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## ABSTRACT

# Risk Taking and Optimal Contracts for Money Managers*

Recent empirical work suggests a strong connection between the incentives money managers are offered and their risk-taking behaviour. We develop a general model of delegated portfolio management, with the feature that the agent can control the riskiness of the portfolio. This represents a departure from the existing literature on agency theory in that moral hazard is not only effort exertion but also risk-taking behaviour. The moral hazard problem with risk taking involves an incentive-compatibility constraint on risk, which we characterize. We distinguish between one period and several periods. In the former case, under mild conditions, there exists a first-best contract, which takes the form of a bonus contract. In the latter, we show that there exists no first-best contract and we use a numerical approximation to study the properties of the second-best contract.

JEL Classification: D82, G11, G24
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# NON-TECHNICAL SUMMARY

Consider a money manager receiving funds from clients who desire to invest their money in financial markets but do not have the time or the knowledge to do it personally. In such a case, the money manager chooses an investment strategy on behalf of the client. While the client or the law can dictate some characteristics of the investment strategy, the manager is usually left with a degree of freedom in determining the composition of the portfolio. In particular, the money manager can control the riskiness of the portfolio.

One can expect that money managers adjust the risk level in order to maximize their (implicit or explicit) compensation rather than acting in clients' best interest. This type of behaviour is sometimes referred to as 'gaming the fee'.

In the present paper, we model the interaction between an investor and a money manager. To focus on the effects of gaming the fee *per se*, we assume that both parties are risk-neutral. To make the problem interesting, we assume that the money manager can refuse negative compensations *ex post*. This limited liability assumption prevents the investor from selling the investment return to the manager in exchange for its expected value, which would be first-best under all circumstances.

The money manager makes two choices: effort and risk. Effort is binary. A low level of effort restricts the manager to invest in a low-yield riskless asset. A high level allows him to access a set of feasible portfolios, which vary in their expected value and in their riskiness. In case of high effort, the agent can select an element of the feasible set of portfolios, which is modelled in a general way. The investor cannot observe the distribution of the portfolio the agent chooses, however, but only the realized return on the portfolio. Hence, the contract between the investor and the agent can only be based on the observed return.

Alternatively, one could assume that the investor observes the composition of the portfolio but that they cannot determine the distribution of the portfolio with the same precision the agent can (if they could, they would not hire the agent in the first place). This more general framework would yield similar results.

We consider two versions of the model: one period and many periods. In the one period case, the money manager makes the portfolio decision only once. Then, the return is realized and compensation is paid. The investor maximizes their expected net return subject to an incentive-compatibility constraint on effort, which is a familiar concept in moral hazard, and to an incentive-compatibility constraint on risk, which is introduced in this work. First, we show that the latter constraint can be written as a first-order condition that does not depend on the shape of the set of feasible portfolios. Thus, the incentive-compatibility constraint on risk can be checked empirically even with limited information on the set of available portfolios (in contrast, checking an incentive-compatibility constraint on effort is very hard because it requires knowing the cost of effort for the agent). Second, we show that, in the one-period case, first-best contracts exist and take the form of bonus contracts.

In the multi-period case, the agent can revise his portfolio choice after observing his performance at intermediate stages. This derives from the assumption that the investor can evaluate the agent's performance less often than the agent can revise his own portfolio decisions. (In general, the delegated money management industry has a well-established practice of end-of-year bonuses depending on performance during the calendar year.)

With several periods, the agent has more room to game the fee. We show that the only first-best contract is a linear contract. The intuition is that an agent who has a non-linear contract and can control risk after observing intermediate performance will have incentives to adjust risk toward the end of the contract in ways that reduce the expected value of the portfolio. Our result strengthens the well-known result on the optimality of linear contracts.

Unfortunately, linear contracts are prevented by limited liability and we conclude that in a multi-period setting a first-best contract does not exist. By means of a numerical example, we study the properties of second-best contracts, which turn out to involve excessive risk-taking on the part of the agent.

# 1 Introduction

A money manager receives funds from clients who desire to invest their money in ¯nancial markets but do not have the time or the knowledge to do it personally. The money manager chooses an investment strategy on behalf of the client. While the client or the law can dictate some characteristics of the investment strategy, the manager is usually left with a degree of freedom in determining the composition of the portfolio. In particular, the money manager can control the riskiness of the portfolio.

One can expect that money managers adjust the risk level in order to maximize their (implicit or explicit) compensation. This type of behavior is sometimes referred to as \gaming the fee." Chevalier and Ellison [2] have analyzed the return and the variance of a sample of mutual funds. They have found that the variance of a fund in the last quarter of the year is negatively correlated to the performance of the same fund in ¯rst three quarters. This suggests that fund managers condition their last quarter strategy on how well they have performed so far. If they have done well they play conservatively, if they have done poorly they \gamble for resurrection."

The problem of the relation between investor and money manager when the latter can play with risk is related to two strands of literature: the microeconomic literature on moral hazard and the ¯nance literature on delegated portfolio management. Unfortunately, neither of the two strands gives a satisfying answer on this point.

The vast literature on moral hazard (See Salani¶e [21] or Hart and Holmstrom [10] for surveys) has mostly focused on the problem of a principal who wants to induce an agent to exert the `right' amount of e®ort. The possibility of the agent controlling the riskiness of the outcome is excluded by the monotone likelihood ratio assumption (Milgrom [19]), which guarantees that the e®ect that di®erent actions have on the expected value of the outcome dominates all other e®ects. Although some works have given explicit consideration to risk taking, we are unaware of any general moral hazard model in which the agent controls the riskiness of the outcome.[1]

The literature on delegated portfolio management includes, among others, Cohen and Starks [3], Grinblatt and Titman [8], Stoughton [23], Heinkel and Stoughton [11], Goetz-

[1]However, see Holmstrom and Ricart i Costa [14], La®ont [17], and Matutes and Vives [18] for agency models in which the agent has some control over the riskiness of the outcome.

mann, Ingersoll, and Ross [6], and Das and Sundaram [4]. In contrast with the moral hazard literature, these authors give a great importance to the agent's incentives for risk-taking. However, these works do not consider full-°edged principal-agent models. Rather, they restrict the set of feasible contracts between the investor and the money manager to limited classes, such as piecewise linear. Instead, our paper makes no limitations on the shape of the contracts the investor can use.

In the present paper, we model the interaction between an investor and a money manager. To focus on the e®ects of gaming the fee per se, we assume that both parties are risk-neutral. The money manager can refuse negative compensations ex post. This limited liability assumption prevents the investor from selling the investment return to the manager in exchange for its expected value, which would be ¯rst-best under all circumstances.

The money manager makes two choices: e®ort and risk. E®ort is binary. A low level of e®ort restricts the manager to invest in a low-yield riskless asset. A high level allows him to access a set of feasible portfolios, which vary in their expected value and in their riskiness. In case of high e®ort, the agent can select an element of the feasible set of portfolios, which is modeled in a general way. The investor, however, cannot observe the distribution of the portfolio the agent chooses but only the realized return on the portfolio. Hence, the contract between the investor and the agent can only be based on the observed return.[2]

We consider two versions of the model: one period and many periods. In the one period case, the money manager makes the portfolio decision only once. Then, the return is realized and compensation is paid. The investor maximizes her expected net return subject to an incentive-compatibility contraint on e®ort, which is a familiar concept in moral hazard, and to an incentive-compatibility constraint on risk, which is introduced in this work. First, we show that the latter constraint can be written as a ¯rst-order condition that does not depend on the shape of the set of feasible portfolios. Thus, the incentive-compatibility constraint on risk can be checked empirically even with limited information on the set of available portfolios (in contrast, checking an incentive-compatibility constraint on e®ort is very hard because it requires knowing the cost of e®ort for the agent). Second, we show that, in the one-period case, ¯rst-best contracts exist and take the form of bonus contracts.

[2] Alternatively, one could assume that the investor observes the composition of the portfolio but she cannot determine the distribution of the portfolio with the same precision the agent can (if she could, she would not hire the agent in the ¯rst place). This more general framework would yield similar results.

In the multi-period case, the agent can revise his portfolio choice after observing his performance at intermediate stages. This derives from the assumption that the investor can evaluate the agent's performance less often than the agent can revise his own portfolio decisions. The situation described by Chevalier and Ellison is a case in point: mutual fund managers control investment volatility continuously, while investors receive performance information once a year. In general, the delegated money management industry has a well-established practice of end-of-year bonuses depending on performance during the calendar year (later in the paper we will point at some puzzling aspects of the this compensation practice).

With several periods, the agent has more room to game the fee. We show that the only ¯rst-best contract is a linear contract. The intuition is that an agent who has a nonlinear contract and can control risk after observing intermediate performance will have incentives to adjust risk toward the end of the contract in ways that reduce the expected value of the portfolio. Our result strengthens the well-known result on the optimality of linear contracts by Holmstrom and Milgrom [13].

Unfortunately, linear contracts are prevented by limited liability and we conclude that in a multi-period setting a ¯rst best contract does not exist. By means of a numerical example, we study the properties of second best contracts, which turn out to involve excessive risk-taking on the part of the agent.

The organization or the paper is as follows. The next subsection reviews related literature. Section 2 presents the model. Section 3 states the principal problem and characterizes the incentive-compatibility condition on risk. Section 4 proves that a ¯rst-best contract exists and that it takes the form of a bonus contract. The following two sections deal with the multi-period case. Section 5 shows that a ¯rst-best contract must be linear and that, therefore, in the problem at hand no forst-best contract exists. Section 6 analyzes, by means of numerical approximation, the properties of second-best contracts. Section 7 concludes.

## Related Literature

Bhattacharya and P°eiderer [1] were the ¯rst to study delegated portfolio management in a principal-agent framework. An investor faces a large number of agents, who vary in their forecasting ability. Thus, the ¯rst problem for the investor is to screen agents. The problem is made more di±cult by the assumption that better forecasters have higher opportunity costs. Once an agent is hired he observes a private signal. Thus, the second problem of the principal is that of eliciting the agent's private signal in order to make the right portfolio decision. Bhattacharya and P°eiderer show the existence of an optimal contract in which agents truthfully report their forecasting ability and their private signal. Compensation is a concave function of return, increasing if return is below the mean and decreasing if it is above the mean. Our work di®ers from Bhattacharya and P°eiderer because it is a hidden action rather than a hidden information model. In their model, the principal is able to verify the level of risk taken by the agent, while in ours she is not. Another di®erence is that we assume that the agent can sabotage the ex post return. This forces the principal to o®er only nondecreasing contracts. In contrast Bhattacharya and P°eiderer's optimal contract is nonmonotonic.[3]

Some authors have considered moral hazard problems in the presence of the limited liability constraint. Sappington [22] does it for a model of hidden information, while Innes [15] does it for hidden action and is therefore more closely related to our work. Innes assumes the Monotone Likelihood Ratio Condition, which in our paper is clearly violated because the agent controls risk. It is interesting to compare our results with his. He shows that the optimal contract (subject to the monotonicity constraint) a debt contract whereby the principal receives the whole return up to a certain level and the residual belongs to the agent. In our framework a debt contract is clearly suboptimal because it gives the agent an incentive to take ine±ciently high levels of risk. Indeed, we prove that any contract that is convex (or concave) in portfolio return is not ¯rst-best.

Gollier, Koehl, and Rochet [7] consider the problem of a risk-averse decision maker with limited liability. The decision maker chooses the size of a risky project. The form of the distribution function of returns is left in a general form. The authors show that the level of risk chosen by the decision-maker is always higher under limited liability than under full liability. The authors also provide comparative static results on the role of the decison-maker initial wealth. While we keep the same level of generality of Gollier, Koehl, and Rochet, our model is clearly di®erent because it is developed in a principal-agent framework.

[3]Bhattacharya and P°eiderer do not make a limited liability assumption. In their optimal contract the agent can incur unbounded losses. However, it is easy to modify the optimal contract in order to allow for a limited liability clause. Thus, limited liability does not seem to be a crucial di®erence between our work and theirs.

The most closely related work is Diamond [5]. Like us, he studies a hidden action moral hazard problem in which the agent controls both effort and the distribution of the outcome. He asks whether, as the cost of effort shrinks relative to the payoffs, the optimal contract converges to the linear contract. The answer is positive if the control space of the agent has full dimensionality (i.e. if the principal has less degrees of freedom in setting the incentives than the agent has degrees of freedom in responding), but not otherwise. Dimensionality formalizes the important intuition that, if the agent has several ways to manipulate the outcome, the principal should offer the simplest possible compensation scheme, that is, the linear contract. There are two important differences between our framework and Diamond's. First, while he considers only three possible outcomes, our model encompasses a continuum of outcomes and is therefore more suited to study financial intermediation. Second, his results are asymptotic, in that they hold when the cost of effort tends to zero, while ours hold in general. In addition, we include the multiperiod case and second-best contracts.

## 2 Model

Let us start by defining the feasible set of risky portfolios. A portfolio is a probability distribution on the support $[\underline{x}; \bar{x}]$, where $\underline{x}$ could be $-\infty$ and $\bar{x}$ could be $+\infty$. The set of feasible portfolios is the family of distribution functions $F$. We assume that an element of $F$ is uniquely identified by its mean $\mu$ and a risk measure $r$. The risk measure need not correspond to variance. A typical element of F is $f(\cdot|\mu; r)$. For simplicity, $f(\cdot|\mu; r)$ is assumed twice continuously differentiable for all $\mu$ and $r$. Also, for any $x \in (\underline{x}; \bar{x})$, $f(x|\mu; r)$ is continuously differentiable in $\mu$ and $r$.

We assume that if $r'' > r'$, $f(\cdot|\mu; r')$ dominates $f(\cdot|\mu; r'')$ in the second-order stochastic sense (SOSD). Thus, if two assets have the same mean, the one with the lower r is less risky than the other. Also, it is clear that, if $\mu'' > \mu'$, then $f(\cdot|\mu''; r)$ dominates $f(\cdot|\mu'; r)$ in a first-order stochastic dominance (FOSD) sense. Our model represents a clear departure from previous models of delegated investment in that the set of feasible portfolios is not completely ordered with respect to first-order stochastic dominance.[4]

[4] Most models of moral hazard consider only two outcomes, in which case the all actions are ordered by first-order stochastic dominance. The models which consider more than one outcome usually assume the Monotone Likelihood Ratio property, which implies FOSD. See for instance Grossman and Hart [9] or Innes [15].

Let $(\mu, r) \in A$ where $A = \{\mu, r | r \geq 0, 0 \leq \mu \leq m(r)\}$. The function $m(\cdot)$ is twice differentiable, strictly concave and has a maximum at $\mu^* = m(r^*)$ with $r^*$ strictly positive.

To interpret A, Figure 1 is useful. A represents the set of feasible risky portfolios. A is bounded above by the curve $\mu = m(r)$, which can be viewed as the efficient portfolio frontier. Each point on the frontier represents the maximum expected value that can be achieved given a certain level of risk. In a typical textbook, only the increasing part of $m(\cdot)$ is depicted. That is because a risk-neutral or risk-averse investor who selects her portfolio without using an agent would never choose portfolios to the right of $r^*$, as they are dominated in both a first- and second-order stochastic sense by the portfolio with $\mu^*$ and $r^*$. However, as we will see, a money manager with the 'wrong' incentive scheme might want to choose a portfolio to the right of $r^*$.[5]

It is also assumed that the agent cannot shortsell, or has limited shortselling power. With unlimited shortselling, the principal-agent problem may not have a solution, because the agent may want to choose unbounded levels of risk.

Both the principal and the agent are risk-neutral. Thus, the efficient portfolio is simply $(\mu^*; r^*)$. The presence of risk-aversion is not central to the argument presented here. The model could be readily extended to risk-averse players. If the principal is risk-averse, $\mu$ can be thought of as the expected value of the principal's utility. The concave shape of $m(r)$ will hold a fortiori.[6]

We want to model the relation between the investor and the money manager as an agency problem with two-dimensional moral hazard: portfolio selection and effort exertion. We have already discussed the former. Let us now turn to the latter. As we are mainly interested in portfolio selection, we will model effort in the simplest way. In order to get access the set of feasible portfolios $F$, the agent must pay a monetary cost $c$. If the agent does not spend $c$, he can only invest in a risk-free asset with return $x_0 \in (\underline{x}; \bar{x})$.

[5]One may object that, from a standard portfolio theory perspective, there cannot exist assets that dominate other assets on both mean and risk. This objection is correct only if all assets carry only systemic risk. A portfolio to the right of $\mu^*$ can exist if, for instance, there is one asset with a low mean, a low systemic risk, and a high idiosyncratic risk. In order to game the fee, the agent may want to select such a portfolio. This situation may capture some of the recent financial crashes, in which the money managers accumulated nonsystemic risk.

[6]In general we can expect that the agency problem becomes more severe if the principal is risk-averse and less severe if the agent is. Both situations fit this model (the individual investor with limited wealth and a large pension fund, or the investment bank and one of its traders).

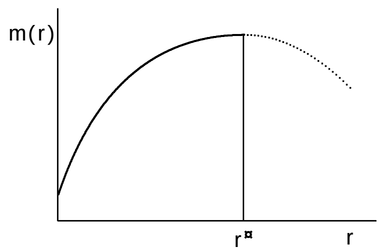


Figure 1: The Feasible Set of Risky Portfolios

The principal offers a contract to the agent. The compensation can only depend on the realized outcome $x$. Thus, let $b(x)$ denote the contract. The agent's opportunity cost of working for this principal is normalized at zero.

**Assumption 1.** The agent can sabotage $x$, that is, given an actual return $x$, he can report any return $x' \in [\underline{x}; x]$.

Then, the optimal contract must be nondecreasing in $x$. Otherwise, if the agent finds that $x$ falls on a decreasing section of $b(x)$, he can increase his compensation by reducing $x$. In the rest of the paper we will consider only nondecreasing contracts.[7]

To summarize, the timing of the principal-agent relationship is:

1. The principal proposes a contract $b(\cdot)$ to the agent.
2. If the agent accepts, he receives a unitary sum to manage for one period.
3. The agent chooses whether or not to spend $c$ in order to access the set of feasible portfolios. If he does not spend $c$, the agent invest in the risk-free asset with return $x_0$.
4. If the agent spends $c$, he chooses $(\mu; r) \in A$. $x$ is realized according to $f(x|\mu; r)$.
5. The principal pays $b(x)$ to the agent and keeps $x - b(x)$.

**Lemma 1.** For any $b(\cdot)$, the agent maximizes $E[b(x)|\mu; r]$ by choosing $\mu$ and $r$ such that $\mu = m(r)$.

*Proof.* Immediate from the $b(\cdot)$ being nondecreasing and first-order stochastic dominance. □

[7] A discussion on the reasons for excluding nonmonotonic contracts is found in Innes [15, p. 46].

**Definition 1.** a contract is first-best if the agent receives no rent and chooses $r = r^*$.

In a first-best contract the principal achieves the same net expected payoff she would receive if there were no asymmetric information. Without limited liability, there exists a linear first-best contract: $b(x) = -B + Ax$, with $A \geq \frac{c}{\mu^* - x_0}$ (incentive compatibility constraint on effort) and $B = A\mu^* - c$ (participation constraint – which must clearly be binding). With this contract, the agent chooses $r = r^*$.

To make the problem interesting and realistic, we assume that the agent has limited liability. With a rescaling, the limited liability of the agent is set at zero:

**Assumption 2.** $b(x) \geq 0$ for $x \in [\underline{x}; \bar{x}]$.

With limited liability, the linear contract $b(x) = -B + Ax$ is not feasible anymore because $B < 0$ and, with positive probability, $x$ is low enough that $-B + Ax < 0$. Of course, the contract $b(x) = Ax$ with $A \geq \frac{c}{\mu^* - x_0}$ will still elicit the first-best action from the agent. However, the participation constraint will be satisfied as an inequality, which means that the principal leaves a positive rent to the agent. Thus, with limited liability, there does not exist a linear first-best contract and we must look for nonlinear first-best contracts.

## 3 One Period: Necessary Conditions

This paper is concerned only with the existence and the properties of first-best contracts. Thus, we will focus on the set of contracts that are feasible, leave no rent to the agent, induce the agent to spend $c$, and induce the agent to choose the optimal amount of risk. A feasible contract is first-best if and only if the following three conditions are satisfied: (i) The agent's participation constraint binds

$$E[b(x)|\mu^*; r^*] = 0; \tag{1}$$

(ii) The incentive-compatibility constraint on effort is satisfied

$$E[b(x)|\mu^*; r^*] - c \geq b(x_0); \tag{2}$$

and (iii) The incentive-compatibility constraint on risk is satisfied

$$E[b(x)|\mu^*; r^*] \geq E[b(x)|m(r); r] \tag{3}$$

for any $r \in r$.

Let $f_r(x|\mu; r) \equiv \frac{\partial}{\partial r} f(x|\mu; r)$ and $f_\mu(x|\mu; r) \equiv \frac{\partial}{\partial \mu} f(x|\mu; r)$. The following proposition provides a simple necessary condition for a contract to be first-best and says what happens if the necessary condition is violated:

**Proposition 1.** Suppose that if the principal offers contract $b(\cdot)$, then the agent chooses $r = \hat{r}$ and $\mu = \hat{\mu}$. Then: (i) A necessary condition for $b(\cdot)$ to be first best is:

$$\int_{\underline{x}}^{\bar{x}} b(x) f_r(x|\hat{\mu}; \hat{r}) dx = 0; \tag{4}$$

and (ii) If, instead, it is the case that

$$\int_{\underline{x}}^{\bar{x}} b(x) f_r(x|\hat{\mu}; \hat{r}) dx > (<) 0;$$

then the principal would lose (gain) from a small increase in $\hat{r}$.

*Proof.* Because the agent only chooses points on the frontier, it must be the case that $m(\hat{r}) = \hat{\mu}$. Recall that $f(x|\mu; r)$ is continuous and differentiable in $\mu$ and $r$ for any fixed $x$. Then, also $E[x|\hat{\mu}; r]$ and $E[b(x)|\hat{\mu}; r]$ are continuous and differentiable in $\mu$ and $r$.

Claim: for any nondecreasing $b(\cdot)$ the expressions

$$\frac{\partial}{\partial r} E[b(x)|\hat{\mu}; r]|_{r=\hat{r}} \tag{5}$$

and

$$\frac{d}{dr} E[x - b(x)|m(r); r]|_{r=\hat{r}} \tag{6}$$

have opposite sign.

Proof of the claim: Given $b(\cdot)$ and $m(\cdot)$ the agent faces an unconstrained maximization problem and sets $\hat{r}$ such that

$$\frac{\partial}{\partial \mu} E[b(x)|\mu; \hat{r}]|_{\mu=\hat{\mu}} m'(\hat{r}) + \frac{\partial}{\partial r} E[b(x)|\hat{\mu}; r]|_{r=\hat{r}} = 0: \tag{7}$$

By definition,

$$\frac{\partial}{\partial r} E[x|\mu; r] = 0: \tag{8}$$

for any $\mu$ and any $r$. By putting together (7) and (8), we have that

$$\frac{d}{dr} E[x - b(x)|m(r); r]|_{r=\hat{r}} = \frac{\partial}{\partial \mu} E[x|\mu; \hat{r}]|_{\mu=\hat{\mu}} m'(\hat{r}):$$

By FOSD and the fact that $b(\cdot)$ is nondecreasing, it must be true that

$$\frac{\partial}{\partial \mu} E[x|\mu;\hat{r}]|_{\mu=\hat{\mu}} m'(\hat{r})$$

and

$$\frac{\partial}{\partial \mu} E[b(x)|\mu;\hat{r}]|_{\mu=\hat{\mu}} m'(\hat{r})$$

have the same sign. But, by (7), the latter expression and (5) have opposite signs, which proves the claim.

As corner solutions are not possible, in a first-best contract (6) must be zero. The claim implies (i) immediately. If (6) is positive, the principal gains from a small increase in $r$. The opposite holds when (6) is negative, and (ii) is proven.[8]

□

Equation (4) is a necessary condition for a contract to be first best. It is easily derived from the incentive-compatibility constraint on risk after applying the Envelope Theorem.

The important point is that condition (4) does not depend on $m(\cdot)$. A condition dependent on $m(\cdot)$ would be very difficult to check in practice because one would need to know the (agent estimated) characteristics of the portfolios that the agent does not choose in equilibrium. Instead testing (4) requires only knowledge of the portfolio chosen in equilibrium and the compensation function.

For example, let us say that portfolios are normally distributed and that we can estimate the mean and the variance of the chosen portfolio.[9] The estimates are respectively $\hat{\mu}$ and $\hat{\sigma}$. Then, (4) becomes

$$\int_{-\infty}^{\infty} b(x) \frac{e^{-\frac{(x-\hat{\mu})^2}{\hat{\sigma}^2}}}{\sqrt{2\pi}\hat{\sigma}} \frac{x^2 - 2x\hat{\mu} + \hat{\mu}^2 + \hat{\sigma}^2}{\hat{\sigma}^3} dx = 0; \tag{9}$$

[8] It is important to stress that the argument in the proof is unrelated to the "first-order approach" (see Grossman and Hart [9] and Rogerson [20]). Here, we are not looking for the optimal contract but only for necessary conditions for a contract to be first-best. Notice that we have not yet proven that a first-best contract exists at all.

[9] Of course, it is impossible to recover the mean and the variance from one observation on return. However they could be estimated if the same contract $b(\cdot)$ is given to the same agent in different times or if we have many agents and returns are not perfectly correlated across agents even if agent choose the same portfolio strategy.

which can be tested. If it does not hold, then we know that the contract offered is not first-best. Moreover if the right-hand side of (9) is positive (negative), we can say that from the principal's viewpoint the agent is taking an excessively risky (conservative) investment strategy.

Proposition 1 allows us to exclude two types of contract from the class of first-best contracts:[10]

**Proposition 2.** A convex (or concave) contract cannot be first-best.

*Proof.* Let us assume that $b(\cdot)$ is convex. By integrating per parts twice,[11]

$$\begin{aligned}
&\int_{\underline{x}}^{\bar{x}} b(x) f_r(x|\mu^*;r^*)dx \\
&= b(\bar{x})F_r(\bar{x}|\mu^*;r^*) - b(\underline{x})F_r(\underline{x}|\mu^*;r^*) - \int_{\underline{x}}^{\bar{x}} b'(x)F_r(x|\mu^*;r^*)dx \\
&= -\int_{\underline{x}}^{\bar{x}} b'(x)F_r(x|\mu^*;r^*)dx \\
&= -b'(\bar{x})\int_{\underline{x}}^{\bar{x}} F_r(x|\mu^*;r^*)dx + \int_{\underline{x}}^{\bar{x}} b''(x)\left(\int_{\underline{x}}^{x} F_r(t|\mu^*;r^*)dt\right)dx;
\end{aligned}$$

where the third equality is due to the fact that $F_r(\bar{x}|\mu^*;r^*) = F_r(\underline{x}|\mu^*;r^*) = 0$ (because $F(\bar{x}|\mu^*;r^*) = 1$ and $F(\underline{x}|\mu^*;r^*) = 0$).

However,

$$\begin{aligned}
\int_{\underline{x}}^{\bar{x}} F_r(x|\mu^*;r^*)dx &= \bar{x}F_r(\bar{x}|\mu^*;r^*) - \underline{x}F_r(\underline{x}|\mu^*;r^*) - \int_{\underline{x}}^{\bar{x}} x f_r(x|\mu^*;r^*)dx \\
&= 0 - 0 - m'(r^*) = 0.
\end{aligned}$$

Moreover, if $b''(\cdot) > 0$ for all x, then $\int_{\underline{x}}^{\bar{x}} b''(x)\int_{\underline{x}}^{x} F_r(t|\mu;r)dtdx > 0$ for any $x \in (\underline{x};\bar{x})$ by the definition of second-order stochastic dominance. Thus, we have proven that

$$\int_{\underline{x}}^{\bar{x}} b(x) f_r(x|\mu^*;r^*)dx > 0$$

Therefore, a convex contract cannot be optimal. The proof for the concave case is identical. □

[10] Notice that Propositions 1 and 2 do not depend on the assumption of limited liability. That is why it is not absurd to consider the possibility of concave contracts.

[11] The proof is done under the assumption that $b(\cdot)$ is twice differentiable. While the fact that $b(\cdot)$ is convex and nondecreasing guarantees that it is continuous, the possibility that $b(\cdot)$ is not differentiable should be taken into account.

With a concave or convex contract, the incentive-compatibility constraint on risk is violated. This result is hardly surprising. With a convex contract, the agent is rewarded for high returns more than he is punished for low returns and he has an incentive to increase risk above the efficient level. The opposite holds for a concave contract.

## 4 One Period: First-Best Contracts

We now ask whether a first-best contract exist.

**Assumption 3.** (i) $F$ is such that, for any $\mu$ and any $r' \neq r''$, $F(x|\mu; r')$ and $F(x|\mu; r'')$ cross exactly once on $(\underline{x}; \bar{x})$. (ii) Let $\hat{x}(\mu; r'; r'')$ denote the point at which they cross. $\hat{x}(\mu; r'; r'')$ is nondecreasing in both $r'$ and $r''$ for all $\mu$.

Part (i) is a technical assumption. For a given $\mu$, $F(x|\mu; r')$ and $F(x|\mu; r'')$ must cross at least once. However, they may cross more than once. We assume they cross exactly once. Part (ii) is central to the results. We assume that the $x$ at which two cumulative distributions cross is nondecreasing in the $r$'s of the two distributions. The assumption is quite natural because increasing $r$ shifts weight on the tails of the distribution, but one can construct examples in which it is violated.

Among families of two-parameter distributions, Assumption 3 is satisfied, among others, by the normal distribution family and by the lognormal distribution family.[12]

A contract $b(\cdot)$ is a bonus contract if

$$b(x) = \begin{cases} B & \text{if } x \geq \hat{x} \\ 0 & \text{if } x < \hat{x} \end{cases}$$

**Proposition 3.** If Assumption 3 is satisfied, there exists a first-best contract, which takes the form of a bonus contract.

*Proof.* By Assumption 3 Part (i) and by second-order stochastic dominance, given $r' < r''$, there exists a $k(r'; r'')$ such that

$$F(x|\mu^*; r') > (<)(=) F(x|\mu^*; r'')$$

[12] We have not found a two-parameter family of distributions which violates Assumption 3, although this is possible in principle. We conjecture that Assumption 3 is satisfied by any $F$ with a constant, nonnegative third moment.

if $x < (>)(=)k(r', r'')$. By Assumption 3 Part (ii), $k(r', r'')$ is nondecreasing in both $r'$ and $r''$. Let

$$\begin{aligned} \bar{k}^- &= \lim_{r' \to r^{*-}} k(r', r^*); \\ \bar{k}^+ &= \lim_{r'' \to r^{*+}} k(r^*, r''). \end{aligned}$$

By continuity, $\bar{k}^- = \bar{k}^+ \equiv \bar{k}$. Also, for any $r' < r^*$, $k(r', r^*) < \bar{k}$ and, for any $r'' > r^*$, $k(r^*, r'') > \bar{k}$. Hence, it must be true that

$$F(\bar{k}|\mu^*, r^*) \le F(\bar{k}|\mu^*, r) \qquad \text{for all } r \ge 0. \tag{10}$$

We will show that a bonus contract in which $\hat{k} = \bar{k}$ is first-best. The agent chooses $r$ to maximize $E[b(x)|r] = F(\bar{k}|m(r), r)$. Let $\psi(\mu, r) = 1 - F(\bar{k}|m(r), r)$. The agent solves the problem: $\max_r \psi(m(r), r)$. We want to show that the unique solution of this problem is $\mu = \mu^*$ and $r = r^*$.

Notice that, because $\mu^* \ge \mu$ for all $\mu$ and by FOSD,

$$\psi(m(r), r) \begin{cases} = \psi(\mu^*, r) & \text{if } r = r^*; \\ < \psi(\mu^*, r) & \text{for any other } r. \end{cases}$$

Thus, if $r^* = \arg\max \psi(\mu^*, r)$, then $r^* = \arg\max \psi(m(r), r)$. But, (10) implies that $\psi(\mu^*, r^*) \ge \psi(\mu^*, r)$ for all $r$. Therefore, $r^* = \arg\max \psi(m(r), r)$. Given a bonus contract in which $\hat{k} = k$, the agent chooses the first-best action. Clearly the principal can gauge the bonus $B$ in order to make the incentive contraint on effort binding. This is done by setting $B = \frac{c}{\psi(\mu^*, r^*)}$. □

With a bonus contract, the agent maximizes the probability of being above the bonus cutoff $\hat{k}$. Proposition 3 says that the principal can find a cutoff level such that maximizing the probability of being above the cutoff level is equivalent to maximizing the expected return of the portfolio.

If the family of distribution functions $F$ only contains distributions that are symmetric around the mean (like the family of normal distributions), then the bonus contract assumes a very simple form:

**Assumption 4.** Let $\underline{x} = -\infty$ and $\bar{x} = \infty$. For all feasible $\mu$ and $r$ and for all $z \in \Re$, $f(\mu - z|\mu, r) = f(\mu + z|\mu, r)$.

Clearly, Assumption 4 implies Assumption 3. In a symmetric distribution, for a given mean $\mu$, all cumulative functions with mean $\mu$ cross at $x = \mu$ because $F(\mu|\mu; r) = 0.5$.

**Corollary 1.** If Assumption 4 is satisfied, then a bonus contract with $\hat{x} = \mu^*$ is first-best.

If Assumption 4 holds, there exists a simple first-best contract. The principal offers the agent a bonus if the performance is above the mean of the optimal distribution and zero if it is below. The bonus is set at the lowest level that satisfies the incentive-compatibility constraint on effort.

Of course, there are other first-best contracts that are not bonus contracts. This is immediate to see in the symmetric case. Suppose that Assumption 4 holds and consider a contract of the form:

$$b(x) = \begin{cases} 0 & \text{if } x < \hat{x} - g \\ B\frac{x - g}{2g} & \text{if } \hat{x} - g \leq x < \hat{x} + g \\ B & \text{if } x \geq \hat{x} + g \end{cases}$$

where $0 < g < \mu - x_0$ (recall that $x_0$ is the return of the risk free asset). It is easy to see that this "smoother" version of the bonus contract achieves first-best as well.

More generally, one could show that there exists a whole class of first-best contracts:[13]

**Proposition 4.** If Assumption 4 is satisfied, then sufficient conditions for a nondecreasing contract to be first-best are that: (i) $b(x_0) = 0$; and (ii) $b(\mu^* - z) = B - b(\mu^* + z)$ for all $z \in \Re$.

## 5 Two Periods: First-Best Contracts

The two-period game is exactly like the one-period game except that the portfolio choice is made twice.[14] Let $x = x_1 + x_2$ denote the return on the investment over the two periods. The principal cannot offer contracts contingent on $x_1$ or $x_2$. That would take us back to the one-period model. The principal can only offer contracts dependent on $x$. The timing of the principal-agent relationship is:

[13] The result that there are many first-best contracts is unsurprising. In moral-hazard models in which both parties are risk-neutral there usually exist a continuum of first-best contracts.

[14] The analysis extends readily to any number of periods.

1. The principal proposes a contract $b(\cdot)$ to the agent. The contract can only depend on $x = x_1 + x_2$.

2. If the agent accepts, he receives a unitary sum to manage for two periods.

3. The agent chooses whether or not to spend $c$ in order to access the set of feasible portfolios. If he does not spend $c$, the agent invest in the risk-free asset with return $x_0$.

4. If the agent spends $c$, he chooses $(\mu_1; r_1)$ within the set of feasible parameters. $x_1$ is realized according to $f(x_1|\mu_2; r_2)$.

5. The agent observes $x_2$ and chooses $(\mu_2; r_2)$ within the set of feasible parameters. $x_2$ is realized according to $f(x_2|\mu_2; r_2)$.

6. The principal observes $x = x_1 + x_2$. She pays $b(x)$ to the agent and she keeps $x - b(x)$.

As before, a contract is first-best if the agent receives no rent and chooses $r_1 = r^*$ and $r_2 = r^*$. We begin with a necessary condition for a contract to be first-best:

**Lemma 2.** A first-best contract is linear in $x$.

*Proof.* Suppose $b(\cdot)$ is first-best. Then,

$$\int_{-\infty}^{\infty} b(x_1 + x_2) f_r(x_2|\mu^*; r^*) dx_2 = 0; \forall x_1 \in \mathbb{R} \tag{11}$$

Also, by definition,

$$\int_{-\infty}^{\infty} f_r(x_2|\mu^*; r^*) dx_2 = 0; \tag{12}$$

and

$$\int_{-\infty}^{\infty} x_2 f_r(x_2|\mu^*; r^*) dx_2 = 0; \tag{13}$$

We want to show that (11), (12), and (13) hold only if $b(\cdot)$ is linear. The proof proceeds as follows. We consider a class of discrete approximations of (11), (12), and (13), and we prove that for all approximations in that class $b(\cdot)$ must be linear. Then, we construct a sequence of approximations and we show that if $b(\cdot)$ must be linear for each approximation, then it must be linear also for the limiting continuous case, which corresponds.

Let $n$ be a positive integer. Let $Y \in \mathbb{R}^n$ and $Z \in \mathbb{R}^n$. Let $z : Y \to Z$. Consider the following system of equations, where $\beta : \mathbb{R} \to \mathbb{R}$ and $s \in \mathbb{R}$ can be seen as a parameter:

$$\sum_{y_i \in Y} \beta(s + y_i) z_i = 0 \tag{14}$$

$$\sum_{y_i \in Y} z_i = 0 \tag{15}$$

$$\sum_{y_i \in Y} y_i z_i = 0 \tag{16}$$

Claim: (14), (15), and (16) are true for any $Y$, $Z$, and $s$ only if $\beta(\cdot)$ is linear.

Proof of the Claim: The system can be rewritten as

$$\begin{bmatrix} \beta(s+y_1) & \beta(s+y_2) & \cdots & \beta(s+y_n) \\ 1 & 1 & \cdots & 1 \\ y_1 & y_2 & \cdots & y_n \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \\ \cdots \\ z_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

or, in a more compact notation,

$$A(s; y) z = 0. \tag{17}$$

Equation (17) is true for generic $Z$, $Y$, and $s$ only if the rows of $A(s; y)$ are linearly dependent. Therefore, for any $y_i \in \mathbb{R}$ and any $s \in \mathbb{R}$, it must be possible to write $\beta(s + y_i)$ as a linear combination of 1 and $y_i$. This is true only if $\beta(\cdot)$ is linear, which proves the Claim.

Now let $P^k$ be a partition of $\mathbb{R}$ with a countable number of elements, such that each element is a segment of the real line. Let $a_i = (a_i'; a_i'')$ denote a generic element of $P^k$. Let $z_i$ be defined as

$$z_i = \int_{a_i} f_r(x_2 | \mu^*; r^*) dx_2 \tag{18}$$

and let $y_i$ be defined as

$$y_i = \frac{\int_{a_i} x_2 f_r(x_2 | \mu^*; r^*) dx_2}{z_i} \tag{19}$$

By construction $y$ and $z$ satisfy (15) and (16). Let $x_1 = s$. Then, (14) is satisfied for all $x_1$ and all $z$ only if $b(\cdot)$ is linear. Notice that the genericity of $y$ is a consequence of the genericity of $f_r$ – an assumption of the present model. Let $s \equiv x_1$. Then, by the Claim, $b(\cdot)$ must be linear.

Let B denote the set of points at which $b(\cdot)$ is discontinuous. The fact that $b(\cdot)$ is monotone nondecreasing implies that B is at most countable. Let $P(B)$ be the partition induced on $\Re$ by B. On any segment a generated by $P(B)$ the function $b(\cdot)$ is continuous. Consider a sequence of partitions $\{P\}_{i=1,2,\ldots} = \{P^1, \ldots, P^k, \ldots\}$, with the property that $P^1 = P(B)$ and $P^j$ is finer than $P^{j-1}$ for any $j$. Let $Y^k$ and $Z^k$ be the sets generated by partition $P^k$, following (18) and (19). Then, for any $x_1$ there exists an appropriately chosen sequence $\{P\}_{i=1,2,\ldots}$ such that

$$\lim_{n \to \infty} \sum_{y_i \in Y} \beta(x_1 + y_i) z_i = \int_{-\infty}^{\infty} b(x_1 + x_2) f_r(x_2 | \mu^*; r^*) dx_2$$

Then, the fact that (14), (15), and (16) are satisfied only if $b(\cdot)$ is linear implies that (11), (12), and (13) are satisfied only if $b(\cdot)$ is linear. A first-best contract must be linear. □

Lemma 2 says that only a linear contract can be optimal. A linear contract has the unique property that the agent's incentives in the second period are independent of the agent's performance in the first period. With a nonlinear contract, the agent may find it optimal to deviate from the optimal portfolio in the second period.

Lemma 2 extends a well-known result by Holmstrom and Milgrom [13, Theorem 6]. Holmstrom and Milgrom consider a continuous time moral hazard problem in which the agent controls the drift rate of a Brownian motion. Hence, the agent chooses an action in any instant from time 0 to time $T$. The principal cannot offer a contract based on the instantaneous drift rate, but can only offer payments based on the final outcome at time $T$. Both parties have an exponential utility function (which comprises the possibility that they are risk-neutral). Holmstrom and Milgrom show that nonlinear contract cannot be optimal. The intuition behind this general result is that in a multi-period model the agent is in a very good position to manipulate the final outcome to his advantage. In contrast the principal can only use a rudimental instrument, rewards on final outcome, to control the agent's actions. The linear contract does the best job at limiting the incentives for the agent of manipulating the final outcome.

Holmstrom and Milgrom's work has been considerably extended by Sung [24] and by Hellwig and Schmidt [12]. Sung allows the agent to control the diffusion rate of the Brownian motion as well as the drift rate and finds that the optimal control is linear. Hellwig and Schmidt consider discrete approximations to the Brownian motion and examine whether a linear contract is asymptotically optimal. They show that if the principal observes only last-period aggregate results and the agent can sabotage outcome, then if the length of the discrete intervals is short enough, then a linear contract is approximately optimal.

Lemma 2 signi¯cantly strengthens the results cited above. The optimality of linear contract is not an asymptotic result. It holds if there are at least two periods. The intuition is simple and has a special relevance to ¯nancial markets. The agent can choose the level of risk in any period. With one period, the agent has only one action and the principal ¯nds it easy to control the agent. As the previous section showed, in the one-period case there is a large number of ¯rst-best contracts. With two periods, the agent has more room to manipulate the ¯nal outcome. He can condition his second-period action on his ¯rst-period outcome. Only if the principal o®ers a linear contract will the agent's incentives be independent of the ¯rst-period outcome. Thus, the linear contract is the only contract in which the agent never games the fee.

The following proposition is immediate from Lemma 2 and the assumption of limited liability:

**Proposition 5.** In the two period-model, there exists no ¯rst-best contract.

The linear contract is the only ¯rst-best contract but, clearly, it con°icts with limited liability. Thus, the main message of this section is a negative one: with more than one period, we cannot hope to ¯nd a ¯rst-best contract. Another simple result that is worth spelling out is:

**Corollary 2.** If Assumption 3 is satis¯ed, the principal is always strictly better if she can o®er contracts on both $x_1$ and $x_2$.

If Assumption 3 is satis¯ed, the principal can o®er a ¯rst-best contract in the one-period model but not in the multiperiod model. Thus, the principal is willing to pay a positive amount to be able to o®er contracts contingent on the performances on two periods rather than one. The general lesson is that being able to o®er contracts on shorter periods is a valuable option. This consideration bids a practical question: Why are most money managers compensated according to their yearly performance (usually from January 1 to December 31)?

Two solutions to the puzzle that come immediately to the mind are: cost of monitoring and agents' risk aversion. However, at a closer look they are both unconvincing, the ¯rst on practical grounds, the second on theoretical grounds. The cost of monitoring money managers is nowadays close to zero given that all their transactions are computerized. The agents' risk aversion cannot play a role because giving the principal more information { $x_1$ and $x_2$ instead of $x$ { cannot hurt him (and, as Grossman and Hart [9, Proposition 13] have shown, it almost always makes him better o®).

## 6 Two Periods: Second-Best Contracts

Once we have established that ¯rst-best cannot be achieved, the next step is to look for second-best contracts. The principal solves

$$\max_{r_1^*; r_2^*(x_1)} E[x_1 + x_2 - b(x_1 + x_2)|r_1^*; r_2^*(x_1)] \tag{20}$$

subject to

$$E[b(x_1 + x_2)|r_2^*(x_1); x_1] \geq E[b(x_1 + x_2)|r_2; x_1] \qquad \text{for a.a. } x_1; r_2 \tag{21}$$

$$E[b(x_1 + x_2)|r_1^*; r_2^*(x_1)] \geq E[b(x_1 + x_2)|r_1; r_2^*(x_1)] \qquad \forall r_1 \tag{22}$$

$$E[b(x_1 + x_2)|r_1^*; r_2^*(x_1)] \geq b(x_0) \tag{23}$$

Conditions (21) and (21) are incentive-compatibility constraints on risk and they correspond to the agent dynamic programming on $r_1$ and $r_2$. Condition (23) is the incentive-compatibility constraint on e®ort.[15] Of course, the contract $b(x)$ must also satisfy limited liability and monotonicity.

The principal-agent problem stated above violates the Monotone Likelihood Ratio Condition. Therefore, we cannot apply any of the known conditions for the validity of the ¯rst-order approach.[16] Being unable to study the problem in its general form, we choose to analyze a much simpler example. As we will see, the example itself has enough complexity that the optimal contract can only be found through numerical approximations.

Let us consider a class of portfolios with only three outcomes: -100, 0, 100. Clearly, any shifting or rescaling of these three values would leave the results inaltered, at least as long as

[15]Notice that e®ort is spent only once {in the beginning of period 1{ rather than twice. This assumption greatly simpli¯es the analysis. The assumption makes sense if we think that agent e®ort has not only short term bene¯ts { like information collection.

[16]See Rogerson [20] and Jewitt [16].

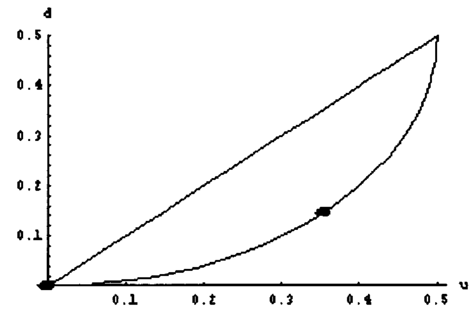


Figure 2:

the principal *must* offer a contract to the agent. If the agent does not incur the information collection cost $c$, then he can only invest in a risk-free asset yielding 0 with probability one. If the agent spends $c$ and acquires information, then he accesses a feasible set of risky portfolios. Each of the risky portfolios is fully described by $u \equiv \Pr[x = 100]$ and $d \equiv \Pr[x = -100]$. Also, let $s \equiv \Pr[x = 0] = 1 - u - d$.

In the example considered in this section, we assume that the feasible set of portfolios is given by:

$$A = \{u, d | 0 \leq u \leq \frac{1}{2}, d \geq \delta(u)\}, \tag{24}$$

where

$$\delta(u) = \frac{1}{2} - \sqrt{\frac{1}{4} - u^2}. \tag{25}$$

The feasible set of portfolios is depicted in Figure 2. The frontier correspond to the South-East quadrant of a circle. This set has the important property that the marginal cost in terms of $d$ of an increase in $u$ is zero when $u = d = 0$ (least risky portfolio) and is infinite when $u = d = \frac{1}{2}$ (riskiest portfolio). These regularity conditions ensure that the nonnegativity constraints of $u$, $d$, and $s$ are never binding. In Figure 2, the black dot represents the efficient portfolio – the one that maximizes the expected value of the outcome. The efficient portfolio has probabilities $(d^*, s^*, u^*) = (0.146, 0.501, 0.353)$. The white dot represents the riskless portfolio. As before, without loss of generality the agent can restrict his attention to portfolios on the frontier $\delta(u)$. Therefore, by selecting $u$, the agent selects a portfolio characterized by the probabilities $(\delta(u), 1 - u - \delta(u), u)$.

In the previous sections we got accustomed to define the set of feasible portfolios in terms of the mean and a measure of risk. Instead, here we have defined it in terms of two

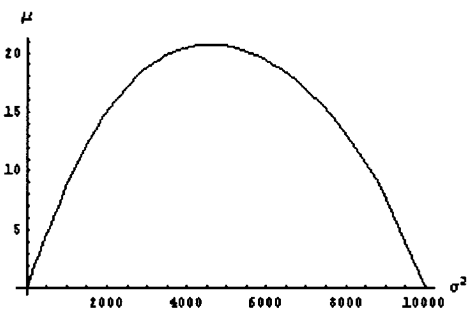


Figure 3:

probabilities. This has been done to make the analysis more transparent. However, it is immediate to see that the set defined in (24) and (25) can also be represented in terms of mean and variance. Figure 3 provides such mean-variance representation. The efficient portfolio is found in correspondence of the global maximum of the expected value $\mu$.

The timing of the game is the same as in the previous section: (i) The principal offers a contract to the agent; (ii) The agent decides whether to incur cost $c$ or not; (iii) The agent chooses a portfolio for the first period; (iv) The agent observes the outcome in the first period, $x_1$, and then chooses a portfolio for the second period; (v) The sum of the outcomes in the two periods, $x \equiv x_1 + x_2$, is observed and payoffs are made.

After two periods there are five possible outcomes $x$: -200, -100, 0, 100, +200. Hence, a contract is a quintuple $b = (b_{-200}, b_{-100}, b_0, b_{+100}, b_{+200})$. From (2), the incentive-compatibility constraint on effort for the agent is

$$\max_u \left(b_{-200} \Pr[x = -200|u] + \ldots + b_{200} \Pr[x = 200|u]\right) - c \geq b_0. \tag{26}$$

If the ICC on effort is not satisfied, the agent does not spend $c$, he chooses the riskless portfolio, and he gets $b_0$ for sure.

In the first-best case, the agent should choose $u = u^*$ in both periods, because this maximizes $E[x_1 + x_2]$. In this case, it can be computed that $E[x] = 41.42$. From Lemma 2 we know that a contract that achieves first-best must be linear in $x$. If the agent had no liability constraint, the principal could offer him a linear contract that leaves the agent no rent. This is easily achieved by letting $b_0 = 0$.

If –as we assume– the agent has the liability constraint, the principal can still offer him contracts that induce the agent to choose the efficient action profile $u_1 = u_2 = u^*$. This class of contracts have two properties: (i) they are linear; (ii) they satisfy the ICC on effort (26).

All these contracts leave a rent to the agent. Let us look for the one that leaves the lowest rent (we will call this the least-cost efficient contract, or efficient for short). Clearly, it must be that $b_{-200} = 0$. Then, we let $b_{-100} = q$, $b_0 = 2q$, $b_{100} = 3q$, and $b_{100} = 4q$. The value of $q$ depends on the parameter $c$. It can be computed that $q = (1 + \sqrt{2})c$. Thus, the least-cost efficient contract is:

$$b^e = (0; (1 + \sqrt{2})c; 2(1 + \sqrt{2})c; 3(1 + \sqrt{2})c; 4(1 + \sqrt{2})c):$$

However, from the principal's viewpoint the least-cost efficient contract is not optimal because it leaves an excessive rent to the agent. As we will shortly see, the principal is better off choosing a contract that is inefficient but reduces the agent's rent.

Consider a generic contract $(b_{-200}; b_{-100}; b_0; b_{+100}; b_{+200})$. The agent's choice is found by backward induction. In the beginning of the second period, there could be three cases: $x_1 = -100$, $x_1 = 0$, or $x_1 = 100$. If $x_1 = -100$, the agent solves a one-period problem over $u_2$ in which he receives: $b_{-200}$ if $x_2 = -2 = 100$, $b_{-100}$ if $x_2 = 0$, and $b_0$ if $x_2 = 100$. If $x_1 = 0$ or $x_1 = 100$, the agent solves analogous problems. To find the agent choice in the two-period case, the following one-period result is useful (in the one-period case, the agent chooses $u$ only once):

**Lemma 3.** Consider the one-period problem for the agent. Suppose that the contract is $(\hat{b}_{-100}; \hat{b}_0; \hat{b}_{100})$. Then, the agent chooses

$$\hat{u}(\hat{b}_{-100}; \hat{b}_0; \hat{b}_{100}) = \frac{\hat{b}_{100} - \hat{b}_0}{2\sqrt{\hat{b}^2_{-100} - 2\hat{b}_{-100}\hat{b}_0 + 2\hat{b}_0^2 + 2\hat{b}_0\hat{b}_{100} + \hat{b}^2_{100}}};$$

and receives the expected compensation

$$\hat{v}(\hat{b}_{-100}; \hat{b}_0; \hat{b}_{100}) = \delta(\hat{u})\hat{b}_{-100} + (1 - \hat{u} - \delta(\hat{u}))\hat{b}_0 + \hat{u}\hat{b}_{100}:$$

Proof. The agent chooses

$$\hat{u}(\hat{b}_{-100}; \hat{b}_0; \hat{b}_{100}) = \max_u \delta(u)\hat{b}_{-100} + (1 - u - \delta(u))\hat{b}_0 + u\hat{b}_{100}:$$

By substituting (25) into $\delta(u)$ and finding the first-order condition, the lemma is proven. □

We can now use Lemma 3 in the backward induction. In the second period, the agent chooses:

$$u_2^*(x_1) = \begin{cases} \hat{u}(\hat{b}_{-200}, \hat{b}_{-100}, \hat{b}_0) & \text{if } x_1 = -100; \\ \hat{u}(\hat{b}_{-100}, \hat{b}_0, \hat{b}_{100}) & \text{if } x_1 = 0; \\ \hat{u}(\hat{b}_0, \hat{b}_{100}, \hat{b}_{200}) & \text{if } x_1 = -100. \end{cases} \tag{27}$$

This determines the expected compensation given the outcome of the first period:

$$E[b_{x_1+x_2}|x_1] = \begin{cases} \hat{v}_{-100} = \hat{v}(\hat{b}_{-200}, \hat{b}_{-100}, \hat{b}_0) & \text{if } x_1 = -100; \\ \hat{v}_0 = \hat{v}(\hat{b}_{-100}, \hat{b}_0, \hat{b}_{100}) & \text{if } x_1 = 0; \\ \hat{v}_{100} = \hat{v}(\hat{b}_0, \hat{b}_{100}, \hat{b}_{200}) & \text{if } x_1 = -100. \end{cases} \tag{28}$$

In the first period the agent knows $\hat{v}_{-100}$, $\hat{v}_0$, and $\hat{v}_{-100}$. The optimal choice is again obtained from Lemma 3:

$$u_1^* = \hat{u}(\hat{v}_{-100}, \hat{v}_0, \hat{v}_{100}). \tag{29}$$

and the expected compensation for the agent (before knowing $x_1$ and $x_2$) is

$$E[b_x|b] = \hat{v}(\hat{v}_{-100}, \hat{v}_0, \hat{v}_{100}). \tag{30}$$

Given a contract $b$, expressions (27), (28), and (29) allow us to compute the agent's optimal choice given any contract $b$. The optimal choice is a quadruple $(u_1^*, u_2^*(-100), u_2^*(0), u_2^*(100))$. In addition, expression (30) gives $E[b_x|b]$.

The choice of u by the agent generates five probabilities $\Pr[x = -200|b], \ldots, \Pr[x = 200|b]$. The expected return given contract $b$ is given by

$$E[x] = b_{-200} \Pr[x = -200|b] + \ldots + b_{200} \Pr[x = 200|b]. \tag{31}$$

The principal chooses $b$ to maximize net return $E[x - b_x|b]$, which is found by subtracting (30) from (31).

While we are not able to find a close-form solution of the optimal contract, we can use a numerical approximation. Three values of the parameter $c$ are considered: 0.2, 1, and 5.[17] The optimal contracts are: for $c = 0.2$, $(0, 0.385, 0.81, 1.275, 1.73)$; for $c = 1$, $(0, 1.11, 2.58, 4.68, 6.68)$; and for $c = 5$, $(0, 1.025, 2.4, 10.4, 18.35)$. Optimal contracts and least-cost efficient contract are graphed in Figures 6.

[17] The program used had a precision of 0.005 for $c = 0.2$, of 0.01 for $c = 1$, and of $0.025$ for $c = 5$ (With a precision of p, one finds the optimal contract under the constraint that all payments to the agent be divisible by p).

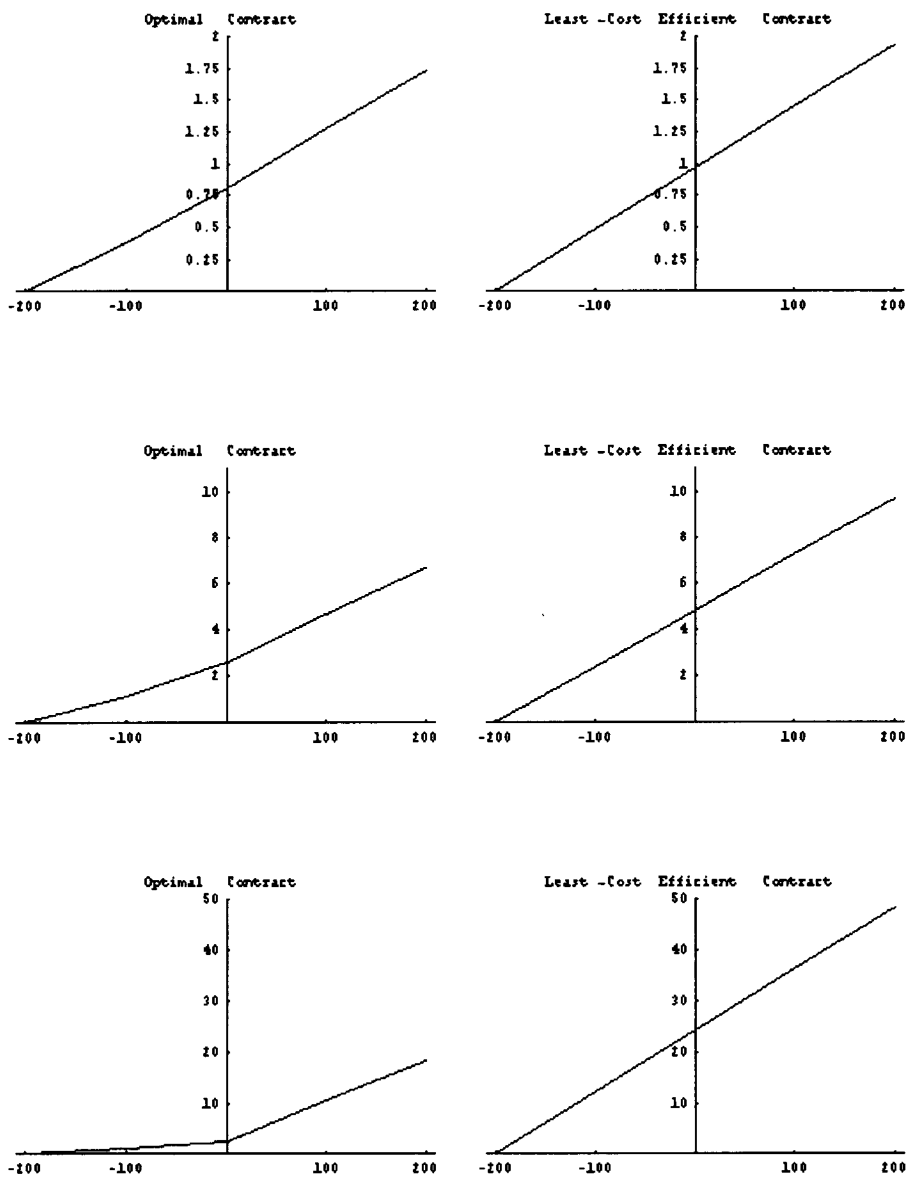
Optimal Contract
Least -Cost Efficient Contract
2
1.75
1.5
1.25
1
0.75
0.5
0.25
-200
-100
100
200
Optimal Contract
Least -Cost Efficient Contract
10
8
6
4
2
-200
-100
100
200
Optimal Contract
Least -Cost Efficient Contract
50
40
30
20
10
-200
-100
100
200

Figure 4:

The principal faces a tradeo® between reducing the agent rent (decrease $E[b_x]$) and inducing the agent to choose a more e±cient risk pro¯le (increase $E[x]$). By switching from the least-cost e±cient contract to the optimal contract, the principal decreases both $E[x]$ and $E[b_x]$. The combined e®ect (the change in $E[x ¡ b_x]$) is positive and can be seen as net savings on compensation. As the next table shows, net saving increases with the information collection cost. A higher c makes the e±cient contract more expensive because the incentive-compatibility constraint on e®ort (26) is more di±cult to meet.

| | contract | $E[x ¡ b_x]$ | $E[b_x]$ |
|---|---|---|---|
| c = 0:2 | optimal | 40.34 | 1.01 |
| | e±cient | 40.26 | 1.17 |
| c = 1 | optimal | 36.99 | 3.58 |
| | e±cient | 35.59 | 5.83 |
| c = 5 | optimal | 27.90 | 7.40 |
| | e±cient | 12.28 | 29.14 |

The optimal contract is nonlinear. By Lemma 2, the agent does not choose the e±cient level of risk. The following table compares the distribution of x under the optimal contract with the ditribution under the least-cost e±cient contract. With the optimal contract, extreme outcomes (nonzero returns) are more likely. Compared to the e±cient contract, the standard deviation of returns is higher and the expected value is lower. Both discrepancies increase with c, which is due {again{ to the increasing di±culty of satisfying the ICC on e®ort.

| | -200 | -100 | 0 | 100 | 200 | E[x] | StDev[x] |
|---|---|---|---|---|---|---|---|
| c = 0:2 | 0.02 | 0.15 | 0.33 | 0.36 | 0.13 | 41.3 | 97.5 |
| c = 1 | 0.04 | 0.17 | 0.29 | 0.38 | 0.13 | 40.6 | 102.0 |
| c = 5 | 0.04 | 0.24 | 0.18 | 0.39 | 0.14 | 35.3 | 112.0 |
| e±cient | 0.02 | 0.15 | 0.35 | 0.35 | 0.12 | 41.4 | 95.6 |

The next table illustrates the dynamic properties of e±cient and optimal contracts over the two periods. The de¯ning characteristic of the e±cient contract is that the agent buys the same portfolio in both periods, irrespective of the ¯rst period outcome. On the other hand, in the optimal contract the agent games the fee. If the ¯rst-period result $x_1$ was not good (-100 or 0), the agent buys an overly risky portfolio. For instance, if $c = 1$ and $x_1 = ¡100$, the agent selects a portfolio with $u = 0{:}382$ (which implies $d = 0{:}210$). The e±cient portfolio would have $u = 0{:}353$ (implying $d = 0{:}146$). On the other hand, if the ¯rst-period result was good, the agent buys a portfolio that is approximately e±cient. Thus, in the optimal contract the opportunties of gaming the fee seem to be mostly on the losing side. A ¯rst-period bad performer takes chooses a suboptimal level of risk while a good performer chooses approximately the right level.

In the ¯rst period, the agent buys an overly risky portfolio (if $c = 1$, he selects $u_1 = 0{:}382$ instead of the e±cient 0.353). This is due to the fact that the shape of the contract is \overall" more convex than concave.

| | $u_1$ | $u_2(¡100)$ | $u_2(0)$ | $u_2(100)$ |
|---|---|---|---|---|
| $c = 0{:}2$ | 0.362 | 0.371 | 0.369 | 0.350 |
| $c = 1$ | 0.382 | 0.399 | 0.410 | 0.345 |
| $c = 5$ | 0.409 | 0.401 | 0.493 | 0.352 |
| e±cient | 0.353 | 0.353 | 0.353 | 0.353 |

To summarize, excessive risk taking occurs in the ¯rst period unconditionally and in the second period conditional on a bad ¯rst-period return. These deviations from e±ciency are increasing in c.

# 7 Conclusions

We have developed a principal-agent model of delegated portfolio management with the central feature that the agent can control the riskiness of the portfolio. This represents a departure from the moral hazard literature in that moral hazard takes also the form of risk taking, rather than only e®ort exertion. First, we have charaterized the incentive-compatibility constraint on risk. This turned out to be a simple ¯rst-order condition which does not depend on the shape of the e±cient portfolio frontier. Second, we have shown that in the one-period case there exists a ¯rst-best bonus contract. As bonus contracts are very widespread in ¯-

nancial institutions, the present work provides a theoretical justi¯cation for them in terms of moral hazard on risk. Third, we have analyzed the multi-period case. The agent has a better opportunity to manipulate risk to his advantage. The only ¯rst-best contract is linear, which con°icts with limited liability. The second-best contract involves excessive risk-taking.

Risk-taking by money manager is a widely perceived problem. Our work is a step toward developing a general theory of moral hazard with risk taking. Our results suggest that { di®erently from moral hazard under pure e®ort exertion { moral hazard with risk taking yield testable implications, which can guide empirical work and, perhaps, contract design.

Our results for the multiperiod model point to a puzzle: Why are money managers offered multi-period contracts when a sequence of single-period contract would be strictly more e±cient? We argued that this question cannot be answered by arguments relating to risk aversion or monitoring costs. We hope that further research will address it.

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