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## TIME-CONSISTENT PROTECTION WITH LEARNING BY DOING

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# TIME-CONSISTENT PROTECTION WITH LEARNING BY DOING

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# ABSTRACT

## Time-Consistent Protection with Learning by Doing*

Can a government induce efficiency gains in domestic industry by protecting it against foreign competition? Would such trade protection be time-consistent? The present Paper builds a dynamic equilibrium model that accounts for learning-by-doing effects that link firms' strategies over time. The model shows that the existence of dynamic economies of scale suffices to overcome the traditional government's lack of commitment to its tariff policy. This Paper compares the infinite horizon Markov perfect equilibria of this game with the dynamic equilibrium under commitment as well as the static Nash equilibrium. Equilibrium strategies are derived in closed form by solving a linear-quadratic differential game. Optimal trade policy involves higher tariff levels than in the static set-up in order to account for future gains in efficiency.

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# NON-TECHNICAL SUMMARY

Infant-industry arguments have long been used to justify protectionist trade policies. The essence of such arguments is that local producers must be allowed some time to overcome a temporary disadvantage with respect to foreign competitors. This disadvantage might come from technological backwardness, lack of access to efficient credit markets, local scarcity of the required human capital, or lack of an established reputation.

Economic theory has traditionally rejected the feasibility of successfully implementing such protection policy even using partial equilibrium analysis. The reason lies in the time inconsistency of such policy. Neither the government nor the local monopolist has the ability to commit to some pricing (or production) and tariff schedule that ensures that protection will only be temporal. The existing models rely on this lack of commitment to conclude that extended protection will arise in the end, and thus, infant-industry arguments should not justify even temporal protection. This result is obtained only because of arbitrary, although common, modelling choices: finite horizon and absence of dynamic linkages between the control and state variables of the game.

This Paper presents a framework where we can identify a time-consistent tariff protection policy and where, most importantly, neither the government nor the monopolist needs any external source of commitment because these pricing-tariff strategies are dynamic best responses to each others' strategy. In order to characterize such equilibrium, I use a linear-quadratic differential game model that enables me to compute the infinite horizon Markov perfect equilibria in linear strategies.

Consumers consider domestic and imported goods imperfect substitutes. The monopolist decides at each time the price of their domestic production to maximize the present value of their profits. They face a linear demand and instantaneous constant returns to scale although marginal cost will be lower in later periods due to learning-by-doing effects. In choosing the optimal tariff, the government maximizes the discounted value of social welfare. Foreign industry is assumed to have exhausted any potential dynamic economies of scale.

The model shows that learning-by-doing effects may effectively induce overproduction in the early stages of development of the industry as a consequence of each player optimal dynamic best response to each other's strategy. The model also shows that this early overproduction is smaller than the achievable level under commitment.

# 1    Introduction

Infant industry arguments have long been used to justify protectionist trade policies. The essence of such arguments is that local producers must be allowed some time to overcome a temporary disadvantage with respect to foreign competitors. This disadvantage might come from technological backwardness, lack of access to efficient credit markets, local scarcity of the required human capital, or lack of an established reputation. However, the existence of temporary disadvantages is not sufficient to justify policy intervention. For this, two additional conditions must be satisfied.

First, overcoming the initial handicap must be socially beneficial although not necessarily privately profitable (at least in the short run). This requires some sort of future positive externality to compensate the current welfare loss associated to any protection policies. One possible externality is due to the existence of dynamic economies of scale in the industry. Both Corden (1974, §9) and Krugman (1984) acknowledge that in the presence of learning by doing, a future cost advantage may justify temporary government protection. Another type of externality is associated to the existence of experience goods in consumption. Governments could create temporary trade barriers against imports with the argument of defense of diversity through the promotion of the local variety. Thus, temporary protection would generate dynamic efficiency gains through increases in production and/or improved management methods, but also in developing a biased taste for domestically produced goods.

Second, policy intervention must be effective. At least two problems may arise here. One difficulty is that protection policies designed to help domestic producers to become internationally competitive may lead to socially costly collusion between foreign and domestic firms [Gruenspecht, (1988)], or among domestic infant–firms in a protective environment. The second difficulty is that protection should only be granted for the shortest period possible required to make domestic firms competitive. In other words, the government must be able to credibly commit to liberalizing trade within a reasonable period of time. Unfortunately, governments can rarely commit credibly to trade policies for more than short periods of time: laws can be changed, treaties can be broken (*e.g.*, Kyoto agreement on emission reduction or the ABM Treaty), and government turnover might be high. In the absence of such "exogenous" commitment power, infant industry protection can easily become permanent. As pointed out by Matsuyama (1990) this lack of commitment of governments to liberalize trade explains the persistence of tariff protection. Given governments' lack of commitment (political, institutional, or due to lack of reputation), local firms prefer not to become internationally competitive, and given that strategic choice, the best policy for the government is to extend trade protection for some additional period.

This paper presents a framework where we can identify a time–consistent tariff protection policy that really incorporates the dynamic issues surrounding infant–industries. In doing so I am addressing some common shortcomings in the current treatment of the topic in the trade literature. The basic elements of the model are the following.

*1. Tariff Protection Policy:* While subsidies or quotas may achieve the same goal of protecting an infant–industry, I choose to study the case of tariff protection policy because this is the instrument most frequently used to protect industries in the early stages of development. Tariff protection was already vindicated by some classic economists, such as List in the 19th Century, as an effective tool to reduce the gap between less developed and industrialized countries.[1]

An additional reason for addressing the case of tariff protection policy is that the model shows how its effectiveness relies on the existence of a taste for variety, which is not required in the case of a subsidy. Thus, protection facilitates local production by introducing a temporary price distortion against imports. An optimally designed tariff should balance the inter–temporal substitution effects of these differentiated goods, as defined by the preferences of domestic consumers, to the future gains of productivity by domestic firms. It might happen that a strongly biased preference for foreign goods turns socially inefficient any kind of protection unless learning takes place almost instantaneously. The paper also shows that accounting for the welfare effects induced by the consumption of a differentiated foreign good is used by the government to grant some market power to the protected domestic monopolist to induce faster learning and depart from the static optimal protection policy when learning effects are present.

*2. Dynamic Economies of Scale:* Infant–industries, as envisioned by Protectionists, will benefit from protection merely by having the possibility to produce. Learning by doing is the only source of marginal cost reductions. Additional investments, while possible, are not critical elements of the model. Learning by doing shifts our attention to a framework with truly dynamic strategies where payoffs of different periods are dependent on previous pricing or production decisions of the domestic firms through the dynamic linkage of its induced marginal cost reductions.

The economic literature has focused frequently in the case where dynamic linkages are not present and thus, investment decisions are independent of the state of the game. For instance in Matsuyama's (1990) model, the domestic firm and the government play a repeated bargaining game where the firm asks for temporary protection in order to develop a cost reducing investment, while the government wants to liberalize trade to maximize welfare. A period later, if the government chose to protect and the firm did not invest, the game remains identical to the one played one period before. However, if learning by doing is considered, there exists at least one state variable, *e.g.*, the level accumulated output, experience, or just the marginal cost, that differs from the previous period due to production. The critical feature of Matsuyama's (1990) model, also present in the work of Miyagiwa and Ohno (1995) and Tornell (1991), is the absence of a dynamic linkage between firms' decisions over time, so that the optimal protection policy becomes time–independent.

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[1] The commonly intended superiority of subsidies over tariffs can only be explained because the shadow cost of rising funds in the rest of the economy is unreasonably assumed to be zero.

By explicitly considering dynamic economies of scale the present approach looses the state–independence of Matsuyama's (1990) model. I therefore require that strategies of the government and the domestic infant–monopolist be dependent on the state of the game. I will therefore restrict my attention to Markov strategies. Focusing in the class of linear–quadratic differential games, the present approach allows me to explicitly characterize an equilibrium where the government's tariff and the monopolist's production schedule are dynamic best responses to each other.

_3. Time Consistency:_ A common criticism to protection policy is the lack of credibility of the government to ending such policy. The fact that investment and tariff protection decisions are independent of the state of the game is the origin of the government's lack of commitment to future liberalization in Matsuyama's (1990) model. Another example of this lack of commitment to future liberalization is Tornell's (1991) model of _'investment–contingent'_ tariffs. In that model, when players reach the supposedly last period of the game, they suddenly become exogenously endowed with asymmetric bargaining power that makes the government to reconsider its liberalization policy, thus extending protection _'beyond the horizon'_ of the game. Two elements of Tornell's (1991) model are worth mentioning here. First, the government's strategy does not fulfill Markov perfection because at the end of the game protection level is not really determined by past investments of firms whose capital accumulation represents the state of the game, but rather by the firm's threat of increased unemployment. A proper Markov strategy should had accounted for this possibility when computing the investment–contingent protection schedule. Second, the game appears to continue beyond the initial two–period horizon that is so unfortunately common in the trade literature whenever time consistency is questioned. In Tornell's (1991) model, firms actually care for protection beyond the formal finite horizon of the game because their expected payoffs are not restricted to these two periods. Similarly, the government has to care about the future rents of domestic producers when, otherwise it would be optimal from a static point of view to liberalize trade in the supposedly last period of the game.

In order to address the time consistency of the tariff I will focus on Markov strategies defined for infinite horizon games where the possibility of extending protection disappears. This modeling approach may need some justification since it is not common in the trade literature. First, Markov perfection requires that the effect of past actions can be summarized by the state of the game along the equilibrium path. In the present model, the state is represented by the level of marginal cost of the domestic monopolist. Thus, both the monopolist's production schedule and the government's tariff are contingent on this level of marginal cost, whose change leads to a dynamic linkage of strategies over time. Second, players' continuation payoffs at the end of the game should have no effect on the equilibrium tariffs. Many models addressing the time–consistency of protection policies assume a two–period game, although players appear to be allowed to have expectations about the world after the end of the second period. This is the case not only of Tornell (1991) but also of Leahy and Neary (1999) and Miyagiwa and Ohno (1999). In all these models the choice of the horizon length is exogenous and serves the role of simplifying the

computation of equilibria. Any subgame perfect equilibria of a finite horizon game is then time consistent by definition. Restricting the attention to Markov strategies, any Markov Perfect Equilibria (MPE) of a finite $T$–period game will also be time consistent according to this common view.

I however do not find this definition of time consistency very compelling for problems of tariff protection. A major issue in the game played between the government and firms is always whether firms may convince the government to extend protection beyond the horizon initially announced. Assigning some positive probability to such event implies that the computation of optimal strategies for a finite horizon of $T$–periods includes potential continuation payoffs beyond such time horizon $T$, thus making firms to behave strategically regarding her output or pricing decisions in order to induce an extension of protection that allows her to increase the present value of her profits.

To illustrate this point, consider the following example. Suppose that the government announces a temporary protection policy for the next ten years, *e.g.*, exempting the exclusive dealerships practices in the automobile industry from the European competition rules. If the government does not alter that policy for the whole announced period of protection, we cannot argue that such policy is not time–consistent according to the common definition used in game theoretical models. However, the possibility that protection could be extended may affect the equilibrium strategies of the firms during the protection phase: domestic firms may engage in lobbying instead of investing in order to induce an extension of protection. Perhaps the European car makers did not reduce capacity enough and did not innovate sufficiently during the ten years of protective regime just to convince the European Commission of extending such policy, what she did for another ten years in 1995, few months before the initial policy expired. There is no doubt that such extension of protection beyond the time horizon initially announced will be seen as reneging of the initial liberalization target, and thus the overall government tariff protection policy will not be considered time–consistent.

The model works as follows. Initially a domestic monopolist faces a very high marginal cost of production that make impossible for her to compete with cost efficient foreign firms that produce a slightly differentiated product. There are however important potential dynamic economies of scale due to learning by doing. The monopolist asks for temporary protection to allow her to reduce her marginal cost and be able to compete once trade is liberalized. The government faces a dynamic trade–off in granting such protection. Consumer surplus will be reduced today due to high cost of domestic production as well as for the high cost of imports induced by the tariff. However, consumer surplus could be enhanced in the future by ensuring the production of the differentiated domestic output and future welfare would be further increased by the profits of a cost efficient domestic industry. The domestic monopolist's pricing decisions over time and the government's schedule of tariffs are made contingent on the marginal cost of the monopolist that captures the state of the game. Strategies are then contingent on the performance of the monopolist through marginal cost reductions induced by pricing and tariff decisions. In equilibrium, both

strategies should be the dynamic optimal best response to each other player's strategy. As learning is exhausted, the state converges to its stationary level and price and tariff remain constant from that moment on.

In this framework, the government will credibly bring the level of protection over time up to the stationary level, which may indeed involve an optimal zero tariff, without any need for some exogenous commitment device. This strategy becomes its best dynamic response to the monopolist's also optimal pricing, or equivalently to her production decisions. Intuitively, this result relies on two features of learning by doing mechanisms. Firstly, the marginal social benefit of future learning decreases as the local firm "goes down its learning curve." Secondly, with learning by doing the domestic firm cannot reap the rents from protection without at the same time becoming progressively more efficient: rents are only obtained for positive levels of output once the domestic firm is competitive against foreign producers, but positive levels of output induce learning and greater efficiency. In other words, learning by doing ties "innovation" and "rent enjoyment" inexorably together.

The paper builds a dynamic equilibrium model of optimal tariff design when learning economies exist and strategies are state contingent. In order to show time–consistency of the equilibrium strategies, the model technical requirements are kept to a minimum. For simplicity, I use a linear demand specification and learning is assumed to reduce marginal costs. Formally, the model is related to the capital accumulation games of Driskill and McCafferty (1989) and Reynolds (1987), although in this case the game is not symmetric. In the present setup the accumulation of capital is translated into a reduction of the monopolist's marginal cost thanks to her own pricing decisions and the tariff decisions of the government. Dynamic considerations lead to domestic prices that are lower or higher than under the static Nash equilibrium depending on the ability of the players to commit to some pricing–tariff schedule. The optimal dynamic tariff is always more protective than myopic static tariff protection policies. The combined effect of higher domestic prices but lower tariff protection induces further cost reductions relative to the static equilibrium.

The paper is organized as follows. In section 2, the model and its assumptions are described. Section 3 solves the static Nash equilibrium that ignores any dynamic effects induced by current production decision. Section 4 incorporates such dynamic considerations in a particular framework where both the monopolist and the government can commit to the announced strategies and characterizes a Nash equilibrium in *open–loop* strategies. This section also discusses the validity of such commitment ability. Section 5 characterizes the Markov perfect equilibrium in *closed–loop* strategies that are dynamic optimal best responses so that some external source of commitment to such policy is not needed. Section 6 concludes.

# 2    Industry Protection with Dynamic Economies of Scale

The game consists of two players: a domestic monopolist and the government of a small country. The problem to be addressed is that of protecting an infant–industry. Initially, foreign firms are much more efficient but the domestic monopolist enjoys dynamic economies of scale. Therefore, the domestic monopolist asks for temporary protection so that today's sales induce cost reductions and thus allows her to compete in the future once protection is lifted. The government may consider temporary protection because today's reduction in consumer surplus due to high tariffs may get compensated with the increase in monopoly profits, but more importantly because this protection induces a more efficient future provision of the differentiated, domestically produced good.

The model is written in continuous time. The only state variable is the level of marginal cost. Technology is characterized with instantaneous constant returns to scale but marginal cost declines with output due to learning by doing. Current output (pricing) decisions incorporate an investment component until learning is exhausted. Successive output decisions reduce marginal costs. In the present model I also consider the possibility that experience depreciates over time so that some minimum production level is required at every period to ensure a net reduction of the monopolist's marginal cost.

Foreign firms are competitive and have exhausted learning effects.[2] Price is the monopolist's only control variable, and the government's control variable is the tariff level. The monopolist's objective in each period is to maximize her discounted profits. On the other hand, the government maximizes the discounted value of the social welfare, *i.e.*, the sum of consumer surplus, total profits, and tariff revenues. Both players take into account the dynamic effects induced by their price and tariff decisions through reductions of the monopolist's marginal cost.

To show that there may exist a time consistent tariff policy and that its existence does not require any external source of commitment, I restrict my attention to the case where both, the monopolist and the government use strategies that are contingent on the state of the game, *i.e.*, the level of marginal cost at each time. The optimal tariff and production schedule is found by characterizing the subgame perfect equilibria in an infinite horizon game where players are restricted to use Markov strategies. The model is solved for the infinite–horizon case to ensure that endpoint transversality conditions are not binding and rule out the possibility that neither the monopolist or the government may be tempted to consider any further extension of protection.

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[2] The assumption of a foreign industry that has exhausted learning is just an extreme case where foreign industries are obviously more developed. But this assumption excludes also the possibility of dynamic strategic effects of current decisions of the domestic firm over foreign firms and *vice versa*, which will surely make the characterization of the solution of the model impossible unless we rely on numerical methods.

## 2.1 Demand System

Assume that domestic and foreign products are considered imperfect substitutes for each other by domestic consumers. Let $X^t$ denote the domestic monopolist's production and let $M^t$ denote imports at time $t$. Assume a quadratic utility function with symmetric cross–effects that is additively separable with respect to money:

$$U[X^t, M^t] = a_x X^t + a_m M^t - \frac{1}{2} \left[ b_x (X^t)^2 + b_m (M^t)^2 + 2k X^t M^t \right], \tag{1}$$

where all parameters $a_x$, $a_m$, $b_x$, $b_m$, $k$, are strictly positive. The sufficient condition $\Delta = b_x b_m - k^2 > 0$ ensures that the utility function is strictly concave. At each time, $t$, consumers maximize $U[X^t, M^t]$ subject to the budget constraint:

$$I^t = Q_0^t + P^t X^t + (1 + \tau^t) M^t, \tag{2}$$

where $Q_0^t$ represents the aggregate consumption of a competitive *numeraire* good, and $\tau^t$ is the import tariff rate. Foreign firms are competitive and have exhausted learning economies, so that the price of foreign products remain constant at the foreign firms' marginal cost. Since the foreign price of imports only plays a residual role in the model, it has been normalized to one. Thus $P^t$ represents the price of domestic production relative to the price of foreign products before the tariff is applied. The solution of this problem can be written as follows:[3]

$$\begin{bmatrix} a_x - P \\ a_m - (1 + \tau) \end{bmatrix} = \begin{bmatrix} b_x & k \\ k & b_m \end{bmatrix} \begin{bmatrix} X \\ M \end{bmatrix}. \tag{3}$$

The solution of this system provides the demand for domestic and imported goods as functions of their prices and import tariff:

$$X(\tilde{P}) = \alpha - \beta P + \gamma(1 + \tau), \tag{4a}$$

$$M(\tilde{P}) = 1 + \gamma P - (1 + \tau) = \gamma P - \tau, \tag{4b}$$

where $\alpha = (a_x b_m - a_m k)/\Delta > 0$, $\beta = b_m/\Delta > 0$, $\gamma = k/\Delta > 0$, and where $\tilde{P} = \{P, \tau\}$ determines the vector of effective prices in the domestic country, once the import tariff has been added.[4] Finally, consumer surplus is given by:

$$CS(\tilde{P}) = U[X(\tilde{P}), M(\tilde{P})] - P X(\tilde{P}) - (1 + \tau) M(\tilde{P}). \tag{5}$$

Observe that demand does not induce any additional dynamic effect because of its stationary linear specification. Welfare gains from protection might be higher than those highlighted by this model if, in addition, consumption have dynamic effects on consumers' utility (experience goods), and/or if demand grows over time.

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[3] Time superscripts are dropped unless their omission may induce ambiguous interpretation.

[4] The intercept and own–price effect of the demand for imports have been normalized to one. In computing equilibria and comparative statics, the values of $\alpha$ and $\beta$ should be understood as relative magnitudes with respect to the demand for imported products. Technical details and most analytical developments of this and other sections are reported in the Appendix.

## 2.2 Marginal Cost Reduction

Technology exhibits instantaneous constant returns to scale but marginal cost, $c$, is reduced over time while the domestic firm accumulates output. The payoff relevant measure of experience is the level of marginal cost. The reduction in marginal cost is described as:

$$\dot{c} = -\lambda[X - \delta c]. \tag{6}$$

Parameter $\lambda \geq 0$ represents the marginal cost reduction effect per unit of output, while $\delta \geq 0$ captures the idea that the value of experience depreciates over time so that recent output decisions have a stronger effect on the current level of marginal cost than early ones.

## 2.3 The Monopolist's Problem

In an infinite horizon game, the monopolist's problem is to maximize the present value of her profits given the government's tariff, while considering the learning effects induced by current production through her pricing decisions. This problem can be stated as:

$$\max_{X} \int_{0}^{\infty} \pi(\tilde{P}, c)e^{-rt}dt \quad \text{s.t.} \quad \dot{c} = -\lambda[X - \delta c] \quad ; \quad c(0) = c^0. \tag{7a}$$

The instantaneous, constrained, monopoly profits are represented by the following Hamiltonian:

$$H^F = [\alpha - \beta P + \gamma(1 + \tau)](P - c - \lambda\mu_f) + \lambda\delta c\mu_f. \tag{7b}$$

## 2.4 The Government's Problem

The government's problem is to maximize the present value of the sum of consumer surplus, monopoly profits, and tariff revenues, given the monopolist's pricing strategy and considering the effects induced by the tariff policy:

$$\max_{\tau} \int_{0}^{\infty} \left[CS(\tilde{P}) + \pi(\tilde{P}, c) + R(\tilde{P})\right] e^{-rt}dt \quad \text{s.t.} \quad \dot{c} = -\lambda[X - \delta c] \quad ; \quad c(0) = c^0, \tag{8a}$$

so that the Hamiltonian associated to the welfare function becomes (see Appendix):

$$H^G = \left[\frac{\alpha + \gamma}{\beta - \gamma^2} - c - \lambda\mu_g\right][\alpha - \beta P + \gamma(1 + \tau)] + \gamma\frac{\alpha + \gamma}{\beta - \gamma^2}[\gamma P - \tau] + \lambda\delta c\mu_g$$

$$- \frac{[\alpha - \beta P + \gamma(1 + \tau)]^2}{2(\beta - \gamma^2)} - \frac{\beta[\gamma P - \tau]^2}{2(\beta - \gamma^2)} - \frac{\gamma[\alpha - \beta P + \gamma(1 + \tau)][\gamma P - \tau]}{\beta - \gamma^2}, \tag{8b}$$

where I made use of the tariff revenue function:

$$R(\tilde{P}) = \tau[\gamma P - \tau]. \tag{9}$$

## 2.5    The Game

The government has to choose the optimal tariff $\hat{\tau}$ that induces marginal cost reductions through increasing of production of the domestic monopolist that suffices to compensate for the reduction in consumer surplus due to the higher prices paid for imports. Although there are several concepts of equilibrium that I will explore in later sections of the paper, the basic idea is that the tariff has to be the dynamic best response to the firm's pricing strategy and *vice versa*. The optimal level of protection will be conditioned by the potential cost reduction that can be achieved with an additional unit of domestic output. Obviously this is determined by the level of marginal cost $c$, which represents the state of the game. Similarly, the firm has to choose the optimal pricing strategy $\hat{P}$ that maximizes the present value of her profits while accounting for the dynamic effects induced by the reduction of her costs as well as the government's tariff policy. Again, the optimal pricing strategy will depend on the level of marginal cost. Consequently, the level of marginal cost in later periods will be determined by the pricing and tariff strategies applied by the monopolist and government respectively.

In games like this one, the state follows a Markov process in the sense that the state of the next period is a function of the current state and actions, and hence, the history at $t$ can be summarized by $c^t$. To solve this model, I assume perfect information, which implies that both the government and the monopolist know the history of the game, *i.e.*, the previous realizations of the state, $c^s$, and the vector of control variables, $\{P^s, \tau^s\}$, $\forall s \leq t$. Markov strategies depend only on the state of the system, and players' information sets include only the payoff–relevant history [Maskin and Tirole (1994)]. Markov perfection requires that these strategies are perfect equilibria for any time and state [Fudenberg and Tirole (1986, §2b)]. A differential game equilibrium of this model is a set of functions $\{\hat{P}(c), \hat{\tau}(c)\}$ such that for any time and state, a player's strategy maximizes its payoff from that time on. Applying dynamic programming, any differential game equilibrium solves a set of generalized Hamilton–Jacobi conditions, *i.e.*, a system of partial differential equations that are first order conditions of the corresponding Hamiltonian for each player. Thus, for the present model, the optimality conditions for the monopolist are:[5]

$$H_P^F = \alpha - \beta(2P - c - \lambda\mu_f) + \gamma(1 + \tau) = 0, \tag{10a}$$

$$\dot{\mu}_f = (r - \lambda\delta)\mu_f + [\alpha - \beta P + \gamma(1 + \tau)] - \gamma[P - c - \lambda\mu_f]\hat{\tau}_c. \tag{10b}$$

Similarly, the optimality conditions for the government are:

$$H_\tau^G = \gamma(P - c - \lambda\mu_g) - \tau = 0, \tag{11a}$$

$$\dot{\mu}_g = (r - \lambda\delta)\mu_g + [\alpha - \beta P + \gamma(1 + \tau)] + [\gamma\tau - \beta(P + c + \lambda\mu_g)]\hat{P}_c. \tag{11b}$$

It is useful to write down the dynamic optimality conditions of this system. In order to obtain them, substitute $\mu_f$ and $\mu_g$ in (10b) and (11b) from (10a) and (11a) respectively.

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[5]  See for instance Fudenberg and Tirole (1991, §13.3.2) or Vives (1999, §9.2.3).

Differentiate (10a) and (11a) making use of (6) to obtain $\dot{\mu}_f$ and $\dot{\mu}_g$ and substitute the expressions for these time derivatives of the co–state variables into (10b) and (11b). After some algebra we get the following characterization of the dynamic trajectories of $P$ and $\tau$ for the general case:

$$
\begin{bmatrix} (\alpha + \gamma)(r - \lambda\delta + \gamma\lambda\hat{\tau}_c) + (r - 2\lambda\delta)\beta c \\ \\ (r - 2\lambda\delta)\gamma c \end{bmatrix} = \begin{bmatrix} 2\beta & -\gamma \\ \\ \gamma & -1 \end{bmatrix} \begin{bmatrix} (r - \lambda\delta)P - \dot{P} \\ \\ (r - \lambda\delta)\tau - \dot{\tau} \end{bmatrix}
$$
$$
+ \lambda \begin{bmatrix} \hat{\tau}_c & 0 \\ \\ 0 & \hat{P}_c \end{bmatrix} \begin{bmatrix} \beta\gamma & -\gamma^2 \\ \\ -2\beta\gamma & \beta + \gamma^2 \end{bmatrix} \begin{bmatrix} P \\ \\ \tau \end{bmatrix}. \quad (12)
$$

Obviously, this system of partial differential equations is not easily solved in closed form except for the case of some particular functional specifications such as the linear–quadratic case of the present model. In addition, in dynamic models like this one, the nature of the solution critically depends on the information set that players use. The following sections present a detailed analysis of three alternative equilibrium strategies: static, dynamic with commitment, and dynamic without commitment. This step–by–step approach is useful to isolate the effect of learning as well as to address the issue of commitment and credibility of the announced tariff protection policy.

## 3    Optimal Static Protection

Suppose that the government and the domestic monopolist play a one–shot game. They will choose the tariff and the price of the domestic product that maximizes total welfare and profits respectively. Dynamic considerations about marginal cost are therefore not relevant since the game will not be played in any later period. Co-state variables do not play any role because the current strategies of the players have do not affect any future payoffs, so that $\mu_f = \dot{\mu}_f = \mu_g = \dot{\mu}_g = 0$. Thus, the static Nash equilibrium (SNE) of this game is found by solving equations (10a) and (11a) while ignoring $\mu_f$ and $\mu_g$. Alternatively, solve (12) assuming $\dot{P} = \dot{\tau} = \hat{P}_c = \hat{\tau}_c = \lambda = 0$.

PROPOSITION 1: *The SNE strategies are:*

$$
P^N = \frac{\alpha + \gamma + (\beta - \gamma^2)c}{2\beta - \gamma^2}, \quad (13a)
$$

$$
\tau^N = \frac{\gamma[\alpha + \gamma - \beta c]}{2\beta - \gamma^2}. \quad (13b)
$$

Concavity of consumers' utility function ensures that $P^N > 0$. The SNE price increases with $\alpha$, $\gamma$, and $c$, *i.e.*, with positive non–price related shifts of demand, the

domestic firm's marginal cost, and the degree of substitution between the domestic and foreign product. A larger $\gamma$ means that domestic and foreign goods are more substitutable and thus, the domestic monopolist is able to charge a higher markup in equilibrium for any given price of the foreign produced good.

The SNE tariff increases with $\alpha$ but decreases with $\beta$ and $c$. However, and contrary to the equilibrium price, the SNE tariff is not ensured to be positive. Large marginal cost $c$ may trigger an optimal negative tariff in equilibrium. In this SNE, learning effects are not considered, and thus, the optimal response to a prohibitively expensive domestic production is to subsidize imports of the foreign good. An elastic domestic demand or a biased taste for the domestic good may also turn a subsidy to be the optimal policy. This is the case with large value of $\beta$ relative to $\alpha + \gamma$. Similarly, if $\gamma$ is low enough, the optimal static tariff may actually be a subsidy. For low values of $\gamma$, consumers would increase their utility much more from the consumption of one unit of the domestic product than from purchasing an additional unit of the foreign one. The role of the subsidy is to reduce the excessive market power of the domestic monopolist by giving incentives to purchase imports. Subsidies are however less likely to happen when the marginal cost of the domestic monopolist is very low. This SNE ignores, by definition, all dynamic effects that link current production to future cost reductions. The solution presented here will be useful to compare the equilibrium impact of these dynamic effects for different informational assumptions.

# 4    Committing to Protect and Produce

In this section we turn our attention to dynamic strategies. I will focus on the equilibrium features that arise exclusively by considering own–induced dynamic effects. In particular, I will focus in situations in which *both* the government and the domestic monopolist have the ability to commit to some strategy announced at the initial state of the game, leading to an *open–loop* equilibrium (OLE). This section characterizes such equilibrium and compares it to the SNE. I also discuss the validity of this commitment assumption.

An OLE is a Nash equilibrium in *open–loop* strategies. In practice this means that at $t = 0$ the firm announces a schedule of prices, $\{P^s\}_{s=0}^{s=\infty}$, while the government announces a schedule of tariffs, $\{\tau^s\}_{s=0}^{s=\infty}$ for the infinite horizon of the game. The game is only dynamic because it involves price and tariff decisions over the whole horizon of the game, but truly dynamic interactions are omitted because both schedules are one–shot decisions announced at the beginning of the game. The concept of OLE assumes that both firm and government can commit to carry the announced strategies regardless of how the game evolves in the future. This is in fact the source of most time–consistency problems that characterizes many dynamic economic models.[6]

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[6] In addition to the trade literature discussed in the Introduction, see for instance Miyagiwa and Ohno (1999), Pearce and Stacchetti (1997), and Staiger and Tabellini (1987).

Since the game involves several periods, players will only account for the effect of their own control strategies on the state of the game. Thus, co–state variables $\mu_f$ and $\mu_g$ now appear in the objective function of the firm and the government respectively. Since they account for payoff consequences of changes in the state, the effect of $H_c^F$ and $H_c^G$ on the co-state variables $\mu_f$ and $\mu_g$ are also considered. But because the announcement is made at the beginning of the game, players cannot take into account any feedback effect of the decisions of their opponents. Thus, the stationary OLE solves (12), although making $\dot{P} = \dot{\tau} = \hat{P}_c = \hat{\tau}_c = 0$.

PROPOSITION 2: *The stationary OLE strategies are:*

$$P^\circ = \frac{(\alpha + \gamma)(r - \lambda\delta) + (r - 2\lambda\delta)(\beta - \gamma^2)c}{(r - \lambda\delta)(2\beta - \gamma^2)}, \tag{14a}$$

$$\tau^\circ = \frac{\gamma[(\alpha + \gamma)(r - \lambda\delta) - (r - 2\lambda\delta)\beta c]}{(r - \lambda\delta)(2\beta - \gamma^2)}. \tag{14b}$$

PROPOSITION 3: *The stationary OLE is globally stable if:*

$$r < \lambda\delta. \tag{15}$$

Observe that one implication of this stability condition is that $P^\circ$ is decreasing in $c$ while $P^N$ increased with $c$. Similarly, the derivative of $\tau^\circ$ with respect to $c$ becomes positive while any increase in $c$ reduced $\tau^N$.

PROPOSITION 4: *The stationary OLE coincides with the SNE whenever $\lambda = 0$, $\delta = 0$, or $c = 0$.*

If $c = 0$ no further cost reduction is possible, and the game becomes necessarily static. When $\lambda = 0$ there is no dynamic linkage between current production and marginal cost reduction, and thus the model becomes static. When $\delta = 0$ experience does not depreciates over time. Therefore, it is not necessary to produce any output just to maintain the same level of marginal cost of the previous period. The combined effect of these two dynamic elements leads to the following result:

PROPOSITION 5: *In the stationary OLE, the equilibrium tariff exceeds the SNE price, but the equilibrium price is lower than in the SNE case:*

$$P^\circ - P^N = \frac{\lambda\delta(\beta - \gamma^2)c}{(r - \lambda\delta)(2\beta - \gamma^2)} < 0, \tag{16a}$$

$$\tau^\circ - \tau^N = \frac{-\gamma\lambda\delta\beta c}{(r - \lambda\delta)(2\beta - \gamma^2)} > 0. \tag{16b}$$

Therefore, the monopolist has a stronger dynamic incentive to underprice (overproduce) relative to the static setting the less substitutable domestic product and imports are,

$\gamma \to \sqrt{\beta}$. This optimal dynamic strategy increases the present value of the marginal revenue of the monopolist while minimizing the investment in marginal cost reduction.

Since the monopolist does not internalize the reduction in consumer surplus due to the high price of the domestic product, the optimal strategy for the government involves a higher tariff than in the static case. By being more protective, the government ensures some additional market power to the domestic monopolist to induce further cost reductions that later compensate the current reduction in consumer surplus due to high prices of the domestic product. Observe that the increase in the optimal OLE relative to the SNE tariff is more important the higher the cost of the domestic monopolist $c$, the more elastic is the demand for the domestic product (larger $\beta$), or the less substitutable domestic and foreign goods are (larger $\gamma$).

The combined effect of these equilibrium pricing and tariff strategies is to lower the stationary equilibrium level of the monopolist's marginal cost relative to the SNE. This is a direct consequence of the monopolist and the government accounting for the future savings induced by their pricing and tariff decisions.

PROPOSITION 6: *In the stable stationary OLE, the domestic monopolist's marginal cost is positive but below the level in the SNE:*

$$0 < c^\circ = \frac{(\alpha + \gamma)(r - \lambda\delta)\beta}{(r - \lambda\delta)(2\beta - \gamma^2)\delta + (r - 2\lambda\delta)\beta^2} < c^N = \frac{(\alpha + \gamma)\beta}{(2\beta - \gamma^2)\delta + \beta^2}. \tag{17}$$

Both $c^N$ and $c^\circ$ are increasing in $\alpha$ and $\gamma$ but decreasing in $\beta$ and $\delta$. Observe however that the difference among them disappears when $\delta = 0$ even if learning effects are significant, *i.e.*, $\lambda > 0$. This result is important because if we ignore the depreciation of experience, dynamic models would lead to exactly the same steady state level of marginal cost in the long run. However, the more important is the effect of the depreciation of experience, the lower the stationary equilibrium level of the marginal cost would be relative to the static equilibrium.

## 4.1  Is Commitment Credible?

Frequently, models of trade protection find that governments have no dynamic incentives to reduce protection. These models, *e.g.*, Miyagiwa and Ohno (1995), only address the case of homogeneous products. The present model replicates these results for the case of independent goods, $\gamma = 0$, and when the stationary *open–loop* tariff coincides with the optimal static one. This result is important because the present model also shows that if domestic and foreign products are slightly differentiated, then governments will more likely consider dynamic effects and increase tariff protection relative to the static case to successfully induce marginal cost reductions by the domestic producer. This is a consideration that has so far been ignored in the trade literature.

The comparison between OLE and SNE of equations $(16a) - (16b)$ also illustrates a common criticism to protection policies. The domestic monopolist is now more aggressive than in the static framework (unless domestic and foreign products behave as perfect substitutes), charging lower prices and speeding up cost reduction. The government's optimal tariff policy is now more protective than in the static case. While the monopolist's optimal pricing is decreasing in $c$, the optimal OLE tariff increases with $c$. The government has to make imports less attractive the higher is the initial disadvantage of the domestic firm. This policy might be very damaging to consumers who bear the cost of protection by facing higher domestic prices of imports as well as higher prices of domestically produced goods until significant learning has been realized. But this policy also ensures that only the minimum additional market power is grantd to the domestic monopolist to effectively induce cost reductions while minimizing the combined loss of consumer surplus due to high prices of domestic good and imports, while allowing the monopolist to keep producing, and therefore further reducing her marginal cost.

Will this protection policy be actually enforced? Most likely not. Both the monopolist and the government receive feedback information as the game evolves. They do not use this information because of the way *open–loop* strategies are defined. But it is not reasonable that they do not make use of such information if in a later state of the game is in their own interest to do so. For instance, the government may find that the monopolist is reducing her cost very slowly. A low tariff or an early liberalization of trade may put in danger the survival of the domestic production and jeopardize the whole intention of the protection policy. Thus, the government will most likely deviate from the announced tariff schedule either by increasing the import tariff or postponing trade liberalization. Similarly, the monopolist may find that the government's tariff is lower than expected and that it does not ensure her survival in the long run. The monopolist then deviates from her announced strategy by increasing her price to maximize the rent extraction while protection lasts.[7]

According to Reynolds (1987, §3), there are two main reasons why we can consider *open–loop* strategies. First, *open–loop* strategies are reasonable when players are unable to observe the state affecting decisions of the other players after the beginning of the game. This is not the case in the present model. The monopolist always observes her marginal cost and the government, even when he cannot observe it directly, can infer it with certainty from the monopolist's price and the known parameters of demand and marginal cost dynamics.

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[7] These are the typical cases commonly documented in the theoretical and trade policy literature. However, they are not the only possible deviations from an OLE. At least in theory, it is also possible that the government discovers that his tariff is too high, thus reducing consumer surplus too much while allowing the domestic monopolist an excessive market power. The government may then find optimal to modify his tariff policy in order to reduce the inefficiency of the announced tariff. Similarly, the monopolist may find that the announced tariff is so high that it shifts demand for the domestic good beyond her expectations. In such a case, the monopolist will also find optimal to deviate from the announced pricing schedule to lower her marginal cost and thus increase her profit margin as much and as long as possible. The lack of commitment is also present in these two latter cases.

The second reason given by Reynolds is that the rules of the game may require precommitment to *open–loop* strategies. This is the case of trade agreements, multilateral agreements through the WTO, and even the passing of constitutional amendments. All these mechanisms are ways to signal that the government is going to commit to certain trade policy, perhaps leading to trade liberalization in the future. But, as pointed out by the examples discussed before, *open–loop* strategies are not dynamic best responses to each other player's strategy because they were decided at the beginning of the game and neglect, by definition, any potential feedback effect, *i.e.*, any payoff relevant action of the other player that may have an unexpected effect on the evolution of the game.

If the difference between the expected and actual payoffs compensates potentially important transaction or reputation costs, the government will break any trade agreement or modify any necessary constitutional amendments, thus easily turning tariff protection permanent. This is the result systematically reported in the trade literature using time–independent strategies in models without dynamic linkage of payoffs over time, *e.g.*, Miyagiwa and Ohno (1995) and Tornell (1991). This section has shown that using state–contingent strategies that account *only* for each player's own–induced dynamic effects is not sufficient to avoid time–consistency problems. The next section studies how to characterize a tariff protection policy that is robust to the time–inconsistency criticism.

# 5    Optimal Dynamic Protection

In this section I address the solution of the model when both players take into account not only the dynamic effects induced by their own strategies, but also recognize the dynamic effects induced by the other player's strategy. Thus, $\hat{P}_c$ and $\hat{\tau}$ in equation (12) are not assumed zero anymore. Now the monopolist's pricing and the government's tariff strategies are going to be made explicit functions of the state of the game $c$, *i.e.*, we will use Markov strategies.

Contrary to the OLE solution of the previous section, the sequence of prices and tariffs that characterize the *closed–loop* solution cannot be announced at the initial stage of the game. Neither the government or the monopolist need to commit to a predetermined sequence of actions, as they will optimally choose their strategies at each state of the game. An MPE is just a subgame perfect equilibrium in Markov strategies. Thus, players take into account each other actions and the equilibrium strategies have to be optimal from any time $t$ onwards. Therefore, the equilibrium strategies are by definition time–consistent.

Solving for MPE is analytically much more involved than characterizing SNE or OLE strategies. Since the model is not symmetric, solving for MPE requires to find several intersections among two hyperbolas. Similarly, the stability condition is also another hyperbola defined in the space of parameters of the model. There are two approaches to deal with such difficult problem: i) Solve the model numerically, find the multiple solutions

and identify the (hopefully) unique stable equilibrium; ii) Analyze how does the solution depends on the parameters of the model.

The first approach is valid if we want to evaluate the model for a given set of parameters, perhaps corresponding to a particular calibration or estimation of demands, costs, and learning effect parameters. This approach however lacks generality and although several sensitivity exercises may confirm the robustness of the features of the found MPE, it is not easy to identify the effect of each parameter on the stability and/or uniqueness of the solution because both result from nonlinear interactions among all parameters.

I will follow the second approach in order to study, among other issues, that an MPE exists for more than a single combination of parameters, thus making the existence and uniqueness result of more general interest. The drawback of this approach is that the resulting conditions are only valid "asymptotically." The nonlinear relations among parameters characterized by the hyperbolas that define the equilibrium and stability conditions are replaced by their corresponding asymptotes, thus characterizing the equilibria through sufficient conditions that depend explicitly on the parameters of the model.

In what follows, I describe how are the relevant parameters computed. I do not characterize every single parameter of the model, but only those who drive the dynamic features of the model, *i.e.*, , $\hat{P}_c = \phi_2^f$ and $\hat{\tau}_c = \phi_2^g$ in equation (12).

## 5.1    Computing Linear Markov Perfect Equilibria

I now solve for MPE. Markov strategies are state dependent and therefore embody the idea of a protection policy that is contingent on the industry's performance. The equilibrium is thus one of simultaneous moves of both players at each time and state.[8] In equilibrium, the government's optimal tariff policy is the optimal dynamic best response to the monopolist's pricing strategy and *vice versa*. Let $P^\star(c) \in S^F$ denote the monopolist's Markov pricing strategy out of the set of all possible state contingent pricing strategies $S^F$. Similarly, $\tau^\star(c) \in S^G$ is the government's Markov tariff strategy out of the set of all possible state contingent tariff strategies. The MPE is a pair of strategies $\{P^\star(c), \tau^\star(c)\} \in S^F \times S^G$ that maximizes each player's value function for any state $c$ given the equilibrium Markov strategy of the other player. The value functions of this game for the monopolist and the government are:

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[8] I do not consider the possible role of the government as Stackelberg leader, neither with total or instantaneous pre–commitment as defined and studied by Cohen and Michel (1988). If the government alone decides at the beginning of the game to set some course of action, *i.e.*, the tariff schedule, regardless of the actions of the other player, that is, ignoring any potential feedback from the actions of the other player, then it is following an *open–loop* Stackelberg strategy that violates the Bellman principle of optimality, and it is therefore time inconsistent.

$$V^F[c(s)] = \int_s^\infty \pi[\tilde{P}, c(t)] \, e^{-rt} dt, \tag{18a}$$

$$V^G[c(s)] = \int_s^\infty W[\tilde{P}, c(t)] \, e^{-rt} dt, \tag{18b}$$

where the marginal cost evolves according to equation (6). As shown by Starr and Ho (1969), the following pair of Bellman equations provides a set of necessary conditions for MPE strategies:

$$rV^F(c) = \max_{P \in S^F} \left\{ \pi(\tilde{P}, c) - \left( \frac{\partial X}{\partial P} + \frac{\partial X}{\partial \tau} \right) \dot{c} V^F \right\}, \tag{19a}$$

$$rV^G(c) = \max_{\tau \in S^G} \left\{ W(\tilde{P}, c) - \left( \frac{\partial X}{\partial P} + \frac{\partial X}{\partial \tau} \right) \dot{c} V^G \right\}. \tag{19b}$$

After the necessary substitutions, the first order conditions for maximizing the terms between curly brackets in (19) become (see Appendix):[9]

$$0 = \alpha - \beta[2P - c + \lambda(\beta - \gamma)V_c^F(c)] + \gamma(1 + \tau), \tag{20a}$$

$$0 = \gamma[P - c + \lambda(\beta - \gamma)V_c^G(c)] - \tau. \tag{20b}$$

Solving this system of first order conditions we can write the optimal strategies as reduced form functions of the state $c$. The optimal state contingent price and tariff strategies are:

$$P^\star(c) = P^N(c) - \frac{\lambda(\gamma - \beta)[\beta V_c^F(c) - \gamma^2 V_c^G(c)]}{2\beta - \gamma^2}, \tag{21a}$$

$$\tau^\star(c) = \tau^N(c) - \frac{\lambda\beta\gamma(\gamma - \beta)[V_c^F(c) - 2V_c^G(c)]}{2\beta - \gamma^2}. \tag{21b}$$

Equations $(21a) - (21b)$ reveal several interesting features of the MPE strategies. First note that the MPE coincides with the SNE when $\lambda = 0$, *i.e.*, there are no potential cost reductions due to learning by doing and neither monopoly pricing or a protective tariff can induce any dynamic effect whenever current production does not carry any investment consideration. We thus need the existence of some dynamic economy of scale to derive equilibrium strategies that differ from the static ones.

Second, MPE and SNE also coincide when $\beta = \gamma$, *i.e.*, when the own and cross–price effects are the same. But even in this case, the inequality $\beta - \gamma^2 > 0$ must be fulfilled to

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[9] The solution to these Bellman equations is subgame perfect because they hold for any state $c$.

ensure the concavity of the utility function of domestic consumers. Combining these two conditions the equality of own and cross–price effects may hold for $\gamma = \beta < 1$. As $\beta$ and $\gamma$ approach 1 the domestic and the imported good becomes almost perfect substitutes, $-e.g.$, Vives (1999, §6.1)–, and dynamic considerations are no longer valid because consumers can buy an "identical" foreign product. In this case, the consumer surplus effect outweights any potential future savings through reduction in the marginal cost. This means that the mere future increase in domestic profits does not suffice to compensate the current reduction in consumer surplus and that the infant–industry argument will not justify protection of the domestic industry unless such protection also leads to the production of a differentiated domestic variety.

Third, in addition to these effects, common to the optimal MPE price and tariff, observe that, as before in the OLE, the optimal MPE tariff also coincides with the SNE tariff when $\gamma = 0$, $i.e.$, when domestic and foreign products have independent demands.

Finally, observe that necessary conditions $(20a) - (20b)$ are sufficient for a maximum because the expressions in curly brackets in $(19a) - (19b)$ are concave in $\{P, \tau\}$ as long as the utility function of domestic consumers is concave, $i.e.$, $\beta - \gamma^2 > 0$ (see Appendix).

Substitution of $(21a) - (21b)$ into $(19a) - (19b)$ produces a system of two partial differential equations that is difficult to solve in general. However, since this is a linear–quadratic differential game, it is reasonable to assume quadratic value functions in the state:

$$V^F(c) = \psi_0^f + \psi_1^f c + \frac{\psi_2^f}{2} c^2, \tag{22a}$$

$$V^G(c) = \psi_0^g + \psi_1^g c + \frac{\psi_2^g}{2} c^2. \tag{22b}$$

One way to proceed is to substitute the quadratic value functions into the Bellman equations $(19a) - (19b)$ and differentiate the expressions between curly brackets. The resulting system of six nonlinear equations in the value function parameters has to be satisfied for any state $c$. Fortunately this system is block–recursive. Only two nonlinear equations determine $\{\psi_2^f, \psi_2^g\}$. Once these parameters have been found, another two nonlinear equations determine only $\{\psi_1^f, \psi_1^g\}$. Finally, the remaining two equations determine $\{\psi_0^f, \psi_0^g\}$. Observe however that after substituting $(22a) - (22b)$ into $(19a) - (19b)$, the quadratic specification for the value functions lead to Markov strategies $(21a) - (21b)$ that are linear in the state. Without loss of generality we can write these strategies as follows:

$$\hat{P}(c) = \phi_1^f + \phi_2^f c, \tag{23a}$$

$$\hat{\tau}(c) = \phi_1^g + \phi_2^g c. \tag{23b}$$

I show in the Appendix that parameters $\psi$'s are uniquely determined through a linear combination of parameters $\phi$'s. Since the Bellman equations are concave in $\{P, \tau\}$, these linear

Markov strategies lead to Bellman equations that are also concave in $c$. Therefore, focusing on the linear first order conditions suffices to characterize the maximizing strategies.

Given the block–recursive system of nonlinear equations in the value function parameters, an alternative but equivalent approach is to solve the two coupled Riccati equations associated to $(23a) - (23b)$ and the optimality conditions $(10) - (11)$. The procedure is to substitute $(23a) - (23b)$ in $(12)$, as well as their derivatives $\hat{P}_c = \phi_2^f$ and $\hat{\tau}_c = \phi_2^g$. Assuming an stationary MPE, impose $\dot{P} = \dot{\tau} = 0$ and equate coefficients of the intercept and slope of these strategies to obtain the following Riccati equations:

$$
\begin{bmatrix} A_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2\beta & -\gamma \\ \gamma & -1 \end{bmatrix} \begin{bmatrix} (r - \lambda\delta)\phi_1^f \\ (r - \lambda\delta)\phi_1^g \end{bmatrix} + \lambda \begin{bmatrix} \phi_2^g & 0 \\ 0 & \phi_2^f \end{bmatrix} \begin{bmatrix} \beta\gamma & -\gamma^2 \\ -2\beta\gamma & \beta+\gamma^2 \end{bmatrix} \begin{bmatrix} \phi_1^f \\ \phi_1^g \end{bmatrix}, \quad (24a)
$$

$$
\begin{bmatrix} A_2\beta \\ A_2\gamma \end{bmatrix} = \begin{bmatrix} 2\beta & -\gamma \\ \gamma & -1 \end{bmatrix} \begin{bmatrix} (r - \lambda\delta)\phi_2^f \\ (r - \lambda\delta)\phi_2^g \end{bmatrix} + \lambda \begin{bmatrix} \phi_2^g & 0 \\ 0 & \phi_2^f \end{bmatrix} \begin{bmatrix} \beta\gamma & -\gamma^2 \\ -2\beta\gamma & \beta+\gamma^2 \end{bmatrix} \begin{bmatrix} \phi_2^f \\ \phi_2^g \end{bmatrix}, \quad (24b)
$$

where $A_1 = (\alpha + \gamma)(r - \lambda\delta + \gamma\lambda\phi_2^g)$ and $A_2 = r - 2\lambda\delta$. Riccati equations are independent of $c$ because MPE should hold for any realization of the state. Furthermore, observe that $\{\phi_1^f, \phi_1^g\}$ can be found as a linear combination of the equilibrium values of $\{\phi_2^f, \phi_2^g\}$. Therefore, we can focus in solving the simpler system of nonlinear equations $(24b)$, whose parameters $\{\phi_2^f, \phi_2^g\}$ characterize the dynamic features of the equilibrium strategies.[10]

## 5.2 Stationary MPE

The existence of cross–products and quadratic terms in the Riccati equations opens the possibility of multiple solutions. Actually, both Riccati equations, as well as the stability condition are hyperbolas on $\{\beta, \gamma\}$ for any given set $\{r, \lambda\, d, \alpha, c\}$. Figures 1–3 summarize the features of this equilibrium for a well behaved set of parameters.[11] Stability and uniqueness results presented here are however not dependent on the particular values given to these parameters to represent the equilibrium in Figures 1–3.

Figure 1 shows that the four intersections of the *'Riccati Hyperbolas.'* Instead of solving numerically the highly nonlinear equations in the model's parameters that characterize the couple of hyperbolas defined in $(24b)$, I focus on the asymptotes of these hyperbolas (also shown in Figure 1) to obtain some qualitative results of this equilibrium. The first result of this section characterizes the stability of the MPE of this model.

---

[10] Riccati equations do not identify $\{\psi_0^f, \psi_0^g\}$ directly, *i.e.*, the parameters related to the intercept of the value function. I will ignore these parameters because they are not relevant for the stability perfection of the equilibria discussed later in the paper. The quadratic value function approach is fully described in Driskill and McCafferty (1989) and Reynolds (1987). Jun and Vives (1999, §4.2) discuss the equivalence between the quadratic value function and the linear Markov strategy approach followed here.

[11] In particular $\alpha = 1$; $\beta = 0.8$; $\gamma = 0.55$; $\lambda = 0.7$; $\delta = 0.01$; and $r = 0.05$ so that the concavity, $\beta > \gamma^2$, and the OLE stability, $r < \lambda\delta$, conditions hold.

PROPOSITION 7: *A stationary MPE is globally stable if for any* $r, \lambda, \delta, \beta, \gamma$, *the coefficients of the optimal linear strategies* $(23a) - (23b)$ *affecting the marginal cost belong to the following restricted set of values:*

$$
\{\phi_2^f, \phi_2^g\} \in \Omega[r, \lambda, \delta, \beta, \gamma] = \left\{ \left\{ (\phi_2^f, \phi_2^g) \middle| \; 2(r-\lambda\delta)(2\beta-\gamma^2) + \lambda[\gamma(\beta-\gamma^2)\phi_2^g - 2\beta^2\phi_2^f] < 0 \right\} \right.
$$

$$
\cap \left\{ (\phi_2^f, \phi_2^g) \middle| \; (r-\lambda\delta)^2(2\beta-\gamma^2)^2 + \lambda(r-\lambda\delta)(2\beta-\gamma^2)[\gamma(\beta-\gamma^2)\phi_2^g - 2\beta^2\phi_2^f] \right.
$$

$$
\left. \left. + \lambda^2\beta\gamma^2[\gamma\phi_2^g + 4\beta\phi_2^f][\gamma\phi_2^g + (\beta+\gamma^2)\phi_2^f] > 0 \right\} \right\}. \tag{25}
$$

Figure 2 represents $\Omega$. I show in the appendix that the region of stable parameters defined in equation (25) almost coincides with, and always includes the shaded area above the upper branch of the hyperbola in Figure 2. Superimposing Figures 1 and 2 in Figure 3, it is easy to see that there is only one solution that fulfills the stability requirement. The following proposition states a single additional condition for this uniqueness result to be true in general and not only as a consequence of a particular numerical example. The proof of this proposition is also presented in the Appendix.[12]

PROPOSITION 8: *A stationary MPE is the unique globally stable equilibrium of this game if the following condition holds:*

$$
\frac{\beta+\gamma}{\sqrt{(\beta+\gamma)^2 + 4\beta^2\gamma^2}} > \max\left\{ \frac{\gamma - \sqrt{\beta^2+\gamma^2}}{(\beta-\gamma) - \sqrt{\beta^2+\gamma^2}} \; , \; \frac{\gamma(5\beta+\gamma^2)}{4\beta(\beta+\gamma^2)+\gamma^2} \right\}. \tag{26}
$$

Formally, condition (26) requires that the upward sloping asymptotes of the *'Riccati Hyperbolas'* in Figure 1 do not intersect the upward sloping asymptote of the stability condition in Figure 2. I show in the Appendix that this is a sufficient condition to ensure the uniqueness of the stable MPE.

However, there is a more interesting interpretation of this condition. Equation (26) is always fulfilled when $\beta > \gamma$. While $\beta - \gamma^2 > 0$ is required for the concavity of the utility function (1), the more restrictive condition $\beta - \gamma > 0$ requires that own–price effects always exceed cross–price effects between imports and domestic production. As I show in the Appendix, the existence of a unique stable MPE is more likely to happen in this model, the more inelastic is the demand for domestic products (small $\beta$ relative to $\alpha$)

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and whenever domestic customers do not have a strongly biased taste for foreign goods $(\beta > \gamma)$.[13]

## 5.3    Results from Simulations

To complete the analysis I should present a comparison of the optimal MPE pricing and tariff strategies, as well as of the level of marginal cost, at the steady–state relative to those of the SNE and OLE. The infinite–horizon MPE of this model has no weakness related to any possibility of extension of the game. Equilibrium strategies are dynamic best response to each other player's and do not need of any external source of commitment. Therefore, comparing these strategies to OLE strategies, we can account for the effect of the lack of commitment of the domestic firm and the government.

To compare the steady–states of the MPE and the OLE, just assume $\dot{P} = \dot{\tau} = 0$ in equation (12). Obviously when $\hat{P}_c \to 0$ and $\hat{\tau} \to 0$, MPE approaches OLE given by equations $(14a) - (14b)$. Unfortunately, as we have seen previously, $\hat{P}_c$ and $\hat{\tau}_c$ are the solution to a set of nonlinear equations on the parameters of the model. This requires computing the unique stable equilibria of Figure 3 several times to establish the following results, that are therefore here only based in numerical simulations and summarized in Figures 4–6.[14]

PROPOSITION 9: *The stable MPE pricing strategy always exceeds OLE pricing.*

Figure 4 shows that $P^\star > P^\circ$. The lack of commitment ability makes the monopolist to behave less aggressive in inducing cost reductions through learning by doing. Actually, $P^\star$ also exceeds $P^N$. Therefore, the monopolist will only price aggressively to induce learning if she and the government are able to commit to a particular pricing–tariff schedule. These differences in pricing strategies are more important the lower are $\beta$ and $\gamma$, and the larger are $\alpha$ and $c$, *i.e.*, in general when there is a stronger biased taste towards domestic production. As before, if $\lambda \to 0$, $P^\star = P^\circ = P^N$, but if $\delta \to 0$ the difference between $P^\star$ and $P^\circ = P^N$ is substantially smaller but does not disappear completely.

PROPOSITION 10: *The stable MPE tariff strategy always exceeds the OLE tariff.*

Figure 5 shows that $\tau^\star > \tau^\circ > \tau^N$, that is, the optimal MPE tariff strategy is always more protective than in any of the other previously studied environments. Observe that these differences are neither monotone in $\beta$ or $\gamma$. The difference between $\tau^\star$ and the other tariffs increases with the stationary level of marginal cost of the domestic firm. The

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[13] It should be stressed that these conditions limit the interactions of control and state variables in players' payoff functions, as well as those of the square of the opponent's control on each player payoff functions. This can be easily shown by analyzing cross–products in the instantaneous payoff functions associated to (7b) and (8b). Lockwood (1996), has shown that these are sufficient conditions for uniqueness of linear infinite horizon MPE.

[14] Model parameters are the same of those presented in footnote 11, but here, in addition, $\beta$ ranges from 0 to 1.3 and $\gamma$ from 0 to $\sqrt{\beta}$, again to enforce the concavity condition of the utility function. I have preferred not to assume that $\beta > \gamma$ to identify cases where this condition the comparative static results.

three tariffs coincide when $\lambda \to 0$, but the difference is still positive if $\delta \to 0$ or $\gamma \to 0$, when we know that $\tau^{\circ} = \tau^{N}$.

PROPOSITION 11: *The stable MPE involves lower levels of marginal costs than either OLE or SNE.*

The lack of commitment of both agents forces the government to design a more protective policy in order to allow the monopolist to induce cost reductions. This makes imports more expensive, thus expanding the demand for domestic products. However, the monopolist also find optimal to charge higher prices, therefore reducing the potential effect of learning by doing on her marginal cost. The net effect of these two forces is represented in Figure 6, that shows that for most combinations of $\beta$ and $\gamma$ the induced effect of the tariff dominates. In particular, this is the case as long as $\beta - \gamma > 0$ and thus, the steady–state level of marginal cost is lower than in a dynamic equilibrium with commitment ability.[15]

Therefore, while both solutions, OLE and MPE, succeed in reducing marginal cost through learning by doing, it appears that the major consequence of the lack of commitment ability of agents is that the government has to grant a higher level of tariff protection to expand domestic demand and thus reduce marginal cost through learning. Consequently, the monopolist is able to increase her discounted profits substantially relative to OLE because such induced expansion of domestic demand when domestic production and imports are close substitutes allows her to charge higher prices in equilibrium. Consequently, consumers will suffer the burden of such policy facing higher prices for domestic and imported products, thus reducing the present value of their consumer surplus.

# 6    Concluding Remarks

This paper presents a major general result and several characterizations of the pricing–tariff dynamic equilibrium when the domestic monopolist reduces her marginal cost through learning by doing and the government designs a tariff to maximize the discounted value of total domestic welfare, thus accounting for future gains induced by current domestic production decisions.

The major result is that contrary to many existing models in the trade literature, the present paper shows that it is possible to characterize a time–consistent tariff protection policy that successfully help the domestic infant–industry become internationally competitive. This is a key question in classical trade theory. The model shows that it is not necessary any external source of commitment to avoid future deviations from this policy. The existence of learning effects makes possible to find an equilibrium in Markov strategies where the government's tariff is the dynamic best response to the domestic monopolist's pricing decisions and *vice versa*. Two modeling choices, also absent in the

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[15] The small region in Figure 6 where $c^{\star} > c^{\circ}$ coincides with that excluded by condition (26) to ensure uniqueness. In the absence of condition (26) more than one stable equilibria are possible and thus the magnitude of $c^{\star}$ relative to $c^{\circ}$ or $c^{N}$ will be ambiguous.

existing literature dealing with time consistency of tariff protection policies, are critical to show this dynamic optimality result: the use of truly state contingent strategies, and solving the infinite horizon version of the game.

The existence of learning by doing could induce overproduction relative to the static equilibrium, thus maximizing the total discounted profits by lowering the marginal cost. However, this result only holds when both the monopolist and the government can commit to a particular schedule of pricing and tariff decisions. To circumvent the lack of commitment of OLE strategies, I characterize the infinite–horizon linear MPE strategies. Since neither the monopolist or the government are able to commit to a specific schedule of actions, the resulting dynamic equilibrium is more damaging for consumers: both the domestic prices and tariffs are higher, although marginal costs reach levels lower than in the OLE case when domestic and imported production are close substitutes. Thus, even with higher domestic prices, the monopolist overproduces because the higher tariff shifts demand significantly from imports to domestic goods.

The model also shows that in the absence of depreciation of experience, the solution of the dynamic OLE mimics the static one, and thus infant–industry tariff protection fails to reduce marginal cost of domestic firms. Similarly, if the domestic and imported good are considered independent, the dynamic OLE and the static equilibrium strategies coincide and infant–industry arguments will also fail.

Finally, something must be said about potential policy implementations of this model. Instead of considering uncertain learning effects, as for instance in Dinopoulos, Lewis, and Sappington (1995), the model assumes perfect information regarding all parameters of the model, including the learning equation. It could be argued that such demanding informational requirement makes the application of the model impractical. In addition, and despite the tedious and complex computations needed to characterize the equilibrium, results are contingent on the specific linear–quadratic structure of this model. While recognizing that both assertions carry some truth, I should emphasize that linear–quadratic differential games are commonly interpreted as a first approximation to MPEs of more complex differential games, for whom closed–form solutions are impractical [Reinganum (1982, §1)]. Furthermore, the linear MPE analized in this paper is robust to zero–mean additive shocks in the learning equation (6) [Vives (1999, §9.2.3)], thus making the informational requirement argument a less striking criticism.

# References

BEAVIS, D. AND I. DOBBS (1990): *Optimization and Stability Theory for Economic Analysis*, Cambridge University Press.

COHEN, D. AND P. MICHEL (1988): "How Should Control Theory Be Used to Calculate a Time–Consistent Government Policy?" *Review of Economic Studies*, 55, 263–274.

CORDEN, W.M. (1974): *Trade Policy and Economic Welfare*. Oxford University Press.

Dinopoulos, E., T.R. Lewis, and D.E.M. Sappington (1995): "Optimal Industrial Targeting with Unknown Learning–by–Doing." *Journal of International Economics*, 38, 275–295.

Driskill, R.A. and S. McCafferty (1989): "Dynamic Duopoly with Adjustment Costs: A Differential Game Approach." *Journal of Economic Theory*, 49, 324–338.

Fudenberg, D. and J. Tirole (1986): *Dynamic Models of Oligopoly*, in J. Lesourne and H. Sonnenschein (eds.): *Fundamentals of Pure and Applied Economics*. Harwood Academic Publishers.

Fudenberg, D. and J. Tirole (1991): *Game Theory*. MIT Press.

Gruenspecht, H.K. (1988): "Dumping and Dynamic Competition." *Journal of International Economics*, 25, 225–248.

Krugman, P.R. (1984): "Import Protection as Export Promotion: International Competition in the Presence of Oligopolies and Economies of Scale," in H. Kierzkowski (ed.): *Monopolistic Competition and International Trade*. Oxford University Press.

Leahy, D. and J.P. Neary (1999): "Learning by Doing, Precommitment and Infant-Industry Promotion." *Review of Economic Studies*, 66, 447-474.

Lockwood, B., and A. Philippopoulos (1994): "Insider Power, Unemployment Dynamics, and Multiple Inflation Equilibria." *Economica*, 61, 59–77.

Lockwood, B. (1996): "Uniqueness of Markov–Perfect Equilibrium in Infinite–Time Affine–Quadratic Differential Games ." *Journal of Economic Dynamics and Control*, 20, 751–765.

Maskin, E. and J. Tirole (1994): "Markov Perfect Equilibrium." Mimeo, Harvard University.

Matsuyama, K. (1990): "Perfect Equilibria in a Trade Liberalization Game." *American Economic Review*, 80, 480–492.

McLenaghan, R. and S. Levy (1996): "Geometry," in D. Zwillinger (ed.): *CRC Standard Mathematical Tables and Formulae*. CRC Press.

Miyagiwa, K. and Y. Ohno (1995): "Closing the Technology Gap Under Protection." *American Economic Review*, 85, 755–770.

Miyagiwa, K. and Y. Ohno (1999): "Credibility of Protection and Incentives to Innovate." *International Economic Review*, 40, 143–163.

Obstfeld, M. (1997): "Dynamic Seigniorage Theory: An Exploration." *Macroeconomic Dynamics*, 1, 588–614.

Pearce, D. and E. Stacchetti (1997): "Time Consistent Taxation by a Government with Redistributive Goals." *Journal of Economic Theory*, 72, 282–305.

Reinganum, J.F. (1982): "A Dynamic Game of R and D: Patent Protection and Competitive Behavior." *Econometrica*, 50, 671–688.

Reynolds, S.S. (1987): "Capacity Investment, Preemption and Commitment in a Infinite Horizon Model." *International Economic Review*, 28, 69–88.

Reynolds, S.S. (1991): "Dynamic Oligopoly with Capacity Adjustment Costs." *Journal of Economic Dynamics and Control*, 15, 491–514.

Staiger, R.W. and G. Tabellini (1987): "Discretionary Trade Policy and Excessive Protection." *American Economic Review*, 77, 823–837.

Starr, A. and Y.C. Ho (1969): "Nonzero-Sum Differential Games." *Journal of Optimization Theory and Applications*, 3, 183–206.

Tornell, A. (1991): "Time Inconsistency of Protectionist Programs." *Quarterly Journal of Economics*, 106, 963–974.

Vives, X. (1999): *Oligopoly Pricing*. MIT Press.

# Appendix 1

**• Demand System**

In order to reduce the number of parameters of the model, the specification of demand normalizes some of them. From equation (3), the demand for domestic and imported products are:

$$X(\tilde{P}) = \frac{(a_x b_m - a_m k) - b_m P + k(1 + \tau)}{b_x b_m - k^2}, \tag{A.1}$$

$$M(\tilde{P}) = \frac{(a_m b_x - a_x k) + kP - b_x(1 + \tau)}{b_x b_m - k^2}. \tag{A.2}$$

Therefore, to obtain the direct demand function system $(4a) - (4b)$ we need:

$$\alpha = [a_x b_m - a_m k]/\Delta = a_x \beta - a_m \gamma, \tag{A.3a}$$

$$\beta = b_m/\Delta, \tag{A.3b}$$

$$\gamma = k/\Delta, \tag{A.3c}$$

$$1 = b_x/\Delta, \tag{A.3d}$$

$$1 = [a_m b_x - a_x k]/\Delta = a_m - a_x \gamma. \tag{A.3e}$$

In addition, the following inequality will be used extensively:

$$\frac{1}{\Delta} = \beta - \gamma^2 > 0. \tag{A.4}$$

Finally, from $(A.3a)$, $(A.3e)$, and $(A.4)$ we have:

$$a_x = \frac{\alpha + \gamma}{\beta - \gamma^2} \quad \text{and} \quad a_m = \frac{\alpha\gamma + \beta}{\beta - \gamma^2}. \tag{A.5}$$

**• Welfare Function**

Equation $(8b)$ presents the government's welfare function in terms of the parameter of the direct demand system. We thus have to make use of the relationships among parameters of the direct and inverse demand systems described above. The three elements of the welfare function are:

$$CS(\tilde{P}) = (a_x - P)X(\tilde{P}) + (a_m - 1 - \tau)M(\tilde{P})$$

$$- \frac{1}{2}\left[b_x X(\tilde{P})^2 + b_m M(\tilde{P})^2 + 2kX(\tilde{P})M(\tilde{P})\right], \tag{A.6a}$$

$$\pi(\tilde{P}, c) = (P - c)X(\tilde{P}), \tag{A.6b}$$

$$R(\tilde{P}) = \tau M(\tilde{P}). \tag{A.6c}$$

Adding these three terms we get:

$$W(\tilde{P}, c) = (a_x - c)X(\tilde{P}) + (a_m - 1)M(\tilde{P}) - \frac{b_x X(\tilde{P})^2 + b_m M(\tilde{P})^2 + 2kX(\tilde{P})M(\tilde{P})}{2}. \quad (A.7)$$

Equation $(8b)$ substitutes $(4a) - (4b)$ and $(A.3) - (A.5)$ into $(A.7)$.

● **Stationary OLE**

After making $\hat{P}_c = \hat{\tau}_c = 0$ in equation $(12)$ we have:

$$\begin{bmatrix} (\alpha + \gamma)(r - \lambda\delta) + (r - 2\lambda\delta)\beta c \\ \\ (r - 2\lambda\delta)\gamma c \end{bmatrix} = \begin{bmatrix} 2\beta & -\gamma \\ \gamma & -1 \end{bmatrix} \begin{bmatrix} (r - \lambda\delta)P - \dot{P} \\ \\ (r - \lambda\delta)\tau - \dot{\tau} \end{bmatrix}. \quad (A.8)$$

The stationary OLE is found by making $\dot{P} = 0$ and $\dot{\tau} = 0$. Then, $P^\circ$ and $\tau^\circ$ can be easily computed from $(A.8)$ using Cramer's Rule. Thus, rewriting $(A.8)$ we have:

$$\begin{bmatrix} (r - \lambda\delta)P^\circ \\ \\ (r - \lambda\delta)\tau^\circ \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{P} \\ \dot{\tau} \end{bmatrix} + \begin{bmatrix} (r - \lambda\delta) & 0 \\ 0 & (r - \lambda\delta) \end{bmatrix} \begin{bmatrix} P \\ \tau \end{bmatrix}. \quad (A.9)$$

Focusing on the homogeneous part of this dynamic system, the Routh–Hurwitz stability condition requires that [Beavis and Dobbs (1990 §5.4)]:

$$\mathrm{TR} \begin{bmatrix} (r - \lambda\delta) & 0 \\ \\ 0 & (r - \lambda\delta) \end{bmatrix} = 2(r - \lambda\delta) < 0, \quad (A.10a)$$

and:

$$\begin{vmatrix} (r - \lambda\delta) & 0 \\ \\ 0 & (r - \lambda\delta) \end{vmatrix} = (r - \lambda\delta)^2 > 0, \quad (A.10b)$$

from which follows that the stationary OLE is globally stable if $r < \lambda\delta$. Finally, the level of marginal cost at the stationary OLE is found making $\dot{c} = 0$ in equation $(6)$ so that:

$$\delta c = (\alpha + \gamma) - \beta P + \gamma\tau. \quad (A.11)$$

The value of $c^\circ$ in equation $(17)$ is found after substituting the OLE strategies $(14a) - (14b)$ in $(A.11)$. The marginal cost in the SNE, $c^N$, is $c^0$ when $\lambda = 0$. The difference between these two stationary equilibrium levels for the state variable of the game is:

$$c^\circ - c^N = \frac{\lambda\delta(\alpha + \gamma)\beta^3}{(r - \lambda\delta)(2\beta - \gamma^2)^2\delta^2 + (r - 2\lambda\delta)\beta^4 + (2r - 3\lambda\delta)(2\beta - \gamma^2)\delta\beta^2} < 0, \quad (A.12)$$

because for any stable equilibria $r < \lambda\delta$.

## • Bellman Equations

To obtain the first order conditions $(20a) - (20b)$ we need to substitute demand production and imports $(4a) - (4b)$ and the marginal cost motion equation $(6)$ into the profit $(A.6b)$ and welfare function $(A.7)$ respectively. Bellman equations can then be written as follows:

$$rV^F(c) = \max_{P \in S^F} \left\{ [P - c - \lambda(\gamma - \beta)V_c^F(c)][\alpha - \beta P + \gamma(1 + \tau)] + \lambda \delta c V_c^F(c) \right\}, \quad (A.13a)$$

$$rV^G(c) = \max_{\tau \in S^G} \left\{ \left[ \frac{\alpha + \gamma}{\beta - \gamma^2} - c - \lambda(\gamma - \beta)V_c^G(c) \right][\alpha - \beta P + \gamma(1 + \tau)] \right.$$

$$+ \gamma \frac{\alpha + \gamma}{\beta - \gamma^2}[\gamma P - \tau] + \lambda \delta c V_c^G(c) - \frac{[\alpha - \beta P + \gamma(1 + \tau)]^2}{2(\beta - \gamma^2)}$$

$$\left. - \frac{\beta[\gamma P - \tau]^2}{2(\beta - \gamma^2)} - \frac{\gamma[\alpha - \beta P + \gamma(1 + \tau)][\gamma P - \tau]}{\beta - \gamma^2} \right\}. \quad (A.13b)$$

To characterize the equivalence between parameters $\psi$'s and $\phi$'s, first differentiate $(22a) - (22b)$ to obtain:

$$V_c^F(c) = \psi_1^f + \psi_2^f c, \quad (A.14a)$$

$$V_c^G(c) = \psi_1^g + \psi_2^g c. \quad (A.14b)$$

After substituting these expressions and $(13a) - (13b)$ in $(21a) - (21b)$ we get the following two system of linear equations defined on the intercepts and slope of the linear Markov strategies respectively:

$$(2\beta - \gamma^2)\phi_1^f = (\alpha + \gamma) + \lambda(\gamma - \beta)[\beta \psi_1^f - \gamma^2 \psi_1^g], \quad (A.15a)$$

$$(2\beta - \gamma^2)\phi_1^g = \gamma(\alpha + \gamma) + \lambda\beta\gamma(\gamma - \beta)[\psi_1^f - 2\psi_1^g], \quad (A.15b)$$

$$(2\beta - \gamma^2)\phi_2^f = (\beta - \gamma^2) + \lambda(\gamma - \beta)[\beta \psi_2^f - \gamma^2 \psi_2^g], \quad (A.15c)$$

$$(2\beta - \gamma^2)\phi_2^g = -\beta\gamma + \lambda\beta\gamma(\gamma - \beta)[\psi_2^f - 2\psi_2^g]. \quad (A.15d)$$

Solving these two systems of linear equations leads to the following equivalence relations:

$$\lambda\beta\gamma(\beta - \gamma)\psi_1^f = \gamma(\alpha + \gamma) - 2\beta\gamma\phi_1^f + \gamma^2\phi_1^g, \quad (A.16a)$$

$$\lambda\beta\gamma(\beta - \gamma)\psi_1^g = -\beta\gamma\phi_1^f + \beta\phi_1^g, \quad (A.16b)$$

$$\lambda\beta\gamma(\beta - \gamma)\psi_2^f = \beta\gamma - 2\beta\gamma\phi_2^f + \gamma^2\phi_2^g, \quad (A.16c)$$

$$\lambda\beta\gamma(\beta - \gamma)\psi_2^g = \beta\gamma - \beta\gamma\phi_2^f + \beta\phi_2^g. \quad (A.16d)$$

## • Riccati Equations

Substitute the proposed linear strategies $(23a) - (23b)$ and its derivatives into $(12)$. A stationary MPE requires $\dot{P} = \dot{\tau} = 0$ so that:

$$\begin{bmatrix} (\alpha + \gamma)(r - \lambda\delta + \gamma\lambda\phi_2^g) + (r - 2\lambda\delta)\beta c \\ \\ (r - 2\lambda\delta)\gamma c \end{bmatrix} = \begin{bmatrix} 2\beta & -\gamma \\ \\ \gamma & -1 \end{bmatrix} \begin{bmatrix} (r - \lambda\delta)(\phi_1^f + \phi_2^f c) \\ \\ (r - \lambda\delta)(\phi_1^g + \phi_2^g c) \end{bmatrix}$$

$$+ \begin{bmatrix} \phi_2^g & 0 \\ \\ 0 & \phi_c^f \end{bmatrix} \begin{bmatrix} \beta\gamma & -\gamma^2 \\ \\ -2\beta\gamma & \beta + \gamma^2 \end{bmatrix} \begin{bmatrix} \lambda(\phi_1^f + \phi_2^f c) \\ \\ \lambda(\phi_1^g + \phi_2^g c) \end{bmatrix}. \quad (A.17)$$

Riccati equations $(24a) - (24b)$ are derived from here by equating the elements on $c$ and those of the independent terms. Finally, observe that because of the block–recursive structure of these Riccati equations, once we know $\{\phi_2^f, \phi_2^g\}$ we can find $\{\phi_1^f, \phi_1^g\}$ through a linear combination implicitly defined in $(24a)$:

$$\begin{bmatrix} (\alpha+\gamma)(r-\lambda\delta+\gamma\lambda\phi_2^g) \\ \\ 0 \end{bmatrix} = \begin{bmatrix} 2\beta(r-\lambda\delta)+\lambda\beta\gamma\phi_2^g & -\gamma(r-\lambda\delta)-\lambda\gamma^2\phi_2^g \\ \\ \gamma(r-\lambda\delta)-2\lambda\beta\gamma\phi_2^f & -(r-\lambda\delta)+\lambda(\beta+\gamma^2)\phi_2^f \end{bmatrix} \begin{bmatrix} \phi_1^f \\ \\ \phi_1^g \end{bmatrix}. \quad (A.18)$$

For convenience, equation $(24b)$ can be rewritten as follows:

$$\lambda\gamma^2(\phi_2^g)^2 - \lambda\beta\gamma\phi_2^f\phi_2^g - 2\beta(r - \lambda\delta)\phi_2^f + \gamma(r - \lambda\delta)\phi_2^g + \beta(r - 2\lambda\delta) = 0, \quad (A.19a)$$

$$2\lambda\beta\gamma(\phi_2^f)^2 - \lambda(\beta + \gamma^2)\phi_2^f\phi_2^g - \gamma(r - \lambda\delta)\phi_2^f + (r - \lambda\delta)\phi_2^g + \gamma(r - 2\lambda\delta) = 0. \quad (A.19b)$$

Each one of these Riccati equations is a particular case of the general quadratic form representation of conic sections:

$$Ax^2 + By^2 + Cxy + Dx + Ey + F = 0. \quad (A.20)$$

McLenaghan and Levy (1996, §4.7.2) show that if the following two conditions hold, the general conic representation $(A.20)$ corresponds to a hyperbola:

$$\begin{vmatrix} A & C/2 & D/2 \\ C/2 & B & E/2 \\ D/2 & E/2 & F \end{vmatrix} \neq 0, \quad \text{and} \quad \begin{vmatrix} A & C/2 \\ C/2 & B \end{vmatrix} < 0. \quad (A.21)$$

This is the case of the Riccati equations $(A.19a) - (A.19b)$:

$$\begin{vmatrix} 0 & \frac{\lambda\beta\gamma}{-2} & \beta(\lambda\delta-r) \\ \\ \frac{\lambda\beta\gamma}{-2} & \lambda\gamma^2 & \frac{\gamma(r-\lambda\delta)}{2} \\ \\ \beta(\lambda\delta-r) & \frac{\gamma(r-\lambda\delta)}{2} & \beta(r-2\lambda\delta) \end{vmatrix} = \frac{\lambda\beta^2\gamma^2[2(r-\lambda\delta)^2+\lambda\beta(r-2\lambda\delta)]}{-4} \neq 0, \quad (A.21a)$$

$$\begin{vmatrix} 0 & \frac{\lambda\beta\gamma}{-2} \\ \frac{\lambda\beta\gamma}{-2} & \lambda\gamma^2 \end{vmatrix} = -\left[\frac{\lambda\beta\gamma}{2}\right]^2 < 0, \tag{A.21b}$$

$$\begin{vmatrix} 2\lambda\beta\gamma & \frac{\lambda(\beta+\gamma^2)}{-2} & \frac{\gamma(r-\lambda\delta)}{-2} \\ \frac{\lambda(\beta+\gamma^2)}{-2} & 0 & \frac{r-\lambda\delta}{2} \\ \frac{\gamma(r-\lambda\delta)}{-2} & \frac{r-\lambda\delta}{2} & \gamma(r-2\lambda\delta) \end{vmatrix} = \frac{\lambda\gamma[(2\beta-\gamma^2)(r-\lambda\delta)^2+\lambda(\beta+\gamma^2)^2(r-2\lambda\delta)]}{-4} \neq 0, \tag{A.21c}$$

$$\begin{vmatrix} 2\lambda\beta\gamma & \frac{(\beta+\gamma^2)}{-2} \\ \frac{\lambda(\beta+\gamma^2)}{-2} & 0 \end{vmatrix} = -\left[\frac{\lambda(\beta+\gamma^2)}{2}\right]^2 < 0. \tag{A.21d}$$

The center of each hyperbola is found solving the following system of linear equations:

$$2Ax + Cy + D = 0, \tag{A.22a}$$

$$Cx + 2By + E = 0. \tag{A.22b}$$

Thus, for the first Riccati equation $(A.19a)$ we have:
$$-\lambda\beta\gamma\phi_2^g - 2\beta(r - \lambda\delta) = 0, \tag{A.23a}$$

$$-\lambda\beta\gamma\phi_2^f + 2\lambda\gamma^2\phi_2^g + \gamma(r - \lambda\delta) = 0, \tag{A.23b}$$

leading to:
$$\hat{\phi}_2^f = -\frac{3(r - \lambda\delta)}{\lambda\beta} > 0, \qquad \text{and} \qquad \hat{\phi}_2^g = -\frac{2(r - \lambda\delta)}{\lambda\gamma} > 0. \tag{A.23c}$$

Similarly, for the second Riccati equation $(A.19b)$ we have:
$$-4\lambda\beta\gamma\phi_2^f - \lambda(b + \gamma^2)\phi_2^g - \gamma(r - \lambda\delta) = 0, \tag{A.24a}$$

$$-\lambda(\beta + \gamma^2)\phi_2^f + (r - \lambda\delta) = 0, \tag{A.24b}$$

leading to:
$$\hat{\phi}_2^f = \frac{r - \lambda\delta}{\lambda(\beta + \gamma^2)} < 0, \qquad \text{and} \qquad \hat{\phi}_2^g = \frac{\gamma(3\beta - \gamma^2)(r - \lambda\delta)}{\lambda(\beta + \gamma^2)^2} < 0. \tag{A.24c}$$

Solving for the intersection of the hyperbolas defined by $(A.19a) - (A.19b)$ in term of the parameters of the model is complicated because it involves nonlinear functions of these same parameters. To characterize the nature of such solution it suffices however to work with the asymptotes of these hyperbolas. This approach does not completely avoid

cumbersome computations. Let start by writing Riccati equation $(A.19a)$ in matrix form as in $(A.20)$:

$$\Phi'\begin{bmatrix} 0 & -\lambda\beta\gamma/2 \\ -\lambda\beta\gamma/2 & \lambda\gamma^2 \end{bmatrix}\Phi + \Phi'\begin{bmatrix} -2\beta(r-\lambda\delta) \\ \gamma(r-\lambda\delta) \end{bmatrix} + \beta(r-2\lambda\delta) = 0, \qquad (A.25a)$$

where $\Phi' = [\phi_2^f \ \phi_2^g]$. To write the hyperbola in standard position I first have to rotate the $\phi_2^f\phi_2^g$ axis so that the equation of the hyperbola in a new basis does not include a cross–product term. Thus, first I find the eigenvalues of the cross–product matrix:

$$\nu_1 = \frac{\lambda\gamma[\gamma \pm \sqrt{\beta^2 + \gamma^2}]}{2}, \qquad (A.25b)$$

which leads to the following transformation matrix made of the corresponding orthonormal basis after normalizing the norm of each eigenvector to 1:

$$T_1 = \begin{bmatrix} \dfrac{\gamma + \sqrt{\beta^2 + \gamma^2}}{\sqrt{\beta^2 + [\gamma + \sqrt{\beta^2 + \gamma^2}]^2}} & \dfrac{\gamma - \sqrt{\beta^2 + \gamma^2}}{\sqrt{\beta^2 + [\gamma - \sqrt{\beta^2 + \gamma^2}]^2}} \\[3ex] \dfrac{\beta}{\sqrt{\beta^2 + [\gamma + \sqrt{\beta^2 + \gamma^2}]^2}} & \dfrac{\beta}{\sqrt{\beta^2 + [\gamma - \sqrt{\beta^2 + \gamma^2}]^2}} \end{bmatrix}, \qquad (A.25c)$$

so that $\mid T_1 \mid = 1$ and the transformation is ensured to be a rotation. After writing $(A.25a)$ in term of the new basis, I still need to translate the new axis to center the hyperbola in its standard position:

$$\left[\frac{\phi_2^f}{a}\right]^2 - \left[\frac{\phi_2^g}{b}\right]^2 = 1. \qquad (A.26)$$

After many computations, $a_1$ and $b_1$ can be written as:

$$a_1^2 = \frac{[\beta^2\gamma^2 - 2\sqrt{\beta^2 + \gamma^2}](r-\lambda\delta)^2 - 2\lambda\beta^3(\beta^2 + \gamma^2)(r-2\lambda\delta)}{\lambda^2\beta^2\gamma[\gamma - \sqrt{\beta^2 + \gamma^2}](\beta^2 + \gamma^2)}, \qquad (A.27a)$$

$$b_1^2 = -\frac{[\beta^2\gamma^2 - 2\sqrt{\beta^2 + \gamma^2}](r-\lambda\delta)^2 - 2\lambda\beta^3(\beta^2 + \gamma^2)(r-2\lambda\delta)}{\lambda^2\beta^2\gamma[\gamma + \sqrt{\beta^2 + \gamma^2}](\beta^2 + \gamma^2)}, \qquad (A.27b)$$

leading to asymptotes that cross at the center $(A.23c)$ with slopes:

$$\pm\frac{b_1}{a_1} = \frac{\pm\beta}{\gamma + \sqrt{\beta^2 + \gamma^2}}. \qquad (A.28a)$$

Finally, to express these slopes in term of the original basis we compute:

$$\begin{bmatrix} a_1' \\ b_1' \end{bmatrix} = T_1\begin{bmatrix} \gamma + \sqrt{\beta^2 + \gamma^2} \\ \pm\beta \end{bmatrix}, \qquad (A.28b)$$

from where we have the following slopes corresponding to $\pm b_1/a_1$:

$$S_1^0 = \frac{\gamma - \sqrt{\beta^2 + \gamma^2}}{(\beta + \gamma) + \sqrt{\beta^2 + \gamma^2}} < 0, \qquad (A.28c)$$

$$S_1^1 = \frac{\gamma - \sqrt{\beta^2 + \gamma^2}}{(\beta - \gamma) - \sqrt{\beta^2 + \gamma^2}} \geq 0. \qquad (A.28d)$$

Similarly, and for the sake of completeness since the model is not symmetric, Riccati equation $(A.19b)$ can be written in matrix form as follows:

$$\Phi' \begin{bmatrix} 2\lambda\beta\gamma & -\lambda(\beta + \gamma^2)/2 \\ -\lambda(\beta + \gamma^2)/2 & 0 \end{bmatrix} \Phi + \Phi' \begin{bmatrix} -\gamma(r - \lambda\delta) \\ (r - \lambda\delta) \end{bmatrix} + \gamma(r - 2\lambda\delta) = 0, \quad (A.29a)$$

where $\Phi$ is defined as before. The eigenvalues of the cross–product matrix are:

$$\nu_2 = \lambda \left[ \beta\gamma \pm \sqrt{\beta^2\gamma^2 + \left(\frac{\beta + \gamma^2}{2}\right)} \right], \qquad (A.29b)$$

leading to the following transformation matrix:

$$T_2 = \begin{bmatrix} \dfrac{1}{\sqrt{2\sqrt{1 + m^2}[\sqrt{1 + m^2} - m]}} & \dfrac{1}{\sqrt{2\sqrt{1 + m^2}[\sqrt{1 + m^2} + m]}} \\[4ex] \dfrac{m - \sqrt{1 + m^2}}{\sqrt{2\sqrt{1 + m^2}[\sqrt{1 + m^2} - m]}} & \dfrac{m + \sqrt{1 + m^2}1}{\sqrt{2\sqrt{1 + m^2}[\sqrt{1 + m^2} + m]}} \end{bmatrix}, \qquad (A.29c)$$

where:

$$m = \frac{2\beta\gamma}{\beta + \gamma^2}. \qquad (A.29d)$$

Since $\mid T_2 \mid = 1$, transformation $(A.29c)$ is also a rotation. After rotating and translating the new axis, the parameters that characterize this second hyperbola in standard form are:

$$a_2^2 = \frac{2\lambda[(r - \lambda\delta)^2(\gamma m - 1) - \lambda(\beta + \gamma^2)(r - 2\lambda\delta)]}{\lambda^2(\beta + \gamma^2)^2[m - \sqrt{1 + m^2}]}, \qquad (A.30a)$$

$$b_2^2 = -\frac{2\lambda[(r - \lambda\delta)^2(\gamma m - 1) - \lambda(\beta + \gamma^2)(r - 2\lambda\delta)]}{\lambda^2(\beta + \gamma^2)^2[m + \sqrt{1 + m^2}]}, \qquad (A.30b)$$

leading to asymptotes that cross at the center $(A.24c)$ with slopes:

$$\pm\frac{b_2}{a_2} = \pm[m - \sqrt{1 + m^2}]. \qquad (A.31a)$$

Finally, to express these slopes in term of the original basis we compute:

$$\begin{bmatrix} a_2' \\ b_2' \end{bmatrix} = T_2 \begin{bmatrix} \pm 1 \\ m - \sqrt{1 + m^2} \end{bmatrix}, \tag{A.31b}$$

from where we have the following slopes corresponding to $\pm b_2 / a_2$:

$$S_2^0 = 0, \tag{A.32c}$$

$$S_2^1 = \sqrt{\frac{(\beta + \gamma)^2}{(\beta + \gamma)^2 + 4\beta^2 \gamma^2}}. \tag{A.32d}$$

## • Stability of Stationary MPE

Pre–multiply both sides of equation (12) by:

$$\begin{bmatrix} 2\beta & -\gamma \\ \gamma & -1 \end{bmatrix}^{-1} = \frac{-1}{2\beta - \gamma^2} \begin{bmatrix} -1 & \gamma \\ -\gamma & 2\beta \end{bmatrix}. \tag{A.33}$$

Then, after substituting the proposed MPE equilibrium strategies $(23a) - (23b)$ we have:

$$\begin{bmatrix} (r - \lambda\delta)P^\circ + \dfrac{\lambda\gamma(\alpha + \gamma)\phi_2^g}{(2\beta - \gamma^2)} \\[2ex] (r - \lambda\delta)\tau^\circ + \dfrac{\lambda\gamma(\alpha + \gamma)\phi_2^g}{(2\beta - \gamma^2)} \end{bmatrix} = - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{P} \\ \dot{\tau} \end{bmatrix} +$$

$$\begin{bmatrix} \dfrac{(r - \lambda\delta) + \lambda\beta\gamma(\phi_2^g + 2\phi_2^f)}{(2\beta - \gamma^2)} & \dfrac{-\lambda\gamma[\gamma\phi_2^g + (\beta + \gamma^2)\phi_2^f]}{(2\beta - \gamma^2)} \\[3ex] \dfrac{\lambda\beta\gamma(\gamma\phi_2^g + 4\beta\phi_2^f)}{(2\beta - \gamma^2)} & \dfrac{(r - \lambda\delta) - \lambda[\gamma^3\phi_2^g + 2\beta(\beta + \gamma^2)\phi_2^f]}{(2\beta - \gamma^2)} \end{bmatrix} \begin{bmatrix} P \\ \tau \end{bmatrix}. \tag{A.34}$$

Stability of this system requires that the trace of the latter matrix is negative while its determinant is positive:

$$2(r - \lambda\delta)(2\beta - \gamma^2) + \lambda[\gamma(\beta - \gamma^2)\phi_2^g - 2\beta^2\phi_2^f] < 0, \tag{A.35}$$

$$\Delta^\star = (r - \lambda\delta)^2(2\beta - \gamma^2)^2 + \lambda(r - \lambda\delta)(2\beta - \gamma^2)[\gamma(\beta - \gamma^2)\phi_2^g - 2\beta^2\phi_2^f]$$

$$+ \lambda^2\beta\gamma^2[\gamma\phi_2^g + 4\beta\phi_2^f][\gamma\phi_2^g + (\beta + \gamma^2)\phi_2^f] > 0. \tag{A.36}$$

While the *trace condition* is a simple linear relation between $\phi_2^g$ and $\phi_2^f$, the *determinant condition* is again a conic section. Actually, as the following two conditions show, it is another hyperbola.

$$\begin{vmatrix} 4(\lambda\beta\gamma)^2(\beta+\gamma^2 & \lambda^2\beta\gamma^3(5\beta+\gamma^2)/2 & -\lambda\beta^2(r-\lambda\delta)(2\beta-\gamma^2) \\ \lambda^2\beta\gamma^3(5\beta+\gamma^2)/2 & \lambda^2\beta\gamma^4 & \lambda\gamma(\beta-\gamma^2)(r-\lambda\delta)(2\beta-\gamma^2) \\ -\lambda\beta^2(r-\lambda\delta)(2\beta-\gamma^2) & \lambda\gamma(\beta-\gamma^2)(r-\lambda\delta)(2\beta-\gamma^2) & (r-\lambda\delta)^2(2\beta-\gamma^2)^2 \end{vmatrix} =$$

$$-(\lambda\beta\gamma^2)[\gamma(r-\lambda\delta)(2\beta-\gamma^2)]^2 \left[\lambda^2\beta^4 + \frac{(3\beta-\gamma^2)^2}{4} + \frac{\lambda^2\beta(\beta-\gamma^2)(7\beta^2+(\beta-2)\gamma^2)}{2}\right] \neq 0,$$

$$(A.37a)$$

$$\begin{vmatrix} 4(\lambda\beta\gamma)^2(\beta+\gamma^2) & \lambda^2\beta\gamma^3(5\beta+\gamma^2)/2 \\ \lambda^2\beta\gamma^3(5\beta+\gamma^2)/2 & \lambda^2\beta\gamma^4 \end{vmatrix} = -\frac{\gamma^2}{4}(\lambda^2\beta\gamma^2)^2(3\beta-\gamma^2)^2 < 0. \qquad (A.37b)$$

The center of this hyperbola is found solving the following system of equations:

$$8\lambda\beta\gamma^2(\beta+\gamma^2)\phi_2^f + \lambda\gamma^3(5\beta+\gamma^2)\phi_2^g - 2\beta(r-\lambda\delta)(2\beta-\gamma^2) = 0, \qquad (A.38a)$$

$$\lambda\beta\gamma^2(5\beta+\gamma^2)\phi_2^f + 2\lambda\beta\gamma^3\phi_2^g + (\beta-\gamma^2)(r-\lambda\delta)(2\beta-g^2) = 0, \qquad (A.38b)$$

leading to:

$$\hat{\phi}_2^f = \frac{(r-\lambda\delta)(2\beta-\gamma^2)(\beta^2-4\beta\gamma^2-\gamma^4)}{\lambda\beta\gamma^2(3\beta-\gamma^2)^2}, \qquad (A.38c)$$

$$\hat{\phi}_2^g = \frac{2(r-\lambda\delta)(2\beta-\gamma^2)(\beta^2+\beta\gamma^3+4\gamma^4)}{\lambda\gamma^3(3\beta-\gamma^2)^2} < 0. \qquad (A.38d)$$

As before, I will only provide with sufficient conditions that hold asymptotically. The *determinant condition* $(A.36)$ in matrix form is:

$$\frac{\lambda^2\beta\gamma^2}{2}\Phi' \begin{bmatrix} 8\beta(\beta+\gamma^2) & \gamma(5\beta+\gamma^2) \\ \gamma(5\beta+\gamma^2) & 2\gamma^2 \end{bmatrix} \Phi$$

$$+\lambda(r-\lambda\delta)(2\beta-\gamma^2)\Phi' \begin{bmatrix} -2\beta^2 \\ \gamma(\beta-\gamma^2) \end{bmatrix} + (r-\lambda\delta)^2(2\beta-\gamma^2)^2 = 0, \qquad (A.39a)$$

where, again, $\Phi' = [\phi_2^f \ \phi_2^g]$. The eigenvalues of the cross–product matrix are:

$$\nu_3 = [4\beta(\beta+\gamma^2)+\gamma^2] \pm \sqrt{[4\beta(\beta+\gamma^2)+\gamma^2]^2 + [\gamma(5\beta+\gamma^2)]^2}, \qquad (A.39b)$$

which leads to the following orthonormal basis:

$$
T_3^1 = \begin{bmatrix} t_3^{11} \\ t_3^{12} \end{bmatrix} = \begin{bmatrix} \dfrac{\gamma(5\beta+\gamma^2)}{\sqrt{[\gamma(5\beta+\gamma^2)]^2+\{[\gamma^2-4\beta(\beta+\gamma^2)]-\sqrt{[4\beta(\beta+\gamma^2)+\gamma^2]^2+[\gamma(5\beta+\gamma^2)]^2}\}^2}} \\[4ex] \dfrac{-[4\beta(\beta+\gamma^2)-\gamma^2]-\sqrt{[4\beta(\beta+\gamma^2)+\gamma^2]^2+[\gamma(5\beta+\gamma^2)]^2}}{\sqrt{[\gamma(5\beta+\gamma^2)]^2+\{[\gamma^2-4\beta(\beta+\gamma^2)]-\sqrt{[4\beta(\beta+\gamma^2)+\gamma^2]^2+[\gamma(5\beta+\gamma^2)]^2}\}^2}} \end{bmatrix}, \qquad (A.39c)
$$

$$
T_3^2 = \begin{bmatrix} t_3^{21} \\ t_3^{22} \end{bmatrix} = \begin{bmatrix} \dfrac{[4\beta(\beta+\gamma^2)-\gamma^2]+\sqrt{[4\beta(\beta+\gamma^2)+\gamma^2]^2+[\gamma(5\beta+\gamma^2)]^2}}{\sqrt{[\gamma(5\beta+\gamma^2)]^2+\{[4\beta(\beta+\gamma^2)-\gamma^2]+\sqrt{[4\beta(\beta+\gamma^2)+\gamma^2]^2+[\gamma(5\beta+\gamma^2)]^2}\}^2}} \\[4ex] \dfrac{\gamma(5\beta+\gamma^2)}{\sqrt{[\gamma(5\beta+\gamma^2)]^2+\{[4\beta(\beta+\gamma^2)-\gamma^2]+\sqrt{[4\beta(\beta+\gamma^2)+\gamma^2]^2+[\gamma(5\beta+\gamma^2)]^2}\}^2}} \end{bmatrix}, \qquad (A.39d)
$$

so that the transformation matrix is ensured to rotate axis $\phi_2^f \phi_2^g$:

$$
|T_3| = \begin{vmatrix} t_3^{11} & t_3^{21} \\ t_3^{12} & t_3^{22} \end{vmatrix} = 1. \qquad (A.39e)
$$

After some algebra, the asymptotes that cross at the center $(A.41)$ are:

$$
\pm \frac{b_3}{a_3} = \pm \frac{[4\beta(\beta+\gamma^2)+\gamma^2]+\sqrt{[4\beta(\beta+\gamma^2)+\gamma^2]^2+[\gamma(5\beta+\gamma^2)]^2}}{\gamma(5\beta+\gamma^2)}. \qquad (A.40a)
$$

These slopes are defined relative to the hyperbola's standard form. To express these slopes in term of the original basis we compute:

$$
\begin{bmatrix} a_3' \\ b_3' \end{bmatrix} = T_3 \begin{bmatrix} \pm\gamma(5\beta+\gamma^2) \\ [4\beta(\beta+\gamma^2)+\gamma^2]+\sqrt{[4\beta(\beta+\gamma^2)+\gamma^2]^2+[\gamma(5\beta+\gamma^2)]^2} \end{bmatrix}, \qquad (A.40b)
$$

from where we have the following slopes corresponding to $\pm b_3/a_3$:

$$
S_3^0 = 0, \qquad (A.40c)
$$

$$
S_3^1 = \frac{\gamma(5\beta+\gamma^2)}{4\beta(\beta+\gamma^2)+\gamma^2} > 0. \qquad (A.40d)
$$

Now, it only remains to show that the the shaded region of Figure 2 suffices to characterize the stable MPE. Notice that the center of the stability hyperbola lies above the downward sloping, dashed line in Figure 2 that represents the *trace condition* $(A.35)$ –equated to zero–. This is a general result. Evaluate the difference between $\hat{\phi}_2^f$ given by $(A.38c)$ and condition $(A.35)$ evaluated at $\hat{\phi}_2^g$ given by $(A.38d)$. Such difference can be written as:

$$
2\lambda\beta\gamma(r-\lambda\delta)(2\beta-\gamma^2)\left[\beta(\beta^2-4\beta\gamma^2-\gamma^4)-\gamma^2(3\beta-\gamma^2)^2-(\beta-\gamma^2)(\beta^2+\beta\gamma^3+4\gamma^4)\right], \qquad (A.41)
$$

which is always positive as shown in Figure A.1 for combinations of $\beta$ and $\gamma$ that ensure concavity of the utility function (1).

## • Uniqueness of Stable MPE

I will take a geometric approach for this *'proof'*, analyzing the relative position of the asymptotes of the *'Riccati Hyperbolas'* in Figure 1. I will make extensive use of the results characterizing the position of centers and slopes of these and the stability asymptotes previously shown in this Appendix. The *'proof'* consists of five steps:

<u>Step 1.</u> The difference between flat asymptotes of the second Riccati equation $(A.19b)$ given by $\hat{\phi}_2^g$ in $(A.24c)$ and that of the stability condition $(A.36)$ given by $(A.38d)$ is always positive. Rewriting this difference we obtain the following equivalent condition:

$$\lambda(r - \lambda\delta)\left[\gamma^4(3\beta - \gamma^2)^3 - 2(\beta + \gamma^2)^2(2\beta - \gamma^2)(\beta^2 + \beta\gamma^3 + 4\gamma^4)\right] > 0, \qquad (A.42)$$

which is is shown to hold in Figure A.2 for any $\beta$ and $\gamma$ such that $\beta - \gamma^2 > 0$.

<u>Step 2.</u> The center of the hyperbola of the second Riccati equation is always above the upward sloping asymptote of the hyperbola of the stability condition. Evaluate the upward sloping asymptote, combining $(A.38c) - (A.38d)$ with $(A.40d)$, at $\hat{\phi}_2^g$ given in $(A.24c)$. The difference of that point with $\hat{\phi}_2^f$ given in $(A.24c)$ should always be positive. After some algebra this condition can be written as:

$$\lambda\gamma^2(r - \lambda\delta)(\beta + \gamma^2)(3\beta - \gamma^2)^2 \cdot \qquad (A.43)$$
$$\left\{\gamma^2(\beta + \gamma^2)(5\beta + \gamma^2)\left[2(2\beta - \gamma^2)(\beta^2 - 4\beta\gamma^2 - \gamma^4)(\beta + \gamma^2) - \lambda\beta\gamma^2(3\beta - \gamma^2)^2\right]\right.$$
$$\left. + \beta\left[\gamma^4(3\beta - \gamma^2)^3 - 2(\beta + \gamma^2)^2(2\beta - \gamma^2)(\beta^2 + \beta\gamma^3 + 4\gamma^4)\right]\left[4\beta(\beta + \gamma^2) + \gamma^2\right]\right\} > 0,$$

which again holds for any $\beta$ and $\gamma$ such that $\beta - \gamma^2 > 0$ as shown in Figure A.3.

<u>Step 3.</u> The center of the hyperbola of the first Riccati equation is always above and to the right of that of the second Riccati equation. Comparing $(A.23c)$ and $(A.24c)$ we have:

$$-\frac{3(r - \lambda\delta)}{\lambda\beta} - \frac{r - \lambda\delta}{\lambda(\beta + \gamma^2)} > 0, \qquad (A.44a)$$

$$-\frac{2(r - \lambda\delta)}{\lambda\gamma} - \frac{\gamma(3\beta - \gamma^2)(r - \lambda\delta)}{\lambda(\beta + \gamma^2)^2} > 0. \qquad (A.44b)$$

<u>Step 4.</u> The upward sloping asymptotes of the Riccati hyperbolas never cross inside the stable cone. This is a direct consequence of condition (26) and the result of previous steps. Step 1 allows that the intersection between the downward sloping asymptote of the first Riccati equation and the flat asymptote of the second one may fall in the shaded stability cone of Figure 2. Step 2, together with Step 1 establishes that the center of the hyperbola

of the second Riccati equation falls outside the stability region. More precisely, that such center is above both asymptotes of the stability condition. Step 3 shows that the same can be said of the center of the first Riccati hyperbola. Then we have the following three intersections to analyze:

#1. Condition (26) requires that the slope of the upward sloping asymptote of the second Riccati hyperbola be steeper than any of the others. Because of the result of Step 3, being this asymptote steeper than any of the other two ensures that the crossing with the upward sloping asymptote of the first Riccati hyperbola will happen above the upward sloping asymptote of the stability condition, *i.e.*, in an unstable region.

#2 The intersection of the upward sloping asymptote of the second Riccati hyperbola with the downward sloping asymptote of the first Riccati hyperbola also falls in an unstable region (above the upward sloping asymptote of the stability condition). This is a direct consequence of condition (26) and the result of Step 3.

#3 The intersection of the upward sloping asymptote of the first Riccati hyperbola with the flat asymptote of the second Riccati hyperbola falls necessarily to the left of the center of the second Riccati hyperbola, *i.e.*, in an unstable region. This follows from the relative magnitude of the slopes of the upward sloping asymptotes imposed by condition (26), the relative position of the centers of they Riccati hyperbolas shown in Step 3, as well as the relative position of the center of the second Riccati hyperbola relative to the stable cone shown in Step 2.

<u>Step 5.</u> There only remains to show that the intersection of the downward sloping asymptote of the first Riccati hyperbola may intersect the flat asymptote of the second one inside the stable cone. A necessary and sufficient condition for this to happen is that there exists a positive difference between the downward sloping asymptote of the first Riccati hyperbola and the upward sloping asymptote of the stability condition, both evaluated at their crossing with the flat asymptote of the second Riccati hyperbola given by $\hat{\phi}_2^g$ in equation $(A.24c)$. The resulting condition is:

$$
\lambda\beta\gamma(r-\lambda\delta)(\beta+\gamma^2)\Big\{2(2\beta-\gamma^2)(\beta^2-4\beta\gamma^2-\gamma^4)(\beta+\gamma^2)(5\beta+\gamma^2) \tag{A.45}
$$

$$
+\gamma^3(3\beta-\gamma^2)(5\beta+\gamma^2)\left[(2\beta^2+7\beta\gamma^2-\gamma^4)[(\beta+\gamma)+\sqrt{\beta^2+\gamma^2}]-3\gamma(\beta+\gamma^2)^2(\gamma-\sqrt{\beta^2+\gamma^2})\right]
$$

$$
-(\gamma-\sqrt{\beta^2+\gamma^2})\left[\beta[\gamma^4(3\beta-\gamma^2)^3-2(\beta+\gamma^2)(\beta^2+\beta\gamma^3+4\gamma^4)]\right][4\beta(\beta+\gamma^2)+\gamma^2]\Big\}>0.
$$

Figure A.4 shows the combinations of $\beta$ and $\gamma$ for which the intersection # 4 falls in the stable cone. Figure A.4 proves two important results. First, it is possible to find more than a single combination of $\beta$ and $\gamma$ for any given value of the other parameters of the model for which there is a unique stable MPE for this game. Second, it is more likely to find a unique stable MPE for low values of $\beta$, and when $\beta-\gamma>0$.
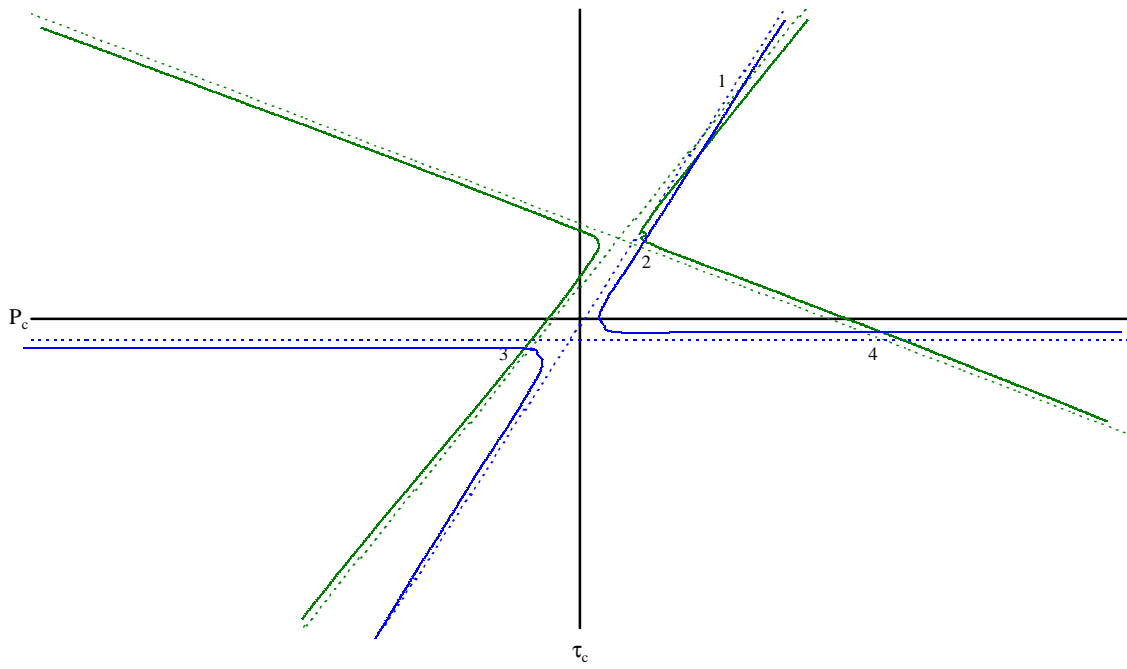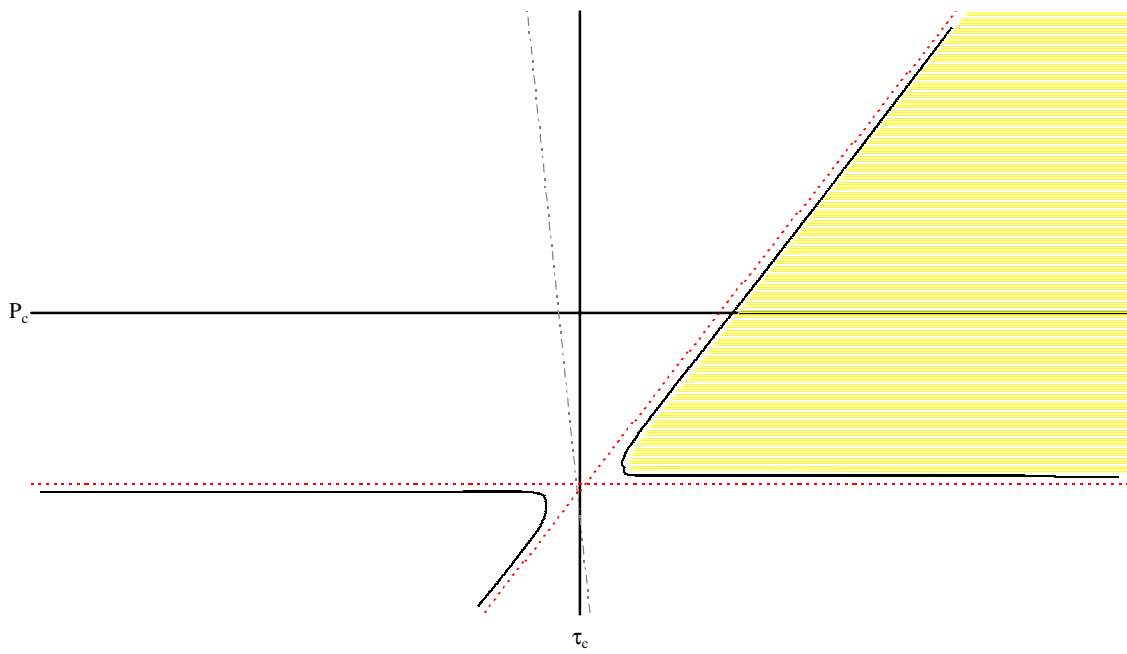
**Figure 1.** Riccati Equations
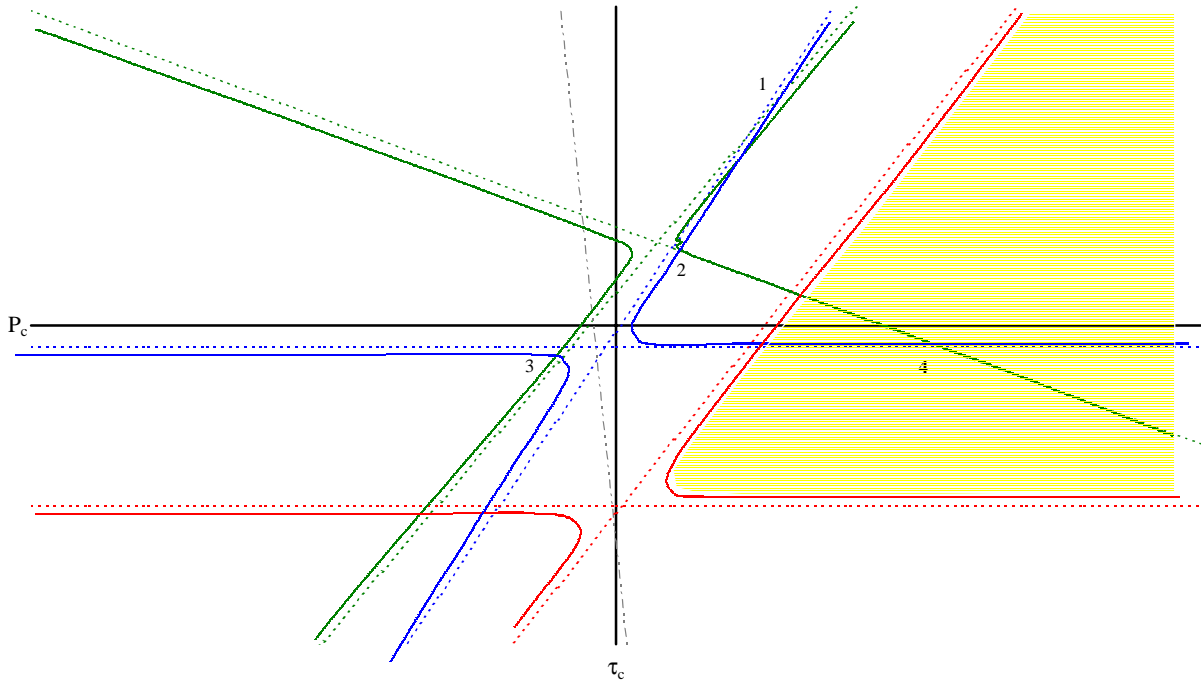


**Figure 2.** Stability Condition
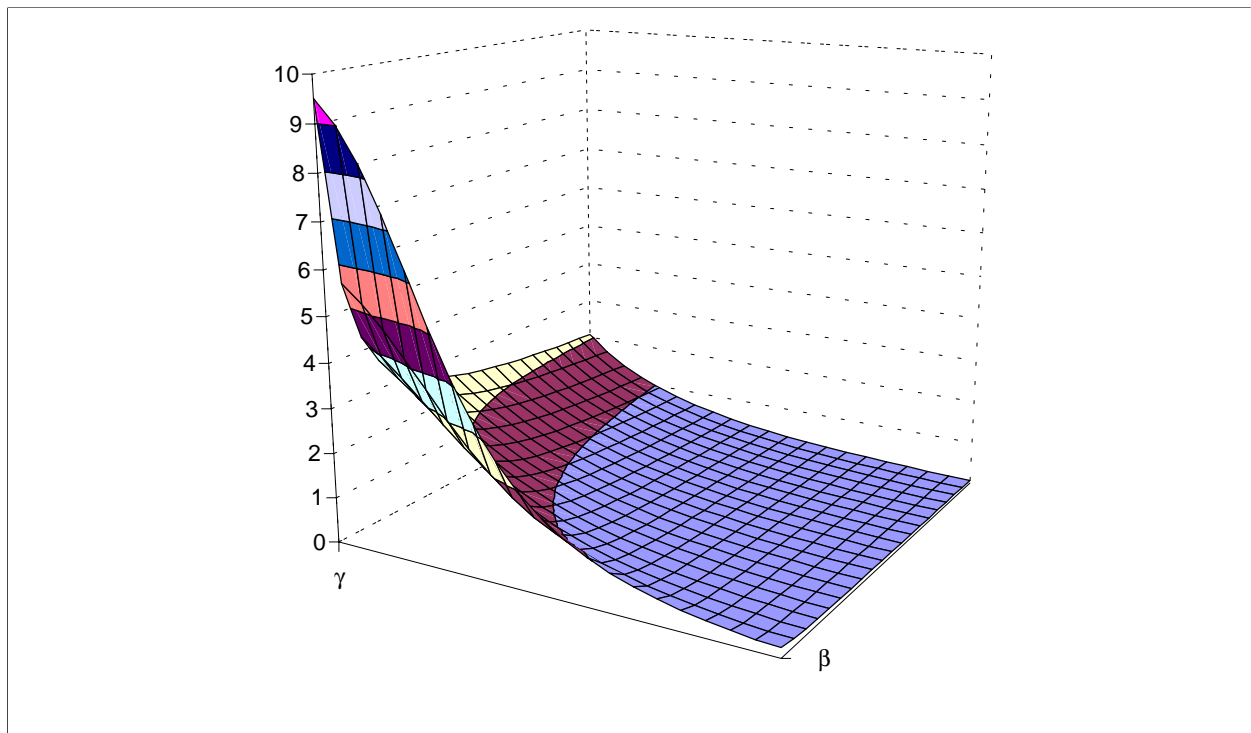
**Figure 3.** Solutions



**Figure 4.** Price Difference Between MPE and OLE Stable Solutions
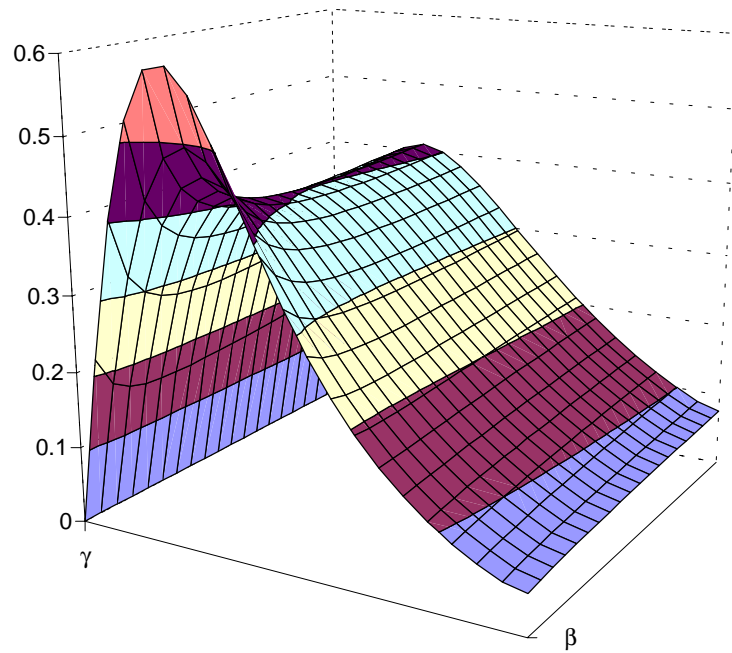
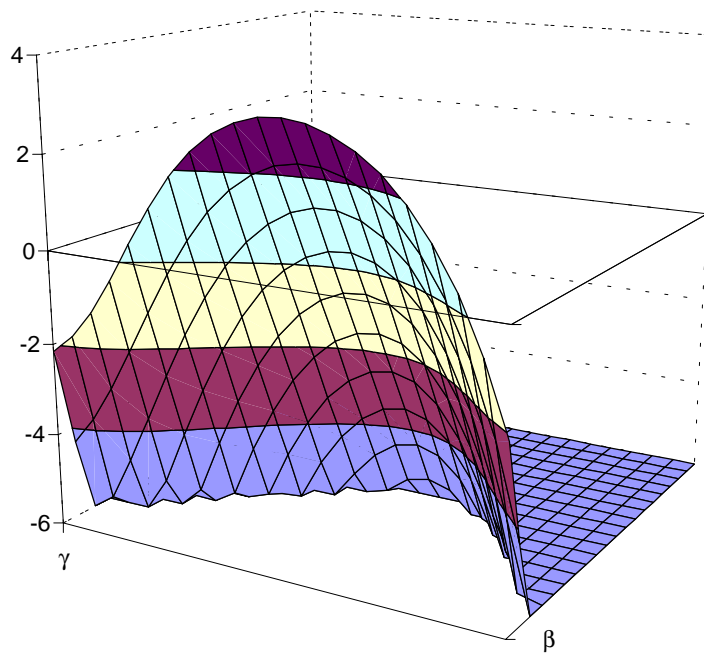**Figure 5.** Tariff Difference Between MPE and OLE Stable Solutions



**Figure 6.** Steady-State Marginal Cost Difference Between MPE and OLE Stable Solutions
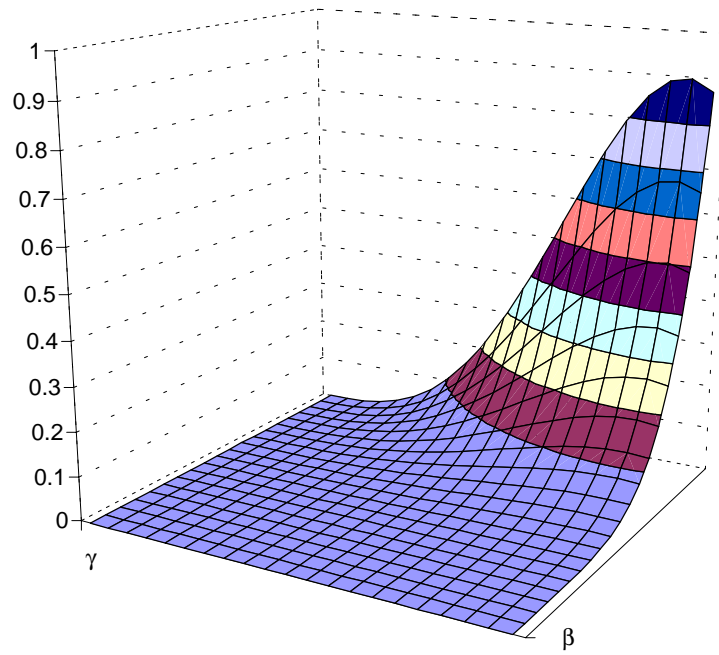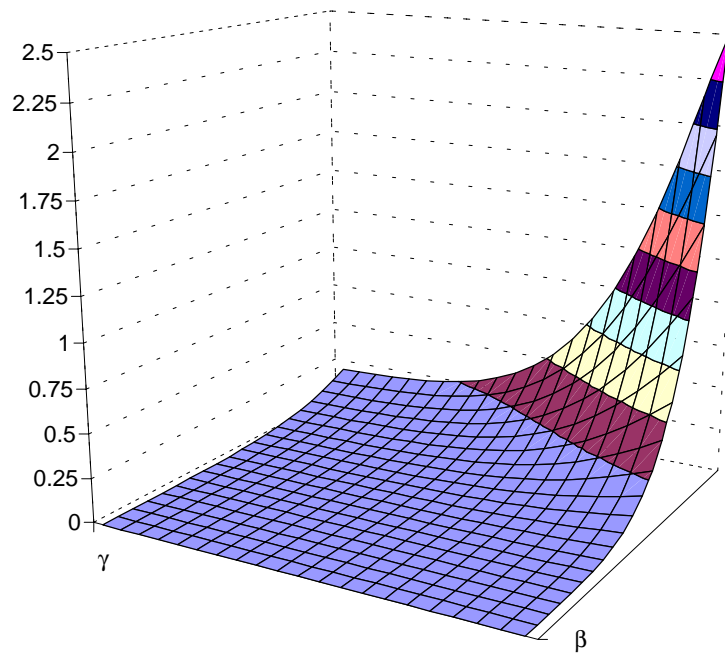
**Figure A.1.** Stability Cone



**Figure A.2.** Distance Between Flat Asymptotes

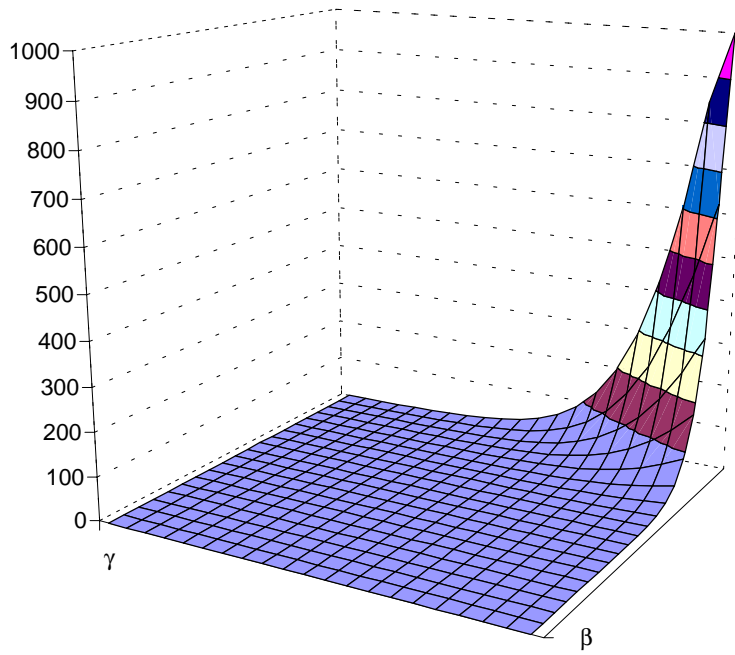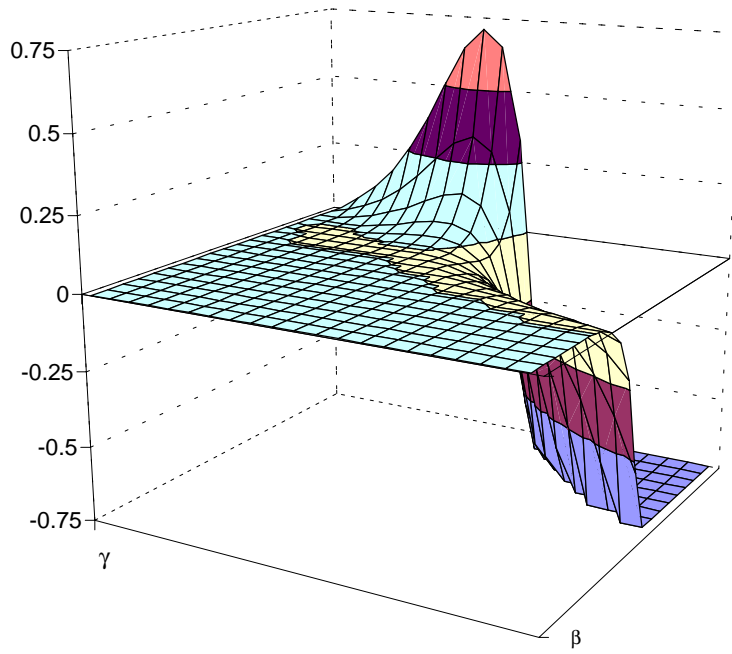**Figure A.3.** Distance Between Center of 2nd Riccati Equation and Stability Cone



**Figure A.4.** Unique Stable MPE