic Growth*

We present a model in which two of the most important features of the long-run growth process are reconciled: the massive changes in the structure of production and employment; and the Kaldor facts of economic growth. We assume that households expand their consumption along a hierarchy of needs and firms continuously introduce new products. In equilibrium industries with an expanding and those with a declining employment share co-exist, and each such industry goes (or has already gone) through a cycle of take-off, maturity, and stagnation. Nonetheless, macroeconomic aggregates grow *pari passu* at a constant rate.

JEL Classification: D91, L16, O11, O31 and O40
Keywords: balanced growth, demand externalities, hierarchic preferences, innovation, Kaldor facts, multiple equilibria and structural change

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1 Introduction

The process of development is characterized by fundamental changes in the structure of production and employment. The emergence of new and the decline of old industries has led to a dramatic reallocation of labor between sectors of production in historical perspective.[1] Despite these large structural changes, the long-term growth process turns out remarkably stable in the aggregate. As mentioned by Kaldor (1961) in his famous stylized facts, a situation where growth rate, interest rate, capital output ratio, and labor share are constant over time is a reasonable approximation of the long-run growth experience of a modern economy.

In this paper we present a model that accounts both for structural change and for the Kaldor facts. On the one hand, industries with a growing share in aggregate production coexist with declining industries, and each such industry is going (or has already gone) through a cycle of take-off, maturity, and stagnation. Hence there is continuous structural change. On the other hand, our model features a situation where all macroeconomic aggregates grow at the same constant rate, and where the interest rate and the labor share are constant over time. Thus, our model meets Kaldor's criteria. In contrast, standard theories of economic growth have been predominantly concerned with models that exhibit a 'balanced' growth path and have almost entirely ignored the issue of structural change.[2]

Generally speaking, changes in the structure of production and employment result either from differences in productivity growth or from differences in the growth of product demand across sectors. In this paper, we focus on the demand side and abstract from technological differences across sectors. Thus the driving force behind structural changes are differences in the income elasticities of demand across sectors.

The basic idea of our analysis is that households expand their consumption along a hierarchy of needs. When the basic needs are saturated, consumers move on to more advanced

[1]The following figures referring to a familiar trichotomy of sectors - agriculture, manufacturing, and services - demonstrate the impressive size of these structural changes (Maddison, 1987): In 1870 the employment share in agriculture amounted to 50 % in the U.S., to 67.5 % in Japan, and to 49.2 % in France. In 1984, the corresponding numbers decreased to 3.3 % in the U.S., 8.9 % in Japan, and 7.6 % in France. During the same period the employment share in the service sector increased from 25.6 % to 68.7 % in the U.S., from 18.7 % to 56.3 % in Japan, and from 23 % to 60.4 % in France.

[2]A noteable exception is Pasinetti (1981) who presents a systematic analysis of economic growth and structural change in the post-Keynesian tradition.

needs. As incomes grow, more and more goods and services enter the consumption bundle, and more and more wants can be satisfied. The empirical motivation for the assumption of a hierarchy structure of preferences is Engel's law, one of the most robust empirical regularities in economics (Houthakker, 1987). Engel (1857) himself saw the implications of this law for economic development very clearly: a declining relative demand for food would inevitably decrease the share of output and employment in the agricultural sector, and would provide the resources for the emergence of new industries.

The supply side of our model has a simple structure. We study a situation where growth is endogenous and driven by industrial R&D. There are interindustry spillovers of knowledge, so innovative activities in one sector add to the economy-wide stock of knowledge and increase productivity in all other sectors. The assumption of economy-wide spillover effects rules out sector-specific technical progress, the second possible source of structural change. The main reason why we disregard uneven technical change is to keep the model tractable and to concentrate on the role of demand. The second reason is that, unlike on the demand side, it is less clear on the supply side how the conditions in expanding relative to stagnating sectors change over time.[3]

The equilibrium outcome of our model has the following features. First, the dynamic equilibrium is characterized by a situation of continuous structural change. At each date, there co-exist goods that have a high income elasticity (luxuries) with goods that have a low income elasticity (necessities). And over time each good starts off as a luxury with a high income elasticity and ends up as a necessity with a low income elasticity. In this sense, each sector goes through the same cycle of take-off, maturity, and stagnation. Hence the equilibrium is characterized by non-linear Engel-curves due to the non-homotheticity of hierarchic preferences.

[3]The literature discusses two important arguments. On the one hand, the transition towards a service eocnomy implies the emergence of new industries with little scope for technical progress (Baumol, 1967, Baumol et al., 1985). On the other hand, the *new* expanding markets consist to large extent of high-tech products and sophisticated services where the potential for technological improvements is high. (In fact, the very recent U.S. experience suggests that technical progress is actually concentrated in these new sectors). Furthermore, while the empirical evidence shows that the service sector has grown more slowly (Maddison, 1987), is not clear to which extent this is due to measurement error. The particular problem is mismeasurement in the quality improvements of services (see Shapiro and Wilcox, 1996, and Hornstein and Krusell, 1996).

Second, the dynamic equilibrium meets Kaldor's criteria. Prima facie reconciling structural change and non-linear Engel-curves with the Kaldor facts seems to be a difficult task. What is the crucial assumption that makes this possible? Necessary conditions for a steady growth path are a constant interest rate on the supply side and a constant elasticity of intertemporal substitution on the preference side. With many goods and a constant interest rate, steady growth is possible if the optimal growth rate of total consumption expenditures is constant over time. The demand and expenditure levels of the various products, however, *need not change in proportion with total expenditures.* It is exactly this pattern that our model generates. With our model of hierarchic preferences, it turns out that the constancy of the optimal growth rate of consumption expenditures depends critically on a function that characterizes the 'steepness' of the hierarchy of needs, that is the willingsness of consumers to move from goods that satisfy needs of higher priority towards goods that satisfy needs of lower priority.

Third, in our model there is an interesting two-way causality between technological progress and the incentives for innovators. On the one hand, the aggregate growth rate is endogenously determined by industrial R&D due to our assumptions regarding productivity improvements. On the other hand, the incentives for innovators depend crucially on the economy-wide growth rate, because all sectors have a positive (albeit non-unitary) income elasticity of demand. This dynamic complementarity between aggregate and sectoral dynamics may give rise to of multiple equilibria. Optimistic (pessimistic) expectations of a high (low) growth rate provide an incentive for a high (low) level of innovative activities that makes expectations come true.

Fourth, the dynamic equilibrium may be characterized by a situation where consumers cannot afford all products that are available on the market. In particular, this means that the non-negativity constraints for the most luxurious (= brand-new) products are binding. For this reason innovators have a 'waiting time' until consumers are rich enough to purchase a new product. Firms may nevertheless incur the R&D costs to get a patent and to prevent potential competitors conquering the market.

Finally, hierarchic preferences imply that incumbent firms have increasing market power as the price elasticities of demand decrease during the product cycle. Rising incomes lead to a higher willingness to pay and hence to higher mark-ups. The growing mark-ups imply strong static price distortions and the socially optimal patent policy is characterized by a finite patent length.

As mentioned above the previous literature has largely ignored to analyze the simultaneity of structural change and steady growth. To our knowledge, the only paper that explicitely addresses this question is the one by Kongsamut, Rebelo, and Xie (2001). They show in the context of a three-goods economy that a 'generalized' balanced growth path is only possible if technology and taste parameters satisfy a certain knife-edge condition. No such link is necessary in our model. In the present set-up new goods are continuously introduced, each of which starts off as a luxury with a high income elasticity and ends up as a necessity with a low income elasticity. Moreover, in Kongsamut et al. (2001) productivity growth is exogenous whereas in our model innovations play a central role and interesting interactions between aggregate and sectoral dynamics arise.[4]

There are several other papers that are related to the present analysis. In Matsuyama (2002) the structure of preferences is similar in spirit to our framework as the various goods are ranked according to priority. In equilibrium, consumer goods industries take off one after another, and new goods are initially luxuries and finally become necessities. Stokey (1988) also analyzes a growth model in which changes in the sectoral structure occur as a result of non-homothetic preferences. Consumers value new goods because they have more characteristics, while old goods with less characteristics disappear. Neither of these papers focuses on the consistency of the changing sectoral structure with the Kaldor facts. Moreover, those papers assume a learning-by-doing mechanism, while in the present paper growth is driven by innovations. Thus, the dynamic demand externalities in our model do not show up there. A further related paper is Laitner (2001) who analyzes changes in the measured savings rate that occur during the process of growth and structural change. Contrary to our model, productivity growth is exogenous and the process of structural change is modeled in a two-sector framework.[5]

The paper is organized as follows. Section 2 presents the general set-up of the model,

[4]Also Echevarria (1997) studies the patterns of structural change in three-sector growth model. However, the focus of her paper is not to reproduce the Kaldor facts.

[5]Other papers where non-homothetic preferences have an impact on macroeconomic outcomes study the role of income inequality and/or unemployment. See, for instance, Murphy, Shleifer and Vishny (1989), Eswaran and Kotwal (1993), Baland and Ray (1991), for static models, and Falkinger (1990, 1994), Chou and Talmain (1996), Zweimüller (1996, 2000), Bertola and Zweimüller (2000), and Zagler (2000) for dynamic models. Flam and Helpman (1987), Stockey (1991) and Matsuyama (2000) study international trade in the context of non-homothetic preferences.

solves the static problems of consumers and firms, and discusses the resulting structure of demand and prices in the static equilibrium. In Section 3 we present our assumptions on technology, the labor market, and the determinants of aggregate savings. Section 4 discusses the equilibrium growth path and describes the patterns of structural changes that occur along this path. Section 5 contains a discussion of multiple equilibria and Section 6 applies the model to optimal patent policy. Section 7 summarizes the results and discusses possible extensions.

2 The Static Equilibrium

2.1 Preferences and consumer demand

Consider a representative agent economy with infinitely many potentially produceable goods ranked by an index i. We study the structure of consumption that is generated by preferences of the form

$$u(\{c(i)\}) = \int_0^\infty \xi(i)\ v(c(i))\ di$$

where $v(c(i))$ is an indicator for the utility derived from consuming good i in quantity c. The 'baseline' utility $v(c(i))$ satisfies the usual assumptions $v' > 0$ and $v'' < 0$; and the 'hierarchy' function $\xi(i)$ is monotonically decreasing in i, $\xi'(i) < 0$, hence low-i goods get a higher weight than high-i goods.

A meaningful specification of hierarchic preferences has to take account of two facts. First, some goods may not be consumed because the consumer cannot afford them. This implies that preferences must be such that the *non-negativity constraints may become binding* and Engel-curves for the various goods are non-linear. Formally, binding non-negativity constraints require that the marginal utility of consuming good i in quantitiy zero, $\xi(i)v'(0)$ is finite for all $i > 0$. If marginal utility at quantity zero were infinitely large, it would always be optimal to consume a (small) positive amount even when prices are very high and/or the budget is very low.[6]

[6] Non-negativity constraints never become binding in the standard monopolistic competition model (Dixit and Stiglitz (1978)) that dominates the macroeconomic literature. In that model $v(c(i)) = \frac{1}{\alpha}c(i)^\alpha$, $\alpha < 1$, and $v'(0) = \infty$. Thus in the standard monopolistic competition model all available goods are consumed in positive amounts.

Second, Engels' law implies that *additional* income is spent primarily on low-priority goods (high income elasticity). This feature is caught by the formulation that the utility of consumption of different goods differs only in the factor $\xi(i)$. As the hierarchy function $\xi(i)$ is decreasing in i the marginal utility of a high priority good (low i) falls quickly. Optimal consumer behavior implies that additional income is spent primarily on the low-priority goods with slowly falling marginal utilities.

To keep the analysis tractable we make two assumptions concerning the functional forms of the weighting function $\xi(i)$ and the baseline utility $v(c(i))$. First we assume that the weighting function is a power function $\xi(i) = i^{-\gamma}$ with $\gamma \in (0,1)$. It will turn out below that it is exactly this assumption which will allow us to study an equilibrium growth path that meets the Kaldor facts. Second, we assume that the baseline utility is quadratic, $v(c(i)) = \frac{1}{2}[s^2 - (s - c(i))^2]$. This allows us to find explicit solutions both for the optimal quantities consumed by the households and for the profit-maximizing prices charged by firms. At the same time this specification features the possibility that non-negativity constraints may become binding, as marginal utility at quantity zero is finite, $\xi(i)v'(0) = i^{-\gamma}\frac{1}{2}s^2 < \infty$ for all goods $i > 0$.

With these assumptions, we can now specifiy the objective function of the consumer's static maximization problem. Assume that only goods with high priority $i \in [0, N]$ are available on the market, whereas all $i > N$ have not yet been invented. In that case the consumers' objective function is[7]

$$u(\{c(i)\}) = \int_0^\infty i^{-\gamma} \; \frac{1}{2}[s^2 - (s - c(i))^2] \; di. \tag{1}$$

which will be maximized subject to the budget constraint $\int_0^N p(i)c(i)di = E$ and the non-negativity constraints $c(i) \geq 0$ for all i. The optimality conditions require that the above constraints and the first order conditions

$$\begin{aligned} c(i)\left[i^{-\gamma}(s - c(i)) - \lambda p(i)\right] &= 0 \qquad \forall i \\ i^{-\gamma}(s - c(i)) - \lambda p(i) &\leq 0 \qquad \forall i. \end{aligned} \tag{2}$$

[7] $v(c(i)) = \frac{1}{2}[s^2 - (s - c(i))^2]$ has been normalized such that $v(0) = 0$. This normalization is necessary to prevent divergence of the utility integral because the consumer's preferences are defined over an *infinite* number of goods. Since only goods in the interval $i \in [0, N]$ can be consumed in positive amounts the consumer's objective can be written as $u(\{c\}) = \int_0^N i^{-\gamma}\frac{1}{2}[s^2 - (s - c(i))^2]di + \int_N^\infty i^{-\gamma}\frac{1}{2}[s^2 - s^2]di$. To prevent divergence of the first integral we must have $\gamma < 1$. By the normalization of $v(.)$ the second integral is zero and does not diverge. We can then restrict our attention to the utility function $u(\{c\}) = \int_0^N i^{-\gamma}\frac{1}{2}[s^2 - (s - c(i))^2]di$.

be satisfied, where λ denotes the Lagrangian multiplier.

2.2 Prices

We assume there are constant marginal cost in production, equal for all goods, and we normalize these marginal costs to unity. Goods $i \in [0, aN]$ are supplied on competitive markets and goods $i \in (aN, N]$ are supplied by monopolistic firms. This means that high priority (low-i) goods are supplied by competitive producers and low priority (high-i) goods are supplied by monopolists.[8]

The prices for goods in the interval $i \in [0, aN]$ are equal to marginal costs which are unity. Determining the prices for the goods $i \in (aN, N]$ is less trivial but straightforward. The market demand function is given by the representative household's optimality conditions (2). The price that the monopolist charges maximizes the objective function $\pi(p(i)) = [p(i) - 1][\max(0,\ s - i^{\gamma} p(i)\lambda)]$. The solution is given by

$$p(i) = \max\left[1,\ \frac{s + i^{\gamma}\lambda}{2i^{\gamma}\lambda}\right] \quad \text{for } i \in (aN, N]. \tag{3}$$

2.3 Equilibrium composition of demand and the structure of prices

We can now characterize the composition of demand and the structure of prices in the static equilibrium, given the representative agent's budget E and the measure of available goods N. This will be done separately for the two scenarios that can occur in equilibrium. In the first case, the consumer cannot afford all supplied goods because the non-negativity constraints for low-priority goods become binding. In the second case, the consumer is rich enough to purchase all goods that are supplied on the market. We discuss these two cases in turn. (The conditions under which the two respective regimes occur are studied in Section 4 below.)

When the consumer does not purchase all available goods, the measure of products consumed in positive amounts falls short of the measure of available goods N. If good i is consumed in positive amounts and supplied at the monopoly price, we know from (2) and (3) that the

[8] While this particular structure is an assumption at this stage, we will see below that it will be the equilibrium outcome of the model. Over time new goods are continuously introduced and the chronological sequence of innovations follows the hierarchy of wants. When innovators are protected by patent with finite duration, the 'new 'goods are protected and charged the monopoly price, whereas the patents for 'old' goods have expired and supplied on competitive markets.

consumed quantity equals $c(i) = \frac{1}{2}\left(s - i^{\gamma}\lambda\right)$. The equilibrium demand is decreasing in i which means that the optimal quantity of low-priority goods is smaller. It also means that there is a good, call it n, such that for goods $i > n$ the optimal level of demand is zero and all goods $i < n$ are consumed in positive amounts. It turns out convenient to express the endogenous variables $c(i)$ and $p(i)$ in terms of the endogenous variable n rather than λ. From $c(n) = \frac{1}{2}\left(s - n^{\gamma}\lambda\right) = 0$ it is straightforward to calculate $\lambda = \frac{s}{n^{\gamma}}$. Substituting this into equations (2) and (3) we get the equilibrium composition of demand, and the equilibrium structure of prices

$$c(i) = \begin{cases} s\left[1 - \left(\frac{i}{n}\right)^{\gamma}\right], & i \in [0, aN] \\ \frac{s}{2}\left[1 - \left(\frac{i}{n}\right)^{\gamma}\right], & i \in (aN, n] \\ 0, & i \in (n, N] \end{cases} \tag{4}$$

and

$$p(i) = \begin{cases} 1, & i \in [0, aN] \\ \frac{1}{2}\left[1 + \left(\frac{n}{i}\right)^{\gamma}\right], & i \in (aN, n] \\ 1, & i \in (n, N]. \end{cases} \tag{5}$$

According to equations (4) and (5), what matters for prices and quantities is the *relative position* in the hierarchy of needs, i/n. We also see that the 'steeper' the hierarchy (the higher is γ) the more important is the relative position. The above expressions for $p(i)$ and $c(i)$ are determined for a given measure of consumed goods n. However, n itself is an endogenous variable. To get the optimal value of n we substitute equations (4) and (5) into the budget constraint to get

$$E = \int_0^n p(i)c(i)di = sn\left[\frac{aN}{n} - \frac{1}{4}\left(\frac{3\left(\frac{aN}{n}\right)^{1+\gamma} + 1}{1+\gamma} - \frac{1 - \left(\frac{aN}{n}\right)^{1-\gamma}}{1-\gamma}\right)\right]. \tag{6}$$

This equation implicitely defines the number of consumed goods n as a function of expenditures E, available goods N, and other parameters of the model.[9] In particular, we note that E and N are exogenous from the point of view of the consumer. Moreover, we see from the above

[9] It is straightforward to verify that the right-hand-side of (6) is monotonically increasing in n. When no intersection occurs for $n \leq N$, the equilibrium is characterized by $n = N$. In that case equation (6') below is relevant. Note that when, respectively, $n = N$ and $p = 1$, the right-hand-sides of the equations (6) and (6') become identical.

equation that n is homogenous of degree one in E and N: when E and N increase by some factor, the equilibrium value of n increases by the same factor.

Now consider the alternative scenario that the consumer chooses to consume all available goods in positive amounts. Obviously, this is the case if $c(N) = \frac{1}{2}(s - N^{\gamma}\lambda) > 0$. Also here it is convenient to replace λ. However, we cannot express λ in terms of the optimal bundle of consumed goods n which is trivially determined by the number of available goods N. Instead we express λ in terms of the price of the good that has least priority in consumption, that is by the endogenous variable $p(N) \equiv p$. From (3) it is straightforward to express the marginal utility of income as $\lambda = \frac{s}{N^{\gamma}(2p-1)}$. The same expression (3) can be used to express the monopoly prices for the goods $i \in (aN, N]$ in terms of p as $p(i) = \frac{1}{2}[1 + \left(\frac{N}{i}\right)^{\gamma}(2p-1)]$. The structure of prices and the equilibrium composition of demand can now be expressed as

$$c(i) = \begin{cases} s[1 - \left(\frac{i}{N}\right)^{\gamma}\frac{1}{2p-1}] & i \in [0, aN] \\ \frac{s}{2}[1 - \left(\frac{i}{N}\right)^{\gamma}\frac{1}{2p-1}], & i \in (aN, N] \end{cases} \tag{4'}$$

and

$$p(i) = \begin{cases} 1, & i \in [0, aN] \\ \frac{1}{2}[1 + \left(\frac{N}{i}\right)^{\gamma}(2p-1)], & i \in (aN, N]. \end{cases} \tag{5'}$$

Note that, in equilibrium, a higher p means higher prices for all goods and this goes hand in hand with higher equilibrium consumption for all goods. The reason for this apparently strange result is that the equilibrium depends crucially on the consumer's budget E relative to the measure of supplied goods N. If E is large relative to N, there is high demand for each good which means that monopolists can charge high prices.

The variables $c(i)$ and $p(i)$ are determined by the endogenous variable p, the profit-maximizing price chosen by the monopolist who supplies the good N. Just like before, the equilibrium depends on the consumer's budget E, the measure of available goods N, and other parameters of the model. To see the relationship between p, E, and N, we insert equations (4') and (5') into the consumer's budget constraint

$$E = \int_0^N p(i)c(i)di = sN\left[a - \frac{1}{4}\left(\frac{3a^{1+\gamma}+1}{(1+\gamma)(2p-1)} - (2p-1)\frac{1-a^{1-\gamma}}{1-\gamma}\right)\right] \tag{6'}$$

This expression implicitely defines p as a function of E, N, and other parameters of the model. We observe that p is homogenous of degree zero in E and N: when E and N grow pari passu, p remains unchanged.

3 The Dynamics of the Economy

3.1 Technical Progress and the Resource Constraint

To keep things simple we assume that labor is the only production factor. Production requires a fixed ('innovation' or 'research') input of $\tilde{F}(t)$ units of labor, and a variable labor input of $\tilde{b}(t)$ per unit of output (t denotes a continuous time index). Denoting by $w(t)$ the wage rate, we have innovation costs $w(t)\tilde{F}(t)$ and marginal costs of production $w(t)\tilde{b}(t)$. We assume $\tilde{b}(t) = \frac{b}{A(t)}$ and $\tilde{F}(t) = \frac{F}{A(t)}$, where $A(t)$ is the aggregate knowledge stock, and $F, b > 0$ are exogenous parameters. These assumptions imply that productivity growth, an increase in $A(t)$, is uniform across sectors and also across activities. Assuming uniform productivity growth across *products* makes sure that all heterogeneity comes from the demand side which is the focus of our analysis. Assuming uniform productivity growth across (production and research) *activities* is important for the existence of a constant growth path. Along this path wages grow with productivity so that marginal production costs $w(t)b(t)$ and innovation costs $w(t)F(t)$ are constant over time. In what follows we take marginal cost as the numeraire, hence $w(t)b(t) = 1$ for all t.

In accordance with much of the endogenous growth literature we assume that the aggregate knowledge stock is proxied by the amount of previous innovations activities. These consist of the measure of goods that are actually available on the market, so we have $A(t) = N(t)$ and $\tilde{b}(t) = \frac{b}{N(t)}$ and $\tilde{F}(t) = \frac{F}{N(t)}$.[10]

The labor force is normalized to 1 and, in equilibrium, there is full employment. At date t, $\dot{N}(t)$ new goods are introduced and the necessary employment level to perform the innovation input is $\dot{N}(t)\frac{F}{N(t)}$. The necessary employment level to produce the demanded consumers goods

[10] Note that our assumption on knowledge spillovers differs from the standard 'love-for-variety' model (Grossman and Helpman, 1992). In that model productivity grows only in research but not in production. In the hierarchical model instead there has to be technical progress otherwise innovations comes to a halt because consumers are not willing to reduce consumption on high-priority goods if new goods come along. Hence without technical progress in production, sooner or later the whole labor force will be employed to satisfy the demand of consumers on the already existing goods. Our assumption can be justified using the argument of Young (1993): If the invention of a new good i leads as a by-product to the discovery of a new intermediate input and if the final goods are produced by combining these inputs using a constant returns to scale CES technology, the productivity of the output sector rises linearly in the number of these inputs.

is $\frac{b}{N(t)}\int_0^{n(t)} c(i,t)di$. Thus with full employment of the labor resources we have

$$1 = \frac{\dot{N}(t)}{N(t)}F + \frac{b}{N(t)}\int_0^{n(t)} c(i,t)di. \tag{7}$$

3.2 The Innovation Process

Innovations occur because firms are granted patents and earn profits as long as their market is protected from competitors. The value of an innovation that occurs at date t, $\Pi(t)$, equals the present value of the profit flow that accrues to the innovating firm. This flow starts at the date when consumers begin to purchase this product and ends when patents have expired. We denote the flow profit at date τ of the date-t innovator (the firm which produces good $N(t)$) by $\pi(N(t),\tau) = [p(N(t),\tau) - 1]\ c(N(t),\tau)$.

When consumers purchase all available varieties $n = N$ the date-t innovator earns positive profits right from the start, that is throughout the interval $[t, t+\Delta]$ where the exogenous policy parameter Δ denotes the duration of the patent. When consumers cannot afford all available varieties $n < N$, the innovator has initially no demand. Consumers purchase only the goods with high priority, that is all goods in the interval $[0, n(t))$ and no goods in the interval $[n(t), N(t)]$. In that case, innovators have a waiting time until consumers are willing to purchase their product. Denoting this waiting time by δ, the profit flow $\pi(N(t),\tau)$ is zero at dates $\tau \in [t, t+\delta]$, positive at all dates $\tau \in (t+\delta, t+\Delta]$, and zero for $\tau > t+\Delta$. To see how δ is determined note that, when consumers start to buy good $N(t)$ at date $t+\delta$, $N(t)$ is the good with least priority in the consumption bundle. Hence δ is given by $n(t+\delta) = N(t)$. In the dynamic equilibrium $n(t)$ grows at the constant rate g and we have $n(t+\delta) = n(t)e^{\delta g} = N(t)$.

Innovation costs are constant over time and given by wF. Assuming free access to the research sector, there is entry as long as innovation costs fall short of the value of an innovation. Hence in equilibrium, when all profit opportunities are exploited, we must have $wF \geq \Pi(t)$, with strict equality whenever innovations take place. The zero-profit condition can be stated as

$$wF = \int_{t+\delta}^{t+\Delta} [p(N(t),\tau) - 1]\ c(N(t),\tau)\ e^{-r(\tau-t)}d\tau. \tag{8}$$

3.3 Optimal Savings

The representative consumer maximizes utility over an infinite horizon. Assuming intertemporal separability of lifetime utility we can apply two-stage budgeting. This means we can treat the dynamic problem (optimal allocation of lifetime expenditures across time) separately from the static problem (optimal allocation of a given amount of expenditures across goods at a given date). In Section 2 above we have studied the solution to the static problem. Now we turn to the consumer's dynamic problem. For the solution of this problem the following Lemma is helpful.

Lemma 1 *In the static equilibrium the maximized instantaneous utility at date* t, $\hat{u}(t)$, *can be written as* $\hat{u}(t) = \frac{E(t)^{1-\gamma}}{1-\gamma}\ K\left(\frac{n(t)}{N(t)}, p(t), a(t); s, \gamma\right)$.

Proof *see Appendix.*

Note that the function $K(.)$ in the Lemma depends on the fraction of consumed relative to available goods $n(t)/N(t)$, the innovators entry price $p(t)$, and the fraction of competitive sectors $a(t)$. These variable can, in principle, change over time, which makes the analysis potentially complicated. We are interested in a growth path that satisfies the Kaldor facts, that is on a situation where expenditures and productivity ($E(t)$ and $N(t)$) grow at the same constant rate. In that case we know from equation (6) and (6') that, in the respective regimes, $n(t)/N(t)$ and $p(t)$ are constant over time. In addition, when $N(t)$ grows at the constant rate g, the fraction of competitive markets $a(t)$ equals $e^{-g\Delta}$ which is independent of t.[11]

Two-stage budgeting implies that, along the equilibrium growth path, the consumers' static and dynamic decisions can be conveniently separated. The static choices determine the equilibrium value of the function $K(.)$, taking $E(t)$ as a constant, and the dynamic choice problem is to decide on the time path of $E(t)$, taking the equilibrium value of $K(.) = \bar{K}$ as a constant. The solution to the latter problem is equivalent to maximizing

$$U(t) = \bar{K} \int_t^{\infty} \left(\frac{E(\tau)^{1-\gamma}}{1-\gamma}\right)^{1-\sigma} \frac{e^{-\rho(\tau-t)}}{1-\sigma}\, d\tau$$

[11]To see the relationship between a, g, and Δ, note that, at date $t+\Delta$ all patents granted at t and before are expired, and all patents granted after t are not yet expired. With $N(t)$ markets at date t, there are $N(t+\Delta) = e^{g\Delta}N(t)$ markets at date $t+\Delta$. Hence, at date $t+\Delta$, the measure of competitive markets equals $N(t)$. Obviously, a fraction $a = e^{-g\Delta}$ is competitive and a fraction $1-a = 1-e^{-g\Delta}$ is monopolistic.

subject to the lifetime budget constraint

$$\int_t^\infty E(s)e^{-R(s)}ds \leq \int_t^\infty w(s)e^{-R(s)}ds + V(t)$$

where ρ is the rate of time preference, σ is a parameter that describes the willingness to shift 'utilities' across periods,[12] $R(s) = \int_t^s r(\tau)d\tau$ is the cumulative interest rate, and $V(t)$ denotes the assets that the consumer owns at date t.

The path of expenditures that maximizes the above objective function has to satisfy the Euler equation

$$\frac{\dot{E}(t)}{E(t)} = g = \frac{r(t) - \rho}{\sigma(1-\gamma) + \gamma}. \tag{9}$$

Clearly, when $E(t)$ grows at a constant rate, the interest rate $r(t)$ is also constant. In the symmetric case ($\gamma = 0$) we get the usual form $g = \frac{r-\rho}{\sigma}$. Note that the effect of γ on the growth rate of consumption is ambiguous (remember $\gamma < 1$). A higher γ raises g when $\sigma > 1$ and it decreases g if $\sigma < 1$. The intuition is subtle: With $\gamma > 0$, the expenditures $E(t)$ enter themselves as a concave function in the utility function. The growth rate of consumption depends on how fast marginal utility falls. In the symmetric case marginal utility declines at rate σ. The asymmetry has two effects. On the one hand, the *inter*temporal substitution effect causes marginal utility to fall only at the rate $\sigma(1-\gamma)$; on the other side the *intra*temporal substitution implies that marginal utility falls at rate γ. In total, marginal utility falls at rate $\sigma(1-\gamma) + \gamma$ which is less than σ if $\sigma > 1$ and bigger than σ if $\sigma < 1$.

4 Long-Run Growth and Structural Change

We now describe the general equilibrium of the model. This equilibrium is characterized by the co-existence of continuous structural change and a growth path that satisfies the Kaldor facts. In this Section we define the equilibrium growth path, establish the conditions under which a unique path exists, and discuss the patterns of structural change along this path. Finally, the critical role of the two preference parameteres, the saturation level s and the steepness of the hierarchy γ is studied.

[12] The reason why we take per-period utility to the power of σ is that this allows us to separate the intra- and intertemporal substitution. Alternatively, think of $u(\{c(i,\tau\})$ as a consumption aggregator and of $\frac{u(\{c(i,\tau\})^{1-\sigma}}{1-\sigma}$ as the instantaneous utility function.

4.1 Definition of Equilibrium Growth Path

The equilibrium growth path is characterized by the following conditions: (i) consumers allocate lifetime expenditures optimally across time and goods, (ii) firms set prices that maximize profits, (iii) research firms leave no profit opportunities unexploited, (iv) the labor force is fully employed and (v) aggregate consumption and investment expenditures and the value of aggregate production grow at the same rate.

When consumers do not purchase all available goods, $n < N$, conditions (i) and (ii) are satisfied when, for each date t, equations (9), (4) and (5) hold.[13] Equation (9) implies that consumer allocate expenditure optimally across time. If equations (4) and (5) are satisfied consumers allocate expenditures optimally across goods, given profit maximizing prices of firms; and firms set profit-maximizing prices given the optimal quantities of consumers. Condition (iii) is satisfied when the resource constraint (7) holds, and condition (iv) is satisfied when the zero-profit equation (8) holds. Condition (v) is satisfied because our specification of preferences boils down to a (maximized) felicity function that is CRRA in total consumption expenditures. The critical underlying assumption is that the weighting factor is a power function $\xi(i) = i^{-\gamma}$ and that technologies are symmetric across industries.[14]

The model has a convenient recursive structure and we can reduce the above system of equations to two equations in two unknows: the economy-wide growth rate g and the innovator's waiting time δ. To obtain the first equation substitute equation (4) into the resource constraint (7) and use the definition $g = \frac{\dot{N}(t)}{N(t)}$. Moreover we make use of the fact that in the dynamic equilibrium we have $n(t) = e^{-\delta g}N(t)$ and $aN(t) = N(t)e^{-\Delta g}$. The former relation says that the relation between consumed and available goods is constant and given by $e^{-\delta g}$. The latter relation says that the fraction of competitive markets among all markets is constant

[13]The time index t enters equations (4) and (5) because n depends on t.

[14]In fact, it can be shown (see Foellmi, 1999) that if (i) utility is given by $\int_0^n i^{-\gamma} v(c(i))di$, (ii) the varieties $i \in (0, n)$ have the same production technology and (iii) are either supplied on perfect or monopolistic markets, any utility function $v(c(i))$ that satisfies $v' > 0$ and $v'' < 0$ leads to an equilibrium such that the utility function is CRRA in the consumer's expenditure level with parameter γ. By this we mean that maximized utility is given by $\frac{E^{1-\gamma}}{1-\gamma}K$, where K is a constant determined by exogenous parameters.

and given by $e^{-\Delta g}$. Thus the resource constraint (7) can be rewritten as

$$1 = gF + \frac{bs}{2}e^{-g\delta}\left[e^{-g(\Delta-\delta)} - \frac{e^{-g(1+\gamma)(\Delta-\delta)}+1}{1+\gamma} + 1\right]. \tag{10}$$

The second equation is obtained by substituting equations (9), (4) and (5) into the zero profit condition (8). Here we note that from (4) $c(N(t),\tau) = \frac{s}{2}\left[1 - \left(\frac{N(t)}{N(\tau)}\right)^{\gamma}\right] = \frac{s}{2}\left[1 - e^{-g\gamma(\tau-t)}\right]$ and from (5) $p(N(t),\tau) = \frac{1}{2}\left[1 + \left(\frac{N(\tau)}{N(t)}\right)^{\gamma}\right] = \frac{1}{2}\left[1 + e^{g\gamma(\tau-t)}\right]$. This yields

$$wF = \frac{s}{4}\left(\frac{1-e^{-\phi(\Delta-\delta)}}{\phi} - 2\frac{1-e^{-(\phi+g\gamma)(\Delta-\delta)}}{\phi+g\gamma} + \frac{1-e^{-(\phi+2g\gamma)(\Delta-\delta)}}{\phi+2g\gamma}\right)\cdot e^{-\delta(\phi+g\gamma)} \tag{11}$$

where where we used the definition $\phi = r - g\gamma$ and the fact that from (9) $r = \rho + g(\sigma(1-\gamma)+\gamma)$.

Similarly, when consumers purchase all available goods, $n = N$, conditions (i) and (ii) are satisfied when equations (9), (4') and (5') hold; and conditions (iii) and (iv) are also given by equations (7) and (8). This system of equations can be conveniently reduced to two equation with two unknowns: the growth rate g and the innovators' entry price p. The solution procedure is analogous to before except that now (4') and (5') are relevant. This yields

$$1 = gF + \frac{bs}{2}\left[e^{-g\Delta} - \frac{e^{-g\Delta(1+\gamma)}+1}{(1+\gamma)(2p-1)} + 1\right] \tag{10'}$$

for the resource constraint, and

$$wF = \frac{s}{4}\left(\frac{1-e^{-\phi\Delta}}{\phi}(2p-1) - 2\frac{1-e^{-(\phi+g\gamma)\Delta}}{\phi+g\gamma} + \frac{1-e^{-(\phi+2g\gamma)\Delta}}{\phi+2g\gamma}\frac{1}{2p-1}\right) \tag{11'}$$

for the zero-profit condition.

4.2 A Unique Equilibrium

To examine existence and uniqueness of the dynamic equilibrium we analyze the respective equilibrium conditions graphically. We denote the resource constraint by R and the zero profit condition by Π and draw R and Π in a (δ, g, p)-diagram. This allows us to discuss the above two scenarios simultaneously (Figure 1). In both parts of Figure 1 the vertical axis measures the growth rate g. In the left part of Figure 1 the horizontal axis measures the innovator's waiting time δ (from right to left, starting at $\delta = 0$), and in the right part of Figure 1 the horizontal axis measures the innovators' entry price p (starting at $p = 1$). Observe that $\delta = 0$ and $p = 1$ is the limiting case where the innovator has neither a waiting time nor enough demand to charge a price above marginal cost.

We now discuss the shape of the two curves in turn. To avoid confusion we denote the resource constraint in (g, δ)-space by $\tilde{R}$ and the one in (g, p)-space by R. Similarly, we have Π and $\tilde{\Pi}$ for the zero-profit condition. The R-curve in (g, δ) space is defined by the equation $1 = \tilde{R}(g, \delta)$ and $\tilde{R}(g, \delta)$ is given by the right-hand-side of equation (10). The Π-curve is defined by the equation $wF = \tilde{\Pi}(g, \delta)$ where $\tilde{\Pi}(g, \delta)$ is given by the right-hand-side of equation (11). Similarly, the R- and the Π-curve in (g, p) space are defined by $1 = R(g, p)$ and $wF = \Pi(g, p)$ where $R(g, p)$ and $\Pi(g, p)$ by given by the right-hand-side of equations (10') and (11').

Figure 1

The shape of the Π-curve Consider first the (g, δ) space. When the consumer does not buy all available goods, $n < N$, innovators have a waiting time $\delta > 0$ until they can sell their product. The slope of the $\tilde{\Pi}$-curve is given by $dg/d\delta = -\tilde{\Pi}_\delta / \tilde{\Pi}_g$. A higher δ decreases profits so $\tilde{\Pi}_\delta < 0$ (where $\tilde{\Pi}_x$ denotes the partial derivative of $\tilde{\Pi}$ with respect to x). This simply results from discounting: the longer one has to wait for a given profit flow, the lower is the present value of this flow. This effect is enhanced by the fact that, due to a fixed patent duration Δ, the period during which the innovator earns positive profits does not only start later but also becomes shorter (recall that we measure δ from left to right).

The impact of the growth rate g on the value of an innovation Π, i.e. the of $\tilde{\Pi}_g$ is ambiguous. In a world with homothetic preferences where all goods enter the utility function in a symmetric way, a higher growth rate always lowers the value of an innovation. This is because in equilibrium, a higher growth rate is always associated with a higher interest rate that discounts future revenues more strongly (see the discussion in Romer, 1990). With hierarchic preferences instead, we have a second effect: a higher growth rate raises demand for the most recent innovator's product and leads to faster growth of the innovator's market. This leads to higher future prices and higher future profits which raises the value of an innovation. The size of the latter effect depends crucially on the value of γ, the steepness of the hierarchy: Lemma 2 below shows that the first effect always dominates if γ is low. Instead, if the hierarchy parameter γ is large, the demand effect of higher growth dominates the interest rate effect at low level of g (see Figure 1). It is important to note that a steep hierarchy is a necessary condition for the regime $n < N$ to be possible at all. When innovators have no initial demand, there are innovation incentives only if, after the waiting period δ, demand grows very quickly.

When consumers purchase all available products, $n = N$, innovators have no waiting time $\delta = 0$ and charge an entry price larger than marginal cost $p \geq 1$. The slope of the Π-curve is given by $dg/dp = -\Pi_p/\Pi_g$. How does the value of an innovation depend on p? We know from (5') that a higher entry price p for the most recent innovator's product means higher prices for all other goods in equilibrium. Moreover, from equation (4') a higher p is also associated with larger equilibrium consumption of each variety. Hence each monopolist has larger profits, so we have $\Pi_p > 0$.

The impact of the growth rate g on the value of an innovation is just like before. The demand effect increases, whereas the interest effect decreases the value of an innovation. The demand effect can dominate at low growth rates when the hierarchy is steep enough, whereas the interest effect dominates at high growth rates.

Lemma 2 *a. The zero profit condition crosses the p-axis at* $p_Z = 1 + \left((1 + \frac{bs}{F} \frac{1-e^{-\Delta\rho}}{\rho})^{\frac{1}{2}} - 1 \right)^{-1}$

b. The value of an innovation falls monotonically in the growth rate if $\gamma \leq \frac{\sigma(p_Z - 1)}{1+\sigma(p_Z-1)}$ (flat hierarchy). *In this case, the zero profit constraint is a monotonically increasing curve in the* (g, p)*-space.*

c. For g sufficiently high, $\Pi_g < 0$ *and* $\tilde{\Pi}_g < 0$.

Proof *see Appendix.*

The shape of the R-curve The slope of the resource constraint R can be derived in an analogous way as before by calculating, respectively, $dg/d\delta = -\tilde{R}_\delta/\tilde{R}_g$ and $dg/dp = -R_p/R_g$ for the two regimes. A higher waiting time δ reduces labor demand. The reason is that a higher δ decreases the demand for each product. (To see this use $n = e^{-\delta g}N$ in equation (4)). This means that $\tilde{R}_\delta < 0$. Similarly, a higher entry price p is associated with higher consumption levels for all goods (see equation (4')), and thus with a larger demand for labor in the whole economy. For this reason $R_p > 0$.

A higher growth rate g has an ambiguous effect on the demand for labor resources. On the one hand, there is the direct effect from a larger demand for workers in the research sector. On the other hand, there is an indirect effect which is due to the increase in the size of the monopolistic sector. (Recall from Section 3 above that, with a given patent duration Δ, a fraction $e^{-g\Delta}$ of all goods is supplied by competitive producers and a fraction $1 - e^{-g\Delta}$ by monopolistic firms). The larger the monopolistic sector, the higher the overall price level, and the lower consumption demand. Hence an increase in g leads to a lower demand for production workers. The following Lemma shows that the latter effect may dominate at low g, whereas the former effect always dominates at high g. We therefore have $R_g > 0$ if g is high and vice versa. We summarize this discussion in the following

Lemma 3 *a. The resource constraint crosses the p-axis at* $p_R = \frac{1}{2}\left[1 + \frac{1}{1+\gamma}\frac{bs}{bs-1}\right]$ *if* $1 < bs \leq \frac{1+\gamma}{\gamma}$.

b. If $bs \leq 1$, *the resource constraint is monotonically falling in the* (g,p)*- and* (δ, g)*-space and reaches asymptotically the growth rate* $\hat{g}$ *implicitly defined by* $1 = \hat{g}F + \frac{bs}{2}\left(1 + e^{-\hat{g}\Delta}\right)$.

c. If $bs > \frac{1+\gamma}{\gamma}$, *even at* $g = 0$ *not all products can be produced, the share of products consumed* $x = \frac{n}{N}$ *is then given by the equation* $1 = \frac{bs}{2}\left[1 - x^{-\gamma}\frac{1}{1+\gamma} + x\frac{\gamma}{1+\gamma}\right]$.

d. For g sufficiently high, $R_g > 0$ *and* $\tilde{R}_g > 0$.

Proof *see Appendix.*

Remark 4 *If* $bs \leq 1$, *the* $R-$*curve never hits the* $p-$*axis. All consumers could consume all varieties at the saturation level and there are still resources available for research.*

Having discussed the shapes of the two curves we can consider the general equilibrium of the model. In this equilibrium both the resource constraint and the zero profit condition have to be satisfied which is the case at the point of intersection E in Figure 1.[15] A sufficient condition for uniqueness is $\gamma \leq \frac{\sigma(p_Z-1)}{1+\sigma(p_Z-1)}$ (flat hierarchy) and $bs \leq 1$, since then the two equilibrium curves are monotonically increasing or falling, respectively.

Proposition 1 *a. If the exogenous parameters satisfy* $p_Z < p_R$ *or if* $bs \leq 1$, *there exists a general equilibrium with positive growth rate.*

b. A sufficient condition for a unique general equilibrium is $bs \leq 1$ *and* $\gamma \leq \frac{\sigma(p_Z-1)}{1+\sigma(p_Z-1)}$.

Proof *Part b. see Appendix.*

Corollary *The general equilibrium is consistent with the Kaldor facts: The growth rate, the interest rate, and the labor share are constant.*

[15] Figure 1 suggests that the equilibrium exists if the Π-curve hits the p-axis to the left of R-curve. For most parameter values, simulations show that the equilibrium is unique.

From the Euler equation (9) constant growth implies a constant interest rate. The labor share remains unchanged since wages grow with the same rate as output.

4.3 Structural Change

The equilibrium growth path exhibits *continuous structural change:* At a given date, many different goods exist and each good has a different income elasticity. Declining sectors with a low income elasticity and a falling share of production and employment co-exist with expanding sectors that have a high income elasticity and expanding share of production and employment. Hence there is uneven development and continuous reallocation of labor across sectors of production. In this Section we describe the pattern of structural change in more detail. We will concentrate on the regime $n < N$ and briefly discuss the regime $n = N$ at the end of the section.

To make the changes in the structure of consumption and employment explicit consider the life cycle of product i. How does demand and employment of an innovator increase over time? To answer this question take equation (4) and note that along the long-run growth path we have $n(t) = e^{-\delta g}N(t)$. Given the initial value of N, the growth rate g, and the innovator's waiting time δ, we know the equilibrium value of $n(t)$. From equation (4) the consumption level $c(i,t)$ and the corresponding level of employment $l(i,t) = b\frac{c(i,t)}{N(t)}$ can be calculated.

Figure 2 shows the Engel-curves for good $i = N(t)$. We draw $c(i,t)$ against total output in the production sector $E(t)$. As $E(t)$, $N(t)$, and $n(t)$ grow at the same rate the shape of the Engel-curve can be derived from (4). Demand is initially zero and the non-negativity constraints are still binding. This means at low income levels consumers cannot afford the product. Once a critical income level has been reached consumers start to buy. Increases in income initially lead to a strong expansion of the market, followed by decreasing growth rates and finally stagnating demand in the long term once consumption approaches the saturation level s. We note further that Engel-curves show a discontinuity at the point of time when patents expire. At this date the market opens up for competition, the price falls to marginal cost, and the demand level jumps up.

Figure 2

The following proposition summarizes the patterns of structural change by referring to the income elasticities of demand and employment. The 'gross' income elasticities take account of both the direct income effect on demand and of the indirect effects due to changes in the own price and the prices of all other (monopolistically supplied) products as incomes grow.

Proposition 2 *a) The 'gross' income elasticity of demand for good i is* $\gamma\frac{s-c(i,t)}{c(i,t)}$.

b) The 'gross' income elasticity for employment is $\gamma\frac{s-c(i,t)}{c(i,t)} - 1$.

This proposition holds for both regimes. As $E(t)$, $N(t)$ and $n(t)$ grow at the same rate, we can calculate the income elasticity as $\frac{dc(i,t)}{dn(t)}\frac{n(t)}{c(i,t)}$ or $\frac{dc(i,t)}{dN(t)}\frac{N(t)}{c(i,t)}$. For both regimes, the expressions in the proposition can be derived, respectively, from equations (4) and (4'). Part b) of the proposition obtains because the employment required to produce $c(i,t)$ is $l(i,t) = b\frac{c(i,t)}{N(t)}$, hence $\frac{dl(i,t)}{dN(t)}\frac{N(t)}{l(i,t)} = \frac{dc(i,t)}{dN(t)}\frac{N(t)}{c(i,t)} - 1$.

The above proposition shows that, for a given product, the demand elasticity is initially high and then decreases monotonically towards zero as consumption approaches the saturation level s.[16] *Despite that the model generates constant growth rates of macroeconomic aggregates, goods with high and low income elasticities coexist and continuous structural change takes place.*

We make two further interesting observations. The first refers to the definition of luxury versus necessary goods. In order to determine whether a good is a necessity or a luxury one frequently refers to the income elasticity of a product. Luxury goods are goods with a high income elasticity (higher than unity), whereas necessities are goods with a low income elasticity. The above proposition shows that, whether or not a good is a luxury or a necessity, depends on the level of development. Income elasticities change as the economy gets richer, and a good that has been a luxury good in the initial period of the product cycle becomes a necessity after incomes have sufficiently grown.

The second interesting observation refers to typical patterns of industry demand. Many writers have suggested that a stylized path of industry demand imply an Engel curve that has a logisitic shape (for an explicit treatment see, for instance, Pasinetti, 1981). Initially demand is low and it also expands slowly. In this initial stage industry growth rates increase, reach a maximum and then start to decrease again. For Figue 2 above we see that, for a single product

[16] Also with respect to demand elasticities note the discontinuity at the date when patents expire. At this date the demand level jumps up and thus there is a sudden decrease in the income elasticity.

such a pattern emerges with the exception that the slowly growing initial stage is not present. However, when we consider a *range of products* (an 'industry'), increasing income effects in the early stage of the product cycle are generated because demand increases both at the intensive and at the extensive margin. As incomes grow consumers purchase more of the same products as well as new products. After a critical income level has been reached all products have positive demand, income effects decrease, and once incomes have sufficiently grown demand approaches the saturation level. Panel a) of Figure 3 simulates industry demand and shows that industry Engel-curves have logistic shape.

Figure 3

In panel b) of Figure 3 we show the corresponding development of industry employment. Whether or not employment increases or decreases, depends on whether demand grows faster or slower than productivity. Hence the employment level reflects the outcome of a race between the growth of demand and productivity as the economy gets richer. Initially the growth of demand is larger than the growth of productivity and employment increases over time. However, after incomes have sufficiently grown, the growth of market demand lags behind the growth of productivity. Hence the industry labor share decreases. We also note that the discontinuities in product and labor demand due to expired patents smooth out as we consider a whole range of products rather than a single variety. We summarize our discussion in the following

Proposition 3 *The Engel curve for a range of products has a logistic shape, and the corresponding Engel curve of employment is bell-shaped.*

4.4 The Impact of Hierarchic Preferences

The assumptions which are crucial for the results of this model refer to the preference side. We have already mentioned the importance of the hierarchy function $\xi(i) = i^{-\gamma}$ to generate constant growth rates for macroeconomic aggregates. We now consider the two parameters that characterize the hierarchy of needs, the steepness γ and the saturation level s.

The impact on growth An *increase in the hierarchy parameter* γ has two competing effects on growth (Figure 4). On the one side, a higher γ raises both prices and demanded quantities.

The resulting higher profits tend to increase the incentive to innovate and raise growth. The zero-profit curve Π shifts to the left. On the other side, with a steeper hierarchy more labor is used in production because a higher γ increases the demand for each good. This raises the demand for production labor and leaves less resources for innovation and growth. The resource constraint R shifts to the left. In general, either effect can dominate so we can conclude that the steepness of the hierarchy γ has no systematic effect on growth.

However, a rise in γ clearly implies that the innovator's waiting time δ increases (regime $n < N$) or that the innovators' entry price p falls (regime $n = N$). This should come to no surprise, as with a steeper hierarchy γ, the low-i goods get more weight in the utility function. Thus the utility drawn from consuming many different goods - the love for variety - becomes less pronounced.

Figure 4

The effects of *a higher saturation level* s are similar to those of a higher γ. As s rises, the demand for each good increases (see equations (4) and (4')). This raises the profitability of an innovation and the Π-curve shifts to the left. But obviously the demand change leads also to an increase in the demand for production labor and leaves less resources for research. As a result, the resource constraint R also shifts to the left. Just like an increase in γ, a larger s has no systematic effect on the growth rate g but leads to an increase in the innovator's waiting time δ (when $n < N$) or to lower prices p (when $n = N$).

The analysis above suggests that we can interpret s and γ as parameters for 'variety-aversion'. This becomes clear when we look at the utility function (1). For a given s, the steepness of the hierarchy says how much weight a certain product gets in the objective function and if γ is larger, the most basic goods get disproportionately high importance. For a given γ, the parameter s is a scaling factor, which determines how many units of good i are to be consumed to achieve a certain utility level. As marginal utility is falling, a large s is equivalent to a slowly falling marginal utility. A consumer does not want to consume a lot of a given product does not want to consume many different goods, in this sense a low s reinforces a given variety-aversion as measured by γ.

The impact on patterns of structural change The preference parameters determine the extent of structural change both directly and indirectly via the growth rate. Below, we will focus on the direct effect and ask how do the hierarchy parameter γ and the saturation level s affect the patterns of structual change, *given the equilibrium values of g and δ (or p).* Of course, the growth rate itself is crucial for structural change. If growth is higher, expenditures rise faster, hence we see from Figure 4 that the velocity of structural change is increased. Without growth no structural change takes place at all.

If the hierarchy is steep, γ is high, the gross income elasticity implies that demand increases strongly and approaches the saturation level quickly. In this sense, we have a lot of structural change. Intuitively, a steeper hierarchy implies a shift of demand away from the most recent (and least priority) goods to necessities. To see this more clearly, consider the other extreme, when no hierarchy exists and $\gamma = 0$. Here structural change is reduced to a minimum: with symmetry across products, the demand for an innovator jumps to its steady-state level at the period when the product is introduced and stays at this level forever. (The 'gross' income elasticity of demand is equal to zero). The whole increase in income takes place at the extensive margin: an increase in consumption means purchasing new goods whereas the consumption level of the old goods is not affected. Reallocation of labor takes the form of a proportional reduction of labor in the existing firms which are employed in the new firms. When $\gamma > 0$ the consumption level of all sectors is affected and the reallocation of labor affects sectors differently. Sectors with a 'gross' income elasticity larger than unity attract workers from sectors with an elasticity lower than unity. Additional income is to a smaller extent directed towards new goods.

The saturation level s only scales up demand but has otherwise no effect on the patterns of structural change. Inserting (4) into $\gamma\frac{s-c(i,t)}{c(i,t)}$, we directly see that s does not affect the gross elasticity of demand.

5 Multiple Equilibria

If the parameter values satisfy $p_Z \geq p_R$ a *stagnation* equilibrium or *multiple* equilibria may arise (see Lemmas 2 and 3 for the definitions of p_Z and p_R). In that case, the Π-curve cuts the p-axis to the right of the R-curve.

Stagnation In a stagnation equilibrium the value of an innovation is (equal or) *smaller* than the costs of an innovation, which implies that no research will be undertaken. Not surprisingly, this outcome is likely if research costs F are high. Also in the stagnatory state, the full employment condition has to be satisfied, hence the equilibrium point lies *on* the R-curve and is located where the R-curve intersects the horizontal axis (at $g = 0$). If $n = N$, the R-curve hits the horizontal axis in the right part of Figure 1. This occurs at $p_R > 1$. When this inequality is violated the regime $n = N$ is not feasible and we are in the regime $n < N$. As the economy does not grow, the waiting time δ is not a meaningful endogenous variable because δ will necessarily be infinite. As stated in Lemma 3c, the resource constraint has to be solved for $x = n/N$, the share of available products that is actually consumed. In such an equilibrium there are firms that know how to produce the goods $i \in (n, N]$, but no production ever takes place since demand given the (constant) income level is too small.

Multiple equilibria If $p_Z \geq p_R$ and if the two curves cross the model exhibits multiple equilibria. We then have three equilibria: the stagnation point and the two points of intersection of the Π- and the R-curve. There are two potential sources of multiplicity: the first is due to *finite patent length*; the second is due to *a hierachic structure of preferences*. To identify the critical assumptions we compare the behavior of an economy where consumers have symmetric preferences ($\gamma = 0$) to the case when preferences are hierarchic ($\gamma > 0$).

Figure 5

With symmetric preferences ($\gamma = 0$) each good faces the same demand, hence all monopolistic prices are equal to $p > 1$. A situation where $p = 1$ and $\delta > 0$ cannot arise in an equilibrium with positive growth since a new good is *immediately* purchased in the same amounts as all other goods supplied by the monopolists. The zero profit condition and the resource constraint, respectively, read

$$\frac{F}{bs} = \frac{1 - e^{-\phi\Delta}}{\phi} \frac{(p-1)^2}{2p-1}, \quad \text{and} \quad 1 = gF + bs\left(1 + e^{-g\Delta}\right) \frac{p-1}{2p-1}.$$

The slope of the zero profit condition is positive because a demand externality does not arise: higher economy-wide growth has no impact on the market demand for previous innovators. Instead demand jumps from zero to a positive level and stays there until the patent has expired.[17] Hence there is always a positive association between the entry price p and the growth rate g.

[17]Thereafter demand makes a further jump due to the fall in prices that occurs as soon as the competitive producers take over the market; needless to say, this demand increase is irrelevant for the innovations incentives

The resource constraint, however, still has an ambiguous slope. A higher growth g not only raises the demand for labor in research but it also decreases the demand for production labor. The larger fraction of monopolistic markets implies high prices on more markets leading to lower aggregate consumption demand. High growth can be sustained due to lower equilibrium employment in production and vice versa. When patent length is *infinite*, this complementarity vanishes. In that case changes in the growth rate do not affect market structure because all markets are monopolized. This point has been made by Laussel and Nyssen (1999) who showed that multiple equilibria can arise in a standard endogenous growth model when patent length is finite.

With hierarchic preferences ($\gamma > 0$) the situation is different. Multiple equilibria can arise even when there are infinitely lived patents because the Π-curve is not necessarily monotonic. With a steep hierarchy (high γ) the Π-curve is backward bending at low levels of g (see Lemma 2b). The reason is a demand externality: when preferences have a hierarchic structure the demand of a previous innovator depends on the economy-wide growth rate. If innovators expect high growth they expect that the demand for their products expands more quickly so that future prices, quantities, and profits are larger. So higher economy-wide growth stimulates the incentive to innovate. If innovators expect low growth, profit expectations and the resulting incentives to innovate are correspondingly low. Hence low growth rates are sustained by pessimistic expectations and vice versa. Obviously, this *demand externality* is at work independently of the particular length of a patent; in particular it holds even when protection is forever.[18]

It is worth noting that the intercept of the resource constraint with the horizontal axis, p_R, shifts to the left with an increase in γ. According to Proposition 1, this implies that multiple equilibria become more likely. We summarize this discussion in the following

Proposition 4 *A hierarchic structure of preferences ($\gamma > 0$) may lead to multiple equilibria even when patent length is infinite.*

[18] A similar mechanism is present in the model of Zweimüller (2000). This paper studies the impact of inequality on the aggregate innovation rate. In that model, consumers buy one unit of each product and a demand effect of higher growth rates arises because the waiting time of the innovator for the demand of the poor becomes shorter. The assumptions in that model are more restrictive than in the present model: consumers buy only one unit of each good, and the mark-up of an innovator is assumed to be exogenous.

6 An Application to Optimal Patent Duration

Patent policy always faces a tradeoff. On the one hand, patents create mark-ups and these mark-ups distort relative prices. On the other hand, patents stimulate R&D by allowing the successful firms to earn profits which may imply a dynamic efficiency gain. In a symmetric world this tradeoff is trivial as there are no relative price distortions when patent length is infinite (see also the discussion in O'Donoghue and Zweimüller, 1998). As long as innovators are not displaced by future innovators (as in models of expanding product variety) it is optimal to set the patent length to infinity. With hierarchic preferences instead such price distortions exist as the mark-ups of innovators increase over time.

In such a set-up it is interesting to study the question whether a higher utility level can be achieved by finite patents. Proposition 5 says that social welfare can always be increased by moving from infinite to finite patent duration.

Proposition 5 *Welfare is maximized at a finite patent length.*

Proof *see Appendix.*

The result in Proposition 5 does *not* depend on the rate of time preference. Intuitively, the static inefficiency of the price distortions is always too strong to make an infinite patent length socially optimal. To illustrate this result graphically, we have plotted the value of intertemporal utility for different hierarchy levels in dependency of the inverse patent duration $1/\Delta$, so that a value of zero for this variable corresponds to infinite patent length (Figure 6). We see that social welfare increases at $1/\Delta = 0$ but the dynamic efficiency loss as a result of lower R&D ultimately dominates the static efficiecy gain from the reduction of price distortions of shorter patents.

Furthermore, it is interesting to note that the optimal patent length becomes shorter as the hierarchy gets steeper. Intuitively, a steeper hierarchy implies stronger price distortions and bigger static inefficiency. For the parameter values chosen in Figure 6, the optimal patent length is about 18 years when $\gamma = 0.7$.

Figure 6

7 Conclusions and Extensions

We have presented a model that captures two of the most important features of the long-run growth process: the dramatic changes in the structure of production and employment; and the Kaldor facts of economic growth. Our model has focused on the demand-explanation of structural change according to which the dramatic reallocation of labor is driven by differences in income elasticities across sectors. The basic idea of our analysis is that household expand their consumption along a hierarchy of needs. If the 'hierarchy function' that characterizes the willingsness of consumers to move from goods with high priority to goods with lower priority takes a particular form, the equilibrium process of growth and structural change is consistent with the Kaldor facts.

Innovations play a crucial role in our model. Innovations drive productivity growth and this leads to interesting interactions between sectoral and aggregate dynamics: Economy-wide growth prospects are of central importance for the emergence of new industries; and the industrial R&D that leads to these new industries is central for improvements in productivity. These complementarities open up the possibility for multiple equilibria. Hence our model is not only capable of yielding insights into the process of growth and structural change, but sheds also light on the question why some countries experience high long-term growth and many industries take off, while in other countries we see neither a change in the production structure nor increases in aggregate productivity.

The way we have discussed the interactions between structural change and economic growth depends on several assumptions and suggests interesting extensions. We want to mention four points. First, our discussion of the model was based on a particular endogenous growth mechanism. However, our main results do not depend on a specific mechanism that drives aggregate productivity. For instance, an exogenous growth mechanism would reproduce similar patterns of structural change as presented in this paper,[19] as would a semi-endogenous growth model in the spirit of Jones (1995) that does not exhibit the scale effects that characterizes the present model. Instead, removing the scale effect by introducing a quality dimension (as in the models survey by Jones, 1999) would add a qualitatively new feature to our model. Structural change could also take place within industries as better goods would replace old goods and

[19] Of course, exogenous growth would only allow for a one-way causality from aggregate to sectoral dynamics without any further feedback mechanism.

incumbent firms may either change their own structure of production or may be displaced by new firms supplying better qualities at lower prices.

Second, our results are based on the assumption that the baseline utility function is quadratic. We have used this specific formulation because it illustrates the idea that consumers get saturated with goods of high priority and move on to goods with lower priority. However, it can be shown (Foellmi, 1999) that this specific formulation of the baseline utility is not crucial. In fact, to reconcile structural changes with the Kaldor facts, any baseline utility function satisfying the usual assumptions works. What is important, however, is that hierarchy function takes a particular form; and that supply conditions are symmetric across sectors (or keep the same relative structure).

A third point concerns the obvious extension of the model to study the role of income *inequality*. Since hierarchic preferences are non-homothetic, rich and poor households will consume different consumption bundles. This opens up a new channel by which income inequality could affect innovation and growth. In that case the pricing decisions of firms with market power depend on the income distribution and these decisions determine whether or not certain groups are excluded from the consumption of certain products (Foellmi and Zweimüller, 2002).

Finally, hierarchic preferences in a world economy with rich and poor countries would imply interesting patterns of *international trade* and growth. First, it is a natural way of modelling the Linder-hypothesis (Linder, 1961) and/or the product-cycle hypothesis (Vernon, 1979). A rich country faces high home-demand and hence will innovate early. The poor country will first import new goods, but later on start to imitate. Hence rich countries will produce *new* goods with a high income elasticity and poor countries will produce *old* goods with a low elasticity. Second, our set-up is also useful to shed light on the Prebisch/Singer-hypothesis (Prebisch, 1950, Singer, 1950) according to which the terms of trade for the poor countries deteriorate as their exports are concentrated on goods with low income elasticities.

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Appendix

Proof of Lemma 1 Consider first the regime $n < N$. To get the maximized utility $\hat{u}$ for a given level of expenditures E and a given menu of goods N, we insert equilibrium quantities (4) into the utility function (1) After some manipulations this yields

$$\hat{u} = \frac{n^{1-\gamma}}{1-\gamma}\frac{s^2}{8}\left(3 + \left(a\frac{N}{n}\right)^{1-\gamma} + 2\left(1-\gamma\right)\frac{aN-n}{n} - \left(1-\gamma\right)\frac{1+3\left(a\frac{N}{n}\right)^{1+\gamma}}{\left(1+\gamma\right)}\right)$$

which we can write as $\hat{u} = \frac{n^{1-\gamma}}{1-\gamma}\psi(\frac{n}{N}, a; \gamma, s)$. We know from (6), that we can write $E = sn\phi(\frac{n}{N}, a; \gamma, s)$ which implies $n = \frac{E}{s\phi(\frac{n}{N},a;\gamma,s)}$. Inserting this latter expression into the above utility function yields

$$\hat{u} = \frac{E^{1-\gamma}}{1-\gamma}K\left(\frac{n}{N}, p = 1, a; s, \gamma\right)$$

where $K\left(\frac{n}{N}, p = 1, a; s, \gamma\right) = \frac{\psi(\frac{n}{N},a;\gamma,s)}{\left[s\phi(\frac{n}{N},a;\gamma,s)\right]^{1-\gamma}}$.

We proceed in a similar way for regime $n = N$. This yields for maximized utility

$$\hat{u} = \frac{N^{1-\gamma}}{1-\gamma}\frac{s^2}{8}\left(3 + a^{1-\gamma} + \frac{2\left(1-\gamma\right)\left(a-1\right)}{2p-1} - \frac{\left(1-\gamma\right)\left(1+3a^{1+\gamma}\right)}{\left(1+\gamma\right)\left(2p-1\right)^2}\right),$$

which can be written as $\hat{u} = \frac{N^{1-\gamma}}{1-\gamma}\Psi(p, a; \gamma, s)$. We know from equation (6') that we can write $E = sN\Phi(p, a; \gamma, s)$. Solving for N and substituting into the above utility expression yields

$$\hat{u} = \frac{E^{1-\gamma}}{1-\gamma}K\left(\frac{n}{N} = 1, p, a; s, \gamma\right),$$

where $K\left(\frac{n}{N} = 1, p, a; s, \gamma\right) = \frac{\Psi(p,a;\gamma,s)}{[s\Phi(p,a;\gamma,s)]^{1-\gamma}}$.

Proof of Lemma 2 We know from the zero-profit condition (11') that

$$\begin{aligned}
\Pi_g &\equiv \frac{\partial \Pi(t)}{\partial g} = \int_t^{t+\Delta} \frac{s}{4} \begin{bmatrix} -\sigma(1-\gamma)\,(\tau-t)\,e^{-[\rho+g\sigma(1-\gamma)](\tau-t)}\,(2p-1) \\ -\left(\sigma(1-\gamma)+2\gamma\right)(\tau-t)\,e^{-[\rho+g\sigma(1-\gamma)+2g\gamma](\tau-t)}\frac{1}{2p-1} \\ +2\left(\sigma(1-\gamma)+\gamma\right)(\tau-t)\,e^{-[\rho+g\sigma(1-\gamma)+g\gamma](\tau-t)} \end{bmatrix} d\tau \\
&= \int_t^{t+\Delta} \frac{s}{4} \begin{bmatrix} -\sigma(1-\gamma)\,(2p-1) \\ -\left(\sigma(1-\gamma)+2\gamma\right)e^{-2g\gamma(\tau-t)}\frac{1}{2p-1} \\ +2\left(\sigma(1-\gamma)+\gamma\right)e^{-g\gamma(\tau-t)} \end{bmatrix} (\tau-t)\,e^{-[\rho+g\sigma(1-\gamma)](\tau-t)} d\tau \\
&=: \int_t^{t+\Delta} \frac{s}{4}\Theta(p,g)\,(\tau-t)\,e^{-[\rho+g\sigma(1-\gamma)](\tau-t)} d\tau
\end{aligned}$$

We will show the following: If $\left.\frac{\partial \Pi(t)}{\partial g}\right|_{g=0} \leq 0$ then $\frac{\partial \Pi(t)}{\partial g} < 0$ for all $g > 0$. Hence, $\left.\frac{\partial \Pi(t)}{\partial g}\right|_{g=0} \leq 0$ is sufficient for Π to have always a positive slope. $\left.\frac{\partial \Pi(t)}{\partial g}\right|_{g=0}$ can be calculated as

$$\frac{s}{4}\left[-\sigma(1-\gamma)\,(2p-1) - \left(\sigma(1-\gamma)+2\gamma\right)\frac{1}{2p-1} + 2\left(\sigma(1-\gamma)+\gamma\right)\right]\int_t^{t+\Delta} (\tau-t)\,e^{-\rho(\tau-t)} d\tau \leq 0$$

$$\begin{aligned}
&\iff \Theta(p,0) = -\sigma(1-\gamma)\,(2p-1) - \left(\sigma(1-\gamma)+2\gamma\right)\frac{1}{2p-1} + 2\left(\sigma(1-\gamma)+\gamma\right) \leq 0 \\
&\iff \gamma \leq \frac{\sigma(p-1)}{1+\sigma(p-1)} \iff p \geq \frac{\sigma(1-\gamma)+\gamma}{\sigma(1-\gamma)} \iff 2p-1 \geq \frac{\sigma(1-\gamma)+2\gamma}{\sigma(1-\gamma)}.
\end{aligned}$$

Note that p depends on g. If $\left.\frac{\partial \Pi(t)}{\partial g}\right|_{g=0} \leq 0$, the zero profit constraint has a positive slope, i.e. we must have $p'(g) \geq 0$. Together with the lower bounds of p above we can determine the sign of the partial derivatives of $\Theta(p,g)$

$$\begin{aligned}
\Theta_p &= -2\sigma(1-\gamma) + \frac{2}{(2p-1)^2}\left(\sigma(1-\gamma)+2\gamma\right)e^{-2g\gamma(\tau-t)} \\
&\leq -2\sigma(1-\gamma) + \frac{2\sigma^2(1-\gamma)^2}{\left(\sigma(1-\gamma)+2\gamma\right)^2}e^{-2g\gamma(\tau-t)} < 0 \\
\Theta_g &= -2\left(\sigma(1-\gamma)+\gamma\right)\gamma\,(\tau-t)\,e^{-g\gamma(\tau-t)} + 2\left(\sigma(1-\gamma)+2\gamma\right)\gamma\,(\tau-t)\,e^{-2g\gamma(\tau-t)}\frac{1}{2p-1} \\
&\leq -2\left(\sigma(1-\gamma)+\gamma\right)\gamma\,(\tau-t)\,e^{-g\gamma(\tau-t)} + 2\sigma(1-\gamma)\gamma\,(\tau-t)\,e^{-2g\gamma(\tau-t)} < 0
\end{aligned}$$

Hence $\Theta(p(g),g) < \Theta(p(0),g) < \Theta(p(0),0) \leq 0$ which completes the proof.

b. Insert $g = 0$ into the zero profit condition (11') (since demand does not grow when $g = 0$, an equilibrium with positive waiting time δ is impossible)

$$\begin{aligned} \frac{F}{b} &= \frac{s}{4}\frac{1-e^{-\rho\Delta}}{\rho}\left(2p_z - 1 + \frac{1}{2p_z - 1} - 2\right) \\ 4\frac{F}{bs}\frac{\rho}{1-e^{-\rho\Delta}} &= \left(\sqrt{2p_z - 1} - \frac{1}{\sqrt{2p_z - 1}}\right)^2 \\ 2\sqrt{\frac{F}{bs}\frac{\rho}{1-e^{-\rho\Delta}}} &= \sqrt{2p_z - 1} - \frac{1}{\sqrt{2p_z - 1}} \end{aligned}$$

Note that only the positive root is relevant since $2p_z - 1 > 1$. We denote $\sqrt{2p_z - 1} = x$, the equation above can then be written as $x^2 - 2\sqrt{\frac{F}{bs}\frac{\rho}{1-e^{-\rho\Delta}}}x - 1 = 0$. This quadratic equation has the solution $x = \sqrt{2p_z - 1} = \sqrt{\frac{F}{bs}\frac{\rho}{1-e^{-\rho\Delta}}} + \sqrt{\frac{F}{bs}\frac{\rho}{1-e^{-\rho\Delta}} + 1}$. Solving for p_z yields the claim of the Lemma.

c.

$$\lim_{g->\infty} \Pi_g = \int_t^{t+\Delta} \frac{s}{4}\Theta(p,g)\,(\tau - t)\,e^{-[\rho+g\sigma(1-\gamma)](\tau-t)}d\tau < 0$$

and

$$\lim_{g->\infty} \tilde{\Pi}_g = e^{-\delta r}\int_t^{t+\Delta-\delta} \frac{s}{4}\Theta(1,g)\,(\tau - t)\,e^{-r(\tau-t)}d\tau - \delta\sigma(1-\gamma)\Pi(t) < 0$$

since

$$\lim_{g->\infty} \Theta(p,g) = -\sigma(1-\gamma)\,(2p-1) < 0.$$

Proof of Lemma 3 **a.** Inserting $g = 0$ into the resource constraint (10') yields

$$1 = bs\left[1 - \frac{1}{(1+\gamma)\,(2p-1)}\right] =: f(p)$$

$1 \leq p < \infty$ implies $bs\frac{\gamma}{1+\gamma} \leq f(p) < bs$. Hence, the equation can only be fulfilled if $1 < bs \leq \frac{1+\gamma}{\gamma}$. Solving the equation yields $p_R = \frac{1}{2}\left[1 + \frac{1}{1+\gamma}\frac{bs}{bs-1}\right]$.

b. If p goes to infinity in equation (10'), the condition follows directly.

c. Replace δ by the new variable $x = \frac{n}{N} = e^{-\delta g}$. Equation (10) now reads:

$$1 = gF + \frac{bs}{2}\left[1 - \frac{e^{-g(1+\gamma)\Delta}}{1+\gamma}x^{-\gamma} + \frac{\gamma}{1+\gamma}x\right].$$

Inserting $g = 0$ yields the condition in the Lemma.

d. The derivatives with respect to g are, respectively, $R_g = F - \frac{bs}{2}\Delta\left[e^{-g\Delta} - \frac{1}{2p-1}e^{-g\Delta(1+\gamma)}\right]$ and $\tilde{R}_g = F - \frac{bs}{2}(\Delta-\delta)e^{-g\delta}\left[e^{-g(\Delta-\delta)} - e^{-g(\Delta-\delta)(1+\gamma)}\right] - \delta\frac{bs}{2}e^{-g\delta}\left[e^{-g(\Delta-\delta)} - \frac{e^{-g(1+\gamma)(\Delta-\delta)}+1}{1+\gamma} + 1\right]$. Hence, $\lim_{g->\infty} R_g = \lim_{g->\infty} \tilde{R}_g = F > 0$.

Proof of Proposition 1 With flat hierarchy, we showed that the Π-curve is monotonically increasing, and so an equilibrium is only possible with $p > 1$. It remains to show that the R-curve monotonically decreasing if $bs \leq 1$, which would guarantee uniqueness. Lemma 2b. says that R approaches $\hat{g}$, defined by $1 = \hat{g}F + \frac{bs}{2}\left(1 + e^{-\hat{g}\Delta}\right)$. The right hand side of the latter expression is a convex function in g and must therefore have a positive slope at $g = \hat{g}$. Thus, $F - \frac{bs}{2}\Delta e^{-\hat{g}\Delta} > 0$. But this implies $R_g = F - \frac{bs}{2}\Delta e^{-g\Delta} + \frac{bs}{2}\Delta\frac{1}{2p-1}e^{-g\Delta(1+\gamma)} > \frac{bs}{2}\Delta\frac{1}{2p-1}e^{-g\Delta(1+\gamma)} > 0$.

Sketch of Proof of Proposition 5 (Full proof available upon request). Consider first regime $n = N$ .The longer the patent protects the higher are profits, $\Pi_\Delta > 0$, thus, Π shifts to the right. A higher Δ is associated with a larger share of the monopolistic sector, which reduces the total amount produced, therefore we see that $R_\Delta < 0$, and R shifts to the left. Applying Cramer's Rule we get: $\frac{dg}{d\Delta} = \frac{\Pi_p R_\Delta - \Pi_\Delta R_p}{\Pi_g R_p - \Pi_p R_g}$ and $\frac{dp}{d\Delta} = \frac{\Pi_g R_\Delta - \Pi_\Delta R_g}{\Pi_g R_p - \Pi_p R_g}$. In a unique equilibrium R has a bigger slope - viewed from the g-axis - than Π at the equilibrium, what implies that the denominator $\Pi_g R_p - \Pi_p R_g$ is negative. This implies that $\frac{dg}{d\Delta}$ is positive: Longer patents increase growth. The sign of $\frac{dp}{d\Delta}$ is ambigous. Intuitively, as Δ increases a lower p satisfies the zero-profit condition, but a higher Δ increases also the share of the monopolistic sector and this implies a higher p to fulfill the resource constraint.

Now calculate the welfare of the representative agent depending on Δ at some date t. $N(t)$ is the inherited number of known designs, $n(t) = e^{-\delta g} N(t)$ is the number of consumed goods. Inserting the value of the instantaneous utility $\hat{u}$ into the intertemporal utility function $U(t)$ (see, respectively Appendix 1 and section 3.3) and evaluating yields

$$U(t) = \frac{K(e^{-(\Delta-\delta)g}, p; \gamma)^{1-\sigma}}{1-\sigma} \left(\frac{\left(N(t)e^{-\delta g}\right)^{1-\gamma}}{1-\gamma} \right)^{1-\sigma} \frac{1}{\rho - g(1-\gamma)(1-\sigma)}$$

Note that $\Delta = \infty$ is associated with $a = 0$ when $g > 0$. To proof that the optimal Δ is finite we take the derivative of the above intertemporal utility function with respect to a holding g constant and then we evaluate the derivative at $a = 0$.

With $p > 1$, δ is equal to zero. We get

$$\frac{dU}{d\Delta} = \Gamma \left(\frac{\partial K}{\partial \Delta} + \frac{\partial K}{\partial p}\frac{dp}{d\Delta} + \frac{\partial K}{\partial g}\frac{dg}{d\Delta} + \frac{(1-\gamma)K}{\rho - g(1-\gamma)(1-\sigma)}\frac{dg}{d\Delta} \right)$$

where Γ is given by $\Gamma = K^{-\sigma} \left(\frac{\left(N(t)e^{-\delta g}\right)^{1-\gamma}}{1-\gamma} \right)^{1-\sigma} \frac{1}{\rho - g(1-\gamma)(1-\sigma)}$. Note that Γ is well defined for every a. The intuition of the derivative $\frac{dU}{d\Delta}$ is: $\frac{\partial K}{\partial \Delta} + \frac{\partial K}{\partial p}\frac{\partial p}{\partial \Delta}$ reflects the static efficiency losses due to an increasing Δ whereas $\frac{\partial K}{\partial g}\frac{\partial g}{\partial \Delta} + \frac{(1-\gamma)K}{\rho - g(1-\gamma)(1-\sigma)}\frac{\partial g}{\partial \Delta}$ measures the corresponding dynamic efficiency gains.Recognize that the sign of $\frac{\partial U}{\partial a} = \frac{dU}{d\Delta}\frac{d\Delta}{da}$ is only determined by the coefficients of the a-terms with the lowest exponent: $a^{-\gamma}$which only arises in the$\frac{\partial K}{\partial \Delta}\frac{d\Delta}{da}$-term.[1] After some manipulations one gets

$$\frac{1}{\Gamma}\frac{\partial U}{\partial a} \approx \frac{s^2}{8}(1-\gamma)a^{-\gamma} > 0$$

The similar reasoning applies to the regime $n < N$. In that case the final expression of the welfare derivative is

$$\frac{1}{\Gamma}\frac{\partial U}{\partial a} \approx \frac{s^2}{8}(1-\gamma)e^{\delta g(1-\gamma)}a^{-\gamma} > 0.$$

[1]Consider the following expression $\lim_{a \longrightarrow 0} Ca^\alpha + Da^{\alpha+\beta} = \lim_{a \longrightarrow 0} a^\alpha \left(C + Da^\beta \right)$ where $\beta > 0$. For a sufficiently small, Da^β is smaller than C in absolute terms. Hence, the sign of the limes is determined by the sign of C.

Figure 1: The equilibrium values of the growth rate g and waiting time δ

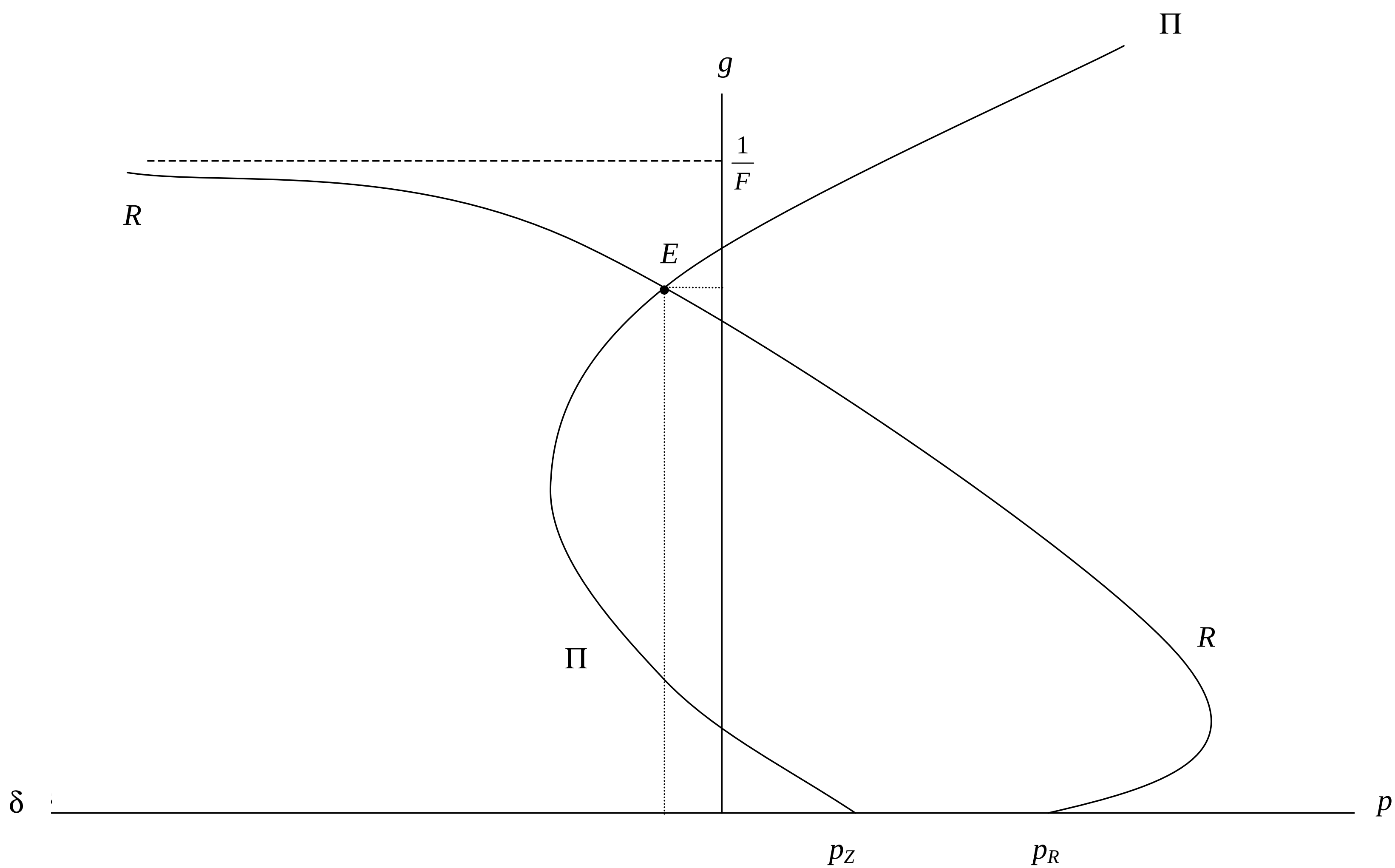


Figure 2: The Engel-curve for good $i = N(t)$

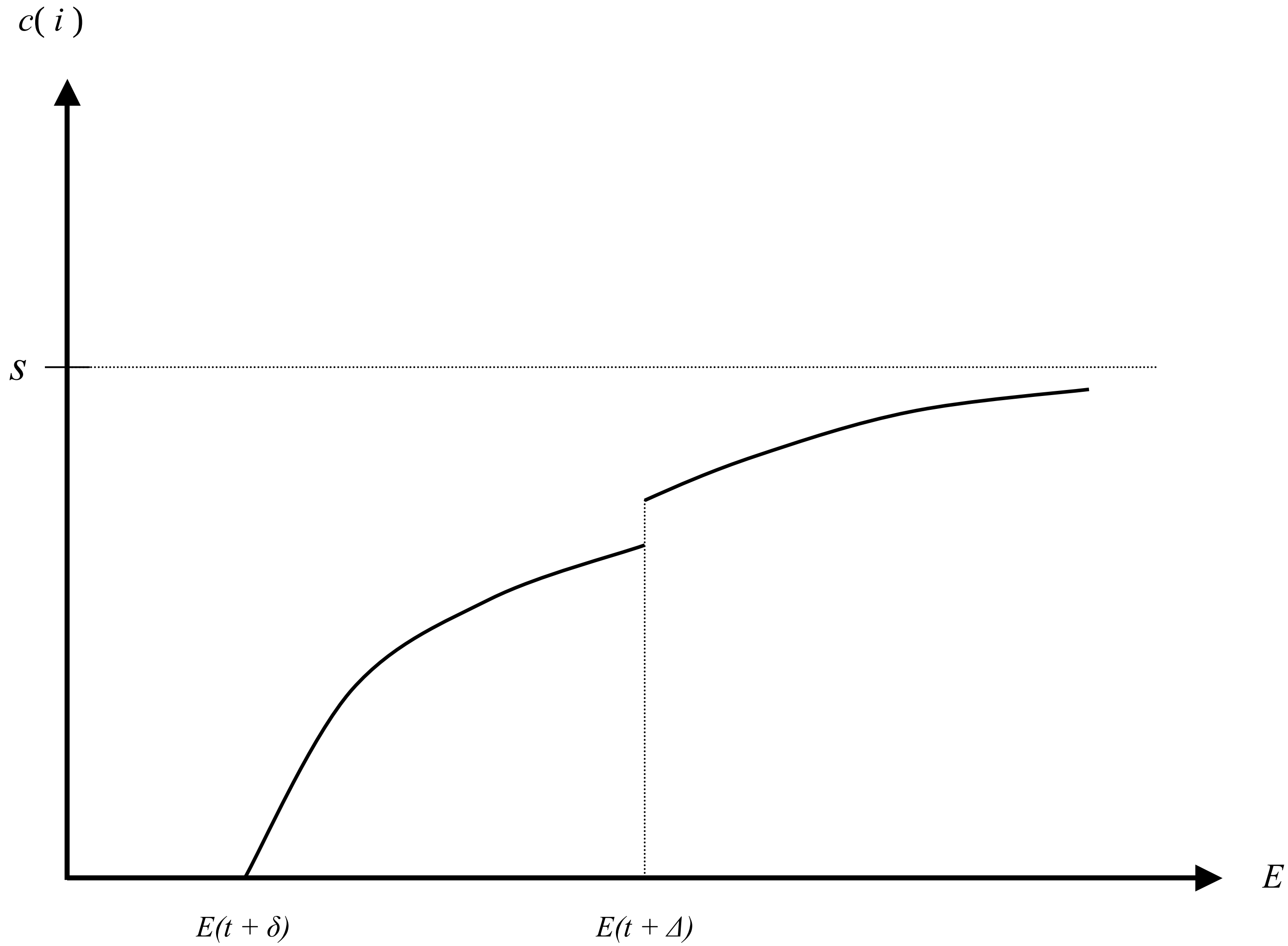


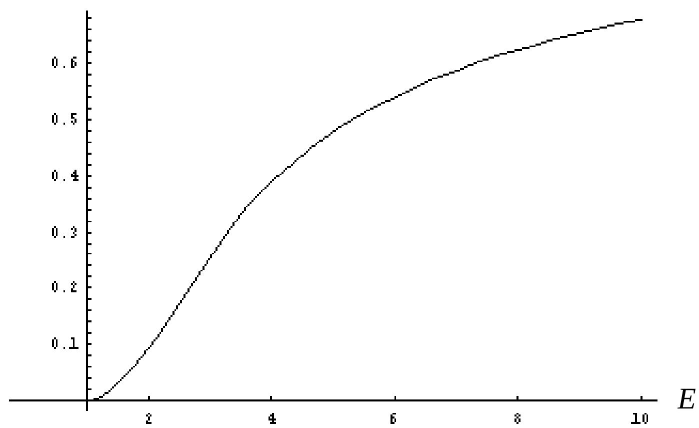
Figure 3: Output and Employment Share of an Industry

Industry Range: [n, kn]

Parameter Values:
$n = s = 1$, $k = 3$, $\gamma = 0.7$, $\delta = 0$, $\Delta = \infty$

Expenditures E, when good $i = n$ starts production, are normalized to 1.

Output



Employment Share

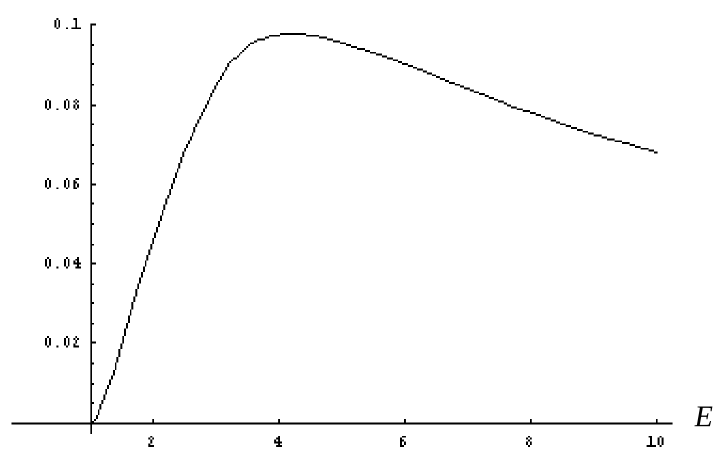


Figure 4: A rise in γ

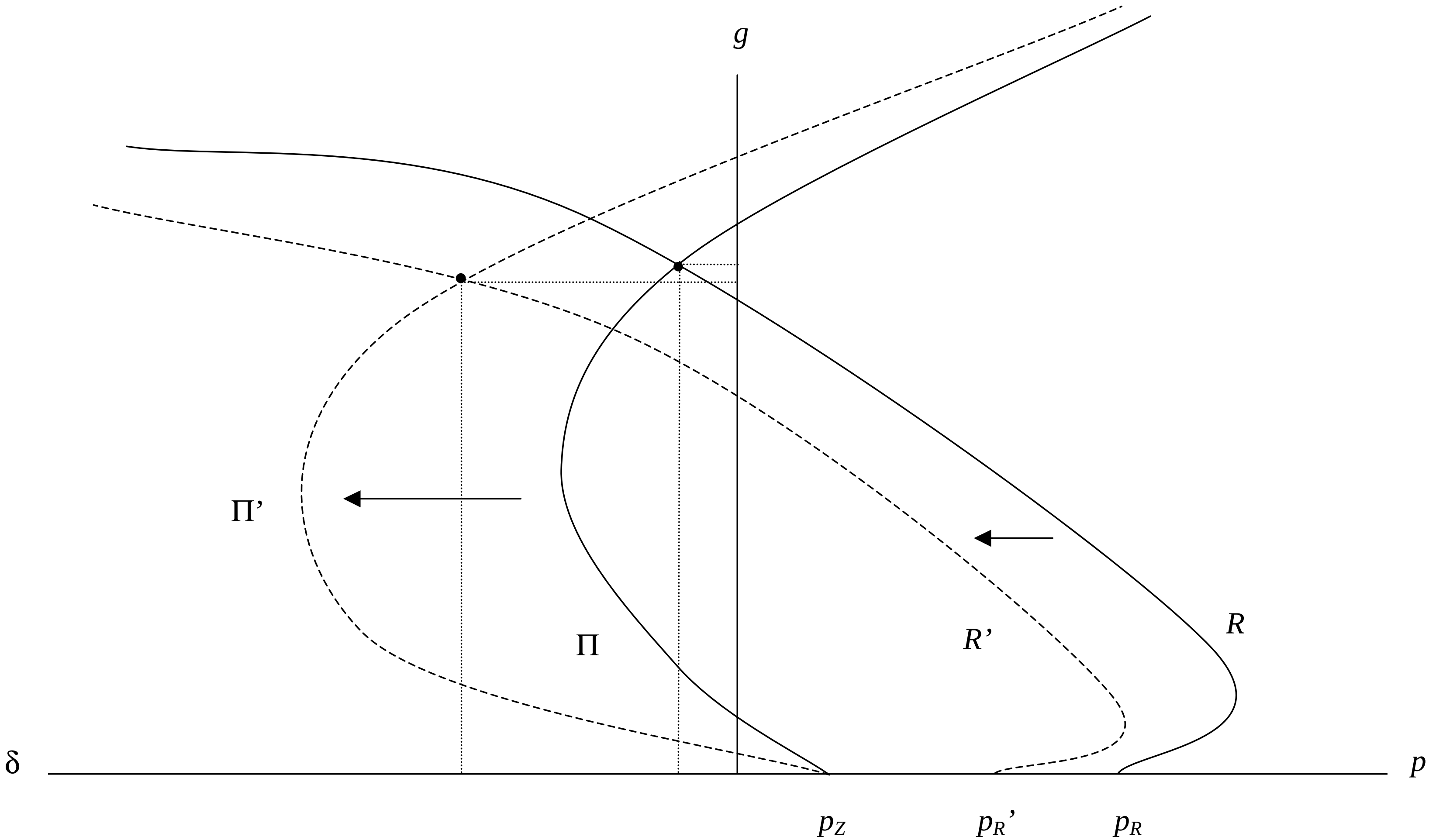


Figure 5: Multiple Equilibria

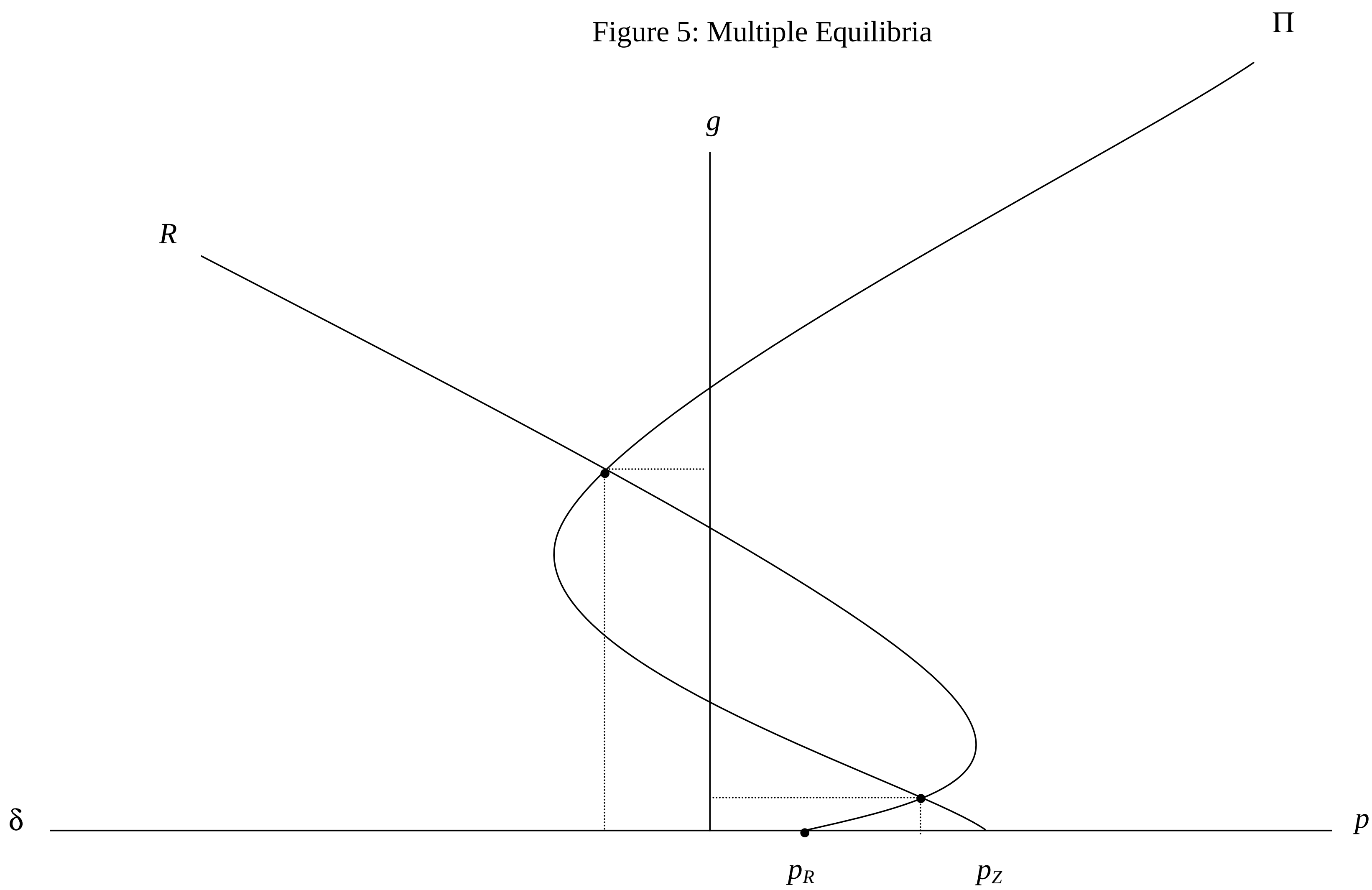
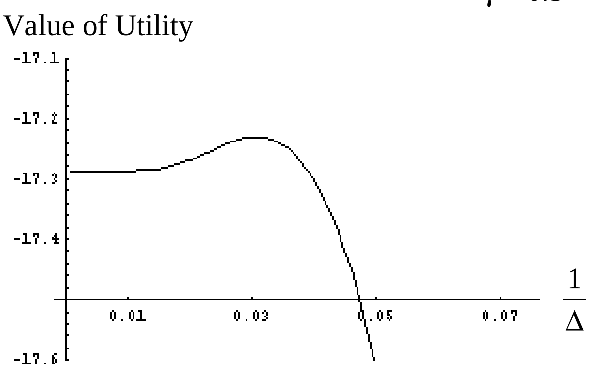


Figure 6: The impact of different patent durations on welfare

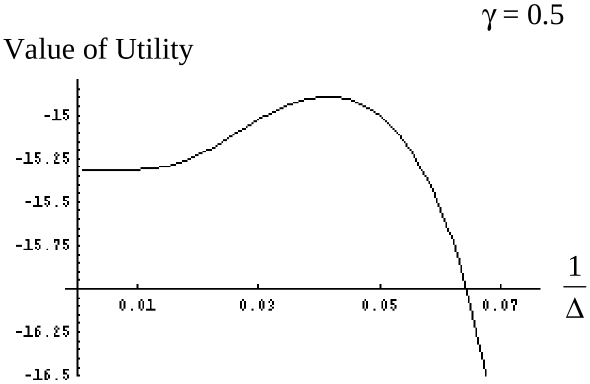
Default Parameter values:

$\sigma = 2 \qquad \rho = 0.02 \qquad \frac{F}{bs} = 1.46 \qquad \frac{1}{bs} = 0.6$

$\gamma = 0.3$



$\gamma = 0.5$



$\gamma = 0.7$