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ABSTRACT

A Two-Country Model of Endogenous Growth*

In this Paper we study the competitive equilibria of a two-country endogenous growth model in which the source of growth is the linearity of technology in reproducible inputs. We begin by showing that in a model with no externalities there is a unique equilibrium; however, there are multiple ways in which the social planner can allocate production plans across countries. We then introduce an externality to human capital and we show that the model has multiple equilibria that can be Pareto-ranked. In many of these equilibria there are perfectly foreseen discrete reallocations of capital from one country to another, accompanied by discrete jumps in growth rates.

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1 Introduction

In this paper we show that indeterminacy arises in a two-country version of a standard two-sector model of endogenous growth as a consequence of a human capital externality of the kind analyzed by Lucas [6]. This indeterminacy is quite different from that which has been shown, by Benhabib and Perli [1], to exist in one country versions of the same model. Instead, indeterminacy in our two-country model is related to the fact that in this class of *international* general equilibrium models, the geographical location of production is indeterminate.[1]

We show that under production externalities the open-economy "Lucas model" has three types of equilibria. First, there is a symmetric equilibrium in which both countries invest and grow at the same rate. Second, there are two asymmetric equilibria in which one of the two countries shuts down its human capital sector completely and, hence, doesn't grow at all. Third, there is a *continuum* of switching equilibria in which an asymmetric equilibrium prevails for a finite length of time followed by a switch to the symmetric equilibrium.

The switching equilibria are particularly interesting since they imply that the model can generate convergence clubs and growth traps as well as sudden and sustained growth takeoffs (miracles). Moreover, the switching equilibria are characterized by sudden swings of capital flows across countries with little change in fundamentals. Two features of our results are worth stressing. First, the existence of multiple equilibria in the model *does not* depend on the size of the production externality: an arbitrarily small externality is sufficient to guarantee the existence of multiple equilibria. Second, the alternative equilibria are not welfare equivalent and can be Pareto ranked.

The rest of the paper is organized as follows. The next section illustrates the main idea of the paper with a simple example. Section 3 lays out the model while Sections 4 and 5 analyzes in detail the equilibria of the model with and without externalities. Section 6 concludes with some final remarks.

[1] In fact, in the closed-economy version of the model that we analyze, the equilibrium is unique. Thus, the indeterminacy that arises in the two-country version of the model is solely due to the model's open economy implications.

2 A Simple Example

The main idea of our paper can be illustrated in a simple AK model of endogenous growth. Suppose that there are two countries and two technologies

$$\begin{aligned} Y &= AK \\ Y' &= A'K' \\ K + K' &= W \end{aligned}$$

where Y and Y' represent output in countries 1 and 2, W is the stock of world wealth and K and K' represent the stocks of wealth used for production in each country. Notice first that if both countries use the same technology, that is if $A = A'$, then the location of production is indeterminate. In this model the welfare theorems hold and one can represent competitive equilibria as the solutions to a social planning problem. Notice that the social planner does not care in which country production is located since the path of output would be identical whichever technology was used.

Now suppose that there are externalities in production represented by the conditions

$$\begin{aligned} A &= \bar{K}^{\gamma}, \\ A' &= \bar{K}'^{\gamma}, \end{aligned}$$

where $\bar{K}$ and $\bar{K}'$ represent aggregate capital in each country. In this case the competitive equilibria of the economy no longer solve a planning problem. There are three kinds of equilibria. In a symmetric equilibrium capital is equally located across countries and

$$A = A' = \frac{W}{2}.$$

The other two equilibria are asymmetric in which either

$$A = W > A' = 0,$$

or

$$A' = W > A = 0.$$

These asymmetric equilibria are more efficient than the symmetric equilibrium since they exploit the non-convexity in production arising from production externalities. The main idea of this paper is that this example can be generalized to the Lucas [6] model of endogenous growth in a two country world.

3 The Model

We begin by studying a social planning problem. We assume that the social planner maximizes the discounted sum of utilities of two representative agents weighted with the welfare weight b,

$$\max V = \int_{t=0}^{\infty} e^{-\rho t} \left[bU(C) + (1-b) U'(C') \right] dt. \tag{1}$$

In this equation C and C' represent consumption of the agent in country 1 and 2 respectively and ρ is the common rate of time preference. We further specialize the utility function to take the form

$$U(C) = \log(C), \quad U'(C') = \log(C').$$

We conjecture that our results will generalize to preferences that are homogenous in consumption but not beyond this class since homogeneity is required for balanced growth.

We allow for perfect mobility of capital but zero mobility of labor and we assume that the social planner faces the following constraints

$$\begin{aligned} \dot{W} &= F(K, HQu) + F(W-K, H'Q'u') - C - C', && (2) \\ \dot{H} &= \delta H (1-u), && (3) \\ \dot{H}' &= \delta H' (1-u'), && (4) \\ W(0) &= W_0, \quad H(0) = H_0, \quad H'(0) = H'_0. && (5) \end{aligned}$$

The notation $F(K, HQu)$ refers to the final goods technology which we assume to be strictly concave and common across countries. We let u and u' refer to the fractions of labor allocated to final goods production in each country. Equations (3) and (4) represent the technologies in each country for producing human capital; the variables $(1-u)$ and $(1-u')$ represent the fractions of labor allocated to production of human capital and the parameter δ is a constant that represents the productivity of the human capital technology. We impose the following assumption:

Condition 1 (Productivity Condition)

$$\rho < \delta.$$

Remark 1 *Condition 1 ensures that the technology for the production of human capital is productive enough, relative to the rate of time preference. It is a necessary condition if the model is to exhibit endogenous growth. If Condition 1 fails, the equilibria of the model will be characterized by stagnation and the stocks of physical and human capital will converge asymptotically to zero.*

The variables Q and Q' refer to human-capital augmenting technical progress. To capture the idea of knowledge spillovers in the process of human capital acquisition, we assume that they are determined by the equations

$$Q = \bar{H}^{\gamma}, \quad Q' = \bar{H}'^{\gamma},$$

where $\bar{H}$ and $\bar{H}'$ represent the economy wide levels of human capital in each country and γ is a parameter that indexes the importance of externalities.

3.1 A Pseudo Planning Problem

Following Kehoe-Levine and Romer [5] we choose to model equilibria in an economy with externalities by studying a *pseudo planning problem*. By this we mean that the social planner solves Problem 1 taking constraints (2)–(5) as given and, importantly, we assume that the social planner ignores the existence of externalities by taking the variables Q and Q' to be exogenous. Given a solution for given Q and Q' we solve a fixed point problem by letting $Q = H^{\gamma}$ and $Q' = H^{'\gamma}$. Kehoe, Levine and Romer show that every solution to a fixed point problem of this kind can be decentralized as a competitive equilibrium by a suitable choice of the initial wealth distribution. Our purpose is not to study welfare issues but to gain insight into a competitive equilibrium with externalities and by setting up the problem in this way we are able to economize on notation and provide simpler proofs of our main results.

We write the planner's Hamiltonian, where Λ, M and M' are the planner's costate variables, as follows,

Problem 1

$$\begin{aligned} \max H &= b \log(C) + (1-b) \log C' \\ &\quad + \Lambda \left(F(K, HQu) + F(W-K, H'Q'u') - C - C' \right), \\ &\quad + \delta M H (1-u) + \delta M' H' (1-u'). \end{aligned}$$

Define the following transformed variables:

$$\psi = \frac{M}{\Lambda Q}, \quad \psi' = \frac{M'}{\Lambda Q'}, \quad Z = W + HQ\psi + H'Q'\psi', \tag{6}$$

$$c = \frac{C}{Z}, \quad \lambda = \Lambda Z, \quad v = \frac{HQ\psi}{Z}, \quad v' = \frac{HQ\psi'}{Z}, \tag{7}$$

$$k = \frac{K}{HQu}, \quad k' = \frac{W-K}{H'Q'u'}, \quad f(k) = F\left(\frac{K}{HQu}, 1\right). \tag{8}$$

The variable ψ represents the shadow price of human capital in country 1 and Z represents the value of world wealth. Throughout the paper, if x is the value of some variable in country 1, we use the notation x' to mean the value of the same variable in country 2.

Notice that in defining world wealth we have multiplied human wealth by Q; this reflects the assumption that human wealth is measured in efficiency units. Individual agents recognize that Q grows but they do not attribute the source of this growth to their own actions. The variable v represents the ratio of the value of human wealth to world wealth and k is the ratio of physical to human wealth located in country 1. λ is the shadow price of consumption.

We also define the following functions.

$$f_H(k) = f(k) - kf_k(k), \quad \phi(\hat{\psi}) = f_H^{-1}(\delta\hat{\psi}), \quad \hat{\psi} = \max[\psi, \psi']. \tag{9}$$

The following table defines our transformed variables for future reference.

Table 1: Definitions of Variables	
Z	World wealth (physical plus the value of human capital).
W	World stock of physical capital.
C	Consumption in country 1 (a prime on a variable denotes country 2).
K	Stock of physical capital in country 1.
H	Stock of human capital in country 1.
HQ	Human capital in efficiency units in country 1.
u	Fraction of labor in country 1 allocated to physical capital production.
$c = \frac{C}{Z}$	The ratio of consumption in country 1 to world wealth.
$\omega = \frac{W}{Z}$	The ratio of world physical wealth to total world wealth
Q	Productivity (taken as exogenous by individual agents).
ψ	The relative price of human capital.
$\hat{\psi}$	The higher of the relative price of H in the two countries.
$k = \frac{K}{HQu}$	The capital-labor ratio in the physical goods sector in country 1.
$v = \frac{\psi QH}{Z}$	The value of human capital in efficiency units as a fraction of world wealth.
λ	The shadow value of c (also of c').

Proposition 1 *Any solution to Problem 1 solves the following set of static first order conditions:*

$$\begin{aligned}
\frac{b}{c} &= \frac{(1-b)}{c'} = \lambda, && (10)\\
f_k(k) &= f_k(k'), && (11)\\
(1-u)(f_H(k) - \delta\psi) &= 0, \quad (1-u) \geq 0, \quad (f_H(k) - \delta\psi) \geq 0, && (12)\\
(1-u')(f_H(k') - \delta\psi') &= 0, \quad (1-u') \geq 0, \quad (f_H(k') - \delta\psi') \geq 0, && (13)\\
1 &= v + v' + \phi(\hat{\psi})\left(\frac{uv}{\psi} + \frac{u'v'}{\psi'}\right), && (14)
\end{aligned}$$

together with the dynamic equations,

$$\frac{\dot{Z}}{Z} = f_k(k) - \frac{1}{\lambda}, \tag{15}$$

$$\frac{\dot{\lambda}}{\lambda} = \rho - \frac{1}{\lambda}, \tag{16}$$

$$\frac{\dot{\psi}}{\psi} = f_k(k) - \frac{u f_H(k)}{\psi} - (1-u)\,\delta\,(1+\gamma), \tag{17}$$

$$\frac{\dot{\psi}'}{\psi'} = f_k(k') - \frac{u' f_H(k')}{\psi'} - (1-u')\,\delta\,(1+\gamma), \tag{18}$$

$$\frac{\dot{v}}{v} = \frac{-u f_H(k)}{\psi} + \frac{1}{\lambda}, \tag{19}$$

$$\frac{\dot{v}'}{v'} = \frac{-u' f_H(k')}{\psi'} + \frac{1}{\lambda}, \tag{20}$$

and the transversality conditions

$$\lim_{T\to\infty} e^{-\rho T}\Lambda W = 0, \quad \lim_{T\to\infty} e^{-\rho T} M H = 0, \quad \lim_{T\to\infty} e^{-\rho T} M' H' = 0. \tag{21}$$

Proof. The proof follows from a straightforward application of the necessary and sufficient conditions for dynamic optimization. The only Equation that requires further explanation is (15) which is derived as follows. From the definition of Z,

$$Z = W + H^{1+\gamma}\psi + H'^{1+\gamma}\psi'.$$

Differentiating gives

$$\begin{aligned} \dot{Z} = {} & \dot{W} + (1+\gamma)\,\delta\,(1-u)\,HQ\psi \\ & + (1+\gamma)\,\delta\,(1-u')\,H'Q'\psi' + HQ\dot{\psi} + H'Q'\dot{\psi}'. \end{aligned} \tag{22}$$

Now use Equation (2) and expand the functions F and F' using Euler's theorem as

$$F = KF_K + HQuF_H,$$

noting that

$$f_H = f - kf_k, \quad f_k(k) = f_{k'}(k'),$$

and

$$k = \frac{K}{HQu}, \quad k' = \frac{K'}{H'Q'u'}.$$

Equation (15) follows from substituting the expressions for $\dot{\psi}, \dot{\psi}', \dot{H}$ and $\dot{H}'$ from equations (17), (18), (3) and (4) into Equation (22) and cancelling terms. ■

Equation (10) is the first order condition for the choice of consumption and (11) ensures that capital is efficiently allocated across countries. Conditions (12) and (13) are the Kuhn-Tucker conditions for allocation of labor across industries in the two countries. If u' is equal to 1 then the production of human capital shuts down in country 2. We explicitly allow for the possibility that this may occur in equilibrium. Equation (14) follows from the condition that the sum of physical capital in each country must equal world physical wealth:

$$K + K' = W. \tag{23}$$

To derive Equation (14), divide both sides of (23) by Z, and use the definitions of v, v', and ϕ to arrive at

$$\phi(\psi)\frac{vu}{\psi} + \phi(\psi')\frac{vu'}{\psi'} = 1 - v - v'. \tag{24}$$

Since the social planner has stocks of human capital in both countries he will choose to produce physical output in both countries (although it is possible that he may choose not to produce *additional* human capital in one or other of the countries). Given that the physical goods industries operate in both countries, the Equation (11) implies that k and k' will be chosen to equal each other. If both countries also produce human capital then the equality of k and k' together with the Kuhn-Tucker conditions (12) and (13) imply that the shadow prices ψ and ψ' are equated.

It is also possible that human capital in one country has a lower shadow price than in the other. In this case the country in which human capital has a lower shadow price will shut down the production of human capital and the country will cease to grow. The existing stock of human capital in this country will be used to produce physical capital and in this case the capital labor ratio in the stagnant country will be a function of the shadow price of human capital in the rest of the world. This discussion leads to the definition

$$\hat{\psi} = \max\{\psi, \psi'\}$$

and to the simplification of Equation (24) as follows

$$\phi\left(\hat{\psi}\right)\left(\frac{vu}{\psi} + \frac{vu'}{\psi'}\right) + v + v' = 1,$$

which is Equation (14) above.

To solve endogenous growth models, it is often useful to find a change of variable such that the transformed variables are stationary along a balanced growth path. We chose to normalize consumption and capital by dividing each variable by Z, a variable that represents "world wealth". Z is equal to the sum of physical wealth, W plus the stocks of human capital in each country valued at the shadow prices ψ and ψ'. In a decentralized version of the model, these shadow prices would be equal to the relative prices of human to physical capital. The choice of Z as a normalization variable is especially convenient in a model in which agents have logarithmic preferences since it allows us to reduce the order of the dynamical system that describes equilibrium trajectories. The following proposition makes this comment explicit.

Proposition 2 *The solution to Problem 1 has the property that the consumption wealth ratio, c, is constant at all times.*

Proof. Notice from Equation (16) that, along a balanced growth path λ^{-1} is equal to ρ, (the rate of time preference) and from Equation (10) the consumption to wealth ratio in each country is proportional to λ^{-1}. Since (16) is an unstable differential equation, there will be unique time path for λ, consistent with the transversality condition (21), in which λ is constant for all t and $\lambda^{-1} = \rho$. It follows from Equation (10) that the consumption to wealth ratio is constant. ■

Definition 1 *An interior solution to Problem 1 is a solution for which*

$$0 < u < 1, \quad 0 < u' < 1.$$

Remark 2 *An interior solution is one in which both industries operate in both countries at all points in time. Since Problem 1 is concave in C and C', there will be a unique solution for the consumption allocation. But since the Lagrangian is only weakly concave in K, there may be many production plans, indexed by different time paths for u, that implement this allocation.*

Proposition 3 *In any interior solution to Problem 1 relative factor prices will be equated across countries at all times.*

Proof. From the Kuhn-Tucker conditions, (12) and (13), it follows that for interior u and u' we must have

$$\begin{aligned} f_H(k) &= \delta\psi \\ f_H(k') &= \delta\psi'. \end{aligned} \tag{25}$$

From Equation (11) we also know that $k = k'$. Hence, it must also be true that $\psi = \psi'$. ■

4 Equilibria in the Absence of Externalities

Since the solutions to the pseudo-social planning problem can be decentralized as competitive equilibria, we refer to them as equilibria. In this section we study equilibria for the case of no externalities, $\gamma = 0$ and hence productivity (equal to Q) is constant and equal to 1.

Proposition 4 *For the case $Q = 1$, if there exists an interior solution to Problem 1, then the variables ω, ψ and ψ' form an autonomous subsystem with dynamics governed by the following equations:*

$$\frac{\dot{\psi}}{\psi} = f_k(\phi(\psi)) - \delta, \tag{26}$$

$$\psi = \psi', \tag{27}$$

$$\frac{\dot{\omega}}{\omega} = \frac{\delta\psi}{\phi(\psi)} - \frac{\rho}{\omega} + \rho. \tag{28}$$

The variables k, k', c, c' and Z are given by the expressions

$$k = f_H^{-1}(\delta\psi) \equiv \phi(\psi), \tag{29}$$

$$k' = f_H^{-1}(\delta\psi') \equiv \phi(\psi'), \tag{30}$$

$$c = b\rho, \tag{31}$$

$$c' = (1-b)\rho, \tag{32}$$

$$\dot{Z} = f_k(\phi(\psi)) - \rho. \tag{33}$$

Proof. Combining Equations (12) and (13) with Equations (17) and (18) it follows that the change of the relative price over time is the same in each country and is governed by the differential equation:

$$\frac{\dot{\psi}}{\psi} = f_k(\phi(\psi)) - \delta.$$

To derive an equation that governs the behavior of ω note that from the definition of ω,

$$\frac{\dot{\omega}}{\omega} = \frac{\dot{W}}{W} - \frac{\dot{Z}}{Z}.$$

Using Equations (2), (15) and the linear homogeneity of F we can rewrite this expression as follows

$$\frac{\dot{\omega}}{\omega} = \left(\frac{HQu}{W} f_H + \frac{K}{W} f_K + \frac{H'Q'u'}{W} f'_H + \frac{(W-K)}{W} f'_K - \frac{C+C'}{W}\right) - \left(f_k(k) - \frac{1}{\lambda}\right),$$

and since $(C+C')/W = 1/\lambda$ from (10) the expression can further be reduced to;

$$\frac{\dot{\omega}}{\omega} = \left(\frac{HQu}{W} f_H + \frac{H'Q'u'}{W} f'_H\right) - \frac{1}{\omega\lambda} + \frac{1}{\lambda}.$$

Now use Proposition 3, Equation (10) the definitions of v and v', and the fact that $\lambda^{-1} = \rho$ to write

$$\frac{\dot{\omega}}{\omega} = \frac{1}{\omega\psi} f_H (uv + u'v') - \frac{\rho}{\omega} + \rho.$$

Equation (12) implies $f_H = \delta\psi$ along an interior path hence,

$$\frac{\dot{\omega}}{\omega} = \frac{\delta}{\omega}(uv + u'v') - \rho\frac{(1-\omega)}{\omega}, \tag{34}$$

and from Equation (14)

$$(uv + u'v') = \frac{\psi\omega}{\phi(\psi)}.$$

Substituting this expression for $(uv + u'v')$ into (34) gives the following differential equation

$$\frac{\dot{\omega}}{\omega} = \frac{\delta\psi}{\phi(\psi)} - \frac{\rho}{\omega} + \rho,$$

which is Equation (28). The equations for k and k' as functions of ψ and ψ' respectively follow directly from the inverse function theorem. The expressions for c and c' follow from (10) and from the fact that $\lambda^{-1} = \rho$. Finally Equation (33) follows from (29) and (15). ■

Remark 3 *Note that along a balanced growth path (26) implies that* $f_k = \delta$ *hence, from (33) the asymptotic growth rate is equal to* $\delta - \rho$. *Condition 1 is required to ensure the existence of positive long-run growth.*

Proposition 5 *If*

$$\lim_{x \to 0} f_k(x) = \infty, \quad \lim_{x \to \infty} f_k(x) = 0,$$

(Inada conditions) the subsystem (26),(28) has a unique stationary solution $\bar{\psi}, \bar{\omega}$ *that satisfies the equations*

$$\begin{aligned} \bar{\psi} &= \frac{f_H\left[f_k^{-1}(\delta)\right]}{\delta}, \\ \bar{\omega} &= \frac{\rho\phi\left(\bar{\psi}\right)}{\delta\bar{\psi} + \rho\phi\left(\bar{\psi}\right)}. \end{aligned} \tag{35}$$

Proof. Follows from the differentiability and strict concavity of $f(k)$. ∎

Proposition 6 *For initial values of* H, H' *and* W, *such that*[2]

$$\frac{H_0 + H_0'}{W_0} \in N\left(\frac{1-\bar{\omega}}{\bar{\omega}\bar{\psi}}\right) \tag{36}$$

there exists a one dimensional manifold of solutions to the dynamical system (26) and (28) that converges asymptotically to $\left\{\bar{\omega}, \bar{\psi}\right\}$.

Proof. The Jacobian of the system (26), (28) is given by

$$J = \begin{vmatrix} f_{kk}\phi' & 0 \\ \frac{\phi\delta - \delta\psi\phi'}{\phi^2} & \frac{\rho}{\omega^2} \end{vmatrix},$$

and the determinant of J evaluated at $\left\{\bar{\psi}, \bar{\omega}\right\}$ is given by

$$\det(J) = \left.\frac{\rho f_{kk}\phi'}{\omega^2}\right|_{\left\{\bar{\psi}, \bar{\omega}\right\}} < 0,$$

where the sign follows from the concavity of $f(k)$. Since $\det(J)$ is equal to the product of the roots of the localized dynamics, one root must be

[2] We use the notation $N(x)$ to mean an epsilon neighborhood of x.

positive and the other negative and the steady state is saddle path stable. It follows (Guckenheimer and Holmes [4] page 13) that locally there is a one dimensional manifold of initial conditions that converges asymptotically to the steady state.

We now establish conditions on H, H' and W for the system to begin close to the steady state. When $\psi = \psi'$ we can write ω as

$$\omega = \frac{W}{W + \psi (H + H')}.$$

Rearranging this expression gives

$$\frac{H + H'}{W} = \left(\frac{1 - \omega}{\omega \psi} \right).$$

Since H and H' are predetermined it follows that once ψ is determined, so are ψ' and ω. Since Equation (35) implies that $0 < \bar{\omega} < 1$, the condition we seek is given by:

$$\frac{H_0 + H_0'}{W_0} \in N \left(\frac{1 - \bar{\omega}}{\bar{\omega} \bar{\psi}} \right)$$

which is condition (36) in the statement of the Proposition . ■

We have established the properties of an interior solution to Problem 1. We now turn to establishing that such solutions exist. To establish existence, we exploit the following three facts:

Fact 1. $\lambda^{-1} = \rho$.

Fact 2. In any interior solution,

$$\frac{f_H (k)}{\psi} = \frac{f_H (k')}{\psi'} = \delta.$$

Fact 3. v, v' and ω are related by the identity,

$$1 - v - v' \equiv \omega. \tag{37}$$

Facts 1–3 imply that we can simplify Equations (14), (19) and (20) as follows:

$$\frac{\dot{v}}{v} = -u\delta + \rho, \tag{38}$$

$$v' \equiv 1 - v - \omega, \tag{39}$$

$$u' = \left(\frac{\omega \psi}{\phi (\psi)} - uv \right) \frac{1}{v'}. \tag{40}$$

Remark 4 *For any choice of u, Equation (38) determines a time path for v. Given the time paths for v,u,ω and ψ, Equations (39) and (40) determine the time paths of v' and u'. Along this path, Equation (20) (a differential equation that determines the evolution of v') also holds since (39) is an identity.*

Proposition 7 *For initial values of H_0, H_0', W_0, that satisfy conditions (36) there exists a continuum of interior solutions to Problem 1.*

Proof. Consider the path for v, u, ω and ψ given by the equations

$$\begin{aligned} u &= \frac{\rho}{\delta}, \quad v = v_0, \quad v' = 1 - v - \omega, \\ u' &= \left(\frac{\omega\psi}{\phi(\psi)} - uv \right) \frac{1}{v'}, \end{aligned}$$

where ω and ψ are determined as solutions to (26) and (28) and ψ_0 is chosen such the system begins on the stable manifold. Since this solution satisfies the transversality conditions, there exists at least one interior solution to Problem 1. To establish that there is a continuum of alternative solutions, let u equal some arbitrary value $\tilde{u}$ for an interval of time $t \in [0, T)$ and let $u = \rho/\delta$, thereafter, $t \in [T, \infty]$. Since this choice of u guarantees that v, v', ω, ψ and ψ' converge to positive finite numbers, transversality is satisfied and hence, by arbitrary choice of $\tilde{u}$, arbitrarily close to any one solution there is another. ■

Remark 5 *All of the above paths implement the same consumption plan. The multiplicity of solutions for u and u' follows from the fact that the location of production is irrelevant.*

In the plan in which $u = \rho/\delta$ for all t, the Social Planner chooses to maintain the relative stocks of human capital to world wealth, equal to their initial levels. But this is not the only way to implement a given consumption plan. As an alternative, the Planner could choose to build up human capital in country 1 by lowering u and allocating more labor time in that country to the production of human capital. To maintain the same production of physical capital the Planner would reallocate labor in country 2 out of human capital production and into physical capital production. During this process the stock of human capital in country 1 will be increasing relative to the stock in country 2. If at date T resources are switched back to human capital production in country 1, the ratio of human capital to world wealth (denoted v_T) will be constant from that date on.

The indeterminacy of the location of production also implies indeterminacy of the pattern of capital flows across countries. If two autarkic economies with different relative factor prices, i.e., $\psi \neq \psi'$, were to open up to trade, then the relative factor price ratios would immediately be equated. But this factor price equalization would not imply an equalization of the physical to human capital ratios across countries since the pattern of capital flows is indeterminate. Any change in the relative factor price ratio (both at the time of opening up to trade as well as over time) can be accommodated through a sectoral reallocation of factors rather than through a cross-country movement of capital.

5 Equilibria with Externalities

In the economy without externalities, there exists a continuum of production plans; but they are all welfare equivalent, since alternative production plans leave the path of world consumption unaffected. In this section we show that this is not the case when there are production externalities in the model. In particular, we set $\gamma > 0$.

To characterize equilibria in the economy with externalities it is convenient to introduce a new variable. We define

$$s = \frac{v'\psi}{v\psi'} \equiv \left(\frac{H'}{H}\right)^{1+\gamma},$$

to be the ratio of human capitals in the two countries, raised to the power $1+\gamma$. We will establish that the world economy can be characterized by a system of differential equations in five variables, ψ, ψ', s , v and Z. The first four of these equations form an autonomous system and the fifth, the $\dot{Z}$ equation, determines the growth rate. Further, we will show that there exist two types of equilibria. In a symmetric equilibrium $\psi = \psi'$ and $\dot{s} = 0$ for all t. In this case the dynamics of an approach to a balanced growth path is characterized by a two dimensional sub-system in ψ and v. In an asymmetric equilibrium $\psi > \psi'$ for all time and the production of human capital is shut down in country 2.

5.1 Interior Equilibria

As in the case of no externalities, we begin by assuming the existence of an interior equilibrium and we characterize its properties. Recall that in Proposition 3 we established, for the economy with no externalities, that $\psi = \psi'$ in any interior equilibrium. We begin by establishing a stronger version of this proposition for the economy with externalities.

Proposition 8 *In any interior solution to Problem 1 with positive externalities; relative factor prices and labor allocated to each industry will be equated across countries. That is; $\psi = \psi'$ and $u = u'$.*

Proof. From the Kuhn-Tucker conditions, (12) and (13), it follows that for interior u and u' we must have $f_H(k) = \delta\psi$ and $f_H(k') = \delta\psi'$. From Equation (11) we also know that $k = k'$. Hence, it must also be true that $\psi = \psi'$. Combining (17), (18), (12) and (13) gives,

$$\begin{aligned} \frac{\dot{\psi}}{\psi} &= f_k(k) - \delta(1+\gamma) + u\delta\gamma, \\ \frac{\dot{\psi}'}{\psi'} &= f_k(k) - \delta(1+\gamma) + u'\delta\gamma, \end{aligned}$$

and since $\psi = \psi'$ for all t, it must be that $u = u'$. ■

Remark 6 *Notice that in the case of no externalities the second part of this proposition does not hold since the differential equations governing the evolution of ψ and ψ' are independent of u and u'.*

Proposition 9 *In any interior equilibrium with positive externalities, $s = (H_0'/H_0)^{1+\gamma}$ is a positive constant.*

Proof. We seek to show that $\dot{H} = \dot{H}'$ for all time. From equations (3) and (4)

$$\begin{aligned} \dot{H} &= \delta H(1-u), \\ \dot{H}' &= \delta H(1-u'), \end{aligned}$$

and from Proposition 8, $u = u'$ hence s is constant in an interior equilibrium with externalities. ■

Proposition 10 *For the case $Q > 1$, any interior equilibrium is characterized by the following pair of autonomous differential equations:*

$$\frac{\dot{\psi}}{\psi} = f_k(\phi(\psi)) - \delta(1+\gamma) + \delta\gamma u, \tag{41}$$

$$\frac{\dot{v}}{v} = -u\delta + \rho, \tag{42}$$

where u is given by the expression

$$u = \frac{\psi(1 - v(1+s))}{\phi(\psi)\, v(1+s)}, \tag{43}$$

and

$$s = \left(\frac{H_0'}{H_0}\right)^{1+\gamma}, \tag{44}$$

is a positive constant determined by the initial levels of human wealth in the two countries. The variables $k, k', c, c', \psi', v', \omega$ and Z are governed by the equations

$$k = \phi(\psi), \tag{45}$$

$$k' = \phi(\psi), \tag{46}$$

$$c = b\rho, \tag{47}$$

$$c' = (1-b)\rho, \tag{48}$$

$$\psi' = \psi, \tag{49}$$

$$v' = sv, \tag{50}$$

$$\omega \equiv 1 - v - v' \tag{51}$$

$$\frac{\dot{Z}}{Z} = f_k(\phi(\psi)) - \rho. \tag{52}$$

Proof. Equations (41) and (42) follow directly from (17) and (19) together with the Kuhn-Tucker condition (12) which implies that $\psi = \phi(\psi)$ in a interior equilibrium. Equation (43) follows from (14), Proposition 8 and the definition of s. Equation (44) follows from Proposition 9, Equations (45) and (46) are implied by the fact that the equilibrium is interior, Equations (45) and (46) follow from (10) and the fact that $\lambda^{-1} = \rho$ is constant (from transversality). Equation (49) is from the fact that equilibrium is interior, Equation (50) is from Proposition (9) and the definition of s, (51) is an identity and Equation (52) follows from (15). ∎

Remark 7 *We have chosen to characterize the system with externalities as a differential equation system in* ψ *and* v *rather than in* ψ *and* ω *since it will facilitate comparison between symmetric and asymmetric equilibria. These two formulations are closely linked since it is identically true that*

$$\omega = 1 - v - v',$$

and in a symmetric equilibrium $v' = sv$ *where* s *is a constant determined by the initial distribution of human capital. For the case of no externalities,* ω *is a better choice of variable to characterize equilibrium since* v *and* v' *are indeterminate. But for the case of an economy with externalities,* v *is a more convenient choice of variable. In this case, in an interior equilibrium,* v *and* v' *are determinate and, as we will see below, the choice of* ψ *and* v *as state variables allows comparison with the asymmetric equilibria that are characterized by a four variable differential equation system in* s, ψ, ψ' *and* v.

Proposition 11 *If*

$$\lim_{x \to 0} f_k(x) = \infty, \quad \lim_{x \to \infty} f_k(x) = 0,$$

(Inada conditions) the subsystem (41)-(43) system has a unique stationary solution $\bar{u}, \bar{\psi}, \bar{v}$ *that satisfies the equations*

$$\begin{aligned} \bar{u} &= \frac{\rho}{\delta}, \\ \bar{\psi} &= \frac{f_H\left[f_k^{-1}\left(\delta + \gamma\left(\delta - \rho\right)\right)\right]}{\delta}, \\ \bar{v} &= \frac{\delta \bar{\psi}}{(1+s)\left(\rho\phi\left(\bar{\psi}\right) + \bar{\psi}\delta\right)}. \end{aligned}$$

Proof. Follows from strict concavity of f plus continuity and the Inada conditions. ■

Proposition 12 *For initial values of* H, H' *and* W, *such that*

$$\frac{H_0^{1+\gamma} + H_0'^{1+\gamma}}{W_0} \quad \in N\left(\frac{\bar{v}\left(1+s\right)}{\left(1-\bar{v}\left(1+s\right)\right)\bar{\psi}}\right)$$

there exists a one dimensional manifold of solutions to the dynamical system (41) and (42) that converges asymptotically to $\left\{\bar{v}, \bar{\psi}\right\}$.

Proof. Consider the subsystem defined by Equations (41) and (42). The Jacobian of this system is given by

$$J = \left| \begin{array}{cc} f_{kk}\phi' + u_{\psi}\delta\gamma & \delta\gamma u_v \\ -u_{\psi}\delta & -u_v\delta \end{array} \right|,$$

where u_v and u_ψ denote the partial derivatives of function

$$u(v, \psi) = \frac{\psi(1 - v(1+s))}{\phi(\psi) v(1+s)},$$

with respect to v and ψ. The determinant of J evaluated at $\{\bar{v}, \bar{\psi}\}$ is given by

$$\det(J) = -u_v \delta f_{kk} \phi' |_{\{\bar{\psi}, \bar{v}\}} < 0,$$

where the sign follows from the definition of ϕ, the concavity of $f(k)$ and the fact that $u_v < 0$. Since $\det(J)$ is equal to the product of the roots of the localized dynamics, one root must be positive and the other negative and the steady state is saddle path stable. It follows (Guckenheimer and Holmes [4]) that locally there is a one dimensional manifold of initial conditions that converges asymptotically to the steady state.

We now establish conditions on H, H' and W for the system to begin close to the steady state. When $\psi = \psi'$ and $v' = sv$ we can write ω as

$$\omega = 1 - v - v' = 1 - v(1+s) = \frac{W}{W + \psi(H^{1+\gamma} + H'^{1+\gamma})}.$$

Rearranging this expression gives

$$\frac{H^{1+\gamma} + H'^{1+\gamma}}{W} = \left(\frac{v(1+s)}{(1 - v(1+s))\psi} \right).$$

Since H and H' are predetermined it follows that once ψ is determined, so are ψ', v' and v. Since $0 < v(1+s) < 1$, the condition for the system to begin close to the steady state is given by:

$$\frac{H_0^{1+\gamma} + H_0'^{1+\gamma}}{W_0} \in N\left(\frac{\bar{v}(1+s)}{(1 - \bar{v}(1+s))\bar{\psi}} \right)$$

which is the condition in the statement of the Proposition . ■

The state of the economy is completely characterized by three variables, W, H and H'. We have established that this economy exhibits a balanced growth path and that for a set of initial conditions in the neighborhood of this path, there exists a locally unique determinate equilibrium that converges to it. We next turn to the possibility that there may exist other kinds of equilibria.

5.2 Asymmetric Equilibria

We now turn to the case in which one country shuts down its human capital sector forever. This could happen if either $0 < u < 1$ and $u' = 1$ or $u = 1$ and $0 < u' < 1$ for all t. We will concentrate on the case $u' = 1$ and hence country 2 ceases to produce human capital although the fact that the model is symmetric implies that there is also another equilibrium in which it is country 1 that follows this route. The following proposition establishes the properties of the dynamics of a solution in the asymmetric case.

Proposition 13 *For the case $Q > 1$, any asymmetric solution is characterized by the following set of four autonomous differential equations:*

$$\begin{aligned} \dot{\psi} &= \psi \left[f_k \left(\phi \left(\psi \right) \right) - \delta \left(1 + \gamma \right) + \delta \gamma u \right], & (53) \\ \dot{\psi}' &= \psi' f_k \left(\phi \left(\psi \right) \right) - \delta \psi, & (54) \\ \dot{v} &= v \left[-u\delta + \rho \right], & (55) \\ \dot{s} &= s \left(u - 1 \right) \delta \left(1 + \gamma \right), & (56) \end{aligned}$$

where u is given by the expression

$$u = u\left(\psi, \psi', v, s\right) \equiv \frac{\psi}{v\phi\left(\psi\right)} - \frac{\psi}{\phi\left(\psi\right)} - s\frac{\psi'}{\phi\left(\psi\right)} - s. \quad (57)$$

The variables k, k', c, c', ω and Z are governed by the equations

$$\begin{aligned} k &= \phi\left(\psi\right), & (58) \\ k' &= \phi\left(\psi\right), & (59) \\ c &= b\rho, & (60) \\ c' &= \left(1 - b\right)\rho, & (61) \\ \omega &\equiv 1 - v - v', & (62) \\ \frac{\dot{Z}}{Z} &= f_k\left(\phi\left(\psi\right)\right) - \rho. & (63) \end{aligned}$$

Proof. Notice first that in an asymmetric equilibrium $u' = 1$ and $0 < u < 1$. Equation (53) then follows from (17)and (54) from (18). Equation (55) follows from (19) and (12). Equation (56) follows from differentiating the definition of s and substituting the equations of motion for ψ, ψ',v (Equations (53), (54) and (55)) and the equation of motion for v' (Equation (20)) recognizing that

$$\frac{-u' f_H(k')}{\psi'} = -\frac{\delta\psi}{\psi'}.$$

Equation (57) follows from (14) and the fact that $u' = 1$. Equations (58)-(63) are as in the proof of the symmetric case. ■

Proposition 14 *If*

$$\lim_{x\to 0} f_k(x) = \infty, \quad \lim_{x\to\infty} f_k(x) = 0,$$

(Inada conditions) the subsystem (53)-(57) system has a unique stationary solution $\left\{\bar{\psi}, \bar{\psi}', \bar{v}, \bar{s}, \bar{u}\right\}$ *that satisfies the equations*

$$\bar{u} = \frac{\rho}{\delta}, \tag{64}$$

$$\bar{s} = 0, \tag{65}$$

$$\bar{\psi} = \frac{f_H\left[f_k^{-1}\left(\delta + \gamma\left(\delta - \rho\right)\right)\right]}{\delta}, \tag{66}$$

$$\bar{\psi}' = \frac{\delta\bar{\psi}}{\delta + \gamma\left(\delta - \rho\right)} < \bar{\psi}, \tag{67}$$

$$\bar{v} = \frac{\delta\bar{\psi}}{\phi\left(\bar{\psi}\right)\rho + \delta\bar{\psi}}. \tag{68}$$

Proof. Follows from solving the steady state conditions (53)-(56). The only point at which existence and uniqueness is an issue is in showing that Equation (66) has a solution. This follows from strict concavity of f plus continuity and the Inada conditions. ■

Proposition 15 *For initial values of H, H' and W, such that*

$$\frac{H_0^{1+\gamma}}{W_0} \in N\left(\frac{\bar{v}}{\left(1-\bar{v}\right)\bar{\psi}}\right), \tag{69}$$

$$\frac{H_0'}{H_0} \in N\left(0\right), \tag{70}$$

there exists a two dimensional manifold of solutions to the dynamical system (53) - (56) that converges asymptotically to $\left\{\bar{\psi}, \bar{\psi}', \bar{v}, \bar{s}\right\}$.

Proof. We begin by establishing that the stable manifold has dimension 2 around the steady state. Define the terms u_s, u_v, u_ψ and $u_{\psi'}$ to be the partial derivatives of the function u evaluated at the point $\left\{\bar{\psi}, \bar{\psi}', \bar{v}, \bar{s}\right\}$ and note that since $\bar{s} = 0$,

$$u_{\psi'} = 0.$$

Using this fact now write the Jacobian of the system (53)-(56) evaluated at the point $\left\{\bar{\psi}, \bar{\psi}', \bar{v}, \bar{s}\right\}$ as follows;

$$J = \left| \begin{array}{cccc} \left[f_{kk}\phi' + \delta\gamma u_\psi\right]\psi & 0 & \delta\gamma\psi u_v & \delta\gamma\psi u_s \\ f_{kk}\phi'\psi' - \delta & f_k & 0 & 0 \\ -v\delta u_\psi & 0 & -v\delta u_v & -v\delta u_s \\ 0 & 0 & 0 & (u-1)\,\delta\,(1+\gamma) \end{array} \right|,$$

where the fourth row contains only one non-zero entry since $\bar{s} = 0$. The characteristic polynomial of J is given by the equation

$$\left[(\bar{u}-1)\,\delta\,(1+\gamma) - \lambda\right]\left[f_k - \lambda\right]\left[\lambda^2 + \lambda\left(\bar{v}\delta u_v - \left[f_{kk}\phi' + \delta\gamma u_\psi\right]\bar{\psi}\right) - \bar{v}\delta u_v f_{kk}\phi'\bar{\psi}\right] = 0$$

which has roots

$$\begin{aligned} \lambda_1 &= f_k, \\ \lambda_2 &= (\bar{u}-1)\,\delta\,(1+\gamma), \end{aligned}$$

plus the roots of the polynomial

$$P(\lambda) = \left[\lambda^2 + \lambda\left(\bar{v}\delta u_v - \left[f_{kk} + u_\psi\right]\bar{\psi}\right) - v\delta u_v f_{kk}\phi'\psi\right] = 0.$$

λ_1 is positive and λ_2 is negative since $0 < \bar{u} < 1$. Let the roots of $P(\lambda)$ be equal to λ_3 and λ_4. Since $\lambda_3\lambda_4 = -\bar{v}\delta u_v f_{kk}\phi'\bar{\psi}$ and $u_v < 0$, (from differentiating u) $f_{kk} < 0$ (by concavity) and $\phi' > 0$ (also from concavity of f) it follows that one root of $P(\lambda)$ is positive and the other is negative. Hence it follows that there is locally a two dimensional manifold of initial conditions that converges asymptotically to the steady state (see Guckenheimer and Holmes [4]).

We now establish conditions on H, H' and W for the system to begin close to the steady state. Write ω as

$$\omega = 1 - v - v' = 1 - v\left(1 + \frac{\bar{\psi}'\bar{s}}{\bar{\psi}}\right) = \frac{W}{W + \left(\bar{\psi}H^{1+\gamma} + \bar{\psi}'H'^{1+\gamma}\right)},$$

and since $\bar{s} = 0$

$$1 - v = \frac{W/H^{1+\gamma}}{W/H^{1+\gamma} + \bar{\psi}}$$

Rearranging this expression gives

$$\frac{H_0^{1+\gamma}}{W_0} = \left(\frac{\bar{v}}{(1-\bar{v})\,\bar{\psi}}\right).$$

which is condition (69). Condition (70) follows directly from the requirement that $s_0 = (H_0'/H_0)^{1+\gamma}$ should begin close to its steady state value of 0. ■

To prove the existence of a symmetric equilibrium we used the assumption that the initial ratio of human wealth to non-human wealth was close to the ratio of this variable along the balanced growth path. In the case of an asymmetric equilibrium we need two initial conditions. The first is identical to the condition required for the symmetric case. The second condition, $H_0'/H_0 \in N(0)$, is equivalent to assuming that country 2 is "small" relatively to country 1. We use both assumptions about the initial state of the economy because we want to appeal to local results in the theory of differential equations. Although we have not been able to prove global stability theorems, we conjecture that these assumptions are much stronger than required for the existence of equilibrium and that more generally, there will exist equilibria, in reasonably parameterized examples, for much larger sets of initial conditions.

Notice that both sets of initial conditions are mutually consistent. We have established that if the world begins with a ratio of human to physical capital that is close to the balanced growth path and that if one economy is much smaller than the other, then there are two possible equilibria. The small economy may grow at the same rate as the large economy and the world may move to a balanced growth path. Or the small economy may cease to grow and, in this case, all future growth will take place in the large economy. Although the small economy will cease to grow; it will continue to produce the physical commodity, in contrast to the one sector AK model that we discussed in Section 2. It is also worth pointing out that the degree of increasing returns to scale does not need to be large in order for there to be asymmetric equilibria; γ must be greater than 0 but can be arbitrarily close to 0.

In the case of externalities, the symmetric and asymmetric equilibria can be Pareto ranked. The asymmetric equilibrium Pareto dominates the symmetric equilibrium since, by concentrating production in a single location, the Social Planner is able to fully exploit the advantages of increasing returns to scale.

5.3 Switching Equilibria

We now turn to a third kind of equilibrium. We will show that there is an equilibrium in which the world economy follows the asymmetric equilibrium for a *finite* length of time before switching to the symmetric equilibrium permanently. We refer to this case as a *switching equilibrium.*

The idea behind the proof of existence of a switching equilibrium is as follows. Suppose that the path for u' (time allocated to the production of physical capital in country 2) is given by

$$u'_t = \begin{cases} 1 & \text{for } t \in [0, T] \\ u_t & \text{for } t \in (T, \infty) \end{cases} \tag{71}$$

If such a path for u' is to be an equilibrium then the relative prices of human capital in the two countries, ψ and ψ' must be continuous at T. If this were not the case, since ψ and ψ' represent relative prices, there would be arbitrage opportunities along a perfect foresight path. To satisfy the transversality conditions, it must also be true that the system hits the stable manifold of the symmetric equilibrium at date T and that it remains on this manifold for all $t \geq T$. In Proposition 10 we established that, along the stable manifold of a symmetric equilibrium, the following two conditions must hold;

$$v' = sv, \tag{72}$$

$$\psi = \psi'. \tag{73}$$

Since neither ψ nor ψ' can jump at time T, v and v' cannot jump either. Hence, Equations (72) and (73) describe two terminal conditions that the dynamic path for the system between $t = 0$ and $t = T$ must satisfy.

Proposition 16 *Let* $\left\{\bar{\psi}, \bar{\psi}', \bar{v}, \bar{s}, \bar{u}\right\}$ *represent the steady state of the asymmetric equilibrium defined in Proposition 14. Let the system begin with the initial conditions* H_0, H_0' *and* W_0 *such that*

$$\frac{H_0^{1+\gamma}}{W_0} \in N\left(\frac{\bar{v}}{(1-\bar{v})\,\bar{\psi}}\right), \tag{74}$$

$$\frac{H_0'}{H_0} \in N(0). \tag{75}$$

Then there exists a continuum of switching equilibria, indexed by T*, characterized by the following system of equations. For* $t \in [0, T]$ *the system is determined by the differential equations:*

$$\dot{\psi} = \psi\left[f_k(\phi(\psi)) - \delta(1+\gamma) + \delta\gamma u\right], \tag{76}$$

$$\dot{\psi}' = \psi' f_k(\phi(\psi)) - \delta\psi, \tag{77}$$

$$\dot{v} = v\left[-u\delta + \rho\right], \tag{78}$$

$$\dot{s} = s(u-1)\,\delta(1+\gamma), \tag{79}$$

where u *is given by the expression*

$$u = u(\psi, \psi', v, s) \equiv \frac{\psi}{v\phi(\psi)} - \frac{\psi}{\phi(\psi)} - s\frac{\psi'}{\phi(\psi)} - s. \tag{80}$$

From dates $t \in [T, \infty]$ *the system is characterized by the equations:*

$$\frac{\dot{\psi}}{\psi} = f_k(\phi(\psi)) - \delta(1+\gamma) + \delta\gamma u, \tag{81}$$

$$\frac{\dot{v}}{v} = -u\delta + \rho, \tag{82}$$

$$\psi' = \psi, \tag{83}$$

$$s = s_T. \tag{84}$$

The variables k, k', c, c', v', ω and Z are governed by the equations

$$\begin{aligned} k &= \phi(\psi), && (85) \\ k' &= \phi(\psi), && (86) \\ c &= b\rho, && (87) \\ c' &= (1-b)\rho, && (88) \\ v' &= sv, && (89) \\ \omega &\equiv 1 - v - v' && (90) \\ \frac{\dot{Z}}{Z} &= f_k(\phi(\psi)) - \rho. && (91) \end{aligned}$$

Proof. We established in Proposition (15) that the stable manifold of the asymmetric steady state has dimension 2. Suppose $T = 0$; then the switching equilibrium is trivially equal to the symmetric equilibrium. Now consider an open interval $T \in (0, \bar{T})$, $\psi_T = \psi'_T$ and choose ψ_T and s_T such that $\{\psi_T, v_T,\}$ is on the stable manifold of the symmetric equilibrium. There is a unique orbit of the differential equation system (76)-(79) that passes through the point $\{\psi_T, \psi_T, v_T, s_T\}$. Let $\{\psi_0, \psi'_0\}$ be chosen such that this orbit passes through the point

$$\begin{aligned} 1 - v_0\left(1 + \frac{\psi'_0 s_0}{\psi_0}\right) &= \frac{W_0}{W_0 + \left(\psi_0 H_0^{1+\gamma} + \psi'_0 H_0'^{1+\gamma}\right)}, \\ \left(\frac{H'_0}{H_0}\right)^{1+\gamma} &= s_0. \end{aligned}$$

It remains to establish that $\psi' < \psi$ for $t \in (0, T]$. Let $p = \psi/\psi'$ and note that

$$\frac{\dot{p}}{p} = \delta(p-1) + \delta\gamma(u-1). \tag{92}$$

Suppose, by contradiction, that $\psi'_0 > \psi$ so that $p_0 < 1$. Then Equation (92) implies that p is falling for all t and p will never equal 1. But if $p > 1$ but $|\gamma(u-1)| > |(p-1)|$ then the second term in Equation (92) will dominate the first, p will be falling over time and there will exist a T such that $p = 1$. ■

Remark 8 *The only requirement that has been imposed on a switching equilibrium is that the switch must happen in finite time. By continuity, if there* *is an equilibrium for a given T, then there is a continuum of nearby switch dates all of which are equilibria as well. This result is similar to the indeterminacy of production locations in the case of the model with no externalities but it differs in one important respect. Since the alternative equilibria in the case of externalities entail different paths for the world production of physical commodities they can be Pareto ranked.*

6 Conclusion

Most existing work on international growth adopts the assumption that the world consists of a collection of closed economies each of which is modeled by the Solow-Swan model [11], [12]. Mankiw, Romer and Weil [7] have argued that when this model is modified to provide a broad definition of capital that it can account for most of the stylized facts. In contrast, the endogenous growth model pioneered by Robert Lucas [6] and Paul Romer [9] [10] has recently fallen from grace.

A key criticism of the endogenous growth model is that it predicts a high correlation between the investment rate and the growth rate. Jones [3] tested this prediction for the OECD countries using post World War II data and concluded that there was no evidence to support it. In contrast, McGrattan [8], in a much longer time series that starts in 1870, finds a strong positive relationship between rates of investment and growth rates for the OECD countries. She finds that the same positive correlation also emerges in the post-war data in a much larger sample of countries. Hence, the evidence is mixed but we think undecided.

A second argument, often levelled against endogenous growth models, is that, controlling for a range of social, political and economic variables, rich countries grow more slowly than poor countries. This phenomenon, known as conditional convergence, can be explained by the Solow model but not by simple versions of endogenous growth models. But Ventura [13] has established that a broader class of endogenous growth models are consistent with conditional convergence and our own previous work (Farmer and Lahiri [2]) establishes a similar result in a two-sector model quite similar to the one presented in this paper. In our current work we have made the case that a relatively standard two-country endogenous growth model, with mild externalities, can display multiple equilibria. Crucially, the equilibrium multiplicity in our model occurs only when capital is internationally mobile. We believe that an endogenous multiplicity of this kind has a number of attractive features among which is the possibility that one may account for the phenomenon of take-off in which a previously stagnant economy begins to grow. We hope in a small way to have advanced the argument that endogenous growth models deserve a closer look.

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