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JEL Classification: C72, C78, D74, N43

Keywords: Bargaining, Coalitions, Conflict

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Feudal Political Economy

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February 9, 2023

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1 Introduction

This paper studies the political economy of societies that lack a monopoly of violence. Specifically, we formally characterize the process of coalition formation in societies where power is weakly institutionalized and the capacity for violence is decentralized. We call such societies feudal and refer to the equilibrium of the game that we study as reflecting a distinctly *feudal* political economy.

Traditionally social scientists have presumed the existence of a functioning state; that is, a state defined as a political entity possessing a monopoly of legitimate violence within a given territory. But such states are a recent phenomenon historically speaking (see Strayer, 1970; Tilly, 1975, 1992). Indeed in many parts of the world the state still does not possess a monopoly of legitimate violence.

Societies lacking a Weberian monopoly of violence confront what North, Wallis and Weingast (2009) call "the problem of violence".[1] Indeed, most people have historically lived in societies where there was no monopoly on legitimate violence and in which power was weakly institutionalized. Many people still live in such societies.

To answer the question, "how is political order achieved and maintained in a world where there is no monopoly of violence?" one has to understand the process of coalition formation. For some time, social scientists have appreciated the shortcoming associated with modeling even autocratic states as unitary actors. North, Wallis and Weingast (2009, 17) write:

> Economists and social scientists concerned with understanding how the state develops and interacts with the larger society have modeled the state as a revenue-maximizing monarch, a stationary bandit, or a single-actor "representative agent." By overlooking the reality that all states are organizations, this approach misses how the internal dynamics of relationships among elites within the dominant coalition affect how states interact with the larger society.

Similarly, de Mesquita, Morrow, Siverson and Smith (2003, 7) observe that no ruler rules alone and that "[e]very leader answers to some group that retains her in power: her winning coalition".[2]

[1]The importance of the problem of violence is also central to the analysis of Bates (2001, 2008, 2017).

[2]In the selectorate theory that they propose, the policies a ruler chooses will depend on the size of this winning coalition. Nonetheless, the formation and composition of the ruling coalition remains a blackbox.

We study the process of coalition formation using a framework where violence is possible. In contrast, the existing literature in political science presumes the existence of coalitions and does not shed light on how coalitions are formed. In particular, it does not allow for the role of the threat of violence in coalition formation.

Coalition formation is critical to understanding the political economy of all societies. It is particularly relevant to our understanding of non-democratic societies of the sort that been dominant for most of recorded history. We focus our analysis on those societies where violence capacity is decentralized. We label these societies "feudal," as some of the best documented examples are from medieval Europe and because medieval Europe provides us with a laboratory for better understanding the process of coalition formation among parties with independent military capabilities.[3] Our framework, however, is applicable to other societies both contemporary and historical. For example, Afghanistan is characterized by decentralized violence and an absence of a modern state (Murtazashvili, 2016), as are parts of sub-Saharan Africa (see, for the example of the Congo, Sánchez de la Sierra, 2020).

We construct a bargaining game played by the elites of a realm, e.g. kings, barons or lords, in which a leader proposes an alliance to every other elite. Under an alliance, the elite commits all her resources – economic and military, to the leader's coalition in exchange for a share in the coalition's rents. The commitment, however, is non-binding, as any member of the coalition can rebel by forcibly taking back what she can from her initial contribution. If the elite rejects the proposal, the leader attempts to force the alliance through battle, with the help of some key members of her coalition which the leader forms to fight for the coalition. Borrowing from Ray (2007), we call this the "approval committee". As the game is infinitely repeated, the coalition expands whenever a player joins, peacefully or through conquest, and contracts whenever a player rebels.

In equilibrium, either the realm is consolidated into one grand coalition, or remains fragmented.

[3] A vast historical literature exists on medieval Europe and specifically on the emergence and key features of feudalism. Indeed historians remain conflicted over whether terms such as feudal or feudalism are useful. Specifically, the contractual aspects of feudalism emphasized by Ganshof (1951) and Bloch (1961, 1964) has been criticized by Brown (1974) and Reynonds (1994). For a good survey of the debate see Abels (2009). From our perspective, the term feudalism is valuable in describing a society where military power is decentralized among competing lords but in which there was also a recognize sovereign (who acts as "proposer" in our model).

The key determinants are the size of the individual resources of the elites, the extent to which the resources are immovable – e.g. land vs. soldiers, and the elites' costs of fighting.

When resources are large, consolidation is more likely as there are more rents to distribute which entices elites to join, and stay in, the coalition. The more immovable these resources are, however, the less likely is consolidation. This is because anyone who rebels can only take back a small amount of resources from the coalition. This weakens the threat of rebellion, making the leader more willing to incur the cost of rebellion rather than distribute rents in order to prevent it. Rebellions are thus more likely to occur, and therefore consolidation less likely.

Fighting costs matter, but only that of the weakest. If the weakest elite is not very weak– that is, its fighting cost is not very low, such that the leader wants her to join and stay in the coalition, then the leader will want everyone else who are stronger to do so. Similarly, only the fighting cost of the weakest approval committee among all committees formed over time matter. When this weakest group is not very weak such that the leader is still willing to prevent any rebellion from its members, then she will also prevent rebellion from any stronger set of members. Thus, if even the weakest fighters are valuable to the coalition, there is a single grand coalition that includes all potential members.

Our work adds to the formal literature on coalition formation, surveyed in Ray and Vohra (2015). We borrow heavily from Ray's (2007) proposal-based model of coalition formation with non-binding agreements. Starting from a state in which individual players are fragmented into several coalitions, a player — the proposer, offers to another player — the responder, a new state in which the latter is included in the former's coalition, which the responder can accept or reject. However, any move to a new state, which changes the composition of the coalition, has to be approved by an approval committee, which is a subset of the proposer's coalition. Members of the approval committee may not approve the proposal and exit the coalition. In this manner, their previous agreement with the proposer to join/stay in the coalition is non-binding. Ray (2007), however, does not specify the process of non-approval and exit, thereby leaving it open to particular applications. In modeling coalition formation in the feudal context where violence plays a dominant role, we interpret exit as rebellion and, in addition, include the possibility of future agreements being forged through conquest. More generally, then, our model is one of coalition formation with violent entry and exit.

A related model is Acemoglu, Egorov and Sonin (2008) who analyze the stability of coalitions in non-democracies where there are no institutions that assign political power but, rather, individuals are endowed with political power and are free to combine their endowments by forming coalitions. In this general setting, the authors derive conditions under which coalitions are self-enforcing.

A large literature on state formation studies the rise of modern states after 1500 (e.g. Tilly, 1975, 1992; Ertman, 1997); and particularly the role of war in this development (see summaries in Dincecco and Onorato, 2015; Johnson and Koyama, 2017). One element of this "bellicist" approach, stressing the granting of taxation for protection was formalized by Bates and Lien (1985). Gennaioli and Voth (2015) develop a model to analyze the role played by military technology in intensifying inter-state conflict and encouraging domestic state building. Empirically, Becker et al. (2020) test the causal impact of warfare on fiscal capacity and political representation.

Within economics and political science a smaller number of papers have explored the distinctive political economy of medieval states. Chaney and Blaydes (2013) document a divergence in the duration of rule in European and the Middle East after 800 CE and they attribute this to the greater stability provided by feudal institutions, particularly those that encouraged bargains between powerful nobles and the monarch.[4] Jia et al. (2021) model the different power structures in premodern China and Europe. Leon (2020) develops a model to explain the size of a ruler's coalition in medieval England and characterize the conditions under which the king will expand his coalition. Acharya and Lee (2018) model the role of economic development in the formation of the European state system. When commerce is underdeveloped, the value of governance is low, and there is no incentive for territorial states to emerge. Territoriality emerges when there are overlapping markets for protection.[5]

[4]For Chaney and Blaydes (2013), the rise of feudalism has implications for the divergence between Europe and the Middle East. In contrast to Western Europe, Islamic states came to rely on slave soldiers. Landlords were alienated from political power as a consequence. Levels of political stability in these two regions of the world thus diverged centuries prior to the divergence in per capita income (Blaydes, 2017).

[5]Another relevant perspective stresses the extent to which the feudal political order was a polycentric one. Volckart (2000, 2002) draws attention to the costs of overlapping and fragmented political order, as does the recent literature on state capacity (Johnson and Koyama, 2017). On the other side of the ledger, Salter and Young (2019) contend that medieval polities were successful to the extent that they aligned the incentives of landowning elites.

Lastly, the literature on conflict studies the reasons for war. On the face of it, war should not occur between rational agents who can negotiate an efficient agreement that avoids any deadweight loss.[6] To explain open conflict, therefore, scholars have pointed to information asymmetries, indivisibilities, commitment problems, and agency problems (Acemoglu, 2003; Jackson and Morelli, 2011). In our model there is complete information, but conflict arises because agreements among coalition members are non-binding: the rents each coalition members receive depend on the composition of coalition which is always subject to change.

The paper is structured as follows. Section 2 characterizes feudal societies and outlines how this was a response to the pervasive problem of violence and introduces the relevant historical background. Two motivating examples are early medieval France and England after the Norman conquest. Section 3 presents the model. Section 5 discusses historical applications while Section 4 the main results. Section 6 concludes.

2 Feudal Coalitions in Medieval Europe

This section describes the key features of the feudal world, features that we seek to capture in the model introduced in Section 3.

The history of Europe from c. 500 CE to 1500 CE was dominated by polities governed by alliances or coalitions among military elites, forged through either war or peaceful means. This coalition-based power structure was precarious and often unstable — elites could move in and out of the ruling coalition. These societies were broadly speaking feudal where feudal is roughly defined as corresponding to governance structures that comprised of alliances forged by mutual legal and military obligations but which were also hierarchical, e.g., there was a king and that king could allocate the resources of the realm, both productive (land) and military; but the lords had their own military forces and hence the power to approve or rebel against the king.

To retain power, a ruler had to maintain a coalition of the major landlords within his territory. This coalition could be continuously changing and a ruler had to be prepared to use violence to

Political rights in medieval societies were bundled with property rights and that consequentially, medieval lords were incentivized to pursue policies that were beneficial to development because they had property rights in their realms. In contrast to governors in centralized empires, they had the political rights to bargain with their sovereigns and to hold them to account.

[6] This is referred to as the "Hicks Paradox" (see Ferández-Villaverde et al., 2023).

maintain his coalition. Political order in this environment rested not on formal institutions but on coalitions between individuals who could mobilize violence. The one long-lasting institution in this period was the Church (see Johnson and Koyama, 2019; Grzymala-Busse, 2020). But in every other respect power was not institutionalized but personal.

These characteristics were the product of European history and they distinguish medieval European polities from other parts of Eurasia. Following the fall of the Western Roman in the 5th century CE, Europe fragmented into many separate kingdoms (Scheidel, 2019; Ferández-Villaverde et al., 2023). Whereas the Roman empire had possessed both a professional army and bureaucracy funded by a centralized fiscal system, its successor kingdoms lacked both of these crucial features. This transformation was complete by 600 CE (Wickham, 2005, 2009).

In the wake of this transformation, military power became significantly more decentralized. The core military resources of the successor kingdoms comprised the personal retinue or *comitatus* of the king. Major landowners formed similar bands of armed retainers. In a world of decentralized violence capabilities, larger polities only formed when ruler were successful in maintaining the loyalty of these landowners.

We further motivate our analysis by considering two feudal polities: early medieval France and Norman England.

Early Medieval France By the late 5th century, Roman power had disintegrated in Northern Gaul. In its place, various warlords, Gallo-Roman aristocrats and Roman generals had established their own petty kingdoms.[7] Among these peoples were the Franks, and a particular sub-tribe, the Salian Franks based in modern Belgium. [8]

Clovis became the leader of the Salian Franks in 482. Beginning with a small number of followers, Clovis sequentially united the various Frankish tribes, and through conquest or alliances consolidated his control over almost the entirety of Roman Gaul. But the coalition he built was transient. His successors controlled smaller territories and over time, political authority was

[7]These included Aegidius and his son Syargius at Soissons, Arbogast at Gaul; Britons fleeing Irish and Saxon invasions had settled in Armorica—what is now Brittany; and several different Germanic peoples occupied other territories (see Dam, 2005). See also Wallace-Hadrill (1982, pp. 159-160); James (1982, pp. 26-28).

[8]Their leader Childeric, like earlier Frankish war leaders, served in the Roman army, fighting under the command of Aegidius during the 460s (James, 1982, 80).

increasingly localized. These centrifugal tendencies were arrested by the rise to power in Francia of the Carolingian dynasty. This period saw major attempts to restore centralized political authority (in addition to territorial expansion) (Collins, 1998; McKitterick, 2008; Wickham, 2009). But it was also relatively short-lived. External threats and internal conflict resulted in the widespread breakdown of political order by the 9th century (see, discussion in Ko et al., 2018, 304-305). The following period saw further decentralization, a period labeled by some as "the feudal revolution" (see the discussions in Barthélemy and White, 1996; Bisson, 1997; Reuter and Wickham, 1997).[9] In the kingdom of the Franks, the authority of the king was restricted to a small area around Paris and local lords entrenched their power (Bisson, 2009).

The resulting political order was one in which authority was local and personal. Centralized power fell to a low ebb. George Duby (1981) emphasized the privatization of justice. Strayer (1970) writes of the absence of the state. Bisson (2009, 27) writes that "[r]oyal order was seldom centralized order". For Hintze (1906, 1975, 192), feudal polities were not states because their rulers 'lacked the attributes of sovereignty—that is, independence beyond its borders and exclusive rights within them". Instead power rested on coalitions. Local lords fought, made peace, married, allied with one another, before falling out and fighting again. Describing 11th century Normandy, Barlow (2000, 6-7) notes that "This bald account of the rise and fall of a feudal principality suppresses the incessant, and to us bewildering, diplomacy and military campaigns which were necessary for its continuing existence. Each ruler competed with the others to construct a superior network of alliances. Princes sought for patrons among the greater powers . . . They had also to make or threaten war against rebel and rivals: there was no court to which they could effectively appeal for the protection of their property".

Norman England England after 1066 was a core consolidated realm than France. Nonetheless, though the kings of England were comparatively powerful, their power rested on their ability to maintain their coalition of lords, each of whom possessed their own lands, castles, and military resources.

[9] Much of the scholarly debate evolves the timing of the feudal revolution and a discussion of the degree to which the experience of northern Francia can be generalized to other parts of Europe. This issues, while important, are not relevant to our analysis.

William the Conquerer (r. 1066-1087) radically changed the distribution and nature of land ownership in England following the Norman Conquest in 1066. He made himself the ultimate lord of all land in the country which was held in fief from him. The majority of the Anglo-Saxon nobility lost their land and were replaced with nobles from Normandy, men who had served and fought with William.

This structure would characterize England's political economy for the remainder of the Middle Ages. The king was the most powerful landowner in country and as feudal overlord, he possessed numerous other rights (in particular over the Royal forests). But he possessed no standing army. Rather beyond his own household knights, he relied on the armed forces of his lords.[10]

The upside of this was that the king's military power and his ability to govern rested on his nobility. These nobles "did not represent sectors of society but pursued their own interests and those of their followers". "Politics was personal, not structural" (Bartlett, 2000, 28). William retained the ability to expropriate or redistribute the land of any of his lords (feudal tenure was not yet secure). There was no rule of law even for elites.[11] Nor was there a codified rules of succession or an institution like a parliament to act as a coordination device. On his death, William I chose to pass England to his second son William "Rufus" (r. 1087-1100). Large-scale rebellions greeted Rufus on his ascension. He defeated the rebellious lords and a prospective invasion from Normandy threatened by his brother. But his rule remained insecure.

In all respects, therefore, Norman England thus remains a "fragile natural state" governed by a fairly loose coalition. Powerful lords had to be coopted through the promise of land and rents. Civil War and rebellion took place between 1139-1154. Major baronial rebellions reoccurred in 1172, 1215, and 1258-1265. Violent rebellions by dissatisfied lords continued to the major source of political instability until the Tudor's consolidated power in the 16th century by effectively outlawing private armies (see Greif and Rubin, 2020). Figure 1 records every year in which there was battle or significant armed conflict within England due to either civil war or minor rebellions. By our estimation, there was at least one significant armed conflict in 14 % of the time between

[10]Note that taxation did not play an important role in Norman England. The right to levy taxes had been established in Anglo-Saxon England as a means of providing defense against Viking attacks but it was allowed to lapse by Henry II in the 12th century.

[11]See discussion in North et al. (2009).

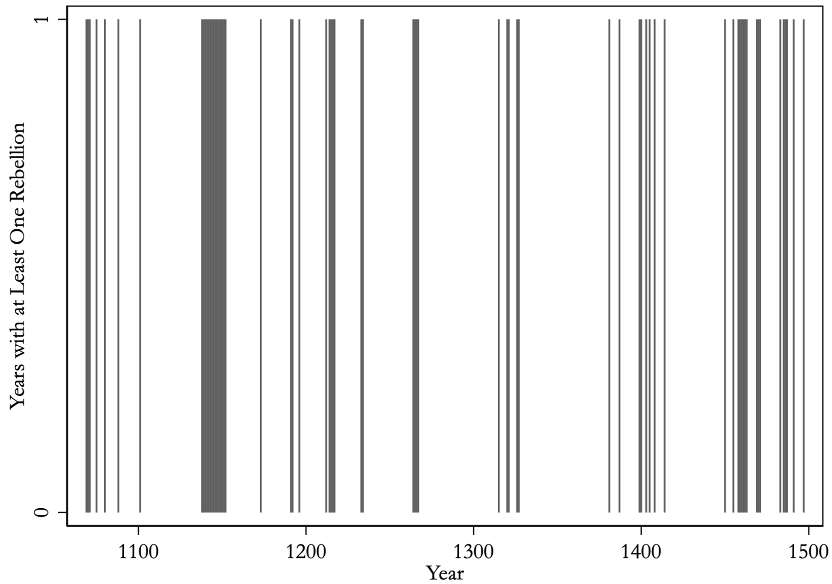

Figure 1: Years during which there was at least one major violent (political) rebellion in England. Data Construction: See main text. We exclude rebellions and wars in Wales, Scotland or in France.

1066 and 1500. If we also include other moments of political conflict and crisis including the purging of major lords or political conflict that did not result in a battle, this increases to 18% of the time. If we exclude the periods when England was at war in France or Scotland, these proportions increase to 20% and 25% respectively.[12]

In the next section we introduce a model of coalition formation in the feudal world, in which alliances, battles, and rebellions all occur.

3 Coalition Formation with Violent Entry and Exit

Motivated by the above examples, we now introduce a formal model of coalition formation with violent entry (conquest) and exit (rebellion).

There is a population $N = \{i\}$ of size N. Each member i of N has, at each time period t, resources e_i, which includes all productive assets, e.g. land, labor, capital, and those used for

[12]We collected this data from several sources including Allmand (1992); Prestwich (1988, 1997); Barlow (1999); Bartlett (2000); Hollister (2001); Carpenter (2003); Rubin (2005); Phillips (2011) which we then cross-checked with Wikipedia.

protection, e.g. military force.[13] We construct a game that describes how, starting from an initial state at $t = 0$ in which each i is its own (singleton) coalition, this coalition structure evolves into one in which there is a single, i.e. grand, coalition, or otherwise into other 'fragmented' structures in which the grand coalition fails to form. Each coalition at t is worth the sum of the resources of the members at t, and a member gets a fraction $\alpha_{i,t} \in (0, 1)$ of that sum.

Suppose, then, that starting from initial state s_0 at time $t = 0$ in which there are N singleton coalitions, the following events occur at each subsequent time period $t = 1, 2, \ldots \infty$:

A pair of players (j, k) is randomly drawn from N, and j makes a proposal to k. Specifically, proposer j offers to responder k a state in which k would join j's existing coalition ω_{t-1} to form coalition $\omega_t = \{\omega_{t-1} \cup k\}$.[14] Under this proposal, k would contribute her own resources e_k to the coalition and, in exchange, would get a share $\alpha_{k,t}$ of the total resources of the coalition at each time period that k remains in the coalition. k can therefore either accept the proposal and be peacefully included in j's coalition, or reject it, in which case j wages battle against k to try to include k by conquest.

Before k decides, however, j forms an approval committee $A_t \subseteq \omega_{t-1}$ from her existing coalition whose unanimous support determines whether or not the proposal is enforced. In particular, should k reject the proposal, j would need the help of A_t to successfully conquer k — that is, a player i that is in A_t would have to incur fighting cost c_i. The total fighting cost of the coalition required to conquer k at t is thus $C_t = \sum c_i \mathbb{1}_{\mathbb{A}_t}$ (where $\mathbb{1}_{\mathbb{A}_t}$ indicates membership in A_t). However, any member of A_t can refuse by rebelling against j. In this case, the rebels exit the coalition, each taking back her own resources, but incurring rebellion cost $r_i \mathbb{1}_{\mathbb{R},t}$ (where $\mathbb{1}_{\mathbb{R},t}$ indicates membership in the set of rebels $R_t \subseteq A_t$). We interpret r_i as a measure of the immovability of i's resources e_i – the higher it is, the less resources i can take back from the coalition. Now if any member of the approval committee rebels, j would then be unable to cover fighting costs C_t and therefore unable to conquer k. In contrast, if every member of A_t supported j, j would be able to conquer k with some probability p which, for simplicity and without loss of generality, we assume to be equal to

[13]We capture in a simple way a low-productivity economy in which any growth in incomes simply replenishes depleted resources. Thus, e_i is maintained at each t.

[14]We specify set union, rather than $\omega_t = \{\omega_{t-1}, k\}$ to capture the possibility that k is already in ω_{t-1}, in which case the proposal is a chance for k to re-affirm its membership in j's coalition.

1.[15]

Faced with j's proposal, and knowing the identities of the members of the approval committee and thus whether each would approve or rebel, k accepts or rejects j's offer, in which case k fights j in battle and incurs cost of fighting c_k.

3.1 The Feudal (Bargaining) Game

Note that the sequence of events previously described assumes perfect information. In choosing a particular proposal and the committee which would approve it, j effectively offers a state to k that would induce either loyalty or rebellion among all or some members of j's approval committee. Since k also knows whether each committee members would support j or rebel, k knows whether or not he would be conquered by j in battle, or probabilistically in case j mixes between inducing loyalty and rebellion (see below). Note, then, that the model does not need to assume incomplete or asymmetric information in order to generate equilibria in which (violent) rebellions and wars occur.

To demonstrate in a simple way, we reduce the events to the following infinite-horizon bargaining game with war and rebellion. At each $t = 1, 2, \dots, \infty$:

1. A pair of players (j, k) is randomly drawn from N, with j the proposer, and k the responder. More specifically, $(j, k) = (a, k)$ where k is randomly drawn (with replacement) from $N \setminus a$ and a is randomly drawn (with replacement) from N and thereafter fixed until all members of $N \setminus a$ have been drawn to play at least once.[16]

2. j proposes to k a state that is either acceptable to the approval committee or not. In other words, j offers a proposal that induces either loyalty or rebellion among members of ω_{t-1}. Denote as L the state that gives the maximum payoff to j under loyalty, and R the state that gives the maximum payoff to j under rebellion. Then j chooses L or R.

3. k either accepts (A) or rejects the offer, in which case she fights (F).

[15]We let $p = 1$ to abstract from exogenous factors and fully endogenize the probability of conquering k, which only depends on whether A_t is loyal or whether there is some rebellion. Note that while $p = 1$ ex ante, it can effectively be less than one ex post. That is, in equilibrium, k may not be conquered with certainty if j mixes actions. See subsequent discussion and results.

[16]Since a and k are randomly drawn with replacement, the time horizon of any proposer and any responder is in effect infinite.

In restricting the proposal of j to L and R, we focus on characterizing equilibria in which either loyalty is induced or rebellion occurs, and are able to abstract from the exact process by which approval committees are formed. To make more tractable the inherent optimization problem of obtaining L and R, we make the following assumption.

Assumption 1 *There exists, for each i, a 'reservation' share $\underline{\alpha}_i$ such that:*

$$\alpha_{i,t} = \begin{cases} [\underline{\alpha}_i, 1] & \textit{if } i \textit{ is in } \omega_t \\ 0 & \textit{otherwise} \end{cases}$$

That is, i has to receive a share of the coalition's resources of at least α_i, below which i will always refuse to join, and therefore get none of those resources. We can think of $\underline{\alpha}_i$ as capturing i's ego, which is distinct from her desire for rents per se. A responder, for instance, that only considers joining a coalition of which she gets at least fifty percent of the pie has more ego than another who considers joining when she gets at least ten percent, irrespective of the size of the pie.

The assumption implies that in equilibrium, $\alpha_{i,t} = \underline{\alpha_i}$ if i is in j's coalition, as this would always give the maximum payoff to j if i is in the coalition, whether or not there is rebellion. This also implies that j gets the 'residual' share – 1 minus the sum of the shares of the members of the coalition.

There are thus four types of states that can be implemented, depending on the chosen actions of j and k. Denote as s_{12} the type of state that is implemented if j chooses $1 = \{L, R\}$ and k chooses $2 = \{A, F\}$. State s_{LA} is the type implemented when j proposes to include k in the coalition that results in payoffs that would not induce a rebellion, which k accepts. Thus, s_{LA} is the type of state in which k is in j's coalition and obtains fraction $\alpha_{k,t}$ of the resources of the coalition at each time t that she remains therein. State s_{LF} is implemented when j proposes to include k in the coalition that results in payoffs, net of fighting costs, that would not induce a rebellion, but which k rejects. Thus, s_{LF} is the type of state in which k incurs fighting cost c_k but is in j's coalition (by conquest) and therefore obtains fraction $\alpha_{k,t}$ of the resources of the coalition at each time t that she remains therein. Note that coalition resources have been reduced temporarily, i.e. at the time of conquest, by the coalition's fighting costs. State s_{RA} is implemented when j proposes to include k in the coalition that results in payoffs that would induce rebellion, which k accepts. Thus, s_{RA} is the type of state in which k is included in j's coalition while the rebels are excluded, and therefore k obtains fraction $\alpha_{k,t}$ of coalition's resources at each time t that she remains therein. Coalition resources have been reduced temporarily, i.e. at the time of rebellion, by the sum of the individual resources of the rebels, net of the rebellion costs they incurred at the time of their exit. (Each rebel takes back her own resources, except what is immovable). Finally, state s_{RF} is implemented when j proposes to include k in the coalition that results in payoffs that would induce rebellion, which k rejects. Thus, s_{RF} is the type of state in which k is excluded in j's coalition and does not share in the coalition's resources, which has been temporarily reduced at the time of rebellion by the sum of the individual resources of the rebels, except what is immovable. Instead, k keeps her own resources, which has been temporarily reduced by her own fighting costs at the time of fighting. (Each rebel takes back her own resources, except what is immovable, incurred at the time of her exit).

The players of this game are thus $\{i\}$, where i can be a proposer j or responder k at a particular state s. Thus, to construct a strategy profile, we specify player i's actions as a proposer and as a responder at s. As proposer, $i = j$ chooses either L or R at s. Let $\mu_j(s)$ denote the probability that the proposer chooses L given s. As responder, $i = k$ chooses either A or F at s. Let $\lambda_k(s)$ be the probability that a responder chooses A given s. A strategy profile $\sigma = \{(\mu_j, \lambda_k)\}_i$ is a collection of pairs of proposer-responder actions over all i. A strategy profile is defined for s, and induces the following expected payoffs for each player $i = \{j, k\}$:

$$V^k(\mu_j, \lambda_k = 1, s) = (1-\delta)u^k(s) + \delta\Big[\mu_j V^k(\mu_j = 1, \lambda_k = 1, s_{LA}) + (1-\mu_j)V^k(\mu_j = 0, \lambda_k = 1, s_{RA})\Big] \tag{1}$$

$$V^k(\mu_j, \lambda_k = 0, s) = (1-\delta)u^k(s) + \delta\Big[\mu_j V^k(\mu_j = 1, \lambda_k = 0, s_{LF}) + (1-\mu_j)V^k(\mu_j = 0, \lambda_k = 0, s_{RF})\Big] \tag{2}$$

$$V^j(\mu_j,\lambda_k,s) = (1-\delta)u^j(s) + \delta\Big[\mu_j\Big(\lambda_k V^j(\mu_j=1,\lambda_k=1,s_{LA}) + (1-\lambda_k)V^j(\mu_j=1,\lambda_k=0,s_{LF})\Big) + (1-\mu_j)\Big(\lambda_k V^j(\mu_j=0,\lambda_k=1,s_{RA}) + (1-\lambda_k)V^j(\mu_j=0,\lambda_k=0,s_{RF})\Big)\Big], \tag{3}$$

where δ is the discount rate and $u^k(s)$, $u^j(s)$ denote one-period payoffs.

The feudal game is thus patterned after Ray's (2007) proposal-based model of coalition formation in which there is a finite set of players, a compact set of states, an infinite time horizon, an initial state, a protocol describing the proposer and order of respondents at each time period, subsets of players that can approve the move from each state to another, and for each player, a continuous one-period payoff function and discount factor common across each players. However, two things are notably different. One is that we give the proposer the option to deliberately choose an approval committee that does not approve and instead rebels. The other is that the proposer wages war against the responder if the latter rejects the proposal.

3.2 The Feudal Political Economy (FPE) Equilibrium

We now define equilibria in the feudal game. To do so, we first define a particular type of pair of proposer-responder actions for $i = \{j,k\}$.

Definition 1 *The pair* (μ_j,λ_k) *of proposer-responder actions for* $i = \{j,k\}$ *is an* ***optimal action pair*** *if:* $\lambda_k = 1$ *if* $V^k(\mu_j,\lambda_k=1,s) > V^k(\mu_j,\lambda_k=0,s)$*, equals* 0 *if the opposite inequality holds, and lies in* $[0,1]$ *if equality holds;* $\mu_j = \arg\max V^j(\mu_j,\lambda_k,s)$.[17]

One can then define equilibria in terms of optimal action pairs:

Definition 2 *A strategy profile* $\sigma = \{(\mu_j,\lambda_k)\}_i$ *is a* ***Feudal Political Economy (FPE) equilibrium*** *if for each* $i = \{j,k\}$*,* (μ_j,λ_k) *is an optimal action pair.*

[17] Note, then, that if $\lambda_k = 1$, $\mu_j = 1$ maximizes $V^j(\cdot)$ if $V^j(\mu_j=1,\lambda_k=1,s_{LA}) > V^j(\mu_j=0,\lambda_k=1,s_{RA})$, 0 if the opposite inequality holds, and lies in $[0,1]$ if equality holds. Analogously, if $\lambda_k = 0$, $\mu_j = 0$ maximizes $V^j(\cdot)$ if $V^j(\mu_j=1,\lambda_k=0,s_{LF}) > V^j(\mu_j=0,\lambda_k=0,s_{RF})$, 0 if the opposite inequality holds, and lies in $[0,1]$ if equality holds. Note that if λ_k lies in $[0,1]$, the value of μ_j that maximizes $V^j(\cdot)$ may be 1 or 0, or may lie in $[0,1]$. For instance, if $V^j(\mu_j=1,\lambda_k=1,s_{LA}) > V^j(\mu_j=0,\lambda_k=1,s_{RA})$ and $V^j(\mu_j=1,\lambda_k=0,s_{LF}) > V^j(\mu_j=0,\lambda_k=0,s_{RF}$, then $\mu_j = 1$ maximizes $V^j(\cdot)$.

Thus, to construct an FPE equilibrium, one needs to construct optimal action pairs. For this purpose, we elaborate on the expected payoffs.

From Definition 1, we know that if a proposer-responder action pair is optimal, then $\lambda_k = 1$ when $V^k(\mu_j, \lambda_k = 1, s) > V^k(\mu_j, \lambda_k = 0, s)$. From (1) and (2), the latter condition is more likely to hold when the differences between $V^k(\mu_j = 1, \lambda_k = 1, s_{LA})$ and $V^k(\mu_j = 1, \lambda_k = 0, s_{LF})$, and between $V^k(\mu_j = 0, \lambda_k = 1, s_{RA})$ and $V^k(\mu_j = 0, \lambda_k = 0, s_{RF})$, are large.

We first look at the difference between $V^k(\mu_j = 0, \lambda_k = 1, s_{RA})$ and $V^k(\mu_j = 0, \lambda_k = 0, s_{RF})$. Without loss of generality, let $j = 1$ and let responders be drawn to play sequentially, i.e. $k = 2$ at $t = 1$, $k = 3$ at $t = 2$, etc.

For responder k drawn to play at $t = 1$, one can construct $V^k(\mu_j = 1, \lambda_k = 1, s_{LA}) = u^k(s_{LA})_0 + \delta u^k(s_{LA})_1 + \delta^2 u^k(s_{LA})_2 + \delta^3 u^k(s_{LA})_3 + \ldots$ or, letting $j = 1$ and $k = 2$:

$$V^2(\mu_1 = 1, \lambda_2 = 1, s_{LA}) = e_2 + \delta\alpha_{2,1}(e_1+e_2) + \delta^2\alpha_{2,2}(e_1+e_2+e_3) + \delta^3\alpha_{2,3}(e_1+e_2+e_3+e_4) + \ldots, \quad (4)$$

where e_2 is $k = 2$'s resources which she owns entirely prior to joining $j = 1$'s coalition, $\alpha_{2,1}$ is 2's share of the coalition's resources at $t = 1$, which is the sum of 1 and 2's resources, and $\alpha_{2,2}, \alpha_{2,3}, \ldots$ are analogously defined. The coalition's resources grow with each draw of responder since under state s_{LA}, each responder drawn joins the coalition (peacefully).

Now, under state s_{LF}, each responder drawn to play fights with j, but is conquered because j induces loyalty among the approval committee. Thus, each responder enters the coalition, but bears cost of fighting c_k. That is, its resources shrink by amount c_k at the period of joining. Cost c_k is temporary, and k's resources are replenished and grows back to e_k by the start of the next period.

A member of the approval committee also incurs cost of fighting $c_i \mathbb{1}_{A,t}$, where $\mathbb{1}_{A,t}$ is an indicator variable equal to one if the player is in the approval committee at t. The coalition fighting cost needed to conquer k at t is thus $C_t = \sum c_i \mathbb{1}_{A,t}$.[18]

Thus, for responder k drawn to play at $t = 1$, one can construct $V^k(\mu_j = 1, \lambda_k = 0, s_{LF}) = u^k(s_{LF})_0 + \delta u^k(s_{LF})_1 + \delta^2 u^k(s_{LF})_2 + \delta^3 u^k(s_{LF})_3 + \ldots$ or, letting $j = 1$ and $k = 2$:

[18] That c_i is fixed per period is without loss of generality – what matters is total cost of fighting C_t against k which varies by period. Thus, how C_t is shared by the coalition members is also immaterial. C_t is large, for instance, when the approval committee has many members with large individual fighting costs.)

$$
\begin{aligned}
V^2(\mu_1 = 1, \lambda_2 = 0, s_{LF}) &= e_2 + \delta\alpha_{2,1}\Big((e_1 - c_1) + (e_2 - c_2)\Big) \\
&+ \delta^2\alpha_{2,2}\Big((e_1 - c_1\mathbb{1}_{A,2}) + (e_2 - c_2\mathbb{1}_{A,2}) + (e_3 - c_3)\Big) \\
&+ \delta^3\alpha_{2,3}\Big((e_1 - c_2\mathbb{1}_{A,3}) + (e_2 - c_2\mathbb{1}_{A,3}) + (e_3 - c_3\mathbb{1}_{A,3}) \\
&+ (e_4 - c_4)\Big) + \ldots,
\end{aligned} \tag{5}
$$

where note that at the start of the first period, $j = 1$ is necessarily in the approval committee as she is the only member of the coalition at $t = 0$. Thus, she always incurs fighting cost in the first period if she goes to war with k. Also note that the responder always incurs fighting costs at the time of joining the coalition — at $t = 1$, $k = 2$ bears cost c_2, at $t = 2$, $k = 3$ bears cost c_3, etc.

With non-zero coalition fighting costs, (4) is always greater than (5). Thus, all else equal, the greater the (positive) difference between (4) and (5), the more likely it is that $k = 2$ chooses $\lambda_2 = 1$.

We next look at the difference between $V^k(\mu_j = 0, \lambda_k = 1, s_{RA})$ and $V^k(\mu_j = 0, \lambda_k = 0, s_{RF})$. One can construct $V^k(\mu_j = 0, \lambda_k = 1, s_{RA}) = u^k(s_{RA})_0 + \delta u^k(s_{RA})_1 + \delta^2 u^k(s_{RA})_2 + \delta^3 u^k(s_{RA})_3 + \ldots$ or, letting $j = 1$ and $k = 2$:

$$
\begin{aligned}
V^2(\mu_1 = 0, \lambda_2 = 1, s_{RA}) &= e_2 + \delta\alpha_{2,1}\Big(e_1 + e_2 - (e_1 - r_1)\Big) \\
&+ \delta^2\alpha_{2,2}\Big(e_1 + e_2 + e_3 - (e_1 - r_1)\mathbb{1}_{R,2} - (e_2 - r_2)\mathbb{1}_{R,2}\Big) \\
&+ \delta^3\alpha_{2,3}\Big(e_1 + e_2 + e_3 + e_4 - (e_1 - r_1)\mathbb{1}_{R,3} - (e_2 - r_2)\mathbb{1}_{R,3} - (e_3 - r_3)\mathbb{1}_{R,3}\Big) \\
&+ \ldots,
\end{aligned} \tag{6}
$$

where r_i is i's cost of rebellion, $\mathbb{1}_{R,t}$ an indicator variable equal to one if i rebels from the coalition at t, with $R_t \subseteq A_t$ denoting the set of approval committee members who rebel. Like the cost of fighting, r_i is temporary and is thus only incurred at the time of rebellion. Thus, the resources of a rebel shrink at the time of rebellion but is fully replenished at the start of the next period. A rebel then takes from the coalition $e_i - r_i$ at the time of rebellion. One can then interpret r_i as the immovable portion of e_i that i cannot take back from the coalition. Note that under state s_{RA}, only $j = 1$ rebels in the first period, taking away $e_1 - r_1$ from the coalition. (Similarly, rebellion cost r_j captures the immovable portion of e_j that j cannot take away from the coalition.) At any time period thereafter, any member of the approval committee can rebel, which excludes the new respondent who peacefully accepts the proposal.

Lastly, one can construct $V^k(\mu_j = 0, \lambda_k = 0, s_{RF}) = u^k(s_{RF})_0 + \delta u^k(s_{RF})_1 + \delta^2 u^k(s_{RF})_2 + \delta^3 u^k(s_{RF})_3 + \dots$ or, letting $j = 1$ and $k = 2$:

$$V^2(\mu_1 = 0, \lambda_2 = 0, s_{RF}) = e_2 + \delta(e_2 - c_2) + \delta^2 e_2 + \delta^3 e_2 + \dots, \tag{7}$$

where $k = 2$ incurs temporary fighting cost at $t = 1$, i.e. when she is drawn to play and fights with $j = 1$. Thereafter, she keeps her entire resources e_2 since she is outside the coalition.

All else equal, the greater the (positive) difference between (6) and (7), the more likely it is that $k = 2$ chooses $\lambda_2 = 1$. Note that (6) is not always larger than (7), but a positive and large difference becomes more likely, when $k = 2$'s resources are small, coalition members' resources are large, and the costs of rebellion are large.

Finally, note that equations (4) to (7) are generalizable to any responder k, and for any order of responders. (One simply changes notation – superscript 2 in $V^2(\cdot)$ and subscript 2 in $\{\alpha_{2,t}\}$ to any k, and the subscripts for the other variables can be easily changed to reflect the order of responders. Similarly for any j, and any order in which j is drawn.)

Next, recall that an optimal action pair also requires $\mu_j = \arg\max V^j(\mu_j, \lambda_k, s)$. We then elaborate on $V^j(\mu_j, \lambda_k, s)$. First, we construct, for $j = 1$ playing at $t = 1$, and assuming a sequential draw of responders, i.e. $k = 2$ at $t = 1$, $k = 3$ at $t = 2$, etc, the following:

$$V^1(\mu_1 = 1, \lambda_2 = 1, s_{LA}) = e_1 + \delta\alpha_{1,1}(e_1 + e_2) + \delta^2\alpha_{1,2}(e_1 + e_2 + e_3) + \delta^3\alpha_{1,3}(e_1 + e_2 + e_3 + e_4) + \dots \tag{8}$$

$$
\begin{aligned}
V^1(\mu_1 = 0, \lambda_2 = 1, s_{RA}) &= e_1 + \delta\alpha_{1,1}\Big(e_1 + e_2 - (e_1 - r_1)\Big) \\
&+ \delta^2\alpha_{1,2}\Big(e_1 + e_2 + e_3 - (e_1 - r_1)\mathbb{1}_{R,2} - (e_1 - r_1)\mathbb{1}_{R,2}\Big) \\
&+ \delta^3\alpha_{1,3}\Big(e_1 + e_2 + e_3 + e_4 - (e_1 - r_1)\mathbb{1}_{R,3} - (e_2 - r_2)\mathbb{1}_{R,3} - (e_3 - r_3)\mathbb{1}_{R,3}\Big) \\
&+ \dots,
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
V^1(\mu_1 = 1, \lambda_2 = 0, s_{LF}) &= e_1 + \delta\alpha_{1,1}\Big((e_1 - c_1) + (e_2 - c_2)\Big) \\
&+ \delta^2\alpha_{1,2}\Big((e_1 - c_1\mathbb{1}_{A,2}) + (e_2 - c_2\mathbb{1}_{A,2}) + (e_3 - c_3)\Big) \\
&+ \delta^3\alpha_{1,3}\Big((e_1 - c_1\mathbb{1}_{A,3}) + (e_2 - c_2\mathbb{1}_{A,3}) + (e_3 - c_3\mathbb{1}_{A,3}) + (e_4 - c_4)\Big) + \dots,
\end{aligned} \tag{10}
$$

$$
V^1(\mu_1 = 0, \lambda_2 = 1, s_{RF}) = e_1 + \delta(e_1 - r_1 - c_1) + \delta^2(e_1 - r_1 - c_1) + \delta^3(e_1 - r_1 - c_1) + \dots \tag{11}
$$

Now suppose $k = 2$ were to accept $j = 1$'s proposal. If $\lambda_2 = 1$, (3) implies that $j = 1$ would choose $\mu_1 = 1$ if $V^1(\mu_1 = 1, \lambda_2 = 1, s_{LA}) > V^1(\mu_1 = 0, \lambda_2 = 1, s_{RA})$; $\mu_1 = 0$ if the reverse inequality holds, and $\mu_1 \in [0, 1]$ if equality holds. Because of non-zero costs of rebellion, (8) is always greater than (9).

Now suppose that $k = 2$ were to reject $j = 1$'s proposal. If $\lambda_2 = 0$, (3) implies that $j = 1$ would choose $\mu_1 = 1$ if $V^1(\mu_1 = 1, \lambda_2 = 0, s_{LF} > V^1(\mu_1 = 0, \lambda_2 = 0, s_{RF}$; $\mu_1 = 0$ if the reverse inequality holds, and $\mu_1 \in [0, 1]$ if equality holds. We thus compare (10) and (11).

Note that (10) may be less than (11), but a positive and large difference becomes more likely, if $j = 1$'s resources are small; its costs of fighting and of rebellion are large; coalition members' resources are large and their costs of fighting small.

Finally, note that equations (8) to (11) are easily generalize to any j and any order of responder, simply by changing the relevant superscript and subscripts.

We can now characterize an optimal action pair, using the following sets of thresholds, conditions, and cut-off points.

Definition 3 *The* ***responder threshold*** *is a vector of (minimum) values* $\{\underline{e_j}_k, \{\underline{e_k}_k\}, \underline{r_j}_k, \{\underline{r_k}_k\}, \underline{c_k}_k\} \equiv \{e_j, \{e_k\}, r_j, \{r_k\}, c_k\}$ *such that, given state* s *and* $\{\alpha_{k,t}\}$, $V^k(\mu_j = 0, \lambda_k = 1, s_{RA}) = V^k(\mu_j = 0, \lambda_k = 0, s_{RF})$.

That is, at the responder threshold, the responder is indifferent between accepting or rejecting the proposal, given that there will be rebellion.

Definition 4 *The* ***proposer threshold*** *is a vector of (maximum) values* $\{\{\bar{C}_t\}_j, \bar{c}_{kj}\} \equiv \{\{C_t\}, c_k\}$ *and (minimum) value* $\underline{r_j}_j \equiv r_j$ *such that, given* s *and* $\{\alpha_{j,t}\}$, $V^j(\mu_j = 1, \lambda_k = 0, s_{LF}) = V^j(\mu_j = 0, \lambda_k = 1, s_{RF})$.

That is, at the proposer threshold, the proposer is indifferent between inducing loyalty or rebellion, given that the responder will fight.

Key to obtaining an optimal action is whether or not these thresholds are met. Consider conditions (a) and (b) below:

Definition 5 ***Condition (a)*** *is met if every element in* $\{e_j, \{e_k\}, r_j, \{r_k\}, c_k\}$ *is greater than or equal to its respective threshold value in* $\{\underline{e_j}_k, \{\underline{e_k}_k\}, \underline{r_j}_k, \{\underline{r_k}_k\}, \underline{c_k}_k\}$.

Definition 6 ***Condition (b)*** *is met if every element in* $\{\{C_t\}, c_k\}$ *is less than or equal to its respective threshold value in* $\{\{\bar{C}_t\}_j, \bar{c}_{kj}\}$, *and* r_j *is greater than or equal to threshold value* $\underline{r_j}_j$.

If conditions (a) and (b) are not met, then the following cut-off points become relevant.

Definition 7 *Suppose that* $V^k(\cdot, s_{RA}) < V^k(\cdot, s_{RF})$. *Then the* ***responder's cut-off*** *is* $\underline{\mu_j} \in \{\mathbb{R} > 0\} \equiv \mu_j$ *such that*

$$\begin{aligned}\mu_j V^k(\mu_j = 1, \lambda_k = 1, s_{LA}) + (1-\mu_j)V^k(\mu_j = 0, \lambda_k = 1, s_{RA}) &= \mu_j V^k(\mu_j = 1, \lambda_k = 0, s_{LF}) \\ &\quad + (1-\mu_j)V^k(\mu_j = 0, \lambda_k = 0, s_{RF}).\end{aligned}$$

Definition 8 *Suppose that* $V^j(\cdot, s_{LF}) < V^j(\cdot, s_{RF})$. *Then the* ***proposer's cut-off*** *is* $\underline{\lambda_k} \in \{\mathbb{R} > 0\} \equiv \lambda_k$ *such that*

$$\begin{aligned}\lambda_k V^j(\mu_j = 1, \lambda_k = 1, s_{LA}) + (1-\lambda_k)V^j(\mu_j = 1, \lambda_k = 0, s_{LF}) &= \lambda_k V^j(\mu_j = 0, \lambda_k = 1, s_{RA}) \\ &+ (1-\lambda_k)V^j(\mu_j = 0, \lambda_k = 0, s_{RF}).\end{aligned}$$

Using the above definitions, one can then construct an optimal action pair from the following lemma.

Lemma 1 *A pair* (μ_j, λ_k) *of proposer-responder actions for* $i = \{j, k\}$ *is an optimal action pair if:*

1. $\lambda_k = 1$ *if condition (a) holds. If (a) does not hold:*

$$\lambda_k = \begin{cases} 1 & \text{if } \mu_j > \underline{\mu_j} \\ [0,1] & \text{if } \mu_j = \underline{\mu_j} \\ 0 & \text{if } \mu_j < \underline{\mu_j} \end{cases}$$

2. $\mu_j = 1$ *if condition (b) holds. If (b) does not hold:*

$$\mu_j = \begin{cases} 1 & \text{if } \lambda_k > \underline{\lambda_k} \\ [0,1] & \text{if } \lambda_k = \underline{\lambda_k} \\ 0 & \text{if } \lambda_k < \underline{\lambda_k}, \end{cases}$$

Proof All proofs are in the Appendix.

One can also refine the FPE equilibrium using optimal action pairs. For reasons that will be obvious in Section 4 – when we derive conditions under which alliances are made and the realm is consolidated, one can consider FPE equilibria in which there is only one optimal action pair for each player. That is, all players are associated with the same optimal action pair, such that the equilibrium, in this specific sense, is "player-proof".

Definition 9 *An FPE equilibrium is* ***player-proof*** *if the optimal action pair* (μ_j, λ_k) *for* $i = \{j, k\}$ *is the same for all* i*.*

While restrictive, player-proof equilibria can serve as benchmark – as we show in Section 4, they can approximately describe the type of polity that is generated by the feudal game.

Lastly, we define a particular type of FPE equilibrium in which all players, when playing as proposer, chooses $\mu_j = 1$. In this equilibrium, no rebellion can occur, which means all respondents join the coalition, whether peacefully or by conquest. Because this equilibrium is characterized by full entry and no exit of players into j's coalition, it gives rise to a consolidated realm.

Definition 10 *An FPE equilibrium is a* ***consolidation equilibrium*** *if the optimal action pair for all* $i = \{j, k\}$ *is* $(\mu_j = 1, \lambda_k)$.

It follows that if for some i, the optimal action pair is $(\mu_j \neq 1, \lambda_k)$, then the equilibrium is not a consolidation equilibrium. Thus, the set of all FPE equilibria consists of the set of consolidation, and the set of non-consolidation, equilibria. Any player-proof equilibrium is either a consolidation or a non-consolidation equilibrium.

4 Alliances and Consolidation

We can now apply the equilibrium concepts in the previous section to answer questions of interest about the feudal world. First, under what set of conditions does a responder k ally with proposer j, and is the alliance peacefull or achieved through violent conquest? Second, what determines the likelihood that a realm consolidates or remains fragmented?

The key variables are resources, the extent to which these resources are immovable, the costs of fighting of approval committees and of reponders.

We formally establish in Theorem 1 that k is more likely to ally with j when the resources of j and all other respondents – actual and potential coalition members, are large and mostly immovable, as this makes the alliance valuable to k. It deters k from fighting and induces her to accept the proposal peacefully. If resources are immovable, the alliance is formed even if some members of j's approval committee rebel. In this case, j would be more likely to let the rebellion occur since rebels cannot take much away from the coalition, but since k is likely to join peacefully, the alliance between j and k is formed, in spite of any rebellion from other members.

Theorem 1 also shows that an alliance is more likely to occur when j's approval committee is good at fighting. In this case, they are less likely to back out or rebel if j calls them to battle; they are also likely to deter k from fighting. Thus, if the approval committee has low fighting costs, the alliance is likely to be made, whether peacefully or by conquest. However, the fighting cost of the responder has an ambiguous effect on the probability of alliance. On the one hand, a responder that is weak is easy to conquer, which could even deter the responder from fighting. Thus, whether peacefully or by conquest, a weak responder may likely join the coalition. On the other hand, the prospect of having a weak member join the coalition and share in its resources could induce rebellion. Thus, a weak responder may avoid getting conquered and may be more likely to remain outside the coalition.

Taking into account all possible alliances, and all possible approval committees that can be formed as the coalition changes with the entry and exit of members, we then analyze the equilibrium in which every member joins, and remains, in a single, grand, coalition. The same variables determine the likelihood of such consolidation, albeit in a different way. This is because the variables not only affect entry in the coalition, or the likelihood that an alliance is made, but also the possibility of exit, or the likelihood of rebellion.

Theorem 2 establishes that large resources make consolidation more likely – large rents are better able to attract and to keep members in the coalition. However, the more immovable they are, the *less* likely consolidation occurs. This is because immovable resources may make joining the coalition attractive, but it also makes rebellion easier (recall Theorem 1). In this case, j is more likely to allow rebellion since it is less harmful to j – rebels can only take back a small amount of resources, and can easily be enticed back into the coalition because immovable resources make the coalition attractive to outsiders.

Theorem 2 also shows that costs of fighting affect the likelihood of consolidation, but the only costs that matter are those of the weakest responder, and of the weakest approval committee that can be formed. The smaller these costs are, the more likely is consolidation. If the worst fighters j can assemble as approval committee are not very weak, then j will always conquer every responder, and everyone eventually joins the coalition. If the worst responder is not too weak such that she is worth keeping in the coalition, then everyone else is valuable and it is always worth preventing rebellion.

Notice that with the exception of the size of resources, which always makes joining and remaining in the coalition attractive, and therefore increases the probability of alliance formation and consolidation, the effect of the other variables are more nuanced and cannot be readily deduced. For this reason, we carefully show the logical progression towards Theorem 1 and Theorem 2 by including other minor results in the following subsections.

4.1 Pairwise Alliance

We first derive conditions under which alliances and describe the nature of the alliances made in equilibrium. To do this, the following result groups optimal action pairs into four types.

Lemma 2 *There exist only four types of optimal action pairs, each obtained by four sets of conditions. For any $i = \{j, k\}$, the optimal action pair (μ_j, λ_k) is determined by the following:*

1. *If* ***condition (a) holds and $\underline{\lambda_k} < 1$, or condition (b) holds and $\underline{\mu_j} < 1$, or both (a) and (b) hold****, then $(\mu_j = 1, \lambda_k = 1)$.*
2. *If* ***condition (a) does not hold, condition (b) holds, and $\underline{\mu_j} \geq 1$****, then $(\mu_j = 1, \lambda_k \in [0, 1))$.*
3. *If* ***condition (a) holds, condition (b) does not hold, and $\underline{\lambda_k} \geq 1$****, then $(\mu_j \in [0, 1), \lambda_k = 1)$.*
4. *If* ***condition (a) does not hold and condition (b) does not hold****, then $(\mu_j = 0, \lambda_k = 0)$.*

In turn, the conditions in Lemma 2 are affected by some key variables, to wit:

Corollary 1 *The following variables determine the likelihood that condition (a) and condition (b) are met, and whether $\underline{\lambda_k} \geq 1$ and $\underline{\mu_j} \geq 1$: $e_j, \{e_k\}, r_j, \{r_k\}, c_k, \{C_t\}$. Specifically:*

1. ***Resources of j, e_j, and of each k, $\{e_k\}$****:*

   The larger e_j is, the more likely that (a) is met, and the less likely that $\underline{\mu_j} \geq 1$. The larger $\{e_k\}$ are, the more likely that (a) is met, the less likely that $\underline{\lambda_k} \geq 1$, and the less likely that $\underline{\mu_j} \geq 1$.
2. ***Immovability of j's resources, r_j, and of each of k's, $\{r_k\}$, from the coalition****:*

   The larger r_j is, the more likely that (a) is met, the more likely that (b) is met, the more likely that $\underline{\lambda_k} \geq 1$, and the less likely that $\underline{\mu_j} \geq 1$. The larger $\{r_k\}$ are, the more likely that (a) is met, the more likely that $\underline{\lambda_k} \geq 1$, and the less likely that $\underline{\mu_k} \geq 1$.
3. ***Cost of fighting of k, c_k, and of the coalition at each t, $\{C_t\}$****:*

   The larger c_k is, the more likely that (a) is met, the less likely that (b) is met, the more likely *that $\underline{\lambda_k} \geq 1$, and the less likely that $\underline{\mu_j} \geq 1$. The larger $\{C_t\}$ are, the less likely that (b) is met, the more likely that $\underline{\lambda_k} \geq 1$, and the less likely that $\underline{\mu_j} \geq 1$.*

The following table summarizes the effect of each variable on the likelihood that each key restriction is met:

	condition (a)	*condition (b)*	$\underline{\lambda_k} \geq 1$	$\underline{\mu_j} \geq 1$
e_j	$\uparrow$			$\downarrow$
$\{e_k\}$	$\uparrow$		$\downarrow$	$\downarrow$
r_j	$\uparrow$	$\uparrow$	$\uparrow$	$\downarrow$
$\{r_k\}$	$\uparrow$		$\uparrow$	$\downarrow$
c_k	$\uparrow$	$\downarrow$	$\uparrow$	$\downarrow$
$\{C_t\}$		$\downarrow$	$\uparrow$	$\downarrow$

We now use the above results to assess the likelihood that a pair of players, when drawn to play, successfully form an alliance. To do this, recall that the FPE equilibrium entails that all players choose optimal action pairs. Thus, in equilibrium, the pairwise outcome – the actions chosen by a randomly drawn pair of players, is also determined by the same sets of conditions that determine the optimal action pair. There are thus four types of pairwise outcomes, corresponding to each set of conditions. That is, denoting a pairwise outcome as $[\mu_j, \lambda_k]$ (to distinguish it from optimal action pair (μ_j, λ_k)), there are also four types:

Proposition 1 ***Pairwise Outcomes***

In equilibrium, the outcome from any pairwise play can be any of the following:

1. ***Peaceful Alliance*** *between j and k, i.e. $[\mu_j = 1, \lambda_k = 1]$ requires* ***only condition (a) to hold and $\underline{\lambda_k} < 1$, or only condition (b) to hold and $\underline{\mu_j} < 1$, or both (a) and (b) to hold.***

2. ***Alliance by Conquest*** *of k by j, i.e. $[\mu_j = 1, \lambda_k \in [0, 1)]$ requires* ***condition (a) not to hold, condition (b) to hold, and $\underline{\mu_j} \geq 1$.***

3. ***Alliance with Unrest*** *in which k accepts to join j's coalition, but some members rebel i.e. $[\mu_j \in [0, 1), \lambda_k = 1]$, requires* ***condition (a) to hold, condition (b) not to hold, and $\underline{\lambda_k} \geq 1$.***

4. ***No Alliance*** *between j and k, i.e. $[\mu_j = 0, \lambda_k = 0]$, requires* ***condition (a) and condition (b) not to hold****.*

It follows from Lemma 2 and Corollary 1 that the the size of resources, the extent of their immovability, and the costs of fighting also affect the likelihood that any of the four pairwise outcomes is obtained from any pairwise play. Thus:

Theorem 1 *The following variables determine the likelihood of each type of pairwise outcome from any pairwise play in equilibrium: $e_j, \{e_k\}, r_j, \{r_k\}, c_k, \{C_t\}$.*

Specifically:

1. ***Resources of j, e_j, and of each k, $\{e_k\}$:***

   e_j increases the likelihood of peaceful alliance and of alliance with unrest, and decreases that of alliance by conquest and of no alliance.

   $\{e_k\}$ increase the likelihood of peaceful alliance, decrease that of alliance by conquest and of no alliance, and have an ambiguous effect on the likelihood of alliance with unrest.

2. ***Immovability from the coalition of j's assets, r_j, and of each of k's, $\{r_k\}$:***

   r_j increases the likelihood of peaceful alliance, decreases that of no alliance, and has ambiguous effects on the likelihood of alliance with unrest and of alliance by conquest.

   $\{r_k\}$ increase the likelihood of peaceful alliance and of alliance with unrest, and decrease that of alliance by conquest and of no alliance.

3. ***Cost of fighting of k, c_k, and of the coalition, $\{C_t\}$:***

   c_k decreases the likelihood of alliance by conquest, increases that of alliance with unrest, and has ambiguous effects on the likelihood of peaceful alliance and of no alliance.

   $\{C_t\}$ decrease the likelihood of peaceful alliance and of alliance by conquest, and increases that of alliance with unrest and of no alliance.

The following table summarizes the effect of each variable on the likelihood of each type of pairwise outcome:

	Peaceful Alliance $[\mu_j = 1, \lambda_k = 1]$	*Alliance by Conquest* $[\mu_j = 1, \lambda_k \in [0,1)]$	*Alliance with Unrest* $[\mu_j \in [0,1), \lambda_k = 1]$	*No Alliance* $[\mu_j = 0, \lambda_k = 0]$
e_j	$\uparrow$	$\downarrow$	$\uparrow$	$\downarrow$
$\{e_k\}$	$\uparrow$	$\downarrow$	$\uparrow / \downarrow$	$\downarrow$
r_j	$\uparrow$	$\uparrow / \downarrow$	$\uparrow / \downarrow$	$\downarrow$
$\{r_k\}$	$\uparrow$	$\downarrow$	$\uparrow$	$\downarrow$
c_k	$\uparrow / \downarrow$	$\downarrow$	$\uparrow$	$\uparrow / \downarrow$
$\{C_t\}$	$\downarrow$	$\downarrow$	$\uparrow$	$\uparrow$

Generally, large and immovable resources make it more likely that accepting j's proposal is the dominant action for k, but makes rebellion more likely since it is less costly for j when rebels can only take a small part of (large) total resources. This makes peaceful alliances and alliances with unrest (rebellion) more likely, and alliance by conquest – which requires zero rebellion, less likely. If the approval committee is strong – its fighting costs small, j is more likely to want to keep them loyal, which increases the likelihood of alliances by conquest and decreases alliance with unrest. (A strong approval committee can also deter k from fighting which increases the likelihood of peaceful alliance.) When responder k has low fighting costs, she may be more likely to fight, which may decrease the probability of a peaceful alliance. A strong responder also induces j to keep the approval committee loyal, thereby decreasing the likelihood of alliance with unrest and increasing that of alliance by conquest.

4.2 Consolidation

Our other main result is concerned with the likelihood of consolidation, and for this we establish conditions under which a consolidation equilibrium is obtained. Since some consolidation equilibria are player-proof, it is useful to derive these more restrictive type of equilibria. They serve as benchmark equilibria that can approximate empirical patterns observed in feudal polities. To see this, note that Lemma 2 implies that there only four types of player-proof equilibria. Each type is obtained whenever the same set of conditions in Lemma 2 hold for every $i = \{j, k\}$. When the first set of conditions holds for all $i = \{j, k\}$, then all proposers choose $\mu_j = 1$ and all responders choose $\lambda_k = 1$. We call this player-proof equilibrium as one of peaceful consolidation, since all proposals are accepted without going to battle, and no one rebels from the coalition. When the second set of conditions hold for all $i = \{j, k\}$, the all proposers choose $\mu_j = 1$ and all responders choose

$\lambda_k \in [0,1)$. In this case, there is always some probability of fighting, but the coalition always wins since approval committees are always loyal. We call this player-proof equilibrium consolidation by conquest. When the third set of conditions hold for all $i = \{j,k\}$, then all proposers choose $\mu_j \in [0,1)$ and all responders choose $\lambda_k = 1$. Every responder (peacefully) joins the coalition, but because there is always some probability of rebellion, this player-proof equilibrium describes a fragmented polity. Lastly, when the fourth set of conditions holds for all $i = \{j,k\}$, then all proposers choose $\mu_j = 0$ and all responders choose $\lambda_k = 0$. No alliance is ever made, and each player remains its own singleton coalition. In other words, this player-proof equilibrium describes independent territories.

Proposition 2 ***Player-proof equilibria***

There exist only four types of player-proof equilibria:

1. ***Peaceful Consolidation****, in which the optimal action pair for all* $i = \{j,k\}$ *is* $(\mu_j = 1, \lambda_k = 1)$.
2. ***Consolidation by Conquest****, in which the optimal action pair for all* $i = \{j,k\}$ *is* $(\mu_j = 1, \lambda_k \in [0,1))$.
3. ***Fragmented Polity****, in which the optimal action pair for all* $i = \{j,k\}$ *is* $(\mu_j \in [0,1), \lambda_k = 1)$.
4. ***Independent Territories****, in which the optimal action pair for all* $i = \jmath, k\}$ *is* $(\mu_j = 0, \lambda_k = 0)$.

Note then that two types of player-proof equilibria — peaceful consolidation and consolidation by conquest, describe a consolidated realm. There are, of course, many non-player proof equilibria, as there is no reason why the same set of conditions in Lemma 2 should hold for all $i = \{j,k\}$. However, some non-player proof equilibria can also give rise to a consolidated realm. In particular, any equilibrium in which the optimal action pair for some players is $(\mu_j = 1, \lambda_k = 1)$, and $(\mu_j = 1, \lambda_k \in [0,1))$ is one in which no rebellion ever occurs. Thus, even if a responder rejects the proposal, she still ends up in the coalition after being conquered. There is full entry into, and no exit from, the coalition.

The following result establishes the necessary and sufficient condition that generates a consolidation equilibrium.

Proposition 3 ***Consolidation Equilibria***

A consolidation equilibrium is obtained if, for all $i = \{j, k\}$, either of the following is true:

1. *condition (b) holds, or*
2. *if (b) does not hold, $\underline{\lambda_k} < 1$.*

In turn, whether these conditions are met depends on the same key variables that determine the type of optimal action pairs and pairwise outcomes. These variables, therefore, affect the likelihood of consolidation. The precise manner is established in our second main result:

Theorem 2 ***Likelihood of Consolidation***

Denote as c_i^* the largest (individual) cost of fighting among all $i \in N$, and C_t^* the largest element in $\{C_t\}$ and, thus, the largest fighting cost among all approval committees that can be formed. Then the likelihood that a consolidation equilibrium is obtained increases with $\{e_i\}$ and decreases with c_i^*, C_t^*, and $\{r_i\}$.[19]

The intuition follows mostly from Theorem 1, to the extent that the variables affect both entry and exit from the coalition. However, since we now consider the entire game, and not just a a single pairwise play, we can generalize to all players by considering, e.g. resources of all players. Large resources, which increases the total rents that can be distributed among coalition members, increases the likelihood of entry and decreases the likelihood of exit (rebellion). Thus, resources increase the likelihood of consolidation. If these resources are largely immovable, however, rebellion becomes less costly to any j. Since any j is now more likely to put forth proposals to responders that can induce rebellion, there is less likelihood of consolidation. Similarly, large fighting costs

[19]Strictly speaking, the likelihood is decreasing in the resource-immovabilities of responders, $\{r_k\}$, but the resource-immovability of any proposer, r_j, has a non-monotonic effect. Specifically, there exist thresholds $r_{jj}^0 < r_{jj}^*$ such that the likelihood of consolidation is decreasing in $r_j \in [0, r_{jj}^0)$, increasing in $r_1 \in [r_{jj}^0, r_{jj}^*]$, and constant in $r_j \in (r_{jj}^*, \infty)$. (See the proof of Theorem 2 in the Appendix for details). That we establish that the likelihood is decreasing in $\{r_i\}$ implies that we consider, for any proposer j, only the region $r_j \in [0, r_{jj}^0)$.

decrease the likelihood of consolidation because while weak fighters are easy to conquer, making entry more likely, they are also easy to let go from the coalition, making exit more likely.

Note, however, that while the entire vector of player resources (and extent of immovability) in the realm determines the likelihood of consolidation, in terms of costs of fighting, only the largest costs matter. That is, Theorem 2 implies the following.

Corollary 2 ***The Weakest Link***

To determine the likelihood of consolidation, one considers the worst, and not the best, members of the realm. In particular, the realm is ***likely to consolidate*** *if it is incentive compatible for the proposer to have* ***the worst fighter join, and the worst approval committee stay****, in the coalition.*

The intuition is simple but powerful – when any leader (proposer) is willing to share rents to keep even the least (militarily) valuable members, then she would be willing to do so for everyone else. The result then implies that even if the rest of the members are just marginally better at fighting, consolidation is still likely. That is, not everyone in the coalition has to be a strong warrior.

In contrast, every player's movable resources contribute to the likelihood of consolidation in that they determine each player's net gain from keeping the coalition. Resources, and therefore rents, cumulate, and so every contribution is relevant to the decision to enter and remain in the coalition. The movability of resources also cumulate in the sense that greater total movability can encourage more members to rebel, increasing the cost of rebellion to any proposer and induces her to allocate rents in order to prevent it. In equilibrium, exit is less likely. One can therefore expect consolidated realms to have large, movable, total resources.

5 Applying the Model

We have explored how a feudal world, the most fundamental elements — resources and military abilities, determine whether alliances were made and territories consolidated. In particular, we shown that the *movability* of resources, that is, the ability of a potential rebel to withdraw resources for the coalition is of critical importance of determining the likelihood of consolidation. We now provide a discussion of how these results can be used to understand patterns of consolidation, conquest, and rebellion in the medieval world.

Our model is quite general, nonetheless it captures several key characteristics of the feudal world. First, politics was personal in the medieval world. The interests of feudal barons did not reflect the interests of the broader population. There were no broader institutions to secure political representation until the emergence of parliaments in the 13th century.

Second, politics was coalitional and personal. States, as we understand them, were weak or nonexistent. The power of the king relative to his most powerful lords was weak: the king ruled in conjunction with his leading lords, on whose combined military power, he depended. Kings governed by making bargains with the most important and powerful lords. As Bartlett (2000, 29) writes: "The court consisted of a group of people enjoying patronage, hoping for it or losing it. It was a constantly changing and competitive environment—'stable only in its mobility."'

Third, the members of the king's coalition were required to support him militarily and to attend court. Our concept of an "approval committee" thus has a direct historical analogue in the king's court. Indeed, "refusal to attend was a defiance that could not be ignored". For example, "in 1095 Robert de Mowbray, earl of Northumberland, 'was unwilling to come to court'. The King was furious and sent messengers to the earl, commanding him to attend his Whitsun court, 'if he wished to be in the king's peace"' (Bartlett, 2000, 29). When de Mowbray failed to attend, the King launched a campaign against him and dispossessed and imprisoned him. Lords who failed to attend a military summons by their king could be similarly punished.

Finally, the predictions of the model can also inform our understanding of the historical record as we can now briefly illustrate. Resources e_i can be understood as primarily referring to the agricultural resources available to each lord. Ownership of land was of fundamental importance in the feudal world. e_i reflects both land quality and access to specific natural resources as well as other location fundamentals such as a costal or riverine location. It also reflects the observability of agricultural output. As argued by Mayshar et al. (2017), Scott (2017), and Mayshar et al. (2022), crops such as wheat that cannot easily be hidden are easier for elites to expropriate. Innovations that improve agricultural productivity, such as the adaption of the iron plough in Northern Europe

c. 800-1200 CE (Andersen et al., 2016), increase e_i and by our model should have led to great levels of consolidation among elites.

The military capabilities of each elite c_i are also important variables. Specifically, the military capabilities of the weakest member of the elite plays a critical role in determining whether or not there is full consolidation. We can understand the distribution of c_i as being partly determined by the existing military technology.

A venerable historical tradition has long associated feudalism with military technologies that favored landed elites such as the stirrup and with economic developments that ensured that central states remained weak (and unable to raise substantial taxes) (e.g. Beeler, 1971, 9-10).[20] That is, the available military and economic technology ensured that the distribution of military capabilities was highly egalitarian. Particularly in the 11th and 12th centuries, the state of military technology favored defense. A single well-fortified lord could resist a king with a much larger army for many months (therefore forcing him to spend tremendous resources in a long siege). Theorem 2 can be interpreted as predicting that consolidation into a single feudal coalition is more likely when the military capabilities of even the weakest elite member are sufficiently great.

Technological innovations like the trebuchet introduced at the end of the 12th century shifted the balance of power towards besiegers (Gravett, 1990, 49-51). But trebuchets required trained engineers and were expensive to construct, so this innovation disproportionately benefited kings or the largest feudal magnates. Our model predicts that, counter-intuitively, such innovations would increase the number of rebellions. The logic is simple: technologies that strengthened the power of the king would encourage him to propose less favorable deals to other elites, generating more rebellions in equilibrium. King John (r. 1199-1216) was the first English king to deploy trebuchets on a large scale and he faced major rebellions during his reign.

Finally, the degree to which elites can costlessly withdraw their resources from the coalition, $r_j, \{r_k\}$ is of critical importance in the model. We refer to this as the movability of resources.

Unfortified or hard to defend land can be thought of as highly immovable. In contrast, investments or technologies that enabled lords to defend their possession, such as castles, can be

[20] This thesis is most strongly associated with White Jr. (1962). For more nuanced modern perspectives see Bachrach and Bachrach (2017).

thought of as increasing movability (because they allow a lord who rebels to retain more of their resources). Our model predicts that a proliferation of castle building should be associated with less consolidation.

This discussion indicates that the assumptions and modeling choices that we have made have a high degree of correspondence and indeed verisimilitude to the feudal world that we have set out to analyze and that our model can generate testable predictions that be assessed against the historical record. Future work can hopefully explore these predictions more systematically.

6 Concluding Comments

In this paper we depict a political economy in which there is no state with a monopoly of violence. Rather, there are elites endowed with resources, both economic and military, who can form alliances in a peaceful way or through battles and conquest. Alliances are also non-binding in that parties can rebel. This political economy captures key features of the feudal world and is relevant for thinking about political order in situations or anarchy or fragmented and weak states.

In such an environment, we derive conditions under which alliances are stable and eventually lead to a consolidated realm, in which all the elites belong to, and stay in, one grand coalition. Consolidation is more likely when elites' resources are large and mostly movable, and when the weakest fighters can remain in the coalition. In contrast, when resources are largely immovable, and if even the elites who are worst at fighting can afford to stay independent, the realm remains fragmented.

Our theoretical analysis has shown that it matters because it makes rebellion less costly to the leader — when members rebel, they can only take back a small part of resources. The leader is then more likely to let these members rebel. In contrast, when resources are movable, any rebel can take back a large amount of resources. This makes rebellion costly to the leader and induces her to allocate rents in a manner that prevents rebellion. With movable resources, then, rebellion is less likely, and consolidation more likely. We leave empirical conceptualizations and measurements of the immovability of resources for future work. Other factors that were of importance in the feudal world, notably the role played by the Church in legitimating political authority (see Rubin, 2017; Johnson and Koyama, 2019), we also leave for subsequent work.

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7 Theory Appendix

An important implication of Assumption 1 first needs to be articulated. Assumption 1 implies that in equilibrium, $\alpha_{i,t} = \underline{\alpha}_i$ if i is in the coalition at t, as any share above it takes away from j's share. Thus, in equation (4), the shares that k obtains at each t are $\alpha_{2,1} = \alpha_{2,2} = \alpha_{2,3} = ... = \underline{\alpha}_2$. Similarly, in (5) and in (6), $\alpha_{2,1} = \alpha_{2,2} = \alpha_{2,3} = ... = \underline{\alpha}_2$.

The shares for j, however, change whenever a member joins or exits the coalition, since j gets all the remaining share, i.e. 1 minus the sum of the reservation shares of the coalition members at t. This implies that $\alpha_{1,1}$ is the same across equation (8), (9), (10), $\alpha_{1,2}$ is the same across (8), (9), (10), etc., but where $\alpha_{1,1} > \alpha_{1,2} > \alpha_{1,3} > ...$, since j relinquishes some share as new members join the coalition.

7.1 Proof of Lemma 1

Suppose condition (a) holds. Then (6) is greater than or equal to (7). Because (4) is always greater than (5), then $\lambda_k = 1$ for any $\mu_j \in [0, 1]$ if (a) holds. If (a) does not hold, then $\lambda_k = 1$ is the best response to any $\mu_j \in (\underline{\mu_j}, 1]$. Otherwise, for $\mu_j \in [0, \underline{\mu_j})$, $\lambda_k = 0$ is the best response of k. Lastly, if $\mu_j = \underline{\mu_j}$, then $\lambda_k \in [0, 1]$.

Now suppose condition (b) holds. Then (10) is greater than or equal to (11). Because (8) is always greater than (9), $\mu_j = 1$ for any $\lambda_k \in [0, 1]$ if (b) holds. If (b) does not hold, then $\mu_j = 1$ is the best response to any $\lambda_k > \underline{\lambda_k}$. It also follows that $\mu_j \in [0, 1]$ is the best response to $\lambda_k = \underline{\lambda_k}$, and $\mu_j = 0$ to any $\lambda_k < \underline{\lambda_k}$.

7.2 Proof of Lemma 2

First note that the conditions in (1) to (4) are exhaustive, since condition (a) can hold or not hold, condition (b) can hold or not hold, $\underline{\lambda_k}$ can be less than one or greater than or equal to 1, and $\underline{\mu_j}$ can be less than one or greater than or equal to 1. These generate the corresponding outcomes in (1) to (4), which are also exhaustive in that they include all the possible combinations of all possible actions taken by j and k. Specifically, j can choose to offer a proposal that will not induce any rebellion, $\mu_j = 1$, or that will certainly induce rebellion, $\mu_j = 0$, or that will induce rebellion with some non-zero probability, $\mu_j \in [0, 1)$. Similarly, k can choose not to fight, $\lambda_k = 1$, to certainly fight, $\lambda_k = 0$, or to fight with some non-zero probability, $\lambda_k \in [0, 1)$.

The proof makes use of Lemma 1.

We first prove (1). When condition (a) holds, then from Lemma 1, $\lambda_k = 1$ is the dominant action for k. When condition (b) holds, then $\mu_j = 1$ is the dominant action for j. Thus, when (a) and (b) both hold, the outcome is $(\mu_j = 1, \lambda_k = 1)$.

When (b) does not hold, $\mu_j = 1$ is j's best response to any $\lambda_k > \underline{\lambda_k}$. The latter implies, given k chooses $\lambda_k = 1$, which is its dominant action if (a) holds, that $\underline{\lambda_k} < 1$. Thus, when (a) holds, (b) does not, and $\underline{\lambda_k} < 1$, the outcome is also $(\mu_j = 1, \lambda_k = 1)$.

When (a) does not hold, $\lambda_k = 1$ is k's best response to any $\mu_j > \underline{\mu_j}$, which implies that $\underline{\mu_j} < 1$, given that j chooses $\mu_j = 1$, which is its dominant action if (b) holds. Thus, when (a) does not hold, (b) holds, and $\underline{\mu_j} < 1$, the outcome is also $(\mu_j = 1, \lambda_k = 1)$.

We next prove (2). The outcome $(\mu_j = 1, \lambda_k \in [0, 1))$ cannot be obtained if condition (a) holds since this would make $\lambda_k = 1$ the dominant action for k. It can only be obtained when (a) does not hold and $\mu_j \leq \underline{\mu_j}$, since this induces $\lambda_k \in [0, 1]$ or $\lambda_k = 0$. Given $\mu_j = 1$ (since (b) holds), it must then be that $\underline{\mu_j} \geq 1$. Thus, when condition (a) does not hold, condition (b) holds, and $\underline{\mu_j} \geq 1$, the outcome is $(\mu_j = 1, \lambda_k \in [0, 1))$.

We next prove (3). The outcome $(\mu_j = 1, \lambda_k = 1)$ cannot be obtained if condition (b) holds since this would make $\mu_j = 1$ the dominant action for j. It can only be obtained when (b) does not hold and $\lambda_k \leq \underline{\lambda_k}$, since this induces $\mu_j \in [0, 1]$, or $\mu_j = 0$. Given $\lambda_k = 1$ (since (a) holds), it must then be that $\underline{\lambda_k} \geq 1$. Thus, when condition (a) holds, condition (b) does not hold, and $\underline{\lambda_k} \geq 1$, the outcome is $(\mu_j = 1, \lambda_k = 1)$.

Lastly, we prove (4). The outcome $(\mu_j = 0, \lambda_k = 0)$ cannot be obtained if condition (a) or (b) holds, for this would make $\lambda_k = 1$ or $\mu_j = 1$ the respective dominant action for k and j. To show that there is no need to place restrictions on $\underline{\lambda_k}$ or $\underline{\mu_j}$, note that with (a) not holding, $\lambda_k = 0$ requires $\mu_j < \underline{\mu_j}$. The latter, however, is already satisfied, since $\lambda_k = 0$ already prompts j to choose $\mu_j = 0$ for any $\underline{\lambda_k} > 0$. Similarly, with (b) not holding, $\mu_j = 0$ requires $\lambda_k < \underline{\lambda_k}$, which is already satisfied since $\mu_j = 0$ already prompts k to choose $\lambda_k = 0$ for any $\underline{\mu_j} > 0$.

7.3 Proof of Corollary 1

The larger e_1 is, the more likely that it surpasses threshold $\underline{e_1}_k$ and thus, from Definition 5 (D.5), that condition (a) is met. It also decreases the difference between equation (7) and (6) and thus, from D.7 and Lemma 1, makes it less likely that $\underline{\mu_j} \geq 1$ (given $V^k(\cdot, s_{RA}) < (V^k(\cdot, s_{RF}))$.

The larger $\{e_k\}$ are, the more likely that they surpass their respective thresholds in $\{\underline{e_k}_k\}$ and thus, from D.5, that (a) is met. It also decreases the difference between equations (11) and (10) and thus, from D.7 and Lemma 1, makes it less likely that $\underline{\lambda_k} \geq 1$ (given $V^j(\cdot, s_{LF}) < V^j(\cdot, s_{RF})$). Lastly, it decreases the difference between (7) and (6) and thus, from D.7 and Lemma 1, makes it less likely that $\underline{\mu_j} \geq 1$ (given $V^k(\cdot, s_{RA}) < V^k(\cdot, s_{RF})$.)

The larger r_1 is, the more likely that it surpasses threshold $\overline{r_1}_j$ and thus, from D.6, that condition (b) is met. It also increases the difference between (11) and (10) and thus, from D.8 and Lemma 1, makes it more likely that $\underline{\lambda_k} \geq 1$ is met (given $V^j(\cdot, s_{LF}) < V^j(\cdot, s_{RF})$). Lastly, it decreases the difference between (7) and (6) and, thus, from D.5 and Lemma 1, makes it less likely that $\underline{\mu_j} \geq 1$.

The larger $\{r_k\}$ are, the more likely that they surpass their respective thresholds in $\{\underline{r_k}_k\}$ and thus, from D.5, that (a) is met. It also decreases the difference between (8) and (9) and thus, from D.7 and Lemma 1, makes it more likely that $\underline{\lambda_k} \geq 1$ (given that $V^j(\cdot, s_{LF}) < V^j(\cdot, s_{RF})$). Lastly, it decreases the difference between (7) and (6) and thus, from D.7 and Lemma 1, makes it less likely that $\underline{\mu_j} \geq 1$ (given that $V^k(\cdot, s_{RA}) < V^k(\cdot, s_{RF})$).

The larger c_k is, the more likely that it surpasses threshold $\underline{c_k}_k$ and thus, from D.5, that (a) is met. At the same time, it is less likely that it is below threshold $\bar{c}_{kj}$ and thus, from D.6, that (b) is met. It also increases the difference between (11) and (10) and thus, from D.8 and Lemma 1, makes it more likely that $\underline{\lambda_k} \geq 1$ (given that $V^j(\cdot, s_{LF}) < V^j(\cdot, s_{RF})$). Lastly, it increases the difference between (4) and (5) and decreases the difference between (7) and (6) and thus, from D.7 and Lemma 1, makes it less likely that $\underline{\mu_j} \geq 1$ (given that $V^k(\cdot, s_{RA}) < V^k(\cdot, s_{RF})$.)

The larger C_t are, the less likely that they are below their threshold $\bar{C}_{tj}$ and thus, from D.6, that (b) is met.

It also increases the difference between (11) and (10) and thus, from D.8 and Lemma 1, makes it more likely that $\underline{\lambda_k} \geq 1$ (given that $V^j(\cdot, s_{LF}) < V^j(\cdot, s_{RF})$). Lastly, it increases the difference between (4) and (5) and thus, from D.7 and Lemma 1, makes it less likely that $\underline{\mu_j} \geq 1$ (given that $V^k(\cdot, s_{RA}) < V^k(\cdot, s_{RF})$).

7.4 Proof of Proposition 1

The result follows directly from Lemma 2. (See discussion in the paragraph before Proposition 1.)

7.5 Proof of Theorem 1

Larger e_1 makes it more likely that (a) holds and, thus, that $\lambda_k = 1$ is the dominant action for k at s. If (a) did not hold, larger e_1 still makes it less likely that $\underline{\mu_j} \geq 1$ and, hence, makes it more plausible that $\mu_j > \underline{\mu_j}$, inducing k to choose $\lambda_k = 1$. Thus, larger e_1 makes more likely pairwise outcomes in which $\lambda_k = 1$ and less likely those which involve otherwise.

Larger $\{e_k\}$ also make it more likely that (a) holds and, thus, that $\lambda_k = 1$ is the dominant action for k at s. If (a) did not hold, larger $\{e_k\}$ still make it less likely that $\underline{\mu_j} \geq 1$ and, hence, makes it more plausible that $\mu_j > \underline{\mu_j}$, inducing k to choose $\lambda_k = 1$. On the other hand, larger $\{e_k\}$ also make it less likely that $\underline{\lambda_k} \geq 1$ and, hence, makes $\lambda_k \geq \underline{\lambda_k}$ more plausible, inducing j to choose at least $\mu_j \in [0, 1]$ should (b) not hold. Thus larger $\{e_k\}$ tends to increase the likelihood of pairwise outcomes in which $\lambda_k = 1$, and decrease those in which $\lambda_k \neq 1$. In turn, this can induce at least $\mu_j \in [0, 1]$ when (b) does not hold and $\lambda_k \geq \underline{\lambda_k}$, or just $\mu_j = 1$ when (b) holds. Together, these increase the likelihood of outcomes $\mu_j = 1, \lambda_k = 1$, decreases that of $\mu_j = 1, \lambda_k = 0$ and of $\mu_j = 0, \lambda_k = 0$, but may increase or decrease the likelihood of $\mu_j \in [0, 1], \lambda_k = 1$.

Larger r_1 makes it more likely that (a) holds and, thus, that $\lambda_k = 1$ is the dominant action for k at s. It also makes it less likely that $\underline{\mu_j} \geq 1$ and, hence, more plausible that $\mu_j \geq \underline{\mu_j}$, inducing at least $\lambda_k \in [0, 1]$. Thus, to the extent that r_1 makes (a) more likely to hold, it increases the likelihood of $\lambda_k = 1$ and decreases that of $\lambda_k \in [0, 1]$. But if (a) does not hold, r_1 increases the likelihood of $\lambda_k \in [0, 1]$. This is why there is an ambiguous effect on the likelihood of allinace by conquest. Similarly, larger r_1 makes it more likely that (b) holds and, hence, that $\mu_j = 1$ is the dominant action for j at s. It also make is $\underline{\lambda_k} \geq 1$ more likely and, hence, makes $\lambda_k \leq \underline{\lambda_k}$ more plausible, inducing j to choose at most $\mu_j \in [0, 1]$ if (b) did not hold. Thus, to the extent that r_1 makes (b) more likely to hold, it makes $\mu_j = 1$ more likely and $\mu_j \in [0, 1]$ less likely. But if (b) did not hold, it make $\mu_j \in [0, 1]$ more likely and $\mu_j = 1$ less likely. This is why r_1 has an ambiguous effect on the likelihood of alliance with unrest. (To the extent that r_1 makes (a) and (b) more likely to hold, it increases the likelihood of peaceful alliance, and decreases that of no alliance.)

Larger $\{r_k\}$ makes it more likely that (a) holds and, thus, that $\lambda_k = 1$ is the dominant action for k at s. It also makes $\underline{\mu_j} \geq 1$ less likely and, hence, more plausible that $\mu_j \geq \underline{\mu_j}$ more likely, inducing at least $\lambda_k \in [0, 1]$. But since $\{r_k\}$ also make (a) more likely, it tends to increase $\lambda_k = 1$ and decrease $\lambda_k \in [0, 1]$. Furthermore, $\{r_k\}$ increases the likelihood that $\underline{\lambda_k} \geq 1$ and, hence, less plausible that $\lambda_k \geq \underline{\lambda_k}$, inducing at most $\mu_j \in [0, 1]$ if (b) did not hold. $\{r_k\}$ do not affect (b), and only tends to make $\mu_j \in [0, 1]$ more likely through increasing the likelihood of $\underline{\lambda_k} \geq 1$.

Larger c_k makes it more likely that (a) holds and, thus, that $\lambda_k = 1$ is the dominant action for k at s. If (a) did not hold, it makes it likely that at least $\lambda_k \in [0, 1]$, since it decreases the likelihood of $\underline{\mu_j} \geq 1$. To the extent that c_k makes (a) more likely, it tends to increase the likelihood of outcomes involving $\lambda_k = 1$ and tends to decrease those involving $\lambda_k \in [0, 1]$ and $\lambda_k = 0$. However, larger c_k also makes (b) less likely to hold and $\underline{\lambda_k} \geq 1$ more likely, inducing at most $\mu_j \in [0, 1]$. Thus, c_k also decreases the likelihood of outcomes involving $\mu_j = 0$. Thus, the effect on peaceful alliance and no alliance is ambiguous, while it decreases the likelihood of alliance by conquest, but decreases that of alliance with unrest.

Larger C_t make it less likely that (b) holds and, hence, that $\mu_j = 1$ is a dominant action for j at s. Also, it makes $\underline{\lambda_k} \geq 1$ more likely and, hence, less plausible that $\lambda_k \geq \underline{\lambda_k}$, inducing at most $\mu_j \in [0, 1]$. Thus, C_t makes less likely outcomes involving $\mu_j = 1$ and more likely those involving $\mu_j \in [0, 1]$ and $\mu_j = 0$.

7.6 Proof of Proposition 2

It is straightforward to see that Proposition 1 gives rise to four types of 'benchmark' equilibra when each set of conditions hold at every state s. One is when, at every s, the conditions in (1) hold and thus, j always chooses $\mu_j = 1$ and every k chooses $\lambda_k = 1$. We call this equilibrium as one of peaceful consolidation, in which every player, whenever drawn, accepts the proposal to join (or stay) in the coalition. The grand coalition is thus formed and remains intact. When the condition in (2) holds at every s, such that j always chooses $\mu_j = 1$, and every k chooses $\lambda_k \in [0, 1)$, then there is always some probability of fighting, but the coalition always wins. Thus, every one joins the coalition, albeit by conquest. We call this consolidation by conquest. When the condition in (3) holds at every s, such that j always chooses $\mu_j \in [0, 1)$, and every k chooses $\lambda_k = 1$, then every responder (peacefully) joins the coalition, but because there is always some probability of rebellion, the grand coalition is not sustainable. We call this a fragmented polity. Lastly, when the condition in (4) holds at every s, such that j always chooses $\mu_j = 0$ and every k chooses $\lambda_k = 0$, then no alliance is ever made, and each player remains its own singleton coalition. We call this equilibrium as one of independent territories.

7.7 Proof of Proposition 3

We want to provide conditions such that $\mu_j = 1$ is (uniquely) chosen at each s, and not $\mu_j \in [0, 1]$. We know that $\mu_j = 1$ for any $\lambda \in [0, 1]$ if (b) holds. Otherwise, if (b) does not hold, then $\mu_j = 1$ if $\lambda_k > \underline{\lambda_k}$. Now, in turn, the latter implies that if (a) holds (such that $\lambda_k = 1 \forall \mu_j \in [0, 1]$), $\underline{\lambda_k}$ must be less than one. (Otherwise, $\lambda_k > \underline{\lambda_k}$ cannot be met.) If (a) does not hold, then either $\lambda_k = 1$ (if $\underline{\mu_j} < 1$), or $\lambda_k \in [0, 1]$ (if $\underline{\mu_j} \geq 1$). Thus, if $\underline{\mu_j} < 1$, it must be that $\underline{\lambda_k} < 1$. We cannot have $\underline{\mu_j} \geq 1$, because λ_k can be $\lambda_k \leq \underline{\lambda_k}$ (in the extreme, λ_k can be zero).

But we also know that if (a) and (b) both hold, that $\mu_j = 1$ is the dominant action of j. One can also obtain $\mu_j = 1$ if (a) holds, but not (b), if $\underline{\lambda_k} < 1$. Thus, for $\mu_j = 1$ at each s, it must be that at each s, either: (1) (a) and (b) hold; (2) (a) holds, (b) does not hold, and $\underline{\lambda_k} < 1$; (3) (a) does not hold, (b) holds, and $\underline{\mu_j} < 1$; and (4) (a) does not hold, (b) holds, and $\underline{\mu_j} \geq 1$. But (3) and (4) can be combined into: (5) (a) does not hold, (b) holds. Thus, (1), (2) and (5) together imply that necessary and sufficient for $\mu_j = 1$ is that either (b) holds or, if (b) does not hold, that $\underline{\lambda_k} < 1$.

7.8 Proof of Theorem 2

From Proposition 3, we know that the relevant conditions are (b) and whether $\underline{\lambda_k} < 1$. From lemma Corollary 1, variables that make it likely for (b) to hold are (small) c_k and C_t, and (large) r_1. Variables that make it likely for $\underline{\lambda_k} < 1$ are (large) $\{e_k\}$, and (small) $r_1, \{r_k\}, c_k, C_t$.

From the foregoing, one can infer that small C_t^* and c_k^* make it likely for consolidation to happen since, in the first place, they make it likely that (b) holds at each state s. That is, if (b) holds for the largest possible costs C_t^*, c_k^*, they also hold for lesser costs. In the second place, even if (b) did not hold at some of all states (e.g. when r_1, C_t, or c_k are too small in that state), they make it more likely that $\underline{\lambda_k} < 1$ in those states (or, equivalently, less likely that $\underline{\lambda_k} \geq 1$.)

Similarly, large $\{e_k\}$ and small $\{r_k\}$ monotonically increase the likelihood of consolidation in that they make it more likely that $\underline{\lambda_k} < 1$, although they are thus only relevant if c_k or C_t are not sufficiently small, or r_1 not sufficiently large, such that (b) does not hold.

In contrast, the effect of r_1 is non-monotonic. The foregoing suggests that increasing r_1 makes (b) more likely to hold and, hence, increase the likelihood of consolidation. However, while decreasing r_1 thus makes (b) less likely to hold, it also makes $\underline{\lambda_k} < 1$ more likely and thereby also increase the likelihood of consolidation. For these to both be true, there must be some $r_{1j}^0 < r_{1j}^*$, where r_{1j}^0 is the smallest possible threshold $\underline{r_1}_j$, and r_{1j}^* the largest, for j across all states, such that in the range $r_1 \in [0, r_{1j}^0)$, the likelihood of consolidation is decreasing in r_1 at all states, while in the range $r_1 \in [r_{1j}^0, r_{1j}^*]$, it is increasing in r_1. For $r_1 \in (r_{1j}^*, \infty)$, given that C_t and C_k are at or below their thresholds, r_1 has no further effect on the likelihood of consolidation, since (b) already holds and $\mu_j = 1$ the dominant action for j, and the likelihood of consolidation is therefore one.

7.9 Proof of Corollary 2

See discussion in text and the proof of theorem 2.