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ORDER FLOW COMPOSITION AND TRADING COSTS IN DYNAMIC LIMIT ORDER MARKETS

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ABSTRACT

Order Flow Composition and Trading Costs in Dynamic Limit Order Markets*

This paper provides a game theoretic model of price formation and order placement decisions in a dynamic limit order market. Investors can choose to post limit orders or to submit market orders. Limit orders result in better execution prices but face a risk of non-execution and a winner's curse problem. The execution probability of a limit order trader is endogenous and depends on the order placement decisions of the other traders. Solving for the equilibrium of this dynamic game, closed form solutions for the order placement strategies are obtained. Thus, testable implications for the cross-sectional behaviour of the mix between market and limit orders and trading costs in limit order markets are derived. It is also shown that the winner's curse problem has a negative impact on the allocative efficiency of these markets.

JEL Classification: C72, D44, G19
Keywords: market microstructure, limit order markets, limit and market orders, bid and ask prices, order flow composition

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NON-TECHNICAL SUMMARY

Several security markets are organized as limit order markets. In these markets, buyers and sellers carry their trades by submitting either limit orders or market orders. A limit order is an offer to buy or to sell a specified quantity at a given price. A market order is an order to buy or to sell a specified quantity at the best available prices posted in the market. Limit orders are stored in a limit order book, waiting for future execution. This execution is triggered by incoming market orders, which are matched with the best offers in the book. In these markets, traders face the following dilemma. With a market order, they are executed immediately at the price posted in the market. They can improve their execution price by posting a limit order. In this case, they run the risk of not being executed. Moreover, limit orders can become mispriced when new public information arrives on the market. In this case, limit order traders run the risk of being executed ('picked off') at a loss.

What is the behaviour of the mix between market and limit orders ('the order flow composition')? Should we expect systematic differences in the proportions of market and limit orders across securities or not? Surprisingly this issue has not been addressed yet, neither empirically nor theoretically. This article proposes a theoretical model of trading in a limit order market. In this framework, it is possible to analyse the determinants of the order flow composition. Furthermore the model has new testable implications for the behaviour of trading costs in limit order markets. Using the framework developed in this paper, we can also address another issue: is the risk of being picked off an impediment to the allocative efficiency of these markets or not? The analysis also has a methodological interest because we are able to solve in closed form for the equilibrium of a dynamic model in which traders can choose both market and limit orders.

A primary finding of the model is that the volatility of the asset is a main determinant of the order flow composition. The intuition is as follows. The larger the volatility, the larger is the risk of being picked off for limit order traders. For this reason, limit order traders shade their offers more and spreads are larger in very volatile markets. But this entails that the cost of trading with a market order is higher. Thus more traders will use limit orders in these markets. It follows that the proportion of limit orders is positively related to asset volatility. Moreover, limit order traders' execution probability is lower in very volatile markets, since less traders submit market orders in these markets. We show that, for this reason, the fill rate (the ratio of filled limit orders to the total number of limit orders) is negatively related to asset volatility, another testable prediction of the model.

We also find that the level of execution risk influences limit order traders' quotes. The intuition is the following. When execution risk is large, traders are more willing to submit market orders, even if execution prices are not very good. But for this reason limit order traders are able to post less competitive prices. This effect of execution risk has implications for trading costs. We argue that it might explain the well-known stylized fact that trading costs (spreads) enlarge at the end of the trading day, in limit order markets. Furthermore, the model provides a more precise prediction: the size of the increase in the trading costs at the end of the trading day should be negatively related to a proxy for the level of competition between limit orders (e.g. the number of markets in which the stock is listed). We also show that execution risk implies that trading costs are related to the ratio of buy to sell orders. More specifically, we obtain that the spread size is a concave function of the ratio of buy order to sell orders, with a maximum when the order flow is balanced (i.e. when the ratio is equal to 1).

Finally, we establish that the risk of being picked off for limit order traders has an adverse effect on the allocative efficiency of limit order markets. The model points to two reasons for this. Limit order traders quote large spreads in order to protect themselves against the risk of being picked off. But this entails that there are instances in which a trade does not occur between two traders, although they could both benefit from trading together. On the other hand, trades, which occur when traders are picked off, can be inefficient; they can result in a transfer of the asset from an agent with a high valuation for the asset to an agent with a lower valuation. We also find that the gains traders anticipate from trading in limit order markets are negatively related to asset volatility. This suggests that limit order markets for assets with high volatility will attract less traders because trading is less profitable in these markets.

1. Introduction.

Several security markets[1] are organized as limit order markets. In these markets, buyers and sellers carry their trades by submitting either limit orders or market orders[2]. Limit orders are stored in a limit order book, waiting for future execution. This execution is triggered by incoming market orders, which are matched with the best offers in the book. Traders face the following dilemma. With a market order, a trader is executed with certainty while accepting the available quoted price. With a limit order, a trader has the possibility to improve his execution price. But a limit order runs the risk of not being executed. Moreover because their prices are fixed over time, limit orders can become mispriced when new public information arrives. This creates a winner's curse problem for limit order traders since they are more likely to be executed ("picked off") at a loss when their orders become mispriced than when they are not.

What is the behavior of the mix between market and limit orders ("the order flow composition") across securities? Surprisingly, this question has not been addressed yet (to our knowledge), neither empirically[3], nor theoretically. The objective of this article is to develop a simple model in which the mix between market and limit orders can be characterized, in equilibrium. As explained below, in this way, we obtain testable predictions concerning the cross-sectional behavior of the order flow composition. Furthermore, the model has new testable implications for the cross-sectional behavior of trading costs in limit order markets. We also consider the welfare

[1]For instance, the NYSE, the Paris Bourse, the Tokyo Stock Exchange and the Toronto Stock Exchange. Domowitz (1993) reports that 35 financial markets have features of limit order markets.

[2]A limit order specifies a limit price and a quantity. For a buy limit order, the limit price is the maximum price that a buyer will pay and for a sell limit order, the limit price is the minimum price that a seller will obtain. Market orders are orders to buy or sell a given quantity at any price. Those orders are the main channels through which liquidity is supplied and consumed in limit order markets. Biais, Hillion and Spatt (1995) report (Table III, p1670) that, for the Paris Bourse, 47.2% of all orders are market orders and 41.3% are limit orders. The other orders are cancellations or applications.

[3]Biais, Hillion and Spatt (1995) and Hedvall and Nyemeyer (1996) focus on the variations in the order flow due to transient changes in the state of the limit order book. Hamao and Hasbrouck (1995) study the supply of liquidity when there is no market-maker. Hamon *et al.* (1993), Handa and Schwartz (1996) and Harris and Hasbrouck (1995) focus on optimal order submission strategies. DeJong, Nijman and Roell (1995) compare trading costs in a limit order market and a dealer market. Finally an interesting approach is developed by Angel (1995) and Harris (1995) who analyze optimal order placement strategies in different market conditions (state of the book, rate of arrival of orders...) exogenously specifying traders' beliefs on their environment and proceeding by simulations.

implications of the risk of being picked off. This risk influences the distribution of trading gains between market order traders and limit order traders (see Brown and Holden (1996)) but does it result in an overall welfare loss?

In order to portray, in a natural way, the execution risk and the risk of being picked off, we consider a dynamic model. Traders arrive sequentially. Upon arrival, a trader can choose to post quotes (place a limit order) or to trade at the quotes previously posted by other traders (place a market order). Execution of limit orders is uncertain and the asset value changes over time, which creates a winner 's curse[4] problem for limit order traders. The optimal choice between a market and a limit order and the optimal prices for limit orders depend on the order submission choices of the future traders. Solving for the equilibrium of this game, the traders' order placement strategies are characterized, in closed form, as a function of traders' valuations and the best offers in the book. This has a methodological interest independent of the issues we address. Actually, to our knowledge, it is the first model with a closed form characterization in equilibrium of *both* quotes *and* order placement decisions in a dynamic limit order market.

Our primary finding is that the volatility of the asset is a main determinant of the mix between market and limit orders. Actually, it determines the intensity of the risk of being picked off. When the asset volatility increases, the probability of being picked off and the losses, which ensue, are larger. For this reason, limit order traders shade more their quotes relative to their reservation prices in markets with high volatility. But this entails that market orders are less attractive and limit order traders' execution probabilities are lower. Thus more traders use limit orders when the asset volatility is high. A testable prediction is that the proportion of limit orders in the order flow increases with the asset volatility. Furthermore the fill rate (the ratio of filled limit orders to the number of submitted limit orders) is negatively related to volatility. In the model, posted spreads are positively related to asset volatility. Consequently, another testable hypothesis is that the proportion of limit orders in the order flow is positively related to the size of the spread. Asset volatility decreases with equity

[4]The "winner's curse" has been extensively studied in auction theory. It describes the fact that a bidder is more likely to win an auction when he overestimates the value of the object being sold than when he underestimates it. This creates an adverse selection bias. Bidders must account for this bias by shading their bids. Otherwise, conditional on the information that he has won the auction, the winning bidder regrets his bid and falls prey to the winner's curse.

capitalization (see, for instance, Hasbrouck (1991, Table 3, p588)). According to our results, small firms should have a larger proportion of limit orders, lower fill rates and larger spreads than large firms, in limit order markets.

We define the increase in execution risk as an exogenous decrease in the execution probabilities of limit orders at all possible price levels. We find that limit order traders react to an increase in execution risk by posting *larger spreads*. Actually, when execution risk is high, traders are under pressure to trade immediately upon arrival because the probability of being executed with a limit order is low. For this reason, traders are willing to place market orders at more defavorable prices. This effect allows limit order traders to capture larger rents in equilibrium. It is a well-known stylised fact that spreads enlarge at the end of the trading day in limit order markets (See McInish and Wood (1992) for the NYSE for instance). The model suggests that this can be due to the fact that execution risk is larger at the end of the trading period. Concerning this empirical finding, the model yields the additional testable hypothesis that the size of the increase in the spread at the end of the trading day is negatively related to the level of competition between limit order traders.

We also obtain that trading costs for buy and sell market orders are related to the ratio of buy to sell orders (limit and market orders), because of execution risk. To see this point, consider a decrease in the proportion of potential sellers (the traders with low valuations for the asset). It results in lower execution probabilities for buy limit orders, at all price levels. Consequently execution risk is higher for potential buyers (the traders with high valuations). Limit orders are less attractive for these traders and the maximum ask prices at which they are willing to submit buy market orders increase. But for this reason, limit order trading is more attractive for potential sellers and bid prices must increase to attract sell market orders. Thus the average trading costs for buy (sell) market orders increase (decrease) with the ratio of buy to sell orders. Moreover, the sum of the average trading costs for buy and sell market orders (for which the spread is a proxy) turns out to be concave in the ratio of buy to sell orders, with a maximum when this ratio is equal to 1. This is another new testable prediction.

Concerning welfare, our main result establishes that the risk of being picked off for limit orders is detrimental to the allocative efficiency of limit order markets. The model points two causes for this. First, limit order traders shade their offers relative to their valuations for the asset in order to protect themselves against the risk of being picked off. But for this reason, some trades, which would improve efficiency, do not take place. On the other hand, trades which are detrimental to allocative efficiency can take place (trades which transfer the asset from a high valuation trader to a low valuation trader). This occurs, for instance, when a trader picks off the buy limit order posted by a trader with a lower valuation. We also show that the expected utility traders derive from trading in limit order markets decreases with asset volatility. This suggests that limit order markets will better perform (e.g. attract more traders) for securities with low volatility.

This paper is the first attempt to analyze the determinants of the order flow composition in a limit order market. Most of the models in the market microstructure literature do not allow traders to choose between market and limit orders. For this reason, these models cannot derive implications concerning the mix of market and limit orders. This is the case for models which focus explicitly on dealer markets (e.g. Glosten-Milgrom (1985)) since in these markets traders cannot choose to trade with market or limit orders. This is also the case for limit order trading models developed by Glosten (1994), Rock (1996), Seppi (1996) or Parlour and Seppi (1997). Kumar and Seppi (1993) (in a static setting) and Parlour (1996) (in a dynamic setting) analyze models of limit order markets in which traders can choose between market or limit orders[5]. However, in these models, limit order traders are not exposed to the risk of being picked off. Here, this risk is at the root of the interactions, between volatility and order flow composition, uncovered by the model. The model is in fact most closely related to the empirical study of Hollifield *et al.*. They empirically relate the order flow and the quotes to the underlying distribution of traders' asset valuations as we do theoretically.

The paper is organized as follows. The model is spelled out in Section 2. In Section 3, the equilibrium of the trading game is defined. In Section 4, the benchmark case in which limit order traders behave competitively is analyzed. In Section 5, the equilibrium of the limit order market is derived. Section 6 derives and discusses the

[5] Cohen, maier, Schwartz and Withcomb (1981) also analyze the choice between a market and a limit order. In their model traders' beliefs on the execution probabilities of their limit orders are exogenous and not derived in equilibrium as we do here.

empirical predictions of the model. Welfare analysis is performed in Section 7. Section 8 discusses robustness issues. We conclude in Section 9. The Appendix contains all the proofs.

2. A Model of Trading with Market and Limit Orders.

In this section, a sequential trading process in which traders can choose between market and limit orders is described.

2.1 The Process of the Asset Expected Value.

Consider the market for a single risky asset. Let $\tilde{V}_T$ be the payoff of the asset at the end of the trading "day". The trading day is divided into discrete time intervals denoted $t = 1, 2, ...\tilde{T}$. We assume that the payoff date is random: At each time t, there is a probability $(1 - \rho) > 0$ that the trading process stops and that the payoff of the asset is realized. It is important to remark that, although the trading stops in finite time with probability 1, the date at which the market is closed and the payoffs are realized is *uncertain*. Let v_t be the expected value of $\tilde{V}_T$ conditional on public information at time t. We refer to v_t as the underlying value of the asset. This value follows a random walk:

$$\tilde{v}_{t+1} = \tilde{v}_t + \tilde{\epsilon}_{t+1} \tag{1}$$

where the innovations, due to the arrival of *public information*, are assumed to be independent and identically distributed. $\tilde{\epsilon}$ can take the values $+\sigma$ or $-\sigma$ with equal probabilities.

2.2 The Traders and the Trading Process.

Following Glosten and Milgrom (1985) or Easley and O'Hara (1992), the trading process is sequential and all transactions are for one unit of the asset. At each time, a new trader arrives in the market. Traders differ by their reservation prices. At time t', the reservation price $\tilde{R}_{t'}$ for the trader who arrives at time $t \leq t'$ is :

$$\tilde{R}_{t'} = \tilde{v}_{t'} + \tilde{y}_t \tag{2}$$

The reservation price is the sum of the asset value and a trader specific component (y_t), which is time invariant[6]. The realization of $\tilde{y}_t$ characterizes a trader 's *type*.

[6]Tauchen and Pitts (1983) and Hollifield *et al.* (1996) use a similar decomposition for reservation prices.

The y 's are assumed independent and identically distributed. Moreover they are independent from the innovations in the asset value. They can take two values $y_h = +L > 0$ or $y_l = -L$ respectively with probabilities k and $1 - k$. The dispersion in reservation prices ($L > 0$), for a given asset value, can be justified by differences of opinions as in Harris and Raviv (1993)[7]. This dispersion creates gains from trade and is necessary to generate trading.

It is worth stressing that there are no "noise traders" in the model. All agents are assumed to maximize their expected utility and form correct expectations on the other traders' trading strategies. They are assumed risk-neutral and the utility of purchasing or selling the asset at price P for an agent of type y, if the final value of the asset turns out to be V_T, is:

$$U(y) = (V_T + y - P)q \tag{3}$$

with $q = +1$ ($q = -1$) if the agent has purchased (sold) the asset. The reservation utility of all the agents if they do not trade is normalized to zero (i.e. $U(y)$ is the surplus obtained by agent y if he trades one unit).

2.3 Market Structure: Orders and Information Sets.

Upon arrival, a trader can choose **a)** to submit either a buy or a sell market order or **b)** to post a buy and a sell limit order for one unit. In case of indifference between the placement of a market order or limit orders, it is assumed that limit orders are chosen. If there is no offer available (the book is empty), the trader posts a buy and a sell limit order[8]. For tractability, limit orders are assumed to expire after one period. As a consequence, a trader 's limit order is not executed if his order is not hit by the next agent. The risk of being picked off exists in real trading situations, because limit order traders do not continuously monitor the market. In order to model this risk in the simplest manner, it is assumed that limit order traders cannot revise (or cancel) their offers once they have been posted.

[7] In a more elaborate framework, it could stem from disparities in endowments or preferences across agents. For instance, in Glosten and Milgrom (1985) or Parlour (1996), the differences in valuations come from differences in agents' discount factors.

[8] There is no loss of generality in assuming that a trader will place both a buy and a sell limit order if he decides not to submit a market order because he can always post a bid (ask) with zero execution probability if he does not want to buy (sell).

Let $s_t = (A_t^m, B_t^m)$ denote the best quotes at time t. For completeness, if the book is empty, we set $A_t^m = +\infty$ and $B_t^m = -\infty$. At the time of his trading decision, an agent observes the current state of the book s_t, the current underlying value of the asset v_t and learns his type y_t. Those variables define the *state of the market at time t* . Let us denote this state as $S_t = (v_t, y_t, s_t)$. At a given point in time, all the traders observe v_t and no trader has superior information[9]. Figure 1, in the Appendix, summarizes our assumptions on the probabilistic structure of the model, the trading process and the market structure. Remark that, in contrast with traditional sequential trade models (e.g. Glosten-Milgrom (1985)), there is no market-maker in our setting. Liquidity is entirely provided by limit order traders, as it is the case in many limit order markets (e.g. the Tokyo Stock Exchange).

Execution risk.

We say that *execution risk increases* if the execution probability of limit orders, at all price levels, decreases. The probability that a limit order will not be executed, *whatever the price chosen for the limit order*, is inversely related to ρ in the model. Thus a lower ρ characterizes a market with a larger execution risk for limit order traders. For this reason, in our setting, we can study the impact of an increase in execution risk on traders' behavior by analyzing the impact of a decrease in ρ.

Winner 's curse problem.

Suppose that the trader who arrives at time t posts a buy limit order. In addition assume that the asset value decreases between time t and time $t+1$ and that, for this reason, the trader's bid price becomes higher than the asset value. In this case, the trader runs the risk of being picked off by the next trader who arrives in the market. Thus limit order traders face the risk of being picked off in our setting.

This discussion shows that, although the model is very stylised, the basic trade-offs to an investor when choosing between a market order or a limit order are present. We can therefore study some implications of these trade-offs. Our strong assumptions

[9]The possibility for some traders to have superior private information concerning $\tilde{V}_T$ at each point in time would add another adverse selection risk for limit order traders. This will make the model more complex without changing the basic results due to execution risk and the risk of being picked off. As the effects of private information on quotes have already been analyzed extensively by the market microstructure literature, we abstract from them. Chakravarty and Holden (1995) consider the case in which informed traders can choose to submit limit or market orders.

however put some constraints on what can be said with the model. First, because all orders are for one unit, we will not be able to derive implications for the depth of limit order markets. As usual with sequential trade models, our focus will be on the quotes and the trading costs. Second because limit orders last only one period, the book has only two possible states: empty or full, in our model. Thus we cannot analyze the interactions between transient changes in the state of the book and the order flow (as in Biais, Hillion and Spatt (1995) for instance). Rather we focus on the cross sectional behavior of the order flow compostion.

2.4 An Example.

The purpose of this section is to consider a special case, which helps to explain intuitively how the model works and the methodology we use to solve for the equilibrium of the limit order market. Suppose the asset value does not change over time ($\sigma = 0$). Thus there is *no winner's curse problem*. Let v be the constant value of the asset in this case. We also assume (only in this section) that traders with type y_h only place buy orders while traders with type y_l only place sell orders. The results hold without this assumption. It is just convenient in order to convey more rapidly the intuition.

First consider a trader of type $y_h$who arrives at time t. Let $B^*(v, +L, t)$ be the bid price chosen by this trader if he posts a buy limit order. This order will be executed only if (i) the game does not stop before the arrival of the next trader (probability ρ) (ii) the next trader has type y_l (probability $(1-k)$) and (iii) the next trader market sells. The probability of the last event is *endogenous*. If the bid price is too low, the trader with type y_l will be better off posting a sell limit order. Denote $C^{s*}(v, -L, t+1)$, the bid price such that the trader with type y_l is indifferent between a market or a limit order. If the trader with type y_h posts a price slightly above this treshold, then his execution probability is $\rho(1-k)$ and he obtains an expected gain equal to $\rho(1-k)[v+L-C^{s*}(v, -L, t+1)]$. In fact, this bid price is optimal. Actually, (a) a higher bid price has the same execution probability and (b) a lower bid price has a zero execution probability.

Now consider the optimal order placement decision of trader y_h. Let denote $C^{b*}(v, +L, t)$ the ask price such that he is indifferent between a buy limit order or a buy market order at this price. This price satisfies:

$$\underbrace{(v+L) - C^{b*}(v,+L,t)}_{Gain\quad With\quad a\quad Market\quad Order} = \underbrace{\rho(1-k)[v+L-C^{s*}(v,-L,t+1)]}_{Expected\quad Gain\quad With\quad a\quad Limit\quad Order} \quad \forall t < \tilde{T} \quad (4)$$

If the best ask price in the market, A^m, is greater than $C^{b*}(v,+L,t)$, the trader with type y_h is better off placing a buy limit order with price $B^*(v,+L,t) = C^{s*}(v,-L,t+1)$, otherwise he submits a buy market order. Now consider a trader with type y_l who arrives at time t. He faces exactly the same type of problem. Proceeding in a symmetric way, we can write:

$$\underbrace{C^{s*}(v,-L,t) - (v-L)}_{Gain\quad With\quad a\quad Market\quad Order} = \underbrace{\rho k[C^{b*}(v,+L,t+1) - (v-L)]}_{Expected\quad Gain\quad With\quad a\quad Limit\quad Order} \quad \forall t < \tilde{T} \quad (5)$$

If the best bid price, B^m, is lower than $C^{s*}(v,-L,t)$, the trader with type y_l posts a sell limit order with price $A^*(v,-L,t) = C^{b*}(v,+L,t+1)$, otherwise he submits a sell market order.

If the payoff date (T) were deterministic, we could compute the functions $C^{s*}(.,-L,.)$ and $C^{b*}(.,+L,.)$, using Equations (4) and (5) recursively, starting from date $T-1$, just before the closing time. *However T is not deterministic in our setting.* For this reason, we use a different method. We look for *stationary solutions* of the system of equations defined by Equations (4) and (5), i.e. functions $C^{s*}(.,-L,.)$ and $C^{b*}(.,+L,.)$, which do not depend on time. It is natural to consider these solutions because the trading environment, in our model, is stationary: The exogenous parameters of the model (L, ρ, σ, k) do not vary over time. Denote $C^{s*}(.,-L)$ and $C^{b*}(.,+L)$ these stationary solutions. Equations (4) and (5) become:

$$(v+L) - C^{b*}(v,+L) = \rho(1-k)[v+L-C^{s*}(v,-L)] \quad (6)$$

and

$$C^{s*}(v,-L) - (v-L) = \rho k[C^{b*}(v,+L) - (v-L)] \quad (7)$$

Equations (6) and (7) form a system of two equations in which the unknowns are the prices $C^{s*}(v,-L)$ and $C^{b*}(v,+L)$. Solving this system, we obtain:

$$B^*(v,+L) = C^{s*}(v,-L) = v + L - \frac{(1-\rho k)}{1-\rho^2 k(1-k)}(2L) \quad (8)$$

$$A^{*}(v,-L) = C^{b*}(v,+L) = v - L + \frac{1-\rho(1-k)}{1-\rho^{2}k(1-k)}(2L) \quad (9)$$

Thus the solutions of the previous system of equations yield *both* a characterization of the order submission choice and the quotes, which will be posted in this market.

We proceed in this way below to solve the equilibrium of the trading game in the more general case in which $\sigma > 0$. This complete and parsimonious characterization of the order placement strategies allows us to compute the mix between market and limit orders and the trading costs, in equilibrium, as a function of the parameters of the model. This is useful in order to derive testable implications and to perform welfare analysis. Throughout the article, we assume $k = 0.5$. We reconsider the results when $k \neq 0.5$ in Section 6.2.

3. Order Placement Strategies.

This section gives a formal definition of the order placement strategies and the equilibrium concept, which is used to solve the trading game.

3.1 Equilibrium Definition.

There are two components to a trader's order placement decision: the order type choice and the limit prices of limit orders. The indicator variable Q takes the value $+1$ (-1) if the trader decides to submit a buy (sell) market order and 0 if he decides to place a buy and a sell limit order. In this case, A and B denote his ask and bid prices respectively. A trader's order placement strategy is then a mapping $O(.)$ from the set of possible values for the state of the market to $\{1, 0, -1\} \times \mathbb{R}^2$. For each possible state of the market, the strategy specifies the order type choice: Market Order ($Q(S_t) \neq 0$) or Limit Orders ($Q(S_t) = 0$) and the quotes ($A(S_t), B(S_t)$) associated to the placement of limit orders. It is worth stressing that we implicitly put two restrictions on the order placement strategies. First remark that they are markovian: the order placement decision just depends on the current state of the market. Second they are stationary: the time of arrival does not directly influence the order placement decision. The first restriction is standard in the analysis of dynamic games. The second restriction is also standard and natural given that, in the model, the trading environment is stationary (see discussion in Section 2.4).

Let $J(S_t, A, B)$ be the expected utility for an agent who arrives in the state of the market S_t if he chooses to place limit orders:

$$J(S_t, A, B) = E(I^s(A)(A - (\tilde{v}_{t+1} + y_t)) \mid S_t) + E(I^b(B)((\tilde{v}_{t+1} + y_t) - B) \mid S_t) \quad (10)$$

where $I^s(A)$ ($I^b(B)$) is an indicator function which takes the value +1 in case of execution of the sell (buy) limit order and 0 otherwise. Let $\Delta\tilde{v}_{t+1}$ denote the change in the asset underlying value between times t and $t+1$. Moreover let $\Gamma(B \mid S_t)$ and $\Psi(A \mid S_t)$ be, respectively, the execution probabilities of a buy limit order with price B and a sell limit order with price A, conditional on the state of the market at time t. Using the definitions of reservation prices and the indicator functions, Equation (10) yields:

$$\begin{aligned} J(S_t, A, B) &= \Psi(A \mid S_t)[A - R_t - E(\Delta\tilde{v}_{t+1} \mid S_t, I^s(A) = +1)] \\ &+ \Gamma(B \mid S_t)[R_t + E(\Delta\tilde{v}_{t+1} \mid S_t, I^b(B) = +1) - B] \end{aligned} \quad (11)$$

The objective function of an agent arriving at time t is then:

$$\max_{O(S_t)=(Q,A,B)} E(U(y_t) \mid S_t) = (v_t + y_t - P(Q))Q + (1 - \mid Q \mid)J(S_t, A, B) \quad (12)$$

with $P(+1) = A_t^m$ and $P(-1) = B_t^m$. The first term is the expected surplus if the agent decides to submit a market order. The second term is the expected surplus with the placement of limit orders. The optimal order placement decision at time t depends on the order placement strategy of the trader who arrives at time $t+1$. Actually both the execution probabilities of limit orders and the conditional expectations in Equation (11) depend on it. The Markov Perfect Equilibria[10] of this game will be analyzed.

Definition 1 : *A Markov Perfect Equilibrium of the limit order market is an order placement strategy $O^*(.)$ such that, for each possible state of the market S_t, $O^*(S_t)$ maximizes the expected utility of a trader who arrives in the state of the market S_t (i.e. is solution of (12)) if the other traders follow the order placement strategy $O^*(.)$.*

A useful formulation of the equilibrium order placement strategy is now proposed. Let $A^*(v_t, y_t)$ and $B^*(v_t, y_t)$ be the quotes, which maximize $J(S_t, ., .)$ in state S_t given

[10] A Markov Perfect Equilibrium is a subgame perfect equilibrium in Markov strategies.

that the future traders will act according to $O^{*}(.)$. Equation (12) implies that the optimal order type choice $Q^{*}(S_t)$ must be solution of:

$$\max_{Q\in\{-1,0,1\}} \quad (v_t + y_t - P(Q))Q + (1- \mid Q \mid)J(S_t, A^{*}(v_t, y_t), B^{*}(v_t, y_t)) \qquad (13)$$

This entails that the optimal order type choice can be described by a simple cutoff rule.

Proposition 1 : *The optimal order type choice depends on the best offers in the book. Upon arrival, a trader submits a buy market order if the ask price is lower than or equal to a given price, called the buy cutoff price (denoted $C^{b*}(.,.)$) or a sell market order if the bid price is greater than or equal to a given price, called the sell cutoff price (denoted $C^{s*}(.,.)$). Otherwise he posts limit orders. Buy and sell cutoff prices are functions of the asset value and the trader's type. Moreover $C^{s*}(v_t, y_t) \geq v_t + y_t \geq C^{b*}(v_t, y_t)$, $\forall v_t, y_t$ and the cutoff price functions increase with the type of the trader.*

The buy (sell) cutoff price is the *highest* ask (lowest bid) price at which an agent who arrives in the market is willing to submit a buy (sell) market order instead of placing limit orders [11]. Cutoff prices are given by:

$$\underbrace{C^{s*}(v_t, y_t) - (v_t + y_t)}_{Gain \quad with \quad a \quad Market \quad Order} \quad = \quad \underbrace{J(S_t, A^{*}, B^{*})}_{Expected \quad Gains \quad with \quad Limit \quad Orders} \qquad (14)$$

$$\underbrace{(v_t + y_t) - C^{b*}(v_t, y_t)}_{Gain \quad with \quad a \quad market \quad order} \quad = \quad \underbrace{J(S_t, A^{*}, B^{*})}_{Expected \quad Gain \quad with \quad Limit \quad Orders} \qquad (15)$$

Let *a quotation strategy* be a pair of functions $\{A(.,.), B(.,.)\}$ and let *an order choice strategy*, be a pair of functions $\{C^{s}(.,.), C^{b}(.,.)\}$. Proposition 1 yields the following corollary.

Corollary 1 : *A Markov Perfect Equilibrium $O^{*}(.)$ of the limit order market is completely characterized by an order choice strategy $\{C^{s*}(.,.), C^{b*}(.,.)\}$ and a quotation strategy $\{A^{*}(.,.), B^{*}(.,.)\}$ such that: (C.1) when the asset value is v_t, the offers $A^{*}(v_t, y_t)$ and $B^{*}(v_t, y_t)$ maximize the expected utility of a trader with type y_t if he places limit orders given that the other traders' order choice strategy is $\{C^{b*}(.,.), C^{s*}(.,.)\}$ and (C.2):*

[11] The cutoff prices are just like reservation prices. But contrary to the R 's, they are endogenous.

$$(v_t + y_t) - C^{b*}(v_t, y_t) = J(S_t, A^*(v_t, y_t), B^*(v_t, y_t)) \quad \forall y_t \quad \forall v_t \tag{16}$$

and

$$C^{s*}(v_t, y_t) - (v_t + y_t) = J(S_t, A^*(v_t, y_t), B^*(v_t, y_t)) \quad \forall y_t \quad \forall v_t \tag{17}$$

Proof : straightforward.

As in the example of Section 2.4, Condition (C.1) and Equations (16) and (17) will be used to derive a system of equations, whose solutions give the equilibrium cutoff prices (the order choice strategy). These cutoff prices can then be used to compute the closed forms solution for the equilibrium quotation strategy. For brevity, only the equilibrium quotation strategy will be reported in the text. The corresponding order choice strategy will be derived in the Appendix.

3.2 Two Important Conditions for Bid and Ask Prices.

Limit order traders can always obtain their reservation utilities by posting orders with zero execution probability. Consequently Equation (11) implies that the quotes posted by a limit order trader satisfy:

$$A(v_t, y_t) \geq R_t + E(\Delta\tilde{v}_{t+1} \mid I^s(A) = +1, S_t) \tag{18}$$

$$B(v_t, y_t) \leq R_t + E(\Delta\tilde{v}_{t+1} \mid I^b(B) = +1, S_t) \tag{19}$$

The Right Hand Side of Equation (18) (resp. Equation (19)) shows that limit order traders post ask (bid) prices at least equal to their initial reservation prices adjusted by the expected change in the asset value conditional on selling (resp. buying) the asset. Actually a sell (buy) limit order is executed when its price is lower (greater) than the buy (sell) cutoff price of the last trader who arrives in the market. Consequently if cutoff prices increase with the asset value (it will be the case in equilibrium), then a sell (buy) limit order trader has a greater probability to be executed when the asset value increases (decreases). Intuitively this entails: $E(\Delta\tilde{v}_{t+1} \mid I^b(A) = +1, S_t) > 0$ and $E(\Delta\tilde{v}_{t+1} \mid I^s(B) = +1, S_t) < 0$ if $\sigma > 0$. As can be seen from Equations (18) and (19), rational traders properly account for this adverse selection bias by shading their offers when they place their limit orders. Combining these two equations, it is straightfoward that the spread posted by limit orders is at least $E(\Delta\tilde{v}_{t+1} \mid I^b(A) = +1, S_t) - E(\Delta\tilde{v}_{t+1} \mid I^s(B) = +1, S_t) > 0$ if $\sigma > 0$. This component of the spread is due to the risk of being picked off. We call it *the reservation spread* since this wedge between ask and bid prices is required for limit orders traders to break-even. Moreover conditions (18) and (19) yield the following:

Lemma 1 : *When the asset value is v_t, the ask price posted by a trader with type y_h is at least equal to: $A(v_t, +L) = v_t + L + \sigma$ and the bid price posted by a trader with type y_l is at most equal to: $B(v_t, -L) = v_t - L - \sigma$.*

The intuition is as follows. Consider a trader with type y_h who arrives at time t. The ask price, A, chosen by this trader must be higher than his reservation price $v_t + L$. If the asset value decreases, the reservation price of the trader who arrives at the next point in time is lower than $v_t + L$. This implies that the trader can be executed only in case of an increase in the asset value. But in this case: $E(\Delta v_{t+1} \mid I^a(A), S_t) = +\sigma$. Thus, according to Equation (18), the ask price posted by y_h must be at least: $v_t + L + \sigma$. A symmetric argument can be used for the bid price of trader with type y_l.

4. A Benchmark: Quotes with Competitive Behaviors.

There is no direct price competition among limit order traders in the model. Thus one concern is that the results are dependent on the imperfectly competitive behavior of the limit order traders[12]. In order to better understand the effects which stem from uncompetitive behaviors, the model is first solved, in this section, under the postulate that limit order traders behave competitively. In this case, the results are completely driven by the risk of being picked off for limit orders. Comparison of the results obtained in this benchmark case and in equilibrium allows to distinguish which of the determinants of traders' quotes are specifically due to imperfect competition from those which are not.

[12]However it is worth stressing that limit order traders' market power is limited because traders can choose to trade with market or limit orders. Consider, for instance, a trader who arrives at time t with type y_h. If he posts a bid price which is too low then the trader who arrives at time $t+1$ will not submit a market order and will instead trade with a limit order. This possibility puts an upper bound on the surplus which can be captured by limit order traders from market order traders despite the fact that there is no direct price competition among limit order traders. In fact Equations (14) and (15) show that a market order trader must obtain trading gains which are at least equal to that he can obtain with limit orders.

Let $\{A^c(.,.), B^c(.,.)\}$ be the quotation strategy when limit order traders are competitive. In this case traders' quotes are such that they break-even: Traders' spreads are equal to their reservation spreads, i.e. Equations (18) and (19) are binding. We already know that for traders with type y_h, the ask price in this case is: $A^c(v_t, +L) = v_t + L + \sigma$ (from Lemma 1). Thus, for these traders, we just have to characterize the competitive bid price. For a symmetric reason, we just have to characterize the competitive ask price posted by traders with type y_l. For this, we can use the fact that Equations (18) and (19) are binding, i.e.:

$$A^c(v_t, y_l) = v_t + y_l + E(\Delta\tilde{v}_{t+1} \mid I^s(A) = +1, S_t) \tag{20}$$

$$B^c(v_t, y_h) = v_t + y_h + E(\Delta\tilde{v}_{t+1} \mid I^b(B) = +1, S_t) \tag{21}$$

The expected change in the asset value conditional on a sell (buy) limit order being executed depends on the price of the limit order. Thus, finding the competitive quotes requires solving for a fixed point. Define $\bar{\sigma}^c = \frac{3}{2}L$. We obtain the following result (details of the computations for the fixed point are in the Appendix).

Proposition 2 : **(zero expected profits quotes)** *When limit order traders behave competitively, their quotation strategy is:*

1. *If* $0 < \sigma < \bar{\sigma}^c$, $A^c(v_t, -L) = v_t - L + \frac{1}{3}\sigma$ *and* $B^c(v_t, +L) = v_t + L - \frac{1}{3}\sigma$. *The execution probability of these offers is equal to* $\frac{3\rho}{4}$.

2. *If* $\bar{\sigma}^c \leq \sigma$, $A^c(v_t, -L) = v_t - L + \sigma$ *and* $B^c(v_t, +L) = v_t + L - \sigma$. *The execution probability of these offers is equal to* $\frac{\rho}{4}$.

3. *In all the cases,* $A^c(v_t, +L)$ *and* $B^c(v_t, -L)$ *are as specified in Lemma 1. The execution probability of these offers is zero.*

The competitive quotation strategy has two interesting properties, which will play a role for deriving the empirical implications and for welfare analysis. First traders shade their offers more, relative to their reservation prices, when the asset volatility increases. Actually, the expected changes in the asset underlying value conditional on execution of a buy order or a sell order (the amounts by which limit order traders shade their offers) increase (in absolute value) with the volatility of the asset. This entails that the spread posted by each type of traders (the reservation spread) enlarges when the volatility of the asset increases.

Second the execution probability of a limit order trader is lower when the volatility is high (larger than $\bar{\sigma}^c$) than when it is low (lower than $\bar{\sigma}^c$). Traders with type y_h, for instance, quote lower bid prices when the volatility is high than when it is low, other things equal. But this implies that their offers are less attractive and the probability of their buy limit order being hit by a market order is lower[13].

5. Equilibrium.

In this section, we give the closed form characterization of limit order traders' quotation strategy in equilibrium. Then we compare the properties of the equilibrium quotation strategy with the competitive quotation strategy. Let $\bar{\sigma}^e$ be equal to $\frac{L}{1+\rho/4}$.

Proposition 3 : (equilibrium quotes) *For all values of the parameters (L, σ, ρ), there exists a unique Markov equilibrium of the limit order market. In this equilibrium, the quotation strategy is:*

1. *If $0 \leq \sigma < \bar{\sigma}^e$, then $A^*(v_t, -L) = v_t - L + (2L - \sigma)(\frac{2}{2+\rho})$ and $B^*(v_t, +L) = v_t + L - (2L - \sigma)(\frac{2}{2+\rho})$. The execution probability of these offers is equal to $\frac{\rho}{2}$.*
2. *If $\bar{\sigma}^e \leq \sigma$, then $A^*(v_t, -L) = v_t - L + \sigma + \frac{8L}{4+\rho}$ and $B^*(v_t, +L) = v_t + L - \sigma - \frac{8L}{4+\rho}$. The execution probability of these offers is equal to $\frac{\rho}{4}$.*
3. *In all the cases, $A^*(v_t, +L)$ and $B^*(v_t, -L)$ are as specified in Lemma 1. The execution probability of these offers is zero.*

The equilibrium quotation strategy is derived using the methodology described in the example of Section 2.4. Details are explained in the Appendix. Using the characterization of traders' cutoff prices, it can be checked that, in equilibrium, only traders with type y_h purchase the asset and only traders with type y_l sell the asset.

Using Proposition 3, we obtain that the spread $(A^*(v_t, y_t) - B^*(v_t, y_t))$ posted by a limit order trader is:

[13] The execution probability does not decrease continuously with the volatility because the probability distributions for the innovations and traders' valuations are discrete.

$$SPREAD = \underbrace{\sigma}_{reservation\ \ spread} + \underbrace{(2L-\sigma)\frac{2}{2+\rho}}_{rent} \quad when \quad \sigma < \bar{\sigma}^c \tag{22}$$

and

$$SPREAD = \underbrace{2\sigma}_{reservation\ \ spread} + \underbrace{\frac{8L}{4+\rho}}_{rent} \quad when \quad \bar{\sigma}^c \leq \sigma \tag{23}$$

In equilibrium, limit order traders shade their offers for two different reasons: (i) the winner's curse problem, as in the competitive case and (ii) uncompetitive behaviors. For the second reason, limit order traders' spreads are larger than in the competitive case and limit order traders' offers have a lower execution probabilities than in the competitive case. The spread posted by a limit order trader can be split in two components. The first component ("reservation spread") is due to the risk of being picked off while the second component ("rent") comes from uncompetitive behaviors and is linked to execution risk.

1. Reservation spread. As explained in Section 3.2, the reservation spread accounts for the winner 's curse problem. A limit order trader's reservation spread is equal to the difference between the R.H.S of Equations (18) and (19). Consider a trader with type y_h for instance (computations for y_l are symmetric). We know from the last part of Proposition 3 that Equation (18) is binding for this trader, so that the R.H.S of Equation (18) is $v_t + y_h + \sigma$. On the other hand, if $\sigma < \bar{\sigma}^c$, the buy limit order posted by the trader with type y_h is executed if the next trader is of type y_l, whatever the evolution of the asset value (See the proof of Proposition 3). Then $E(\Delta v_{t+1} \mid I^b(B^*(v_t, +L)), S_t) = 0$ and the R.H.S of Equation (19) is $v_t + y_h$. Thus the reservation spread of the trader in this case is σ. If $\bar{\sigma}^c \leq \sigma$, the bid price chosen by the limit order trader is executed only if the asset value decreases and thus $E(\Delta v_{t+1} \mid I^b(B^*(v_t, +L)), S_t) = -\sigma$. Then the reservation spread is 2σ. In all the cases the reservation spread increases with the volatility. This is the reason why limit order traders' execution probability is weakly decreasing with the asset volatility. These two properties are obtained, for the same reasons, in the competitive case.

2. Execution risk component. The difference between the spread posted by limit order traders and their reservation spread is a measure of their rents in case of execution. This difference is equal to $2(2L-\sigma)/(2+\rho)$ if $\sigma < \bar{\sigma}^e$ and $8L/(4+\rho)$ otherwise. Thus the rents captured by limit order traders decrease as ρ increases. An increase in ρ improves the execution probability of limit order traders for all possible quotes, i.e. unambiguously decreases the risk of non-execution. As a consequence, a trader who arrives in the market is less under the pressure to trade immediately because of the threat of not being executed with a limit order. Thus the minimum bid price at which a trader is willing to submit a sell market order (his sell cutoff price) increases. In the same way the maximum ask price (the buy cutoff price) at which he is willing to submit a buy market order decreases. This obliges limit order traders to improve their offers in equilibrium. Actually bid (ask) prices are equal to sell cutoff prices in equilibrium (see the proof of Proposition 3). Through this channel, execution risk determines the rent component of the spread posted in the limit order market.

To our knowledge, this effect of execution risk on the spread has not been mentioned in the previous literature. It is specific to limit order markets. Actually in a dealer market, (e.g. the London Stock Exchange), a trader does not have the option to post a limit order instead of submitting a market order. Execution risk is therefore irrelevant to his decision of whether or not to trade at the quotes posted by dealers.

Non execution is not a cost in our setting. Thus limit order traders do not need to be compensated for this risk. This is the reason why it does not influence traders' reservation spreads and it does not play a role when traders are competitive (ρ does not influence traders' quotes in the competitive case). As explained above, execution risk determines the fraction of the gains from trade ($2L$), limit order traders can obtain when they behave strategically. Because of these interactions between limit order traders' rents and execution risk, the limit order market can feature a spread even when the risk of being picked off is not an issue. This can be seen by considering the particular case in which $\sigma = 0$ and $k = 0.5$. In this case, the spread posted by limit order traders, is (using Equation (22)): $SPREAD = \frac{4L}{2+\rho} > 0$.

6. Testable implications.

6.1 Implications for the Order Flow Composition.

Let $\tilde{M}_{\tilde{T}}$ be the number of market orders in the interval of time $[0, \tilde{T}-1]$. $T-1$ is the total number of orders (limit orders and market orders) in this period. The proportion of market orders in the order flow over this period is then $\tilde{m}_{\tilde{T}} = \tilde{M}_{\tilde{T}}/(\tilde{T}-1)$ and $1 - \tilde{m}_{\tilde{T}}$ is the proportion of limit orders. Let $\bar{m}_t = E(\tilde{m}_{\tilde{T}} \mid \tilde{T} = t)$ be the expected proportion of market orders conditional on the total number of orders being $(t-1)$. Finally let us define $\bar{m} = lim_{t \to +\infty} \bar{m}_t$. The proportions of market orders and limit orders in the order flow over a long period of time are then: $\bar{m}$ and $(1-\bar{m})$ respectively. These proportions can be computed using the execution probabilities of limit orders given in Propositions 2 and 3.

Proposition 4 : *In equilibrium,* 80% *of all the orders are limit orders if* $\sigma \geq \bar{\sigma}^e$. *Otherwise* 66.66% *of all the orders are limit orders. In the competitive case,* 80% *of all the orders are limit orders if* $\sigma \geq \bar{\sigma}^c$. *Otherwise the proportion of limit orders in the order flow is* 57.14%.

All the implications of Proposition 4, which are derived below, are valid both in equilibrium and in the competitive case. Actually they derive from the fact that when the volatility increases, limit order traders' reservation spreads enlarge. A property, which is obtained both in equilibrium and in the competitive case. An immediate implication of Proposition 4 is that cross-sectional variations in the asset volatility must generate cross-sectional variations in the mix of market and limit orders. More specifically:

Corollary 2 : *Other things equal, the proportion of limit orders in the order flow increases with the asset volatility.*

When the asset volatility increases, limit order traders are more exposed to the risk of being picked off and they post offers which are less attractive (their reservation spreads enlarge). Consequently more traders find it optimal to carry their trades using limit orders instead of market orders. According to Corollary 2, in a regression of the proportion of limit orders on volatility, the sign of the coefficient for the volatility should be negative. The asset volatility (σ) is not directly observable but techniques have been proposed to estimate it. For instance, Hasbrouck (1991) decomposes the mid-quote into a random walk and a residual discrepancy term. He interprets the random walk component as the asset efficient value (v_t) and shows how to estimate its volatility using changes in the mid-quotes and trade innovations. Corollary 4 below offers an alternative way to test the previous corollary.

The fill rate (fr) of limit orders is defined as the total number of limit orders executed divided by the total number of limit orders submitted. Fill rates are a measure of likelihood of execution for limit orders and offer an alternative characterization of the order flow (the mix between filled and unfilled limit orders). Let $E(\tilde{fr} \mid \tilde{l} = l, \tilde{T} = t-1)$ be the expected fill rate, conditional on the total number of orders being $t-1$ and the total number of limit orders being l. We obtain the following corollary.

Corollary 3 : *Other things equal, the expected limit order fill rate decreases with the asset volatility.*

In markets with high volatility, limit order traders shade more their offers because the risk of being picked off is larger. For this reason, their execution probabilities are low when volatility is high (See Propositions 2 and 4). This results in lower fill rates. Corollary 3 leads to the testable hypothesis that the average fill rate is negatively related to asset volatility. Hasbrouck (1991) (Table 3) shows that volatility is negatively related to equity capitalization. Thus, according to our results, the proportion of limit orders for stocks with small capitalization must be larger than for stocks with large capitalization. Moreover fill rates must be lower for stocks with small capitalization.

We denote $t(n)$ as the time of the n^{th} transaction and P_n as the associated transaction price. $\tilde{Q}_n$ is an indicator variable which takes the values $+1$ if this transaction is triggered by a buy market order and -1 if it is triggered by a sell market order. In equilibrium, if the n^{th} transaction is triggered by a buy market order, then the trade is consumed at price $A^*(v_{t(n)-1}, -L)$ (remember that the execution probability of an ask price posted by a trader with type y_h is zero). Conversely if it is triggered by a sell market order, the trade is consumed at price $B^*(v_{t(n)-1}, +L)$. Using the closed form solutions given in Proposition 3, $\tilde{P}_n$ can then be written:

$$\tilde{P}_n = \tilde{v}_{t(n)-1} + [(2L - \sigma)(2/2 + \rho) - L]\tilde{Q}_n \quad if \quad \sigma < \bar{\sigma}^c \tag{24}$$

and

$$\tilde{P}_n = \tilde{v}_{t(n)-1} + [\sigma + 8L/(4 + \rho) - L]\tilde{Q}_n \quad if \quad \sigma \geq \bar{\sigma}^c \tag{25}$$

Proceeding in the same way, similar expressions for the transaction prices can be derived in the competitive case. Then the variance of changes in transaction prices

$Var(\tilde{P}_{n+1} - \tilde{P}_n)$ can be computed in equilibrium and in the competitive case. Proposition 4 has the following corollary.

Corollary 4 : *For a given L, the variance of changes in transaction prices is positively related to the proportion of limit orders and negatively related to the expected limit order fill rate.*

The intuition is as follows. When the volatility of the asset underlying value is large, the proportion of limit orders is large and the fill rate is low. At the same time, the variance of changes in transaction prices is large because (i) the volatility of the asset value, per period, is large, (ii) the average interval of time between two transactions is large (because there are less market orders) and (iii) the difference between the prices at which buy and sell market orders are executed is large because traders shade more their offers. The variance of changes in transaction prices is another characterization of the asset volatility. Thus Corollary 4 reinforces the conclusion that more volatile markets should feature more limit orders. Moreover it shows that the variance of changes in transaction prices, in place of an estimation of the unobservable volatility, can be used to test the predictions of Corollaries 2 and 3.

The following result is a direct implication of Proposition 4 and Equations (22) and (23).

Corollary 5 : *For a given L, the spread in the limit order market is positively related to the proportion of limit orders in the order flow.*

When the volatility increases, limit order traders shade more their offers, which entails a decrease in the proportion of market orders. This creates a positive correlation between the size of the spread in the limit order market and the proportion of limit orders. Remark that this entails a negative relationship between the spread and the proportion of market orders and thereby a negative relationship between the spread and transaction frequency. This observation is consistent with empirical observation (e.g. McInish and Wood (1992) for the NYSE). The traditional explanation is that greater trading activity leads to lower spreads because of economies of scale in trading costs. For limit order markets, the model shows that the winner 's curse problem provides an equally plausible interpretation. In limit order markets with high volatility, spreads are larger because the risk of being picked off is larger. This entails that traders place less market orders and transactions are less frequent.

6.2. Implications for the Trading Costs.

The trading cost for the n^{th} transaction, denoted $\tilde{TC}_n$, is defined as the premium (discount) between the asset value and the price at which the n^{th} market order is executed (as in Hasbrouck (1993), for instance)[14]:

$$\tilde{TC}_n = (\tilde{P}_n - \tilde{v}_{t(n)})\tilde{Q}_n \tag{26}$$

Using Equations (24) and (25), it follows that:

$$\tilde{TC}_n = -\epsilon_{\tilde{t(n)}}\tilde{Q}_n - (\frac{2}{2+\rho})\sigma + (\frac{2-\rho}{2+\rho})L \quad if \quad \sigma < \bar{\sigma}^c \tag{27}$$

and

$$\tilde{TC}_n = -\epsilon_{\tilde{t(n)}}\tilde{Q}_n + \sigma + (\frac{4-\rho}{4+\rho})L \quad if \quad \sigma \geq \bar{\sigma}^c \tag{28}$$

Using these equations, we obtain the following result.

Proposition 5 *: In equilibrium, the expected trading cost in the limit order market is:* $E(\tilde{TC}_n) = (\frac{4-\rho}{4+\rho})L$ *if* $\sigma \geq \bar{\sigma}^c$ *and* $E(\tilde{TC}_n) = (\frac{2-\rho}{2+\rho})L - (\frac{2}{2+\rho})\sigma$ *otherwise. It decreases with* ρ.

As explained in Section 5, the larger the execution risk for limit order traders, the larger the wedge between their posted spread and their reservation spread. For this reason, trading costs enlarge when ρ decreases (i.e. when the execution risk of limit order traders increases).

The lower ρ in the model, the larger the probability that a trader will be the last trader of the trading "day" and will, for this reason, not be executed. In real trading situations, this probability is larger at the end of the trading day. Thus comparing the size of the trading costs when ρ is small and ρ is large in the model, is like comparing the size of the trading costs at the end of the trading day and at an earlier point in time during the trading day. Thus the model predicts that trading costs must increase at the end of the trading day. This is consistent with the empirical findings concerning limit order markets (e.g. McInish and Wood (1992), Kleidon and Werner

[14]Here the spread posted by the traders is not a good measure of actual trading costs for market order traders. First, because the asset value fluctuates over time, a limit order price can be stale relative to the fair value of the asset at the time of the transaction. Second, a quote posted by a limit order trader is not necessarily a price at which a transaction will take place. The measure of trading costs, which is defined here, overcomes these two problems. It 's worth stressing however that the same results are obtained when we use the quoted spread as a proxy for execution costs.

(1993) and Biais *et al.* (1995)). The interpretation provided here is that execution risk is larger at the end of the trading day. For this reason, traders are willing to trade at more defavorable prices, to avoid execution risk. But this allows limit order traders to extract larger rents from market order traders.

As explained in Section 5, execution risk (ρ) does not influence traders' quotes when limit order traders post zero expected profits quotes. For this reason, the expected trading costs do not depend on ρ in the competitive case (the computations are skipped for brevity). Thus, according to the model, the increase in trading costs at the end of the trading day relies on the possibility for limit order traders to extract rents from market order traders. This has the implication that the size of the increase in trading costs at the end of the trading day must be negatively related to the level of competition between limit order traders. This could be tested in the following way. The posted spread is a proxy for trading costs[15]. The difference, $\Delta SPREAD$, between the spread at the end of the trading day and the spread, say, in the middle of the day measures the extent by which trading costs increase at the end of the day. Empirical studies (e.g. McInish and Wood (1992)) have used different proxies for the level of competition between liquidity providers. Examples of such proxies are the number of markets in which a stock is traded or, in the case of the NYSE, the proportion of trades, for a stock, occuring on regional exchanges. The testable hypothesis is that, in a cross-sectional analysis, $\Delta SPREAD$ is negatively related to the proxy chosen for the level of competition between limit order traders.

Consider now the case in which the proportions of traders of type y_h and traders of type y_l are not equal (i.e. $k \neq 0.5$) and there is no winner 's curse problem ($\sigma = 0$). The equilibrium quotes in this case have been derived in Section 2.4. Using Equations (8) and (9), we obtain the following result.

Proposition 6 : *In equilibrium, when $\sigma = 0$, the ask prices posted by traders with type y_l and the bid prices posted by traders with type y_h increase with k.*

This result is also due to execution risk. Actually, an increase in the proportion of traders with type y_h exogenously decreases the execution probability of these traders

[15]Alternatively, Hasbrouck (1993) shows how to get a linear estimate of $\bar{P}_n - \tilde{v}_{t(n)}$ using data on transaction prices and trades. His method can be used to estimate $E(\tilde{TC}_n)$.

(because they trade only with traders of type y_l in equilibrium). As a consequence they become more "impatient" and the maximum ask price at which they are willing to submit a buy market order increases. This allows traders with type y_l to increase their ask prices. For this reason, traders with type y_l become more "patient" (because their gains of trading with limit orders and their execution probability increase) and the minimum bid price at which they are willing to submit a sell market order increases. This forces traders with type y_h to improve their bid prices. As a result, the rents of traders with type y_l increase whereas the rents of traders with type y_h decrease. The result generalizes to the case in which $\sigma > 0$ [16].

In fact, in a market in which there are few sellers (traders with type y_l), the execution risk faced by the buyers (traders with type y_h) is higher. Consequently they must leave larger gains from trade to the sellers. This is reflected in the expected trading costs for *buy market orders* ($E(\tilde{TC}_n \mid Q_n = +1)$) and *sell market orders* ($E(\tilde{TC}_n \mid Q_n = -1)$). These expected trading costs are (Using Equations (8) and (9)):

$$E(\tilde{TC}_n \mid Q_n = +1) = A^*(v, -L) - v = L(\frac{1 - \rho(1-k)(2-\rho k)}{1 - \rho^2 k(1-k)}) \tag{29}$$

$$E(\tilde{TC}_n \mid Q_n = -1) = v - B^*(v, +L) = L(\frac{(1 - \rho k(2 - \rho(1-k))}{1 - \rho^2 k(1-k)}) \tag{30}$$

Finally we denote by STC, the sum of the expected trading costs for buy market order traders and sell market order traders, i.e.:

$$STC = E(\tilde{TC}_n \mid Q_n = +1) + E(\tilde{TC}_n \mid Q_n = -1) = A^* - B^* = \frac{2 - 2\rho + 2\rho^2 k(1-k)}{1 - \rho^2 k(1-k)} \tag{31}$$

This yields the following corollary.

Corollary 6 :

1. *Expected trading costs for buy market orders increase with k while expected trading costs for sell market orders decrease with k.*

2. *STC is a concave function of k and is maximum in $k = 0.5$.*

[16] Proposition 6 and the next corollary were proven in the case $\sigma > 0$ in a previous version of this paper. Since the derivations are quite long, they have been omitted in this version. It is also possible to show that a unique equilibrium exists for all the values of the parameters when $k \neq 0.5$ and $\sigma > 0$.

Now remark that k is the proportion of buy orders (buy limit orders and buy market orders) in the order flow and that $(1-k)$ is the proportion of sell orders. Thus the ratio of buy to sell orders is given by $i = k/(1-k)$. The previous corollary leads to the following testable predictions:

1. The average trading costs for buy (sell) market orders are positively (negatively) related to the ratio of buy to sell orders.

2. The sum of the average trading costs for buy market orders and for sell market orders increases with i when $i < 1$ and decreases with i when $i > 1$.

Equation (31) shows that the spread is a proxy for STC. Thus the second implication can be tested in the following way. Define $D(x)$ an indicator dummy variable, which takes the value +1 if $x > 1$ and 0 otherwise. Then consider the two variables: $i * D(i)$ and $i * D(1/i)$. According to the second implication, a cross-sectional regression of the spread on these two variables should yield a negative coefficient for the first independent variable $(i*D(i))$ and a positive coefficient for the second one $(i*D(1/i))$.

7. Welfare.

In this section we define a measure of the allocative efficiency of the limit order market. Our main result is that the risk of being picked off for limit orders is an impediment to the allocative efficiency of limit order markets.

In experimental economics, it is usual to measure the allocative efficiency of a market by the ratio of the actual total gains from trade earned by the traders (the sum of buyers and sellers' surplus) to the maximum gains from trade that could have been earned by the traders (had the trading allocation been efficient). We use such a measure here. Consider, for instance, a sequence of traders' arrivals over two periods $(t, t+1, t+2)$ starting with a situation in which there is no offer in the limit order book at time t and *at least* two traders have different types. The potential gains from trade for this sequence of arrival are equal to $(2L)$[17]. Define $\tilde{G}$ as the realized gains

[17]It is possible to include the sequences in which all the traders arriving at t, $t+1$, $+2$ have the same type. But for such a sequence the gains from trade are zero and this will make the maximum gains from trade random (they are equal to $2L$ or zero). Then one would have to measure efficiency by the ratio of the expected gains from trade to the expected maximum gains from trade. Moreover a longer period of time could be used. In all these cases, the computations are more involved but the results are qualitatively similar. Remark that any price in the interval $(v_{t+2} - L, v_{t+2} + L)$ is a walrassian equilibrium of a market organized at time $t+2$ between the 3 traders considered in the sequence. In this case, the trading outcome is efficient. Trades occur between two traders with

from trade conditional on this sequence of arrivals. It will be either $(2L)$ if there is a trade between the two traders who have different types and 0 otherwise. The allocative efficiency of the limit order market is measured by $e = \frac{E(\tilde{G})}{2L}$, the proportion of the maximum gains from trade which are realized, on average. The closer e is to 1, the greater the allocative efficiency of the limit order market.

We start by considering the benchmark case in which traders post zero expected profits. Then we show that the results are similar in equilibrium.

Proposition 7 : *In the competitive case, the allocative efficiency of the limit order market is $e^c = 100\%$ if there is no winner 's curse problem (i.e. if $\sigma = 0$). Otherwise $e^c = 83.33\%$ if $\sigma < \sigma^c$ and $e^c = 58.33\%$ if $\sigma \geq \sigma^c$.*

Only when $\sigma = 0$, all the gains from trade in the limit order market are realized. When the volatility increases, the allocative efficiency decreases. Remember that the larger the asset volatility, the larger the intensity of the winner 's curse problem. Thus the proposition establishes that the winner 's curse problem is a main source of inefficiency for limit order markets. The model shows that there are two reasons for this inefficiency.

First, when the asset volatility increases, the winner's curse problem becomes more acute. Limit order traders react by enlarging their spreads, but this entails that profitable opportunities to trade are missed. This explains that the efficiency of the limit order market decreases with asset volatility. To see this effect, consider two traders with types y_h and y_l, who arrive respectively at times t and $t+1$. Clearly a trade between these two traders is welfare improving. If $\sigma \geq \bar{\sigma}^c$, the trader with type y_h posts a very low bid price, equal to $v_t + L - \sigma$ (See Proposition 2), in order to limit his exposure to the risk of being picked off. But as a result, if the asset value increases, the bid price posted by the trader with type y_h is lower than the reservation price of the trader with type y_l $(v_t + \sigma - L)$. This trader finds it unprofitable to market sell and he posts limit orders. Gains from trade are lost since this trader is not certain whether his limit orders will be executed. On the contrary, if $0 < \sigma < \bar{\sigma}^c$, the bid price posted by the trader with type y_h is larger than the reservation price of the trader with type y_l, whether the asset value increases or not. Thus the welfare improving

different types and a third trader does not trade.

trade between the two traders will take place. This explains why the efficiency is higher in this case.

A second cause of inefficiency is that trades, which are not desirable for allocative efficiency can occur. Consider, for instance, the case in which two traders with the same type y_h arrive at time t and $t+1$ respectively, while a trader with type y_l arrives at time $t+2$. Furthermore assume that $0 < \sigma < \bar{\sigma}^c$. The first trader with type y_h posts a bid price equal to $v_t + L - \frac{1}{3}\sigma$ in this case. If the asset value decreases between time t and $t+1$, this price is above the reservation price $(v_t - \sigma + L)$ of the second trader with type y_h. This implies that the second trader picks off the buy limit order posted by the first trader. But this trade does not create any gains from trade since the two traders have the same type. It is even detrimental to allocative efficiency since the possibility of a better allocation (the second trader of type y_h trading with the trader of type y_l) has been lost[18]. This explains why, when $0 < \sigma < \bar{\sigma}^c$, all the gains from trade are not realized, despite the fact that the first source of inefficiency is not at work.

Proposition 8 :

1. *In equilibrium, the allocative efficiency of the limit order market is* 100% *if* $0 \leq \sigma < \bar{\sigma}^c$ *and* 58.33% *if* $\sigma \geq \bar{\sigma}^e$.

2. *Moreover, when* $\bar{\sigma}^c < \sigma \leq \bar{\sigma}^e$, *the allocative efficiency of the limit order market in equilibrium is strictly lower than when traders post competitive quotes.*

As in the competitive case, the winner 's curse problem results in a negative relationship between volatility and the allocative efficiency of the limit order market. In equilibrium, traders shade more their offers than in the competitive case. For this reason, there is no case in which two traders of the same type can trade together and the second source of inefficiency is not present. This explains that as long as $\sigma < \bar{\sigma}^e$, all the gains from trade are realized, while they are not in the competitive case. This

[18]With a continuous probability distribution for traders' traders' types, this effect would be reinforced by the fact that the bid price of a trader could be picked off by traders with *srictly* higher types in case of a decrease in the asset value. Such a trade transfers the asset from traders with high valuations to traders with low valuations, which results in a welfare loss.

property of the equilibrium is not general. In fact the competitive case clearly indicates that it comes from the absence of competition between limit order traders. Uncompetitive behaviors in equilibrium is a new cause of inefficiency. Strategically limit order traders post spreads larger than their reservation spreads. This decreases their execution probabilities, implying that welfare improving opportunities to trade are lost relative to the competitive case. This explains the last part of the proposition.

It is intuitive that the risk of being picked off determines the distribution of trading gains between limit order traders and market order traders (Brown and Holden (1996) provides an interesting analyzis). However, to our knowledge, the negative impact of the risk of being picked off on the allocative efficiency of limit order markets has not been stressed in discussions concerning the effects of this risk.

Consider a trader at time t with type y. Denote $U^p(y)$ the expected utility of this trader, before observing the state of the book, i.e. before knowing the asset value v_t and whether he will use a limit order or a market order to carry his trade. If the trader had to pay a fee in order to participate to the limit order market, he would compare $U^p(y)$ to the fee and would participate to the trading process only if $U^p(y)$ is larger than the fee. The previous propositions suggest that the utility traders derive from trading in limit order markets is negatively related to asset volatility. This is confirmed by the next proposition.

Proposition 9 : *Consider the case in which $\rho = 1$. In equilibrium, $U^p(y_l) = U^p(y_h) = 2L/3$ if $\sigma < \bar{\sigma}^c$ and $U^p(y_l) = U^p(y_h) = 2L/5$ if $\sigma \geq \bar{\sigma}^c$. Thus traders' utility from taking part to the trading process decreases with the volatility of the asset.*

For brevity, $U^p(y)$ has been computed only when $\rho = 1$. In the other cases ($\rho < 1$), the expected utility of an agent who enters at time t in the market will be dependent on ρ. However the result is qualitatively unchanged. Moreover, for brevity again, we reported the values of $U^p(y)$ only in equilibrium. Similar results are obtained in the competitive case.

When volatility increases, traders shade more their offers and, for this reason, the best offers' execution probability decreases. Thus traders anticipate that they have a lower chance to carry their trades and this lowers the utility they expect from participating to the trading process. The result implies that the maximum fee limit order traders will be willing to pay to take part to the trading process decreases with asset volatility in limit order markets. It also implies that markets with high volatility will attract less trading because traders anticipate less gains from trade in these markets, because of the winner 's curse problem. These implications are important for market organizers.

8. Robustness.

For tractability, we have assumed that limit orders expire after one period. One concern is that the empirical and the welfare implications derived in the model depends on this assumption. We have emphasized the fact that the empirical implications concerning the order flow composition comes from a very basic effect. Namely, an increase in volatility forces limit order traders to shade more their offers, because it increases the probability of being picked off and the associated losses. Increasing the maturity of limit orders will not suppress their exposure to the risk of being picked off and therefore will not remove the effect driving the empirical implications for order flow composition. In the same way, the welfare implications derive only from limit order traders' exposure to the risk of being picked off. We have also shown that the empirical implications concerning order flow composition and welfare were obtained whether limit order traders were competitive or not.

Another basic effect drives our empirical implications concerning trading costs. When the probability of being able to trade with a limit order decreases, traders are more willing to submit market orders, even if execution prices are not too good. But this allows limit order traders to capture larger rents, which are reflected in larger trading costs. We have shown that this effect of execution risk depends on the possibility to post non zero-expected profits quotes without losing any chance to trade[19]. Whatever the maturity of limit orders, a decrease in their execution probabilities will decrease the gains expected from trading with these orders and will make traders more willing to submit market orders at defavorable prices. Thus as long as competition between limit order traders is not perfect, this effect will allow limit order traders to capture larger rents and our implications for trading costs will be obtained.

[19]Handa and Schwartz (1996) and Biais, Hillion and Spatt (1995) provide evidences that limit order traders' profits are larger than zero on average.

9. Conclusion.

This article extends sequential trade models, which are used in the literature on market microstructure, in two key aspects. Firstly traders can choose to trade with limit orders or market orders. Secondly we consider the case in which liquidity is entirely supplied by limit order traders. A complete characterization, in closed form, of the traders' order placement strategies is derived in equilibrium.

The closed form characterization of the equilibrium is useful in order to examine how the risk of being picked off as well as the execution risk faced by limit order traders influence (a) the order flow composition and (b) the trading costs, in limit order markets. We summarize below the main testable hypotheses derived in Section 6:

- **H1.** The proportion of limit orders in the order flow is positively related to asset volatility.
- **H2.** The fill rate (the ratio of filled limit orders to total number of limit orders) is negatively related to asset volatility.
- **H3.** The proportion of limit orders is positively related to the average size of the spread.
- **H4.** The increase in trading costs at the end of the trading day is negatively related to the level of competition between limit order traders.
- **H5.** The size of the sum of trading costs for buy and sell orders (for which the spread is a proxy) is concave in the ratio of buy to sell orders, with a maximum when this ratio is equal to one.

Identifying the sources of inefficiency specific to a limit order market is of interest for market design. We establish that the risk of being picked off for limit order traders is a cause of inefficiency for the limit order market. We also show that the level of inefficiency is positively related to asset volatility. Finally we find that traders expect larger trading gains in limit order markets for assets with low volatility.

Appendix.

Proof of Proposition 1. Consider a given state of the market S_t. In this state, Equation (13) implies that an agent must submit a buy market order if :

$$v_t + y_t - A_t^m \geq Max\{J(S_t, A^*(v_t, y_t), B^*(v_t, y_t)), B_t^m - v_t - y_t\} \tag{32}$$

In the same way, an agent must submit a sell market order if:

$$B_t^m - v_t - y_t \geq Max\{J(S_t, A^*(v_t, y_t), B^*(v_t, y_t)), v_t + y_t - A_t^m\} \tag{33}$$

Consider first the two following inequalities:

$$v_t + y_t - A_t^m \geq J(S_t, A^*(v_t, y_t), B^*(v_t, y_t)) \tag{34}$$

$$B_t^m - v_t - y_t \geq J(S_t, A^*(v_t, y_t), B^*(v_t, y_t)) \tag{35}$$

Denote by $C^b(v_t, y_t)$ ($C^s(v_t, y_t)$) the ask (bid) price such that the first (second) inequality holds as an equality (the expressions are given Equations (14) and (15) in Section 3.2). $J \geq 0$ because a trader has always the possibility to get a zero expected surplus by posting quotes with zero execution probabilities. This implies $C^s(v_t, y_t) \geq v_t + y_t \geq C^b(v_t, y_t)$. Since $A_t^m > B_t^m$, if $A_t^m < C^b$ then $v_t + y_t - A_t^m \geq J(S_t, A^*(v_t, y_t), B^*(v_t, y_t)) > B_t^m - v_t - y_t$. In the same way, if $B_t^m > C^s$ then $B_t^m - v_t - y_t > J(S_t, A^*(v_t, y_t), B^*(v_t, y_t)) \geq v_t + y_t - A_t^m$. Consequently, an agent must submit a buy (sell) market order if the ask (bid) price is lower (greater) than C^b (C^s) and place limit orders otherwise. Now consider two market states S_t and $S_t^{'}$ which differ only by the realization of the agent 's type, with $y_t > y_t^{'}$.

$$C^b(v_t, y_t) - C^b(v_t, y_t^{'}) = y_t - y_t^{'} - [J(S_t, A^*(v_t, y_t), B^*(v_t, y_t)) - J(S_t^{'}, A^*(v_t, y_t^{'}), B^*(v_t, y_t^{'})] \tag{36}$$

Now remark that:

$$J(S_t^{'}, A^*(v_t, y_t), B^*(v_t, y_t)) = J(S_t, A^*(v_t, y_t), B^*(v_t, y_t)) \tag{37}$$

because the type of the agent who arrives at time t does not influence the execution probability of his limit order. Therefore:

$$C^b(v_t, y_t) - C^b(v_t, y_t^{'}) = y_t - y_t^{'} - [J(S_t^{'}, A^*(v_t, y_t), B^*(v_t, y_t)) - J(S_t^{'}, A^*(v_t, y_t^{'}), B^*(v_t, y_t^{'})] \tag{38}$$

which is positive, using the definition of $\{A^*(v_t, y_t^{'}), B^*(v_t, y_t^{'})\}$. A similar argument would prove that $C^s(v_t, y_t) > C^s(v_t, y_t^{'})$. **Q.E.D.**

Proof of Lemma 1. Consider a trader with type y_l who arrives at time t. The bid price, $B(v_t, -L)$, posted by this trader must be lower than his reservation price $v_t - L$. If the asset value increases, the reservation price $v_t + \sigma + y_t$ of the trader who arrives at the next point in time is necessarily larger than $v_t - L$ since $-L \leq y_t$. But then the bid price posted by the trader with type y_l has a zero execution probability when the asset value increases. Thus execution can occur only if the asset value decreases. But then, using Equation (19), the bid price must be at most $v_t - \sigma - L$ for the trader with type y_l to break-even. The proof is symmetric for the ask price posted by a trader with type y_h.**Q.E.D**

Proof of Proposition 2. In the competitive case, limit order traders get zero expected profits, i.e. $J = 0$. Thus their cutoff prices are just equal to their reservation prices. The execution probability of a given offer depends on its position relative to the possible reservation prices of the future traders, as described in Table 1.

Step 1. We first look for solutions of $B^c(v_t, +L) = v_t + L + E(\Delta\tilde{v}_{t+1} \mid I^b(B^c(v_t, +L)) = +1, S_t)$. Denote $\pi_t(B)$ the probability that an increase in the asset value has occured between time t and $t+1$, conditional on the execution of a buy limit order with price B at time $t+1$. It is the case that:

$$E(\Delta v_{t+1} \mid I^b(B) = +1, S_t) = \pi_t(B)\sigma - (1 - \pi_t(B))\sigma = \sigma(2\pi_t(B) - 1)$$

Using Bayes' law:

$$\pi_t(B) = \frac{Prob(I^b(B) = +1 \mid \epsilon_{t+1} = +\sigma) * Prob(\epsilon_{t+1} = +\sigma)}{Prob(I^b(B) = +1)} \tag{39}$$

Since $\pi_t(B) \geq 0$, the competitive bid price is at least equal to $B^c(v, +L) = v + L - \sigma$. Suppose first that $L > \sigma > 0$ so that: $v_t + L - \sigma > v_t - L + \sigma$. For a bid price higher than $v_t + L - \sigma$, Equation (39) yields:

$$\pi_t(B) = \frac{\frac{\rho}{2}\frac{1}{2}}{\frac{\rho}{2}\frac{1}{2} + \frac{\rho}{2}} = \frac{1}{3}$$

Thus $E(\Delta\tilde{v}_{t+1} \mid I^b(B^c(v_t, +L)) = +1, S_t) = -\frac{1}{3}\sigma$. This gives $B^c(v_t, +L) = v_t + L - \frac{1}{3}\sigma$. Suppose now that $\frac{2}{3}\sigma < L \leq \sigma$ so that $v_t + L - \sigma \leq v_t - L + \sigma < v_t + L - \frac{1}{3}\sigma$. The previous bid is a solution again because (i) it is still the case $\pi_t(v_t + L - \frac{1}{3}\sigma) = 1/3$ and (ii) a higher bid cannot break-even. Finally if $\frac{2}{3}\sigma \geq L$ then $\pi(v_t + L - \frac{1}{3}\sigma) = 0$ because $v_t + L - \frac{1}{3}\sigma \leq v_t - L - \sigma$. Thus $B^c(v_t, +L) = v_t + L - \frac{1}{3}\sigma$ is not solution. It is direct that a higher bid cannot break-even. A lower bid price B is executed only if the asset value decreases, i.e. $\pi(B) = 0$. Consequently $B^c(v_t, +L) = v_t + L - \sigma$ is the only solution if $L \leq \frac{2}{3}\sigma$. Finally remark (using Table 1) that if $\frac{3}{2}L > \sigma > 0$, $Prob(I^b(B^c(v_t, +L) = +1) = \frac{3}{4}\rho$ and if $\frac{3}{2}L \leq \sigma$, $Prob(I^b(B^c(v_t, +L) = +1) = \frac{1}{4}\rho$. This gives the execution probabilities of the bid price posted by the trader with type y_h according to the position of σ relative to $\bar{\sigma}^c$. The same arguments can be developed to find the ask prices solutions of $A^c(v_t, -L) = v_t - L + E(\Delta\tilde{v}_t \mid I^b(A^c(v_t, -L)) = +1, S_t)$ and their execution probabilities.

Step 2. Lemma 1 implies that the best possible bid price posted by a trader with type y_l who arrives at time t is $v_t - \sigma - L$. It has a zero execution probability since it is lower than the possible reservation prices for the trader who arrives at time $t+1$ (See Table 1). The same argument holds for the ask price posted by a trader with type y_h. This gives the last part of the proposition.**Q.E.D**

Proof of Proposition 3. In what follows, quotation strategies are derived under the conjecture that cutoff prices are increasing in the asset value in equilibrium. Then, it is checked, using the closed-form solutions, that this conjecture is indeed correct. Consider a possible candidate $\{C^{s*}(.,.), C^{b*}(.,.)\}$ for the equilibrium order choice strategy. Table 2 gives the execution probability of a bid price posted at time t according to its position relative to the sell cutoff prices of the trader who arrives at time $t+1$.
Step 1. From Proposition 1, we know that $C^{s*}(v_t - \sigma, -L) \geq v_t - \sigma, -L$. Then the proof of the last part of the proposition is as Step 2 in the previous proof.

Step 2. Consider a trader with type y_h who arrives at time t. He must optimally choose a bid price slightly higher than $C^{s*}(v_t - \sigma, -L)$ or slightly higher than $C^{s*}(v_t + \sigma, -L)$. Actually, other bids could be decreased without changing the execution probability or by decreasing the execution probability only in states in which the order is executed at a loss (See Table 2). A symmetric argument implies that a trader with type y_l in state S_t must choose an ask price, which is either slightly lower than $C^{b*}(v_t - \sigma, +L)$ or slightly lower than $C^{b*}(v_t + \sigma, +L)$. We consider two cases now.

Case 1: Assume $\frac{L}{1+\rho/4} \leq \sigma$. Choosing a buy limit order with a bid price slightly higher than $C^{s*}(v_t - \sigma, -L)$ instead of $C^{s*}(v_t + \sigma, -L)$ is optimal for a trader with type y_h if:

$$\frac{1}{4}[v_t + L - \sigma - C^{s*}(v_t - \sigma, -L)] \geq \frac{1}{2}[v_t + L - C^{s*}(v_t + \sigma, -L)] \tag{40}$$

(remark that we proceed as if the bid prices were just equal to $C^{s*}(v_t - \sigma, -L)$ and $C^{s*}(v_t + \sigma, -L)$ since they can be chosen as closed as desired to these cutoff prices.) This condition is necessary and sufficient if $C^{s*}(v_t + \sigma, -L) \leq C^{s*}(v_t - \sigma, +L)$. If not, this condition is sufficient but not necessary. Actually, in that case, with a bid price slightly above $C^{s*}(v_t + \sigma, -L)$, the agent obtains:

$$\frac{1}{2}[v_t + L - C^{s*}(v_t + \sigma, -L)] + \frac{1}{4}([v_t + L - \sigma - C^{s*}(v_t + \sigma, -L)]$$

and this is lower than the R.H.S of (40) since $C^{s*}(v_t + \sigma, -L) \geq C^{s*}(v_t - \sigma, +L) \geq v_t - \sigma + L$. Proceeding in the same way, it is optimal for agents of type y_l to quote an ask price slightly below $C^{b*}(v_t + \sigma, +L)$ if:

$$\frac{1}{4}[C^{b*}(v_t + \sigma, +L) - (v_t + \sigma - L)] \geq \frac{1}{2}[C^{b*}(v_t - \sigma, +L) - (v_t - L)] \tag{41}$$

If $C^{b*}(v_t - \sigma, +L) \geq C^{b*}(v_t + \sigma, -L)$ this condition is necessary and sufficient and only sufficient otherwise, for symmetric reasons as with the agent of type y_h. Suppose these two conditions are satisfied. Using Equations (16) and (17) of Corollary 1 and Step 1, the two following equalities must be satisfied:

$$v_t + L - C^{b*}(v_t, +L) = \frac{\rho}{4}[v_t + L - \sigma - C^{s*}(v_t - \sigma, -L)] \tag{42}$$

$$C^{s*}(v_t, -L) - (v_t - L) = \frac{\rho}{4}[C^{b*}(v_t + \sigma, +L) - (v_t + \sigma - L)] \tag{43}$$

The last equation implies:

$$C^{s*}(v_t - \sigma, -L) - (v_t - \sigma - L) = \frac{\rho}{4}[C^{b*}(v_t, +L) - (v_t - L)] \tag{44}$$

Using (42) and (44), an equation with unknown $C^{b*}(v_t, +L)$ is obtained. Solving for $C^{b*}(v_t, +L)$, yields:

$$C^{b*}(v_t, +L) = v_t + L - \frac{\rho}{4+\rho}(2L) \tag{45}$$

Proceeding in the same way:

$$C^{s*}(v_t, -L) = v_t - L + \frac{\rho}{4+\rho}(2L) \tag{46}$$

Using the expressions for $C^{s*}(., -L)$ and $C^{b*}(., +L)$ given by (45) and (46), one obtains that (40) and (41) are satisfied iff $\frac{L}{1+\rho/4} < \sigma$ as supposed. Consequently, the quotation strategy $A^*(v_t, -L) = C^{b*}(v_t + \sigma, +L)$, $B^*(v_t, +L) = C^{s*}(v_t - \sigma, -L)$, $A^*(v_t, +L) = v_t + L + \sigma$ and $B^*(v_t, -L) = v_t - L - \sigma$ and the associated order type choice strategy is an equilibrium under this condition on the parameters $\{L, \sigma, \rho\}$. Moreover, Equations (45) and (46) can be used to obtain closed forms for the equilibrium ask prices of traders with type y_l and the equilibrium bid prices of traders with type y_h. They can be written as in Proposition 3 after straightforward manipulations. A trader with type y_h chooses a bid price slightly above $C^{s*}(v_t - \sigma, -L)$. Thus the execution probability of his bid price is $\rho/4$ (See Table 2). By symmetry, the execution probability of the ask price posted by a trader with type y_l is also $\rho/4$.

Case 2: $\frac{L}{1+\rho/4} > \sigma$. Conjecture that $C^{s*}(v_t+\sigma, -L) \leq C^{s*}(v_t-\sigma, +L)$ and $C^{b*}(v_t+\sigma, -L) \leq C^{b*}(v_t - \sigma, +L)$ for all possible values for the asset. Under this conjecture, necessary and sufficient conditions for $A^*(v_t, -L) = C^{b*}(v_t - \sigma, +L)$ and $B^*(v_t, +L) = C^{s*}(v_t + \sigma, -L)$ to be optimal in equilibrium are:

$$\frac{1}{4}[v_t + L - \sigma - C^{s*}(v_t - \sigma, -L)] \leq \frac{1}{2}[v_t + L - C^{s*}(v_t + \sigma, -L)] \tag{47}$$

and

$$\frac{1}{4}[C^{b*}(v_t + \sigma, +L) - (v_t + \sigma - L)] \leq \frac{1}{2}[C^{b*}(v_t - \sigma, +L) - (v_t - L)] \tag{48}$$

If these conditions are satisfied, then according to Equations (16) and (17), cutoff prices are given by:

$$v_t + L - C^{b*}(v_t, +L) = \frac{\rho}{2}[v_t + L - C^{s*}(v_t + \sigma, -L)] \tag{49}$$

$$C^{s*}(v_t, -L) - (v_t - L) = \frac{\rho}{2}[C^{b*}(v_t - \sigma, +L) - (v_t - L)] \tag{50}$$

Using the same procedure as in case 1, this system can be solved for the cutoff price functions to get $C^{s*}(v_t, -L) = v_t - L + (2L - \sigma)\frac{\rho}{2+\rho}$ and $C^{b*}(v_t, +L) = v_t + L - (2L - \sigma)\frac{\rho}{2+\rho}$. Since $L > \sigma$, in Case 2, it can be checked that our initial conjecture on cutoff prices in this case is satisfied. Moreover, using closed form solutions for cutoff prices, it turns out that Conditions (47) and (48) are satisfied if $\frac{L}{1+\rho/4} > \sigma$ as supposed. Consequently the quotation strategy: $A^*(v_t, -L) = C^{b*}(v_t - \sigma, +L)$, $B^*(v_t, +L) = C^{s*}(v_t + \sigma, -L)$, $A^*(v_t, +L) = v_t + L + \sigma$ and $B^*(v_t, -L) = v_t - L - \sigma$ and the associated order type choice strategy is an equilibrium under this condition on the parameters $\{L, \sigma, \rho\}$. As in Case 1, the closed form solution for cutoff prices can be used to derive directly the closed form solution for the quotation strategy. Using Table 2 (Case 2), the execution probability of the bid price (ask price) of a trader with type y_h (y_l) is $\rho/2$.

Existence and Uniqueness. Remark that for each case above, the conjecture that cutoff price functions are increasing in v_t is satisfied. On the other hand, there is no set of parameters for which no equilibrium can be obtained. This proves existence. Moreover, for a given set of parameters, it is possible, proceeding as above and by direct computations, to show that no other equilibrium than those derived above, in each case, can be obtained. This proves uniqueness. As the computations are quite long and do not further explain the intuition behind the model, they are omitted.**Q.E.D.**

Proof of Proposition 4. Using the definition of $\tilde{Q}_i$ (given in Section 3), remark that: $|\tilde{Q}_i|$ takes the value +1 if a market order is submitted at time i (Event 1) and 0 if a limit order is submitted at time i (Event 2). It follows that: $\tilde{M}_t = \sum_{i=0}^{i=t-1} |\tilde{Q}_i|$. Then $E(\tilde{m}_{\tilde{T}} \mid \tilde{T} = t) = (\sum_{i=0}^{i=t-1} E(|\tilde{Q}_i|))/(t-1)$, for $t \geq 2$. Call π_{ij} the probability of each of the two previous events ($j = \{1, 2\}$) at time $i \leq t-1$, conditional on the game stopping at time t (note that $\pi_{02} = 1$). It follows that: $\bar{m}_t = (\sum_{i=0}^{i=t-1} \pi_{i1})/(t-1)$ and that:

$$\pi_{i1} = Prob(|Q_i| = +1 \mid \tilde{T} > i, Q_{i-1} = 0) Prob(Q_{i-1} = 0 \mid \tilde{T} > i) +$$

$$Prob(|Q_i| = +1 \mid \tilde{T} > i, |Q_{i-1}| = 1) Prob(|Q_{i-1}| = 1 \mid \tilde{T} > i) \quad \forall i \leq t-1$$

Now remark that, if a market order is placed at time $i-1$, then the book is empty at time i. In this case, no market order can be placed at time i and $Prob(|Q_i| = +1 \mid \tilde{T} > i, |Q_{i-1}| = 1) = 0$. On the other hand, $Prob(Q_{i-1} = 0 \mid \tilde{T} > i) = \pi_{(i-1)2}$. Thus the previous equation is rewritten:

$$\pi_{i1} = Prob(|Q_i| = +1 \mid \tilde{T} > i, Q_{i-1} = 0)\pi_{(i-1)2} \tag{51}$$

$Prob(|Q_i| = +1 \mid \tilde{T} > i, Q_{i-1} = 0)$ is the execution probability of a limit order trader who places limit orders at time $i-1$, conditional on the trading process not being stopped at time i. From Proposition 3, we know that this conditional execution probability is $\frac{1}{2}$ if $\sigma < \bar{\sigma}^e$. Since $\pi_{(i-1)2} = 1 - \pi_{(i-1)1}$, we obtain: $\pi_{i1} = (1/2) - (1/2)\pi_{(i-1)1}$. Straightforward manipulations finally yield:

$$\bar{m}_t = \frac{1}{2} - \frac{1}{2}\frac{t-2}{t-1}\bar{m}_{t-1}$$

Then taking the limit on both side yields $\bar{m} = 1/3$. The same types of computation can be used in the case $\sigma \geq \bar{\sigma}^e$. The only difference is that $Prob(|Q_i| = +1 \mid \tilde{T} > i, Q_{i-1} = 0) = \frac{1}{4}$. One obtains $\bar{m} = \frac{1}{5}$ in this case. The reasoning is the same in the competitive case. The proportion of market orders in this case is $\bar{m} = \frac{3}{7}$ when $0 < \sigma < \sigma^c$ and 1/5 if $\sigma \geq \sigma^c$. **Q.E.D**

Proof of Corollary 2. Consider two levels of volatility σ_h and σ_l with $\sigma_h > \sigma_l$. If σ_h and σ_l are such that $\sigma_h \geq \bar{\sigma}^e > \sigma_l$ then, from Proposition 4, we know that the proportion of limit orders is higher when the volatility is σ_h than when it is σ_l. In all the other cases, the proportion of limit orders is the same when the volatility is σ_h and when it is σ_l. This shows that in equilibrium the proportion of limit orders increases with volatility. The reasoning is exactly the same in the competitive case. **Q.E.D**

Proof of Corollary 3. Remember that each limit order trader posts 2 limit orders (a buy limit and a sell limit). Thus the arrival of l limit orders entails that $l/2$ traders have decided to place limit orders. Using this remark, conditional on the arrival of l limit order traders until the end of the game, the fill rate $\tilde{fr}$ can be written:

$$\tilde{fr} = \sum_{i=1}^{i=\frac{l}{2}} \tilde{F}_i / l$$

with $\tilde{F}_i = 0$ if none of the limit orders placed by the i^{th} limit order trader is executed and $\tilde{F}_i = 1$ if one of the order is executed (the model is such that at most one can be executed).

Consider the case in which $\sigma < \bar{\sigma}^c$. From Proposition 3, we know that each limit order trader has a probability $\rho/2$ of being executed in this case. *Conditional on the game not stopping before the arrival of the next trader*, the execution probability of each trader is then 1/2. Thus $E(\tilde{F}_i \mid \tilde{l} = l, \tilde{T} = t) = \frac{1}{2}$ and $E(\tilde{fr} \mid \tilde{l} = l, \tilde{T} = t) = \frac{1}{4}$. If $\bar{\sigma}^c \leq \sigma$, the execution probability of a limit order is $\rho/4$. Following the same reasoning, we obtain that $E(\tilde{fr} \mid \tilde{l} = l, \tilde{T} = t) = \frac{1}{8}$ in this case. Then we can consider two levels of volatility σ_h and σ_l and proceed as in the previous proof to prove the corollary. The methodology of the proof and the result are the same in the competitive case. **Q.E.D**

Proof of Corollary 4.

In all cases, the n^{th} transaction price can be written $\tilde{P}_n = \tilde{v}_{t(n)-1} + (D - L)\tilde{Q}_n$. The constant D varies with the parameters and has different values in equilibrium and in the competitive case. For instance, in equilibrium, $D = (2L - \sigma)(2/(2+\rho))$ if $\bar{\sigma}^c > \bar{\sigma}$ and $D = \sigma + 8L/(4+\rho)$ otherwise. Thus:

$$\Delta\tilde{P}_n = \tilde{P}_{n+1} - \tilde{P}_n = \tilde{v}_{t(n+1)-1} - \tilde{v}_{t(n)-1} + (D - L)(\tilde{Q}_n - \tilde{Q}_{n+1}) \tag{52}$$

The symmetry of the model when $k = 0.5$ implies that in all cases: $Prob(\tilde{Q}_n = +1) = Prob(\tilde{Q}_n = -1) = 0.5$. Some algebra gives:

$$Var(\Delta\tilde{P}_n) = E(\tilde{t}(n+1) - \tilde{t}(n))\sigma^2 + 2(D-L)^2 - 2E(\tilde{\epsilon}_{t(n)}\tilde{Q}_n) \tag{53}$$

$E(\tilde{t}(n+1) - \tilde{t}(n))$ is the average time between two transactions. It depends on the parameters values since the transaction frequency depends on them. Using the characterization of order placement strategies in equilibrium, the average time between two transactions in equilibrium is shown to be 3 periods if $\bar{\sigma}^c \geq \sigma$ and 5 periods otherwise. Moreover $E(\tilde{\epsilon}_{t(n)}\tilde{Q}_n) = 0.5E(\tilde{\epsilon}_{t(n)} \mid \tilde{Q}_n = +1) - 0.5E(\tilde{\epsilon}_{t(n)} \mid \tilde{Q}_n = -1)$. Now consider the case in which $\sigma \geq \bar{\sigma}^c$. The quotes chosen by the limit order traders in equilibrium are such that a buy (sell) market order is observed only if the asset innovation is positive (negative) (See proof of Proposition 3). This entails: $E(\tilde{\epsilon}_{t(n)}\tilde{Q}_n) = \sigma$ in this case. When $\sigma < \bar{\sigma}^c$, buy (sell) market orders are placed only by traders of type y_h (y_l) whatever the innovation in the asset value. This entails: $E(\tilde{\epsilon}_{t(n)}\tilde{Q}_n) = 0$ in this case.

Here again, consider two levels of volatility σ_h and σ_l with $\sigma_h > \sigma_l$. If $\sigma_h \geq \bar{\sigma}^c > \sigma_l$, using the expressions for D, $E(\epsilon_{t(n)}Q_n)$ and $E(\tilde{t}(n+1) - \tilde{t}(n))$, the variance of transaction prices is greater when the volatility is σ_h than when it is σ_l. It is also the case that the proportion of limit orders is higher when the volatility is σ_h. In the other cases, the variance of transaction prices and the proportion of limit orders when the volatility is high are the same as when the volatility is low. Thus, overall there is a positive relationship between the volatility of transaction prices and the proportion of limit orders. We can proceed in the same way to show that there is a negative relationship between the variance of transaction prices and the fill rate. In the competitive case, the average time between two transactions is 7/3 periods if $\sigma < \bar{\sigma}^c$ and 5 periods otherwise. Moreover $E(\tilde{\epsilon}_{t(n)}\tilde{Q}_n)$ takes the values $\sigma/3$ and σ, respectively, in the different cases for σ. Finally $D = (\sigma/3)$ if $\sigma < \bar{\sigma}^c$ and $D = \sigma$ otherwise. The result is then proved as in the equilibrium case.**Q.E.D**

Proof of Corollary 5. Equations (22) and (23) give the possible sizes for the spread posted by limit order traders. Then we can consider two levels of volatility σ_h and σ_l with $\sigma_h > \sigma_l$ and proceed exactly as in the proof of Corollary 4 to obtain the result.**Q.E.D**

Proof of Proposition 5. Take the expectations in Equations (27) and (28). They depend on $E(\tilde{\epsilon}_{t(n)}\tilde{Q}_n)$. We have shown in the proof of Corollary 4 that $E(\tilde{\epsilon}_{t(n)}\tilde{Q}_n) = \sigma$ if $\sigma \geq \bar{\sigma}^c$ and 0 otherwise. Then it is then straightfoward to obtain the expressions for the expected trading costs. **Q.E.D.**

Proof of Proposition 6. Using Equations (8) and (9), it is obtained that:

$$\frac{\partial A^*(v,-L)}{\partial k} = \frac{\rho}{(1-\rho^2 k(1-k))^2}[1-k\rho+\rho(1-k)(1-(1-k)\rho^2)] \geq 0$$

and

$$\frac{\partial B^*(v,+L)}{\partial k} = \frac{\rho}{(1-\rho^2 k(1-k))^2}[1-\rho+k\rho(2-\rho k)] \geq 0$$

Q.E.D

Proof of Corollary 6. The first part of the proposition is straightforward. On the other hand:

$$\frac{\partial STC}{\partial k} = \frac{2(2-\rho)\rho^2(1-2k)}{(1-\rho^2 k(1-k))^2}$$

This is positive for $k < 0.5$, equal to zero for $k = 0.5$ and negative for $k > 0.5$. Moreover:

$$\frac{\partial^2 STC}{\partial^2 k} = \frac{4(2-\rho)\rho^2}{(1-\rho^2 k(1-k))^2}[\rho^2 k(1-k)(1-2k)-1] < 0$$

which proves the second part of the proposition. **Q.E.D.**

Proof of Proposition 7. It is the case that $E(\tilde{G}) = E(\tilde{G} \mid \tilde{y}_t = +L)\frac{1}{2} + E(\tilde{G} \mid \tilde{y}_t = -L)\frac{1}{2}$. Since the order placement strategies for the type $y_h = +L$ and $y_l = -L$ are symmetric, it turns out that: $E(\tilde{G} \mid \tilde{y}_t = +L) = E(\tilde{G} \mid \tilde{y}_t = -L)$. In order to compute $E(\tilde{G} \mid \tilde{y}_t = +L)$, remark that conditional on the arrival of an agent with type y_h at time t and the fact that there are at least two traders with different types who arrive over $[t, t+2]$, three configurations (with equal probabilities) can be observed for the types of the traders who arrive in the market at times $\{t, t+1, t+2\}$: $\{+L, -L, +L\}$, $\{+L, -L, -L\}$, $\{+L, +L, -L\}$.

Case 1). When $\sigma = 0$ and traders post zero expected profit quotes, they just quote their reservation prices. Then it is immediate that a trade occurs if and only if two successive traders have different types. This entails that for each of the possible configurations, the gains from trade are equal to $2L$ and thus $e^c = 100\%$.
Case 2). When $0 < \sigma < \bar{\sigma}^c$, the buy limit order chosen by a trader with type y_h is such that it is executed by the next trader if this trader has type y_l, whether the asset value increases or not. Thus the expected gains from trade of the two first configurations are $2L$. If the trader who arrives at time $t+1$ has type y_h and the asset value decreases, he picks off the buy limit order of the trader who arrives at time t. This does not create gains from trade. Otherwise he trades with the trader with type y_l who arrives at $t+2$. Thus the expected gains from trade for the last configuration are equal to $0.5 * (2L)$. In this case, we obtain finally: $E(\tilde{G} \mid \tilde{y}_t = +L) = \frac{2}{3}(2L) + \frac{1}{3} * \frac{1}{2}(2L) = \frac{5}{6}(2L)$ and thus $e^c = 5/6 = 83.33\%$.
Case 3. When $\sigma \geq \bar{\sigma}^c$, the buy limit order chosen by a trader with type y_h (y_l) is such that it is executed by the next trader if and only if this trader has type y_l (y_h) and the

asset value decreases (increases). The probability that *no trade* occurs is thus $\frac{1}{4}$ for the first sequence of arrivals and $\frac{1}{2}$ for the two other sequences. Moreover if a trade takes place the gains from trade are $2L$. Thus $E(\tilde{G} \mid \tilde{y}_t = +L) = \frac{1}{3} * \frac{3}{4}(2L) + \frac{1}{3} * \frac{1}{2}(2L) + \frac{1}{3} * \frac{1}{2}(2L) = \frac{7}{12}(2L)$. This gives $e^c = \frac{7}{12} = 58.33\%$.**Q.E.D**

Proof of Proposition 8. We can proceed exactly as in the proof of Proposition 7.**Q.E.D**

Proof of Proposition 9. Suppose $\sigma < \bar{\sigma}^c$ and consider an agent with type y_h who arrives at time t. Let $\pi(y_l)$ be the probability that the book is not empty at time t and that the quotes posted in the book are from a trader with type y_l (Event 1). Using the same methodology as in the proof of Proposition 4, it is obtained that: $\pi(y_l) = 1/3$ (remark that this is the long run probability that the book is not empty $(1 - \bar{m})$ time the probability (0.5) that the limit orders standing in the book have been placed by a trader with type y_l). In this event, since $\sigma < \bar{\sigma}^c$, the trader with type $+L$ will place a buy market order. If the asset value increases between $t-1$ and t, the trader with type y_h obtains an expected gain, with the buy market order, equal to $v_{t-1} + \sigma + L - A^*(v_{t-1}, -L) = 2L/3 + 2\sigma/3 + \sigma$. If the asset value decreases, he obtains an expected gain equal to $v_{t-1} - \sigma + L - A^*(v_{t-1}, -L) = 2L/3 + 2\sigma/3 - \sigma$. On average the expected gain conditional on the placement of a market order for this trader is: $2L/3 + 2\sigma/3$. If the book is empty or if the quotes posted in the book are from a trader with type y_h (Event 2, which occurs with probability $1 - \pi(y_l)$), the trader with type y_h places a limit order. The expected gain with a limit order for this trader is $J(S_t, A^*(v_t, +L), B^*(v_t, +L))$, which is equal to $2L/3 - \frac{\sigma}{3}$ in equilibrium. Finally the expected utility from participating to the trading process for a trader with type y_h in equilibrium is:

$$U^p(y_h) = \pi(y_l) \underbrace{(2L/3 + 2\sigma/3)}_{Gain\ with\ a\ market\ Order} + (1 - \pi(y_l)) \underbrace{(2L/3 - \frac{\sigma}{3})}_{Gain\ with\ limit\ orders}$$

Using $\pi(y_l) = 1/3$, this gives $U^p(y_h) = 2L/3$. Computations for type y_l are symmetric and this gives the first part of the proposition. Computations in the case $\sigma \geq \bar{\sigma}^c$ are similar with the exception that $\pi(y_l) = 1/5$ in equilibrium and that a trader with type y_h submits a market order only if (a) the quotes posted in the book are from a trader with type y_l (probability $\pi(l)$) *and* (b) the asset value increases between $t-1$ and t (probability 0.5). This implies that the probability that he trades with a market order is just 1/10. Finally we obtain $U^p(y_h) = U^p(y_l) = 2L/5$ in this case. **Q.E.D**

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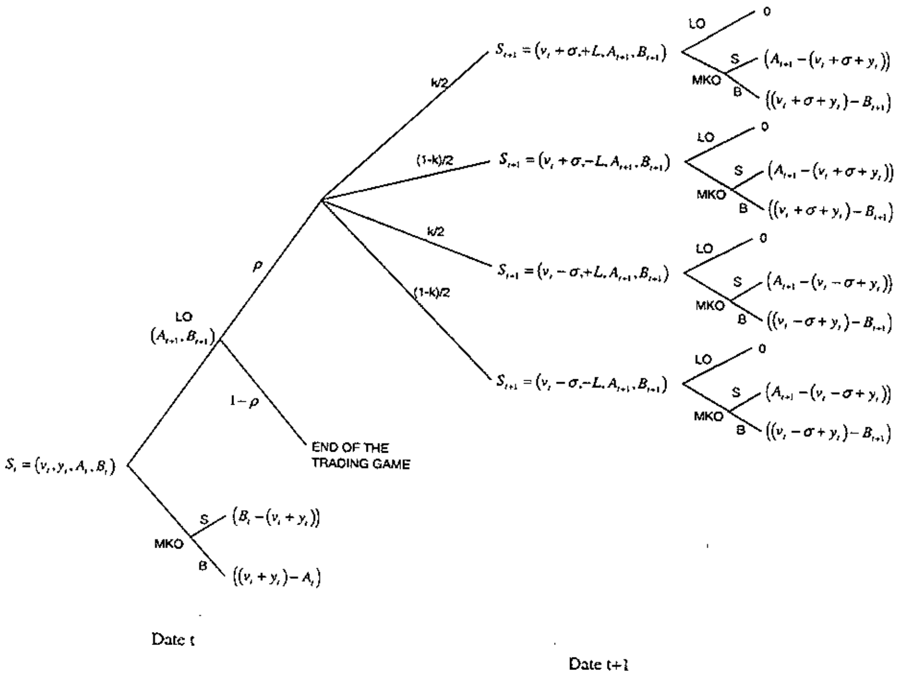
Figure 1



THE ORDER PLACEMENT PROBLEM:

This tree represents the possible trading decisions, at time t, for a trader of type y_t and the possible payoffs for *this trader* according to the decision of the trader arrived at time t+1.

LO = Limit Order
MKO = Market Order
B = Buy S = Sell

Table 1

Buy Limit Orders Execution Probabilities in the competitive case

Case 1: $L < \sigma$

Bid price	Execution Probability
$\leq v_t - \sigma - L$	0
$\in]\, v_t - \sigma - L, v_t - \sigma + L]$	$\rho/4$
$\in]\, v_t - \sigma + L, v_t + \sigma - L]$	$\rho/2$
$\in]\, v_t + \sigma - L, v_t + \sigma + L]$	$3\rho/4$
$> v_t + \sigma + L$	ρ

Case 2: $L \geq \sigma$

Bid price	Execution Probability
$\leq v_t - \sigma - L$	0
$\in]\, v_t - \sigma - L, v_t + \sigma - L]$	$\rho/4$
$\in]\, v_t + \sigma - L, v_t - \sigma + L]$	$\rho/2$
$\in]\, v_t - \sigma + L, v_t + \sigma + L]$	$3\rho/4$
$> v_t + \sigma + L$	ρ

Table 2

Buy Limit Orders Execution Probabilities in equilibrium

Case 1: $L/(1+\rho/4) \leq \sigma$

Bid price	Execution Probability
$\leq C^{s^*}(v_t - \sigma, - L)$	0
$\in]\, C^{s^*}(v_t - \sigma, - L), C^{s^*}(v_t - \sigma, + L)]$	$\rho/4$
$\in]\, C^{s^*}(v_t - \sigma, +L), C^{s^*}(v_t + \sigma, - L)]$	$\rho/2$
$\in]C^{s^*}(v_t + \sigma, - L), C^{s^*}(v_t + \sigma, + L)]$	$3\rho/4$
$> C^{s^*}(v_t + \sigma, + L)$	ρ

Case 2: $L/(1+\rho/4) > \sigma$

Bid price	Execution Probability
$\leq C^{s^*}(v_t - \sigma - L)$	0
$\in]\, C^{s^*}(v_t - \sigma, - L), C^{s^*}(v_t + \sigma, - L)]$	$\rho/4$
$\in]\, C^{s^*}(v_t + \sigma, -L), C^{s^*}(v_t - \sigma, + L)]$	$\rho/2$
$\in]C^{s^*}(v_t - \sigma, + L), C^{s^*}(v_t + \sigma, + L)]$	$3\rho/4$
$> C^{s^*}(v_t + \sigma, + L)$	ρ