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Abstract

We study a class of preferences that we call Heterothetic Cobb-Douglas (HCD). They feature unitary own-price elasticity and non-unitary income effects so that differences in expenditure shares for a given good are solely due to income effects. HCD preferences generate a tractable demand system that can be introduced in standard general equilibrium models, yielding rich results. We illustrate HCD's properties with different applications. First, we show that under HCD preferences, the money-metric cost of inflation in a cross-section of households can be computed with information on prices, expenditure shares, and total expenditures. Second, applied to growth theory, we show that HCD preferences strengthen and generalize the classic results by Kongsamut et al. (2001) and Foellmi et al. (2008). Third, when applied to economic geography and international trade, we show how HCD preferences yield new insights in the Krugman (1991) core-periphery model, the class of spatial economy models of Allen and Arkolakis (2014) and Redding (2016), and the monocentric city model (Alonso, 1964; Mills, 1967; Muth, 1969). Here, the combination of unitary own-price elasticity and equalization of utility over space plays a crucial role in making the analysis tractable.

JEL Classification: D11, F11, O40, R10

Keywords: N/A

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Heterothetic Cobb-Douglas: Theory and Applications

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April 6, 2023

Abstract

We study a class of preferences that we call Heterothetic Cobb-Douglas (HCD). They feature unitary own-price elasticity and non-unitary income effects so that differences in expenditure shares for a given good are solely due to income effects. HCD preferences generate a tractable demand system that can be introduced in standard general equilibrium models, yielding rich results. We illustrate HCD's properties with different applications. First, we show that under HCD preferences, the money-metric cost of inflation in a cross-section of households can be computed with information on prices, expenditure shares, and total expenditures. Second, applied to growth theory, we show that HCD preferences strengthen and generalize the classic results by Kongsamut et al. (2001) and Foellmi et al. (2008). Third, when applied to economic geography and international trade, we show how HCD preferences yield new insights in the Krugman (1991) core-periphery model, the class of spatial economy models of Allen and Arkolakis (2014) and Redding (2016), and the monocentric city model (Alonso, 1964; Mills, 1967; Muth, 1969). Here, the combination of unitary own-price elasticity and equalization of utility over space plays a crucial role in making the analysis tractable.

Keywords: Cobb-Douglas, Nonhomothetic Preferences.

JEL Codes: D11, F11, O40, R10.

1 Introduction

Rich and poor households consume very different baskets of goods. For example, the expenditure share in food, energy, and other essential goods is well known to be larger for poor than for rich households. In technical terms, this implies that their demands are nonhomothetic. A vast amount of literature has documented the importance of nonhomotheticities in demand, starting from the seminal work of Engel (Engel, 1857).[1] Despite nonhomotheticities being extensively well-documented, their applications to fields that rely on applied general equilibrium models, such as

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[1] An ensuing extensive literature confirmed Engel's initial findings on food consumption and extended it to other items in the consumption basket (e.g., Houthakker, 1957). Modern neoclassical demand systems have also extensively documented the presence of nonhomotheticities, e.g., through the so-called (q)aids demand systems (Banks et al. 1997; Deaton and Muellbauer 1980a).

macroeconomics or international trade, have remained relatively marginal. This inconspicuous usage of nonhomothetic demands is probably due to a large extent to the technical difficulty of dealing with them, in contrast to homothetic demands—which assume that prices are the only determinant of relative consumption choices.[2]

In this paper, we study a nonhomothetic demand system that we call Heterothetic Cobb-Douglas (HCD) and illustrate its tractability by presenting various applications. The distinctive feature of this demand system is that it features a unitary own-price elasticity and non-unitary income effects. As a result, differences in expenditure shares for a given good are solely due to income effects. HCD preferences generate a highly tractable demand system that can be introduced in standard general equilibrium models, yielding rich results. Despite having been noticed as early as in Hanoch (1975), to our knowledge, it was only first used in an applied general equilibrium model by Bohr et al. (2021).

The goal of this paper is two-fold. First, we present this demand system in detail and provide an analysis complementary to Bohr et al. (2021). Second, to illustrate its analytical simplicity, we apply HCD to various classic frameworks in economic growth, international trade, and economic geography. When augmented with HCD preferences, we can show how the foundations of these well-known models are, in some cases, strengthened and generalized (e.g., Kongsamut et al., 2001), also yielding new insights in most cases (e.g., Krugman, 1991). We also show its convenience for inferring welfare-relevant measures for the cost of inflation.

The roadmap for this paper is as follows. Section 2 lays out the foundations of Heterothetic Cobb-Douglas preferences. We show its basic properties and provide sufficient conditions for HCD preferences to be well-defined. Section 3 applies HCD preferences to computing a money-metric welfare change in the cost of living. We show that under HCD preferences, it is possible to compute the monetary change required to attain a certain level of utility due to a change in the cost of living (price changes) for a cross-section of households, using only data on prices, household expenditure shares, and total household expenditures.

Sections 4 and 5 apply HCD preferences to the growth models of Kongsamut et al. (2001) and Foellmi et al. (2008), respectively. We show that HCD preferences extend the celebrated result in Kongsamut et al. (2001) on the existence of a balanced growth with underlying structural change in three sectors driven by nonhomothetic preferences. Incorporating HCD preferences circumvents a main criticism of the original paper of relying on a knife-edge condition that links preferences and relative prices. Instead, under HCD preferences, we obtain that a balanced growth path (BGP) exists generically without making any other change to the original setup. In Section 5, we show that the model of Foellmi et al. (2008) can be adapted to feature the consumption cycles emphasized in the paper under endogenous technological change along the lines of Bohr et al. (2021).

[2] Abstracting from nonhomotheticities in demand can come at the substantial cost of distorting our conclusions in many quantitative analyses. Indeed, during the last decade, several papers have shown the shortcomings of abstracting from nonhomotheticities in the study of different economic phenomena, e.g., structural change (Boppart, 2014; Alder et al., 2022; Comin et al., 2021), the patterns of international trade (Fieler, 2011; Caron et al., 2014; Fajgelbaum et al., 2011) or measuring welfare differences across agents due to price changes (Fajgelbaum and Khandelwal, 2016; Jaravel, 2019; Argente and Lee, 2021; Jaravel and Lashkari, 2022).

Sections 6 through 9 apply HCD preferences to canonical international trade and geography models: the gains-from-trade result of Arkolakis et al. (2012), the general-equilibrium spatial economy of Allen and Arkolakis (2014) and Redding (2016), the core-periphery model of Krugman (1991), and the monocentric city model of Alonso (1964), Mills (1967), and Muth (1969). We demonstrate how HCD preferences can seamlessly incorporate income effects in standard set-ups and yield new intuitions for each application. For example, for the monocentric city model, we show that it provides a natural explanation for why rich people live closer to the Central Business District if housing is a necessity. In the core-periphery model, we can perform comparative statics on the level of productivity (rather than the commonly done transportation cost) to assess under which conditions structural change leads to the emergence of regional disparities. In the spatial economy model, we show how to extend the standard framework to nonhomothetic preferences. A key simplification that HCD preferences afford is that since agents, in equilibrium, are indifferent across locations, they must obtain the same utility and thus have the same expenditure shares across locations. Section 10 concludes.

2 Heterothetic Cobb Douglas: Definition and Properties

We start introducing the Heterothetic Cobb-Douglas (HCD) utility function and its associated demand derived from the consumer's problem. Let $\mathcal{N} \equiv \{1, ..., N\}$ denote the set of goods available to a consumer, and $(c_1, \cdots, c_N)$ the associated consumption vector. The Heterothetic Cobb-Douglas (HCD) utility function $U(c_1, \cdots, c_N)$ is implicitly defined by

$$\ln U = \sum_{i \in \mathcal{N}} \alpha_i (U) \ln (c_i), \tag{1}$$

where $\alpha_1(U), \cdots, \alpha_N(U)$ denote N functions of U.[3] Note that, if for all $i \in \mathcal{N}$, $\alpha_i(U)$ is a constant independent of U, then we recover the standard homothethic Cobb-Douglas preferences. To simplify our analysis, we assume throughout that $\alpha_i(\cdot)$ is continuously differentiable. We start by characterizing the solution to a consumer's problem with HCD utility and derive the demand for each good i. We proceed under the assumptions that *i)* $(\alpha_i(U))_{i=1}^N$ ensure that the solution to (1) is unique, and *ii)* that U is monotone and quasi-concave in $(c_i)_{i=1}^N$. We provide sufficient conditions for this to be the case below.

Consider the consumer's expenditure minimization problem of choosing a consumption vector

[3] For the sake of concreteness, we consider a set of finite and countable goods in our baseline exposition. The results extend to uncountable sets $\mathcal{N} \subseteq \mathbb{R}$, as we show in some of the examples below. In this case, additional assumptions may be imposed to ensure that the integration involving the definition of HCD is well-defined. In particular, the weighting functions $\alpha_i(\cdot)$ may be required to be Lipschitz continuous or uniformly continuous.

$(c_i)_{i=1}^N$ taking prices $(p_i)_{i=1}^N$ as given,[4] subject to obtaining a certain utility level $\bar{U}$,

$$\min_{(c_i)_{i=1}^N} \sum_{i\in\mathcal{N}} p_i c_i \quad \text{s.t.} \quad \ln \bar{U} = \sum_{i\in\mathcal{N}} \alpha_i\left(\bar{U}\right)\ln\left(c_i\right).$$

The first order condition of this problem for good i yields $p_i c_i = \lambda\alpha_i(\bar{U})$, where λ is the Lagrange multiplier on the constraint. Denoting total expenditure by $E = \sum_{i\in\mathcal{N}} p_i c_i$, we find that the equilibrium expenditure share on good i and the Lagrange multiplier are equal to

$$s_i\left(\bar{U}\right) \equiv \frac{p_i c_i}{E} = \frac{\alpha_i\left(\bar{U}\right)}{\sum_{j\in\mathcal{N}}\alpha_j\left(\bar{U}\right)}, \qquad \lambda\left(\bar{U}\right) = \frac{E}{\sum_{j\in\mathcal{N}}\alpha_j\left(\bar{U}\right)}. \tag{2}$$

The result in Equation 2 highlights the key property of this demand system. Expenditure shares only depend on the level of utility $\bar{U}$. Thus, conditional on reaching the same level of utility, differences in the prices of goods $(p_i)_{i=1}^N$ do not affect consumers' expenditure shares.

For example, if we set $\alpha_i(U) = \exp(a_i U)$, then expenditure shares in (2) take a "multinomial logit" form,

$$s_i(U) = \frac{\exp(a_i U)}{\sum_{j\in\mathcal{N}}\exp(a_j U)}. \tag{3}$$

If we instead set $\alpha_i(U) = U^{a_i}$, we obtain the expenditure shares,

$$s_i(U) = \frac{U^{a_i}}{\sum_{j\in\mathcal{N}} U^{a_j}}. \tag{4}$$

These expenditure shares coincide with those characterized for the nonhomothetic CES in Comin et al. (2021) when the elasticity of substitution tends to one. They would also coincide with the "multinomial logit" shares derived above, Equation (2), under the utility transformation $\ln U = V$ and are thus observationally equivalent.

2.1 Expenditure Function

The expenditure function associated to HCD preferences is derived by substituting the demand function (2) in the definition of HCD (1), yielding

$$\ln E = \ln\tilde{U} + \sum_{i\in\mathcal{N}} s_i\left(U\right)\ln\left(\frac{p_i}{s_i\left(U\right)}\right), \tag{5}$$

where $\tilde{U} \equiv U/\sum_{i\in\mathcal{N}}\alpha_i(U)$. This formula implies that the indirect utility function for HCD coincides with that of homothetic Cobb Douglas preferences (conditional on taking the expenditure shares as given).[5] Also, Equation (5) makes clear that a useful cardinalization of preferences (as in

[4] We follow Berthold Herrendorf's suggestion of presenting the derivation using cost minimization on the grounds of being the most transparent way to derive the demand system of implicitly-defined preferences. Maximizing utility subject to the budget constraint yields the same demand. We also note that it is possible to have alternative definitions of HCD preferences instead of Equation (1), some of these are presented in the applications below.

[5] Furthermore, the indirect utility function $V = V(p, E)$ is defined as the fixed point of $V =$

its homothetic counterpart) is $\sum_{i\in\mathcal{N}}\alpha_i(U)=1$. In this case, $\tilde{U}=U$ and the demand equation (2) and the expenditure function (5) further simplify to

$$s_i(U)=\alpha_i(U), \qquad \ln E=\ln U+\sum_{i\in\mathcal{N}} s_i\left(U\right)\ln\left(\frac{p_i}{s_i\left(U\right)}\right). \tag{6}$$

Note also that the expenditure function (5) implies that it is possible to obtain the level of utility U with knowledge of total expenditure, expenditure shares, and prices. See Section 3 for further discussion on this point and how to use this result to assess the cost of inflation.

Expenditure Elasticities It is useful to decompose the expenditure elasticity of this demand system as follows[6]

$$\eta_E^{s_i}\equiv\frac{\partial\ln s_i}{\partial\ln E}=\frac{\partial\ln s_i}{\partial\ln U}\frac{\partial\ln U}{\partial\ln E}\equiv\eta_U^{s_i}\eta_E^U. \tag{7}$$

This decomposition is intuitive: the expenditure elasticity is the product of the elasticity of expenditure shares to utility (by which percent expenditure shares change if one increases U by one percent) times the elasticity of utility with respect to total expenditure (by which percent U increases if one increases total expenditure by one percent). The decomposition in Equation (7) implies that

$$\frac{\eta_E^{s_i}}{\eta_E^{s_j}}=\frac{\eta_U^{s_i}}{\eta_U^{s_j}}.$$

That is, the ratio of expenditure elasticities is the same as the ratio of utility elasticities. Thus, we can use the expression of $s_i(U)$ from Equation (2) to recover all relative behaviors for a given increase in U. For example, under Equations (3) and (4) we have that

$$\frac{\eta_E^{s_i}}{\eta_E^{s_j}}=\frac{a_i-\bar{a}}{a_j-\bar{a}}, \qquad \text{where} \qquad \bar{a}\equiv\sum_{i\in\mathcal{N}} s_i a_i.$$

Remark: HCD as a Production Function We may also apply this functional form to non-homothetic production functions, as in Hanoch (1975). Appendix A provides the details.

2.2 Monotonicity and Quasi-concavity of HCD Preferences

Next, we turn our attention to the monotonicity and quasi-concavity of HCD preferences. A sufficient condition for the consumer problem to be well defined is that preferences are monotone and quasi-concave (Mas-Colell et al., 1995). To be sure, less stringent conditions can be imposed,

$E\prod_{i\in\mathcal{N}}\left(s_i\left(V\right)/p_i\right)^{-s_i(V)}$. Also, it is readily verified that the indirect utility function is homogeneous of degree zero in $\left(\left(p_i\right)_{i=1}^N, E\right)$, decreasing in each p_i, and increasing in E.

[6] For convenience, we work with the expenditure share elasticity. Note that this elasticity is related to the expenditure elasticity in levels by $\eta_E^{s_i}=\eta_E^{p_i c_i}-1$.

e.g., local non-satiation of preferences.[7]

First, we analyze under which conditions we have monotonicity of $U(\cdot)$. Given the assumption of continuous differentiability of $\alpha_i(\cdot)$, we can generically apply the implicit function theorem to (1). We find that strict monotonicity holds,[8] i.e., $\frac{\mathrm{d}\ln U}{\mathrm{d}\ln c_i} > 0$ for all $i \in \mathcal{N}$ if and only if

$$\frac{1}{\alpha_i(U)} - \sum_{i\in\mathcal{N}} \frac{\mathrm{d}\ln \alpha_i(U)}{\mathrm{d}\ln U} \ln c_i > 0, \qquad \forall i \in \mathcal{N}. \tag{8}$$

This condition can hold under different sets of mutually exclusive assumptions. For example, a sufficient set of assumptions is that $\alpha_i(U) > 0$, $\alpha_i'(U) < 0$ and $c_i \geq 1$. This condition highlights that the choice of units for consumption measurement may not be innocuous. In practice, this requires ensuring that, at the equilibrium prices and total expenditure, the household choices satisfies $c_i \geq 1$ for all $i \in \mathcal{N}$.[9] This condition is analogous to the requirements of attaining a sufficient level of consumption in the case of Stone-Geary preferences with a subsistence requirement, or of PIGL preferences, in the sense that it imposes conditions linking preferences with prices and expenditure levels. Importantly, note that the monotonicity condition also ensures that the indirect utility function is increasing in total expenditures, and that the expenditure function is increasing in utility (see Propositions 3.D.3 and 2.E.2 in Mas-Colell et al., 1995).

In addition to other sets of sufficient conditions (which we refrain from listing here to avoid a taxonomic analysis), it is worth mentioning the result presented in Bohr et al. (2021), which points out to the existence of alternative assumptions that ensure monotonicity. They consider the case in which $\mathcal{N} = R_+$ and the weights α_i correspond to weights of an exponential distribution, $\alpha_i(U) = \frac{\mathrm{e}^{-i/U}}{U}$, with $i \geq 0$. In their setup, it can be verified that $\frac{\mathrm{d}\ln U}{\mathrm{d}\ln c_i} > 0$ and thus monotonicity is satisfied (note that $\alpha_i(\cdot)$ does not satisfy the example of sufficient condition stated above).[10]

Second, we turn to the analysis of quasi-concavity. We assume that monotonicity is satisfied, i.e., Equation (8) holds. Notice that for any given conjectured level of utility U, the indifference curves between any pair of goods c_i and c_j correspond to that of a homothetic Cobb-Douglas utility

[7] Monotonicity ensures that we can globally disregard the existence of "bads". However, allowing for this possibility can be potentially interesting.

[8] It can be verified that $\frac{\mathrm{d}\ln U}{\mathrm{d}\ln c_i} = \frac{1}{\frac{1}{\alpha_i(U)} - \sum_{i\in\mathcal{N}} \frac{\mathrm{d}\ln \alpha_i(U)}{\mathrm{d}\ln U} \ln c_i}$ if $\alpha_i(U) \neq 0$.

[9] The choice of units is without loss of generality in the following sense. Consider the following specification of HCD preferences

$$\ln U = \sum_{i\in\mathcal{N}} \alpha_i(U) \ln\left(\frac{c_i}{z_i}\right), \tag{9}$$

where $z_i > 0$ for all $i \in \mathcal{N}$ denotes a normalization constant. The demand for each good i is still given by (2), and it is thus independent from z_i. This implies that, for given any vector of prices and total expenditure, it is always possible to find z_i so that condition (8) is satisfied.

[10] In light of this discussion around monotonicity, another potentially interesting generalization of HCD preferences is to add a minimum, non-divisibility requirement in consumption, by defining

$$\ln U = \sum_{i\in\mathcal{N}} \alpha_i(U) \ln\left(\max\left\{c_i, 1\right\}\right). \tag{10}$$

This formulation allows to have zero consumption of certain goods, while preserving the unit price elasticity and optimal choices, e.g., Equation (1), for the set of goods with consumption greater than one.

function with expenditure shares given by Equation (1). Homothetic Cobb-Douglas preferences are well-known for displaying convex upper envelopes of the indifference curves. Moreover, note that monotonicity implies that increasing consumption of any good shifts the indifference curves outwards from zero. These two facts taken together imply quasi-concavity of preferences.

3 Exact Price Index and Money Metric of Inflation Costs

In this section, we start from the observation that HCD preferences yield a price index that can be constructed using readily available data on total expenditure, expenditure shares, and prices. Since these preferences are nonhomothetic, knowledge of household price indices is not sufficient to directly assess welfare changes. This is in contrast to homothetic aggregators (see, e.g., Deaton and Muellbauer 1980b). We then show how to make the information contained in the price indices operational to compute a money metric of welfare changes in a cross-section of households.

From the household problem, we have already established in Equation (5) that, under the normalization $\sum_{i\in\mathcal{N}}\alpha_i(U)=1$, the expenditure function satisfies

$$\ln E-\sum_{i=1}^{N}s_i\ln\left(\frac{p_i}{s_i}\right)=\ln U.$$

Thus, we can interpret the second term in the left-hand side above as a price index:

$$\ln P\equiv\sum_{i=1}^{N}s_i\ln\left(\frac{p_i}{s_i}\right). \tag{11}$$

Note that Equation (11) is the Kullback-Leibler divergence or relative entropy between the expenditure shares and prices (since the sum of prices can be normalized to one, it can also be interpreted as a distribution). Moreover, we note that $\ln P=\ln P^{Stone}+\ln S$, where the first term is the well-known Stone price index (which is widely used in empirical work, e.g., in the estimation of the AIDS demand as suggested by Deaton and Muellbauer, 1980a), while the second is an entropy measure of the dispersion of the expenditure shares

$$\ln P^{Stone}\equiv\sum_{i=1}^{N}s_i\ln p_i,\qquad \ln S\equiv-\sum_{i=1}^{N}s_i\ln s_i.$$

Recovering the Money-Metric True Cost-of-Living Index Number Suppose we observe a panel of households for two periods (or two repeated cross sections). This panel contains information on household total expenditures, expenditure shares, and the prices they face at two different points in time. To simplify our analysis, assume that total expenditures follow a distribution, $E\sim F(E)$, with support going from zero to infinity and dense in its entire support (e.g., the lognormal distribution). Assuming that all households have the same preferences, given by Equation

(5), we now show how to recover the household-specific money-metric cost of price changes. We denote with superindices the time dimension in the panel.

It follows from our previous discussion that the distribution of log utilities at time t can be computed using:

$$\ln U^t = \ln E^t - \ln P\left(E^t, (p_i^t)_{i=1}^N, (s_i^t)_{i=1}^N\right), \qquad \forall E \in (0, \infty).$$

We assume that $\ln P$ is bounded above and below, so that, by construction, $\ln U^t \in (-\infty, +\infty)$ for all t (since expenditures go from zero to infinity). Thus, the range of utilities is *time-invariant.* This result simplifies the exposition (without loosing the economic insight of the exercise). Note that even though the range of utilities is time-invariant, it does not mean that household utility remains constant over time.

We are interested in quantifying the change in the cost of living for different households associated with a change of prices from $(p_i^1)_{i=1}^N$to $(p_i^2)_{i=1}^N$. To fix ideas, suppose we are interested in computing the change in the cost of living that household x with expenditures at time 2, E_x^2, has experienced. We can proceed as following three steps.

1. Compute the utility level associated with E_x^2 at time $t = 2$ prices. This is done by carrying out the operation $\ln U = \ln E_x^2 - \ln P\left(E_x^2, (p_i^2)_{i=1}^N, (s_{x,i}^2)_{i=1}^N\right)$, where the price index is given by Equation (11).

2. Using that the mapping of E to U is exhaustive on all the reals, find the expenditure at time 1, E_y^1, yielding the same utility level as in step 1 at time $t = 1$ prices. That is, the utility satisfying $\ln U = \ln E_y^1 - \ln P\left(E_y^1, (p_i^1)_{i=1}^N, (s_{y,i}^2)_{i=1}^N\right)$. We have found the expenditure required at time 1 to attain U. Note that, this expenditure at time 1 can correspond to that of another household, denoted y.

3. The change in the cost of living is given by $\frac{E_x^2}{E_y^1}$. Note that we have constructed E_y^1 so that we are holding the level of utility at household's x in period $t = 2$. That is, $U\left(E_y^1, (p_i^1)_{i=1}^N\right) = U\left(E_x^2, (p_i^2)_{i=1}^N\right)$. We note that this is what Deaton and Muellbauer (1980b) refer to as the true cost-of-living index, $E^2(U, (p_i^2)_{i=1}^N)/E^1(U, (p_i^1)_{i=1}^N)$ where U refers to the utility level attained at time $t = 2$ by household x.

In sum, we have shown how we can recover the change in the cost of living for all households between two periods. This method can be analogously applied if we take as reference the utility level in the first period instead of the second. Importantly, this procedure *only* requires observed information available in the standard data (household expenditures and expenditure shares, and a vector of prices) under the assumption, customary in this literature, that households face the same prices.

In this derivation, we have imposed that the set of goods available to the consumer is constant over time. However, it is immediate to show that the same result goes through if the set of goods is also indexed by time—in this sense, this derivation bypasses the Feenstra (1994) correction for new goods. We have made some convenient assumptions of having households at all income levels in both periods. In practice, it may be necessary to interpolate (and extrapolate) the data on expenditures and utilities since an exhaustive mapping between E and U is, by definition, unavailable with finite data.

4 Structural Change under Exogenous Balanced Growth

Kongsamut et al. (2001) provided a simple model of the US economy which was both consistent with Kaldor's facts on economic growth, and with sectoral reallocation of labor from agriculture to services along the growth path. This model is still often used as a baseline for understanding structural change that is driven by nonhomotheticities in demand. However, the model relies on an exact cross-restriction on preferences and technology. Moreover, the income effects which drive the structural transformation are due to the initial conditions of the model and disappear asymptotically. Replacing the original Stone-Geary preferences with HCD preferences provides an equally simple model which does not rely on a knife-edge condition, while also featuring income effects in the long run (if desired). We briefly review the original model and compare it to our version with HCD preferences.

Production The production side features three sectors $i \in \{A, M, S\}$. Each sector produces its own output Y^i using an identical production function $F(\cdot, \cdot)$, up to a factor of total productivity B_i, that is continuous, homogenous of degree one, concave, and increasing in both of its arguments, capital K and labor N.

$$Y_t^i = B_i F(\phi_t^i K_t, N_t^i X_t).$$

Here, ϕ^i and N^i denote the shares of the total labor and capital stock that are used by sector i (the labor stock is normalized to one). X_t denotes labor augmenting technological progress, which grows at an exogenous fixed rate g. The agricultural and service outputs are used purely for consumption, $Y^i = C^i$ for $i \in A, S$, while the manufacturing output is used for both consumption and investment, $Y_t^M = C_t^M + \dot{K}_t + \delta K_t$.

Since all sectors use the same production function up to a scalar and inputs are fully mobile, the cost-minimizing allocation implies an identical relative use of inputs by each sector. Sectors are competitive and prices are equal to their marginal costs, which only differ by their total factor productivities, B_i. Letting the manufacturing good be the unit of account, the prices become $P_A = B_M/B_A$, $P_M = 1$, and $P_S = B_M/B_S$.

Balanced growth with Stone-Geary Preferences (Kongsamut et al., 2001) Preferences across time are discounted and aggregated according to

$$U = \int_0^\infty \mathrm{e}^{-\rho t} \frac{C_t^{1-\sigma} - 1}{1 - \sigma} dt. \tag{12}$$

The intratemporal utility function for the consumption bundle C_t is given by the Stone-Geary aggregator

$$C_t = \left(C_t^A - \bar{A}\right)^{\beta} \left(C_t^M\right)^{\gamma} \left(C_t^S + \bar{S}\right)^{\theta},$$

where $\bar{A}, \bar{S}, \rho, \sigma, \beta, \gamma, \theta$ are all strictly positive, and $\beta + \gamma + \theta = 1$, as laid out in the original paper. Using the fact that production across sectors uses the same relative inputs and only differ by their total scale, we can write the budget constraint the household faces as

$$C_t^M + \dot{K}_t + \delta K_t + P_A C_t^A + P_S C_t^S = B_M F\left(K_t, X_t\right).$$

Using this budget constraint, Kongsamut et al., 2001 characterize the optimal consumption allocations over time. They show that a balanced growth path (defined as an equilibrium with a constant real interest rate) exists under the preference-technology cross-restriction that

$$\frac{\bar{A}}{B_A} = \frac{\bar{S}}{B_S}. \tag{13}$$

Along the balanced growth path, the sectoral growth rates are the following:

$$\frac{\dot{C}_t^M}{C_t^M} = g, \qquad \frac{\dot{C}_t^A}{C_t^A} = g\frac{C_t^A - \bar{A}}{C_t^A}, \qquad \frac{\dot{C}_t^S}{C_t^S} = g\frac{C_t^S + \bar{S}}{C_t^S}.$$

This balanced growth path features initially declining labor share in agriculture and a rising labor share in services. Asymptotically the shares stabilize, so the structural change is a feature of the initial conditions of the economy. Though some primitive relation between preferences and technology may be justified, Ngai and Pissarides (2007) point out that the cross-restriction imposed in (13) is particularly sharp and does not hold generically since it requires the ratio of Stone-Geary parameters, $\bar{A}/\bar{S}$, to be equal to the ratio of relative productivities, B_A/B_S.

Balanced Growth with HCD Preferences We can formulate a heterothetic Cobb-Douglas function to mimic the features of the Stone-Geary balanced growth path, with the advantage of balanced growth existing for a range of parameter values rather than the singleton implied by Equation (13). Preferences across time are the same as in the original paper and are defined in (12). Instead, intra-temporal preferences are given by the following HCD specification

$$\ln C_t = \beta\left(C_t\right) \ln C_t^A + \gamma \ln C_t^M + \theta\left(C_t\right) \ln C_t^S,$$

where $\beta(C_t)$ is decreasing in C_t, and $\beta(C_t)+\gamma+\theta(C_t)=1$, which implies that $\theta(C_t)$ is increasing.[11] The optimal consumption choices imply the following balanced growth path:

$$\frac{\dot{C}_t^M}{C_t^M}=g, \qquad \frac{\dot{C}_t^A}{C_t^A}=g+\frac{\dot{\beta}_t}{\beta_t}, \qquad \frac{\dot{C}_t^S}{C_t^S}=g+\frac{\dot{\theta}_t}{\theta_t},$$

which is again defined by a constant interest rate.

This balanced growth path also features a declining labor share in agriculture and a rising labor share in services. Whether this income effect asymptotically disappears or not is freely chosen based on the functional forms of $\beta(\cdot)$ and $\theta(\cdot)$. As anticipated, the cross-restriction (13) required in the original paper is no longer needed.

5 Continuous Sectoral Take-off under Endogenous Growth

Foellmi et al. (2008) provided the first endogenous growth model featuring continuous unbalanced growth across an ever-expanding number of sectors while maintaining balanced growth in the aggregate. In their model, using hierarchical preferences, newly innovated products start as luxuries and are consumed in small amounts. As income grows, these products are consumed in increasing amounts and gradually evolve into necessities. At the same time, newer, more income-elastic products are created. Moreover, as more sectors come into existence, the consumption shares of the initial sectors eventually begin to decline. The model beautifully features an ever-expanding array of sectors that come into existence, and whose expenditure shares rise and fall. We briefly outline their model to then juxtapose our extension, which substitutes hierarchical with HCD preferences.

Engel's Consumption Cycles under Hierarchical Preferences (Foellmi et al., 2008) Household intratemproal preferences over an infinite continuum of sectors, $\varepsilon\in[0,\infty)$ is given by

$$u\left(\{c_{\varepsilon,t}\}_{\varepsilon=0}^{\infty}\right)=\int_0^{\infty}\varepsilon^{-\gamma}\frac{1}{2}\left[s^2-(s-c_{\varepsilon,t})^2\right]\mathrm{d}\varepsilon,$$

where $\gamma\in[0,1]$ controls the degree of income effects across the sectors, and s defines the saturation level for any specific sector. Intertemporal preferences are the same as in the previous section, Equation (12), with an elasticity of intertemporal substitution $\frac{1}{\sigma}$ and time discounting ρ. Households can allocate consumption intertemporally via an asset with an interest rate r_t, which represents the total value of all firms.

[11] An example of a parametrization is $\beta(C_t)=(1-\gamma)\exp(-C_t)$ and $\theta(C_t)=(1-\gamma)(1-\exp(C_t))$. The sufficient condition for monotonicity, Equation (8), is satisfied if $\min\left\{C_t^A,C_t^S\right\}\geq 1$. Satisfying this monotonicity condition imposes a restriction on the initial wage and price level (note that if it is satisfied at the initial time, the condition is going to be satisfied going forward because there is aggregate growth and monotonicity is ensured). This condition is analogous to the condition imposed in Kongsamut et al. (2001) to ensure that the minimum subsistence requirement is met in the Stone-Geary demand system.

The technologies governing production and innovation are linear in the single factor of production, labor, whose supply is inelastic and normalized to one. Both technologies feature aggregate knowledge spillovers from the total number of innovated sectors at time t, N_t,

$$Y_{\varepsilon,t} = N_t L^Y_{\varepsilon,t}, \qquad \dot{N}_t = \beta N_t L^R_t,$$

where β is a positive parameter, and $L^Y_{\varepsilon,t}, L^R_t$ are the amount of labor allocated to production in sector ε or to innovation, respectively. In their model a firm acquires a patent of finite duration Δ by innovating in a new product. Once the patent expires, the sector becomes perfectly competitive. There is free entry into innovation. Thus, new products will be innovated until the sum of the future discounted profit from the patent equals the cost of innovating it. We refer the reader to the original paper for further details on their framework.

This model features multiple balanced growth path equilibria. Foellmi and Zweimuller argue that this is inherent to the hierarchical preferences and finite duration of patents. Also, their clean results on how any sector's income elasticity devolves from being a luxury into a necessity rely on an alternative version of their model with exogenous growth. The model also features innovation only on the extensive margin; that is, no innovation takes place within a sector, only for the development of new sectors, circumventing the concept of within-sector innovation.

Using a Logit-HCD functional form instead of hierarchical preferences, we can augment their model to feature endogenous sector-specific innovation of varying intensities across all sectors. Furthermore, the Logit-HCD formulation provides clean analytical results on each sector's unbalanced growth dynamics and the evolution of income elasticities that do not rely on the alternative exogenous growth version offered in the original paper.

Engel's Consumption Cycles with Directed Technical Change under HCD Preferences We impose a few changes to the framework of Foellmi et al. (2008), which despite making the framework seemingly more complex, lead to simpler analytical results. Arguably, our adjustments also benefit from being more closely aligned to what is "standard" in the endogenous growth literature. First, we let firms claim a perpetual patent on a newly innovated product (an assumption that is customary but counterfactual, unlike Foellmi and Zweimuller's). Second, we expand the product space to have a within-sector margin as well, $(\varepsilon, i) \in [0,\infty) \times [0, N_{\varepsilon,t}]$, where $N_{\varepsilon,t}$ is the number of products in sector ε.[12] Lastly, we let households have logit-HCD preferences across sectors, where the sectoral expenditure shares are given by $\alpha_\varepsilon(u_t) = u_t^{-\gamma}\exp(-u_t^{-\gamma}\varepsilon)$, where $\gamma > 0$ also the controls the strength of the non-homotheticicities as in Foellmi et al. (2008),

In sum, we have the following nested demand system,

$$0 = \int_0^\infty \alpha_\varepsilon(u_t)\ln\left(\frac{c_{\varepsilon,t}}{u_t}\right)\mathrm{d}\varepsilon, \qquad c_{\varepsilon,t} = N_{\varepsilon,t}^{-\frac{1}{\theta-1}}\left(\int_0^{N_{\varepsilon,t}} c_{\varepsilon i,t}^{(\theta-1)/\theta}\mathrm{d}i\right)^{\theta/(\theta-1)},$$

[12] In Bohr et al. (2021) we allow for a sector-specific mechanism rather than the aggregate knowledge spillovers in production, making the existence of balanced growth more demanding. However, this generates sectoral price dynamics that are consistent with those observed in the data.

where the within-sector price elasticity of substitution is given by $\theta > 1$. We mute the gains-from-variety effect in the CES aggregator above through the term $N_{\varepsilon,t}^{-\frac{1}{\theta-1}}$ to maintain a closer parallel with Foellmi et al. (2008). Instead we maintain their aggregate knowledge spillovers in production,

$$Y_{\varepsilon i,t} = N_t L_{\varepsilon i,t},$$

where $L_{\varepsilon i,t}$ denotes the amount of labor used in production of product εi.

The results of the model are sketched out below. The price of each product i in any sector ε is a markup over the marginal cost of production, $p_{\varepsilon i,t} = \frac{\theta}{\theta-1}\frac{W_t}{N_t}$, where W_t is the wage rate. Given the symmetry of prices and the lack of gains from variety, the sectoral price-index coincides with the price of products in the sector, $p_{\varepsilon,t} = p_{\varepsilon i,t}$.

There is free entry into innovation, and the cost of innovation is identical across sectors. Firms enter into all sectors until the future discounted profits of an additional new product in every sector are the same and equal to the labor cost of innovating. Since this will be true at any moment in time, profits at any moment are equalized across sectors too (i.e., the Hamilton Jacobi Bellman equation holds). The Hicksian demand for any given sector is given by $c_{\varepsilon,t} = p_{\varepsilon,t}^{-1}\alpha_\varepsilon(u_t)E_t$, where E_t denotes total expenditures. Given the free entry condition, there will be more innovation in sectors where demand is increasing the most. Since this entry happens until profits π_t are equalized across sectors, we can invert a firm's profit condition to find exactly how many products will be innovated in each sector. The distribution of products across sectors ends up perfectly mirroring the expenditure share distribution,

$$N_{\varepsilon,t} = N_t \alpha_\varepsilon(u_t), \qquad N_t = \int_0^\infty N_{\varepsilon,t}\mathrm{d}\varepsilon = \frac{1}{\theta}\frac{E_t}{\pi_t}.$$

As a result, the sectoral price distribution is the inverse of the exponential distribution of the number of products across sectors. Since there is less demand for higher ε sectors due to their higher income elasticities, less innovation has occurred in them, and therefore they feature higher prices. The ideal price index implied by the logit-HCD preferences delivers a log-linear mapping between utility and aggregate innovation, $u_t^{1+\gamma} = e^{-1}L_{Y,t}N_t$, where $L_{Y,t}$ is the total labor allocated to production aggregated across sectors. Under the normalization of the price index to 1,

$$\ln P = \int_0^\infty \alpha_\varepsilon(C)\ln\left(\frac{P_\varepsilon}{\alpha_\varepsilon(C)}\right)d\varepsilon = 0,$$

the household Euler equation is identical to that in standard one-sector growth models:

$$\frac{\dot{E}_t}{E_t} = \frac{1}{\sigma}(r_t - \rho).$$

We can then define an aggregate balanced growth path, as in standard endogenous growth models, which is characterized by a constant interest rate and constant shares of labor allocated to production and innovation.

Under this aggregate behavior, however, innovation will be directed toward more income-elastic sectors, as demand is shifted toward these more income-elastic sectors relative to less income-elastic sectors, making them grow faster. The expenditure elasticity of sector ε, $\eta_{\varepsilon,t} \equiv \frac{\partial \ln p_\varepsilon c_\varepsilon}{\partial \ln E}$, the sector's growth rate, and the moment the sector's value-added share peaks are given by

$$\eta_{\varepsilon,t} = 1 - \frac{\gamma}{1+\gamma}(1 - u_t^{-\gamma}\varepsilon), \qquad \frac{\dot{N}_{\varepsilon,t}}{N_{\varepsilon,t}} = \eta_{\varepsilon,t}\frac{\dot{N}_t}{N_t}, \qquad \varepsilon^{peak} = u_t^{\gamma},$$

respectively. As utility grows, the income elasticity of any sector falls. While aggregate growth is constant along the balanced growth path, the sectoral growth rate is determined by its income elasticity at that time. When the expenditure elasticity has fallen and equals unity, i.e., the moment the sector pivots from a luxury to a necessity, is also when the a sector's value-added share in the economy peaks.[13] In sum, by introducing HCD preferences, we can parsimoniously characterize the sectoral behavior, which is qualitatively similar to that in Foellmi and Zweimuller where sectors take off one after another according to their sectoral index.

6 Gains From Trade in a Ricardian Trade Model with Income Effects

In this section, we revisit the celebrated gains-from-trade formula from Arkolakis et al. (2012) (ACR) and Eaton and Kortum (2002), applied to the multi-sector Ricardian model of Costinot et al. (2012) (CDK). The introduction of nonhomotheticities to the CDK set-up implies that the effects of trade policy changes on welfare in a country depend on the interaction between its comparative advantage and its endogenous expenditure share. At a broad level, this has been already pointed out, for example by Caron et al. (2014)—see for example Equation (16) in their paper.

In this case, the introduction of HCD preferences allows us to obtain a very crisp result. Let us write the indirect utility function V as a function of income w

$$\ln V = \ln w - \sum_{i\in\mathcal{N}} s_i(V)\ln p_i = \sum_{i\in\mathcal{N}} s_i(V)\ln\frac{A_i}{\pi_i},$$

where A_i denotes the exogenous TFP of sector i, and π_i denotes the own trade shares in the country under study. Let x' denote the counterfactual value of x under the unobserved autarky regime, and let and let $\hat{V} \equiv V'/V$. We obtain the following augmented ACR formula

[13] To see how this model can be further generalized to other expenditure distributions, non-unitary own-price elasticity across sectors, as well as alternative drivers of growth and prosperity, see Bohr et al. (2021), who lay out a broader growth framework. In fact, the model here may be viewed as a simplified, alternative version of this paper, which can be pieced together from its appendices.

$$
\begin{aligned}
\ln \hat{V} &= \sum_{i \in \mathcal{N}} s_i \left(V'\right) \ln \frac{A_i}{\pi_i'} - \sum_{i \in \mathcal{N}} s_i \left(V\right) \ln \frac{A_i}{\pi_i} \\
&= \underbrace{\sum_{i \in \mathcal{N}} s_i \left(V\right) \ln \pi_i}_{\text{standard ACR}} + \underbrace{\sum_{i \in \mathcal{N}} \left[s_i \left(V'\right) - s_i \left(V\right)\right] \ln A_i}_{\text{novel}},
\end{aligned}
\tag{14}
$$

where the second equality uses $\pi_i' = 1$ (the autarky trade shares are equal to one).

The novel effect arises because expenditure shifts with utility. Importantly, the sign of the novel term, which results from nonhomothetic preferences, depends on the correlation between TFP (i.e., absolute advantage) and the income elasticity of demand for the various goods. The standard ACR term is negative (returning to autarky reduces welfare by foregoing gains from trade). The sign of the novel term is ambiguous.

In the generalized ACR formula of Equation (14), the s_i's and the π_i's are observed, and we can infer the A_i's by assuming perfect competition and observing goods and factor prices, but the counterfactual expenditure shares are not observed. However, we can bound this novel effect since the shares are, by definition, a number in the unit interval and sum up to one. We plan to explore this quantitative question in the future.

7 General Spatial Equilibrium with Income Effects

Virtually all quantitative general spatial equilibrium models assume homothetic Cobb-Douglas preferences for its tractability. The key property of these preferences that make such models tractable is the unitary own-price elasticity of demand. Though useful, the unitary income elasticity of demand is unnecessary for that purpose. We show that HCD preferences, combined with utility equalization over space, lead to a nearly as tractable a framework because this combination of assumptions ensures that expenditure shares are equalized across space regardless of the spatial variation of consumer prices.

We illustrate this point by encapsulating HCD preferences into a general spatial equilibrium model a-la Allen and Arkolakis (2014) and Redding (2016) in this section and into Krugman (1991) in Section 8.

7.1 Topography of the Spatial Economic Geography with Income Effects

Housing is a major component of virtually any household's expenditure or portfolio, and the income elasticity of housing is well below unity (Combes et al., 2018). It follows, then, that the equilibrium mobility response of households, which depends on this key elasticity, depends, in turn, on their economic well-being. To make this point, we build a quantitative general spatial equilibrium model a-la Allen and Arkolakis (2014) and Redding (2016) featuring housing as a necessity.

We denote by C the set of cities / regions. We reset our notation and use subscript i to denote cities. Consider two goods, housing (not traded), whose unit price is r_i, and the traded good, with consumer price index P_i. We also allow for consumption amenities $B_i > 0$, which enter the expenditure function in a multiplicatively separable way; that is to say, higher amenities imply that the level of expenditure required to attain utility level u_i at prices (P_i, r_i) is lower. The expenditure function is thus:

$$\ln E\left(P_i, r_i, u\right) = -\ln B_i + s_i \ln\left(\frac{P_i}{s_i}\right) + \left(1 - s_i\right) \ln\left(\frac{r_i}{1 - s_i}\right) + \ln u_i, \qquad s_i \in (0, 1)\,, \tag{15}$$

where s_i denotes the expenditure share on the traded good.

Individuals supply labor, are free to move across cities, and consume housing and the traded good; local landowners own land and consume the traded good only. Let L_i denote the (endogenous) population in i, with

$$\sum_{i \in C} L_i = \overline{L}, \tag{16}$$

where $\overline{L}$ is total population (exogenous). Let w_i denote the wage in location i, and let H_i denote the inelastically supplied stock of housing in i.[14] Then, local housing markets clear if and only if the value of supply is equal to the value of demand:

$$r_i H_i = \left(1 - s_i\right) w_i L_i. \tag{17}$$

In equilibrium, expenditure E_i is equal to labor wage w_i. Together with (15), (17) this yields

$$s_i \ln w_i + \ln B_i = s_i \ln\left(\frac{P_i}{s_i}\right) + \left(1 - s_i\right) \ln\left(\frac{L_i}{H_i}\right) + \ln u_i. \tag{18}$$

Clearly, a higher population in i or a higher price for tradables must be compensated by a higher wage in the same location to offset higher living costs.

For convenience, let the traded good be differentiated by origin (the Armington hypothesis), with a constant elasticity of substitution of $\sigma > 0$. The sector is competitive and produces with a constant unit cost of w_i/A_i, where A_i denotes TFP. Let $T_{ij} > 1$ denote the iceberg transportation cost from i to j. The value of exports from i to j is equal to

$$X_{ij} = \left(\frac{T_{ij} w_i}{A_i P_j}\right)^{1-\sigma} w_j L_j, \tag{19}$$

which requires $\sigma > 1$ for trade volumes to fall with trade costs (and thus distance), as is empirically relevant. The price index obeys

$$P_j^{1-\sigma} = \sum_{i \in C} \left(T_{ij} \frac{w_i}{A_i}\right)^{1-\sigma}. \tag{20}$$

[14] We assume an inelastically supplied housing stock for simplicity. This assumption can be easily relaxed to make housing supply elastic.

7.2 Equilibrium

Housing markets clear if (17) holds. Utility of mobile workers is equalized across space if it exists a $u > 0$ such that $w_i = E(P_i, r_i, u)$ whenever $L_i > 0$, and $w_i \leq E(P_i, r_i, u)$ otherwise.[15] That is, in a spatial equilibrium, wages must cover expenditure given local prices in all locations with a positive population. It also follows from $u_i = u$ that $s_i = s$, for some $s \in (0, 1)$. Thus, at any spatial equilibrium:

$$L_i > 0 \;\; \Rightarrow \;\; \ln E(P_i, r_i, u) = \ln w_i = -\frac{1}{s}\ln B_i + \ln P_i + \frac{1-s}{s}\ln\left(\frac{L_i}{H_i}\right) + \frac{1}{s}\ln u - \ln s. \tag{21}$$

The market of traded good i clears if the following expression holds:

$$w_i L_i = \sum_{j \in C} X_{ij} = \left(\frac{w_i}{A_i}\right)^{1-\sigma} \sum_{j \in C}\left(\frac{T_{ij}}{P_j}\right)^{1-\sigma} w_j L_j, \tag{22}$$

where the second equality uses (19). Finally, full employment (16) holds.

Together, (21) and (22) yield what is known as the *wage equation* in the "New Economic Geography":[16]

$$\begin{aligned} L_i w_i^{\sigma} A_i^{1-\sigma} &= \sum_{j \in C}\left(\frac{T_{ij}}{P_j}\right)^{1-\sigma} w_j L_j \\ &= s^{\sigma-1} u^{-\frac{\sigma-1}{s}} \sum_{j \in C} T_{ij}^{1-\sigma} w_j^{\sigma} B_j^{\frac{\sigma-1}{s}} H_j^{\frac{(\sigma-1)(1-s)}{s}} L_j^{1-\frac{(\sigma-1)(1-s)}{s}}. \end{aligned} \tag{23}$$

By the same token, combining (18) with the price index (20) yields the so-called *consumer market access* equation:

$$w_i^{1-\sigma}\left(\frac{L_i}{H_i}\right)^{\frac{(\sigma-1)(1-s)}{s}} B_i^{-\frac{\sigma-1}{s}} = s^{\sigma-1} u^{-\frac{\sigma-1}{s}} \sum_{j \in C}\left(T_{ji}\frac{w_j}{A_j}\right)^{1-\sigma}. \tag{24}$$

We show in Appendix B that, if trade costs are quasi-symmetric, then dividing both sides of

[15] That is, $u_i = u$ for all i and some $u > 0$, in a spatial equilibrium with free mobility. Thus the expenditure share is constant across space, too (i.e., $s_i = s$ for all i and some $0 < s < 1$), which is broadly consistent with the empirical observation of Davis et al. (2011), even though preferences are not homothetic Cobb-Douglas.

[16] Guide to calculations:

$$\begin{aligned} L_i w_i^{\sigma} A_i^{1-\sigma} &= s \sum_{j \in C}\left(\frac{T_{ij}}{P_j}\right)^{1-\sigma} w_j L_j = s^{\sigma} u^{-\frac{\sigma-1}{s}} \sum_{j \in C} T_{ij}^{1-\sigma} w_j^{\sigma-1} B_j^{\frac{\sigma-1}{s}} \left(\frac{H_j}{L_j}\right)^{\frac{(\sigma-1)(1-s)}{s}} w_j L_j \\ &= s^{\sigma} u^{-\frac{\sigma-1}{s}} \sum_{j \in C} T_{ij}^{1-\sigma} w_j^{\sigma} B_j^{\frac{\sigma-1}{s}} H_j^{\frac{(\sigma-1)(1-s)}{s}} L_j^{1-\frac{(\sigma-1)(1-s)}{s}}. \end{aligned}$$

(23) by those of (24) yields:[17]

$$L_i^{\frac{\tilde{\sigma}}{s}} H_i^{-\frac{\sigma\tilde{\sigma}(1-s)}{s}} A_i^{-(\sigma-1)\tilde{\sigma}} B_i^{-\frac{\sigma\tilde{\sigma}}{s}} = s^{\sigma-1} u^{-\frac{\sigma-1}{s}} \sum_{j\in C} T_{ij}^{1-\sigma} A_j^{\sigma\tilde{\sigma}} B_j^{\frac{(\sigma-1)\tilde{\sigma}}{s}} H_j^{\frac{(\sigma-1)\tilde{\sigma}(1-s)}{s}} L_j^{[1-\sigma(1-s)]\frac{\tilde{\sigma}}{s}}, \qquad (25)$$

where

$$\tilde{\sigma} \equiv \frac{\sigma-1}{2\sigma-1} \in \left(0, \frac{1}{2}\right).$$

Three results follow. First, given u (and hence s), a unique equilibrium exists by $\sigma(1-s)) > 0$.[18] Second, with s as an endogenous outcome, the conditions for equilibrium existence uniqueness are different from these papers. We conjecture that conditions for existence are rather generic, but that conditions for uniqueness may be quite involved; we leave them for future work. Third, the equilibrium elasticity of L_i with respect to fundamentals (A_i, B_i, H_i) is governed by σ and s; treating the right-hand side of (25) as constant in a first-order approximation

$$\frac{\partial \ln L_i}{\partial \ln A_i} \approx (\sigma-1)\, s, \qquad \frac{\partial \ln L_i}{\partial \ln B_i} \approx \sigma, \qquad \frac{\partial \ln L_i}{\partial \ln H_i} \approx \sigma(1-s).$$

That is to say, the migration response to a local improvement in productivity (or to a reduction of the housing supply) is increasing in s (and decreasing in the housing expenditure share $1-s$). When economic well-being increases, then s increases and the absolute value of these elasticities increases. This argument implies that *mobile workers become more reactive to heterogeneous local conditions.*

8 Regional Disparities and Structural Change

Krugman (1991) introduces a two-region, two-sector, two-factor model featuring factor mobility, trade costs, and increasing returns to scale internal to (manufacturing) firms. He shows in a parsimonious way how regional disparities can emerge when transportation costs fall, even among otherwise similar regions. In Krugman's model, and in the entire so-called New Economic Geography (henceforth NEG) literature that ensued, a high share of manufacturing in GDP is associated with stronger agglomeration forces (since only manufacturing displays localized external scale economies by assumption) and higher regional disparities. Krugman's paper uses homothetic preferences., Hence the share of manufacturing in national income is invariant in real per capita incomes. As a result, the model cannot be used to study the interaction between structural change and regional disparities. Here, we show how we can use heterothetic Cobb-Douglas preferences to do just this exercise in a parsimonious way. In particular, we show that as technical progress lifts incomes, the expenditure share on manufacturing increases, and, in turn, regional disparities emerge.

[17] Trade costs are said to be quasi-symmetric if $T_{ij} = 1$ for $j = i$, and, for $i \neq j$, (i) $T_{ji} = T_i T_j D_{ji}$ and (ii) $D_{ji} = D_{ij}$.

[18] That is, the exponent of L_i on the left-hand side of (25) is larger than that of L_j on its right-hand side. The proof follows that of Theorem 1 in Allen and Arkolakis (2014) and Proposition 1 in Redding (2016).

NEG models that venture beyond the combination of the following assumptions are unwieldy, and only a few analytical results are available. These assumptions are two symmetric regions, iceberg transportation costs, and Cobb-Douglas preferences. It turns out that the key property of the latter that leads to analytically tractable result is the *unit price elasticity of demand*. The HCD preference system is the only system of nonhomothetic preferences featuring a unit price elasticity of demand. Hence it is also unique in enabling us to study the endogenous joint emergence of regional disparities and structural change in a NEG framework.

8.1 Model

Consider two regions, $i, j \in \{1, 2\}$ having access to identical technologies, two sectors, agriculture A and manufacturing M, and two factors, unskilled labor L and skilled labor or human capital H (alternatively, L can be interpreted as land and H as labor).

Preferences For analytical simplicity, unskilled workers consume the manufacturing good only. Skilled workers have Heterothetic Cobb-Douglas preferences over A and M, with equilibrium expenditure shares $1 - s(u)$ and $s(u)$, respectively, where u denotes the level of utility of skilled workers. Manufacturing is the luxury good (Agriculture is the necessary good) so that $s'(u) > 0$. They also have CES preferences over the varieties produced by M, and $\sigma > 2$ denotes the elasticity of substitution.[19]

Endowments Both regions host $L/2$ immobile unskilled workers. Region 1 hosts λH skilled workers (Region 2 hosts $(1 - \lambda) H$ skilled workers). Skilled workers move to the region that offers the highest real returns, hence λ is an endogenous variable of the model; as we shall see in Subsection 8.2 below, $\lambda \in \left\{0, \frac{1}{2}, 1\right\}$ are typical equilibrium outcomes in this model.[20] Finally, we set $L = H = 1$ without loss of generality.

Technology, trade, and migration Agriculture uses labor L only, under constant returns to scale, and $A = aL_A$, where A denotes agriculture TFP, $a > 0$ is labor productivity, and L_A is labor used in agriculture. Using labor in region 1 as the numeraire, we get $p_1 = 1/a$. Its output is freely traded, hence $p_j = 1/a$ and $w_j = 1$, $j = 1, 2$. Workers are geographically immobile, but move freely across sectors. Manufacturing is composed of firms producing a differentiated good. Each firm needs one unit of human capital as a startup cost, and each unit of output requires a_M units of labor, hence the cost function of any manufacturing firm is $C_j(x) = \pi_j + a_M w_j x$, where π_j denotes human capital wage and x denotes output; without loss of generality, we choose units

[19]Two comments are in order. First, if $s(u) = \mu$, for some $\mu \in (0, 1)$, then preferences correspond to homothetic Cobb-Douglas and this version of the model features the same equilibrium properties as Krugman's original model, as well as many others (Robert-Nicoud, 2005). Thus, the only meaningful departure from Krugman's model is the introduction of nonhomothetic Cobb-Douglas preferences. Second, we impose $\sigma > 2$ to avoid a typology of cases.

[20]The model assumes two regions endowed with identical fundamentals, hence $\lambda = \frac{1}{2}$ is always an equilibrium. The model also famously features self-enforcing agglomeration mechanisms, hence $\lambda = 0$ and its mirror-image $\lambda = 1$ are also equilibrium outcomes under some parameter configurations. Subsection 8.2 provides details.

for M such that $a_M = 1 - 1/\sigma$, so that the FOB price is 1 in equilibrium. This good is traded at an iceberg cost $T > 1$. Let $\phi \equiv T^{1-\sigma}$ denote a free-ness of trade measure, where $\phi = 0$ when trade costs are prohibitive, and $\phi = 1$ when Regions 1 and 2 are fully integrated. Then, the price-index for manufacturing varieties in Region j, P_j, obeys

$$P_j^{1-\sigma} = \lambda_j + T^{1-\sigma}\left(1 - \lambda_j\right) = \lambda_j + \phi\left(1 - \lambda_j\right). \tag{26}$$

Technical progress We model technical progress as an exogenous increase in a. We show below that an increase in labor productivity a will result in an increase in the manufacturing expenditure share s at any given equilibrium spatial configuration.[21]

8.2 Equilibrium

As is customary in this literature, we solve for the equilibrium in two steps. First, take the location of mobile skilled workers as given, i.e., treat λ as parametric. Under the assumptions of Section 8.1, it is straightforward to show that market clearing in manufacturing in Region j requires:

$$\sigma\pi_j = \frac{\lambda_j \pi_j s_j + \frac{1}{2}}{\lambda_j + \phi\left(1 - \lambda_j\right)} + \phi\frac{\lambda_i \pi_i s_i + \frac{1}{2}}{\lambda_i + \phi\left(1 - \lambda_i\right)}, \quad \lambda_i + \lambda_j \equiv 1. \tag{27}$$

The left hand side of this expression is the revenue of a typical manufacturing firm in j (with Dixit-Stiglitz monopolistic competition, the return to the fixed factor is a constant fraction $1/\sigma$ of this revenue), and the right-hand side is sales in Regions j and $i \neq j$, respectively. Using (6) and letting $u = Us^{-s}\left(1 - s\right)^{-1+s}$, expenditure in Region j is equal to

$$\ln E\left(p_j, P_j, u\right) = -\left(1 - s_j\right)\ln\left(a\right) + s_j \ln\left(P_j\right) + \ln u, \tag{28}$$

where s_j is increasing in u by assumption. Since E is increasing in u, it follows that an exogenous increase in a results in an increase in u, given E. In turn, s increases since manufacturing is the income-elastic good.

Second, we let skilled workers search for the highest real wage, and hence let λ be an equilibrium outcome. A spatial equilibrium arises whenever $u_j \leq u$, some $u > 0$, for all $j \in \{1, 2\}$, with equality if $\lambda_j > 0$. In order to boost economic intuition, it is useful to introduce some change of variables before we start with the conditions for the latter outcome to be an equilibrium.

Let

$$\delta\left(u\right) \equiv \frac{L/2}{\pi_j H + L/2} = \frac{\sigma - s\left(u\right)}{\sigma + s\left(u\right)} \tag{29}$$

denote expenditure in the periphery relative to expenditure in the core for $\lambda_j = 1$, with $0 < \delta\left(u\right) < 1$ by $\sigma > 2$ and $s\left(u\right) \in (0, 1)$. The second equality above comes from using (27) and from setting

[21] Note that modeling technical progress as an increase in the productivity in agriculture only is without loss of generality. All meaningful results are isomorphic if one considers homogeneous increases in both Agriculture and Manufacturing TFP. The reason for this result is the unitary elasticity of substitution between the two consumption goods.

$H = L = 1$. Above, $\delta(u)$ is a measure of the strength of dispersion forces in the model that stem from immobile demand; $\delta(u)$ is equal to unity in the absence of agglomeration forces, and it is decreasing in the expenditure share on the manufacturing good, $s(u)$. Hence, δ is decreasing in u. Let also

$$\theta(u) \equiv \frac{s(u)}{\sigma - 1} \tag{30}$$

capture agglomeration economies that arise from cost linkages, with $0 < \theta(u) < 1$ by $\sigma > 2$ and $s(u) \in (0, 1)$. High-skilled workers are mobile and hence they care about both nominal factor returns and local prices, and $\theta(u)$ captures the importance of the latter (if these workers cared only about nominal returns, then $\theta(u)$ would be equal to zero). Here, θ is increasing in u.[22]

In such symmetric models, a symmetric equilibrium with $\lambda_1 = \lambda_2 = \frac{1}{2}$ always exists. It may not always be stable, though; we return to this question below. But a Core-Periphery equilibrium with all skilled workers agglomerated in a single region, with $\lambda_j = 1$ and $\lambda_i = 0$, may also exist. We turn to this case first.

Sustain Point What are the conditions that ensure $\lambda \in \{0, 1\}$ is an equilibrium outcome? Without loss of generality, assume $\lambda = 1$. If the Core-Periphery configuration $\lambda = 1$ is an equilibrium, then $\pi_1(u) = E_1(u)$ (high-skill income in Region 1 covers the expenditure required to achieve utility level u) and $\pi_2(u) \leq E_2(u)$ (high-skill income in Region 2 is insufficient to cover the level of expenditure required to achieve u). The standard practice in the NEG is to find a threshold value for ϕ in the unit interval, known as the "Sustain Point" ϕ^{Sust}, such that the latter inequality holds strictly for $\phi > \phi^{\text{Sust}}$ and is violated for $\phi < \phi^{\text{Sust}}$. We show in Appendix C that such a $\phi^S \in (0, 1)$ exists if $\sigma > 2$ (which ensures $0 < \theta < 1$). The (implicit solution) for the sustain point is $\phi^{\text{Sust}} \in (0, 1)$, where

$$\phi^{\text{Sust}} = \left\{ \phi \in (0, 1) : 0 = \delta + \phi^2 - [1 + \delta] \, \phi^{1-\theta} \right\}. \tag{31}$$

We show in Appendix C that utility u of mobile workers is increasing in the level of technology a in the Core-Periphery equilibrium. Thus, the higher labor productivity a, the lower the trade cost threshold (corresponding to ϕ^{Sus}) below which regional disparities can emerge.

Break Point The symmetric configuration $\lambda = 1/2$ is always an equilibrium of this symmetric model, but in may not be stable, in the following sense. The symmetric equilibrium is said to be unstable if, following an exogenous migration shock $\hat{\lambda}$, the change in income, $\hat{\pi}$, is larger than the change in expenditure required to maintain utility, $\hat{E}$, and to be stable otherwise (here we use "hats" to denote log changes). We are interested in the parameter configuration such that the two effects are exactly equal at the margin, $\hat{\pi} = \hat{E} = s\hat{P}$, in which case utility (and hence expenditure

[22] The attentive reader may have noticed that expenditure shares generically differ across locations. However, when studying the symmetric or Core-Periphery equilibriums, there is a unique such share: in the symmetric equilibrium, $s_j(u) = s$, some u, all $j \in \{1, 2\}$, whereas in the Core-Periphery equilibrium, all mobile workers locate in a single region, and the expenditure share of mobile workers in the other region is moot.

shares) are invariant. The break point for ϕ is equal to

$$\phi^{\text{Break}} = \frac{1-\theta}{1+\theta}\delta, \tag{32}$$

where the second equality comes from (29) and (30). The symmetric equilibrium is unstable if $\phi > \phi^{\text{Break}}$ and stable otherwise. We show in Appendix C that this break point belongs to the interior of the unit interval if a is high enough ($\phi^{\text{Break}} = 1$ and hence the symmetric equilibrium is stable for any ϕ otherwise), and it is decreasing in u by $\partial\theta/\partial u > 0$ and $\partial\delta/\partial u < 0$.[23] Finally, we show in Appendix C that utility u of mobile workers is increasing in the level of technology a and of trade free-ness ϕ in the symmetric equilibrium. Thus, the higher labor productivity a, the lower the trade cost threshold (corresponding to ϕ^{Break}) below which regional disparities must emerge is.

8.3 Regional Disparities and Structural Change

In our model, exogenous technical change in agriculture leads to both structural change and to the emergence of regional disparities. To summarize the results of this section, we have shown:

Proposition 1. *Regional Disparities and Structural Change. Consider the model in Section 8.1. (i) The symmetric equilibrium is stable for any level of trade costs when labor productivity is low enough. (ii) When labor productivity is high enough, the Core-Periphery outcome is an equilibrium outcome if and only if trade costs are low enough. (iii) The higher labor productivity, the lower the trade cost threshold below which regional disparities emerge.*

Proof. In the text and in Appendix C. □

A few remarks are in order. First, the expressions characterizing the spatial equilibria in (31) and (32) are identical to those in Robert-Nicoud (2005) and Ottaviano and Robert-Nicoud (2006), which establish that Krugman's original model and many others are isomorphic once written in the appropriate state space. The only difference between our current model and the models studied in these papers is encapsulated in s alone: s is an endogenous variable that captures income effects in our model, whereas s is a parameter under homothetic Cobb-Douglas preferences in all models considered in these papers—in which income effects are not present by construction. Second, using standard algebra, it is straightforward to show that the break and sustain points are both increasing in the strength of dispersion forces $\delta(u)$ and decreasing in the strength of agglomeration forces $\theta(u)$, making both the conditions for a Core-Periphery equilibrium to emerge and those for the symmetric equilibrium be unstable more likely to hold as u rises. Thus, as technical progress advances, s rises and the range of parameters other than TFP a that make the symmetric equilibrium unstable increases: *structural change fosters regional disparities.* Third, if $1 < \sigma \leq 2$ then the symmetric equilibrium is unstable (and the Core-Periphery equilibrium exists) for all values of s in the unit

[23] Importantly, (32) contains endogenous variables on both sides of the equation. The so-called break point is a fixed point satisfying both (32) and (43). We show in Appendix (C) that a unique fixed point exists.

interval. Finally, the model may also admit two interior, asymmetric equilibriums (symmetric to one another) but, when they exist, they are unstable (Robert-Nicoud, 2005).

9 The Monocentric City Model with HCD Preferences

In this section, we study the standard monocentric city model of Alonso (1964); Mills (1967); Muth (1969) (AMM). Each worker works at the Central Business District (CBD), and earns an idiosyncratic, exogenous wages w drawn from a twice-continously differentiable cumulative distribution $F(w)$, with probability density function denoted $f(w)$, and support $\Omega \equiv [1, W]$, with $W > 1$ (the normalization of the lower bound to unity is without loss of generality). There are N workers in this city.

Preferences Workers supply their labor in the CBD, and consume housing services and a numeraire good. This good is freely traded and, thus, the law of one price holds. Each worker residing at distance $x > 0$ from the CBD has to incur a utility cost $T(x)$ of commuting, with $T(0) = 0$ and $T'(x) > 0$. The unit price of housing, $r(x)$, depends on distance to the CBD and it is determined in equilibrium. The indirect utility function $V(p, r(x), w, x)$ is the implicit solution for u in

$$\ln u = \ln w - s(u) \ln r(x) - T(x). \tag{33}$$

Housing is a necessity, hence, $s' < 0$.

Bid Rent Curves and Spatial Income Sorting What is the willingness to pay for living at distance x from the CBD of a worker with wage w and arbitrary level of utility u? Her bid rent curve $\Psi(x, u, w)$ provides the answer to this question. Substituting $\Psi(x, u, w)$ for $r(x)$ in (33), differentiating the resulting expression , and setting $\mathrm{d}u = 0$, leads to the following slope of her bid rent curve:[24]

$$\frac{\partial \ln \Psi(x, u, w)}{\partial x} = -\frac{1}{s(u)} T'(x), \tag{34}$$

The right-hand side of this expression comprises an idiosyncratic component, $s(u)$, and a common component, $T'(x)$. Observe that the right-hand side of (34) does not depend on w, which is a consequence of HCD preferences. In what follows, we thus write the bid-rent curve as $\Psi(x, u)$ for short. The bid-rent curve of high-u individuals (who, in equilibrium, are high-w workers) is steeper than the bid-rent curve of low-wage workers:

$$\frac{\partial^2 \ln \Psi(x, u)}{\partial \ln x \partial u} = \frac{1}{s(u)} \frac{\partial \ln s(u)}{\partial u} T'(x),$$

[24] Or, equivalently, to its elasticity:

$$\frac{\partial \ln \Psi(x, u, w)}{\partial \ln x} = -\frac{1}{s(u)} \frac{\partial T(x)}{\partial \ln x}.$$

which is decreasing by $T'(x) > 0$ and $s'(u) < 0$. In equilibrium, then, high-wage workers live closer to the CBD than low-wage workers. This pattern prevails in most European cities.

To close the model, we have to integrate the slope of the bid rent curve and to solve for the assignment of workers, w, to locations, x. Appendix D provides the details.

10 Conclusion

This paper studies a class of preferences that we call Heterothetic Cobb-Douglas (HCD). These preferences feature unitary own-price elasticity and non-unitary income effects. As a result, differences in expenditure shares for a given good are solely due to income effects. HCD preferences yield a simple demand whose tractability mirrors in many respects that of homothetic Cobb-Douglas preferences. We illustrate the potential usefulness of HCD in various applications in different fields (economic growth, international trade, economic geography, and inflation costs). We believe that HCD preferences can help incorporate nonhomotheticicities and deliver new economic insights in many other settings—including some for which the use of nonhomothetic demand systems has proven hard and remains anecdotical. We look forward to these applications, and we hope to contribute to this endeavor in the future.

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A HCD as a Production Function

We may also apply the functional form introduced in Section 2 to nonhomothetic production functions, as in Hanoch (1975).

For example, with two factors, K, L, we can write

$$\min_{K,L} wL + rK \quad \text{s.t.} \quad 0 = \alpha(Y) \ln\left(\frac{K}{\tau_K(Y)\,Y}\right) + \beta(Y) \ln\left(\frac{L}{\tau_L(Y)\,Y}\right),$$

where Y denotes output, and $\alpha, \beta, \tau_K, \tau_L$ are functions of Y (the τ's provide a degree of freedom to be used below). Let λ denote the Lagrange multiplier associated with the problem above.[25] The total cost of production is equal to

$$C \equiv wL + rK = \lambda(\alpha + \beta),$$

[25] We obtain as first order conditions $wL = \lambda\beta; rK = \lambda\alpha$.

and output is equal

$$\ln Y = \frac{\alpha}{\alpha+\beta}\ln\left(\frac{C}{\tau_K r}\frac{\alpha}{\alpha+\beta}\right)+\frac{\beta}{\alpha+\beta}\ln\left(\frac{C}{\tau_L w}\frac{\beta}{\alpha+\beta}\right),$$

Let $s \equiv \alpha/\left(\alpha+\beta\right)$, $\tau_K \equiv s$, and $\tau_L \equiv 1-s$. We can invert this expression to obtain the cost function:

$$\ln C = \ln Y + s\ln r + (1-s)\ln w \quad \Leftrightarrow \quad C\left(Y,w,r\right) = Y r^s w^{1-s}.$$

The average and marginal costs are equal to

$$AC \equiv \frac{C}{Y} = r^s w^{1-s}, \quad MC \equiv \frac{\partial C}{\partial Y} = AC + Y\frac{\partial}{\partial Y} r^s w^{1-s}.$$

In this case, we have increasing returns to scale, $MC < AC$ if and only if

$$\frac{\partial}{\partial Y} r^s w^{1-s} < 0 \quad \Leftrightarrow \quad \ln\left(\frac{r}{w}\right)\frac{\partial s}{\partial Y} < 0.$$

Thus, if we assume that production becomes more K-intensive as output grows, $\partial s/\partial Y > 0$, this condition requires $r < w$ (which, being set in equilibrium, is external to the firm).

B General Spatial Equilibrium with Income Effects

B.1 Guide to Calculations of Section 7

If trade costs are quasi-symmetric, then dividing both sides of (23) by those of (24) yields:[26]

$$w_i^{2\sigma-1} H_i^{\frac{(\sigma-1)(1-s)}{s}} L_i^{1-\frac{(\sigma-1)(1-s)}{s}} B_i^{\frac{\sigma-1}{s}} A_i^{1-\sigma} = \kappa, \tag{35}$$

some $\kappa \in \mathbb{R}_{++}$.

Using (35) to substitute for w_i in either (23) or (24) yields:[27]

$$L_i^{\frac{\tilde{\sigma}}{s}} H_i^{-\frac{\sigma\tilde{\sigma}(1-s)}{s}} A_i^{-(\sigma-1)\tilde{\sigma}} B_i^{-\frac{\sigma\tilde{\sigma}}{s}} = s^{\sigma-1} u^{-\frac{\sigma-1}{s}} \sum_{j\in C} T_{ij}^{1-\sigma} A_j^{\sigma\tilde{\sigma}} B_j^{\frac{(\sigma-1)\tilde{\sigma}}{s}} H_j^{\frac{(\sigma-1)\tilde{\sigma}(1-s)}{s}} L_j^{[1-\sigma(1-s)]\frac{\tilde{\sigma}}{s}},$$

which is (25) in the text, and where

$$\tilde{\sigma} \equiv \frac{\sigma-1}{2\sigma-1} \in \left(0, \frac{1}{2}\right).$$

[26] Guide to calculations. Let $\tilde{s} \equiv 1 - \frac{(\sigma-1)(1-s)}{s}$, where $\tilde{s} < 1$. Dividing both sides of (23) by those of (24) yields:

$$\begin{aligned}
w_i^{2\sigma-1} H_i^{1-\tilde{s}} L_i^{\tilde{s}} B_i^{\frac{\sigma-1}{s}} A_i^{1-\sigma} &= \frac{\sum_{j\in C} T_{ij}^{1-\sigma} w_j^{\sigma} H_j^{1-\tilde{s}} L_j^{\tilde{s}} B_j^{\frac{\sigma-1}{s}}}{\sum_{j\in C} T_{ji}^{1-\sigma} w_j^{1-\sigma} A_j^{\sigma-1}} \\
&= \frac{w_i^{\sigma} H_i^{1-\tilde{s}} L_i^{\tilde{s}} B_i^{\frac{\sigma-1}{s}} + T_i^{1-\sigma} \sum_{j\neq i} D_{ij}^{1-\sigma} T_j^{1-\sigma} w_j^{\sigma} H_j^{1-\tilde{s}} L_j^{\tilde{s}} B_j^{\frac{\sigma-1}{s}}}{w_i^{1-\sigma} A_i^{\sigma-1} + T_i^{1-\sigma} \sum_{j\neq i} D_{ij}^{1-\sigma} T_j^{1-\sigma} w_j^{1-\sigma} A_j^{\sigma-1}} \\
&= w_i^{2\sigma-1} H_i^{1-\tilde{s}} L_i^{\tilde{s}} B_i^{\frac{\sigma-1}{s}} A_i^{1-\sigma} \frac{1 + T_i^{1-\sigma} \sum_{j\neq i} \frac{D_{ij}^{1-\sigma} T_j^{1-\sigma} w_j^{\sigma} H_j^{1-\tilde{s}} L_j^{\tilde{s}} B_j^{\frac{\sigma-1}{s}}}{w_i^{\sigma} H_i^{1-\tilde{s}} L_i^{\tilde{s}} B_i^{\frac{\sigma-1}{s}}}}{1 + T_i^{1-\sigma} \sum_{j\neq i} \frac{D_{ij}^{1-\sigma} T_j^{1-\sigma} w_j^{1-\sigma} A_j^{\sigma-1}}{w_i^{1-\sigma} A_i^{\sigma-1}}} \\
\Leftrightarrow \quad 1 &= \frac{1 + T_i^{1-\sigma} \sum_{j\neq i} \frac{D_{ij}^{1-\sigma} T_j^{1-\sigma} w_j^{\sigma} H_j^{1-\tilde{s}} L_j^{\tilde{s}} B_j^{\frac{\sigma-1}{s}}}{w_i^{\sigma} H_i^{1-\tilde{s}} L_i^{\tilde{s}} B_i^{\frac{\sigma-1}{s}}}}{1 + T_i^{1-\sigma} \sum_{j\neq i} \frac{D_{ij}^{1-\sigma} T_j^{1-\sigma} w_j^{1-\sigma} A_j^{\sigma-1}}{w_i^{1-\sigma} A_i^{\sigma-1}}} \\
\Leftrightarrow \quad 0 &= \sum_{j\neq i} D_{ij}^{1-\sigma} T_j^{1-\sigma} \left(\frac{w_j^{\sigma} H_j^{1-\tilde{s}} L_j^{\tilde{s}} B_j^{\frac{\sigma-1}{s}}}{w_i^{\sigma} H_i^{1-\tilde{s}} L_i^{\tilde{s}} B_i^{\frac{\sigma-1}{s}}} - \frac{w_j^{1-\sigma} A_j^{\sigma-1}}{w_i^{1-\sigma} A_i^{\sigma-1}} \right) \\
&= \sum_{j\neq i} D_{ij}^{1-\sigma} T_j^{1-\sigma} \frac{w_j^{1-\sigma} A_j^{\sigma-1}}{w_i^{1-\sigma} A_i^{\sigma-1}} \left(\frac{w_j^{2\sigma-1} H_j^{1-\tilde{s}} L_j^{\tilde{s}} B_j^{\frac{\sigma-1}{s}} A_j^{1-\sigma}}{w_i^{2\sigma-1} H_i^{1-\tilde{s}} L_i^{\tilde{s}} B_i^{\frac{\sigma-1}{s}} A_i^{1-\sigma}} - 1 \right).
\end{aligned}$$

[27] Guide to calculations:

$$\begin{aligned}
LHS \equiv L_i A_i^{1-\sigma} & \left[\frac{H_i^{\frac{(\sigma-1)(1-s)}{s}} L_i^{1-\frac{(\sigma-1)(1-s)}{s}} B_i^{\frac{\sigma-1}{s}} A_i^{1-\sigma}}{\kappa} \right]^{-\frac{\sigma}{2\sigma-1}} \\
&= L_i^{\frac{1}{2\sigma-1}\left[2\sigma-1-\sigma+\frac{(\sigma-1)(1-s)}{s}\right]} H_i^{-\frac{\sigma(\sigma-1)(1-s)}{(2\sigma-1)s}} A_i^{-\frac{(\sigma-1)^2}{2\sigma-1}} B_i^{-\frac{\sigma(\sigma-1)}{(2\sigma-1)s}} \kappa^{\frac{\sigma}{2\sigma-1}} \\
&= L_i^{\frac{\sigma-1}{(2\sigma-1)s}} H_i^{-\frac{\sigma(\sigma-1)(1-s)}{(2\sigma-1)s}} A_i^{-\frac{(\sigma-1)^2}{2\sigma-1}} B_i^{-\frac{\sigma(\sigma-1)}{(2\sigma-1)s}} \kappa^{\frac{\sigma}{2\sigma-1}}
\end{aligned}$$

B.2 Endogenous Amenities

Following Allen and Arkolakis, we now allow for spillovers in production (we omit spillovers in consumption, since housing plays this role already):

$$A_i = \overline{A}_i L_i^{\alpha}, \tag{36}$$

where $\overline{A}_i$ is exogenous, and $\alpha > 0$ captures positive spillovers. Note that we are abusing parameters, since α here has nothing to do with the preferences parameters α_i.

Remark 2. In an urban context, we can interpret α as the external returns to density by setting $A_i = \overline{A}_i \left(L_i/H_i\right)^{\alpha}$ and redefining H_i and $\overline{A}_i$ accordingly.

We conjecture that the equilibrium is unique under conditions similar to those in Theorem 2 of Allen and Arkolakis.

Imposing the conditions for a spatial equilibrium yields $u_i = u$, $U_i = U$, and $s_i = s$ for all i and some $\{s, u, U\}$. Inserting (36) and ((25)) into (24) yields

$$w_i^{1-\sigma} \left(\frac{L_i}{H_i}\right)^{(\sigma-1)(1-s_i)} U^{\sigma-1} B_i^{1-\sigma} = \sum_{j\in C} \left(T_{ji} \frac{w_j}{\overline{A}_j}\right)^{1-\sigma} L_j^{\alpha(\sigma-1)}$$

and

$$L_i w_i^{2\sigma-1} \overline{A}_j^{1-\sigma} L_j^{\alpha(1-\sigma)} = \phi \left(\frac{L_i}{H_i}\right)^{(\sigma-1)(1-s_i)} U^{\sigma-1} B_i^{1-\sigma} \tag{37}$$

---

and

$$\begin{aligned}
RHS &\equiv s^{\sigma-1} u^{-\frac{\sigma-1}{s}} \sum_{j\in C} T_{ij}^{1-\sigma} \left[\frac{H_i^{\frac{(\sigma-1)(1-s)}{s}} L_i^{1-\frac{(\sigma-1)(1-s)}{s}} B_i^{\frac{\sigma-1}{s}} A_i^{1-\sigma}}{\kappa}\right]^{-\frac{\sigma}{2\sigma-1}} B_j^{\frac{\sigma-1}{s}} H_j^{\frac{(\sigma-1)(1-s)}{s}} L_j^{1-\frac{(\sigma-1)(1-s)}{s}} \\
&= s^{\sigma-1} u^{-\frac{\sigma-1}{s}} \sum_{j\in C} T_{ij}^{1-\sigma} A_j^{\frac{\sigma(\sigma-1)}{2\sigma-1}} B_j^{\frac{(\sigma-1)^2}{(2\sigma-1)s}} H_j^{\frac{(\sigma-1)^2(1-s)}{(2\sigma-1)s}} L_j^{\left[1-\frac{(\sigma-1)(1-s)}{s}\right]\frac{\sigma-1}{2\sigma-1}} \kappa^{\frac{\sigma}{2\sigma-1}} \\
&= s^{\sigma-1} u^{-\frac{\sigma-1}{s}} \sum_{j\in C} T_{ij}^{1-\sigma} A_j^{\frac{\sigma(\sigma-1)}{2\sigma-1}} B_j^{\frac{(\sigma-1)^2}{(2\sigma-1)s}} H_j^{\frac{(\sigma-1)^2(1-s)}{(2\sigma-1)s}} L_j^{[s-\sigma+1+s\sigma-s]\frac{\sigma-1}{(2\sigma-1)s}} \kappa^{\frac{\sigma}{2\sigma-1}}.
\end{aligned}$$

Thus:

$$L_i^{\frac{\sigma-1}{(2\sigma-1)s}} H_i^{-\frac{\sigma(\sigma-1)(1-s)}{(2\sigma-1)s}} A_i^{-\frac{(\sigma-1)^2}{2\sigma-1}} B_i^{-\frac{\sigma(\sigma-1)}{(2\sigma-1)s}} = s^{\sigma-1} u^{-\frac{\sigma-1}{s}} \sum_{j\in C} T_{ij}^{1-\sigma} A_j^{\frac{\sigma(\sigma-1)}{2\sigma-1}} B_j^{\frac{(\sigma-1)^2}{(2\sigma-1)s}} H_j^{\frac{(\sigma-1)^2(1-s)}{(2\sigma-1)s}} L_j^{[1-\sigma(1-s)]\frac{\sigma-1}{(2\sigma-1)s}}.$$

so that

$$
\begin{aligned}
&= U^{1-\sigma} B_i^{\sigma-1} H_i^{(\sigma-1)(1-s)} w_i^{\sigma-1} \sum_{j \in C} T_{ji}^{1-\sigma} \overline{A}_j^{\sigma-1} L_j^{\alpha(\sigma-1)} w_j^{1-\sigma} \\
&= U^{1-\sigma} B_i^{\sigma-1} H_i^{(\sigma-1)(1-s)} \left[\phi \left(\frac{L_i}{H_i} \right)^{(\sigma-1)(1-s)} U^{\sigma-1} B_i^{1-\sigma} \overline{A}_i^{1-\sigma} L_i^{-1+\alpha(\sigma-1)} \right]^{\frac{\sigma-1}{2\sigma-1}} \\
\times &\sum_{j \in C} T_{ji}^{1-\sigma} \overline{A}_j^{\sigma-1} L_j^{\alpha(\sigma-1)} \left[\phi \left(\frac{L_j}{H_j} \right)^{(\sigma-1)(1-s)} U^{\sigma-1} B_j^{1-\sigma} \overline{A}_j^{1-\sigma} L_j^{-1+\alpha(\sigma-1)} \right]^{\frac{1-\sigma}{2\sigma-1}} \\
&= U^{1-\sigma} \left(B_i H_i^{1-s} \right)^{\frac{\sigma}{2\sigma-1}} L_i^{[(\sigma-1)(1-s+\alpha)-1]\frac{\sigma-1}{2\sigma-1}} \overline{A}_i^{(1-\sigma)\frac{1-\sigma}{2\sigma-1}} \\
&\times \sum_{j \in C} T_{ji}^{1-\sigma} \overline{A}_j^{(\sigma-1)\frac{3\sigma-2}{2\sigma-1}} \left(B_j H_j^{1-s} \right)^{(\sigma-1)^2/(2\sigma-1)} L_j^{(\alpha\sigma+\sigma-1)\frac{\sigma-1}{2\sigma-1}}
\end{aligned}
$$

so that

$$
\begin{aligned}
L_i^{[\sigma(1-s)+1-\alpha(\sigma-1)]\frac{\sigma-1}{2\sigma-1}} &= U^{1-\sigma} \left(B_i H_i^{1-s} \right)^{\frac{\sigma}{2\sigma-1}} \overline{A}_i^{(1-\sigma)\frac{1-\sigma}{2\sigma-1}} \\
&\times \sum_{j \in C} T_{ji}^{1-\sigma} \overline{A}_j^{(\sigma-1)\frac{3\sigma-2}{2\sigma-1}} \left(B_j H_j^{1-s} \right)^{(\sigma-1)^2/(2\sigma-1)} L_j^{(\alpha\sigma+\sigma-1)\frac{\sigma-1}{2\sigma-1}}.
\end{aligned}
$$

Remark 3. This expression is similar to Equation (13) in Allen and Arkolakis. Our model nests theirs and, at the spatial equilibrium with an expenditure share that is uniform across space, conditions for existence and uniqueness should mimic theirs; we have a housing market so that $s < 1$ (where as $s = 1$ in their model), but no congestion spillovers (whereas they write $U_i = \overline{U}_i L_i^{\epsilon}$, with $\epsilon < 0$ when congestion dominates). Thus, our s plays the role of their ϵ.

C Spatial Structural Change and Geography with Income Effects

This Appendix contains the proofs and details to calculations pertaining to Section 8.

Sustain Point. If the Core-Periphery configuration $\lambda = 1$ is an equilibrium, then $\pi_1(u) = E_1(u)$ and $\pi_2(u) \leq E_2(u)$. The standard practice in the NEG is to find a threshold value for ϕ in the unit interval, known as the "Sustain Point" ϕ^{Sust}, such that the latter inequality holds strictly for $\phi > \phi^{\text{Sust}}$ and is violated for $\phi < \phi^{\text{Sust}}$.[28] We may rewrite the conditions $\pi_1(u) = E_1(u)$ and

[28] Guide to calculations. Using (27), we get the following expressions for the equilibrium return to high-skilled workers in the Core (Region 1) and for the equilibrium shadow return to high-skilled workers in the Periphery (Region 2):

$$
\pi_1(u) = \frac{1}{\sigma} \left[\pi_1(u) s(u) + 1 \right] = \frac{1}{\sigma - s(u)}, \qquad \pi_2 = \frac{1}{\sigma} \left[\frac{1}{2\phi} + \phi \left(\pi_1(u) s(u) + \frac{1}{2} \right) \right] = \frac{1}{\sigma} \left[\frac{1}{2\phi} + \phi \frac{1}{\delta(u)} \right].
$$

Turning to expenditure, and using (26) and (28), we get:

$$
E_1(u) = \frac{u}{a^{1-s(u)}}, \qquad E_2(u) = \frac{u}{a^{1-s(u)}} T^{s(u)} = \frac{u}{a^{1-s(u)}} \phi^{-\theta(u)}.
$$

$\pi_2(u) \leq E_2(u)$ for $\lambda = 1$ as:

$$0 \geq \frac{\pi_2/E_2}{\pi_1/E_1} - 1 \propto \delta(u) + \phi^2 - [1+\delta(u)]\,\phi^{1-\theta(u)} \equiv f(\phi;\delta,\theta),$$

where "$\propto$" stands for "of the same sign of."[29] By Laguerre's general rule of signs, f admits at most two positive roots for ϕ; by inspection, $\phi = 1$ is one root. To find the other root, we use standard algebra to find:

$$0 < \theta < 1 \quad \Rightarrow \quad \lim_{\phi\to 0} f = \delta(u) > 0, \quad \lim_{\phi\to 0}\frac{\partial f}{\partial \phi} = -\infty; \quad \lim_{\phi\to 1}\frac{\partial f}{\partial \phi} > 0.$$

It then follows that there exists a $\phi^{\text{Sust}} \in (0,1)$ such that the Core-Periphery configuration is an equilibrium outcome if $\phi \geq \phi^{\text{Sust}}$, where

$$\phi^{\text{Sust}}(u) = \left\{\phi \in (0,1) : 0 = \delta(u) + \phi^2 - [1+\delta(u)]\,\phi^{1-\theta(u)}\right\},$$

which is 31 in the text.

Plugging $\lambda = 1$ into (27), using (28), and $\pi_1 = E_1$, the level of technology a and equilibrium utility u are related by

$$\ln a = \frac{\ln(\sigma - s(u)) + \ln u}{1 - s(u)}. \tag{38}$$

We show in Appendix (C) that the right-hand side of (38) is increasing in s (and thus in u). We show in Appendix (C) that utility u of mobile workers is increasing in the level of technology a in the Core-Periphery equilibrium:

$$\left.\frac{\partial u}{\partial a}\right|_{\lambda=1} > 0.$$

Here, we are more interested in the role of the expenditure share s (which is parametric in most of the NEG, including in Krugman 1991). We may rewrite the conditions for a sustainable Core-Periphery equilibrium as

$$(1-\theta)\ln\phi \geq \ln\left(\frac{\delta + \phi^2}{\delta + 1}\right). \tag{39}$$

The left-hand side of this expression monotonically increases from $\ln\phi$ when $s = \theta = 0$, to $\frac{\sigma-2}{\sigma-1}\ln\phi$

[29] Intermediate steps:

$$\begin{aligned}
0 &\geq \frac{\pi_2/E_2}{\pi_1/E_1} - 1 \\
&= \frac{\sigma - s(u)}{2\sigma\delta(u)}\phi^{-1+\theta(u)}\left[\delta(u) + \phi^2\right] - 1 \\
&= \frac{\sigma + s(u)}{2\sigma}\phi^{-1+\theta(u)}\left[\delta(u) + \phi^2 - \frac{2\sigma}{\sigma + s(u)}\phi^{1-\theta(u)}\right] \\
&\propto \delta(u) + \phi^2 - [1+\delta(u)]\,\phi^{1-\theta(u)} \equiv f(\phi;\delta,\theta).
\end{aligned}$$

when $s = 1$. The right-hand side of this expression monotonically decreases from $\ln\left(\frac{1+\phi^2}{2}\right)$ when $s = 0$ (and hence $\delta = 1$), to $\ln\left(\frac{1+\phi^2}{2} - \frac{1-\phi^2}{2\sigma}\right)$ when $s = 1$. The inequality in (39) is then violated for $s = 0$.[30] Thus, there exists at most one threshold value of s, call it s^{Sust} such that this inequality holds for $s > s^{\text{Sust}}$ and is violated otherwise. The threshold value s^{Sust} belongs to the unit interval if and only if $\phi > \phi_1^{\text{Sust}}$, where $\phi_1^{\text{Sust}} \in (0, 1)$ is the value of the sustain point derived in (31) for $s = 1$.[31] Hence, as labor productivity increases, the range of values for parameters T and σ (recall $\phi \equiv T^{1-\sigma}$) that fulfill this inequality expands.

To sum-up, we have shown:

Lemma 4. *The Core-Periphery configuration $\lambda = 1$ is an equilibrium outcome if trade free-ness ϕ and labor productivity a are high enough.*

Proof. In the text above. □

Break Point. The symmetric configuration $\lambda = 1/2$ is always an equilibrium of this symmetric model, but in may not be stable, in a sense that we make precise shortly. In the symmetric equilibrium, (26), (27), and (28) become:

$$P^{1-\sigma} = \frac{1+\phi}{2}, \qquad \pi = \frac{1}{\sigma}(\pi s + 1) = \frac{1}{\sigma - s}, \qquad \ln E = -(1-s)\ln(a) + s\ln(P) + \ln u,$$

where we have dropped region indices since $x_j = x_i = x$ for all equilibrium variables x.

Here we use "hats" do denote log changes, and we define

$$Z \equiv \frac{1-\phi}{1+\phi} = \frac{1-T^{1-\sigma}}{1+T^{1-\sigma}} \in (0,1).$$

Starting from the symmetric configuration $\lambda = \frac{1}{2}$, an increase in $\lambda_j$increases local market crowding (i.e., reduces the market share of local firms and thence the returns to local skilled labor),

$$(1-\sigma)\hat{P}_j = Z\hat{\lambda}_j \tag{40}$$

[30] Indeed, $\phi < \frac{1+\phi^2}{2}$ for any $\phi \in (0, 1)$.

[31] The threshold value s^{Sust} belongs to the unit interval if and only if

$$\max_s \left(1 - \frac{s}{\sigma - 1}\right)\ln\phi > \min_s \ln\left[\frac{\sigma\left(1+\phi^2\right) - s\left(1-\phi^2\right)}{2\sigma}\right],$$

that is, if and only if

$$\frac{\sigma-2}{\sigma-1}\ln\phi > \ln\frac{\phi^2(\sigma+1) + (\sigma-1)}{2\sigma}.$$

Equivalently, if and only if

$$0 < (\sigma+1)\phi^2 - 2\sigma\phi^{(\sigma-2)/(\sigma-1)} + (\sigma-1).$$

The smaller positive root, $\phi_1^{\text{Sust}} \in (0,1)$, is the value of the sustain point derived in (31) for $s = 1$.

from (26); it increases local demand relative to exports,

$$(\sigma - Zs)\,\hat{\pi}_j = Z\,(s - Z\sigma)\,\hat{\lambda}_j + \frac{1}{1+\phi}\,(\mathrm{d}s_j + \phi\mathrm{d}s_i) \tag{41}$$

from (27),[32] and the minimum expenditure required to reach the equilibrium level of utility follows the change in prices and the change in utility level,

$$\hat{E}_j = s\hat{P}_j + \hat{U}_j - \ln a\mathrm{d}s_j \tag{42}$$

by (28). The symmetric equilibrium is said to be unstable if, following an exogenous migration shock $\hat{\lambda}$, the change in income,$\hat{\pi}$, is larger than the change in expenditure required to maintain utility, $\hat{E}$, and to be stable otherwise. We are interested in the parameter configuration such that the two effects are exactly equal at the margin, $\hat{\pi} = \hat{E} = s\hat{P}$, in which case utility (and hence expenditure shares) are invariant. Hence, we set $\hat{u}_j = 0 = \mathrm{d}s_j = \mathrm{d}s_i$; plugging these expressions is (40) to (42) yields

$$(\sigma - Zs)\,s\frac{Z}{1-\sigma} = Z\,(s - Z\sigma)\,.$$

Taking s as given, this equation admits two roots: $Z = 0$ (and hence $\phi = 1$), and $Z = s\,(2\sigma - 1) / \left(\sigma^2 - \sigma + s^2\right)$, which corresponds to the so-called break point. The break point for ϕ is equal to

$$\phi^{\text{Break}} = \frac{1-\theta}{1+\theta}\delta = \frac{(\sigma - s)\,(\sigma - 1 - s)}{(\sigma + s)\,(\sigma - 1 + s)},$$

which is (32) in the text, and where the second equality comes from (29) and (30). This break point belongs to the unit interval by inspection, and it is decreasing in u by $\partial\theta/\partial u > 0$ and $\partial\delta/\partial u < 0$.[33]

Plugging for $\lambda = \frac{1}{2}$ into (27), using (28), and $\pi = E$, we obtain that equilibrium utility u is

[32] Note that $\mathrm{d}s_j$ and $\mathrm{d}s_i$ are of the opposite sign but, unlike the other variables, $\mathrm{d}s_j + \mathrm{d}s_i$ is not necessarily equal to zero at the symmetric equilibrium.

[33] Importantly, (32) contains endogenous variables on both sides of the equation. The so-called break point is a fixed point satisfying both (32) and (43). Let us invert (32) in order to get an expression relating the expenditure share s to the level of trade free-ness ϕ, denoted by $s^{\text{break}}\,(\phi)$, with

$$s^{\text{break}}\,(0) > 1, \qquad s^{\text{break}}\,(1) = 0, \qquad \frac{\partial s^{\text{break}}}{\partial\phi} < 0.$$

Let us also invert (43) to get an expression relating the expenditure share s to the level of technology a and the level of trade free-ness ϕ, denoted by $s\,(a,\phi)$, with

$$\frac{\partial s\,(a,\phi)}{\partial a} > 0, \qquad \frac{\partial s\,(a,\phi)}{\partial\phi} > 0.$$

Unlike $s^{\text{break}}\,(\phi)$, which holds at the break point only, $s\,(a,\phi)$ holds for any combination of parameters at the symmetric equilibrium. It thus follows that we can write the break point is a combination of parameters satisfying both $s^{\text{break}}\,(\phi)$ and $s\,(a,\phi)$. This break point is unique, and we can write it as a frontier in the (a,ϕ)-space, denoted by $B\,(a,\phi) = 0$, with

$$\left.\frac{\mathrm{d}\phi}{\mathrm{d}a}\right|_{B(a,\phi)=0} < 0.$$

related to the level of technology a and trade free-ness by

$$\ln a = \frac{\frac{s(u)}{\sigma-1}\ln\left(\frac{2}{1+\phi}\right) + \ln\left(\sigma - s\left(u\right)\right) + \ln u}{1 - s\left(u\right)}. \tag{43}$$

The left-hand side of this expression is increasing in a, while its right-hand side is decreasing in ϕ and increasing in s (and thus in u).[34] We thus conclude that utility u of mobile workers is increasing in the level of technology a and of trade free-ness ϕ in the symmetric equilibrium:

$$\left.\frac{\partial u}{\partial a}\right|_{\lambda=\frac{1}{2}}, \quad \left.\frac{\partial u}{\partial \phi}\right|_{\lambda=\frac{1}{2}} > 0.$$

D The Monocentric City Model with HCD Preferences

To close the model of Section 9, we have to integrate the slope of the bid rent curve and to solve for the assignment of workers, w, to locations, x.

Let us first integrate the slope of the bid rent curve:

$$\ln \Psi\left(x, u\right) = \ln \Psi\left(0, u\right) - \frac{1}{s\left(u\right)} T\left(x\right),$$

or, equivalently,

$$\Psi\left(x, u\right) = \Psi\left(0, u\right) \exp\left\{-\frac{T\left(x\right)}{s\left(u\right)}\right\}.$$

The housing price gradient, $r\left(x\right)$, is the upper envelope of these individual bid rent curves. Land at distance x is allocated to the highest bidder, denoted as $w\left(x\right)$. Inverting this expression, we obtain the assignment mapping from wages to any given location, $x\left(w\right)$.

For simplicity, assume that households consume land directly (i.e., land is converted one-for-one into housing services), and let $h\left(x, w\right)$ denote the quantity of housing consumed at distance x by worker w. The housing market clears locally if and only if $\mathrm{d}x = -Nh\left(x, w\right) f\left(w\right) \mathrm{d}w$, which gives the slope of the assignment function. Integrating:

$$x\left(w\right) = N \int_{W}^{w} h\left(x, \omega\right) \mathrm{d}F\left(\omega\right),$$

[34] Indeed, the right-hand side of (43) is equal to the right-hand side of (38), which we have shown to be increasing in $s\left(u\right)$ in the previous footnote, plus a term that is increasing in $s\left(u\right)$ by $0 < \phi < 1$,

$$\frac{\frac{s(u)}{\sigma-1}\ln\left(\frac{2}{1+\phi}\right)}{1 - s\left(u\right)}.$$

Thus, the right-hand side of (43) is increasing in $s\left(u\right)$ as well. As a corollary, note that if parameter values are such that the Core-Periphery outcome and the symmetric equilibrium are stable equilibriums, then mobile workers are better off in the former than in the latter, as they face a lower cost of living.

with

$$h(x,w) = s(w, r(x)) \frac{w}{r(x)}.$$

Above, $s(w, r(x)) = s(u(w, r(x)))$, where $u(w, r(x))$ is from Equation (33).

Some Results under Particular Functional Forms With only two goods, there is a one-to-one relationship between the level of utility, u, and the expenditure shares, s. Let $p = 1$ by choice of the numeraire. Hence, we may rewrite equation (33), for instance, as

$$\frac{1-s}{s} = \ln w - s \ln r(x) - T(x)$$

so that s is the solution of

$$1 - s[1 + \ln w - T(x)] + s^2 \ln r = 0.$$

Hence,

$$s = \frac{1 + \ln w - T(x) \pm \sqrt{[1 + \ln w - T(x)]^2 - 4 \ln r}}{2 \ln r}.$$

Alternatively, if $1 - s = \ln w - s \ln r(x) - T(x)$,then

$$s = \frac{1 - \ln w + T(x)}{1 - \ln r},$$

which is decreasing in w if and only if $1 > \ln r$, and belongs to the unit interval if and only if $0 < \ln w - T(x) < 1$.