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DYNAMIC GAMES AND THE TIME INCONSISTENCY OF OPTIMAL POLICY IN OPEN ECONOMIES

MARCUS H. MILLER

MARK H. SALMON

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Centre for Economic Policy Research
6 Duke of York Street
London SW1Y 6LA

Tel: 01 930 2963

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ABSTRACT

In this paper the Maximum Principle is used to derive optimal policies for linear-quadratic, continuous-time economic systems where there may be more than one policy-maker and where the private sector may have rational expectations. The analogy between solving full-information differential games and designing policy in the presence of forward-looking expectations is explored first, before these two problems are considered in combination.

Both the "time inconsistent" optimal policy which arises from strategic asymmetries, and various time consistent alternatives are discussed; and the approach is illustrated with an application to fiscal stabilisation policy in a Common Market.

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Marcus H. Miller and Mark H. Salmon
Department of Economics
University of Warwick
Coventry
CV4 7AL
0203 24011

SUMMARY

As a consequence of the "rational expectations revolution in macroeconomics" the conventional theory of economic policy has had to be revised to take account of forward-looking expectations formed in the private sector, see Calvo (1978). For policy makers in an open economy, in particular, expectations formed by participants in foreign currency markets play a vital role and John Driffill (1982) showed how this affects the design of optimal monetary policy in a small open economy. The foreign exchange rate being the relative price of two currencies, however, implies that policy in one country will be judged in relation to that pursued elsewhere. Recognising this, national policy makers find themselves in a strategic relationship with each other, as well as with private speculators.

The purpose of this paper is to present an analytically tractable framework for considering both of these aspects of policy at the same time. The relationship between national policy makers is represented as a full-information, open-loop differential "game"; and the results obtained by considering a variety of strategic relationships also prove useful when it comes to deriving optimal macroeconomic policy with a private sector which forms "rational expectations".

Throughout, the method used to calculate optimal policy is Pontryagin's Maximum Principle, so the first section shows briefly how to solve the "linear-quadratic" optimal control problem for a single decision-maker using this technique. (The dynamic system is linear and the costs, to be minimised over an infinite horizon, are quadratic). It is the case that cooperative policy, where all players co-ordinate their actions to minimise a common cost function, can be treated as just such a single-controller problem.

In section 2 it is shown how this technique can be applied to

some non-cooperative dynamic games. The assumption that each decision-maker takes the others' policy actions (now and in the future) as given, defines a symmetric Nash solution. When one player (the leader) is assumed to choose his policy actions knowing how the others will react to his announced plan of action, one obtains an asymmetric Stackelberg solution, cf. Simaan and Cruz (1973).

Optimal policy in such an (open loop) Stackelberg dynamic game is not "time consistent" - i.e. it is not optimal for the leader to continue with the announced plan as time moves on from the date at which the plan was formulated. Two alternative approaches which do yield time consistent equilibria are then considered. The first is the computation of a "recursive" Stackelberg equilibrium, obtained using dynamic programming techniques, in a recent paper by Cohen and Michel (1984). The second approach is simply to avoid the strategic asymmetry - which, in this context, involves simply reverting to the Nash game.

A national government designing optimal policy in an environment where the private sector has rational expectations is in a position analogous to that of a Stackelberg leader; hence it is not surprising to find that optimal macroeconomic policy is "time inconsistent" given such forward-looking behaviour. After deriving this optimal policy in section 3, two time-consistent alternatives are described - the first based directly on the recursive Stackelberg equilibrium of Cohen and Michel; and the second, the "loss of leadership solution" proposed by Willem Buiter (1983), being related to the Nash game.

Section 4 indicates how to construct solutions to problems, characteristic of open-economy macroeconomics, involving several national policy makers and private markets with rational expectations. The design of fiscal policy in a Common Market, with fixed internal cross-rates but a floating external exchange rate, is discussed by way of illustration.

Introduction

In the conduct of macroeconomic policy in open economies where there are significant economic "spillover" effects it is appropriate to consider the strategic response of policy makers overseas, as the history of tariff wars indicates. If, in addition, exchange rates are floating and capital is internationally mobile, account must also be taken of the effect of policy announcements on expectations in private markets. In this paper we analyse the design of macroeconomic policy under a variety of different strategic relationships between rational policy makers, in an environment where there is forward looking behaviour on the part of private agents, so that expectations of future policy actions influence the present.

To characterise rational behaviour of decision makers in an interdependent (but static) world, Koichi Hamada (1974) called upon ideas from game theory, specifically the notion of Nash reactions and Stackelberg leadership. For dynamic economies, one turns to analgous developments in the dynamic game theory. As Simaan and Cruz (1973) demonstrated, however, strategic dominance in such a dynamic setting leads to the "time inconsistency" of optimal policy - it not being optimal for the Stackelberg leader to continue with the initial plan if policy is subsequently reoptimised. One of these authors consequently described a class of Stackelberg equilibria, Cruz (1975), where the leader is constrained to implement optimal time consistent strategies obtained recursively and one such recursive Stackelberg equilibrium has recently been analysed in an illuminating fashion by Cohen and Michel (1984). Time inconsistency may, of course, also be avoided by ruling out strategic asymmetry.

After a brief review of linear quadratic "optimal control" theory in section 1, the relevant solutions for differential games are derived in section 2. Only (linear-quadratic) open loop games are examined here, closed loop games being discussed in a later paper, Miller and Salmon (1984).

As for the design of economic policy in "rational expectations" models, this too was found to be time inconsistent, because effectively the policy maker acts as a Stackelberg leader, see Kydland and Prescott (1977) and Lucas and Sargent (1981, Introduction). Subsequently Calvo (1978) and Driffill (1982) have derived time inconsistent optimal policy in continuous time perfect foresight models of a closed and small open economy respectively. After characterising the time inconsistent optimal policy for a general linear model with forward-looking markets in section 3 we consider two time consistent alternatives, the first relating directly to the recursive Stackelbersg equilibrium of Cohen and Michel (1984) and the second, proposed by Buiter (1983), being related to the Nash differential game.

The technique used to derive all the equilibria in the paper is Pontryagin's maximum principle. After a schematic review of these solutions, the analytical tractability of the resulting framework for the analysis of economic policy in open economies is illustrated by an example of fiscal policy in a Common Market.

1. A review of policy optimisation with a single controller

The analysis of dynamic games given in this paper may usefully be prefaced by a brief review of the optimisation problem for a single controller with a quadratic criterion defined on a linear dynamic system. Table 1 introduces the notation and outlines the steps involved in solving this problem using Pontryagin's maximum principle, cf. Intriligator (1971) Chapter 14 or Wiberg (1971) Chapter 10.

Given initial values $x(0)$ for the state variables, the quadratic cost function V defined in (1.1) is to be minimised over an infinite time horizon by choice of a time path for the control variables u subject to the state equation (1.2), which describes the inherent dynamics of the system. The maximum principle technique involves introducing costate variables p and defining the Hamiltonian function as in (1.3), with first order necessary conditions for optimisation as in (1.4). The first of these produces a relationship between the control variables and the costate variables; the second is the state equation and the third describes the dynamic behaviour of the costate variables, which can be interpreted as "forward-looking" shadow prices measuring the marginal contribution to V^*, the minimised value of V arising from changes in the state variables. These necessary conditions are succinctly summarized in the adjoint system of differential equations (1.5), giving the dynamic behaviour of x and p along an optimal path.

Solving this adjoint system, subject to $x(0)$ and the transversality condition that $\text{Lim}\ p(t) \to 0$ as $t \to \infty$, see Michel (1982), is a two-point boundary-value problem. Since the

TABLE 1 : The single controller Linear Quadratic problem

Cost function: $$V = \frac{1}{2}\int_{t_0}^{\infty} (x^T(s)Qx(s) + u^T(s)Ru(s))ds \qquad (1.1)$$

State Equation: $$Dx = Ax + Bu \qquad (1.2)$$

Hamiltonian: $$H = \frac{1}{2}x^TQx + \frac{1}{2}u^TRu + p^T(Ax + Bu) \qquad (1.3)$$

First Order Conditions:
$$\left.\begin{aligned} &\partial H/\partial u = Ru + B^Tp = 0 \rightarrow u = -R^{-1}B^Tp \\ &Dx = \partial H/\partial p = Ax + Bu \\ &Dp = \partial H/\partial x = -Qx - A^Tp \end{aligned}\right\} \qquad (1.4)$$

Adjoint System: $$\begin{bmatrix} Dx \\ Dp \end{bmatrix} = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix}\begin{bmatrix} x \\ p \end{bmatrix} \equiv \begin{bmatrix} A & -J \\ -Q & -A^T \end{bmatrix}\begin{bmatrix} x \\ p \end{bmatrix} \equiv M\begin{bmatrix} x \\ p \end{bmatrix} \qquad (1.5)$$

Solution of 2 point boundary value problem:

(a) Definition of canonical variables

$$\begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}\begin{bmatrix} z_s \\ z_u \end{bmatrix} \; ; \quad \begin{bmatrix} Dz_s \\ Dz_u \end{bmatrix} = \begin{bmatrix} \Lambda_s & 0 \\ 0 & \Lambda_u \end{bmatrix}\begin{bmatrix} z_s \\ z_u \end{bmatrix} \qquad (1.6)$$

(b) Trajectories of states and costates

$$\left.\begin{aligned} &x = C_{11}z_s \; ; \qquad p = C_{21}z_s = C_{21}C_{11}^{-1}x \\ &x(t) = C_{11}e^{\Lambda_s(t-t_0)}C_{11}^{-1}x(t_0) \; ; \; p(t) = C_{21}e^{\Lambda_s(t-t_0)}C_{11}^{-1}x(t_0) \end{aligned}\right\} \qquad (1.7)$$

Notation

Variables

- x n-vector of states
- p n-vector of costates
- u m-vector of controls
- z 2n-vector of canonical variables (z_s stable, z_u unstable)

Operators

$Dx \equiv dx(t)/dt$; $e^{\Lambda t} \equiv \mathrm{diag}(e^{\lambda_i t})$

Parameters

- Q nxn symmetric pos.semi.def.
- R mxm symmetric pos.def.
- M 2nx2n matrix
- C matrix of (col.)eigenctors of M
- Λ (diagonal) matrix of eigenvalues of M

solution for this infinite time problem involves only the stable roots of the adjoint system, it is convenient to define the canonical variables z each associated with a single root, see (1.6). C is a matrix of column eigenvectors of M, so that $MC = C\Lambda$, Λ being the diagonal matrix of eigenvalues of M (Λ_s stable, Λ_u unstable). Partitioning C conformably as shown and setting $z_u = 0$ provides the solution for the states and costates with trajectories given explicitly in terms of $x(0)$ in (1.7).

For this problem, it can be shown that the minimised cost, V^*, is easily evaluated given the states and costates as $V^* = \frac{1}{2}p^T x$. The "closed loop representation" of the solution for x which follows from (1.6) and (1.7) is $Dx = C_{11}\Lambda_s C_{11}^{-1} x$.

Since this path for x could equally well be derived using dynamic programming, it is clear that the policy is "time consistent" and satisfies Bellman's Principle of Optimality, see Intriligator (1971), Chapter 13. To introduce notation to be used below, let us denote by a prefix to the state variable the date at which the open loop plan was formed and include explicitly the time index. Then we find that reoptimising at t_1 given the initial condition ${}_{t_0}x(t_1)$ (achieved by following the optimal plan formed at t_0) yields an optimal plan

$$
\begin{aligned}
{}_{t_1}x(t) &= C_{11}e^{\Lambda_s(t-t_1)} C_{11}^{-1} \; {}_{t_0}x(t_1) \quad \text{for} \quad t \geq t_1 \\
&= C_{11}e^{\Lambda_s(t-t_0)} C_{11}^{-1} \, x(t_0)
\end{aligned}
$$

so the optimal plan found at t_1 is simply a continuation of the plan found at t_0 provided the system arrives at ${}_{t_0}x(t_1)$. The same logic applies to the shadow prices and controls so for a time consistent plan

$$ {}_{t_1}p(t) = {}_{t_0}p(t), \quad {}_{t_1}u(t) = {}_{t_0}u(t), \quad \text{for} \quad t_1 > t_0. $$

i.e. the shift of "origin" for optimisation from t_0 to t_1 leaves the planned values for costate and policy variables unaffected.

No discount factor is included in the cost function in Table 1, or elsewhere in the paper, as the integrals converge. Where discounting is necessary, for convergence or otherwise, use of the "current-value" Hamiltonian and "current value" shadow prices leaves the adjoint equations described in the paper virtually unchanged, see Kamien and Schwarz (1981, Part II, Section 8).

2. DYNAMIC GAMES

2.1 Nash and Pareto Efficient Solutions

The simplest extension of a policy optimisation to the case of many decision makers is the Nash equilibrium of a differential game. In this basic characterisation of noncooperative behaviour all the participants are on an equal strategic footing, unlike the situation where one player may be viewed as a leader, which we discuss in the next section.

Consider the dynamic economic system with the state equation

$$Dx = Ax + B_1u_1 + B_2u_2 \tag{2.1}$$

where the u_i represent the instruments under the control of player i. Each player then seeks to minimise a cost function of the form

$$V^i = \frac{1}{2}\int_{t_0}^{\infty} x^T(s)Q_i x(s) + u_1^T(s)R_{i1}u_1(s) + u_2^T(s)R_{i2}u_2(s) \tag{2.2}$$

where the weighting matrices are symmetric and satisfy $Q_i \geq 0$, $R_{ii} > 0$, $R_{ij} \geq 0$ $i \neq j$. The corresponding Hamiltonian for each player may then be written as

$$H^i = x^TQ_ix + u_1^TR_{i1}u_1 + u_2^TR_{i2}u_2 + p_i^T(Ax+B_1u_1+B_2u_2) \tag{2.3}$$

where p_i represent the costates for that player. On the open loop Nash assumption that each player takes the other's policy time path as given, the first order conditions for cost minimisation by each player are exactly as described in the previous section for the single controller. Collecting these together we find that the dynamics of the system under control may be expressed as

$$\begin{bmatrix} Dx \\ Dp_1 \\ Dp_2 \end{bmatrix} = \begin{bmatrix} A & -J_1 & -J_2 \\ -Q_1 & -A^T & 0 \\ -Q_2 & 0 & -A^T \end{bmatrix} \begin{bmatrix} x \\ p_1 \\ p_2 \end{bmatrix} \equiv M^{OLN} \begin{bmatrix} x \\ p_1 \\ p_2 \end{bmatrix} \qquad (2.4)$$

where $J_i \equiv B_i R_{ii}^{-1} B_i^T$, $i = 1,2.$

With the solution being restricted by transversality conditions to involve only the stable roots and with all the state variables predetermined at t_0, the open loop path for the latter is

$$x(t) = C_{11} e^{\Lambda_s (t-t_0)} C_{11}^{-1} x(t_0)$$

where C_{11} and Λ_s are now defined with respect to the matrix M^{OLN} above. The decision rules can be characterised as

$$u_i = -R_{ii}^{-1} B_i^T p_i = -R_{ii}^{-1} B_i^T C_{i+1,1} C_{11}^{-1} x.$$

where $C_{i+1,1}$ refers to the appropriate block of column eigenvectors associated with the stable eigenvalues Λ_s. In practice, therefore, all that is required to solve the Open Loop Nash game is the computation of the eigenvectors and eigenvalues of M^{OLN}.

As decision makers are acting noncooperatively, the Nash solution is typically not Pareto optimal. A Pareto efficient cooperative solution may be calculated by assuming a single controller who seeks to minimise some weighted average, with weights w_i, of the individual cost functions i.e.

$$\min_{u_1, u_2} \quad V = w_1 V^1 + w_2 V^2$$

Under this form of centralised control the dynamics of the system are shown by

$$\begin{bmatrix} Dx \\ Dp \end{bmatrix} = \begin{bmatrix} A & -(J_1+J_2) \\ -(w_1Q_1+w_2Q_2), & -A^T \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} = M^{PO} \begin{bmatrix} x \\ p \end{bmatrix} \qquad (2.5)$$

where $J_i \equiv B_i R_i^{*-1} B_i^T$ and $R_i^* = (w_1 R_{1i} + w_2 R_{2i})$.

Once again the system may be solved by calculating the eigenvectors of the adjoint matrix, M^{PO}; the decision rules are then given by

$$u_i = -R_i^{*-1} B_i^T C_{21} C_{11}^{-1} x.$$

While it may be straightforward to solve for such Pareto efficient policy, it is not so obvious how such policies may be sustained in dynamic games. In full information, infinite horizon <u>repeated</u> games it has been argued that such cooperative outcomes may be achieved in a non cooperative equilibrium where the players adopt "punishment" or "trigger" strategies. In such cases, control actions are not simply related to the current state but depend also on past states of the system, i.e. involve "memory"; see, for example, J.W.Friedman (1984). The applicability of such strategies to dynamic games is an important area for research.

2.2 Stackelberg solutions

While both players in the preceding Nash games are treated entirely symmetrically, in many strategic situations one player is leader and the other player a follower. For this asymmetric case the two person Stackelberg differential game may provide a useful characterisation of behaviour. In the open loop Stackelberg game the follower takes the leader's policy path as given, so his behaviour will be governed by the same first order conditions discussed above for the Nash game. The leader, however, calculates his optimal policy taking into account the reaction of the follower to his own policy path.

As the leader seeks to minimise a cost function as in (2.2) subject to constraints that now include the first order condition for the follower, namely

$$Dp_2 = -Q_2x - A^Tp_2 \tag{2.6}$$

(where subscript 1 denotes the leader and 2 the follower) the leader's Hamiltonian becomes

$$H^1 = x^TQ_1x + u_1^TR_{11}u_1 + u_2^TR_{12}u_2 + p_1^T(Ax + B_1u_1 + B_2u_2) + p_2^*(-Q_2x - A^Tp_2). \tag{2.7}$$

where p_2^* is the costate attached by the leader to p_2, the follower's costate. As the latter is a "forward looking" variable, however, $p_2^*(t_0)$ is not, like $x(t_0)$, predetermined but will depend systematically on the leader's announced policy path.

Applying the maximum principle yields an adjoint system of the form

$$\begin{bmatrix} Dx \\ Dp_2^* \\ Dp_1 \\ Dp_2 \end{bmatrix} = \begin{bmatrix} A & 0 & -J_1 & -J_2 \\ 0 & A & J_2 & -J_{12} \\ -Q_1 & Q_2 & -A^T & 0 \\ -Q_2 & 0 & 0 & -A^T \end{bmatrix} \begin{bmatrix} x \\ p_2^* \\ p_1 \\ p_2 \end{bmatrix} \equiv M^S \begin{bmatrix} x \\ p_2^* \\ p_1 \\ p_2 \end{bmatrix} \qquad (2.8)$$

where $J_i = B_i R_{ii}^{-1} B_i^T \quad i = 1,2; \quad J_{12} = B_2 R_{22}^{-1} R_{12} R_{22}^{-1} B_2^T$.

The transversality conditions include $x(t_0) = x_0$, $\lim_{t\to\infty} p_1(t) = 0$, $\lim_{t\to\infty} p_2(t_2) = 0$ as before and in addition $p_2^*(t_0) = 0$, as the leader chooses his announced policy so as to minimise his integrated costs.

The dynamics of x and p_2^* will thus be characterised by

$$\begin{bmatrix} Dx \\ Dp_2^* \end{bmatrix} = C_{11} \Lambda_s C_{11}^{-1} \begin{bmatrix} x \\ p_2^* \end{bmatrix} \qquad (2.9)$$

where $\begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix}$ and Λ_s are stable eigenvectors and values of M^S above, which, for scalar x describes a stable node such as that portrayed in Figure 1.

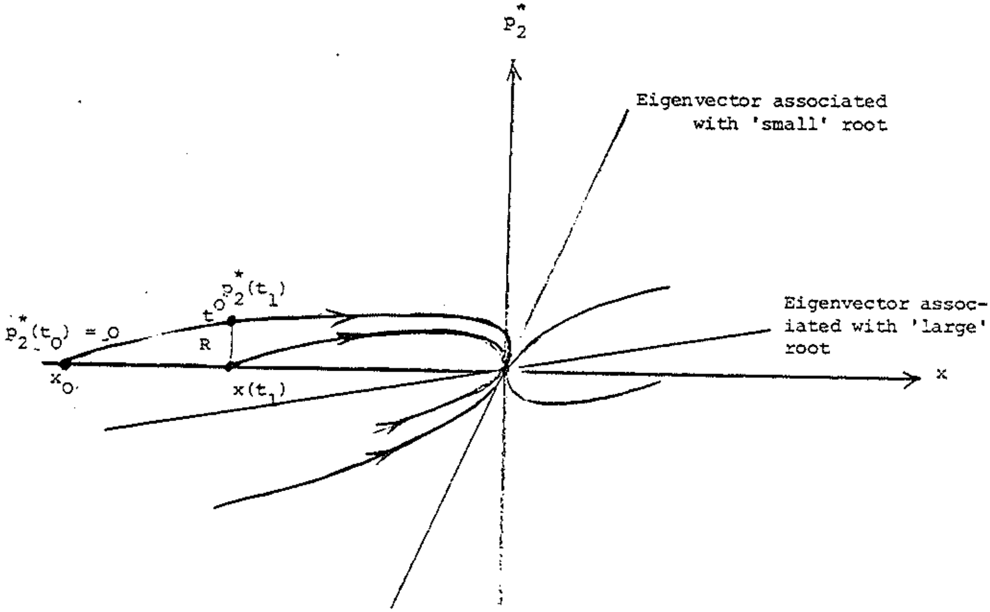

FIGURE 1 : The "time inconsistency" of optimal policy

The unique optimal trajectory associated with the initial condition $x(t_0) = x_0$ is shown starting with $p_2^*(t_0) = 0$. It is evident from the figure, however, such a path is not "time consistent" as the conditions for reoptimising at $t_1 > t_0$ involve setting ${}_{t_1}p_2^*(t_1) = 0 \neq {}_{t_0}p_2^*(t_1)$. The change of plan implicit in resetting p_2^* to zero in this way reflects the temptation upon the leader to "treat bygones as bygones" and renege on those long term plans whose strategic effects on the follower have already been achieved, in favour of those which promise greater strategic effects from t_1 onwards, see Kydland and Prescott (1977).

In the absence of precommitment only time consistent plans are credible, as such reoptimisation can be expected continuously, and the appropriate time consistent solution for this Stackelberg game obtained by dynamic programming is analysed by Cohen and Michel (1984). To characterise the recursive solution using the maximum principle they show that the follower minimises a Hamiltonian of the form (2.3) treating the leader's control path as given, while the leader minimises his corresponding Hamiltonian on the assumption of closed loop behaviour by the follower, specifically treating $p_2 = \Theta^* x$ with Θ^* parametric to the leader but endogenous to the problem. (In this formulation, unlike (2.7), p_2^* does not appear.)

By extending their univariate analysis to the multivariate case, the adjoint equations for this "time consistent" solution to the Stackelberg game can be written

$$\begin{bmatrix} Dx \\ Dp_1 \\ Dp_2 \end{bmatrix} = \begin{bmatrix} A & -J_1 & -J_2 \\ -Q_1 & -A^T+\Theta^{*T}J_2 & -\Theta^{*T}J_{12} \\ -Q_2 & 0 & -A^T \end{bmatrix} \begin{bmatrix} x \\ p_1 \\ p_2 \end{bmatrix} = M^{TC} \begin{bmatrix} x \\ p_1 \\ p_2 \end{bmatrix} \quad (2.10)$$

where J_i, J_{12} are as defined in (2.9) above and $\Theta^* = C_{13}C_{11}^{-1}$ depends on the stable eigenvector of M^{TC}. The transversality conditions are as before, except p_2^* does not appear. Since the coefficients of M^{TC} involve the eigenvectors, it is evident that to obtain the latter it will in general be necessary to proceed numerically with an iterative scheme to obtain the eigenvectors and values required to generate the "time consistent" trajectories for x, p_1, p_2.

Of course if Θ^* in equation (2.10) were not constrained as shown but were simply set to zero, we would obtain the adjoint equations of the Nash game of the previous section, with each player taking the other's control path as a predetermined open loop.

3. Optimal control of an economy with forward looking behaviour

Macroeconomic models with agents exhibiting "forward looking" behaviour typically possess a "saddlepoint" dynamic structure; the state variables are consequently partitioned between those which are historically pre-determined and those which are free to jump so as to satisfy conditions for dynamic stability. Such non-predetermined state variables are characteristically forward-looking asset prices which depend on the (expected) future evolution of the system in much the same way as do the shadow prices of an optimal control problem, cf. Dornbusch (1980, Chapter 11), Blanchard (1981) Summers (1981) and Hayashi (1982).

Indeed the Government seeking to minimise a social cost function given such a macroeconomic structure appears to be in much the same position as the Stackelberg leader in a dynamic game, a point argued with some force by Lucas and Sargent (1981, Introduction). Consequently procedures analogous to those used to find the leader's optimal control path can be used for determining optimal policy in these economic models. Policies derived using the maximum principle on the assumption that the leader can precommit himself are derived for a closed economy in Calvo (1978) and for an open economy with a freely floating exchange rate by Driffill (1982). In the absence of credible commitments, of course, such plans will not be time consistent, but dynamic programming methods may be used to derive optimal time consistent policies, as described below.

If the government is not acting as a leader, however, the appropriate analogy may rather be the Nash differential game, and

Buiter (1983) describes what this "loss of leadership" implies for policy design in an economy with forward-looking private agents.

We start with the characterisation of the optimal "time inconsistent" policy for a government with the quadratic cost function (1.1) above seeking the optimal time path for its controls u in an economy with a dynamic structure described by (1.2). We now assume, however, that the linear economic system has a "saddle point" structure and that the state vector x is partitioned between those variables, x_1, which are predetermined at t_0 and those variables, x_2, which are free to adjust at the time of optimisation. (The corresponding costate variables are denoted p_1 and p_2.)

The differential equations (1.5) for the adjoint system of state and costate variables, described in Section 1 still apply, although the assumption made there, that the entire initial state vector is predetermined, is clearly no longer appropriate. Instead, at t_0, we take x_1 to be predetermined (given by history) and p_2, the costates for x_2, to be set to zero by the transversality conditions (a restriction analogous to setting $p_2^*(t_0) = 0$ for the Stackelberg game in the previous section).

To describe the trajectories under the optimal plan it is convenient to <u>reorder</u> the system so that the variables whose initial conditions are known come first, so the adjoint equations become:

$$\begin{bmatrix} Dx_1 \\ Dp_2 \\ Dx_2 \\ Dp_1 \end{bmatrix} = \begin{bmatrix} A_{11} & -J_{12} & A_{12} & -J_{11} \\ -Q_{21} & -A_{22}^T & -Q_{22} & -A_{12}^T \\ A_{21} & -J_{22} & A_{22} & -J_{21} \\ -Q_{11} & -A_{12}^T & -Q_{12} & -A_{11}^T \end{bmatrix} \begin{bmatrix} x_1 \\ p_2 \\ x_2 \\ p_1 \end{bmatrix} = M \begin{bmatrix} x_1 \\ p_2 \\ x_2 \\ p_1 \end{bmatrix} \qquad (3.1)$$

where A, Q and J are partitioned to conform with x and p. The path to be followed by x_1 and p_2 from these initial conditions can now be described as a function of the stable eigen-values and their associated eigenvectors in the usual way, so

$$\begin{bmatrix} {}_{t_0}x_1 \\ {}_{t_0}p_2 \end{bmatrix} = C_{11} \; e^{\Lambda_s(t-t_0)} \; C_{11}^{-1} \begin{bmatrix} x_1(t_0) \\ 0 \end{bmatrix} \qquad (3.2)$$

with the remaining variables determined by

$$\begin{bmatrix} p_1 \\ x_2 \end{bmatrix} = C_{12}C_{11}^{-1} \begin{bmatrix} x_1 \\ p_2 \end{bmatrix} \qquad (3.3)$$

where $\begin{bmatrix} C_{11}, C_{12} \end{bmatrix}^T$, Λ_s denote the stable eigenvectors and roots of M.

In the absence of credible commitments, optimisation is restricted to time consistent plans found recursively. By analogy with the recursive Stackelberg equilibrium, the policy maker proceeds "as if" $x_2 = \Theta x_1$, an assumption which makes the jump variables effectively predetermined and so removes the time inconsistency. The trajectories for x_1 (and the value for Θ) can be determined as usual from the adjoint system:

$$\begin{bmatrix} Dx_1 \\ Dx_2 \\ Dp_1 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & -J_{11} \\ A_{21} & A_{22} & -J_{21} \\ -Q_{11}-\Theta_1^T Q_{21} & -Q_{12}-\Theta^T Q_{22} & -A_{11}^T-\Theta^T A_{12}^T \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ p_2 \end{bmatrix} = M^{TC} \begin{bmatrix} x_1 \\ x_2 \\ p_1 \end{bmatrix} \quad (3.4)$$

where the elements of M are defined as for (3.1) except that Θ is defined by the stable eigenvector of M^{TC} by $\Theta = C_{12}C_{11}^{-1}$.

If the policy maker were to "lose his leadership" altogether and to act as if the value of all jump variables were exogenously given, the "Nash" solution proposed by Buiter (1983) will apply, with an adjoint system obtained by simply setting Θ to zero in (3.4).

4. Solving dynamic games between open economies

4.1 The nature of solutions

Since the lack of credible pre-commitment renders the optimal policy of a Stackelberg leader "time inconsistent", alternative policies have been described which may be obtained recursively. But way of summary, we outline in Table 2 - in highly schematic fashion - various possible solutions to full information dynamic games in open economies with forward looking financial markets, before proceeding to describe a (particularly simple) example.

The rows of the Table contain the various "inter-governmental" relationships which have been considered. The columns show likewise the relations between these governments and private forward looking markets: for simplicity, it is here assumed that all governments stand in the same relationship to all such markets. In the body of the Table the nature of the solution to be calculated is indicated symbolically, using the variables and parameters occurring earlier. Thus the symbols p_2^* and p_2 flag the sources of time inconsistency, while Θ^* and Θ refer to the constraints to be observed in computing time consistent leadership solutions.

It should be noted,however, that several of the economic boxes appearing in the Table may well prove empty (see below); and furthermore that a number of interesting papers on economic interdependence which treat the exchange rate as determined by the current account only are not strictly covered (as they contain no foward looking variables). See, for example, Hamada and Sakurai (1978), Sachs (1983) and Turner (1984), the last two of which treat policy making as an explicit Nash dynamic game using discrete and continuous time methods respectively.

Where the capital account plays a role and expected movements of the exchange rate depend on interest differentials as in Dornbusch (1976, 1980 Chap.11) problems of time inconsistency and of preserving credibility are unavoidable. John Driffill (1982) was the first to our knowledge to characterise the time inconsistent optimal policy for a small open economy. (The single controller approach he used is formally almost identical to that indicated in the last column of the Table.) The example below (a much reduced version of what appears in Annex 2 of our 1983 paper) also treats the exchange rate as forward looking and illustrates the various solution concepts indicated in the last two columns of the Table.

TABLE 2 : Key to constructing solutions

Relationships between Governments

			Non-Cooperative			Cooperative
			Asymmetric Stackelberg (a)	Asymmetric Stackelberg (b)	Symmetric: Nash	Single Controller
Relations between governments & markets with forward looking variables	Stackelberg Leadership	(a)	p_2^*, p_2	Θ^*, p_2	$0, p_2$	p_2
	Stackelberg Leadership	(b)	p_2^*, Θ	Θ^*, Θ	$0, \Theta$	Θ
	No Leadership		$p_2^*, 0$	$\Theta^*, 0$	$0, 0$	0

Notes:
- (a) time inconsistent (only credible with pre-commitment).
- (b) time consistent (solved recursively).
- p_2^* costate for follower's costate required.
- p_2 costate for forward looking asset prices required.
- Θ^* relation between followers costates and state variables constrained
- Θ relation between forward-looking variables and state variables constrained.

4.2 An example

As an illustration of the various solutions described in this paper, we include an exceedingly simplified example of fiscal stabilisation policy in a (fix-price) Common Market. The dynamic equations determining Market output and the price of foreign ("Rest of World") currency are given in Table 3(a), where the variables used are also defined. Output is determined in a Keynesian fashion by government expenditure, the exchange rate and past output, see equation (4.1). Given the free movement of goods inside the Market, fiscal expenditure has effects spread widely over partner countries, so the objective of each policy maker is to help stabilise Market output, subject to a quadratic cost attached to the use of fiscal expenditure.

Anticipated movements in the exchange rate (defined as the price of Rest of World currency) offset interest differentials between the Market and the Rest of World, with Market countries locking their cross-rates and pursuing a Market money supply target under conditions of perfect capital mobility. Consequently, high Market output, will (via high interest rates) be associated with an anticipated rise in the price of foreign currency, see equation (4.2).

In principle, each Market member, acting as a Nash player, will have shadow prices for y and e; but as they share a common purpose in controlling y and have identical private costs of using fiscal policy, all such shadow prices will be identical! We can thus represent the adjoint equations using only one set of costate variables (the very same, indeed, as would guide a central controller minimising simple sum of partner costs).

TABLE 3(a) : Fiscal Policy in a Common Market

Behavioural Equations for the Private Sector

(Output) $$Dy = -\alpha y + \beta e + \gamma \sum_{i=1}^{n} g_i / n \qquad (4.1)$$

(Exchange Rate) $$De = \delta y \qquad (4.2)$$

Official Objectives

$$\min_{g_i} \int_{t_0}^{\infty} \frac{y^2 + g_i^2}{2}$$

Definition of Variables

y	log of real output of Common Market
e	log of exchange rate (price of Rest of World Currency)
p_y	shadow price of y
p_e	shadow price of e
g_i	log of government expenditure in country i

The adjoint equations for the various solutions to this Nash game are shown in Table 3(b). The time inconsistent policy involves a costate for e, and has four differential equations, see [1]. Two forms of time consistent policy are specified, [2] in which the countries assume $e = \Theta y$ and [3] in which the exchange rate is taken to be exogenous in designing fiscal stabilisation policy. At the foot of the Table, the inherent "saddlepoint" dynamics of output and the exchange rate in the absence of policy is indicated, see [4].

Setting $\alpha = 1$, $\beta = \gamma = \delta = \frac{1}{2}$, for example, one finds stable roots for the four cases as follows: [1] -1.573, -0.159, [2] -1.534, [3] -1.553, [4] -1.207, with the biggest root associated with time inconsistent policy and the smallest with the absence of policy. Integrated welfare costs increase monotonically (though not by much) from [1] to [4], as one might expect.

Since the shadow prices in each partner country are the same as they would be for a centralised policy maker using equal weights, policy coordination hardly appears worthwhile. However, the plausibility of those solutions ([1] and [2]) which involve strategic effects on the exchange rate is surely greater for centrally cordinated policy than it is for n countries each pursuing its own objectives. Helping to attain such strategic effects on the exchange rate may thus provide a rationale for coordination in this case.

TABLE 3(b) : Dynamic Adjustment

[1] Time Inconsistent Optimal Policy

$$\begin{bmatrix} Dy \\ Dp_e \\ De \\ Dp_y \end{bmatrix} = \begin{bmatrix} -\alpha & 0 & \beta & -\gamma^2 \\ 0 & 0 & 0 & -\beta \\ \delta & 0 & 0 & 0 \\ -1 & \delta & 0 & \alpha \end{bmatrix} \begin{bmatrix} y \\ p_e \\ e \\ p_y \end{bmatrix}$$

[2] Time Consistent Optimal Policy $(\Theta = C_{12}C_{11}^{-1})$

$$\begin{bmatrix} Dy \\ De \\ Dp_y \end{bmatrix} = \begin{bmatrix} -\alpha & \beta & -\gamma^2 \\ \delta & 0 & 0 \\ -1 & 0 & \alpha-\Theta\beta \end{bmatrix} \begin{bmatrix} y \\ e \\ p_y \end{bmatrix}$$

[3] Time Consistent Optimal Policy $(\Theta = 0)$

$$\begin{bmatrix} Dy \\ De \\ Dp_y \end{bmatrix} = \begin{bmatrix} -\alpha & \beta & -\gamma^2 \\ \delta & 0 & 0 \\ -1 & 0 & \alpha \end{bmatrix} \begin{bmatrix} y \\ e \\ p_y \end{bmatrix}$$

[4] Absence of Policy $(g_1 = g_2 = 0)$

$$\begin{bmatrix} Dy \\ De \end{bmatrix} = \begin{bmatrix} -\alpha & \beta \\ \delta & 0 \end{bmatrix} \begin{bmatrix} y \\ e \end{bmatrix}$$

5. Conclusion

In this paper we have analysed some of the implications of dynamic games and of "rational expectations" for the theory of economic policy in inter-dependent economies. Various non-cooperative relationships between rational policy makers were considered under alternative assumptions as to the credibility of official policy announcements. The maximum principle has been used to characterise both optimal time inconsistent policies, which may arise when there is strategic dominance, and various time consistent alternatives.

Although the resulting framework is analytically fairly straightforward for linear structures and quadratic criteria, the number of differential equations can rapidly escalate. As Aoki (1981) has pointed out with respect to the positive analysis of open economy macroeconomics, however, considerable simplification is to be had by setting parameters the same in different countries (as the canonical variables can then be specified independently of the values of these parameters). This simplification carries over to the analysis of policy design so long as the coefficients of objective functions are similarly constrained, cf. Miller and Salmon (1983) Appendix 2.

It has to be recognised, however, that a purely algebraic treatment will probably need to be supplemented with numerical analysis in order to obtain definite results. The examination of policy design and policy conflict on econometrically estimated macromodels is probably essential to arrive at a realistic estimate of the potential gains from policy co-ordination; see Oudiz and Sachs (1984) for example.

In addition, recent study by game theorists of how such co-operative behaviour may be sustained by explicit threats of punishment or by loss of "reputation" is likely further to illuminate the theory of policy.

Even without this, we would hope that the above account has indicated the potential of dynamic games for examining important issues of policy formation in open economies and of the maximum principle for characterising optimal policy in a variety of strategic settings.

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