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ABSTRACT

Incentives in Dynamic Duopoly*

We compare steady states of open loop and locally stable Markov perfect equilibria (MPE) in a general symmetric differential game duopoly model with costs of adjustment. Strategic incentives depend on whether an increase in the state variable of a firm hurts or helps the rival and on whether there is intertemporal strategic substitutability or complementarity at the MPE. Furthermore, we characterize completely strategic incentives in the linear-quadratic specification of the model and find that when production (price) is costly to adjust there is intertemporal strategic substitutability (complementarity) and the steady state of the Markov perfect equilibrium is more (less) competitive than the steady state of the open-loop equilibrium, which coincides with the static outcome. In particular, in a differentiated product duopoly market with price competition and costly production adjustment, each firm's leadership attempts turn into Stackelberg price warfare, yielding a MPE steady state outcome more competitive than static Bertrand competition. The static strategic complementarity in the price game is turned into intertemporal strategic substitutability.

JEL Classification: C73
Keywords: adjustment costs, Bertrand, Cournot, differential game, Markov perfect equilibrium, open-loop equilibrium, product differentation and Stackelberg warfare point

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NON-TECHNICAL SUMMARY

We study the strategic incentives arising in duopolistic interaction over an infinite horizon in the presence of adjustment costs. The analysis is cast in the context of a differential game of a differentiated product duopoly. Strategic incentives are characterized comparing steady states of open-loop and Markov perfect equilibria of the dynamic game. We explain the role of adjustment costs in preserving, or reversing, short-run (static) strategic substitutability or complementarity in the intertemporal framework. The Paper provides a taxonomy of strategic incentives in a general dynamic duopoly game. This establishes an infinite horizon differential game counterpart of the classification of strategic incentives in two-stage games of Fudenberg and Tirole. The linear-quadratic case is fully characterized, extending previous work. An important step in the analysis is the study of price competition when production is costly to adjust. This model turns out to provide a dynamic equilibrium rationalization of the 'Stackelberg warfare point'.

We study first a general (non-linear) symmetric duopoly model with adjustment costs and compare Markov perfect with open-loop equilibria. The general model allows for either Cournot or Bertrand competition with production or price adjustment costs. Mixed cases where the adjustment cost falls on a variable different from the strategic variable of the firm are allowed. At any point in time a firm controls the rate of change of its strategic variable (price or output). The open-loop strategies are those in which the firms commit to a path for the game. Markov strategies depend on the payoff-relevant variables, that is, the state variables. At a Markov perfect equilibrium (MPE) strategies are optimal for a firm for any state of the system given the strategy of the rival. MPE therefore capture the strategic incentives that firms face.

The first result is that the steady state of open loop equilibria (OLE) are in one-to-one correspondence with the (interior) static Nash equilibria of the duopoly provided that adjustment costs are minimized when there is no adjustment. The OLE provides a benchmark against which the strategic incentives at the MPE can be explained. The second result characterizes the effects of such incentives at symmetric and locally stable MPE on the difference between the steady state of the MPE, y^*, and that of the OLE, the (assumed unique now) static Nash equilibrium y^N. It is shown that if an increase in the state variable of firm j hurts firm i the sign of $\{y^* - y^N\}$ is positive or negative depending on whether, at y^*, there is intertemporal strategic substitutability or complementarity. That is, whether, respectively, an increase in the state variable of firm i decreases or increases the action of firm j.

Equipped with these results we turn to the linear-quadratic model and we provide a complete characterization of the stable symmetric linear MPE (LMPE). Our contribution completes the map of possible strategic interaction

in continuous time dynamic duopoly models with one strategic variable per firm and adjustment costs. We find that when production (price) is costly to adjust there is intertemporal strategic substitutability (complementarity) and the steady state of the LMPE is more (less) competitive than the static outcome (and this holds irrespective of whether competition is in prices or quantities).

The rest of the Paper is devoted to the mixed case of price competition with production adjustment costs. It is assumed that firms are committed to supply whatever demand is forthcoming at the set price and there are no inventories. That is, we consider the classical Bertrand competition case. We find that the steady state LMPE outcome with price competition and costly production adjustment is more competitive than Bertrand and that LMPE price trajectories involve lower prices uniformly than the OLE trajectory. By cutting its price today a firm will make the rival smaller and therefore less aggressive in the future because its short-run marginal cost will have increased due to costly production adjustment. To raise a rival's cost a firm has to cut prices today. This will push the rival firm towards setting higher prices. When firms face symmetric (or not too asymmetric) production adjustment costs the strategy is self-defeating and firms are locked into a price war.

Price leadership can be understood as an attempt by a firm to soften the price policy of the rival (or rivals). When firms are symmetric (and therefore have symmetric commitment capacities) the leadership attempt by each firm turns into Stackelberg warfare yielding a steady state outcome more competitive than static Bertrand competition. The strategic complementarity of the static price game is transformed into an intertemporal strategic substitutability in the presence of costly production adjustment. In consequence, the MPE steady state of our dynamic market provides an equilibrium story for the 'Stackelberg warfare point' where each firm in a duopoly attempts to be leader in a quantity-setting game.

We have shown that what drives the competitiveness of a market in relation to the static benchmark is whether production or prices are costly to adjust and not the character of the competition (Cournot or Bertrand). Indeed, when output (price) is costly to adjust the MPE steady state is more (less) competitive than the static Nash equilibrium. In particular, the static strategic complementarity characterizing price competition is turned into intertemporal strategic substitutability whenever firms face similar adjustment production costs. The outcome is fierce competition and a steady state below the static Bertrand benchmark.

The consideration of adjustment costs has implications for empirical work. Adjustment costs are indeed important in quite a few industries. The importance of taking into account the dynamic structure of the market when estimating product differentiation models cannot be underscored. For example, it is typically assumed that firms compete according to a static Bertrand model. From this assumption sophisticated estimates of patterns of elasticities and cross-elasticities of substitution among products are derived building on discrete choice theory. An obvious problem is that if a dynamic structure exists in the industry then there will be biases in the estimation of the degree of product differentiation. If the true model of an industry corresponds to our case of price competition with production adjustment costs, the estimates based on static Bertrand competition would systematically overestimate the degree of substitutability of the products. The lesson to draw is that, even when the modeller is reasonably certain that collusion is not an issue in an industry to neglect the dynamic structure is dangerous and leads to biases in the estimation.

1. Introduction

We study the strategic incentives arising in duopolistic interaction over an infinite horizon in the presence of adjustment costs. The analysis is cast in the context of a differential game of a differentiated product duopoly. Strategic incentives are characterized comparing steady states of open-loop and Markov perfect equilibria of the dynamic game. We explain the role of adjustment costs in preserving, or reversing, short-run (static) strategic substitutability or complementarity in the intertemporal framework.

The paper provides a taxonomy of strategic incentives in a general dynamic duopoly game. This establishes an infinite horizon differential game counterpart of the classification of strategic incentives in two-stage games of Fudenberg and Tirole (1984).[1] The linear-quadratic case is fully characterized extending previous work of Reynolds (1987) and Driskill and McCafferty (1989), who examined the case of Cournot competition with production adjustment costs. An important step in the analysis is the study of price competition when production is costly to adjust. This model turns out to provide a dynamic equilibrium rationalization of the "Stackelberg warfare point" (Stackelberg (1952)).

We study first a general (nonlinear) symmetric duopoly model with adjustment costs and compare Markov perfect with open-loop equilibria. The general model allows for either Cournot or Bertrand competition with production or price adjustment costs. Mixed cases where the adjustment cost falls on a variable different from the strategic variable of the firm are allowed. A firm controls at any point in time the rate of change of its strategic variable (price or output). The open-loop strategies are those in which the firms commit to a path for the game. Markov strategies depend on the payoff-relevant variables, that is, the state variables. At a Markov Perfect Equilibrium (MPE) strategies are optimal for a firm for any state of the system given the strategy of the rival. MPE therefore capture the strategic incentives that firms face.

The first result (Proposition 2.1) is that the steady state of Open Loop Equilibria (OLE) are in one-to-one correspondence with the (interior) static Nash equilibria of the duopoly provided that adjustment costs are minimized when there is no adjustment. The OLE provides a benchmark against which the strategic incentives at the MPE can be explained. The second result (Proposition 2.2) characterizes the effects of such incentives at symmetric and locally stable MPE on the difference between the steady state of the MPE, y^*, and that of the OLE,

1 See also Bulow et al (1985). Lapham and Ware (1994) provide a taxonomy of the strategic incentives in discrete time dynamic games for small adjustment costs. Strategic incentives in the choice of capacity followed by collusive pricing, supported with repeated competition, have been analyzed by Benoit and Krishna (1987) and Davidson and Deneckere (1990).

the (assumed unique now) static Nash equilibrium y^N. It is shown that if an increase in the state variable of firm j hurts firm i the sign of $\{y^* - y^N\}$ is positive or negative depending on whether, at y*, there is intertemporal strategic substitutability or complementarity (that is, whether, respectively, an increase in the state variable of firm i decreases or increases the action of firm j).

Equipped with these results we turn to the linear-quadratic model and we provide (Proposition 3.1) a complete characterization of the stable symmetric linear MPE (LMPE). Our contribution completes the map of possible strategic interaction in continuous time dynamic duopoly models with one strategic variable per firm and adjustment costs. We find that when production (price) is costly to adjust there is intertemporal strategic substitutability (complementarity) and the steady state of the LMPE is more (less) competitive than the static outcome (and this holds irrespective of whether competition is in prices or quantities). This follows from the results obtained for the Cournot model with production adjustment costs by Reynolds (1987) and Driskill and McCafferty (1989), from a duality result which yields the case of Bertrand competition with price adjustment costs, and from the analysis of this paper which studies the "mixed" case of price competition with production adjustment costs (and again by duality the result for quantity competition with price adjustment costs follows). The study of the mixed cases pushes the frontier in deriving explicit results in linear-quadratic differential games by allowing for the adjustment cost of a firm to depend on the controls of both firms.

The rest of the paper is devoted to the mixed case of price competition with production adjustment costs. It is assumed that firms are committed to supply whatever demand is forthcoming at the set price and there are no inventories. That is, we consider the classical Bertrand competition case.[2] We find that the steady state LMPE outcome with price competition and costly production adjustment is more competitive than Bertrand and that LMPE price trajectories involve lower prices uniformly than the OLE trajectory. A firm by cutting its price today will make the rival smaller and therefore less aggressive in the future because its short-run marginal cost will have increased due to costly production adjustment. To raise rival's cost a firm has to cut prices today. This will push the rival firm towards setting higher prices.

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2 This may come about because of an (implicit) contract with customers or because of regulation. For example, common carrier regulation in utilities (like electricity, gas, and local phone) typically requires an obligation to serve and firms have to fulfil all the demand forthcoming at the set price (see Spulber (1989)). In terms of our model and thinking in the electricity market the products of each firm could be differentiated because of the different package of services they provide or because one firm offers "green power" (with hydro, biomass conversion, or solar systems).

When firms face symmetric (or not too asymmetric) production adjustment costs the strategy is self-defeating and firms are locked into a price war.[3]

Price leadership can be understood as an attempt by a firm to soften the price policy of the rival (or rivals). When firms are symmetric (and therefore have symmetric commitment capacities) the leadership attempt by each firm turns into Stackelberg warfare yielding a steady state outcome more competitive than static Bertrand competition. The strategic complementarity of the static price game is transformed into an intertemporal strategic substitutability in the presence of costly production adjustment. In consequence, the MPE steady state of our dynamic market provides an equilibrium story for the "Stackelberg warfare point" where each firm in a duopoly attempts to be leader in a quantity setting game.[4]

The plan of the paper is as follows. Section 2 provides a general framework and derives the results on the steady state of OLE (Proposition 2.1) and the relationship between steady states of OLE and MPE (Proposition 2.1). Section 3 turns attention to a linear-quadratic specification of the game and provides a complete characterization of strategic incentives extending the results of the literature with the results obtained in the present paper. Section 4 is devoted to study price competition with production adjustment costs. Concluding remarks close the paper.

2. A general dynamic duopoly framework

Two firms, $i = 1, 2$, produce differentiated products, compete in continuous time with an infinite horizon, $t \in [0, \infty)$ and discount the future at rate r. The action ("control") of firm i, u_i, belongs to a subset of the real line U_i. Let y_i denote the state variable under the control of player i. The law of motion of the system is given by $\dot{y}_i(t) \equiv \frac{dy_i}{dt} = u_i(t)$ with $y_i(0) = y_i^0$, i =

[3] The basic force is present also when there is a learning curve (with no industry spillovers). Then a decrease in the price of a firm raises its output and lowers the rival's output, with the effect of lowering the marginal cost of the firm and increasing the marginal cost of the rival firm. For a strategic analysis of the learning curve see Fudenberg and Tirole (1983), Dasgupta and Stiglitz (1988), and Cabral and Riordan (1994). Miravete (1997) examines a differential game model where learning by doing reduces fixed costs of production.

[4] Stackelberg (p.194-195, 1952) thought that his leader-follower solution "is unstable, for the passive seller can take up the struggle at any time......It is possible, of course, that the duopolists may attempt to supplant one another in the market so that 'cut-throat' competition breaks out." The are several attempts in the literature to endogenize leadership (see Hamilton and Slustky (1990), Deneckere and Kovenock (1992) and Mailath (1993)).

1, 2. [5] The state variables are thus (y_1, y_2). The instantaneous profit of firm i is given by $\pi_i = R_i(y_1, y_2) - F_i(u_1, u_2)$, where R_i stands for revenue net of production costs, and F_i is the adjustment cost, which is a function of the controls (u_1, u_2). The duopoly game is symmetric. That is, $U_1 = U_2$ and $\pi_1(y_1, y_2; u_1, u_2) = \pi_2(y_2, y_1; u_2, u_1)$). Furthermore, we assume that R_i is smooth and concave in y_i, and F_i is smooth and convex in (u_1, u_2) with $\frac{\partial F_i}{\partial u_j}(0,0) = 0$, and $F_i(0, 0) = 0$, i, j = 1, 2. The first assumption (subject to standard boundness conditions) implies that there is a Nash equilibrium of the static simultaneous move game in which firm i has payoff R_i and strategy y_i, i = 1, 2. The second assumption implies that adjustment costs are minimized when there is no adjustment. We have therefore that π_i is concave in (y_i, u_i).

This formulation encompasses both quantity or price competition with quantity or price adjustment costs. To ease notation suppose that production costs are zero. We have Cournot competition where $R_i = P_i(x_1, x_2)\, x_i$, with $P_i(x_1, x_2)$ the inverse demand of firm i, and $u_i = \dot{x}_i$, the rate of change of its output (the state variable $y_i = x_i$). We have Bertrand competition where $R_i = p_i D_i(p_1, p_2)$, with $D_i(p_1, p_2)$ the demand of firm i, with $u_i = \dot{p}_i$, the rate of change of its price (the state variable $y_i = p_i$).[6] With production adjustment costs we have $F_i(\dot{x}_i)$ and with price adjustment costs, $F_i(\dot{p}_i)$. Note that we allow also the "mixed" cases in which there is quantity (price) competition and price (quantity) is costly to adjust. In the mixed cases the adjustment cost borne by firm i depends also on the control of the rival firm j. We make the convention that with Cournot competition the state variables are quantities and with Bertrand competition prices.

We will study both the open-loop and Markov perfect equilibria of this (stationary) differential game. When a firm's strategy is only a function of time, being independent of the state variables, it is called an open-loop strategy. An open-loop equilibrium (OLE) is an open-loop strategy profile such that each firm's strategy is a best response to the other's choice. The open-loop strategy space for firm i will consist of the piecewise continuous functions of time. In general, firms' strategies can depend on past histories. Markov strategies are the strategies which depend only on payoff-relevant state variables (in our case the vector (y_1, y_2)). A Markov perfect equilibrium (MPE) is a Markov strategy profile such that each strategy is a best response to the others for any state. Hence, a Markov perfect equilibrium is a subgame-perfect equilibrium. We will restrict attention to Markov strategies which are stationary (that is, time

5 A variable with a dot represents the time derivative of the variable. Two dots represent the second time derivative.

6 It is assumed that the firms must supply all the demand at the going prices. The products are not storable or the cost of holding inventory is infinite.

independent), continuous, and (almost everywhere) differentiable functions $u_i(y_1, y_2)$, i = 1, 2, of the state variables.

From the necessary conditions for $u_i(y_1, y_2)$, i =1, 2 to form an MPE, at a steady state we obtain easily that (see Appendix I) for i, j = 1, 2, $j \neq i$, $r - \frac{\partial u_j}{\partial y_j} \neq 0$ and

(*) $$\frac{\partial R_i}{\partial y_i} + \frac{\frac{\partial R_i}{\partial y_j}\frac{\partial u_j}{\partial y_i}}{(r - \frac{\partial u_j}{\partial y_j})} = 0.$$

Now, at an OLE firms do not take into account the effect of changes of the state variables on the strategies, that is, there is no "feedback" and $\frac{\partial u_j}{\partial y_i} = 0$, i, j = 1, 2. An (interior) static Nash equilibrium is defined by the FOC, $\frac{\partial R_i}{\partial y_i} = 0$. In consequence:

<u>Proposition 2.1.</u> Stationary states of open-loop equilibria are in one-to-one correspondence with interior Nash equilibria of the static duopoly game.

The intuition for the result is based in that adjustment costs are minimized when there is no adjustment. At a stationary state the strategy of rival firm j is not to change the current action. Firm i can make the marginal cost of adjustment arbitrarily small by choosing u_i small enough. It follows that not changing is a best response only if the net marginal revenue of a change in action (namely, $\frac{\partial R_i}{\partial y_i}$) is equal to 0. This holds only at a static interior Nash equilibrium

At an MPE in general there is feedback and the steady state differs from the stationary OLE or static Nash equilibrium. It is difficult to characterize MPE in differential games. However, it is possible to ascertain the effects of strategic incentives at a locally stable steady state of MPE at least in the symmetric version of the model (with symmetric product differentiation and symmetric adjustment costs).

Consider a symmetric MPE $u_i(y_1, y_2)$ (that is, $u_1(y_1, y_2) = u_2(y_2, y_1)$), and a symmetric steady state $y_1 = y_2 = y^*$ of the dynamical system $\dot{y}_j = u_j(y_1, y_2)$, j = 1, 2. We will say that (y^*, y^*) is regular if $\frac{\partial u_i}{\partial y_i}(y^*, y^*) \neq \frac{\partial u_i}{\partial y_j}(y^*, y^*) \neq 0$, $j \neq i$, i = 1, 2. If the static game is symmetric and the Nash equilibrium is unique then the equilibrium will be symmetric also. Say that a symmetric Nash equilibrium of the (symmetric) static game is regular if it is interior and $\left(\frac{\partial^2 R_i}{(\partial y_i)^2}\right)^2 - \left(\frac{\partial^2 R_i}{\partial y_j \partial y_i}\right)^2 \neq 0$ at the equilibrium, for i = 1, 2. Using (*) strategic incentives at a locally stable MPE can be characterized (see Appendix I for a proof):

<u>Proposition 2.2.</u> Suppose that there is a unique regular Nash equilibrium of the static game, (y^N, y^N). Consider a locally stable, regular, and symmetric steady state, (y^*, y^*), of a given symmetric MPE of the dynamic game, $u_i(y_1, y_2)$, i = 1, 2. Then sign $\{y^* - y^N\}$= sign $\{\frac{\partial R_i}{\partial y_j}(y^*, y^*)\ \frac{\partial u_j}{\partial y_i}(y^*, y^*)\}$.

The proposition extends the taxonomy of strategic behavior due to Fudenberg and Tirole (1984) to the differential game duopoly. Strategic incentives to under or over invest in a state variable at a locally stable MPE, with respect to the OLE benchmark, depend on whether there is <u>intertemporal</u> strategic substitutability ($\frac{\partial u_j}{\partial y_i} < 0$) or complementarity ($\frac{\partial u_j}{\partial y_i} > 0$) as well as whether "investment" of a firm in its state variable makes the rival worse ($\frac{\partial R_i}{\partial y_j} < 0$) or better ($\frac{\partial R_i}{\partial y_j} > 0$) off. In the Cournot case we have that $\frac{\partial R_i}{\partial y_j} < 0$, and in the Bertrand case that $\frac{\partial R_i}{\partial y_j} > 0$. In the linear quadratic model we will see that intertemporal strategic substitutability (complementarity) obtains when production (price) is costly to adjust.

3. The linear-quadratic model: Overview of results

Given the difficulty in solving general differential games attention has concentrated in the linear-quadratic model. In our duopoly we have that $R_i = (A - B y_i - C y_j)y_i$ with $B > |C| \geq 0$ and the unique (and symmetric) Nash equilibrium of the static game is given by $y^N = A/(2B+C)$. If the adjustment costs are borne by the strategic variable of the firm (for example, production in a Cournot model or price in a Bertrand model) then $F_i (u_1, u_2) = \lambda(u_i)^2/2$, $\lambda > 0$. In a mixed model $F_i (u_1, u_2) = \lambda(-Bu_i - Cu_j)^2$. For example, with quantity competition ($R_i = (A - B x_i - C x_j)x_i$) and price adjustment costs: $F_i = \lambda(\dot{p}_i)^2/2$, where $\dot{p}_i = -B\dot{x}_i - C\dot{x}_j$.[7] It is worth noting that with Cournot competition the case of homogenous product and increasing marginal cost can be accommodated. Indeed, let C > 0 and note that $R_i = (A - B x_i - C x_j)x_i = (A - C (x_i + x_j))x_i - (B - C) (x_i)^2$. The slope of marginal cost is (B - C) /2.

7 Nothing substantive would change in the analysis by including a linear term in the adjustment.

We will investigate linear Markov perfect equilibria (LMPE). Namely, equilibria in which the strategies are linear (affine) functions of the state variables.[8] Existence and other properties of LMPE of Cournot dynamic duopoly games with homogenous product and production adjustment costs were studied by Reynolds (1987 and 1991) and Driskill and McCafferty (1989). We provide here an overview of their results, a reinterpretation covering the case of Bertrand competition with differentiated products and price adjustment costs, and an overview of the results obtained in Section 4 below for the mixed case of price competition with production adjustment costs. This introduces further complexity in the analysis because now the action of a firm affects the adjustment cost of the rival.

In the linear-quadratic model the existence of a unique stable symmetric LMPE can be ascertained from received results in the literature and the results in Section 4.[9] Given MPE strategies, $u_i = \alpha + \beta y_i + \gamma y_j$, i, j = 1, 2, $j \neq i$, with $\beta < 0$ and $\beta^2 - \gamma^2 > 0$ for stability, the steady state is symmetric and given by $y^* = A/(2B + C(1 - \gamma(\beta - r)^{-1}))$. This follows from (*) by setting $\frac{\partial R_i}{\partial y_j}(y, y) = -Cy, \frac{\partial u_j}{\partial y_i} = \gamma, \frac{\partial u_j}{\partial y_j} = \beta$, and $\frac{\partial R_i}{\partial y_i}(y, y) = A-(2B+C)y$ and obtaining $A-(2B+C)y - (C\gamma y/(r-\beta)) = 0$. Obviously, the equilibrium parameters β and γ depend on the exogenous parameters of the model (B, C, λ and r; A is a scale parameter and does not affect β or γ). We have that $\text{sign}\{y^* - y^N\} = \text{sign}\{-C\gamma\}$.

Reynolds (1987, 1991) and Driskill and McCafferty (1989) have characterized the case of Cournot competition with homogenous product, quadratic production costs ($C > 0$) and production adjustment costs. (As we have seen, this is equivalent to the product differentiation model.) They find that $\gamma < 0$ and, indeed, the steady state output is larger than the Cournot static output $A/(2B + C)$ because $-C\gamma > 0$. The reason is the presence of intertemporal strategic substitutability. A larger output by firm i today leads the firm to be more aggressive tomorrow. With symmetric adjustment costs both firms are in the same situation and quantities are pushed beyond the Cournot level.

Using the duality between price and quantity competition in the duopoly model with product differentiation (see Singh and Vives (1984)) we can characterize price competition with price

8 Equilibria in linear strategies can be rationalized as the limit as the horizon lengthens of the strategies used in finite horizon games. In a linear-quadratic differential finite horizon dynamic game the linear solution is unique in the class of strategies which are analytic functions of the state variables (Papavassilopoulos and Cruz (1979)). We do not explore potential nonlinear equilibria in our model.

9 Reynolds (1987) needed some restrictions on parameter values in order to prove existence and uniqueness of a stable LMPE. Driskill and McCafferty (1989) introduce a graphical apparatus which allows the analysis without parameter restrictions. We follow a similar method in Section 4.

adjustment costs. The model is formally identical to the one before but now $C < 0$. The effect of this is that now at the unique stable LMPE, the rate of change of prices of each firm is increasing in the price of the rival ($\gamma > 0$) and $-C\gamma > 0$. This makes the steady state price larger than the Bertrand static price. The economic force behind the results is intertemporal strategic complementarity. A firm by pricing high today will elicit high prices from the rival tomorrow. The cost of adjusting the price lends credibility to the strategy.

In Section 4 below we consider the "mixed" case in which there is price competition ($C < 0$) and production is costly to adjust. We show that there is a unique stable LMPE. At this equilibrium the rate of change of price of each firm is decreasing in the price of the rival ($\gamma < 0$). (See Proposition 4.2 below.) Now the steady state price is smaller than the Bertrand static price because $-C\gamma < 0$. Here a firm has to cut its price in order to induce the rival to price softly in the future, intertemporal strategic substitutability prevails. A duality argument gives us the results for Cournot competition ($C > 0$) with costly price adjustment. Now at the unique stable LMPE the rate of change of production of each firm is increasing in the output of the rival ($\gamma > 0$) and $-C\gamma < 0$.

The following proposition summarizes the results. We will say that a steady state is "more competitive" when it is lower (higher) in price (quantity) competition.

Proposition 3.1. In the linear-quadratic model there is a unique (globally) stable symmetric LMPE. The strategies are given by $u_i = \alpha + \beta y_i + \gamma y_j$, $i, j = 1, 2$, $j \neq i$, where $\beta < 0$ and $|\beta| > |\gamma| > 0$. The steady state is symmetric and given by $y^* = A/(2B + C(1 - \gamma(\beta - r)^{-1}))$. When production (price) is costly to adjust, $\gamma < 0$ ($\gamma > 0$) and y^* is more (less) competitive than the static Nash equilibrium $A/(2B + C)$.

We see thus that what determines the competitiveness of a market is whether the (symmetric) adjustment costs are supported by prices or quantities. With price adjustment costs intertemporal strategic complementarity ($\gamma > 0$) prevails and this pushes prices up. With production adjustment costs intertemporal strategic substitutability ($\gamma < 0$) prevails and this pushes prices down.

Our results show in a standard model that the outcomes of dynamic competition, even when firms condition only on payoff-relevant variables, need not be bounded between the Cournot and Bertrand static long-run outcomes. This result may come at first as a surprise because of thinking that static strategic complementarity or substitutability will translate, respectively, into intertemporal strategic complementarity or substitutability. This happens in Cournot (Bertrand) competition with production (price) adjustment costs. In the Cournot case a larger output by firm 1 today leads the firm to be more aggressive tomorrow. With symmetric production adjustment costs both firms are in the same situation and quantities are pushed beyond the Cournot level.[10] Furthermore, in models of price competition with switching costs Markovian equilibria yield steady state prices above the one-shot level (Beggs and Klemperer (1992)). The reason is that intertemporal strategic complementarity holds since a higher price today, increases the market share of the rival and therefore makes the rival price more softly tomorrow.[11]

We can also show the following comparative static result.

Proposition 3.2. As the adjustment cost λ tends to zero y^*-y^N does not tend to zero and as the discount rate r tends to infinity y^* tends to y^N.

The comparative static result of y^* with respect to r is intuitive. The discrepancy between y^* and y^N is governed by $|\gamma|/(r-\beta)$. When the discount rate is low the future matters more and the strategic incentive increases. It is not surprising then that when r grows unboundedly, and the future does not matter, the steady state converges to the static Nash level. The result that when λ tends to zero y^* does not tend to y^N needs more explanation. One effect is that when λ is low the strategic incentive, as measured by $|\gamma|$, should be larger. This is so since for low λ it will be less costly for the rival firm to change its action and then firm i has more incentive to change his state variable to influence the behavior of the rival. However, a low λ should also increase the response to the own state variable, $|\beta|$. In fact, as λ tends to zero both $|\beta|$ and $|\gamma|$ tend to infinity but $|\gamma| / |\beta|$, and therefore $|\gamma|/(r-\beta)$, tend to a number between 0 and 1. [12]

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10 Fershtman and Kamien (1987) consider a quantity setting game with slowly adjusting prices in which the steady state price is below the Cournot price. In their model there is intertemporal strategic substitutability (a higher output of firm 1 today leads to lower price and lower output for firm 2 tomorrow). Similarly, in the alternating move quantity-setting duopoly game of Maskin and Tirole (1987) the (MPE) dynamic reaction functions of the firms are monotone decreasing and there is intertemporal strategic substitutability.

11 However, in the alternating move Markov price game with homogenous product of Maskin and Tirole (1988b) different types of equilibria can be supported due to the fact that the (equilibrium) dynamic reaction function of a firm is not monotonic. See also Eaton and Engers (1990) for results with product differentiation.

12 When $\lambda = 0$ there does not exist a LMPE but there is a nonlinear MPE yielding y^N (firm i jumps to y^N if its state variable is not at the Nash level and stays put otherwise). There is thus a discontinuity of LMPE as the friction in the market disappears (this has been found also in Fershtman and Kamien (1987), Reynolds (1987) and Driskill and McCafferty (1989)).

Remark: With Cournot (Bertrand) competition and production (price) asymmetric adjustment costs strategic incentives are similar to the symmetric case. In the first (second) instance there is intertemporal strategic substitutability (complementarity) independently of whether adjustment costs are symmetric or asymmetric. Furthermore, in both cases when the adjustment cost of firm 2 is very small the LMPE steady state is close to the Stackelberg outcome with firm 1 as leader. These results show the emergence of the Stackelberg equilibrium (the first, with quantity leadership and the second with price leadership) as the steady state of dynamic competition where the leader is the firm that faces an adjustment cost, and therefore can commit. [13] However, in the mixed cases the strategic incentives with symmetric or asymmetric adjustment costs may differ. In Section 4.2 below it is shown that with production adjustment costs and price competition if the adjustments costs are asymmetric enough intertemporal strategic complementarity can be restored.

4. Price competition with costly production adjustment

The rest of the paper concentrates on the linear-quadratic model with Bertrand competition and production adjustment costs. Let us set the notation for this model. The instantaneous profit of firm i is given by $\pi_i(t) = p_i(t)D_i(p_1(t), p_2(t)) - F(\dot{x}_i(t))$, $i = 1, 2$, where $D_i(p_1, p_2) = a - bp_i + cp_j$, with $b > c \geq 0$, and $F(\dot{x}_i) = \lambda(\dot{x}_i)^2/2$, $\lambda > 0$, with $\dot{x}_i = -b\dot{p}_i + c\dot{p}_j$. The state variables are prices (p_1, p_2). We require the initial state $(p_1(0), p_2(0)) = (p_1^0, p_2^0)$ to belong to the region in price space P for which the demand for both firms is non-negative.[14]

Demands can be derived from the maximization problem of a representative consumer who has a quadratic and symmetric utility function for the differentiated goods (and utility is linear in money): $U(x_1, x_2) = A(x_1 + x_2) - \left[B(x_1^2 + x_2^2) + 2Cx_1x_2\right]/2$. This yields inverse demands, $P_i(x_1, x_2) = A - B x_i - C x_j$, and demands, $D_i(p_1, p_2) = a - b p_i + c p_j$. Then, $a = A/(B+C)$, $b = B/(B^2 - C^2)$, and $c = C/(B^2 - C^2)$ with $B > |C| \geq 0$. When B = C the two products are homogeneous from the consumer's view point, when C = 0 the products are independent and when C < 0 they are complements.

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[13] The result is based on simulations. Hanig (1986) shows it for the Cournot case and we have done it for the Bertrand case. In both cases at the steady state firm 2 will necessarily be very close its static best response function since the firm faces almost no adjustment cost (and has almost no commitment power). Firm 1 will optimize accordingly and consequently will be close to its Stackelberg level.

[14] The region P is given by the intersection of a cone-shaped region with vertex (a/(b-c), a/(b-c)) and the non-negative orthant: $a - bp_i + cp_j \geq 0$, $i \neq j, i = 1, 2$, and $p_i \geq 0$.

4.1. Open-loop equilibria

Let $u_i \equiv \dot{p}_i$. We know already that the Bertrand equilibrium a/(2b-c) is the unique stationary state of OLE (Proposition 2.1). We claim now that there is a unique OLE which yields a stable trajectory. The following proposition states the result (the proof is omitted):

Proposition 4.1. There is a unique pair of OLE strategies which yield stable price trajectories. These strategies are given by, for i = 1, 2,

$$u_i(t) = \left\{(p_i^0 + p_j^0)/2 - p^B\right\}\phi_1 e^{\phi_1 t} + \left\{(p_i^0 - p_j^0)/2\right\}\phi_2 e^{\phi_2 t}$$

where $\phi_1 = \frac{1}{2}\{r - \sqrt{r^2 + \frac{4(2b-c)}{\lambda b(b-c)}}\}$, and $\phi_2 = \frac{1}{2}\{r - \sqrt{r^2 + \frac{4(2b+c)}{\lambda b(b+c)}}\}$. We have that $\phi_1 < \phi_2 < 0$. Whenever (p_1^0, p_2^0) is in P, $(p_1(t), p_2(t))$ is also in P for all t.

Remark: A similar result can be derived when adjustment costs are asymmetric.[15]

4.2. Markov perfect equilibria

Markov strategies depend only on payoff-relevant variables, the level of prices in our case. We restrict attention to strategies which are stationary (that is, time independent), continuous, and (almost everywhere) differentiable functions $u_i(p_1, p_2)$, i = 1, 2, of the prices. We will characterize stable LMPE. To do so we follow a standard approach and find a quadratic value function for the optimization problem firms face.

Proposition 4.2 summarize the characterization of the stable LMPE. A stable LMPE generates a stable stationary state starting from any initial condition in P. Proofs are provided in Appendix II.

Proposition 4.2. There exists a unique symmetric stable LMPE:

$$u_i^* = \alpha + \beta p_i + \gamma p_j,$$

with $\beta < \gamma < 0$ and $\alpha > 0$. It corresponds either to solution #3 or solution #5 in Table 1. The steady state is symmetric, with prices equal to $p^* = a/(2b-c(1-\gamma(\beta-r)^{-1}) < p^B = a/(2b-c)$.[16]

Remark: Table A in Appendix II provides closed-form expressions for the LMPE parameters α, β and γ as (complicated) functions of the underlying parameters of the model. The stable

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15 See Fershtman and Muller (1984) for general results on stability of OLE of Cournot-type models.

16 In terms of the utility parameters we have $p^* = A(B-C)/(2B-C(1-\gamma(\beta-r)^{-1}) < p^B = A(B-C)/(2B-C)$.

LMPE switches between branches of the solution depending on parameters (see Result A.1 in Appendix II).

Remark: The described MPE remains an equilibrium when the law of motion of the system is subject to additive shocks with mean zero. More precisely, suppose that the demand intercept is time dependent according to a shock that follows a Brownian motion, $a(t) = a + \sigma w(t)$ where $dw(t)$ is normally distributed with mean zero and variance dt. Consider outputs as the state variables, then the law of motion is given by $dx_i(t) = (-bu_i(t) + cu_j(t))\, dt + \sigma\, dw(t)$, $i = 1,2$, where $\sigma > 0$. Under these conditions our deterministic LMPE is also an equilibrium for the stochastic game (and it is independent of σ).[17]

It is possible to compare the OLE and the LMPE price paths using numerical simulations (see Appendix III):

Result 4.1.[18] Prices at the stable LMPE trajectory are (strictly) lower than the prices at the OLE trajectory for all initial states that belong to P.

Firms are more aggressive at the MPE than at the OLE due to a strategic incentive. Firm i would like rival firm j to price softly but the rate of price change u_j depends negatively on p_i (since, in equilibrium, $\gamma < 0$) and therefore there is an incentive to cut prices. This happens to both firms which get trapped in trying to elicit soft behavior of the rival cutting prices. By cutting prices today the rival (j) is made softer tomorrow because he is smaller and this raises his (short-run) marginal cost to increase output due to the production adjustment cost. This happens even though firm i has become more aggressive meanwhile because it is larger and therefore its (short-run) marginal cost is smaller. The first effect (the increase in firm j's marginal costs) dominates the indirect effect through the decrease in marginal costs of firm i.

If the initial state is the static Bertrand equilibrium both firms have an incentive to decrease prices. This did not happen at the OLE since then the rival would not react to a price cut by the firm over an interval (and then such a move would not have a first order effect on profit). At the LMPE a price cut represents a commitment because of costly production adjustment. Once deviated from the Bertrand equilibrium it is costly to go back to it. A firm then has the incentive to cut prices to make the rival softer tomorrow transforming the static strategic complementarity of static price competition into intertemporal strategic substitutability. Given that both firms have symmetric commitment capacities (adjustment costs) this attempt of leadership is self-

17 See, for example, Basar and Olsder (Section 6.5, 1982).

18 The term "result" refers to the proposition the "proof" of which relies to some extent in numerical analysis.

defeating and the outcome is price warfare. In this sense the steady state represents the "Stackelberg warfare point" in which the leadership attempt of both firms turns into very aggressive behavior.[19]

4.3. Comparative dynamics

4.3.1. Comparative statics of the steady state.

A number of comparative static and comparative dynamic results can be obtained. Recall that $p^* < p^B$. The following proposition is proved in Appendix III:

Proposition 4.3. As $\psi \equiv \lambda r^2$ tends to infinity p^B - p* tends to zero; as ψ tends to zero p^B - p* tends to a strictly positive number

When the adjustment cost is large (or the future does not matter much) the strategic incentive of a firm is small since it is very costly for the rival to change prices. We have in particular that the more costly to adjust output is the closer we are in steady state to the Bertrand equilibrium.[20] The explanation for the result when ψ tends to zero has been provided in Section 3.

Furthermore, using numerical methods we can obtain the following:

Result 4.2. The ratio $(p^B - p^*)/p^B$ is decreasing in $\psi \equiv \lambda r^2$ and increasing in the degree of product substitutability C.

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19 With asymmetric enough commitment capacities intertemporal strategic complementarity may be preserved and soft pricing induced. Consider a two period version, t = 0, 1, of the game with production adjustment costs equal to $F_i(x_i^t - x_i^{t-1}) = \lambda_i(x_i^t - x_i^{t-1})^2 / 2,\ \lambda_i \geq 0$. Suppose that $\lambda_1 > 0$ and $\lambda_2 = 0$. At the last period (period 1) firm 2 will price according to its static Bertrand best reply function since neither firm can manipulate the costs of firm 2. However, an increase in the price of firm 1 in period 0 induces a decrease in its output and therefore an increase in its marginal cost in period 1. This moves the best response function of firm 1 to the right in period 1. The outcome is higher prices for both firms giving a strategic incentive for firm 1 to raise its price in period 0. The described incentives will be the same whenever λ_2 is close to zero, in which case the period 1 best reply of firm 2 will also be affected by changing prices in period 0.

20 This seems to provide a counterpoint to the idea that "quantity precommitment and price competition yields Cournot outcomes" (Kreps and Scheinkman (1983)) since the source of the precommitment value of quantity is that quantity is more costly to adjust than price. However, in our model firms have only one strategic variable (the rate of price change) meanwhile in Kreps and Scheinkman (1983) firms make both quantity/capacity and price choices.

The closer substitutes the products are the larger the (relative) strategic incentive, because a cut in prices is more effective in inducing softer behavior of the rival (as C increases we have that $|\gamma|/|\beta|$ increases). The ratio $(p^B - p^*)/p^B$ can get as large as 35% when ψ is close to zero and C close to one. Table 1 gives $(p^B - p^*)/p^B$ for different values of C and λr^2. For example, even with high adjustment costs (λ = 100) and r = 10% (yielding ψ = 1), we have that $(p^B - p^*)/p^B$ = 21% if C = .95; if C = .7, then the ratio goes down to 7.6%.[21]

Table 1

Values of $(p^B - p^*)/p^B$

Ψ =	C = .1	C = .3	C = .5	C = .7	C = .9	C =.95
0	.13%	1.38%	4.73%	12.11%	28.42%	35.58%
0.01	.13%	1.33%	4.55%	11.65%	27.33%	34.21%
0.1	.12%	1.23%	4.18%	10.65%	24.81%	30.89%
1	.09%	.91%	3.07%	7.63%	16.98%	20.6%
10	.03%	.34%	1.10%	2.58%	5.26%	6.21%

4.3.2. Price dynamics

Price changes at the OLE (rewriting the strategies in terms of state variables[22]) are given by:

$$u_i(t) = \phi_1\left\{((p_1(t) + p_2(t))/2) - p^B\right\} + \phi_2\left\{(p_i(t) - p_j(t))/2\right\}$$
$$= \frac{\phi_1 + \phi_2}{2}(p_i(t) - p^B) + \frac{\phi_1 - \phi_2}{2}(p_j(t) - p^B)$$

and at the LMPE by:

$$u_i^*(t) = \beta(p_i(t) - p^*) + \gamma(p_j(t) - p^*)$$
$$= (\beta + \gamma)\left\{((p_1(t) + p_2(t))/2) - p^*\right\} + (\beta - \gamma)\left\{(p_i(t) - p_j(t))/2\right\}.$$

The OLE and LMPE trajectories have the following properties:

(i) A higher adjustment cost or discount rate slow down convergence to the steady state. Indeed, a price change towards the steady state today increases adjustment costs today but

21 Note that, a fortiori, p^* is decreasing in C (equalling the monopoly price A/2B when products are independent, C = 0, and the competitive price 0 when products are perfect substitutes, C = B).

22 Note, however, that this yields the open-loop price changes only along the equilibrium price trajectories.

decreases them in the future and when r increases the future is discounted more. ($-\phi_1$ and $-\phi_2$ as well as $-(\beta+\gamma)$ and $-(\beta-\gamma)$ decrease with λ and r.)[23]

(ii) There is *decreasing dominance*. Starting from an asymmetric initial position the system converges to the symmetric steady state. This happens because the larger firm is softer in pricing: $u_i - u_j = \phi_2\ (p_i - p_j)$ and $u_i^* - u_j^* = (\beta - \gamma)\ (p_i - p_j)$ are positive when $p_i - p_j < 0$.[24]

(iii) Convergence to the steady state is slower in the LMPE than in the OLE case. (Indeed, we have that $-(\beta+\gamma) < -\phi_1$ and $-(\beta-\gamma) < -\phi_2$.)[25]

(iv) There are trajectories for which there is *overshooting* with respect to the steady state prices. The initially larger firm is the one that overshoots the steady state level. For example, this happens both at the OLE and at the LMPE, for firm 2 when: A = B = 1, C= 0.5, r = 0.05, λ = 0.1, $p_1(0) = 0.7$ and $p_2(0) = 0.4$. We have then $p^B = 1/3$ and $p^* = 0.3177$. Firm 2 starts with a price larger than the steady state, to decrease below it after a while (reaching below 0.314) and then the price increases again towards the steady state level.[26]

5. Concluding remarks

We have shown that what drives the competitiveness of a market in relation to the static benchmark is whether production or prices are costly to adjust and not the character of competition (Cournot or Bertrand). Indeed, when output (price) is costly to adjust the MPE steady state is more (less) competitive than the static Nash equilibrium. In particular, the static strategic complementarity characterizing price competition is turned into intertemporal strategic substitutability whenever firms face similar adjustment production costs. The outcome is fierce competition and a steady state below the static Bertrand benchmark.

The consideration of adjustment costs has implications for empirical work. Adjustment costs are indeed important in quite a few industries. The importance of taking into account the dynamic structure of the market when estimating product differentiation models can not be underscored. In the work of Berry (1994) and Berry, Levinshon and Pakes (1995) it is assumed that firms compete according to a static Bertrand model. From this assumption sophisticated estimates of patterns of elasticities and cross-elasticities of substitution among

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23 The results for the LMPE equilibrium parameters are checked numerically.

24 Cabral and Riordan (1994) who find conditions for increasing dominance in a learning by doing model. See also Rosenthal and Spady (1989) and Budd, Harris, and Vickers (1993).

25 The inequalities are checked numerically (see the derivation of Result 4.2).

26 This contrasts with switching costs models where convergence is monotone (Beggs and Klemperer (1992)).

products are derived building on discrete choice theory. An obvious problem is that if a dynamic structure exists in the industry then there will be biases in the estimation of the degree of product differentiation. For example, if the true model of an industry corresponds to our case of price competition with production adjustment costs, the estimates based on static Bertrand competition would systematically overestimate the degree of substitutability of the products. The lesson to draw is that, even when the modeler is reasonably certain that collusion is not an issue in an industry to neglect the dynamic structure is dangerous and leads to biases in the estimation.[27]

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27 For the empirical implementation of dynamic models with adjustment costs (and evidence of adjustment factors in the rice and coffee export markets) see Karp and Perloff (1989, 1993a, b). The dynamic pricing implications of the learning curve are considered in Benkard (1997) building on the work of Ericson and Pakes (1995). Slade (1998) provides estimates of price adjustment costs in the retail-grocery sector.

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APPENDIX I

At an MPE firm i chooses $u_i(\cdot)$ to maximize the discounted sum of profits, $\int_0^{\infty} \pi_i(y_1(t), y_2(t), u_i(t), u_j(y_1(t), y_2(t)))e^{-rt}dt$, given $u_j(\cdot)$, for any possible initial condition $y_j(0) = y_j^0$, where $\dot{y}_j(t) = u_j(y_1(t), y_2(t))$, j = 1, 2.

Let $H_i = R_i(y_1, y_2) - F_i(u_1, u_2) + \mu_{ii} u_i + \mu_{ij} u_j$ be the (current value) Hamiltonian of firm i, where $\mu_i = (\mu_{ii}, \mu_{ij})$ is the vector of co-state variables. The following are necessary conditions (i = 1, 2) for (u_1, u_2) to form an MPE pair:

$u_i(y_1, y_2) \in \arg\max_{v_i} H_i(\mu_i, y_1, y_2, v_i, u_j(y_1, y_2))$,

(I.1) $$\frac{\partial H_i}{\partial u_i} = -\frac{\partial F_i}{\partial u_i} + \mu_{ii} = 0,$$

(I.2) $$\dot{\mu}_{ii} = r\mu_{ii} - \frac{\partial H_i}{\partial y_i} - \frac{\partial H_i}{\partial u_j}\frac{\partial u_j}{\partial y_i} = r\mu_{ii} - \frac{\partial R_i}{\partial y_i} + \left(\frac{\partial F_i}{\partial u_j} - \mu_{ij}\right)\frac{\partial u_j}{\partial y_i}, \text{ and}$$

$$\dot{\mu}_{ij} = r\mu_{ij} - \frac{\partial H_i}{\partial y_j} - \frac{\partial H_i}{\partial u_j}\frac{\partial u_j}{\partial y_j} = r\mu_{ij} - \frac{\partial R_i}{\partial y_j} + \left(\frac{\partial F_i}{\partial u_j} - \mu_{ij}\right)\frac{\partial u_j}{\partial y_j}.$$

At a steady state we will have that $u_i = \dot{\mu}_{ij} = 0$ for i, j = 1, 2, and therefore $\mu_{ii} = 0$ (because $\frac{\partial F_i}{\partial u_j}(0,0) = 0$, i, j = 1, 2). Note that $r - \frac{\partial u_j}{\partial y_j} = 0$ would be inconsistent with (I.2) at the steady state whenever $\frac{\partial R_i}{\partial y_j} \neq 0$. Solving for μ_{ij} in the necessary conditions for firm i we obtain for i, j = 1, 2, $j \neq i$

(*) $$\frac{\partial R_i}{\partial y_i} + \frac{\frac{\partial R_i}{\partial y_j}\frac{\partial u_j}{\partial y_i}}{r - \frac{\partial u_j}{\partial y_j}} = 0.$$

<u>Proof of Proposition 2.2.</u> Let $\gamma \equiv \frac{\partial u_i}{\partial y_j}(y^*, y^*)$ and $\beta \equiv \frac{\partial u_i}{\partial y_i}(y^*, y^*)$. If the steady state (y*, y*) is regular then $\det\begin{pmatrix} \beta & \gamma \\ \gamma & \beta \end{pmatrix} \neq 0$. This insures, in particular, that (y*, y*) is an isolated rest point of the dynamical system. It follows then from Poincaré's linearization result that if (y*, y*) is locally stable necessarily the matrix $\begin{pmatrix} \beta & \gamma \\ \gamma & \beta \end{pmatrix}$ has negative trace, $2\beta < 0$, and positive determinant, $\beta^2 - \gamma^2 > 0$.[1] Rewriting the necessary conditions (*) for the steady state of an MPE we obtain

$$\frac{\partial R_i}{\partial y_i}(y^*, y^*) = -\frac{\frac{\partial R_i}{\partial y_j}(y^*, y^*)\gamma}{(r-\beta)} .$$

We show that sign{$y^* - y^N$} = sign $\{\frac{\partial R_i}{\partial y_j}\gamma\}$. Since there is a unique interior and regular equilibrium which is symmetric, it is necessary (with a direct application of the index approach to uniqueness using the Poincaré-Hopf Index Theorem, see Gillemin and Pollack (p.134, 1974)) that $(\frac{\partial^2 R_i}{(\partial y_i)^2})^2 - (\frac{\partial^2 R_i}{\partial y_j \partial y_i})^2 > 0$ at the equilibrium. Because $\frac{\partial^2 R_i}{(\partial y_i)^2} < 0$ this means that $\frac{\partial^2 R_i}{(\partial y_i)^2} + \frac{\partial^2 R_i}{\partial y_j \partial y_i} < 0$ at the equilibrium. Hence, if we define $\phi(z) \equiv \frac{\partial R_i}{\partial y_i}(z,z)$, then $\phi'(y^N) = \frac{\partial^2 R_i}{(\partial y_i)^2}(y^N, y^N) + \frac{\partial^2 R_i}{\partial y_j \partial y_i}(y^N, y^N) < 0$. Since y^N is the unique solution to the equation $\phi(z) = 0$, we have sign {$y^* - y^N$}= sign $\{-\frac{\partial R_i}{\partial y_i}(y^*, y^*)\}$. From this and the fact that $\beta < 0$ and so $r-\beta > 0$ the conclusion follows.

Q.E.D.

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1 Indeed, if the nonlinear dynamical system $\dot{y}_i = u_i(y_1, y_2)$, i = 1, 2, has a locally stable steady state (y*, y*) then the trace (determinant) of the matrix of the linearized system is negative (positive) except if the roots of the matrix are pure imaginary or if the roots are real and equal (see, for example, Takayama (p.405-411, 1993)). In our case the roots of $\begin{pmatrix} \beta & \gamma \\ \gamma & \beta \end{pmatrix}$ are real and different: $\beta + \gamma$ and $\beta - \gamma$, with $\gamma \neq 0$. (Note also that the stated conditions on the matrix are necessary and sufficient for a linear system to be stable.) This means that we have two possible situations: Either $\beta < \gamma < 0$ or $|\beta| > \gamma > 0$.

APPENDIX II

Let (u_i^*, u_j^*) be an MPE. We have then that $\dot{x}_i = -b\dot{p}_i + c\dot{p}_j = -b\,u_i^* + c\,u_j^*$. The value function for firm i $V_i(p_1, p_2)$ is the present discounted value of profits at the MPE with initial conditions $(p_1(0), p_2(0)) = (p_1, p_2)$ and law of motion $\dot{p}_j = u_j^*$, j = 1, 2.

Given u_j^* the Bellman equation for firm i, i = 1, 2, is given by

(II.1) $$rV_i(p_1,p_2) = \max_{u_i} [H_i(\frac{\partial V_i}{\partial p_i}(p_1,p_2), \frac{\partial V_i}{\partial p_j}(p_1,p_2), p_1, p_2, u_i, u_j^*(p_1,p_2)),$$

where the maximand on the right hand side is the current Hamiltonian:

$$H_i = \pi_i + \frac{\partial V_i}{\partial p_i} u_i + \frac{\partial V_i}{\partial p_j} u_j^*,$$

where $\pi_i = p_i D_i(p_1,p_2) - \lambda(-bu_i + cu_j^*(p_1,p_2))^2/2$ is the instantaneous profit and $\frac{\partial V_i}{\partial p_k}$ is the shadow value of state variable p_k for firm i. The equation (II.1) has to hold for any price level vector (due to the perfection requirement). Since u_i^* is a maximizer of the current Hamiltonian, the first order condition

$$\lambda b(-bu_i^* + cu_j^*) + \frac{\partial V_i}{\partial p_i} = 0$$

must hold for i = 1, 2 and j ≠ i. This defines i's instantaneous best response. The first order condition is also sufficient for a maximum given the concavity of the objective function on the right hand side of (II.1) with respect to u_i. We can derive therefore the equilibrium of the instantaneous game given p_1 and p_2:

(II.2) $$u_i^* = \frac{1}{\lambda b\left(b^2 - c^2\right)}(b\frac{\partial V_i}{\partial p_i} + c\frac{\partial V_j}{\partial p_j}).$$

<u>Proof of Proposition 4.2:</u>

In order to solve for symmetric LMPE we follow a standard procedure[2]:

(1) Posit a quadratic value function for firm i (and a symmetric function for firm j):

2 For examples of computation and characterization of LMPE see Starr and Ho (1969), Reynolds (1987), Driskill and McCafferty (1989) and Fudenberg and Tirole (Ch.13, 1991).

(II.3) $$V_i(p_1,p_2) = z + vp_i + wp_j + \frac{m}{2}p_i^2 + np_ip_j + \frac{s}{2}p_j^2.$$

(2) Obtain a system of partial differential equations substituting the instantaneous equilibrium u_i^* and u_j^* in the necessary conditions for equilibrium (II.1):

(II.4) $$rV_i = p_iD_i - \frac{1}{2\lambda b^2}\left(\frac{\partial V_i}{\partial p_i}\right)^2 + \frac{1}{\lambda b(b^2-c^2)}\left(b\frac{\partial V_i}{\partial p_i} + c\frac{\partial V_j}{\partial p_j}\right)\frac{\partial V_i}{\partial p_i}$$
$$+ \frac{1}{\lambda b(b^2-c^2)}\left(b\frac{\partial V_j}{\partial p_j} + c\frac{\partial V_i}{\partial p_i}\right)\frac{\partial V_i}{\partial p_j}.$$

Both the left hand side and the right hand side are functions of (p_1, p_2). If V_i is a quadratic function, then the right hand side of (II.4) will also be a quadratic function. Furthermore, the equation must hold for any pair (p_1, p_2) according to the definition of MPE.

(3) Find the coefficients of the value function taking derivatives of V_i and V_j with respect to p_i and p_j, substituting the result in the right hand side of (II.4), and comparing the coefficients using the left hand side of (II.3). Candidate LMPE strategies follow from (II.2). If we write

$$u_i^* = \alpha + \beta p_i + \gamma p_j,$$

then (in terms of the utility parameters A, B, C, and normalizing so that B = 1) $\alpha = \lambda^{-1}(1-C^2)(1+C)v$, $\beta = \lambda^{-1}(1-C^2)(m+Cn)$, and $\gamma = \lambda^{-1}(1-C^2)(Cm+n)$. The exercise, after tedious computations summarized in AII.1 below, yields *six solutions in closed-form,* listed in Table 1, corresponding to the six candidate value functions. Note that α is linear in A because v is.[3]

(4) Identify a unique stable solution. This necessarily is a MPE (because the stable solution to the partial differential equation system (II.1) fulfills the transversality condition). In Appendix A.II.2 we show (using a method similar to Driskill and McCafferty (1989)) that only one of the solutions generates a stable stationary state. The unique LMPE which generates a stable solution has $\alpha > 0$ and $\beta < \gamma < 0$. Solutions #3 and #5 in Table A are the only candidates to generate the stable LMPE.

---

[3] From (A.5) below we can see that w is linear in v. After susbtituting w out in (A.4) v can be checked to be a linear function of A.

(5) Check that when the initial state belongs to P the LMPE path stays in P. The LMPE path can be obtained by solving the differential equations $u_i^* = \alpha + \beta p_i + \gamma p_j, i, j = 1, 2, i \neq j$. It is given by

$$\text{(II.5)} \quad p_i^{MP}(t) = p^* + \left(\frac{p_i^0 + p_j^0}{2} - p^* \right) e^{(\beta+\gamma)t} + \left(\frac{p_i^0 - p_j^0}{2} \right) e^{(\beta-\gamma)t},$$

where p^* is the stationary state of the stable LMPE path, $p^* = -\alpha/(\beta+\gamma) = a/(2b-c(1-\gamma(\beta-r)^{-1})$, the last equality following from Proposition 3.1). This can be done using a similar argument to the one used for OLE.

A.II.1 Derivation of the LMPE

By substituting for the derivatives in (II.4) using (II.3) and comparing the coefficients (in terms of A, B = 1, and C) we obtain the following equations:

$$\text{(A.1)} \quad 0 = 2\lambda + \lambda r(1-C^2)m - (1-C^2)^2\{(1+C^2)m^2 + 4Cmn + 2n^2\}$$

$$\text{(A.2)} \quad 0 = \lambda C - \lambda r(1-C^2)n + 1(1-C^2)^2\{(Cm+n)s + Cm^2 + (2+C^2)mn + 2Cn^2\}$$

$$\text{(A.3)} \quad 0 = \lambda rs - (1-C^2)\{2(m+Cn)s + 2Cmn + (1+C^2)n^2\}$$

$$\text{(A.4)} \quad 0 = \lambda A - (1+C)\left[\lambda rv - (1-C^2)\{(1+C+C^2)mv + (1+2C)nv + (Cm+n)w\}\right]$$

$$\text{(A.5)} \quad 0 = \lambda rw - (1-C^2)\{Cmv + (1+C+C^2)nv + (1+C)sv + (m+Cn)w\}$$

$$\text{(A.6)} \quad 0 = 2\lambda rz - (1-C^2)(1+C)\{(1+C)v^2 + 2vw\}$$

In order to solve for linear strategies we need to solve for m, n, and v. Using (A.3) one can substitute s out in (A.2) and (A.5). Also using (A.5) one can substitute w out in (A.4). Then transform the resulting equations into equations in α, β, and γ, using $\alpha = \lambda^{-1}(1-C^2)(1+C)v$, $\beta = \lambda^{-1}(1-C^2)(m+Cn)$, and $\gamma = \lambda^{-1}(1-C^2)(Cm+n)$.The following system of equations emerges:

$$\text{(A.7)} \quad 0 = 2(1-C^2) + \lambda r(\beta - C\gamma) - \lambda\beta^2 + 2\lambda C\beta\gamma - \lambda(2-C^2)\gamma^2$$

$$\text{(A.8)} \quad 0 = rC(1-C^2) - \{2(1-C^2) - \lambda r^2\}C\beta - \lambda r^2\gamma - 3\lambda rC\beta^2 + \lambda r(4-C^2)\beta\gamma + 2\lambda C\beta^3 - \lambda(4-C^2)\beta^2\gamma + \lambda\gamma^3$$

$$\text{(A.9)} \quad 0 = A(1-C^2)(1+C)\left(r^2 - 3r\beta + 2\beta^2\right) - \lambda\alpha\{(1-C)(r^3 - 4r^2\beta + 5r\beta^2 - 2\beta^3) - r^2(1-C^2)\gamma + r(3-2C^2)\beta\gamma - (2-C^2)\beta^2\gamma - r\gamma^2 + 2\beta\gamma^2 - \gamma^3\}$$

Equation (A.9) determines α from β and γ. From (A.7) and (A.8) the following equations for β and γ are derived

$$\text{(A.10)}\quad 0 = -\frac{-\{8(1-C^2)+\lambda r^2\}C^2}{\lambda^3(9-C^2)^2} + \frac{8(8-11C^2+5C^4)+4\lambda r^2(4-3C^2)+\lambda^2 r^4}{\lambda^2(9-C^2)^2}\gamma^2$$

$$-\frac{8(18-13C^2+3C^4)+2\lambda r^2(9+C^2)}{\lambda(9-C^2)^2}\gamma^4+\gamma^6$$

$$\text{(A.11)}\quad \beta = \frac{r}{2} + \frac{(8-10C^2+\lambda r^2)\gamma - \lambda(9-5C^2)\gamma^3}{2C(1-6\lambda\gamma^2)}$$

We can solve (A.10) for explicit values of γ because it is a cubic equation in γ^2. Below we sketch how one obtains explicit solutions for a cubic equation when there are three real roots.

Suppose a cubic equation is given in the following form:

$$0 = a_0 + a_1 z + a_2 z^2 + z^3.$$

Define $\Gamma = \frac{3a_1 - a_2^2}{9}, \Lambda = \frac{a_0}{2} - \frac{a_1 a_2}{6} + \frac{2a_2^3}{27}$, and $\Delta = \Gamma^3 + \Lambda^2$. Δ is the discriminant of the above equation. If $\Delta < 0$, then the above cubic equation has three real roots. The three roots are given by

$$z_k = -\frac{a_2}{3} + 2\sqrt{-\Gamma}\cos\frac{\theta + 2k\pi}{3}, k = 0,\ 1,\ 2,$$

where $\theta = \arctan\left(\frac{\sqrt{-\Delta}}{-\Lambda}\right) \in [0,\pi)$. Notice that $-\Gamma > 0$ if $\Delta < 0$. In our model Δ is a complicated function of λ, r, and C, which is given in Table 1. One can check analytically that Δ is negative in our model (unless $(39-55C^2+6\lambda r^2)^2=0$). According to the above formula the solutions to (A.10) for γ^2 can be obtained. Then the solutions for γ are obtained by taking positive and negative square roots. The six solutions are summarized in Table A.

Table A

Solution #	γ	Solution #	γ
1	$\gamma_1 = -\gamma_2$	2	$\gamma_2 = \sqrt{-\frac{a_2}{3} + 2\sqrt{-\Gamma}\cos\frac{\theta}{3}}$
3	$\gamma_3 = -\gamma_4$	4	$\gamma_4 = \sqrt{-\frac{a_2}{3} + 2\sqrt{-\Gamma}\cos\frac{\theta+2\pi}{3}}$
5	$\gamma_5 = -\gamma_6$	6	$\gamma_6 = \sqrt{-\frac{a_2}{3} + 2\sqrt{-\Gamma}\cos\frac{\theta+4\pi}{3}}$

$$\beta = \frac{rC + (8 - 10C^2 + \lambda r^2)\gamma - 6\lambda rC\gamma^2 - \lambda(9 - 5C^2)\gamma^3}{2C(1 - 6\lambda\gamma^2)},$$

$$\alpha = -\frac{A(B-C)(\beta+\gamma)}{2B - C(1 - \frac{\gamma}{\beta - r})}, \text{ where}$$

$$a_2 = -\frac{8(18 - 13C^2 + 3C^4) + 2\lambda r^2(9 + C^2)}{\lambda(9 - C^2)^2}, \quad \theta = \arctan(\sqrt{-\Delta}/(-\Lambda)) \in [0, \pi),$$

$$\Gamma = -\{8(648 - 639C^2 + 383C^4 - 321C^6 + 57C^8) + 4\lambda r^2(324 + 171C^2 + 215C^4 - 95C^6 + 33C^8) + \lambda^2 r^4(81 + 126C^2 + C^4)\}\{9\lambda^2(9 - C^2)^4\}^{-1}$$

$$\Delta = -C^2(39 - 55C^2 + 6\lambda r^2)^2 \{64(1 - C^2)(256 - 139C^2 - 15C^4 + 31C^6 - 5C^8) + 16\lambda r^2(512 - 539C^2 + 169C^4 - 17C^6 + 3C^8) + \lambda^2 r^4(1536 - 971C^2 + 214C^4 - 11C^6) + 4\lambda^3 r^6(32 - 9C^2 + C^4) + 4\lambda^4 r^8\}\{108\lambda^6(9 - C^2)^8\}^{-1}, \text{ and}$$

$$\Lambda = \{8(93312 - 507627C^2 + 615087C^4 - 197854C^6 - 9810C^8 + 17433C^{10} - 2349C^{12}) + 3\lambda r^2(93312 - 428409C^2 + 308772C^4 - 78166C^6 + 17316C^8 - 17433C^{10}) + 48\lambda^2 r^4(729 - 2916C^2 + 945C^4 - 218C^6 - 12C^8) + 2\lambda^3 r^6(9 + C^2)(9 - 18C + C^2)(9 + 18C + C^2)\}\{54\lambda^3(9 - C^2)^6\}^{-1}$$

Bounds on γ^2

We want to compare the sizes of γ^2 for the six solutions. We will use this information later. First, γ^2 can be rewritten as

(B.1) $$\gamma^2 = -\frac{a_2}{3}\left(1 + t\cos\frac{\theta + 2k\pi}{3}\right), k = 0, 1, 2,$$

where $t = 6\sqrt{-\Gamma / a_2{}^2}$ and a_2 is as given in Table 1. The parameter θ is the angle associated to the complex number $-\Lambda + \sqrt{-\Delta}i$. We have that $0 \le \theta \le \pi$ since $\sqrt{-\Delta} \ge 0$. Hence,

(B.2) $$-1 < \cos\frac{\theta + 2\pi}{3} \le -0.5 \le \cos\frac{\theta + 4\pi}{3} \le 0.5 \le \cos\frac{\theta}{3} \le 1.$$

Since - $a_2 > 0$ and $t > 0$, we know from (B.2) that #3 and #4 are the smallest in absolute value and #1 and #2 are the largest.

AII.2 Uniqueness of stable LMPE

We show that there is a unique solution to (A.7) and (A.8) which satisfies the stability condition. First, (A.7) can be rewritten as follows.

(A.7') $$\gamma = \frac{C(\beta - r/2)}{2 - C^2} \pm \frac{\sqrt{(2 - C^2)\{8(1 - C^2) + \lambda r^2\} - 8\lambda(1 - C^2)(\beta - r/2)^2}}{2\sqrt{\lambda}(2 - C^2)}$$

Using (A.7), we can derive the following from (A.8):

(A.8') $$\gamma = \frac{2C(3 - C^2)(\beta - r/2)}{18 - 7C^2 + C^4} + \frac{F_1(\beta - r/2)}{F_2(18 - 7C^2 + C^4)},$$

where

$$F_1 = 8C\{24 - 76C^2 + 46C^4 - 11C^6 + C^8 + (12 - 7C^2 + C^4)\lambda r^2\},$$

$$F_2 = (2 - C^2)\{8 + (1 - C^2)\lambda r^2\} - 4\lambda(18 - 7C^2 + C^4)(\beta - r/2)^2.$$

The graphs of (A.7’) and (A.8’) including the asymptotes are drawn in Figure A1. The two graphs have six intersections corresponding to the six solutions listed in Table A. The graphs are symmetric with respect to point (r/2, 0) in the β–γ plane. The derivative of the second term of the RHS of (A.8') with respect to β is either positive for all values of β or negative for all values of β. The sign of the slope is the same as the sign of F_1 in (A.8’). In Figure A1 the slope is positive. We deal with the two cases separately. In both cases, all we need to show is that the two graphs representing (A.7’) and (A.8') intersect each other only once in the region defined by the stability condition, namely $\gamma > \beta$ and $\gamma < -\beta$.

Case 1: $F_1 \geq 0$.

The proof can be most clearly explained graphically. Refer to Figure A2 in the subsequent argument. Point *A* is the intersection of the graph of (A.7’) and that of $\gamma = -\beta$. Point *B* is the point on the graph of (A.8’) that has the same β-coordinate as *A*. Point *C* is the intersection of the horizontal asymptote of (A.8’) (the straight line represented by the first term of the RHS of (A.8')) and the graph of (A.7’). Point D is the intersection of the graph of (A.7’) and the graph of $\gamma = \beta$. Finally point S is the intersection of the graphs of (A.7’) and (A.8’) with the smallest β-coordinate. We claim that point S is the unique stable solution. The claim is proved by checking that:

(1) Point *A* lies to the left of the smaller vertical asymptote.

(2) Point *B* lies below point *A*.

(3) Point *C* lies above the line $\gamma = \beta$.

We skip the details. When point A lies to the left of the smaller vertical asymptote, so does point D. Since the stable solution must lie on arc AD, and the graph of (A.8’) intersects arc AD only once, uniqueness follows.

Case 2: $F_1 < 0$.

Again the proof can be most clearly explained graphically. Refer to Figure A3 in the subsequent argument. Points A, C, D, and S are defined as before. Points E and F are the points on the graph of (A.8’) that have the same β-coordinate as D and A, respectively. Again we prove that point S is the unique stable solution The proof is done by showing that:

(1) Point *D* lies to the left of the smaller vertical asymptote.

(2) Point *E* lies above point *D*.

(3) If point *A* lies to the right of the smaller vertical asymptote, then point *F* lies above point *A*.

We skip the details again. (1) – (3) imply that point S is the only solution on arc AD, which proves the uniqueness of the stable solution

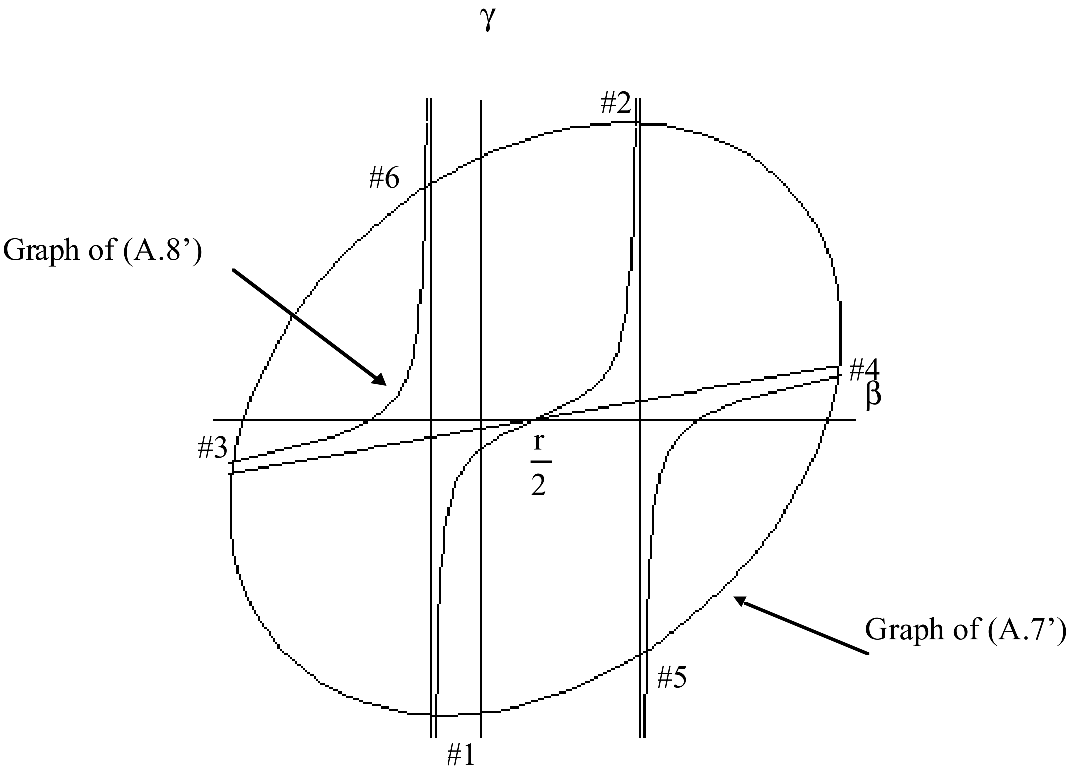


Figure A1

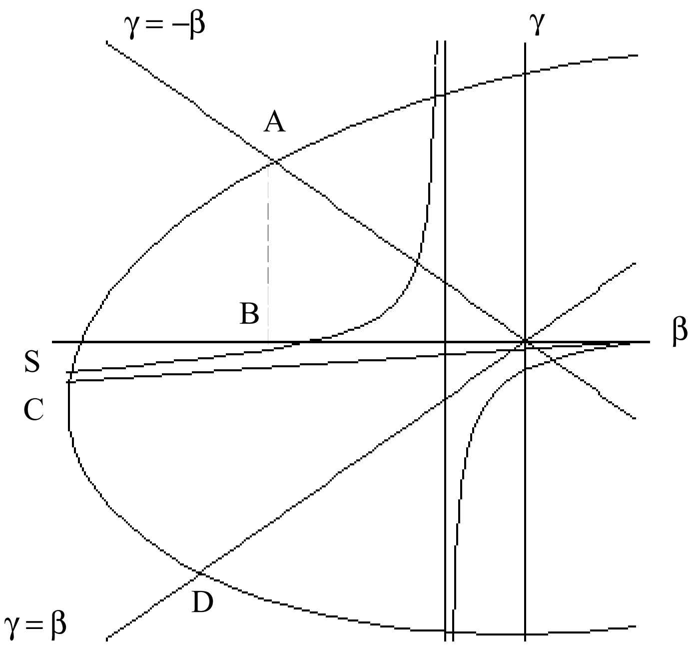


Figure A2

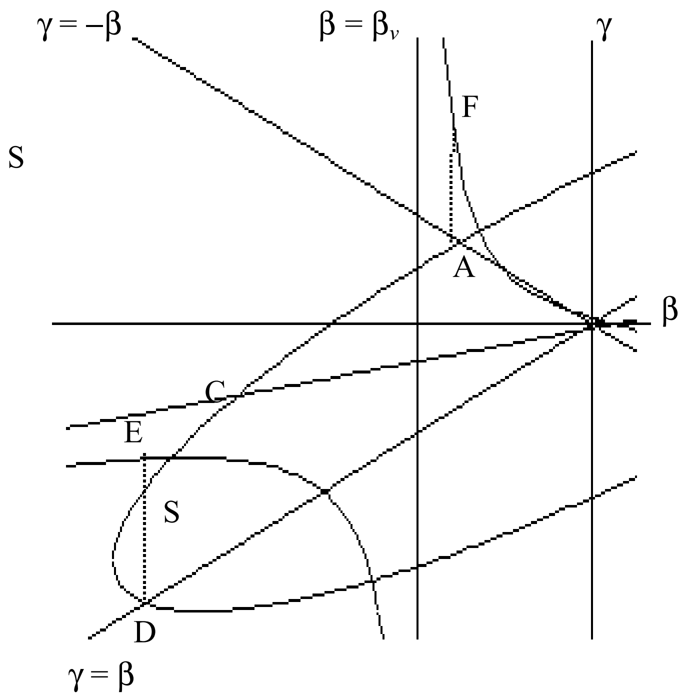


Figure A3

A.II.3 **<u>Result A.1.</u>** **(Identifying the stable solution)** Let B = 1. If C > $\sqrt{39/55}$ and λr^2 < (-39 + 55 C^2)/6, then solution #5 in Table A is the stable LMPE. Otherwise, solution #3 is the stable LMPE.

Derivation:

We can check that #3 is the stable solution when $F_1 \geq 0$. We sketch the proof. Refer to Figure A4. Point C is defined in the same way as in Figure A2. Points G and H are the intersections of the graphs of (A.7') and (A.8') with the β-axis, respectively, on the left of the smaller vertical asymptote. Point I is the point on the graph of (A.7') that has the same β coordinate as point H. Points K and L are the intersections of the vertical asymptotes of (A.8') and the graph of (A.7'). First one can show that the β-coordinate of point H is larger than that of point G. This proves that the γ-coordinate of S is negative since the slope of the graph of (A.8') is positive. Secondly, one can check that the γ coordinate of point I is larger than the absolute value of the γ coordinate of point C. Thirdly, the γ coordinate of point L is larger than that of point K, because (A.8') is the sum of an ellipse with center at (r/2, 0) and a strait line through (r/2, 0) with positive slope. This proves that the stable solution is the smallest in absolute value with negative γ coordinate, namely #3.

We can show that the stable solution is either #3 or #5 when $F_1 < 0$. It is clear from Figure A3 that S has $\gamma < 0$ (i.e., S is either #1 or #3 or #5 in Table 1). We can show also that the γ-coordinate of S can not be the largest in absolute value. We sketch the proof. Refer to Figure A5. Point M is the intersection of the graph of (A.8') and the graph of $\gamma = \beta - r/2$ on the left of the smaller vertical asymptote of (A.8'). Point M' is the point on the lower part of (A.7') that has the same β-coordinate as point M. Point N is the point on the graph of (A.8') whose β-coordinate is the minimum of the points on the graph of (A.7'). One can check that γ-coordinate of point N is larger than that of point M, which is in turn larger than that of point M' and that the slope of (A.8') at M is negative. These together with the fact that (A.8') is concave on the left of the smaller vertical asymptote proves that the γ-coordinate of S is not the largest in absolute value. Numerical analysis shows that #3 is the stable solution if $\lambda r^2 > (55C^2 - 39)/6$ and that #5 is the stable solution otherwise

Figure A6 summarizes the results. For C < $\sqrt{39/55} \approx 0.842075$, #3 is the coordinate of S. For larger values of C and for $\lambda r^2 < (-39 + 55C^2)/6$ (region II), #5 is the coordinate of S. Figure A7 depicts 3 solutions of (A.10), a cubic equation in γ^2 as functions of C for $\lambda r^2 = 1/100$ and $\lambda = 1$. The stable branch starts out the smallest when C is small. As C increases it increases and crosses one of the unstable branches, to become the second largest. The crossing occurs when $\theta = 0$, namely when $\lambda r^2 = (-39 + 55C^2)/6$.

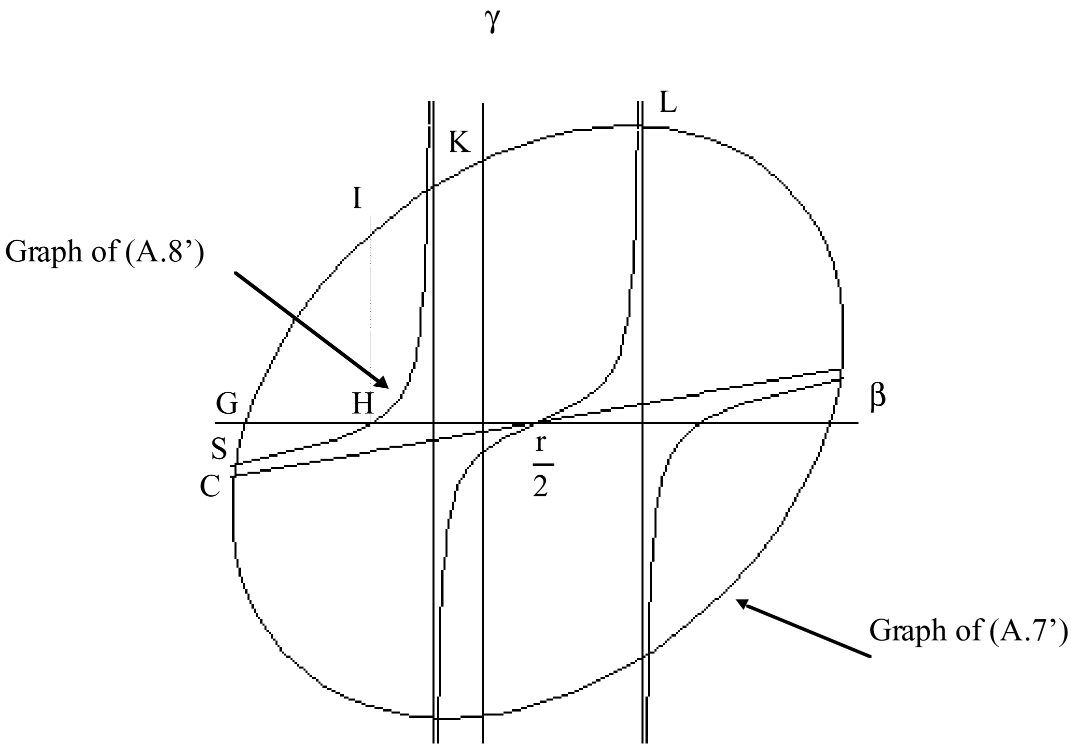


Figure A4

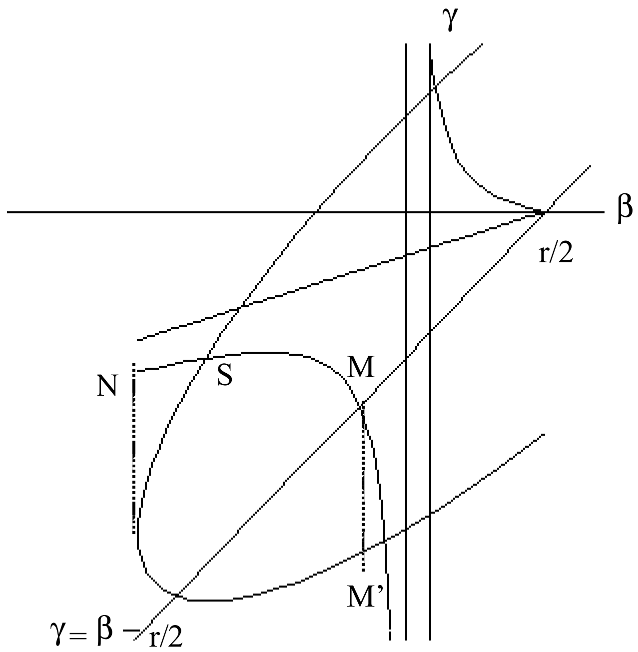
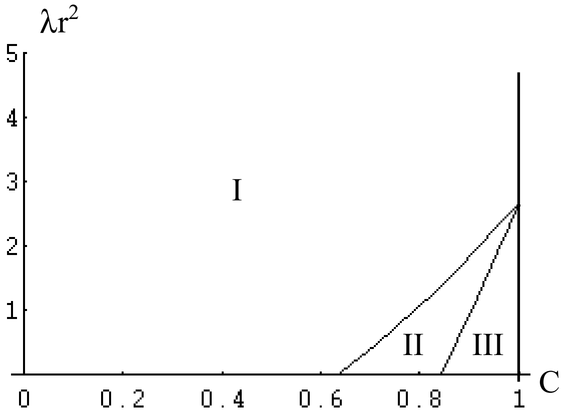


Figure A5



Region I: $F_1 \geq 0$, stable solution is #3
Region II: $F_1 < 0$, stable solution is #3
Region III: $F_1 < 0$, stable solution is #5

Figure A6

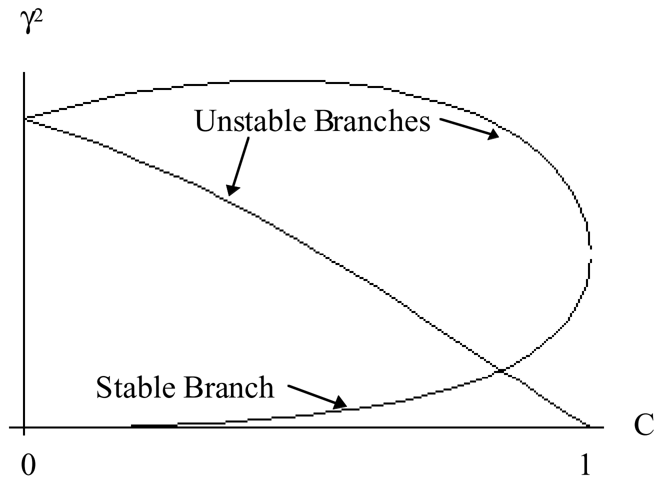


Figure A7 Graphs of γ^2

APPENDIX III

Derivation of Result 4.1

We show that $p_i^{OL}(t) - p_i^{MP}(t) > 0$ for $t \geq 0$, i = 1, 2 for prices in the region P. From Proposition 4.1 we obtain

$$\text{(III.1)} \quad p_i^{OL}(t) = p^B + \left(\frac{p_i^0 + p_j^0}{2} - p^B\right)e^{\phi_1 t} + \left(\frac{p_i^0 - p_j^0}{2}\right)e^{\phi_2 t}$$

We get $p_i^{OL}(t) - p_i^{MP}(t)$ by subtracting (II.5) from (III.1). First define a new set P_t for a given t to be the set of prices (p_1^0, p_2^0) for which $p_i^{OL}(t) - p_i^{MP}(t) > 0$, i = 1, 2. We want to show that P is included in P_t for all $t \geq 0$. This inclusion is illustrated in Figure A8. In order to prove the inclusion it is enough to show (1) the coordinates of the vertex N are larger than that of M, (2) the horizontal coordinate of point L is larger than that of K.

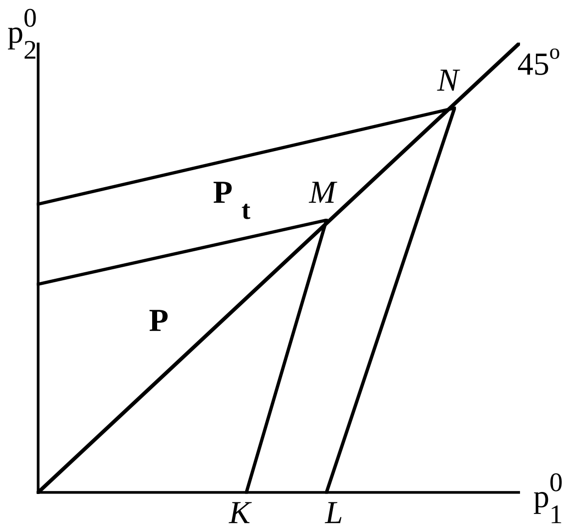


Figure A8

The horizontal (and vertical) coordinate of N is $\dfrac{(1-\exp[\phi_1 t])p^B - (1-\exp[(\beta+\gamma)t])p^*}{\exp[(\beta+\gamma)t] - \exp[\phi_1 t]}$, whereas the horizontal (and vertical) coordinate of M is A. We want to show that $(1-\exp[\phi_1 t])(p^B/A) - (1-\exp[(\beta+\gamma)t])(p^*/A) > \exp[(\beta+\gamma)t] - \exp[\phi_1 t]$. Define function f(t) by

$$f(t) = (1-\exp[\phi_1 t])(p^B/A) - (1-\exp[(\beta+\gamma)t])(p^*/A) - (\exp[(\beta+\gamma)t] - \exp[\phi_1 t]).$$

We want to show that $f(t) > 0$. First, $f(0) = 0$. $f'(t)$ and hence $f(t)$ would be positive if $(\beta + \gamma)(1 - p^* / A) < \phi_1 (1 - p^B / A)$ and $\phi_1 < \beta + \gamma$. Since $p^* = -\alpha / (\beta + \gamma)$ and $p^B = A(1-C)/(2-C)$ normalizing $B = 1$, the first inequality is equivalent to

$$\phi_1 - (2 - C)(\alpha / A + \beta + \gamma) > 0.$$

We check this inequality and the inequality $\phi_1 < \beta + \gamma$ numerically.

The horizontal coordinate of L is $\dfrac{2\{(1-\exp[\phi_1 t])p^B - (1-\exp[(\beta+\gamma)t])p^*\}}{\exp[(\beta+\gamma)t] - \exp[\phi_1 t] + \exp[(\beta-\gamma)t] - \exp[\phi_2 t]}$, whereas the coordinate of K is $A(1 - C)$ after normalizing $B = 1$. Assuming $\phi_1 < \beta + \gamma$ and $\phi_2 < \beta - \gamma$, as we will show later, we want to show that $(1-\exp[\phi_1 t])(p^B / A) - (1-\exp[(\beta + \gamma)t])(p^* / A) - ((1 - C)/2)\{\exp[\phi_1 t] - \exp[(\beta + \gamma)t] + \exp[\phi_2 t] - \exp[(\beta - \gamma)t]\} > 0$.

In summary, we check numerically that

(i) $\phi_1 - (2 - C)(\alpha / A + \beta + \gamma) > 0$,

(ii) $\phi_1 < \beta + \gamma$ and $\phi_2 < \beta - \gamma$, and

(iii) $(1-\exp[\phi_1 t])(p^B / A) - (1-\exp[(\beta + \gamma)t])(p^* / A) - ((1 - C)/2)\{\exp[\phi_1 t] - \exp[(\beta + \gamma)t] + \exp[\phi_2 t] - \exp[(\beta - \gamma)t]\} > 0$.

We describe the numerical method below. p^B/A is a function of C only. p^*/A, $\sqrt{\lambda\phi_1}$, $\sqrt{\lambda\phi_2}$, $\sqrt{\lambda\alpha}$, $\sqrt{\lambda\beta}$, and $\sqrt{\lambda\gamma}$ are functions of C and λr^2. For (i) and (ii) we check the graphs of $\sqrt{\lambda}\,\{\phi_1 - (2 - C)(\alpha/A + \beta + \gamma)\}$, $\sqrt{\lambda}\{\phi_1 - (\beta + \gamma)\}$, and $\sqrt{\lambda}\{\phi_2 - (\beta - \gamma)\}$ as functions of C (or λr^2) fixing the value of the other. λr^2 ranges from 0 to 1000 and C ranges from 0.001 to 1. For (iii) we substitute $t = \sqrt{\lambda}\tan\omega$ (so that $\exp[\phi_1 t] = \exp[\sqrt{\lambda\phi_1}\tan\omega]$ for example) and check the graph of the above expression as a function of ϖ for various values of C and λr^2, where $0.001 \le \varpi \le (\pi/2) - 0.0001$ (which corresponds to $0.001/\sqrt{\lambda} \le t \le 10{,}000/\sqrt{\lambda}$ approximately), and $0 \le \lambda r^2 \le 1000$. All the graphs are either monotone or single-peaked and satisfy the required sign conditions.

Proof of Proposition 4.3 (As λr^2 tends to infinity $p^B - p^*$ tends to zero; as λr^2 tends to zero $p^B - p^*$ tends to a strictly positive number.)

It is enough to show that as λr^2 tends to infinity both β and γ tend to 0 and that as λ or r tend to 0 $\gamma / (\beta - r)$ converges to a number strictly between 0 and 1. We prove each of these separately in the following.

(i) We show that as λr^2 tends to infinity the graphs of (A.7') and (A.8') have the following properties in the limit (see Figure A1):

(a) The vertical asymptotes have positive β-coordinates.
(b) F_1 in (A.8') is positive. Hence the graph of (A.8') is increasing in β.
(c) Both the graphs of (A.7') and (A.8') go through (0, 0).
(d) The slope of (A.7') is larger than that of (A.8') at (0, 0).

These imply that among the six solutions five have positive β-coordinates and one is (0, 0). Since we know that $\beta < \gamma < 0$ for the stable solution (Proposition 5.1), it must be the case that (0, 0) is the limit of the stable solution.

The β-coordinates of the vertical asymptotes are the values of β that make F_2 of (A.8') equal to 0. As λr^2 tends to infinity they converge to $(r/2)\left[1 \pm \sqrt{\frac{(1-C^2)(2-C^2)r^2}{18-7C^2+C^4}}\right]$ which are positive. (b) is trivial to check. (c) It is easy to see that (0, 0) lies on the positive part of the graph of (A.7') and on the graph of (A.8') in the limit. (d)The slope of (A.7') at (0, 0) is 1/C which is larger than that of (A.8') which is $\frac{C(3-C^2)}{4-C^2}$ in the limit.

(ii) First, define $\hat{\gamma} = \sqrt{\lambda}\gamma$ and $\hat{\beta} = \sqrt{\lambda}\beta$, and rewrite (A.7') as follows.

$$\text{(A. 7'')}\ \hat{\gamma} = \frac{C(\hat{\beta} - \sqrt{\lambda}r/2)}{2-C^2} \pm \frac{\sqrt{(2-C^2)\left\{8(1-C^2)+\lambda r^2\right\} - 8(1-C^2)(\hat{\beta}-\sqrt{\lambda}r/2)^2}}{2(2-C^2)}$$

(A.8') can be rewritten as follows

$$\text{(A.8'')}\ \hat{\gamma} = \frac{2C(3-C^2)(\hat{\beta}-\sqrt{\lambda}r/2)}{18-7C^2+C^4} + \frac{F_1(\hat{\beta}-\sqrt{\lambda}r/2}{F_2(18-7C^2+C^4)},$$

where

$$F_1 = 8C\{24\text{-}76C^2 + 46C^4 - 11C^6 + C^8 + (12 - 7C^2 + C^4)\lambda r^2\},$$

$$F_2 = (2 - C^2)\{8 + (1 - C^2)\lambda r^2\} - 4\,(18 - 7C^2 + C^4)\,(\hat{\beta} - \sqrt{\lambda}r/2)^2$$

Comparing (A.7'') and (A.8'') with (A.7') and (A.8') one can see that $\hat{\gamma}$ and $\hat{\beta}$ are the solutions to the equations (A.7') and (A.8') when $\lambda = 1$ and r is replaced by $\sqrt{\lambda}\gamma$. Hence all the properties we show about γ and β apply to $\hat{\gamma}$ and $\hat{\beta}$. Especially $\hat{\gamma}$ and $\hat{\beta}$ non-positive in the limit. In the limit as λ tends to 0, (A.7") becomes

$$\hat{\gamma} = \frac{C\hat{\beta}}{2 - C^2} \pm \frac{\sqrt{8(1 - C^2)\left\{(2 - C^2) - \hat{\beta}^2\right\}}}{2(2 - C^2)},$$

and (A.8") becomes

$$\hat{\gamma} = \frac{2C(3 - C^2)\hat{\beta}}{18 - 7C^2 + C^4} + \frac{2C(24 - 76C^2 + 46C^4 - 11C^6 + C^8)\hat{\beta}}{(18 - 7C^2 + C^4)\left\{2(2 - C^2) - (18 - 7C^2 + C^4)\hat{\beta}^2\right\}},$$

The same argument given in Appendix II applies here to show that $\hat{\gamma}$ and $\hat{\beta}$ have finite negative values such that $\hat{\beta} < \hat{\gamma} < 0$ for the stable solution. Hence, $\lim_{\lambda r^2 \to 0} \gamma / \beta = \lim_{\lambda r^2 \to 0} \hat{\gamma} / \hat{\beta}$ belongs to the open interval (0, 1). Since $\hat{\gamma}\sqrt{\lambda}\gamma$ and $\hat{\beta}\sqrt{\lambda}\beta$ converge to finite negative numbers as λ tends to 0, γ and β must tend to $-\infty$. Hence $\lim_{\lambda \to 0} \gamma / (\beta - r) = \lim_{\lambda \to 0} \gamma / \beta$. Also clearly, $\lim_{r \to 0} \gamma / (\beta - r) = \lim_{r \to 0} \hat{\gamma} / (\hat{\beta} - \sqrt{\lambda} r) = \lim_{r \to 0} \hat{\gamma} / \hat{\beta}$.