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JEL Classification:

Keywords:

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External Instrument SVAR Analysis for Noninvertible Shocks

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29th January 2023

Abstract

We propose a novel External-Instrument SVAR procedure, the Generalised External-Instrument SVAR, to identify and estimate the impulse response functions, regardless of the shock being invertible or recoverable. When the shock is recoverable, we also show how to estimate the unit variance shock and the 'absolute' response functions. When the shock is invertible, the method collapses to the standard External-Instrument SVAR procedure. We show how to test for recoverability and invertibility. We apply our techniques to a monetary policy VAR. It turns out that, using standard specifications, the monetary policy shock is not invertible, but is recoverable. When using our procedure, results are plausible even in a parsimonious specification, not including financial variables. Contrary to previous findings, monetary policy has significant and sizeable effects on prices.

JEL classification: C32, E32.

Keywords: Proxy-SVAR, SVAR-IV, Impulse response functions, Variance Decomposition, Historical Decomposition, Monetary Policy Shock.

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1 Introduction

Since the seminal contributions of Stock (2008), Mertens and Ravn (2013) and Stock and Watson (2018), SVAR-IV (or Proxy-SVAR) methods have become a popular approach to structural macroeconomic analysis. A very partial list of recent noticeable applications include Stock and Watson (2012), Mertens and Ravn (2014), Gertler and Karadi (2015), Mertens and Montiel-Olea (2018), Paul (2020) and Miranda-Agrippino and Ricco (2021). In this paper we refer to the method proposed in Mertens and Ravn (2013) as the 'standard External-Instrument SVAR'.

A severe limitation of the method is that it requires invertibility (Stock and Watson, 2018, Miranda-Agrippino and Ricco, forthcoming).[1] A shock is invertible if it is a linear combination of the present and past values of the VAR variables, or, equivalently, a contemporaneous linear combination of the VAR residuals. Invertibility is a demanding property, unlikely to be satisfied in the presence of 'news' technology shocks (Forni et al., 2014), forward guidance (Ramey, 2016) or fiscal foresight (Mertens and Ravn, 2010, Ramey, 2011, Leeper et al., 2013) and necessarily failing for the so-called 'noise' shocks (Blanchard et al., 2013, Forni et al., 2017). The problem is that current and past values of macroeconomic variables do not convey enough information to recover the shock. Here we show that indeed invertibility does not hold for a few standard monetary-policy VAR specifications.

Plagborg-Møller and Wolf (2021) have proposed to overcome this limitation by adopting an 'Internal-Instrument SVAR' approach (see also Ramey, 2016). The approach consists of a standard Cholesky identification where the instrument enters the VAR as the first variable, and the IRFs to the first shock are the IRFs of interest. With this procedure, the impulse response functions are consistently estimated even in absence of invertibility. Other approaches which are valid under noninvertibilty are the LP-IV (discussed in Stock and Watson, 2018), based on Jordà (2005)'s Local Projections, and the VARX where the proxy plays the role of the exogenous variable (Paul, 2020). With these alternative methods, however, an important flexibility of the original External-Instrument approach is lost: the possibility of having different time spans for the model variables and the proxy, a property which is very useful when the time span of the proxy is relatively short, as it is often in practice.

The main methodological contribution of this paper is to generalise the External-Instrument SVAR approach to the case of noninvertibility. We refer to our proposed method as the Generalised External-Instrument SVAR. We show that all types of analysis that can be conducted with the above alternative methods can be also conducted, under similar validity

[1] Stock and Watson (2018) provide validity conditions under global invertibility. Miranda-Agrippino and Ricco (forthcoming) show the method can be generalised to the case of partial invertibility, at the cost of reinforcing the validity conditions for the instrument.

conditions, with our approach. Furthermore, as compared to the Internal-Instrument SVAR approach, our method maintains the full flexibility of the original External-Instrument methodology. First, the important time-span flexibility cited above is retained. Second, the conditional exogeneity of the instrument can be ensured by using an information set different from the VAR information set. Third, by keeping the instrument out of the VAR, the dynamics of the instrument are not constrained by those of the VAR variables.

Our proposed method is the following. Instead of regressing the VAR residuals onto the current proxy only (the standard External-Instrument method), we regress the VAR residuals onto the current proxy and its lags. The impulse response functions are then estimated by combining the coefficients of this regression with the reduced-form impulse response functions obtained from the VAR. Moreover, we show how to implement, within our setting, the upper and lower bounds for the relative IRFs and the variance decomposition proposed by Plagborg-Møller and Wolf (2022).

While the 'relative' impulse-response functions can be estimated independently of the shock of interest being invertible or recoverable, the 'absolute' response functions and the structural shock itself cannot, unless the shock is recoverable. Recoverability (Chahrour and Jurado, 2021) is much less demanding than invertibility.[2] A shock is recoverable if it is a linear combination of the present, past and future values of the VAR variables, or, equivalently, it is a linear combination of the present and future values of the VAR residuals.

Here we show how to test for recoverability and estimate the structural shock of interest when it is recoverable. We regress the instrument onto the present and future values of the VAR residuals. If the shock is recoverable, the fitted value is a consistent estimate of the shock and therefore must be serially uncorrelated. Following a valuable suggestion of Plagborg-Møller and Wolf (2022), we then propose to test for recoverability by testing for serial uncorrelation of such projection. If the test does not reject the null, the above fitted value provides an estimate of the structural shock. Moreover, we show how to estimate the corresponding 'absolute' impulse-response functions.

Having an estimate of the shock, historical decomposition can be performed as usual. Standard variance decomposition is downward biased at short horizons when the model is not globally invertible. However, we can get an unbiased variance decomposition by looking at the integrals of the spectral densities over specific frequency bands, as suggested in Forni et al. (2019).[3]

The regression of the proxy onto the present and future values of the VAR residuals

[2]Consider that, if the number of shocks is equal to the number of variables, recoverability is always fulfilled for all shocks, whereas invertibility requires special conditions on the impulse-response functions (see e.g. Lippi and Reichlin, 1993, Lippi and Reichlin, 1994 and Fernandez-Villaverde et al., 2007).

[3]The method is the same used, for different purposes, in Angeletos et al. (2020).

also allows us to test for invertibility. For, under the invertibility assumption, the shock is a linear combination of current VAR residuals only. Hence a simple F-test for the null of zero coefficients for future residuals tells us whether the shock is invertible or not. Both the recoverability test and the invertibility test provide a valuable guidance for the choice of the VAR variables. If the null of invertibility is not rejected, our proposed procedure collapses to the standard External-Instrument SVAR procedure. Stock and Watson (2018) and Plagborg-Møller and Wolf (2022) also propose invertibility tests based on the proxy. The former requires estimation of LP-IV. The latter consists in testing whether the proxy Granger causes the VAR variables. In our setting, the F-test on the joint significance of future VAR residuals seems more appropriate than the Granger causality test, being a natural by-product of the estimation procedure.

A few Monte Carlo exercises validate our proposed estimation and testing method in small samples. Two of these simulations show that our procedure has an excellent performance in estimating the structural impulse response functions in comparison to the Internal-Instrument SVAR.

From a theoretical point of view, our contribution is twofold. On the one hand, we have representation results relating the structural impulse response functions and shocks to the reduced form VAR coefficients and residuals. Such results do not involve the instrument. They can be regarded as a generalisation of Lippi and Reichlin (1994) to the important case in which there are more shocks than variables. On the other hand, we present identification results relating the structural impulse response functions (and the shock of interest) to the proxy. For these results, we are closely connected to Plagborg-Møller and Wolf (2022). The basic difference between our formulas and the ones in that paper is that, in order to develop our approach, we write such relations in terms of the reduced form VAR coefficients and residuals, using the representation results cited above.

In the empirical application, we study the effects of US monetary policy using the proxy of Gertler and Karadi (2015). Such instrument is based on surprises in federal fund futures with three-month maturity, so that it is likely to capture both conventional monetary policy shocks and shocks to forward guidance about the path rate at short horizons. This news component might induce noninvertibility, thus providing a strong motivation for our analysis.

Our main findings are the following. First, in standard VAR specifications the monetary policy shock turns out to be noninvertible according to our test. This is true even for specifications including the excess bond premium or other financial variables. Hence, the results obtained so far with the standard External-Instrument SVAR approach should be taken with caution.

Second, when using our External-Instrument method, a contractionary shock reduces inflation and output consistently across different VAR specifications and independently of the inclusion of financial variables. By contrast, when using the standard approach, results are dramatically different across VAR specifications, with large price and real activity puzzles emerging when the financial variables are not included.

Finally, the monetary policy shock is recoverable, so that we can perform variance decomposition. The variance decomposition shows that the contribution of the monetary policy shock on both output and prices is sizeable and larger than previously reported. This is a noticeable result, in that it suggests that monetary policy can be effective in controlling prices, contrary to what found in most of the existing literature.

The remainder of the paper is organised as follows. Sections 2 present our structural MA model and our representation results. In Section 3 we present our identification results and our proposed estimation and testing procedure, which is summarised in Section 4. Section 5 collects our Monte Carlo exercises. Section 6 presents our empirical application. The last section provides some conclusions. The Online Appendix provides the proofs of all propositions, examples, additional simulations and a few robustness checks for the empirical application.

2 Representation theory

In this section we introduce our theoretical framework and study the relation between the structural representation and the VAR representation when the structural shock of interest is not recoverable, recoverable but not invertible, and invertible.

2.1 The model

Let us start from our assumptions about the structural macroeconomic model and the VAR representation.

Assumption 1. (Structural MA representation) *The observable macroeconomic variables in the n-dimensional vector y_t (possibly after suitable transformations) have the representation*

$$y_t = B(L)u_t, \tag{1}$$

where (i) $B(L) = B_0 + B_1 L + B_2 L^2 + \cdots$ is an $n \times q$ matrix of rational impulse-response functions in the lag operator L; (ii) $n \leq q$ and $B(L)$ has maximum rank n; (iii) u_t is a q-dimensional white noise vector including the structural shocks, whose variance covariance *matrix is* I_q.[4]

The above model is sometimes referred to as the Slutsky-Frisch representation of the macro economy. It can be thought of as resulting from the linearisation of a DSGE model and can easily be derived from its state-space representation. Notice that we do not assume that the number of shocks is equal to the number of variables, so that the matrix $B(L)$ is not necessarily square. However, we assume that the number of variables cannot be larger than the number of shocks ($n \leq q$). This assumption can be justified by recognising that the variables are observed with error and such errors are allowed to enter the vector u_t together with the structural shocks. Since the entries of $B(L)$ are rational functions, the assumption that $B(L)$ has maximum rank n implies that $B(z)$, z being a complex variable, has rank n almost everywhere in the complex plane. This is tantamount to assuming that the spectral density matrix of y_t, i.e. $S^y(\theta) = \frac{1}{2\pi} B(e^{-j\theta})B'(e^{j\theta})$, j being the imaginary unit, is nonsingular a.e. in $[\pi \ \pi)$.[5]

In this paper we are concerned with identification of a single shock of interest, say u_{it}. In order to highlight the shock of interest and the corresponding response functions, it is convenient to re-write (1) in the form

$$y_t = b_i(L)u_{it} + \tilde{B}(L)\tilde{u}_t, \tag{2}$$

where $b_i(L) = b_{i0} + b_{i1}L + b_{i2}L^2 + \cdots$ is the i-th column of $B(L)$, $\tilde{B}(L)$ includes the other columns of $B(L)$ and $\tilde{u}_t = (u_{1t} \ \cdots \ u_{i-1,t} \ u_{i+1,t} \ \cdots \ u_{qt})'$.

On the other hand, being stationary and purely nondeterministic by (1), y_t necessarily admits the Wold representation

$$y_t = C(L)\varepsilon_t \tag{3}$$

where $C(L) = C_0 + C_1L + C_2L^2 + \cdots$ and ε_t is a vector white-noise process with covariance matrix Σ_ε. Of course, this representation is square.

For VAR estimation, it is convenient to add the following assumption.

Assumption 2. (VAR representation) *The matrix of the Wold representation* $C(L)$ *has an inverse in the non-negative powers of* L*, possibly of infinite order. Hence, letting its inverse be* $A(L)$*,* y_t *has the VAR representation*

$$A(L)y_t = \varepsilon_t. \tag{4}$$

[4]We omit the constant term for notational simplicity. A rational impulse-response function is the ratio of two polynomials in L.

[5]A treatment of the case $q < n$, where the variables have a singular spectral density matrix, can be found in Forni et al. (2020).

We remark that we do not require invertibility of the structural representation, but only invertibility of the Wold representation. Moreover, we do not assume that the VAR has a finite order.

2.2 Structural shocks and VAR residuals

What is the relation between the structural shocks u_t and the VAR residuals ε_t? What is the relation between the structural response functions $B(L)$ and the VAR coefficients $A(L)$? To begin, we present a very general result, not requiring either invertibility or recoverability. Then we define these important properties of the structural shocks and specialise our results to the recoverability and the invertibility cases.

The results in the present section can be regarded as a generalisation of the representation results in Lippi and Reichlin (1994), which refer to a square system, to the case of possibly "short" structural models, where $q \geq n$ and recoverability does not hold for all structural shocks.

2.2.1 A general result

Clearly, the VAR residuals ε_t are linear combinations of the current and lagged structural shocks u_t, since from Equations (1) and (4) we see that

$$\varepsilon_t = A(L)y_t = A(L)B(L)u_t = Q(L)u_t. \tag{5}$$

But what about the inverse relation, i.e. the one relating u_t to ε_t? Unfortunately, in the general case we cannot write u_t as an exact function of the ε's. To see this, consider the case $q > n$, i.e. there are more structural shocks than VAR residuals: intuition suggests that in this case we cannot have an exact linear mapping relating all of the shocks in u_t to those in ε_t. We show below that this is in fact the case. The best we can do, within a linear setting, is to approximate u_t by taking the projection of u_t onto $\mathcal{H}$, the space spanned by the present, past and future of the Wold shocks ε_t: $\mathcal{H} = \overline{\text{span}}(\varepsilon_{j,t-k}, j = 1, \ldots, n, k \in \mathbb{Z})$. Denoting with P the linear projection operator, we have

$$u_t = P(u_t|\mathcal{H}) + s_t = D'(F)\varepsilon_t + s_t, \tag{6}$$

where s_t is the residual of the projection, $F = L^{-1}$ is the forward operator such that $F\varepsilon_t = \varepsilon_{t+1}$ and $D'(F)$ is a $q \times n$ matrix of linear filters. Existence of the above infinite sum representation is guaranteed by the fact that ε_t is a vector white noise.

The following result shows that there is a precise relation between (5) and (6).

Proposition 1. (Basic representation theorem)
(i) $D(F)$ is one-sided in the non-negative powers of F.
(ii) $Q(L)$ defined in (5) is linked to the projection coefficients in (6) by the relations

$$Q(L) = \Sigma_\varepsilon D(L); \qquad D(L) = \Sigma_\varepsilon^{-1} Q(L). \tag{7}$$

(iii) The structural impulse-response functions are linked to the Wold impulse response functions by the relation

$$B(L) = C(L)Q(L) = C(L)\Sigma_\varepsilon D(L).$$

In particular, for the impulse-response functions of interest the relation is

$$b_i(L) = C(L)q_i(L) = C(L)\Sigma_\varepsilon d_i(L). \tag{8}$$

Proposition 1 establishes a mapping between the Wold impulse-response functions $C(L)$ and the structural impulse-response functions $B(L)$ which holds true independently of invertibility or recoverability of the structural shocks.

2.2.2 Recoverable shocks

An important special case is the one of u_{it} being recoverable. In this case, we have an exact mapping relating u_{it} to the present and future of ε_t.

Definition 1. (Recoverability) *Let $\mathcal{H}^y$ be the closed linear space spanned by present, past and future values of y_t: $\mathcal{H}^y = \overline{\text{span}}(y_{j,t-k}, j = 1, \dots, n, k \in \mathbb{Z})$. We say that the structural shock u_{it} is recoverable with respect to y_t if and only if $u_{it} \in \mathcal{H}^y$. We say that u_t is (globally) recoverable with respect to y_t if and only if all of the structural shocks are recoverable.*

Proposition 2. (Structural shocks and VAR residuals) *If u_{it} is recoverable with respect to y_t,*

$$u_{it} = d_i'(F)\varepsilon_t = q_i'(F)\Sigma_\varepsilon^{-1}\varepsilon_t, \tag{9}$$

where $d_i(F) = d_{i0} + d_{i1}F + d_{i2}F^2 + \cdots$ is the i-th column of $D(F)$ and $q_i(F) = q_{i0} + q_{i1}F + q_{i2}F^2 + \cdots$ is the i-th column of $Q(F)$. Moreover

$$d_i'(F)\Sigma_\varepsilon d_i(L) = q_i'(F)\Sigma_\varepsilon^{-1} q_i(L) = 1.$$

Remark 1. (Global recoverability and square systems) An immediate consequence of the above result is that, if u_t is globally recoverable, then $s_t = 0$ and $u_t = D'(F)\varepsilon_t$. Hence from

(5) we see that $D'(F)$ is a left-inverse of $Q(L)$. This can be true in our setting, where $n \leq q$, only if $q = n$ (so that $D'(F)$ is the inverse of $Q(L)$). The converse is also true, i.e. if $q = n$, then u_t is recoverable. For, from (5) we get $\Sigma_\varepsilon = Q(z)Q'(z^{-1})$. Hence, if $\det Q(z)$ vanishes for $z = z^*$, then $\det Q(z^{-1})$ must have a pole in z^*, which is impossible if $|z^*| = 1$, since ε_t is stationary. Hence $Q(L)$ is invertible (possibly toward the future) and $u_{it} \in \mathcal{H}$ for any i. In conclusion, u_t is globally recoverable if and only if $q = n$.[6]

Remark 2. (Global recoverability and Blaschke matrices) Lippi and Reichlin (1994), in the context of a square model, show that the structural IRFs and shocks are related to the Cholesky IRFs and shocks by a Blaschke matrix, i.e. a matrix $M(z)$ of rational functions in z such that $M(z)M'(1/z) = I$. This can be easily seen using the above results. From (3) we see that the Cholesky representation of y_t is $y_t = [C(L)H]\eta_t$, where H is the Cholesky factor of Σ_ε, such that $HH' = \Sigma_\varepsilon$, and $\eta_t = H^{-1}\varepsilon_t$. We have seen in Remark 2 that, under global recoverability, $u_t = D'(F)\varepsilon_t$. Hence $u_t = M'(F)\eta_t$, where $M'(F) = D'(F)H$. Moreover, $M(z)$, z being a complex variable, is a Blaschke matrix. For, we have seen in Remark 2 that, under global recoverability, $D'(F)Q(L) = I_n$. Since $Q(L) = \Sigma_\varepsilon D(L)$ by (7), we have $M'(F)M(L) = M(L)M'(F) = I_n$. Hence $\eta_t = M(L)u_t$ and the structural impulse response functions are related to the Cholesky impulse-response functions by the equation $B(L) = [C(L)H]M(L)$.

Remark 3. (Recoverability measure) Let us consider the spectral density function of the projection of u_{it} onto $\mathcal{H}$, i.e. $d_i'(F)\varepsilon_t = q_i'(F)\Sigma_\varepsilon^{-1}\varepsilon_t$. We have

$$R_r^2(\theta) = d_i'(e^{j\theta})\Sigma_\varepsilon d_i(e^{-j\theta}) = q_i'(e^{j\theta})\Sigma_\varepsilon^{-1} q_i(e^{-j\theta}),$$

where j denotes the imaginary unit. This quantity represents the fraction of the total variance of u_{it} explained by the projection, decomposed by frequency, and is equal to 1 at all frequencies if u_{it} is recoverable. Correspondingly, the variance of the projection, i.e.

$$R_r^2 = \sum_{k=0}^{\infty} d_{ik}'\Sigma_\varepsilon d_{ik} = \sum_{k=0}^{\infty} q_{ik}'\Sigma_\varepsilon^{-1} q_{ik}$$

is the fraction of total variance explained by the ε's and is equal to 1 when u_{it} is recoverable. Hence $R_r^2(\theta)$ and R_r^2 can be regarded as measures of recoverability.

[6] See also Chahrour and Jurado (2021).

2.2.3 Fundamental/invertible shocks

If u_{it} is recoverable, it may be the case that it fulfils a more demanding property, that is fundamentalness. In the literature, fundamentalness is often regarded as a synonymous of invertibility. Indeed, fundamentalness is somewhat weaker than invertibility, in that invertibility requires that u_{it} can be written as a linear combination of the present and past history of y_t, whereas fundamentalness does not. For instance, if $y_t = (1 - L)u_t$, then u_t is fundamental but does not have a VAR representation since $(1 - L)$ is not invertible. In our setting however we are assuming that y_t has a VAR representation, so that fundamentalness and invertibility coincide.

Definition 2. (Fundamentalness) *Let $\mathcal{H}_t^-$ be the closed linear space spanned by present and past values of y_t: $\mathcal{H}_t^- = \overline{\text{span}}(y_{j,t-k}, j = 1, \dots, n, k \geq 0)$. We say that the structural shock u_{it} is fundamental with respect to y_t if and only if $u_{it} \in \mathcal{H}_t^-$. We say that u_t is fundamental with respect to y_t if and only if all structural shocks are fundamental with respect to y_t.*

From the definition of fundamentalness we see that if u_{it} is fundamental for y_t, then u_{it} is recoverable, whereas the converse is not necessarily true. The following well-known result holds.

Proposition 3. (Structural shocks and VAR residuals) *If u_{it} is fundamental for y_t, then $d_i(F) = d_{i0} = d_i$ and $q_i(F) = q_{i0} = q_i$, so that*

$$u_{it} = d_i'\varepsilon_t = q_i'\Sigma_\varepsilon^{-1}\varepsilon_t. \tag{10}$$

Remark 4. (Global fundamentalness) An immediate consequence of Proposition 3 is that, if u_t is globally fundamental for y_t, then it is a contemporaneous linear combination of the ε's, that is $u_t = D'\varepsilon_t$. Hence using Proposition 1 we get $Q(L) = \Sigma_\varepsilon D'$, so that $Q(L) = Q = B_0$ (since $A_0 = I_n$, see Equation 5). Moreover, we see from Equation (5) that $D'\varepsilon_t = u_t = D'Qu_t$, so that $D'Q = I_n$. Since we have global fundamentalness, B_0, Q and D must be square (see Remark 1), so that D' is the inverse of the matrix of the impact effects B_0.

Remark 5. (Fundamentalness measure) In analogy with Remark 3, we can define a measure of fundamentalness as $R_f^2 = d_i'\Sigma_\varepsilon d_i$. This measure corresponds to the fundamentalness measure proposed by Forni et al. (2019).

Regarding the impulse-response functions, we see from Proposition 3 that, if u_{it} is fundamental, Equation (8) reduces to

$$b_i(L) = C(L)q_i = C(L)\Sigma_\varepsilon d_i, \tag{11}$$

where $d_i'\Sigma_\varepsilon d_i = q_i'\Sigma_\varepsilon^{-1}q_i = 1$.

2.3 A general representation

From the above result we can derive a general representation of y_t in the fundamental shocks, the recoverable but nonfundamental shocks and the nonrecoverable shocks.[7]

Proposition 4. (General Representation) *Any vector process y_t satisfying Assumptions 1 and 2 can be represented as*

$$\begin{aligned} y_t &= B^f(L)u_t^f + B^r(L)u_t^r + B^n(L)u_t^n \\ &= C(L)Q^f u_t^f + C(L)Q^r(L)u_t^r + C(L)Q^n(L)u_t^n \\ &= C(L)\Sigma_\varepsilon D^f u_t^f + C(L)\Sigma_\varepsilon D^r(L)u_t^r + C(L)\Sigma_\varepsilon D^n(L)u_t^n. \end{aligned} \tag{12}$$

where u_t^f is the sub-vector of the fundamental structural shocks, u_t^r of the recoverable (but nonfundamental) shocks, and u_t^n of the nonrecoverable ones. Σ_ε is the covariance matrix of the Wold innovations ε_t and $C(L)$ is the matrix of the Wold representation. $Q^h(L)u_t^h$, for $h = f, r, n$, is the projection of ε_t onto u_{t-k}^h, with $k \geq 0$; and $D^h(L)$, for $h = f, r, n$, is such that $D^h(F)\varepsilon_t$ is the projection of u_t^h onto ε_{t+k}, with $k \geq 0$. Moreover, the following properties hold:

(i) D^f and Q^f are such that $D^{f\prime}\Sigma_\varepsilon D^f = Q^{f\prime}\Sigma_\varepsilon^{-1}Q^f = I_{q_f}$, q_f being the number of fundamental shocks;

(ii) $D^r(L)$ and $Q^r(L)$ are such that $D^{r\prime}(F)\Sigma_\varepsilon D^r(L) = Q^{r\prime}(F)\Sigma_\varepsilon^{-1}Q^r(L) = I_{q_r}$, q_r being the number of recoverable but nonfundamental shocks.

3 Identification, estimation, and testing

We now discuss how the structural impulse-response functions and the structural shocks can be identified and estimated by using an external proxy in the VAR framework discussed in the previous section.

3.1 The instrument

Let us start by introducing the instrument.

[7] Proposition 4 generalises representation results provided in Lippi and Reichlin (1994) and Miranda-Agrippino and Ricco (2021).

Assumption 3. (The Instrument) *The researcher can observe the proxy* $\tilde{z}_t$*, following the relation*

$$\tilde{z}_t = \beta(L)\tilde{z}_{t-1} + \mu'(L)x_{t-1} + \alpha u_{it} + w_t, \tag{13}$$

where w_t *is an error orthogonal to* $u_{j,t-k}$*,* $j = 1, ..., q$*, for any integer* k *and to* z_{t-k}, x_{t-k}*,* $k \geq 0$*, and* $\beta(L)$*,* $\mu(L)$ *are rational functions in the lag operator* L *(for simplicity we omit the constant term).*

This assumption is similar to the one used in Plagborg-Møller and Wolf (2022), which in turn is essentially equivalent to the weak LP-IV condition of Stock and Watson (2018). A remarkable difference, however, is that the vector x_t appearing on the right side of (13) is not necessarily equal to the VAR vector y_t.

Stock and Watson (2018) stresses that Equation (13) is more restrictive than the standard validity conditions for the SVAR-IV, i.e. (i) $\text{cov}(\tilde{z}_t, u_{it}) \neq 0$ and (ii) $\text{cov}(\tilde{z}_t, u_{kt}) = 0$ for $k \neq i$. Let us observe, however, that (i) and (ii) are sufficient only when assuming *global* invertibility, which of course is more demanding than invertibility of u_{it} alone. If we require only partial invertibility, we need a stronger validity condition, similar to Equation (13) above (see Miranda-Agrippino and Ricco, forthcoming).[8]

The above proxy cannot be used directly in our setting unless $\beta(L) = 0$. Hence, in place of $\tilde{z}_t$, we shall consider the residual of the projection of $\tilde{z}_t$ onto the past history of $\tilde{z}_t$ and x_t, i.e.

$$z_t = \alpha u_{it} + w_t. \tag{14}$$

Indeed the first step of our proposed procedure is to 'clean' $\tilde{z}_t$ by estimating (13) and then using the residual z_t in place of $\tilde{z}_t$.

3.2 The IRFs and the shock

In this subsection, we present our main identification results. First we consider the case of nonrecoverable shocks. In this case the shock and the impulse response functions corresponding to a unit-variance shock (the absolute IRFs) are not identified, but the relative response functions are identified. Moreover, we can estimate upper and lower bounds for the absolute IRFs. Then we turn to the case of a recoverable shock. In this case both the shock and the absolute IRFs are identified. Finally we consider the case of a fundamental shock. In this case the IRFs and the shock are identified by the standard External-Instrument SVAR formulas.

[8] As an example, consider the proxy $\tilde{z}_t = u_{1t} + u_{2,t-1} + w_t$, where u_{2t} is not fundamental. If the shock of interest is the first one ($i = 1$), this proxy fulfills (i) and (ii). But in general $u_{2,t-1}$ will be correlated with ε_t, thus inducing a bias in the estimated response functions.

Throughout this and the following subsections, we present results for the 'cleaned' proxy $z_t = \alpha u_{it} + w_t$ (see Equation 14). Let us observe, however, that all results hold true also for the original proxy $\tilde{z}_t$, provided that $\beta(L) = 0$ in Equation (13).

The identification results of the present subsection are essentially equivalent to those of Plagborg-Møller and Wolf (2022); the difference is that our identification formula, basing on the representation results of the previous section, relate the structural IRFs to the coefficients of the Wold representation of y_t, on the one hand, and the projection of the VAR residuals on the current and lagged instrument, on the other hand. As for the structural shock, we relate it to the coefficients of the projection of the instrument on the present and future of the VAR residuals. This is useful to introduce our proposed estimation method, where, unlike Plagborg-Møller and Wolf (2022), the instrument is kept outside the VAR.

3.2.1 Nonrecoverable shocks

Let us consider the projection of ε_t onto the present and past of the proxy:

$$\varepsilon_t = \psi(L) z_t + e_t. \tag{15}$$

The following result holds.

Proposition 5. (Relative IRFs) *The coefficients of the projection (15) are related to $q_i(L)$ appearing in (8) by the equation*

$$\psi(L)\sigma_z^2 = q_i(L)\alpha. \tag{16}$$

Hence the impulse-response functions fulfil the relation

$$b_i(L)\alpha = C(L)\psi(L)\sigma_z^2. \tag{17}$$

A consequence of Proposition 5 is that a possible strategy to estimate the impulse response functions is to perform the OLS regression of ε_t onto the present and past values of z_t until a maximum lag r to get an estimate of $\psi(L) = q_i'(L)\alpha/\sigma_z^2$, say $\widehat{\psi}(L)$, and estimate $b_i(L)\alpha$ as

$$\widehat{b_i(L)\alpha} = \widehat{C}(L)\widehat{\psi}(L)\widehat{\sigma}_z^2 = \widehat{\gamma}(L)\widehat{\sigma}_z^2. \tag{18}$$

Remark 6. (Consistency) The estimator (18), as well as the estimators suggested below, are elementary functions of the estimators appearing on the right-hand side of the equation, so that consistency is ensured by consistency of such estimators. Notice that the population VAR is possibly of infinite order and projection (15) is necessarily of infinite order, so that, when setting p and r, we truncate such relations. The approximation works since the coefficients shrink the further you go into the past; however, consistency requires that p and r go to infinity with T at a suitable rate.

Unfortunately, in the general case α cannot be estimated consistently so that we cannot estimate the impulse response functions corresponding to a unit-variance shock. However, we can estimate the relative IRFs, by normalising the IRFs obtained according to (18) by dividing by the effect on a pre-specified variable at a given lag, as suggested in Stock and Watson (2018). For instance we can normalise the impulse-response functions by dividing by the impact effect on the first variable:

$$\widehat{\frac{b_i(L)}{b_{i1}(0)}} = \frac{\widehat{\gamma}(L)}{\widehat{\gamma}_1(0)}, \tag{19}$$

where $\widehat{\gamma}_1(L)$ is the first entry of $\gamma(L)$. The IRFs are then the ones corresponding to a shock having impact effect 1 on the first variable.

Remark 7. (An alternative estimator for the IRFs) An alternative strategy for the estimation of the relative impulse response functions is to perform the opposite projection, i.e. the OLS projection of z_t onto the present and future of ε_t, until a maximum lead r, to get an estimate of $\delta(F) = \alpha d_i(F)$ (see Equation 6) and use the equality $b_i(L) = C(L)\Sigma_\varepsilon d_i(L)$ (see Formula 8). This strategy however implies estimation of a single equation with $(r+1)n$ regressors as against the $r+1$ regressors of each one of the n equations in (15). Our simulations, not reported here, confirm that the procedure proposed above performs better than this alternative.

Plagborg-Møller and Wolf (2022) show that, while it is impossible to estimate the absolute response functions, it is nonetheless possible to compute upper and lower bounds for the parameter α; such upper and lower bounds can be derived in our setting and provide lower and upper bounds, respectively, for the absolute response functions $b_i(L) = \gamma(L)\sigma_z^2/\alpha$.

Proposition 6. (Upper and lower bounds) *Letting $\overline{\alpha}^2$ and $\underline{\alpha}^2$ be the upper and the lower bound of α^2, respectively, we have*

$$\begin{aligned} \alpha^2 &\leq \sigma_z^2 = \overline{\alpha}^2 \\ \alpha^2 &\geq \alpha^2 \sup_{\theta\in(0\ \ \pi]} R_r^2(\theta) = \sigma_z^4 \sup_{\theta\in(0\ \ \pi]} \psi'(e^{j\theta})\Sigma_\varepsilon^{-1}\psi(e^{-j\theta}). \end{aligned} \tag{20}$$

In practice, the upper bound can be easily estimated as $\widehat{\overline{\alpha}} = \widehat{\sigma}_z$; the lower bound can be estimated by taking the maximum of $\widehat{\psi}'(e^{j\theta})\widehat{\Sigma}_\varepsilon^{-1}\widehat{\psi}(e^{-j\theta})$ over the Fourier frequencies

$\theta_f = f(2\pi/T)$,for $f = 1, \ldots, T$, T being the number of observations:

$$\underline{\widehat{\alpha}} = \widehat{\sigma}^2 \max_{0<f\leq T} \sqrt{\widehat{\psi}'(e^{j\theta_f})\widehat{\Sigma}_\varepsilon^{-1}\widehat{\psi}(e^{-j\theta f})}.$$

Having an estimate of the two bounds, the lower bound for the absolute impulse response functions can be estimated as

$$\underline{b}_i(L) = \widehat{\gamma}(L)\widehat{\sigma}_z^2/\widehat{\overline{\alpha}} = \widehat{\gamma}(L)\widehat{\sigma}_z = \widehat{C}(L)\widehat{\psi}(L)\widehat{\sigma}_z. \tag{21}$$

Similarly, the upper bound for the impulse response functions can be estimated as

$$\overline{b}_i(L) = \frac{\widehat{\gamma}(L)\widehat{\sigma}_z^2}{\underline{\widehat{\alpha}}} = \frac{\widehat{C}(L)\widehat{\psi}(L)}{\max_{0<f\leq T}\sqrt{\widehat{\psi}'(e^{jf\pi/T})\widehat{\Sigma}_\varepsilon^{-1}\widehat{\psi}(e^{-jf\pi/T})}}. \tag{22}$$

The upper and lower bounds for α can also be used to get upper and lower bounds for the fraction of variance accounted for by the shock of interest. We explain in detail how to perform variance decomposition in subsection 3.3 below.

3.2.2 Recoverable but nonfundamental shocks

If the shock is recoverable, we can estimate the absolute IRFs and the shock itself. Let us begin with the absolute IRFs. The following results holds.

Proposition 7. (Absolute IRFs) *If u_{it} is recoverable, its (absolute) impulse response functions are given by the equation*

$$b_i(L) = \gamma(L)\sigma_z^2/\alpha = \frac{C(L)\psi(L)}{\sqrt{\sum_{k=0}^{\infty}\psi_k'\Sigma_\varepsilon^{-1}\psi_k}}.$$

From the above proposition we see that $b_i(L)$ can be estimated as

$$\hat{b}_i(L) = \widehat{\gamma}(L)\widehat{\sigma}_z^2/\widehat{\alpha} = \frac{\widehat{C}(L)\widehat{\psi}(L)}{\sqrt{\sum_{k=0}^{r}\widehat{\psi}_k'\widehat{\Sigma}_\varepsilon^{-1}\widehat{\psi}_k}}. \tag{23}$$

Coming to the shock, let us consider the projection of z_t onto the space spanned by the present and the future of the VAR residuals:

$$z_t = \delta'(F)\varepsilon_t + v_t. \tag{24}$$

The following proposition holds.

Proposition 8. (The structural shock) *If* u_{it} *is recoverable, then*

$$u_{it} = \frac{\delta'(F)\varepsilon_t}{\sqrt{\sum_{k=0}^{\infty} \delta_k' \Sigma_\varepsilon \delta_k}}.$$

From the above proposition we see that, if the shock is recoverable, it can be estimated as

$$\hat{u}_{it} = \frac{\hat{\delta}'(F)\hat{\varepsilon}_t}{\sqrt{\sum_{k=0}^{r} \hat{\delta}_k' \widehat{\Sigma}_\varepsilon \hat{\delta}_k}}. \tag{25}$$

Having an estimate of the shock, we can perform historical and variance decomposition as explained in Section 3.3 below.

Remark 8. (An alternative estimator for the shock) An alternative strategy for the estimation of the shock is to use the equality $q_i(F) = \psi(F)\sigma_z/\alpha$ (see (17)), which, coupled with Equation (9), gives $\tilde{u}_{it} = \widehat{\psi}(F)\widehat{\Sigma}_\varepsilon^{-1}\widehat{\sigma}_z/\widehat{\alpha}$. We do not follow this alternative route since our simulations, available on request, show that the estimator $\hat{u}_{it}$ performs better than $\tilde{u}_{it}$.

Remark 9. (Measuring instrument validity) Having an estimate of the shock, we can measure the Instrument Relevance (IR), and test for it, by using the correlation coefficient of $\hat{u}_{it}$ and z_t:

$$\widehat{\mathrm{IR}} = \mathrm{corr}(\hat{u}_{it}, z_t).$$

3.2.3 Fundamental shocks

If we have fundamentalness, the following result holds.

Proposition 9. (IRFs and shocks under fundamentalness)
(i) Let us consider the projection equation $\varepsilon_t = \psi' z_t + e_t$*. If* u_{it} *is fundamental, then*

$$b_i(L) = \frac{C(L)\psi}{\sqrt{\psi'\widehat{\Sigma}_\varepsilon^{-1}\psi}}.$$

(ii) Let us consider the projection equation $z_t = \delta'\varepsilon_t + e_t$*. If* u_{it} *is fundamental, then*

$$u_{it} = \frac{\delta'\varepsilon_t}{\sqrt{\delta'\Sigma_\varepsilon\delta}}.$$

From Proposition 9 (i) we see that, if the shock is fundamental, we can estimate (15) without including the lags of z_t, i.e. we can estimate by OLS the projection $\varepsilon_t = \psi' z_t + e_t$ to get an estimate of ψ. The impulse-response functions can then be estimated as

$$\hat{b}_i(L) = \frac{\widehat{C}(L)\widehat{\psi}}{\sqrt{\widehat{\psi}'\widehat{\Sigma}_\varepsilon^{-1}\widehat{\psi}}}. \tag{26}$$

Notice that the above procedure is nothing else that the standard estimation procedure, which is usually applied without testing (see below for our proposed fundamentalness test).

Turning to estimation of the shock, by Proposition 9 (ii) we can estimate (24) including only the current ε_t among the regressors, in order to estimate δ; having $\hat{\delta}$, the unit variance shock can be estimated as

$$\hat{u}_{it} = \frac{\hat{\delta}'\hat{\varepsilon}_t}{\sqrt{\hat{\delta}'\widehat{\Sigma}_\varepsilon\hat{\delta}}}. \tag{27}$$

3.3 Historical and variance decomposition

In this subsection we discuss historical decomposition and variance decomposition. First we consider the case of recoverability, then we turn to nonrecoverability.

3.3.1 Recoverable shocks

We have shown that, if the shock of interest is recoverable, it can be estimated. Having an estimate of the shock and the corresponding impulse-response functions, historical decomposition can be performed in the standard way.

Variance decomposition is more problematic. The standard forecast error variance decomposition (FVD) can be computed only for globally invertible models. This is because the forecast error depends on all structural shocks and the corresponding impulse response functions. Having an estimate of the IRFs of u_{it} we can of course compute the numerator of the ratio, but we cannot estimate the denominator without estimating the whole structural model.

Plagborg-Møller and Wolf (2022) replace this denominator with the forecast error variance based on present and past values of y_t, which can be estimated, and name this ratio FVR. At long horizons, these denominators are equal, since the forecast error coincides with the variable itself; but in the short run the past values of y_t are less informative than the structural shocks and have larger forecast errors. As a consequence, the FVR underestimates the variance contribution of the structural shock of interest as given by the unfeasible FVD.[9] In the Online Appendix, Subsection B.3, we provide an example showing that the downward bias can be

[9]See Forni et al. (2019), Section 3.4.

very large, so that a low FVR at short horizons is scarcely informative.[10]

To solve this problem, in place of the FVR we suggest to use the variance decomposition given by the integral of the spectral density over suitable frequency bands. This decomposition has been proposed in Forni et al. (2019) for the case of partial fundamentalness and has become popular with Angeletos et al. (2020), where it is used to identify the so called Main Business-Cycle Shock. It is the one referred to as VD in Plagborg-Møller and Wolf (2022).[11] The relation linking the VD to the FVD is not simple. However, the VD for the whole interval $[-\pi \;\; \pi)$ is the total variance of the variable accounted for by the shock of interest, divided by the total variance of the variable, so that it is equivalent to both FVD and FVR at horizon infinity.

Let $b_{ih}(L)$ be the h-th element of $b_i(L)$. The total variance of the component of y_{ht} which is attributable to u_{it} can be computed as $\int_0^\pi b_{ih}(e^{-j\theta})b_{ih}(e^{j\theta})d\theta/\pi$, where j denotes the imaginary unit. But we can also compute the variance on a specific frequency band $[\theta_1 \;\theta_2)$. If we are interested for instance in the variance of waves of business cycle periodicity, say between 8 and 32 quarters, the corresponding angular frequencies (with quarterly data) are $\theta_1 = \pi/16$ and $\theta_2 = \pi/4$ and the corresponding variance is $\int_{\pi/16}^{\pi/4} b_{ih}(e^{-j\theta})b_{ih}(e^{j\theta})d\theta/\pi$. In practice, the integral must be approximated by averaging over a suitable frequency grid within the relevant interval (for instance, the Fourier frequencies $\theta_f = f(2\pi/T)$, f being a natural number such that $\theta_1(T/2\pi) \leq f \leq \theta_2(T/2\pi)$). This is the numerator of the suggested variance decomposition. As for the denominator, we need the spectral density of y_{ht}, say $\frac{1}{2\pi}S_h(\theta)$, where

$$\widehat{S}_h(\theta) = \widehat{C}_h(e^{-j\theta})\widehat{\Sigma}_\varepsilon\widehat{C}_h(e^{j\theta})',$$

$C_h(L)$ being the h-th row of the matrix $C(L)$ appearing in the Wold representation. The total variance of y_{ht} on the frequency band $[\theta_1 \;\theta_2)$ is given by $\int_{\theta_1}^{\theta^2} \widehat{S}_h(\theta)d\theta/\pi$. Hence the contribution of u_{it} to the variance of y_{ht} on the frequency band $[\theta_1 \;\theta_2)$, say $c_{ih}(\theta_1, \theta_2)$ can be estimated as the ratio

$$\hat{c}_h(\theta_1, \theta_2) = \frac{\int_{\theta_1}^{\theta_2} \hat{b}_{ih}(e^{-j\theta})\hat{b}_{ih}(e^{j\theta})d\theta}{\int_{\theta_1}^{\theta^2} \widehat{S}_h(\theta)d\theta}. \tag{28}$$

We can also evaluate the fraction of total variance explained by the shock of interest at each

[10]Were it possible to have an estimate of all structural shocks, the sum of the variance contributions of these shocks, computed with the FVR, is smaller than 100% (in the example of the Online Appendix it is just 20%).

[11]We recommend to use this variance decomposition even if fundamentalness of the shock of interest is not rejected according to the test described in the following subsection, since of course we might have fundamentalness of this shock but not global fundamentalness; and, as stated above, the standard forecast error variance decomposition requires global fundamentalness.

frequency, i.e.

$$\hat{c}_h(\theta) = \frac{\hat{b}_{ih}(e^{-j\theta})\hat{b}_{ih}(e^{j\theta})}{\widehat{S}_h(\theta)}. \tag{29}$$

We stress that the above formulas provide a variance decomposition for the variable itself. In this respect, its interpretation is more direct than the one of the standard variance decomposition, which refers to the variance of the forecast errors.

3.3.2 Nonrecoverable shocks

In the case of nonrecoverability we can only estimate $\gamma(L)\sigma_z^2 = \alpha b_i(L)$ according to Equation (18). Hence we can get an estimate of

$$\alpha^2 c_h(\theta_1, \theta_2) = \frac{\sigma_z^4 \int_{\theta_1}^{\theta_2} \gamma_h(e^{-j\theta})\gamma_h(e^{j\theta})d\theta}{\int_{\theta_1}^{\theta 2} S_h(\theta)d\theta}. \tag{30}$$

Since α cannot be estimated, we cannot perform the variance decomposition. However, the upper and lower bounds for α^2 provided in subsection 3.2.1 provide respectively lower and upper bounds for $c_h(\theta_1, \theta_2)$. Precisely, we can estimate the lower bound as

$$\hat{\underline{c}}_h(\theta_1, \theta_2) = \widehat{\sigma}_z^2 \frac{\int_{\theta_1}^{\theta_2} \widehat{\gamma}_h(e^{-j\theta})\widehat{\gamma}_h(e^{j\theta})d\theta}{\int_{\theta_1}^{\theta 2} \widehat{S}_h(\theta)d\theta} \tag{31}$$

and the upper bound as

$$\hat{\bar{c}}_h(\theta_1, \theta_2) = \widehat{\sigma}_z^4 \frac{\int_{\theta_1}^{\theta_2} \widehat{\gamma}_h(e^{-j\theta})\widehat{\gamma}_h(e^{j\theta})d\theta}{\underline{\widehat{\alpha}}^2 \int_{\theta_1}^{\theta 2} \widehat{S}_h(\theta)d\theta}, \tag{32}$$

where $\theta_f = f(2\pi/T)$, $f = 1, \ldots, T$. Finally, the lower and upper bounds for the frequency-by-frequency decomposition of Equation (29) can be computed as

$$\hat{\underline{c}}_h(\theta) = \widehat{\sigma}_z^2 \frac{\widehat{\gamma}_h(e^{-j\theta})\widehat{\gamma}_h(e^{j\theta})}{\widehat{S}_h(\theta)} \tag{33}$$

$$\hat{\bar{c}}_h(\theta) = \frac{\widehat{\sigma}_z^4 \widehat{\gamma}_h(e^{-j\theta})\widehat{\gamma}_h(e^{j\theta})}{\underline{\widehat{\alpha}}^2 \widehat{S}_h(\theta)}, \tag{34}$$

where $\theta_f = f(2\pi/T)$, $f = 1, \ldots, T$.

3.4 Testing for recoverability and fundamentalness

In this subsection we propose a test for recoverability and a test for fundamentalness.

3.4.1 Recoverability test

In the proof of Proposition 8 we have seen that, if u_{it} is recoverable, then the projection in (24), $\delta'(F)\varepsilon_t$, is equal to αu_{it} and therefore is a white noise process. By contrast, if recoverability does not hold, $s_{it} \neq 0$ and the projection is not equal to αu_{it}. Being a Moving Average of present and future VAR residuals, the projection will in general be autocorrelated.

Hence, to check whether the shock is recoverable or not, following a suggestion of Plagborg-Møller and Wolf (2022), we propose to test for zero serial correlation of the projection $\delta(F)\varepsilon_t$.[12] Precisely, we propose to perform the OLS regression of z_t onto the present and future values of $\hat{\varepsilon}_t$, until a maximum lead r, to get an estimate of $\delta(F)$. Then apply the Ljung-Box Q-test to the estimated projection $\hat{\delta}(F)\hat{\varepsilon}_t$. The null hypothesis is recoverability (serial uncorrelation) and the alternative is nonrecoverability (serial correlation). In the following section we present a Monte Carlo exercise in which the autocorrelation test has a reasonably good power in rejecting recoverability when it is false.

If recoverability is rejected, we have two options: (1) estimate the relative impulse response functions and the upper and lower bounds for variance decomposition; (2) amend the VAR specification by adding variables (or use a FAVAR model in place of the VAR) and perform the test with the novel VAR specification.

It is worth noticing that the above test is valid under the maintained hypothesis that Equation (13) is fulfilled. If it is not, the lags of u_{it} may appear in Equation (14) and the test may reject serial uncorrelation even if the shock is recoverable. Hence the serial uncorrelation test is indeed a joint test about recoverability and instrument validity.

If recoverability is not rejected, we can estimate the 'absolute' response function, the unit variance shock and the variance decomposition as explained above.

3.4.2 Fundamentalness test

If u_{it} is fundamental with respect to y_t we see from Proposition 9 that in Equation (24) $\delta_k = 0$ for all positive k and $\delta(F)$ reduces to $\delta_0 = \delta$. Hence, we can test for the null of fundamentalness against the alternative of nonfundamentalness by estimating (24) by OLS as explained above and perform a standard F-test for the joint significance of the coefficients of the leads. Notice that the test is valid even if recoverability does not hold; hence, in principle we can test directly for fundamentalness without testing for recoverability.[13]

Remark 10. (Estimating the degree of fundamentalness) If fundamentalness does not hold, but recoverability does, we can estimate consistently the degree of fundamentalness R^2_f (see

[12]In Plagborg-Møller and Wolf (2022), the practical implementation of the test is left for future research.

[13]However, the recoverability test can provide a useful check in that, if recoverability is rejected, fundamentalness is rejected as well.

Remark 5). From Proposition 8 we see that the fraction of the variance of u_{it} explained by the current VAR residuals is $R_f^2 = \delta_0'\Sigma_\varepsilon\delta_0 / \sum_{k=0}^{\infty}\delta_k'\Sigma_\varepsilon\delta_k$. Hence we can estimate R_f^2 as

$$\hat{R}_f^2 = \hat{\delta}_0'\widehat{\Sigma}_\varepsilon\hat{\delta}_0 / \sum_{k=0}^{r}\hat{\delta}_k'\widehat{\Sigma}_\varepsilon\hat{\delta}_k.$$

Stock and Watson (2018) and Plagborg-Møller and Wolf (2022) also propose invertibility tests based on the proxy. The latter paper, in particular, proposes to test whether the proxy Granger causes the VAR variables.[14] By using Sims' theorem, it can be shown that their method is asymptotically equivalent to ours. It would be interesting to compare the small sample performances of all these tests; this however is left for further research. In the online appendix we present a Monte Carlo exercise showing that the proposed F-test has a good performance in small samples.

If fundamentalness is not rejected, we can estimate the 'absolute' response function, the unit variance shock and the variance decomposition as explained in the previous subsections.

3.5 Inference

For inference purposes, we suggest the following bootstrap procedure. For simplicity, we assume that the sample size T is the same for z_t and y_t. The generalisation to the case of different time spans is straightforward.

First, draw with reintroduction $T-(p+r)$ integers $i(t)$, $t = 1, \dots, T-(p+r)$, uniformly distributed between 1 and $T-(p+r)$, and construct the artificial sequences of shocks $\varepsilon_t^1 = \hat{\varepsilon}_{i(t)}$ and $v_t^1 = \hat{v}_{i(t)}$, $t = p+1, \dots, T-r$, $\hat{v}_t$ being the estimated residual of regression (24). Set the final conditions $\varepsilon_t^1 = \hat{\varepsilon}_t$ for $t = T-r+1, \dots, T$. Repeat the procedure H times to get the sequences ε_t^h, $t = p+1, \dots, T$ and v_t^h, $t = p+1, \dots, T-r$, for $h = 1, \dots, H$.

Second, compute y_t^h, $h = 1, \dots, H$, according to the VAR Equation (4). Precisely, set the initial conditions $y_t^h = y_t$, $t = 1, \dots, p$, for all h. Then compute

$$y_t^h = -\sum_{k=1}^{p}\widehat{A}_k y_{t-k}^h + \varepsilon_t^h$$

for $t = p+1, \dots, T$. As for the proxy, set the initial and final conditions $z_t^h = z_t$, $t = 1, \dots, p$

[14]Fundamentalness test based on Granger causality have been previously proposed, in the context of standard VAR identification, in Giannone and Reichlin (2006) and Forni and Gambetti (2014).

and $t = T - r + 1, \dots, T$, for all h. Then compute

$$z_t^h = \sum_{k=0}^{r} \hat{\delta}_k \varepsilon_{t+k}^h + v_t^h$$

for $t = p + 1, \dots, T - r$.

Finally, repeat the estimation procedure for any one of the artificial data sets $y_1^h, \dots, y_T^h$, $h = 1, \dots, H$ to get the sequences of absolute IRFs $b_i^h(L)$, $h = 1, \dots, H$, or the corresponding sequence of relative IRFs. Compute the confidence band as usual, by taking appropriate percentiles of the distribution of b_{ik}^h, for each lag k.

4 The proposed procedure

On the basis of the above considerations and results, we propose the following estimation and testing procedure.

1. As a first step, regress the available proxy $\tilde{z}_t$ onto the first m lags (notice that m is not necessarily equal to p) of $\tilde{z}_t$ itself and a set of regressors x_t —which can in principle be different from y_t— to get an estimate of the residual z_t, say $\hat{z}_t$. If the F-test does not reject the null that the coefficients of past $\tilde{z}_t$'s are all zero, i.e. $H_0 : \beta(L) = 0$, step 1 is unnecessary and can be skipped.

2. Estimate a VAR(p) with OLS to obtain $\widehat{A}(L)$, $\widehat{C}(L) = \widehat{A}(L)^{-1}$, $\hat{\varepsilon}_t$ and $\widehat{\Sigma}_\varepsilon$.

3. Regress with OLS the proxy $\hat{z}_t$ on the current value and the first r leads of the Wold residuals:
$$\hat{z}_t = \sum_{k=0}^{r} \hat{\delta}_k' \hat{\varepsilon}_{t+k} + \hat{v}_t = \hat{\delta}(F)\hat{\varepsilon}_t + \hat{v}_t.$$
Save the fitted value of the above regression, let us call it $\hat{\eta}_t$. Test for invertibility by performing the F-test for the null $H_0 : \delta_1 = \delta_2 = \cdots = \delta_r = 0$ against the alternative that at least one of the coefficients is non-zero.

4. Case 1: invertibility is not rejected. In this case estimate (24) without the leads of ε_t to get an estimate of δ and estimate the unit-variance shock according to (27). To estimate the corresponding IRFs, apply the standard procedure, i.e. estimate (15) without the lags of z_t to get $\hat{\psi}$ and estimate the IRFs according to (26). Estimate the variance decomposition according to equation (28) or (29).

4′. Case 2: invertibility is rejected. In this case perform the recoverability test by testing for the null of serial uncorrelation of the fitted value $\hat{\eta}_t$ by using the Ljung-Box Q-test.

5. Case 1: recoverability is not rejected. In this case estimate the unit-variance shock according to (25) and the corresponding IRFs according to (23). Estimate the variance decomposition according to equation (28) or (29). Historical decomposition can be performed in the standard way.

5′. Case 2: recoverability is rejected. In this case, either amend the VAR specification and repeat steps 2-4, or estimate (15) with a maximum lead r and the 'relative' IRFs according to (19). Estimate lower and upper bounds according to (21) and (22) and the corresponding variance contributions according to (31) and (32) or (33) and (34).

Let us stress that the above External-Instrument SVAR procedure is more flexible than the Internal-Instrument method, recently re-proposed in Plagborg-Møller and Wolf (2021, 2022), and the VARX method, proposed in Paul (2020).[15] This is mainly because the sample span of the proxy can be different from the sample span of the VAR (or the VARX). Moreover, with respect to the internal proxy SVAR method, our procedure is more flexible because the number of lags m and the regressors x_t used in step 1 of the above procedure are not necessarily equal to the number of lags p and the regressors y_t used in the VAR. In particular, if the regression in step 1 is not significant, or the coefficient in the lags of the proxy are not significant, the proxy can be used without treatment ($m = 0$). When including the proxy into the VAR model, this would require to impose restrictions on the VAR parameters. Finally, notice that the number of lags r used in the regression of the VAR residuals ε_t onto the current and lagged proxy can be different from p – the number of lags used in the VAR. In Simulation 5 below we show that this flexibility may translate into better small-sample estimation performances.

5 A simulated economy with fiscal foresight

In this section, we assess the small sample performance of the proposed estimation and testing procedure in two Monte Carlo exercises. The simulations show that the method works and outperforms the Internal-Instrument method in estimating the relative IRFs, conditional on the data generating process studied here. Additional Monte Carlo exercises are provided in Section C of the Online Appendix.

[15] The internal proxy method consists in including the proxy into the VAR model, ordered first, and identifying the shock as the first one in a Cholesky scheme; the VARX method consists in estimating a VARX with the proxy used as the exogenous variable.

5.1 Simulation 1: A recoverable shock

In this simulation exercise we use the fiscal foresight model of Leeper et al. (2013) (LWY henceforth). The model is a simple Real Business Cycle model with log preferences, inelastic labor supply and two shocks: $u_{a,t}$, a technology shock, and $u_{\tau,t}$, a tax shock. A nonstandard feature of the model is the fact that the tax shocks are announced to the agents before being implemented, thus inducing fiscal foresight. The equilibrium capital accumulation is

$$k_t = \alpha k_{t-1} + a_t - \kappa \sum_{i=0}^{\infty} \theta^i E_t \tau_{t+i+1} \tag{35}$$

where $0 < \alpha < 1$, $0 < \theta < 1$, $\kappa = (1-\theta)\tau/(1-\tau)$, τ being the steady state tax rate, $0 \leq \tau < 1$, and a_t, k_t and τ_t are the log deviations from the steady state of technology, capital and the tax rate, respectively. Technology and taxes are given by

$$\begin{aligned} a_t &= u_{a,t} \\ \tau_t &= u_{\tau,t-2}, \end{aligned}$$

where $u_{\tau,t}$ and $u_{a,t}$ are i.i.d. shocks. Solving for k_t we obtain the following equilibrium MA representation for capital and taxes:

$$\begin{pmatrix} \tau_t \\ k_t \end{pmatrix} = \begin{pmatrix} L^2 & 0 \\ \dfrac{-\kappa(L+\theta)}{1-\alpha L} & \dfrac{1}{1-\alpha L} \end{pmatrix} \begin{pmatrix} u_{\tau,t} \\ u_{a,t} \end{pmatrix} = B(L) u_t. \tag{36}$$

The determinant of the above matrix vanishes for $L = 0$; hence the shocks are not fundamental. However, they are recoverable, since the system is square (see Remark 1). In particular, the tax shock is equal to taxes two periods ahead: $u_{\tau,t} = \tau_{t+2}$.

We generate 1000 different dataset with 240 time observations from model (36) using the parameterisation in Leeper et al. (2013): $\alpha = 0.36$, $\theta = 0.2673$ and $\tau = 0.25$ and $u_t \sim N(0, I)$. The proxy is generated according to the equation

$$\tilde{z}_t = u_{\tau,t} + 0.5 z_{t-1} + 0.4 k_{t-1} - 0.6 \tau_{t-1} + v_t,$$

where $v_t \sim iid\mathcal{N}(0,1)$; here the parameters are arbitrarily chosen. For each dataset, we test for invertibility and recoverability, and estimate the tax shock along with its response functions as explained in the previous section, by setting $p = m = 2$, $r = 0$ and $p = m = 2$, $r = 4$. Invertibility is correctly rejected in all cases. Recoverability is (wrongly) rejected at the 5% level in 10% of the cases, showing that the test is somewhat oversized in small samples with this DGP. Simulation 4 and 5 in the Online Appendix provide additional information about the empirical size and power of the invertibility and recoverability tests.

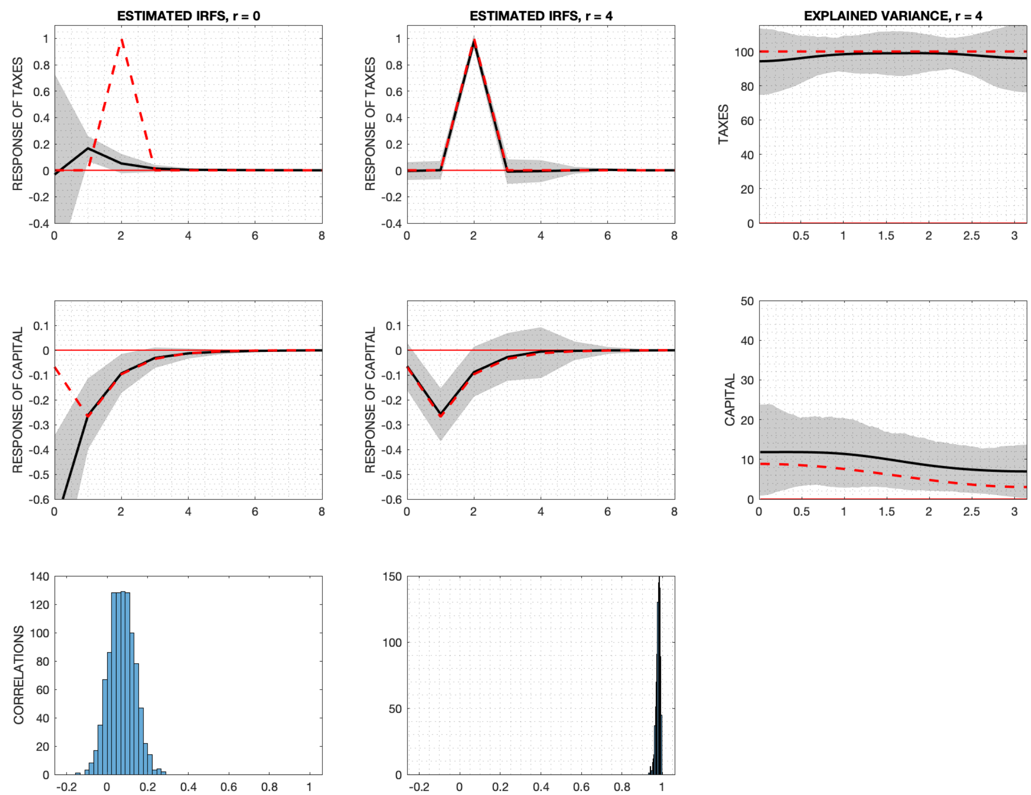

Figure 1: Simulation 1: LWY fiscal foresight model (recoverable shock). Left panels: IRF estimation with $r = 0$ (standard method). Middle panels: IRF estimation with $r = 4$. Right panels: estimation of the variance contribution according to Equation (29), expressed in percentage terms. Upper and middle panels: red dashed line, true response functions and variance decomposition; black solid line, mean of the 1000 estimated response function and variance decompositions; grey area, 16th to 84th percentiles. Lower panels: frequency distribution of the correlation coefficients between the estimated shock and the true shock.

Estimation results are shown in Figure 1. The first column shows the estimates obtained with the standard method ($r = 0$). The upper and middle panels show the estimated response functions: the black solid line is the mean across the 1000 experiments; the grey area shows the estimates between the 16th and the 84th percentiles; the red dashed lines are the true IRFs. The bottom panel shows the frequency distribution of the correlation coefficients between the estimated shock and the true shock. As expected, the estimates are dramatically wrong, since $r = 0$ requires invertibility, which does not hold in this case. The second column shows the estimates obtained with $r = 4$. Here the estimates are very good, both because black and red lines are almost perfectly overlapping, and because the grey area is not large, particularly for taxes. The correlations coefficients between the estimated shock and the true shock are very close to 1. The third column shows the variance decomposition obtained according to Equation (29), expressed in percentage terms. The estimates are fairly good, albeit somewhat less precise than those of the IRFs.

5.2 Simulation 2: A comparison with the Internal-Instrument SVAR

In this second Monte Carlo exercise we compare the small-sample performance of our proposed procedure in estimating the relative IRFs with the one of the Internal-Instrument SVAR approach, using the LWY model of Simulation 1. In particular, we want to see whether the flexibility of the Generalised External-Instrument SVAR translates into smaller estimation errors. The instrument is generated as in the previous exercise. We focus on the relative IRFs of the tax shocks. Since $b_{i1}(0) = 0$, the normalisation in Equation (19) cannot be used. Hence we normalise the IRFs by dividing by the effect on the first variable (taxes) at lag 2, i.e. we estimate

$$\mu(L) = \frac{\gamma(L)}{\gamma_{1,2}} = \begin{pmatrix} L^2 \\ \dfrac{-\kappa(L+\theta)}{1-\alpha L} \end{pmatrix}.$$

For our procedure, the instrument is preliminarily 'cleaned' by setting $x_t = y_t$ and the number of lags m according to the BIC. For the Internal-Instrument method, we estimate a VAR for the vector $(\tilde{z}_t \;\; y_t')'$; the IRFs are identified as the ones corresponding to the first shock of a Cholesky scheme. As in the previous exercise, we generate 1000 different dataset with 240 time observations. The estimation error of the two competing methods is measured as the sum of the squared errors divided by the sum of the squared coefficients of the true IRFs:

$$100 \times \frac{\sum_{h=1}^{n} \sum_{k=0}^{K} (\hat{\mu}_{hk} - \mu_{hk})^2}{\sum_{h=1}^{n} \sum_{k=0}^{K} \mu_{hk}^2}. \tag{37}$$

We set $K = 10$ and $n = 2$. This ratio is equal to 100 for the flat estimate $\hat{\mu}(L) = 0$.

Results are shown in Table 1. The first column report results for the Internal-Instrument method. The second column reports results obtained with our procedure, with r selected according to the BIC, applied to Equation (17). Columns 3-7 report results obtained by setting arbitrarily $r = 3, 4, 5, 6, 7$. Boldface numbers are the best estimates of each column.

Let us focus first on columns 3-7. Clearly, $r = 3$ is the best choice for our DGP. Column 2 shows that the BIC criterion is successful in detecting the correct value of r, since the errors are very close to those of the best column (indeed, using just one decimal figure, we cannot see any difference).

VAR order	Internal IV	External IV					
		$r = BIC$	$r = 3$	$r = 4$	$r = 5$	$r = 6$	$r = 7$
$p = 1$	410.8	**4.3**	**4.3**	**5.0**	**6.4**	**7.9**	**9.4**
$p = 2$	34.8	5.2	5.2	5.9	**6.4**	**7.9**	**9.4**
$p = 3$	**7.6**	6.0	6.0	6.8	7.3	8.0	9.5
$p = 4$	9.5	7.1	7.1	7.8	8.4	9.1	9.6
$p = 5$	11.2	8.0	8.0	8.8	9.3	10.0	10.6
$p = 6$	12.9	8.9	8.9	9.6	10.2	10.9	11.5
$p = BIC$	**7.6**	**4.3**					

Table 1: Results of Simulation 2. Estimation errors for the relative IRFs, measured acording to (37), for the internal instrument SVAR (first column) and our proposed procedure, with r determined by the BIC (second column) and $r = 3, 4, 5, 6, 7$ (columns 3-7). Boldface numbers are the best results obtained for each column.

The Generalised External-Instrument SVAR performs better than the Internal-Instrument SVAR for all values of p (small values of p seem particularly effective with our method). When using the BIC for the VAR order p (last row), our method performs better than the internal proxy SVAR and the relative improvement is large (about 40%).

Our interpretation is the following. We have three dynamic relations: the one used to clean the proxy $\tilde{z}_t$, the VAR for y_t and the relation linking the VAR residuals and the cleaned proxy z_t. Such relations may have different optimal orders. With the External-Instrument method we can set different values for such orders (m, r and p): a flexibility that the Internal-Instrument method does not have.

A similar Monte Carlo exercise with a different DGP, but similar results, is provided in Section C of the Online Appendix (see Simulation 6). A more extensive comparison between our proposed procedure and existing alternatives would be interesting but is beyond the aim of the present paper.

6 The effects of monetary policy shocks

In this section, we provide an empirical application of our method and study the propagation of monetary policy shocks. We first show that the monetary policy shocks identified with standard high-frequency surprises at the short end of the yield curve are likely to be nonfundamental but recoverable, in a few routinely used VAR specifications. We then show that the standard Internal-Instrument SVAR procedure delivers price and output puzzles. Conversely, when using our suggested procedure for noninvertible shocks, we get results in line with the textbook effects of monetary policy.

6.1 Data, VAR specification, instruments

Our baseline VAR specification includes three variables at monthly frequency: the 1-year government bond rate (1YB), industrial production (IP) in growth rates and CPI inflation (Specification I). We also present results for two additional specifications, one including Gilchrist and Zakrajšek (2012)'s excess bond premium (EBP) (Specification II); the other including EBP along with the mortgage spread (MS) and the commercial paper spread (CPS) (Specification III). We use 1YB as the policy indicator variable.[16]

Our benchmark sample spans the period 1983:1-2008:12. In two robustness exercises, we consider alternative initial dates, i.e. 1979:7, 1987:8, and 1990:1, as well as two alternative ending dates, i.e. 2012:6 and 2019:6.[17] The trending variables, CPI and IP, are taken in differences since these variables are unlikely to be cointegrated. In a robustness exercise, we consider a VAR specification with all the trending variables in levels.

The instrument for monetary policy shocks consists of the Gürkaynak et al. (2005)'s intra-daily monetary policy surprises triggered by Federal Open Market Committee (FOMC) decisions in the three month ahead monthly Fed Funds futures (FF4), as proposed in Gertler and Karadi (2015) (GK from now on). The use of this instrument provides scope for testing our approach to noninvertibility since, as discussed in Gertler and Karadi (2015) and Ramey (2016), surprises in futures with a three month maturity are likely to capture both conventional monetary policy shocks, and shocks to forward guidance about the path rate at short horizon. We 'clean' the instrument by regressing it onto its own lags and the lags of the three variables of Specification I, using 6 lags.[18] The instrument turns out to be relevant, with a measure of relevance $\widehat{\mathrm{IR}}$ between 0.4 and 0.6 depending on the specification adopted.

6.2 Fundamentalness and recoverability

We start our analysis by applying our fundamentalness test (Table 2a) to verify whether our specifications turns out to be fundamental or not when using our instrument GK. The main takeaway is that the results obtained with the standard proxy-SVAR approach in the monetary

[16]We use monthly data taken from the FRED-MD data set of McCracken and Ng (2015). Specifically, we use industrial production (FRED mnemonic INDPRO, IP from now on), taken in log differences, the CPI index (FRED mnemonic CPIAUCSL), taken in log differences, and the 1-year government bond rate (FRED mnemonic GS1, 1YB from now on). In addition, we use the excess bond premium (EBP), the mortgage spread (MS) and the commercial paper spread (CPS) taken from the replication files of Gertler and Karadi (2015).

[17]The samples starting in 1983:1 and 1987:8 are chosen in line with Sims and Zha (2006). Moreover, 1979:7 is the beginning of Volcker's mandate; 1987:8 is the beginning of Greenspan's mandate; 2008:12 is the first month in which the 1-year bond rate falls below 1%, so that cutting our sample to 2008:12 excludes the zero lower bound period.

[18]The regression is significant at the 5% level and the residual is serially uncorrelated according to the Ljung-Box Q-test.

	Number of leads r					
	$r=4$	$r=5$	$r=6$	$r=7$	$r=8$	$r=9$
Specification I						
$p=6$	0.008	0.028	0.002	0.003	0.001	0.001
$p=9$	0.016	0.051	0.003	0.003	0.002	0.001
$p=12$	0.011	0.045	0.003	0.002	0.001	0.000
Specification II						
$p=6$	0.080	0.195	0.027	0.001	0.000	0.000
$p=9$	0.180	0.351	0.034	0.002	0.000	0.000
$p=12$	0.221	0.457	0.059	0.003	0.000	0.000
Specification III						
$p=6$	0.060	0.184	0.089	0.003	0.001	0.002
$p=9$	0.184	0.362	0.220	0.020	0.002	0.003
$p=12$	0.215	0.353	0.250	0.060	0.031	0.027

(a) Fundamentalness test

	Number of leads r					
	$r=4$	$r=5$	$r=6$	$r=7$	$r=8$	$r=9$
Specification I						
$p=6$	0.619	0.662	0.251	0.469	0.037	0.060
$p=9$	0.350	0.571	0.114	0.435	0.050	0.042
$p=12$	0.880	0.944	0.324	0.820	0.466	0.285
Specification II						
$p=6$	0.441	0.473	0.308	0.777	0.394	0.357
$p=9$	0.119	0.186	0.104	0.517	0.222	0.193
$p=12$	0.472	0.558	0.269	0.913	0.701	0.575
Specification III						
$p=6$	0.034	0.315	0.446	0.608	0.738	0.546
$p=9$	0.005	0.064	0.148	0.046	0.391	0.103
$p=12$	0.032	0.037	0.065	0.057	0.343	0.022

(b) Recoverability test

Table 2: P-values for the invertibility (a) and the recoverability tests (b), for different values of p and r. Specification I includes the 1-year bond rate (1YB, industrial production growth (IP) and CPI inflation (CPI). Specification II includes 1YB, IP, CPI and the excess bond premium (EBP). Specification III includes 1YB, IP, CPI, EBP, the mortgage spread and the commercial paper spread. The proxy is the one of Gertler and Karadi (2015).

policy literature should be taken with caution, since the VAR specification might be affected by nonfundamentalness. In fact, results in Table 2a show that, for $r \geq 6$, fundamentalness is rejected at the 1% level with Specifications I and II and for $r \geq 7$ is rejected either at the 5% level or the 10% level with Specification III. The degree of fundamentalness $\widehat{R}_f^2$ is below 0.5 for all specification for $r \geq 6$. We conclude that the inclusion of financial variables in the VAR may not be sufficient to solve fundamentalness problems. These results are in line with the findings of Plagborg-Møller and Wolf (2022) and the arguments in Ramey (2016), who cautions against the standard SVAR-IV approach.

Yet, the good news is that the shock is recoverable. The p-values of the Ljung-Box Q-test for serial correlation of the estimated monetary policy shock, for our three specifications, with different values of p and r (maximum lag 24) are reported in Table 2b. The result is that recoverability cannot be rejected at the 5% level for most parameter configurations. We conclude that, at least for our time span, the monetary policy shock is recoverable, even with the three-variable Specification I, and that financial variables are not needed to find the policy shock.

6.3 The three-variable VAR

In this subsection we compare the impulse response functions obtained with the standard method with those obtained with our proposed method. We choose Specification I with the

GK instrument as our benchmark. We set the number of lags in the VAR equal to 12 ($p = 12$) and the number of leads of the VAR residuals included in the regression of the proxy equal to 6 ($r = 6$). In a robustness exercise, we try different values of p and r.[19]

The basic insight delivered by our exercise is that by incorrectly assuming invertibility without testing one can get dramatically misleading results. On the contrary, when the proposed procedure is applied, the estimation delivers results in line with textbook effects of monetary policy, even with a small VAR, and not including the EBP or other financial variables.

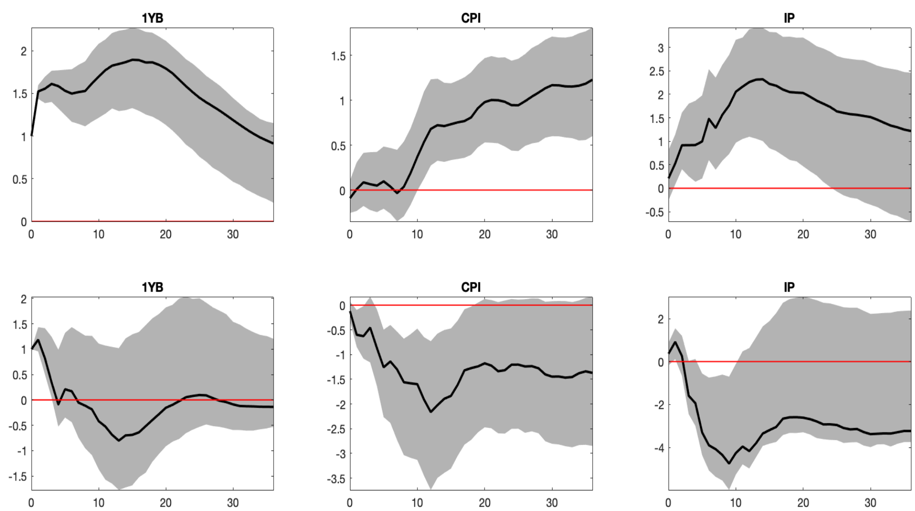

Figure 2: VAR results: Specification I, $p = 12$, GK instrument. Top panels: estimated response functions with $r = 0$ (standard method). Bottom panels: estimated response functions with our proposed method $r = 6$. Black line: point estimate. Grey area: 68% confidence bands.

The results from the baseline model are reported in Figure 2. All responses are normalised to have an impact effect of 100 basis points on the 1-year bond rate. The top panels show the estimated impulse response functions obtained with $r = 0$, i.e. the standard proxy SVAR procedure. The response of 1YB is hump-shaped and very persistent (the zero line is not reached after 3 years). Both prices and industrial production significantly increase after a tightening shock, so that we have both a large price puzzle and a large real activity puzzle.

The bottom panels show the result obtained with our proposed procedure with $r = 6$. Results are completely different and much more plausible. The reaction of the policy variable is much less persistent: after the significant impact effect it reaches the zero line in about 4 months and further on it is no longer significant. Both inflation and output puzzles disappear:

[19]For simplicity, here we do not make any attempt at cleaning the instrument from the information effects recently discussed in the literature (see, for example, Jarociński and Karadi, 2020 and Miranda-Agrippino and Ricco, 2021). This is left for future research.

prices and industrial production reduce significantly after an impact effect which is very close to zero. The effects on real activity are no longer significant after about one year, showing that the effects of monetary policy on real activity are transitory, in line with the consensus. In Section D of the Online Appendix, we report several robustness checks. The overall conclusion is that the above results are reasonably robust.

6.4 Medium-size VAR specifications

Are results sensitive to the VAR information set? To answer this question, in this subsection we examine results for Specification II, that includes EBP, and Specification III, that incorporates the mortgage spread and the commercial paper spread, as well as EBP. We set $p = 12$, $r = 0$ and $r = 6$, as in the previous subsection. The main conclusion of this exercise is that results obtained with our proposed method are reasonably robust, whereas results obtained with the standard method are not.

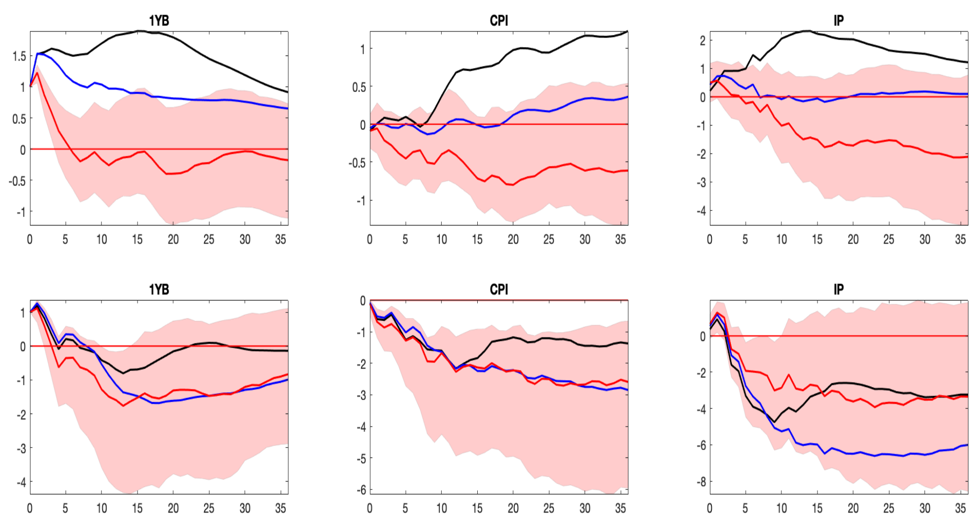

Figure 3: IRFs for Specification III (1YB, CPI inflation, IP growth, EBP, MS CPS). The instrument is GK. Red line: point estimates; blue line: point estimates for Specification II; black line: point estimates for Specification I. Top panels: estimated response functions with $p = 12$, $r = 0$ (standard method). Bottom panels: estimated response functions with our proposed method, $p = 12$, $r = 6$. Pink shaded area: 68% confidence bands for Specification III.

In the top panels, Figure 3 reports the case $r = 0$ (standard method). The lines are respectively the point estimates for Specification I (black), Specification II (blue), and Specification III (red). We also report the confidence bands of Specification III (pink shaded areas). Results appear to be very sensitive to the set of variables included in the VAR. With Specification II, the effects on prices and real activity are essentially zero. With Specification

III, the sign of the IRFs of prices and industrial production are negative, and the puzzles of Specification I disappear. Still, the effects are quantitatively small, especially for prices, and not significant.

The IRFs obtained with the Generalised External-Instrument approach ($r = 6$) of the first three variables are similar to those obtained with Specifications I and II (bottom panels in Figure 3). The reaction of prices is large and significant. The reaction of IP is barely significant, since the confidence bands are very large, however the point estimates consistently show a sizeable reduction of about 3-5 percentage points of production after one year, depending on the VAR specification.

6.5 Variance decomposition

Do monetary policy shocks account for a sizeable share of the variance of prices and output? To answer this question, it is useful to evaluate the variance decomposition VD of CPI inflation and industrial production growth obtained with our VAR specifications ($p = 12$, $r = 6$). We report results for waves of periodicity 2-18 months (short run), 18-96 month (business cycle) and 2+ months (overall variance). The main finding is that the effects of monetary policy on prices are much larger than previously reported, suggesting that it can be used successfully in controlling inflation.

Table 3 reports the point estimates and the 68% confidence bands (in brackets) for the percentage of variance explained by the monetary policy shock. The estimates for the cyclical variance (18-96 months) are not very reliable because of the large confidence bands, so that we focus mainly on the short-run and overall variances. The point estimates of the short-run volatility contributions range from 12.3% to 19.2% for inflation and from 16.1% to 27.7% for industrial production growth. As for the overall variance, the estimates range from 12.5% to 20.8% for inflation and from 13.0% to 28.3% for industrial production.

With Specification III, which provides the smallest estimates, the monetary policy shocks explains 12.5% of the overall variance of inflation and 13% of the overall variance of production, with the 68% confidence bands ranging between a minimum of around 10% and a maximum of around 20% for both variables. We conclude that, contrary to previous findings, the effects of discretionary monetary policy on inflation are far from negligible. These results are at odds with the ones in Plagborg-Møller and Wolf (2022), where, according to FVR estimates, the contribution of policy shocks to inflation fluctuations is negligible at all horizons between 0 and 24 months.

To understand the sources of the difference, we compute the point estimates of the FVR of inflation, reported in Table 4. In the lower part of the table, we also report results for CPI in

	Waves of periodicity		
	2 – 18 months	18 – 96 months	2+ months
Specification I			
CPI inflation	19.2	27.6	20.8
	(13.5—29.1)	(12.8—64.2)	(16.2—35.1)
IP growth	27.7	33.8	28.3
	(19.1—36.4)	(13.1—55.4)	(20.0—37.6)
Specification II			
CPI inflation	12.3	12.9	13.2
	(10.4—23.1)	(9.7—45.1)	(13.4—26.8)
IP growth	20.3	29.5	22.5
	(15.8—28.2)	(11.4—51.5)	(16.7—31.3)
Specification III			
CPI inflation	12.5	10.3	12.5
	(10.2—19.5)	(6.9—34.2)	(11.2—21.5)
IP growth	16.1	5.2	13.0
	(12.2—22.2)	(4.2—22.0)	(11.2—20.7)

Table 3: Percentage of variance accounted for by the monetary policy shock, for waves of periodicity 2-18 months (short run), 18-96 months (business cycle), 2+ months (overall variance). 68% confidence bands in brackets.

	FVR Horizon					VD
	impact	3 months	6 months	12 months	24 months	2+ months
CPI inflation						
Specification I	0.5	7.2	15.3	18.4	20.7	20.8
Specification II	0.2	4.7	9.1	13.3	13.4	13.2
Specification III	0.3	5.6	7.4	12.5	12.4	12.5
CPI index in levels						
Specification I	0.5	4.2	9.9	20.0	21.5	
Specification II	0.2	2.6	5.3	13.7	22.5	
Specification III	0.3	4.4	7.1	13.8	18.5	

Table 4: Percentage of variance of CPI inflation and prices accounted for by the monetary policy shock, according to the FVR measure of Plagborg-Møller and Wolf (2022), on impact and at 3,6, 12, 24 months horizons.

levels,[20] as is common practice in the literature. At short horizons, the variance contributions are small. As argued in Subsection 3.3, these numbers should be taken with caution because of the downward bias. By contrast, the estimate of the FVR at the 24-month horizon is reliable, since we see in the upper part of the table that the numbers are almost identical

[20]This is obtained simply by taking the cumulated sums of the IRFs.

Time span	VD: waves of periodicity 2 − 18 months	 18 − 96 months	 2+ months	FVR: horizon 24 months
1983:1–2008:12	10.4	22.0	16.1	15.5
1990:1–2012:6	6.3	15.5	8.0	8.1
1987:1–2008:12	7.3	15.4	11.3	10.6
1983:1–2012:6	10.0	24.6	12.7	12.8
1979:7–2012:6	17.2	19.3	17.4	17.5
1979:7–2019:6*	15.7	18.2	15.3	15.1

Table 5: Variance decomposition of inflation for different time spans, Specification IV: FFR, CPI inflation, IP growth, EBP. VD: percentage of inflation variance accounted for by the monetary policy shock, for waves of periodicity 2-18 months (short run), 18-96 months (business cycle), 2+ months (overall variance). FVR: percentage of forecast error variance of inflation accounted for by the monetary policy shock at the 2-year horizon. For the sample 1979:7–2019:6 in place of the EBP series we use three financial variables: the 10-year treasury bond rate, the BAA corporate bond yield and the S&P500 stock price index.

to those of the overall VD, reported in the last column for convenience. This means that at horizon 24 all of the IRFs of inflations are already close to zero and the bias has disappeared. We conclude that the use of FVR in place of VD cannot explain the inconsistency of estimates. Coming to the bottom part of the table and focusing on the 2-year horizon, we see that the variance contribution of monetary policy to the forecast error of prices is even larger when considering the price index taken in levels.

Another potential source of differences in the empirical estimates is the policy indicator. Following Plagborg-Møller and Wolf (2022), we consider a model (Specification IV) incorporating the same variables of Specification II but with the federal funds rate (FFR) in place of 1YB. In addition, we set $p = 6$, as in Plagborg-Møller and Wolf (2022) (instead of $p = 12$). We retain $m = 6$ and set $x_t = y_t$ for the preliminary treatment of the instrument. We compute the point estimates of VD and FVR at horizon 24 for different time spans (Table 5). The fist row reports the estimates for the full sample, which are somewhat larger than the ones obtained for Specification II with $p = 12$. We conclude that the number of lags used in VAR estimation and the use of FFR in place of 1YB cannot explain the difference.

In the second row, we report the estimates for the time span 1990:1–2012:6, the same used in Plagborg-Møller and Wolf (2022). The explained variances for this sample are sizeably smaller than the ones of our time span: the overall VD is 8.0% as against 16.1%. This points to the fact that the different time spans explain part of the discrepancy. The remaining difference can only be due to the estimation methods.

The span of the GK instrument, 1990:1–2012:6, by excluding the 80's and including the first years of the zero-lower-bound period, exhibits little variation of both inflation and interest rates, which could be detrimental to the reliability of the estimates. To verify how different spans affect the estimates, we report in the bottom part of the table the results for four additional time spans: 1987:1–2008:12, 1983:1–2012:6, 1979:7–2012:6 (the same of Gertler and Karadi, 2015) and 1979:7–2019:6.[21] Notice that, for these time spans (as well as our benchmark 1983:1–2008:12) the Internal-Instrument method cannot be used, at least with the GK instrument. This is a nice illustration of the advantages of our proposed method. Despite results vary considerably across different samples, the overall picture emerging from Table 5 confirms our main finding: discretionary monetary policy has non-negligible effects on prices.

7 Concluding remarks

In this paper we propose a new estimation procedure for structural VARs with an external instrument. The procedure includes a test for invertibility and a test for recoverability, a method to estimate the relative impulse response functions when the shock is not recoverable and a method to estimate the absolute response functions and the shock itself when the shock is recoverable but not invertible. The procedure reduces to the standard method when the shock is invertible. Results reported in this paper indicate that all procedures work remarkably well under simulation, when the sample size is comparable with those typical of macroeconomic empirical analyses.

An application to monetary policy shocks, using the instrument of Gertler and Karadi, 2015, indicates that the policy shocks are not invertible in a few popular monetary policy VAR specifications. While the standard method produces puzzling results, our procedure delivers results in line with textbook effects. Finally, we find that the policy shock is recoverable, so that we can estimate its variance contributions. Variance decomposition shows that monetary policy has sizeable effects on both real activity and inflation, suggesting that monetary policy can be effective in controlling prices.

References

Angeletos, George-Marios, Fabrice Collard, and Harris Dellas, "Business-Cycle Anatomy," *American Economic Review*, October 2020, *110* (10), 3030–70.

Bernanke, Ben S. and Kenneth N. Kuttner, "What Explains the Stock Market's Reaction to Federal Reserve Policy?," *Journal of Finance*, June 2005, *60* (3), 1221–1257.

[21] For the latter time span in place of the EBP series, which is not available, we use three financial variables: the 10-year treasury bond rate, the BAA corporate bond yield and the S&P500 stock price index.

Blanchard, Olivier J., Jean-Paul L'Huillier, and Guido Lorenzoni, "News, Noise, and Fluctuations: An Empirical Exploration," *American Economic Review*, December 2013, *103* (7), 3045–70.

Chahrour, Ryan and Kyle Jurado, "Recoverability and Expectations-Driven Fluctuations," *The Review of Economic Studies*, 03 2021, *89* (1), 214–239.

Fernandez-Villaverde, Jesus, Juan F. Rubio-Ramirez, Thomas J. Sargent, and Mark W. Watson, "ABCs (and Ds) of Understanding VARs," *American Economic Review*, June 2007, *97* (3), 1021–1026.

Forni, Mario and Luca Gambetti, "Sufficient information in structural VARs," *Journal of Monetary Economics*, 2014, *66* (C), 124–136.

_ , _ , and Luca Sala, "No News in Business Cycles," *Economic Journal*, December 2014, *124* (581), 1168–1191.

_ , _ , and _ , "Structural VARs and noninvertible macroeconomic models," *Journal of Applied Econometrics*, 2019, *34* (2), 221–246.

_ , _ , Marco Lippi, and Luca Sala, "Noisy News in Business Cycles," *American Economic Journal: Macroeconomics*, October 2017, *9* (4), 122–152.

_ , _ , _ , and _ , "Common Component Structural VARs," CEPR Discussion Papers 15529, C.E.P.R. Discussion Papers December 2020.

Gertler, Mark and Peter Karadi, "Monetary Policy Surprises, Credit Costs, and Economic Activity," *American Economic Journal: Macroeconomics*, 2015, *7* (1), 44–76.

Giannone, Domenico and Lucrezia Reichlin, "Does information help recovering structural shocks from past observations?," *Journal of the European Economic Association*, 04-05 2006, *4* (2-3), 455–465.

Gilchrist, Simon and Egon Zakrajšek, "Credit Spreads and Business Cycle Fluctuations," *American Economic Review*, 2012, *102* (4), 1692–1720.

Gürkaynak, Refet S, Brian Sack, and Eric Swanson, "Do Actions Speak Louder Than Words? The Response of Asset Prices to Monetary Policy Actions and Statements," *International Journal of Central Banking*, May 2005, *1* (1).

Jarociński, Marek and Peter Karadi, "Deconstructing Monetary Policy Surprises– The Role of Information Shocks," *American Economic Journal: Macroeconomics*, April 2020, *12* (2), 1–43.

Jordà, Òscar, "Estimation and Inference of Impulse Responses by Local Projections," *American Economic Review*, March 2005, *95* (1), 161–182.

Leeper, Eric M., Todd B. Walker, and Shu-Chun Susan Yang, "Fiscal Foresight and Information Flows," *Econometrica*, May 2013, *81* (3), 1115–1145.

Lippi, Marco and Lucrezia Reichlin, "The Dynamic Effects of Aggregate Demand and Supply Disturbances: Comment," *American Economic Review*, June 1993, *83* (3), 644–652.

_ **and** _ , "VAR analysis, nonfundamental representations, Blaschke matrices," *Journal of Econometrics*, July 1994, *63* (1), 307–325.

McCracken, Michael W. and Serena Ng, "FRED-MD: A Monthly Database for Macroeconomic Research," Working Papers 2015-12, Federal Reserve Bank of St. Louis June 2015.

Mertens, Karel and José Luis Montiel-Olea, "Marginal Tax Rates and Income: New Time Series Evidence," *The Quarterly Journal of Economics*, 2018, *133* (4), 1803–1884.

_ **and Morten O. Ravn**, "Measuring the Impact of Fiscal Policy in the Face of Anticipation: A Structural VAR Approach*," *The Economic Journal*, 2010, *120* (544), 393–413.

_ **and** _ , "The Dynamic Effects of Personal and Corporate Income Tax Changes in the United States," *American Economic Review*, June 2013, *103* (4), 1212–47.

_ **and** _ , "A reconciliation of SVAR and narrative estimates of tax multipliers," *Journal of Monetary Economics*, 2014, *68*, S1–S19. Supplement issue: October 19-20, 2012 Research Conference on âĂIJFinancial Markets, Financial Policy, and Macroeconomic ActivityâĂİ Sponsored by the Study Center Gerzensee and the Swiss National Bank.

Miranda-Agrippino, Silvia and Giovanni Ricco, "The Transmission of Monetary Policy Shocks," *American Economic Journal: Macroeconomics*, July 2021, *13* (3), 74–107.

_ **and** _ , "Identification with External Instruments in Structural VARs," *Journal of Monetary Economics*, forthcoming.

Paul, Pascal, "The Time-Varying Effect of Monetary Policy on Asset Prices," *The Review of Economics and Statistics*, 2020, *102* (4), 690–704.

Plagborg-Møller, Mikkel and Christian K. Wolf, "Local Projections and VARs Estimate the Same Impulse Responses," *Econometrica*, 2021, *89* (2), 955–980.

_ **and** _ , "Instrumental Variable Identification of Dynamic Variance Decompositions," *Journal of Political Economy*, 2022, *130* (8), 2164–2202.

Ramey, Valerie A., "Identifying Government Spending Shocks: It's all in the Timing," *The Quarterly Journal of Economics*, 2011, *126* (1), 1–50.

_ , "Macroeconomic Shocks and Their Propagation," in John B. Taylor and Harald Uhlig, eds., *Handbook of Macroeconomics*, Vol. 2 of *Handbook of Macroeconomics*, Elsevier, 2016, chapter 2, pp. 71 – 162.

Sims, Christopher A. and Tao Zha, "Were There Regime Switches in U.S. Monetary Policy?," *American Economic Review*, March 2006, *96* (1), 54–81.

Stock, James H., "What's New in Econometrics: Time Series, Lecture 7," NBER Summer Institute, Short course lectures 2008.

_ **and Mark W. Watson**, "Disentangling the Channels of the 2007-09 Recession," *Brookings Papers on Economic Activity*, 2012, *44* (1 (Spring), 81–156.

_ **and** _ , "Identification and Estimation of Dynamic Causal Effects in Macroeconomics Using External Instruments," *The Economic Journal*, 2018, *128* (610), 917–948.

Online Appendix:
External Instrument SVAR Analysis for Noninvertible Shocks

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Abstract

This Online Appendix provides proofs, and additional materials for the paper 'External Instrument SVAR Analysis for Noninvertible Shocks'. Section A contains all the proofs to the results reported in the main text. Section B and C report additional examples and simulations, respectively. Section D contains robustness exercises for the empirical section.

JEL classification: C32, E32.

Keywords: Proxy-SVAR, SVAR-IV, Impulse response functions, Variance Decomposition, Historical Decomposition, Monetary Policy Shock.

A Proofs

A.1 Proof of Proposition 1

Let $\mathcal{H}_t^+$ be the 'half-space' spanned by present and future ε's, i.e. $\mathcal{H}_t^+ = \overline{\text{span}}(\varepsilon_{j,t-k}, j = 1, \ldots, n, k \leq 0)$, and $\mathcal{H}_t^-$ be the half-space spanned by present and past ε's, i.e. $\mathcal{H}_t^- = \overline{\text{span}}(\varepsilon_{j,t-k}, j = 1, \ldots, n, k \geq 0)$. Then $\mathcal{H}$ is the direct sum of the orthogonal subspaces $\mathcal{H}_t^+$ and $\mathcal{H}_{t-1}^-$, so that $P(u_t|\mathcal{H}) = P(u_t|\mathcal{H}_t^+) + P(u_t|\mathcal{H}_{t-1}^-)$. But the latter projection is zero, since u_t is white noise, and therefore is orthogonal to $\varepsilon_{t-k} = Q(L)u_{t-k}$ for any $k > 0$. This establishes (i). To prove (ii) we use the cross-covariance generating function of ε_t and u_t, say $G_{\varepsilon u}(z)$, z being a complex variable.[1] From (5) we get $G_{\varepsilon u}(z) = Q(z)$. Now let us consider the opposite covariance, i.e. the cross-covariance generating function of u_t and ε_t: from (6) we get $G_{u\varepsilon}(z) = D'(z^{-1})\Sigma_\varepsilon$. But $G_{\varepsilon u}(z) = G'_{u\varepsilon}(z^{-1}) = \Sigma_\varepsilon D(z)$. The first Equation in (7) follows. As for the second, we have just to show that Σ_ε is invertible. Now, the rank of the spectral density matrix of y_t, which is equal to $B(e^{-j\theta})B'(e^{j\theta})$, j being the imaginary unit, is equal to n a.e. in $[-\pi, \pi)$, since $B(L)$ has maximum rank n by Assumption 1 (ii). But $B(e^{-j\theta})B'(e^{j\theta}) = C(e^{-j\theta})\Sigma_\varepsilon C'(e^{j\theta})$, so that Σ_ε has rank n. Finally, (iii) is obtained by pre-multiplying Equation (5) by $C(L) = A(L)^{-1}$ and using (ii).

A.2 Proof of Proposition 2

We see from (4) and (3) that $\varepsilon_{t-k} \in \mathcal{H}^y$ for any k and $y_{t-k} \in \mathcal{H}$ for any k. Hence $\mathcal{H}^y = \mathcal{H}$, so that u_{it} is recoverable if and only if it belongs to $\mathcal{H}$. It follows that, if u_{it} is recoverable, in Equation (6) $s_{it} = 0$ and $u_{it} = d_i'(F)\varepsilon_t$. This proves (9). Finally, by taking the autocovariance generating function of all members of (9) we get $1 = d_i'(z^{-1})\Sigma_\varepsilon d_i(z) = q_i'(z^{-1})\Sigma_\varepsilon^{-1} q_i(z)$.

A.3 Proof of Proposition 3

We see from (3) and (4) that ε_{t-k}, $k \geq 0$, span the same linear space as y_{t-k}, $k \geq 0$. Hence if u_{it} is fundamental it belongs to $\mathcal{H}_t^-$. On the other hand, if u_{it} is fundamental, it is also recoverable, so that, by Proposition 2, we have $u_{it} = d_i(F)\varepsilon_t$. By projecting both sides of this equation onto $\mathcal{H}_t^-$ we get $u_{it} = P(u_{it}|\mathcal{H}_t^-) = d_{i0}\varepsilon_t$, since $d_{ik}\varepsilon_{t+k}$ is orthogonal to $\mathcal{H}_t^-$ for $k > 0$.

[1] Let a_t and e_t be two zero-mean weakly stationary variables. Let $\Gamma_k = \mathrm{E}(a_t b'_{t-k})$. The cross-covariance genearating functions is defined as the z-transform of the cross-covariance function, i.e. $G_{ae}(z) = \sum_{k=-\infty}^{\infty} \Gamma_k z^k$. A well-known result in time series theory is the following. Let v_t be a vector white noise with covariance matrix Σ_v. Let $a_t = M(L)v_t + p_t$, where p_t is orthogonal to v_t at all leads and lags and $M(L) = \sum_{-\infty}^{\infty} M_k L^k$. Finally, let $e_t = N(L)v_t + s_t$, where p_t is orthogonal to v_t and p_t at all leads and lags and $N(L) = \sum_{-\infty}^{\infty} N_k L^k$. Then the cross-covariance generating function of a_t and e_t is given by $G_{ae}(z) = M(z)\Sigma_v N'(z^{-1})$.

A.4 Proof of Proposition 4

Let us reorder u_t in the structural representation in Equation (1) in such a way that $u_t = (u_t^{f\prime} \;\; u_t^{r\prime} \;\; u_t^{n\prime})'$. Correspondingly, we partition $B(L)$ as $B(L) = (B^f(L) \;\; B^r(L) \;\; B^n(L))$, $Q(L)$ as $Q(L) = (Q^f \;\; Q^r(L) \;\; Q^n(L))$, and $D(L)$ as $D(L) = (D^f \;\; D^r(L) \;\; D^n(L))$. The equations in (12) are obtained from (1), (5), (8) and (11). Properties (i) and (ii) follow from Propositions 2 and 3.

A.5 Proof of Proposition 5

Notice first that the residual of (15) e_t is orthogonal to z_{t-k} for $k \geq 0$ by construction; moreover, it is orthogonal to z_{t-k} for $k < 0$ since the future of z_t is orthogonal to both ε_{t-k} and z_{t-k}, $k \geq 0$. Hence e_t is orthogonal to z_t at all leads and lags. Therefore, by taking the cross-covariance generating functions of ε_t and z_t we get $G_{\varepsilon z}(L) = \psi(L)\sigma_z^2$. Similarly, from Equation (5) we get $G_{\varepsilon u_i}(L) = q_i(L)$, $q_i(L)$ being the i-th column of $Q(L)$. But it is easily seen from (14) that $G_{\varepsilon z}(L) = \alpha G_{\varepsilon u_i}(L) = \alpha q_i(L)$. Equation (16) follows. Equation (17) is an immediate consequence of (8).

A.6 Proof of Proposition 6

The first inequality is easily obtained from (14). As for the second, from (16) we get $(\sigma_z^2)^2 \psi'(e^{j\theta})\Sigma_\varepsilon^{-1}\psi(e^{-j\theta}) = R_r^2(\theta)\alpha^2$, where $R_r^2(\theta)$ is the measure of recoverability defined in Remark 2. Inequality (20) follows from $R_r^2(\theta) \leq 1$.

A.7 Proof of Proposition 7

From (16) and Remark 2 we get $\alpha^2 \sum_{k=0}^{\infty} q_{ik}'\Sigma_\varepsilon^{-1}q_{ik} = \alpha^2 R_r^2 = (\sigma_z^2)^2 \sum_{k=0}^{\infty} \psi_k'\Sigma_\varepsilon^{-1}\psi_k$. If u_{it} is recoverable, we have $R_r^2 = 1$, so that

$$\alpha = \sigma_z^2 \sqrt{\sum_{k=0}^{\infty} \psi_k'\Sigma_\varepsilon^{-1}\psi_k}. \tag{1}$$

The statement follows from Proposition 5.

A.8 Proof of Proposition 8

We have $P(z_t|\mathcal{H}_t^+) = \alpha P(u_{it}|\mathcal{H}_t^+) + P(w_t|\mathcal{H}_t^+)$. But the latter projection is zero, since w_t is orthogonal to $u_{j,t-k}$ $j = 1, \ldots, q$, for any integer k, so that it is orthogonal to $\mathcal{H}$. Hence from (6) we get

$$z_t = \delta'(F)\varepsilon_t + v_t = \alpha d_i'(F)\varepsilon_t + (\alpha s_{it} + w_t).$$

where $\alpha d_i'(F)\varepsilon_t$ is the projection and $v_t = \alpha s_{it} + w_t$ is the residual. If u_{it} is recoverable, $s_{it} = 0$, so that $v_t = w_t$ and $\delta'(F)\varepsilon_t = \alpha u_{it}$.

A.9 Proof of Proposition 9

Let us first notice that, if u_{it} is fundamental, ε_t is orthogonal to z_{t-k} for $k > 0$, since z_{t-k} is a contemporaneous linear combination of ε_{t-k} (see Proposition 3) and ε_t is a vector white noise. It follows that in Equation (15) the coefficients of lagged z_t's are all zero and $\psi(L) = \psi_0 = \psi$. Result (i) then follows from Proposition 7. For the same reason, in Equation (24) the coefficients of leaded ε_t's are all zero and $\delta(F) = \delta_0 = \delta$. Result (ii) then follows from Proposition 8.

B Examples

B.1 A square system

Consider the structural model

$$\begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = B(L)u_t = \begin{pmatrix} L & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix}.$$

with $u_t \sim \mathcal{WN}(0, I_2)$. Here the system is square, i.e. $n = q = 2$. The model is noninvertible (toward the past) since $\det B(z) = 0$ for $z = 0$. Noninvertibility arises from the fact that u_{1t} has a delayed effect on y_{1t}. However, by inverting the matrix on the right-hand side toward the future we get

$$\begin{pmatrix} L^{-1} & 0 \\ -L^{-1} & 1 \end{pmatrix} \begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = \begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix}.$$

Hence both shocks are recoverable with respect to y_t (see Remark 2). By projecting y_t onto its past values one can find the VAR(1) representation[2]

$$A(L) \begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = \begin{pmatrix} 1 & -0.5L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}.$$

By inverting the VAR we get the MA(1) Wold representation

$$\begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = C(L)\varepsilon_t = \begin{pmatrix} 1 & 0.5L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix},$$

where $\Sigma_\varepsilon = \begin{pmatrix} 0.5 & 0 \\ 0 & 2 \end{pmatrix}$. Hence

$$\begin{aligned} u_{1t} &= y_{1,t+1} = \varepsilon_{1,t+1} + 0.5\varepsilon_{2t} \\ u_{2t} &= -y_{1,t+1} + y_{2t} = -\varepsilon_{1,t+1} + 0.5\varepsilon_{2t} \end{aligned} \tag{2}$$

so that both u_{1t} and u_{2t} are linear combinations of the present and future values of ε_{1t} and ε_{2t} as stated in Proposition 2.

[2] The projection of y_{2t} on y_{t-1} is zero, so $\varepsilon_{2t} = y_{2t}$. The projection of y_{1t} onto y_{t-1} is $\frac{Var(u_{1,t-1})}{Var(y_{2,t-1})} y_{2,t-1} = 0.5 y_{2,t-1}$.

Now let us assume that

$$z_t = u_{1t} + w_t,$$

where w_t is orthogonal to u_t at all leads and lags and $\sigma_w^2 = 1$. By projecting the proxy onto the present and the future of ε_t we get $\delta(L^{-1}) = (L^{-1} \;\; 0.5)'$. The implied standard deviation α is 1. Hence $\delta(L) = (L \;\; 0.5)'$. According to Proposition 8 we get $u_{1t} = \delta'(L^{-1})\varepsilon_t = (L^{-1} \;\; 0.5)\varepsilon_t$, which corresponds to Equation (2).

Turning to the response functions, from (2) and the definition of z_t we get

$$\begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix} = \begin{pmatrix} 0.5L & -0.5L \\ 1 & 1 \end{pmatrix} \begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix} = \begin{pmatrix} 0.25L \\ 0.5 \end{pmatrix} z_t + \begin{pmatrix} 0.25z_{t-1} - 0.5w_{t-1} - 0.5u_{2t} \\ 0.5z_t - w_t + u_{2t} \end{pmatrix}. \tag{3}$$

It can be easily verified that the second term on the right side is orthogonal to z_t and its past history, so that (3) is the projection equation of ε_t onto z_t and its lags. Hence $\psi(L) = (0.25L \;\; 0.5)'$. Applying formula (17) we get the first column of the matrix $B(L)$:

$$b_1(L)\alpha = b_1(L) = C(L)\psi(L)\sigma_z^2 = \begin{pmatrix} 1 & 0.5L \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0.25L \\ 0.5 \end{pmatrix} 2 = \begin{pmatrix} L \\ 1 \end{pmatrix},$$

which provides the response functions of interest.

B.2 Example 2: A 'short' system

Now let us modify the example above by adding a third structural shock affecting only y_{2t}:

$$\begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = B(L)u_t = \begin{pmatrix} L & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} u_{1t} \\ u_{2t} \\ u_{3t} \end{pmatrix}.$$

with $u_t \sim \mathcal{WN}(0, I_3)$. Now the system is 'short', since $n = 2$ and $q = 3$. By projecting y_t onto its past values one can find the Wold representation, which is now[3]

$$\begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = C(L)\varepsilon_t = \begin{pmatrix} 1 & L/3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix}$$

[3] The projection of y_{2t} on y_{t-1} is zero, so $\varepsilon_{2t} = y_{2t}$. The projection of y_{1t} onto y_{t-1} is $\frac{Var(u_{1,t-1})}{Var(y_{2,t-1})} y_{2t-1} = y_{2,t-1}/3 = \varepsilon_{2,t-1}/3$ and the innovation is $\varepsilon_{1t} = y_{1t} - \varepsilon_{2,t-1}/3$.

where $\Sigma_\varepsilon = \begin{pmatrix} 2/3 & 0 \\ 0 & 3 \end{pmatrix}$. Since $q > n$, we cannot have global recoverability (see Remark 1). However, u_{1t} is still recoverable (whereas u_{2t} and u_{3t} are not), since

$$u_{1t} = y_{1,t+1} = \varepsilon_{1,t+1} + \varepsilon_{2t}/3. \tag{4}$$

Assuming again $z_t = u_{1t} + w_t$, when projecting z_t onto the present and future of ε_t we find $\delta(L^{-1}) = (L^{-1} \;\; 1/3)'$ and $\alpha = 1$. Hence $\delta(L) = (L \;\; 1/3)'$. According to (25) we get $u_{1t} = \delta'(L^{-1})\varepsilon_t = (L^{-1} \;\; 1/3)\varepsilon_t$, which corresponds to Equation (4). The response functions can be found following the lines of Example 1.

Now let us assume that the shock of interest is u_{2t}, which is not recoverable, and we have the proxy $Z_t = u_{2,t} + v_t$, where v_t is a unit-variance white noise process orthogonal to u_t at all leads and lags. Notice that $\sigma_Z^2 = 2$. Combining the Wold representation and the structural representation we find

$$\varepsilon_t = C(L)^{-1}B(L)u_t = \begin{pmatrix} 2L/3 & -L/3 & -L/3 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} u_{1t} \\ u_{2t} \\ u_{3t} \end{pmatrix}.$$

Hence the projection of ε_t onto the present and past of u_{2t} is $(-L/3 \;\; 1)'u_{2t}$. By replacing the regressor u_{2t} with Z_t the variance of the regressor is doubled whereas the covariances are the same, so that we have an attenuation bias equal to one half and the projection is $\psi(L)Z_t = (-L/6 \;\; 1/2)'Z_t$. Applying formula (17) we get

$$b_2(L)\alpha = C(L)\psi(L)\sigma_z^2 = \begin{pmatrix} 1 & L/3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -L/6 \\ 1/2 \end{pmatrix} 2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We can then normalize by dividing by the impact effect of the shock on the second variable, which is 1, to get the correct relative IRFs.

B.3 Example 3: FVR and FVD

The following example shows that, as stated in Subsection 3.3, the sum of the FVR over all shocks can be much smaller than 100% (indeed it can be any number between 0% and 100%), and, for the shock of interest, the FVR can be any number between 0% and the FVD.

Let us consider the following square structural model:

$$\begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = B(L)u_t = \begin{pmatrix} 1 & 0 \\ -1 & 1+3L \end{pmatrix} \begin{pmatrix} u_{1t} \\ u_{2t} \end{pmatrix},$$

with $u_t \sim \mathcal{W}N(0, I_2)$. According to the structural model, for the information set given by the past of u_t, the one step ahead forecast error of y_{2t} is $-u_{1t} + u_{2t}$, whose variance is 2. The contribution of both shocks to the forecast error variance is FVD(0)= 50%.

Since the model is square, all shocks are recoverable (see Remark 2). The shock u_{1t} is invertible, since it can be written as a linear combination of the present and past of the variables, being equal to y_{1t}. However, the model is not globally invertible, since the determinant $1 + 3L$ vanishes for $L = 1/3$, which is within the unit disk in the complex plane. The noninvertible shock is u_{2t}. The fundamental Cholesky representation is

$$\begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = B(L)u_t = \begin{pmatrix} 1 & 0 \\ -1 & 1+L/3 \end{pmatrix} \begin{pmatrix} \varepsilon_{1t} \\ \varepsilon_{2t} \end{pmatrix},$$

where

$$\varepsilon_{1t} = u_{1t} \qquad \varepsilon_{2t} = \frac{1+3L}{1+L/3} u_{2t} \qquad \Sigma_\varepsilon = \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}.$$

Hence, with respect to the information set given by the past of ε_t, the one step ahead forecast error is $-\varepsilon_{1t} + \varepsilon_{2t}$, whose variance is 10. The contribution to the forecast error variance, according to the ratio FVR, is FVR(0)= 10% for both shocks, since the numerator is 1 for both shocks, whereas the denominator is 10. The sum of the two contributions is just 20%, even though there are no other shocks in the system.

It is easily seen that, by replacing $1 + 3L$ with $1 + \alpha L$ in the structural MA matrix and choosing a suitable value for α we can get arbitrarily low values for FVR(0), which therefore can be any number between 0% and 50%, the FVD. Indeed, the only one useful information provided by the FVR about the relative importance of the shock of interest is given by the fact that it is a lower bound for the FVD. Hence an high FVR is informative, but a low FVR is not.

C Additional simulations

C.1 Simulation 3: a nonrecoverable shock

Our third Monte Carlo exercise adopts the LWY model of Simulation 1 (see the main text), modified by adding a demand shock that affects taxes. Precisely, the data generating process is now

$$\begin{pmatrix} \tau_t \\ k_t \end{pmatrix} = \begin{pmatrix} L^2 & 0 & \frac{1}{1-\gamma L} \\ \frac{-\kappa(L+\theta)}{1-\alpha L} & \frac{1}{1-\alpha L} & 0 \end{pmatrix} \begin{pmatrix} u_{\tau,t} \\ u_{a,t} \\ u_{d,t} \end{pmatrix} = B(L)u_t.$$

We arbitrarily set $\gamma = 0.7$. Now the system is no longer square an the tax shock is no longer recoverable. As usual, we generate 1000 different dataset with 240 time observations. As in Simulations 1 and 2, the proxy is generated according to the equation

$$\tilde{z}_t = u_{\tau,t} + 0.5z_{t-1} + 0.4k_{t-1} - 0.6\tau_{t-1} + v_t,$$

where $v_t \sim iid\mathcal{N}(0,1)$. For all estimates we set $p = m = 2$ and $r = 4$.

First, we test for recoverability and fundamentalness. Invertibility is correctly rejected in all cases; recoverability is correctly rejected in 85.5% of the cases. Then we estimate the relative IRFs of the tax shock. Since $b_{i1}(0) = 0$, the normalisation in Equation (19) cannot be used, so that we normalise by dividing by $\hat{\gamma}_{1,2}$ i.e. by imposing that the effect on taxes at lag 2 is equal to 1. In addition, we estimate upper and lower bounds for the absolute IRFs, according to equations (22) and (21). Finally, we compute the lower and upper bounds for the variance decomposition according to equations (33) and (34).

In the left panels of Figure 1, one can observe that the estimates of the relative IRFs are consistently very close to the true relative IRFs.[4] In the middle panes are reported the estimated upper and lower bounds for the IRFs. The black solid lines are the average upper bounds; the blue solid lines are the average lower bounds; as before, the red dashed lines are the true IRFs and the shaded areas are bounded by the 16th and the 84th percentile of the IRFs distribution. Similarly, in the right panels are reported the upper and lower bounds for the explained variance, decomposed by frequency, expressed in percentage terms (equations (33)-(34)). The basic message is that upper and lower bounds provide some valuable information about the importance of the shock for the two variables.

[4]Here we do not report results for the shock, since the shock cannot be estimated consistently.

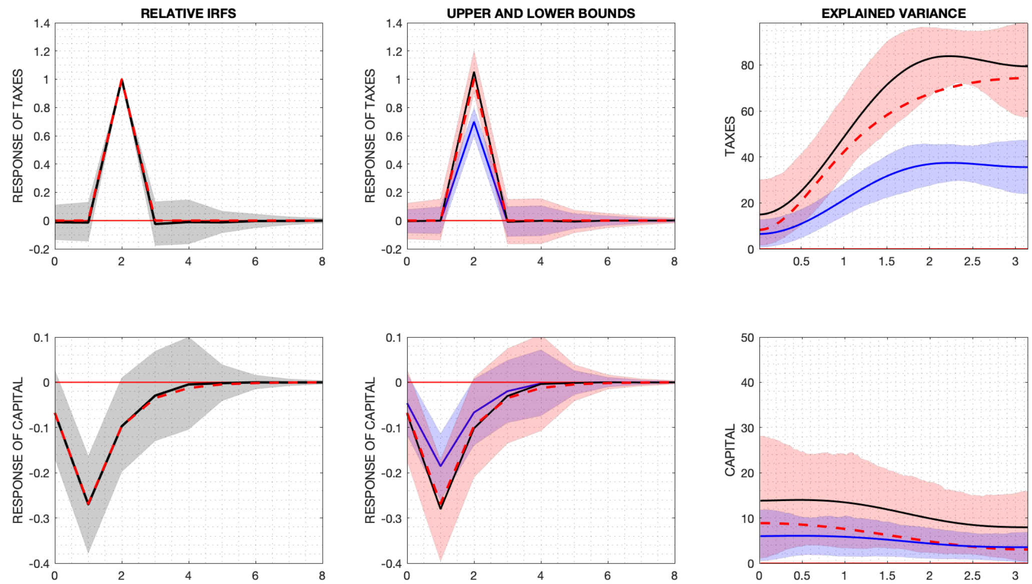

FIGURE 1: Simulation 3: Modified fiscal foresight model (nonrecoverable shock). Left panels: Estimated relative IRFs. Middle panels: estimated upper and lower bounds for the absolute IRFs. Right panels: estimated upper and lower bounds for the variance contribution according to equations (33)-(33), expressed in percentage terms. Red dashed lines: true response functions and variance decomposition; black solid lines, average of the estimated upper bounds; blue solid lines: average of the estimated lower bounds; shaded areas: 16th to 84th percentiles.

C.2 Simulation 4: fundamentalness test

The goal of our fourth experiment is to evaluate the small-sample performance of the fundamentalness test. In this simulation the DGP for y_t is the ARMA(1,1) model

$$(I - AL)\begin{pmatrix} y_{1t} \\ y_{2t} \\ y_{3t} \end{pmatrix} = (I - BL)\begin{pmatrix} u_{1t} \\ u_{2t} \\ u_{3t} \end{pmatrix},$$

where A and B are generated as follows. First, we produce a 3×3 matrix with random entries, uniformly distributed in the interval $(0 \quad 1)$. Next we divide all entries by the maximum eigenvalue. Finally, to generate A we multiply by 0.5, whereas to generate B we multiply by β, with $\beta = 0.8, 1.1, 1.2, 1.3, 1.4, 1.5$. Hence the maximum eigenvalue of A is 0.5, whereas the maximum eigenvalue of B is β, implying that $I - AL$ is invertible, whereas $I - BL$ is invertible only when $\beta = 0.8$; for the other values of β it is not, and the larger is β the larger is the departure from invertibility. The shocks are again $u_t \sim iid\mathcal{N}(0, I_3)$. The proxy is generated as in Simulations 1, 2 and 3.

Again, we generate 1000 dataset with $T = 240$ for each value of β, and estimate the shock u_{1t} as described in the main text, using $p = m = 6$ and $r = 1, 2, 3, 4, 5$. Then we test for fundamentalness as explained in the main text.

	Fundamentalness parameter					
	$\beta = 0.8$	$\beta = 1.1$	$\beta = 1.2$	$\beta = 1.3$	$\beta = 1.4$	$\beta = 1.5$
$r = 1$	7.1	36.9	68.3	87.2	95.9	98.3
$r = 2$	6.5	48.9	79.2	94.4	98.9	99.5
$r = 3$	7.1	52.4	82.8	96.1	99.5	99.7
$r = 4$	8.1	53.0	83.8	96.0	98.8	99.7
$r = 5$	7.9	53.5	84.6	95.4	98.7	99.7

TABLE 1: Results of Simulation 3. Percentage of rejection of the fundamentalness test over 1000 replications for different values of the parameter governing invertibility.

Table 1 shows the percentage of rejections obtained at the 5% level with each r, β configuration. In the first column we have $\beta = 0.8$, so that we have invertibility. Hence the percentage of rejections shows the empirical size of the test, which is reasonably close to the theoretical value (5%), even if there is some evidence that the test is oversized for this DGP. In the other columns we do not have invertibility; hence the percentage of rejections shows the power of the test. With $\beta = 1.1$ the model is close to invertibility, so that the power is relatively low; but with $\beta = 1.2$ the test exhibits a reasonably good power, which becomes excellent for $\beta = 1.3, 1.4, 1.5$.

C.3 Simulation 5: recoverability test

In this Monte Carlo exercise, we evaluate the performance of the recoverability test. The DGP is given by the 'short' model

$$\begin{pmatrix} y_{1t} \\ y_{2t} \end{pmatrix} = B(L)u_t = \begin{pmatrix} \frac{\sum_{j=0}^{4} \psi_j L^j}{1 - \psi_5 L} & 0 & \beta \\ \frac{\psi_6 L}{1 - \psi_7 L} & \frac{1 + \psi_8 L}{1 - \psi_7 L} & \beta \end{pmatrix} \begin{pmatrix} u_{1t} \\ u_{2t} \\ u_{3t} \end{pmatrix}.$$

The shocks are $u_t \sim iid\mathcal{N}(0, I_2)$. We set $\psi_0 = 0.1$, $\psi_1 = 0.3$, $\psi_2 = 0.5$, $\psi_3 = 1$, $\psi_4 = -0.3$, $\psi_5 = 0.7$, $\psi_6 = 1$, $\psi_7 = 0.8$, $\psi_8 = 0.5$. These parameters are set arbitrarily. Moreover, we set $\beta = \sqrt{(\sigma_1^2 + \sigma_2^2)\theta/(1 - \theta)}$, where σ_1^2 and σ_2^2 are the variances of the components of y_{1t} and y_{2t}, respectively, driven by u_{1t} and u_{2t}. In this way, the parameter θ is the fraction of the total variance of the variables explained by the shock u_{3t}, so that the larger is θ the larger is the deviation of the model from recoverability. We set $\theta = 0, 0.2, 0.4, 0.6, 0.8$ and test for serial correlation of $\hat{u}_{1t}$ using the Liung-Box Q-test with maximum lag 6, 12, 18, and 24.

	Recoverability parameter				
	$\theta = 0$	$\theta = 0.2$	$\theta = 0.4$	$\theta = 0.6$	$\theta = 0.8$
max lag = 6	2.9	68.1	77.2	86.9	93.6
max lag = 12	2.9	51.6	65.5	80.7	88.3
max lag = 18	3.7	47.9	59.7	73.7	83.1
max lag = 24	4.0	45.5	56.6	69.4	80.7

TABLE 2: Results of Simulation 4. Percentage of rejection of the recoverability test over 1000 replications for different values of the parameter governing recoverability.

Table 2 shows the percentage of rejections over 1000 experiments. In the first column we have $\theta = \beta = 0$, so that the model is square and we have recoverability (see Remark 1). Hence the percentage of rejections shows the empirical size of the test, which is somewhat undersized for this DGP (the theoretical size is 5%). In the other columns we do not have recoverability; hence the percentage of rejections shows the power of the test. Of course, with small values of θ the power is relatively low; but with $\theta \geq 0.4$ we have a fairly good power of the test, especially when the maximum lag is set to 6.

C.4 Simulation 6: additional comparison with the Internal-Instument method

In this exercise we compare the small-sample performance of our proposed procedure in estimating the relative IRFs with the one of the Internal-Instrument SVAR approach, using the same DGP as in Simulation 5, with $\theta = \beta = 0$. The instrument is generated as in the previous simulations. We focus on the relative IRFs of the first shock. We normalise the IRFs by dividing by the effect on the first variable at lag 3, i.e. we estimate

$$\mu(L) = \frac{\gamma(L)}{\gamma_{1,3}} = \begin{pmatrix} \dfrac{\sum_{j=0}^{4} \psi_j L^j}{\psi_3(1 - \psi_5 L)} \\ \dfrac{\psi_6 L}{\psi_3(1 - \psi_7 L)} \end{pmatrix}.$$

For our procedure, the instrument is preliminarily "cleaned" by setting $x_t = y_t$ and the number of lags m according to the BIC. For the Internal-Instrument method, we estimate a VAR for the vector $(\tilde{z}_t \;\; y_t')'$; the IRFs are identified as the ones corresponding to the first shock of a Cholesky scheme. As in the previous exercises, we generate 1000 different dataset with 240 time observations. The estimation errors are measured according to Equation (37) in the main text.

VAR order	Internal IV	External IV					
		$r = BIC$	$r = 3$	$r = 4$	$r = 5$	$r = 6$	$r = 7$
$p = 1$	250.8	11.6	21.3	11.4	11.5	11.7	12.6
$p = 2$	55.7	**9.8**	18.3	**9.5**	**10.2**	**10.9**	**12.3**
$p = 3$	41.8	11.6	15.5	11.4	11.7	12.1	13.1
$p = 4$	**12.1**	11.5	**14.5**	11.3	11.7	12.2	13.0
$p = 5$	13.5	12.2	14.7	12.0	12.5	12.8	13.6
$p = 6$	14.3	12.6	15.1	12.3	12.7	13.2	13.9
$p = BIC$	12.5	11.8					

TABLE 3: Results of Simulation 6. Estimation errors for the relative IRFs, measured acording to (37), for the Internal-Instrument SVAR (first column) and our proposed procedure, with r determined by the BIC (second column) and $r = 3, 4, 5, 6, 7$ (columns 3-7). Boldface numbers are the best results obtained for each column.

Results are reported in Table 3. The first column report results for the Internal-Instrument method. The second column reports results obtained with our procedure, with r selected according to the BIC, applied to Equation (17). Columns 3-7 report results obtained by setting arbitrarily $r = 3, 4, 5, 6, 7$. Boldface numbers are the best estimates of each column.

Let us focus first on columns 3-7. Clearly, $r = 4$ is the best choice for this DGP. Column 2 shows that the BIC criterion is fairly successful in detecting the correct value of r, since the errors are close to those of the best column.

The results of Simulation 2 in the main text are confirmed. The Generalised External-Instrument SVAR performs better than the Internal-Instrument SVAR for all values of p. Comparing the best results of columns 1 and 2 the relative improvement is about 20%. When using the BIC (last column) the improvement is small; still, the External-Instrument method has the best performance.

D Empirics: robustness checks

D.1 Changing p, r and the instrument

In this subsection, we explore the robustness of our baseline VAR to changes of the number of lags p, the number of leads r of the VAR residuals used in the regression of the proxy and to the use of a different instrument.

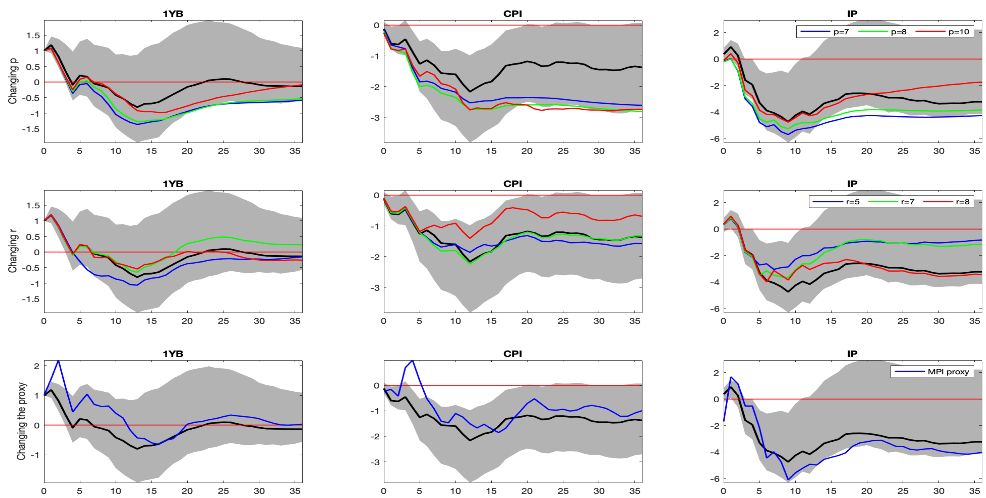

FIGURE 2: Top panels: changing p. Black line: $p = 12$; blue line: $p = 7$; green line: $p = 8$; red line: $p = 10$. Middle panels: changing r. Black line: $r = 6$; blue line: $r = 5$; green line: $r = 7$; red line: $r = 8$. Bottom panels: changing the instrument. Black line: GK instrument; blue line MPI instrument. Grey areas: 68% confidence bands for the baseline specification.

Figure 2, top panels, reports results for $p = 7$ (blue lines), $p = 8$ (red lines) and $p = 10$ (green lines), whereas $r = 6$ is kept fixed. The baseline IRFs $p = 12$ are the black solid lines; the shaded grey area is the 68% confidence region of the baseline. In the middle panels we show results obtained with $p = 12$ and different values of r, i.e. the benchmark specification ($r = 6$, black lines), $r = 5$ (blue lines), $r = 7$ (green lines), $r = 8$ (red lines). In the bottom panel we report the results obtained by using the MPI instrument (blue lines) in place of the GK instrument (black lines).

Our conclusion is that results are remarkably robust to these changes.

D.2 Using levels and changing the VAR specification and the time span

In this subsection, we explore the robustness of our baseline VAR to changes of the treatment of the variables, the VAR specification and the time span. The parameters are kept fixed to $p = 12$ and $r = 6$; the instrument is GK.

Results are reported in Figure 3. The top panels refer to the results obtained with CPI and IP in log-levels (blue lines). The effects on prices is similar to that of the benchmark, but the effect on IP is much smaller. The middle panels show the results obtained with different time spans, namely 1987:8-2008:12 (blue line); 1979:7-2008:12 (green line); 1979:7-2019:6 (red line). For the last period the effects on prices are sizeably smaller than the benchmark (black lines). Finally, the bottom panels show what happens when changing the VAR specification by adding the financial variables. The blue lines correspond to Specification II and the red lines to Specification III. Results are qualitatively similar to those of the benchmark.

Our overall conclusion is that results are reasonably robust to these important changes.

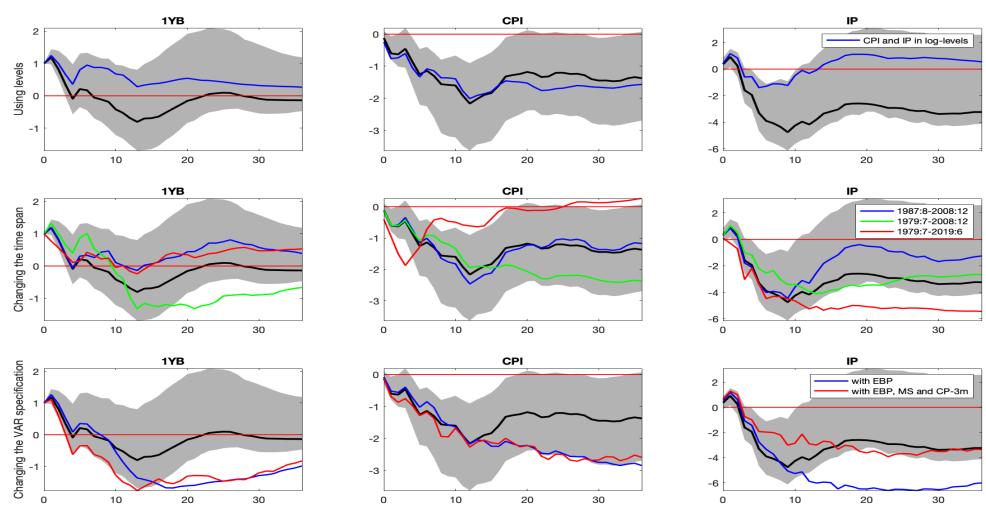

FIGURE 3: Top panels: changing the treatment of the variables. Black line: benchmark; blue line: CPI and IP are taken in log-levels. Middle panels: changing the time span. Black line: benchmark; blue line: 1987:8-2008:12; green line: 1979:7-2008:12; red line: 1979:7-2019:6. Bottom panels: changing the VAR specification. Black line: benchmark (Specification I); blue line: Specification II; red line: Specification III. Grey areas: 68% confidence bands for the baseline specification.