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Centre for Economic Policy Research
33 Great Sutton Street, London EC1V 0DX, UK
Tel: +44 (0)20 7183 8801
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Abstract

We develop a theory of labor markets in a monetary economy with four realistic features: search frictions, worker productivity shocks, wage rigidity, and two-sided lack of commitment. Due to the non-Coasean nature of labor contracts, inefficient job separations occur in the form of endogenous quits and layoffs that are unilaterally initiated whenever a worker's wage-to-productivity ratio moves outside an inaction region. We derive sufficient statistics for the aggregate labor market response to a monetary shock based on the distribution of workers' wage-to-productivity ratios. These statistics crucially depend on the incidence of inefficient job separations, which we show how to identify using readily available microdata on wage changes and worker flows between jobs.

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Andres Blanco - jablanco@umich.edu
Department of Economics, University of Michigan

Andres Drenik - andres.drenik@austin.utexas.edu
University of Texas at Austin

Christian Moser - c.moser@columbia.edu
Columbia University and CEPR

Emilio Zaratiegui - ez2292@columbia.edu
Columbia University

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1 Introduction

The classical idea that inflation "greases the wheels of the labor market" (Keynes, 1936; Tobin, 1972; Card and Hyslop, 1997) forms the bedrock of many Keynesian theories: After the onset of a recession, nominal wage rigidities cause real wages to be inefficiently high, which creates a role for inflation in reducing real wages and restoring the efficient level of employment. However, models in the Keynesian tradition are usually silent on the micromechanics of the labor market, including which jobs are saved, destroyed, or created through inflation. Conversely, models in the search-theoretic tradition (Diamond, 1982; Pissarides, 1985; Mortensen and Pissarides, 1994) yield rich predictions for the distribution of wages and employment in the labor market, but usually do not distinguish between nominal versus real wages due to the simplifying assumption that contracts can be continuously and costlessly renegotiated.

Our objective is to bridge the Keynesian and search-theoretic traditions by explicitly modeling the equilibrium distribution of wages and employment in a frictional labor market subject to wage rigidity. To this end, we study a monetary economy with four realistic features. First, job search is frictional. Second, workers are subject to idiosyncratic productivity shocks. Third, wages are rigid within a match. Fourth, neither workers nor firms can credibly promise to remain in a match. Together, these four features—i.e., search frictions, productivity shocks, wage rigidity, and two-sided lack of commitment—imply that labor contracts are non-Coasean, in the sense that not all gains from trade are exploited. As a result, inefficient job separations occur in the form of endogenous quits and layoffs that are unilaterally initiated whenever a worker's wage-to-productivity ratio moves outside an inaction region. In this setting, wages are allocative in the sense that they matter over and above match surplus for job creation and destruction. We derive sufficient statistics for the aggregate labor market response to a monetary shock based on the distribution of workers' wage-to-productivity ratios. These statistics crucially depend on the prevalence of inefficient job separations, which we show how to identify using readily available microdata on wage changes and worker flows between jobs.

We first consider a nonmonetary economy set in continuous time. The labor market is populated by a unit mass of heterogeneous workers with risk-neutral preferences and an endogenous mass of identical firms. A worker's income depends on their employment state and idiosyncratic productivity, which follows a Brownian motion with drift. New matches are created as a result of unemployed workers and firms with vacancies who direct their search across labor submarkets segmented by the wage rate and worker productivity (Moen, 1997). Importantly, matches are subject to two contractual frictions. First, wages are determined at the time of match formation and rigid thereafter.[1] Second, neither workers nor

[1] Wage rigidity may capture prohibitive renegotiation costs, fairness concerns, or labor market institutions (Bewley, 1999).

firms can commit to a match, which may be endogenously dissolved in the form of unilateral quits and layoffs in addition to being exogenously dissolved at a fixed Poisson rate.

In this environment, a worker-firm match can be characterized as a *nonzero-sum stochastic differential game with stopping times* (Bensoussan and Friedman, 1977). Forward-looking workers and firms play a *game* due to the strategic choice of their own *stopping times*, or when to unilaterally separate from the match. The game is *stochastic* and *differential* because worker productivity follows a Brownian motion. The game is *nonzero-sum* since the match surplus is positive along the equilibrium path. To characterize the equilibrium of this problem, we leverage powerful tools from the literature on variational inequalities (Lions and Stampacchia, 1967).

While workers and firms engage in complex forward-looking behavior, we show that their decisions depend only on a single state variable: the wage-to-productivity ratio. A match is dissolved when the wage-to-productivity ratio moves outside an inaction region bounded by two thresholds. On one side, workers resign when their wage-to-productivity ratio falls below a quit threshold. On the other side, firms dismiss workers whose wage-to-productivity ratio exceeds a layoff threshold. Endogenous job separations due to quits and layoffs are unilateral in the sense that they occur voluntarily in the eyes of one party but involuntarily in the eyes of the other party. Thus, a novel implication of our framework is that it distinguishes between quits and layoffs through the lens of an equilibrium search model.

Our analysis yields three main results. First, we prove the existence and uniqueness of a block recursive equilibrium (BRE). This result is noteworthy, in that it extends the seminal insights of Menzio and Shi (2010a) to a game-theoretic setting with inefficient job separations under two-sided limited commitment. Second, we provide a novel characterization of competitively determined entry wages in our non-Coasean setting. Specifically, we link match surplus to the expected discounted duration of a match, which itself is linked to the degree of rent sharing between a worker and a firm implied by the entry wage. Third, we characterize the endogenous job-separation thresholds and demonstrate that two-sided limited commitment has profound consequences for the impact of uncertainty on the labor market. Unlike in classical models of inaction (e.g., Barro, 1972; Bernanke, 1983), workers' and firms' ability to unilaterally separate renders the option value of remaining matched bounded, even as the volatility of productivity shocks grows unboundedly.

Having characterized the equilibrium of the nonmonetary economy, we introduce monetary shocks to the model. To this end, we assume that incumbents' wages are nominally rigid while the aggregate price level fluctuates. As a result, an expansionary monetary shock lowers incumbents' real wages and therefore affects the rate of job separations (i.e., quits and layoffs). In case of flexible entry wages, the search behavior of the unemployed reflects the price increase. Given that the wages of new hires are critical for the job-finding rate (Pissarides, 2009), we also allow for the possibility of rigid entry wages, in which case the unemployed keep searching at the same nominal wage schedule as before the expansionary monetary shock, thereby increasing vacancies and the job-finding rate. Thus, inflation "greases the wheels of the labor market" by affecting both job-separation and job-finding rates in our model.

To study the effects of a monetary shock on aggregate employment and real wages, we analyze the *cumulative impulse response (CIR)* as the area under the respective impulse response function. To this end, we build on the seminal work of Alvarez *et al.* (2016) using sufficient statistics in the product pricing literature. Under flexible entry wages, the CIR of employment is fully described by three observable moments: the steady-state job-finding rate, the variance of workers' wage changes across jobs, and a measure of skewness of wage changes across jobs. The relevance of skewness is a novel result. Intuitively, wage changes reflect deviations between a worker's wage and their productivity in their old job. Therefore, the skewness of the distribution of wage changes reflects the relative mass of workers near the quit threshold versus the layoff threshold of the inaction region in the wage-to-productivity-ratio space. For example, a positively skewed distribution of wage changes—i.e., a more truncated left tail of the distribution—reflects a higher incidence of layoffs than quits, meaning that an inflationary shock increases employment.

The anticipation of inefficient job separations in the future also has consequences at the time of match formation. Under rigid entry wages, the CIR of employment depends on the elasticity of the job-finding rate with respect to the monetary shock, which itself is a function of the prevalence of inefficient job separations. Intuitively, an increase in the price level incentivizes firms to post more vacancies by reducing real entry wages. The magnitude of this effect decreases in the share of inefficient job separations as a consequence of two-sided lack of commitment: Firms choose when to lay off workers but do not control workers' quit decisions, and vice versa. Therefore, the share of inefficient job separations indirectly measures a firm's expected returns from posting an additional vacancy.

While our theoretical analysis highlights the relevant mechanisms at play in a non-Coasean labor market, the true value of our sufficient statistics lies in their potential to be measured. Such an enterprise might seem daunting, given that the share of endogenous or inefficient job separations is unobserved. Indeed, we show that aggregate statistics cannot be used as a model discrimination device between Coasean and non-Coasean features of our economy, including the prevalence of inefficient job separations.[2] We overcome this challenge by proving identification of our model based on readily available microdata on wage changes and worker flows between jobs. In a nutshell, the model predicts that endogenous

[2]For example, a zero CIR of employment is consistent with either all separations being exogenous or, alternatively, all separations being endogenous as long as the inaction region is symmetric.

job separations lead to selection into unemployment based on wage-to-productivity ratios, which can be recovered from data on wage changes across consecutive jobs. Intuitively, an economy with "excess mass" in the tails of the wage change distribution indicates a larger prevalence of inefficient quits and layoffs. More formally, we provide an identification proof that allows us to recover the labor market's latent state—namely, the unobserved distribution of cumulative productivity shocks in employment.

Related Literature. We highlight two contributions. Our first contribution is to develop a framework with the non-Coasean feature that nominal fluctuations affect the split of match surplus and thus both job-finding and job-separation rates. This approach sets us apart from two traditions. On one hand, models in the Keynesian tradition have highlighted a staggered wage setting (Erceg *et al.*, 2000), downward nominal wage rigidity (Schmitt-Grohé and Uribe, 2016), or real wage rigidity (Blanchard and Galí, 2010) as important propagation mechanisms in the transmission of aggregate shocks. We contribute to this literature a model of how the equilibrium distribution of wages and employment is determined in a frictional labor market subject to nominal wage rigidity, which we show how to discipline using microdata. On the other hand, models in the search-theoretic tradition have studied the role of wage rigidity in amplifying unemployment fluctuations in models of search and matching, following Shimer (2005a). These models restrict attention to the effects of wage rigidity on hiring due to the assumption of Coasean labor contracts, which preclude bilaterally inefficient job separations. For instance, Hall (2003) and Elsby *et al.* (2022) assume that infrequent wage renegotiations prevent the dissolution of matches with positive surplus. Similarly, the wage setting protocols assumed by Hall (2005), Hall and Milgrom (2008), Christiano *et al.* (2016), Ravn and Sterk (2020), Gornemann *et al.* (2021), Moscarini and Postel-Vinay (2022), and Birinci *et al.* (2022) are all bilaterally efficient. Related work by Gertler and Trigari (2009) and Gertler *et al.* (2020) study environments with aggregate productivity shocks subject to wage rigidities, which they ex post verify lead to no inefficient job separations. All aforementioned models steer clear of the seminal Barro (1977) critique of inefficient outcomes under long-term contracting. While models in this tradition have produced important insights, our theory of non-Coasean labor markets represents a significant departure from this tradition as a result of the inclusion of the four realistic features: search frictions, productivity shocks, wage rigidity, and two-sided lack of commitment. We show that such a theory yields rich predictions for labor market dynamics in steady state and over the business cycle. An appealing aspect of our theory lies in its ability to speak to the mounting empirical evidence of inefficient job separations, which we analyze in relation to macroeconomic aggregates and fluctuations.[3]

[3] A growing body of evidence based on surveys of managers and workers investigates why firms lay off workers instead of cutting their wages (see Kaufman, 1984; Blinder and Choi, 1990; Bewley, 1999, and the literature that follows). Recently, Davis and Krolikowski (2022) document that most new recipients of unemployment insurance in the U.S. would have accepted

We also make a methodological contribution with respect to two prominent literatures. Relative to the search-and-matching literature, we introduce the powerful tools of nonzero-sum stochastic differential games with stopping times (Bensoussan and Friedman, 1977). Such continuous-time methods are well suited to handle strategic interactions involving stopping times and offer two distinct advantages. First, they allow us to prove the existence and uniqueness of a BRE in an environment with non-Coasean labor contracts. Second, they admit convenient mathematical properties of value functions (e.g., continuity) and allow for the analysis of equilibrium conditions using variational inequalities. Third, they allow for sharp comparative statics (e.g., the anticipatory and option value effects due to the drift and volatility of productivity). In their foundational work, Menzio and Shi (2010a) study the BRE in a discrete-time model of directed search under bilaterally efficient labor contracts. We complement their work by characterizing the BRE of a continuous-time model of directed search that features privately inefficient labor contracts due to the interaction between productivity shocks, wage rigidity and two-sided limited commitment.[4] Relative to the product pricing literature, we extend the sufficient statistic approach from heterogeneous-agent models of inaction to an environment with endogenous transitions between discrete states. We build on the important insight of Alvarez *et al.* (2016) that the CIR of output is linked to the ratio of the kurtosis and frequency of price changes in a large class of product pricing models.[5] In contrast to the product pricing context, a central feature of our labor market environment is that workers endogenously switch between employment and unemployment. Taking this into account, we derive sufficient statistics to summarize the response of aggregate labor market outcomes to a monetary shock.

Outline. The rest of the paper is organized as follows. Section 2 develops a model of non-Coasean labor markets and characterizes its equilibrium. Section 3 derives sufficient statistics for the aggregate labor market response to a monetary shock. Section 4 proves identification of the model based on labor market microdata, and Section 5 concludes.

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significant wage cuts instead of being laid off, while Bertheau *et al.* (2022) find that employers do not consider pay cuts to be a viable substitute for layoffs during crises. Jäger *et al.* (2022) provide quasi-experimental evidence of inefficient job separations by exploiting the introduction and subsequent repeal of an unemployment insurance reform in Austria. Schmieder and von Wachter (2010) provide evidence that downward wage rigidities amplify job displacement. Relatedly, Calvo *et al.* (2012), Blanco *et al.* (2022b), and Adamopoulou *et al.* (2022) provide evidence consistent with the notion that inflation "greases the wheel of the labor market" in a variety of countries.

[4] Previous search models have restricted attention to privately efficient contracts under full commitment (e.g., Moen, 1997; Acemoglu and Shimer, 1999a,b), one-sided lack of commitment (e.g., Shi, 2009; Menzio and Shi, 2010a,b, 2011; Schaal, 2017; Herkenhoff, 2019; Fukui, 2020; Balke and Lamadon, 2022), and two-sided lack of commitment (e.g., Sigouin, 2004; Rudanko, 2009, 2021; Bilal *et al.*, 2021, 2022).

[5] For additional references, see Carvalho and Schwartzman (2015), Alvarez *et al.* (2021), and Baley and Blanco (2021, 2022).

2 A Model of Non-Coasean Labor Markets

In this section, we develop a model of non-Coasean labor markets with search frictions, productivity shocks, wage rigidity, and two-sided lack of commitment. We first introduce the nonmonetary economy.

2.1 Environment

A unit mass of workers and an endogenously determined mass of firms meet in a frictional labor market. Time is continuous and indexed by t.

Preferences. Both workers and firms discount the future at a common rate $\rho > 0$. Firms maximize profits. Workers have risk-neutral preferences over an expected discounted consumption stream $\{C_t\}_{t=0}^{\infty}$:

$$\mathbb{E}\left[\int_0^{\infty} e^{-\rho t} C_t \, \mathrm{d}t\right].$$

Technology. A worker's flow income depends on the worker's employment state E_t, which can be either employed (h) or unemployed (u), and the worker's productivity Z_t. While employed, a worker produces $Y_t = Z_t$ and consumes wage flow W_t. While unemployed, a worker produces and consumes flow value $B(Z_t)$ from home production. Henceforth, lower-case letters denote the natural logarithm of variables in upper-case letters. For example, z_t denotes the worker's log productivity and w_t denotes the log wage.

Stochastic Process. A worker's idiosyncratic productivity follows a Brownian motion in logs:

$$dz_t = \gamma \, \mathrm{d}t + \sigma \, \mathrm{d}\mathcal{W}_t^z,$$

where γ is the drift, σ is the volatility, and $\mathcal{W}_t^z$ is a Wiener process. For now, we focus on a stationary environment with only idiosyncratic worker productivity shocks, but Section 3 adds aggregate shocks.

Search Frictions. Unemployed workers search for jobs and vacant firms search for workers in a frictional labor market. Search is directed, as in Moen (1997) and Menzio and Shi (2010a), and segmented across submarkets indexed by the wage w and worker productivity z. In each submarket (w, z), firms post vacancies $\mathcal{V}$ at flow cost $K(Z_t)$. Given $\mathcal{U}$ unemployed workers and $\mathcal{V}$ vacancies, a Cobb-Douglas matching function with constant returns produces $m(\mathcal{U}, \mathcal{V}) = \mathcal{U}^{\alpha}\mathcal{V}^{1-\alpha}$ matches, where α is the elasticity of matches to the unemployment rate. Thus, a worker's job-finding rate is $f(w,z) = m(w,z)/\mathcal{U}(w,z) = \theta(w,z)^{1-\alpha}$ and a firm's job-filling rate is $q(w,z) = m(w,z)/\mathcal{V}(w,z) = \theta(w,z)^{-\alpha}$, where $\theta(w,z) := \mathcal{V}(w,z)/\mathcal{U}(w,z)$

denotes market tightness in submarket (w, z). Existing matches are exogenously dissolved at Poisson rate δ, or they can be endogenously and unilaterally dissolved by either the worker or the firm.[6]

Wage Determination. We assume that entry wages are competitively set at match formation and remain constant throughout a match. We assume prohibitive renegotiation costs that prevent wage bargaining within the match. This assumption can be thought of as a technological friction, akin to adjustment costs in models of pricing (e.g., Barro, 1972) and investment (e.g., Cooper and Haltiwanger, 2006).

Agents' Choices. An unemployed worker's choice of submarket (w, z) is associated with job-finding rate $f(w, z)$, which induces a stochastic job offer arrival time τ^u. Given the (fixed) wage w and current productivity z, a matched worker chooses when to quit, which induces a stopping time τ^h. Based on the same (w, z) pair, a matched firm chooses when to lay off the worker, which induces a stopping time τ^j. Given the choices by workers and firms in addition to the exogenous stopping time τ^δ, the actual match duration is the minimum stopping time in the vector $\vec{\tau}^m = (\tau^h, \tau^j, \tau^\delta)$, denoted $\tau^m = \min\{\tau^h, \tau^j, \tau^\delta\}$.

Value Functions. Agents' value functions depend on the worker's productivity z and, if matched, the fixed wage rate w. In theory, they may also depend on the aggregate state, which consists of the joint distribution of worker productivities, wages, and employment states. However, we show that there exists a unique BRE, as in Menzio and Shi (2010a), in which equilibrium objects do not depend on the distribution of workers' idiosyncratic states. Thus, we omit the aggregate state from all notation.

The value of an unemployed worker with productivity z is

$$U(z) = \max_{\{w_t\}_{t=0}^{\tau^u}} \mathbb{E}_0 \left[\int_0^{\tau^u} e^{-\rho t} B(e^{z_t})\, \mathrm{d}t + e^{-\rho \tau^u} H(w_{\tau^u}, z_{\tau_u}, \vec{\tau}^m(w_{\tau^u}, z_{\tau^u})) \right]. \tag{1}$$

That is, an unemployed worker searches for a job in submarket (w_t, z_t) at time $t \leq \tau^u$ until becoming employed at wage w_{τ^u} and receiving the value of employment $H(w_{\tau^u}, z_{\tau_u}, \vec{\tau}^m(w_{\tau^u}, z_{\tau^u}))$ at time τ^u. Given a vector of stopping times $\vec{\tau}^m$, the value of a worker employed at wage w with productivity z is

$$H(w, z, \vec{\tau}^m) = \mathbb{E}_0 \left[\int_0^{\tau^m} e^{-\rho t} e^{w}\, \mathrm{d}t + e^{-\rho \tau^m} U(z_{\tau^m}) \right]. \tag{2}$$

That is, an employed worker consumes a constant wage w until time τ^m when she either endogenously or exogenously transitions to unemployment. Similarly, given a vector of stopping times $\vec{\tau}^m$, the value of a

[6] The exogenous separation shock can be interpreted as a permanent shock to the productivity of the match that renders the match unproductive forever.

firm matched with a worker with wage w and productivity z is

$$J(w, z, \vec{\tau}^m) = \mathbb{E}_0 \left[\int_0^{\tau^m} e^{-\rho t} \left[e^{z_t} - e^w \right] \mathrm{d}t \right]. \tag{3}$$

That is, the match produces e^{z_t}, of which e^w is paid to the worker until it gets dissolved at time τ^m.

Free Entry. In choosing the number of vacancies to post in each submarket, firms trade off the expected benefit—i.e., the product of the filling rate $q(w, z)$ and the value of a filled job $J(w, z, \vec{\tau}^m(w, z))$—with the flow cost $K(e^{z_t})$ of posting a vacancy. In each submarket, firms post vacancies up to the point at which the marginal vacancy posting cost exceeds its expected benefits. Thus, free entry requires that

$$K(e^{z_t}) - q(w, z) J(w, z, \vec{\tau}^m(w, z)) \geq 0 \tag{4}$$

and $\theta(w, z) \geq 0$, with complementary slackness, for all (w, z).

Equilibrium Definition. We are now ready to define an equilibrium. Let $\mathcal{T}$ be the set of all stopping times for a given match. Given the state (w, z), staying in the match is a *weakly dominant strategy* for the worker if there exists a stopping time $\tau^{h\star}(w, z) \in \mathcal{T}$ such that $\Pr(\tau^{h\star}(w, z) > 0) = 1$ and

$$H(w, z, \tau^{h\star}(w, z), \tau^j, \tau^\delta) \geq H(w, z, \tau^h, \tau^j, \tau^\delta), \quad \forall \tau^h, \tau^j \in \mathcal{T},$$

with strict inequality for some τ^j. Similarly, given (w, z), staying in the match is a weakly dominant strategy for the firm if there exists a stopping time $\tau^{j\star}(w, z) \in \mathcal{T}$ such that $\Pr(\tau^{j\star}(w, z) > 0) = 1$ and

$$J(w, z, \tau^h, \tau^{j\star}(w, z), \tau^\delta) \geq J(w, z, \tau^h, \tau^j, \tau^\delta), \quad \forall \tau^h, \tau^j \in \mathcal{T},$$

with strict inequality for some τ^h.

Definition 1. *An equilibrium consists of a set of value functions $\{H(w, z, \vec{\tau}^m), J(w, z, \vec{\tau}^m), U(z)\}$, a market tightness function $\theta(w, z)$, and policy functions $\left\{\tau^{h*}(w, z), \tau^{j*}(w, z), w^*(z_t)\right\}$, such that:*

1. *Given $H(w, z, \vec{\tau}^{m*}(w, z))$, $U(z)$, and $\theta(w, z)$, the search strategy $\{w^*(z_t)\}_{t=0}^{\tau^{u*}}$ solves equation (1).*

2. *Given $J(w, z, \vec{\tau}^{m*}(w, z))$, market tightness $\theta(w, z)$ solves the free-entry condition (4).*

3. *Given* $U(z)$, $(\tau^{h*}(w,z), \tau^{j*}(w,z))$ *is a nontrivial Nash equilibrium with stopping times* (τ^h, τ^j) *that satisfy*

$$H(w,z,\tau^{h*}(w,z),\tau^{j*}(w,z),\tau^{\delta}) \geq H(w,z,\tau^{h},\tau^{j*}(w,z),\tau^{\delta}), \quad \forall (w,z) \tag{5}$$

$$J(w,z,\tau^{h*}(w,z),\tau^{j*}(w,z),\tau^{\delta}) \geq J(w,z,\tau^{h*}(w,z),\tau^{j},\tau^{\delta}), \quad \forall (w,z) \tag{6}$$

and $Pr(\tau^{h*}(w,z) > 0) = 1$ *(resp.* $Pr(\tau^{j*}(w,z) > 0) = 1$*) whenever staying in the match is a weakly dominant strategy for the worker (resp. the firm) given the state* (w,z)*.*

Part 1 of Definition 1 requires the optimality of unemployed workers' search strategies and Part 2 requires free entry. Part 3 requires that agents' strategies form a Nash equilibrium in weakly dominant strategies. That is, the worker's optimal quit strategy τ^{h*} is the best response to the firm's layoff strategy τ^{j*}, and vice versa—see equations (5) and (6). Our equilibrium definition rules out the trivial Nash equilibrium, in which both the worker and the firm choose to dissolve the match immediately. To see the need for an equilibrium refinement, assume that time is discrete, a period's length is $\mathrm{d}t$, and the match will end in the following period with certainty. Table 1 lists the game's payoffs in this environment. Assume that current productivity z is such that flow payoffs in the match exceed flow payoffs from the outside options for both the worker and the firm—i.e., $(e^z - e^w)\,\mathrm{d}t > 0$ and $e^w\,\mathrm{d}t + \mathbb{E}_{z'}[e^{-\rho\,\mathrm{d}t}U(z')|z] > U(z)$. Under these assumptions, there are two Nash equilibria: one in which both agents choose to dissolve the match and one in which both players decide to stay in the match. However, the first equilibrium does not survive the *iterated elimination of weakly dominated strategies* since, independent of what the other agent does, it is weakly better to continue in the match.

TABLE 1. PAYOFFS IN PERIOD GAME

	Worker stops	**Worker continues**
Firm stops	$(0, U(z))$	$(0, U(z))$
Firm continues	$(0, U(z))$	$((e^z - e^w)\,\mathrm{d}t, (e^w\,\mathrm{d}t + \mathbb{E}_{z'}[e^{-\rho\,\mathrm{d}t}U(z')|z])$

Notes: This table shows the payoffs in a discrete-time approximation of the game played between a worker and a firm under the assumption that in the next period the probability of an exogenous separation is 1.

As we take the limit $\mathrm{d}t \to 0$, we obtain the continuous-time conditions that render remaining in the match a weakly dominant strategy. That is, if $(e^z - e^w)\,\mathrm{d}t > 0$ and $e^w\,\mathrm{d}t + \mathbb{E}_{z'}[e^{-\rho\,\mathrm{d}t}U(z')|z] > U(z)$, then, as $\mathrm{d}t \to 0$, $e^z - e^w > 0$ and $\rho u(z) < e^w + \gamma \partial u(z)/\partial z + (\sigma^2/2)\partial^2 u(z)/\partial z^2$, respectively.

Allocative Wages and Inefficient Job Separations. Here, we define two key concepts that play an important role in our analysis. We refer to wages as ex post *allocative* whenever they affect the expected duration of the match—that is, whenever $\exists\, w, w' \in \mathbb{R} : \mathbb{E}[\vec{\tau}^m(w,z)] \neq \mathbb{E}[\vec{\tau}^m(w',z)]$. We also refer to a job separation as *inefficient* whenever a match is dissolved in spite of a strictly positive joint match surplus $S(w,z,\vec{\tau}^m) := H(w,z,\vec{\tau}^m) - U(z) + J(w,z,\vec{\tau}^m) > 0$. That wages are allocative and job separations may be inefficient reflects the non-Coasean nature of labor contracts, which arise due to the interaction between search frictions, worker productivity shocks, wage rigidity, and the two-sided lack of commitment.[7] It is important to highlight that inefficiencies that arise ex post (i.e., once the match is formed) due to lack of commitment also have ex ante effects on unemployed workers' policies through $H(w,z,\vec{\tau}^{m*}(w,z))$ as well as firms' policies and market tightness through $J(w,z,\vec{\tau}^{m*}(w,z))$. Therefore, in equilibrium, lack of commitment affects *both* job-finding and job-separation rates.

Homotheticity. Shocks to worker productivity affect agents' choices because they change the relative values of three margins: wages while employed, w; home production while unemployed, $B(Z_t)$; and vacancy posting costs, $K(Z_t)$, all relative to a worker's productivity level Z_t. In order to focus on the margin that pertains to the relative value of wages, which is the main focus of this paper, we assume that the search cost and unemployment income are homothetic in workers' productivity (thereby abstracting from the other two margins). That is, $B(Z_t) = \tilde{B}Z_t$ for $\tilde{B} \in (0,1)$ and $K(Z_t) = \tilde{K}Z_t$ for $\tilde{K} > 0$.

Discussion of Assumptions. For expositional clarity, we imposed several assumptions that are not essential for the main insights that emerge from our theory of non-Coasean labor markets: (i) full rigidity of wages within a match, (ii) no on-the-job search, (iii) homotheticity of the technology for home production and search, and (iv) the absence of alternative sources of heterogeneity.

Regarding (i), this assumption means that we only need to track productivity fluctuations, but not wages, during a match. Online Appendix C presents an extension of our model with staggered wage renegotiations. There, we assume that renegotiations arrive according to a Poisson process with constant hazard rate à la Calvo (1983) and that renegotiations follow a Nash bargain with worker weight α. Integrating more complex wage-adjustment frictions could speak to empirical patterns of wage rigidity, such as those documented in Grigsby *et al.* (2021) and Blanco *et al.* (2022a), and is left for future research.

Regarding (ii), this assumption allows us to abstract from employed workers directing their search across markets according to their current wage and productivity. Even with on-the-job search, inefficient separations into unemployment would occur for the same reasons spelled out above. Adding on-the-job search, while numerically tractable, would force us to take a stance on how wages are renegotiated when

[7] The lack of commitment is reflected in the equilibrium definition: Stopping times depend on the history of shocks, and (τ^j, τ^h) are optimal for every history and each player. Naturally, we require an agent's stopping times to be measurable with respect to the agent's information set (including the entire history of shocks).

a worker receives an outside offer (Postel-Vinay and Robin, 2002).

Regarding (iii), this assumption implies that workers of different productivity levels face the same job-finding rate, separation rate, and wage per efficiency unit. Homotheticity is not crucial for any parts of our theory, but allows for a much simpler exposition by focusing on the economic mechanisms within, rather than across, worker types.

Finally, regarding (iv), this assumption simplifies the analysis by ignoring firm and match heterogeneity in productivity and wages. It is motivated by the empirical observation that worker heterogeneity explains the lion's share of empirical wage variation (Bonhomme *et al.*, 2019). Future work could study alternative sources of heterogeneity in a non-Coasean labor market.

2.2 Equilibrium Characterization

Let $u(z)$, $h(z;w)$, $j(z;w)$, and $\theta(z;w)$ denote the values of an unemployed worker, an employed worker, a filled vacancy, and the market tightness function evaluated at equilibrium policies, where w indexes the (constant) wage. We now derive necessary and sufficient conditions for a BRE. We break our equilibrium characterization into two steps. In the first step, firms post vacancies and workers search for jobs. This problem is characterized by the Hamilton-Jacobi-Bellman (HJB) equation for unemployed workers,

$$\rho u(z) = \tilde{B}e^z + \gamma \frac{\partial u(z)}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 u(z)}{\partial z^2} + \max_w f(w,z)[h(z;w) - u(z)], \tag{7}$$

and the free-entry condition for firms, which requires that the equilibrium market tightness satisfies

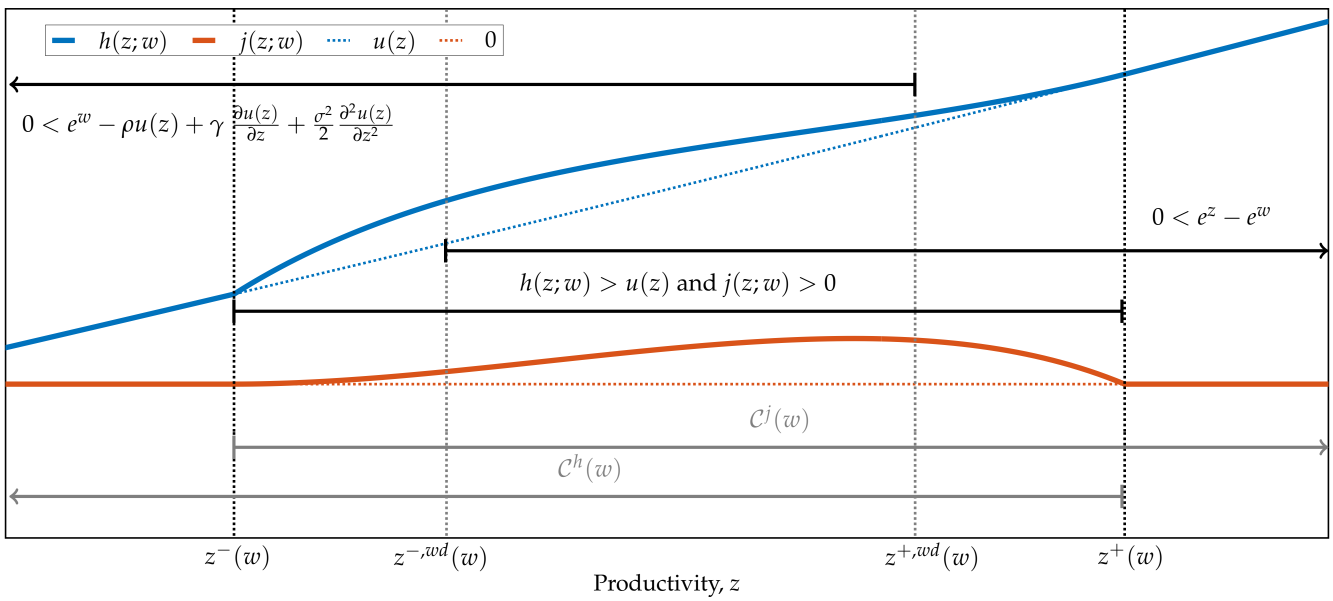
$$\tilde{K}e^z - q(w,z)j(z;w) \geq 0 \tag{8}$$

and $\theta(w,z) \geq 0$, with complementary slackness, for all (w,z).

In the second step, which is the novel focus of this paper, matched workers and firms play a *game* due to the strategic choice of their own *stopping times*, or when to unilaterally separate from the match. The strategic worker-firm interaction has three features. First, agents play a *nonzero-sum* game, since the match value, e^z, exceeds the outside option value $\tilde{B}e^z$ for $\tilde{B} < 1$. Second, agents' payoffs are stochastic and move with productivity, which follows a Wiener process. Third, agents' strategies consist of stopping times. Thus, the strategic interaction between workers and firms can be formulated as a nonzero-sum stochastic differential game with stopping times (Bensoussan and Friedman, 1977). To characterize the equilibrium, we make use of quasi-variational inequalities—a methodological approach that we import from the literature on variational inequalities (Lions and Stampacchia, 1967). We highlight that the application of these state-of-the-art tools in an economic context is an important contribution of this paper.

Before deriving the quasi-variational inequalities that characterize this problem, we describe the equilibrium conditions for a worker-firm match with the aid of Figure 1, which illustrates the equilibrium values, outside options, and optimal policies for both agents.

FIGURE 1. EQUILIBRIUM VALUES AND OPTIMAL POLICIES



Notes: The figure plots the value functions of workers and firms for a given log wage w as a function of log productivity z. The blue and red solid lines show the value functions for the worker and the firm, respectively. The blue and red dashed lines show the opportunity costs for the worker and the firm, respectively. The firm's optimal job-separation trigger based on weakly dominant strategies is $z^{-,wd}(w) := w$. The worker's optimal job-separation trigger under weakly dominant strategies is $z^{+,wd}(w^*)$ and satisfies $e^{w^*} = \tilde{B}e^{z^{+,wd}(w^*)} + f(z^{+,wd}(w^*);w^*)[h(z^{+,wd};w^*) - u(z^{+,wd})]$. The optimal job-separation triggers for the worker and the firm are $z^+(w) := \sup_z\{z : h(z;w) > u(z)\}$ and $z^-(w) := \inf_z\{z : j(z;w) > 0\}$, respectively.

The possibility that both the worker and the firm can unilaterally dissolve a match imposes lower bounds on the values $h(z;w)$ and $j(z;w)$. *Individual rationality* of the worker and the firm requires that

$$h(z;w) \geq u(z) \quad \forall z, \tag{9}$$

$$j(z;w) \geq 0 \quad \forall z. \tag{10}$$

Let $\mathcal{C}^h(w)$ denote the interior of the set of productivities for which the worker prefers to stay in the match with wage w under our equilibrium definition. Importantly, this set is made up of two productivity ranges: one in which both the firm and the worker opt to continue the match and one in which only the worker prefers to continue the match. Let $\mathcal{C}^j(w)$ denote the analogous object for the firm.

Optimality Conditions. Workers' and firms' optimal policies for productivity levels inside and outside the other agent's continuation set are characterized by variational inequalities. The HJB equation of a worker employed at wage w with productivity $z \in \mathcal{C}^j(w)$, at which the firm opts to continue, is given by

$$\rho h(z;w) = \max\left\{ e^w + \gamma \frac{\partial h(z;w)}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 h(z;w)}{\partial z^2} + \delta\left[u(z) - h(z;w)\right],\ \rho u(z) \right\}. \tag{11}$$

This value satisfies $h(\cdot;w) \in \mathbb{C}^1(\mathcal{C}^j(w)) \cap \mathbb{C}(\mathbb{R})$. That is, it is continuously once-differentiable on the continuation set and continuous everywhere. These continuity and differentiability conditions correspond to the *value matching* condition and the *smooth pasting* condition, respectively, in the worker's best response. Importantly, a smooth pasting condition holds whenever the worker has to choose between staying or leaving the match (i.e., in the firm's continuation set). Similarly, the HJB equation of a firm employing a worker at wage w with productivity $z \in \mathcal{C}^h(w)$, at which the worker prefers to continue, is given by

$$\rho j(z;w) = \max\left\{ e^z - e^w + \gamma \frac{\partial j(z;w)}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 j(z;w)}{\partial z^2} - \delta j(z;w),\ 0 \right\}, \tag{12}$$

which must be continuously differentiable on the worker's continuation set: $j(\cdot;w) \in \mathbb{C}^1(\mathcal{C}^h(w)) \cap \mathbb{C}(\mathbb{R})$.

On the other hand, if any one agent chooses to dissolve the match, then the other agent receives the value of the corresponding outside option. Therefore, the worker's and the firm's values of a match with productivity z and wage w satisfy the following conditions:

$$h(z;w) = u(z) \quad \forall z \in (\mathcal{C}^j(w))^c, \tag{13}$$

$$j(z;w) = 0 \qquad \forall z \in (\mathcal{C}^h(w))^c, \tag{14}$$

where $X^c := \mathbb{R}\backslash X$. Equations (13)–(14) define the game's *value matching conditions*, which imply the continuity of one agent's value function at the boundary of the other agent's continuation set.

Continuation Sets. Two conditions characterize the agents' continuation sets. First, the match continues whenever both agents strictly prefer to remain in the match:

$$h(z;w) > u(z) \quad \forall z, \tag{15}$$

$$j(z;w) > 0 \qquad \forall z. \tag{16}$$

Second, each agent prefers to continue whenever staying in the match is a weakly dominant strategy. To understand this last condition, note that for *any* worker's policy, the firm strictly prefers to continue the match if flow profits are strictly positive (i.e., $e^z - e^w > 0$). This preference results from the fact that the firm's continuation value is nonnegative because the firm always has the option to fire the worker in the future and receive an outside option value of zero. Therefore, the firm's continuation set is

$$\mathcal{C}^j(w) := int\left\{z \in \mathbb{R} : j(z;w) > 0 \text{ or } e^z - e^w > 0\right\}. \tag{17}$$

Similarly, the worker's weakly dominant continuation set includes all productivity levels for which the sum of the current wage and the discounted capital gains from unemployment is positive:

$$\mathcal{C}^h(w) := int\left\{z \in \mathbb{R} : h(z;w) > u(z) \text{ or } 0 < e^w - \rho u(z) + \gamma\frac{\partial u(z)}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 u(z)}{\partial z^2}\right\}. \tag{18}$$

Intuitively, from the HJB equation of the unemployed worker in (7), we have that

$$0 < e^w - \rho u(z) + \gamma\frac{\partial u(z)}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 u(z)}{\partial z^2} \iff 0 < e^w - \underbrace{\left(\tilde{B}e^z + \max_w f(z;w)[h(z;w) - u(z)]\right)}_{\text{flow opportunity cost}}.$$

That is, if the wage exceeds the flow opportunity cost, then staying matched strictly dominates quitting.

Figure 1 depicts the two agents' continuation sets. The firm's continuation set is $\mathcal{C}^j(w) = \{z \in (z^-(w), \infty)\}$ and includes the range of productivities for which the firm makes positive flow profits—i.e., $z > z^{-,wd} := w$—so it is weakly dominant to retain the worker. Moreover, the set $\mathcal{C}^j(w)$ also includes the range of productivities for which both the firm and the worker find it optimal to remain matched despite either party's flow profits being negative—i.e., $z \in (z^-(w), z^{-,wd})$. For z in this range, the firm's continuation value is positive and large enough to compensate for current flow losses. An analogous intuition applies to the worker's continuation set $\mathcal{C}^h(w)$. The existence and uniqueness of a threshold that characterizes the separation policy is not an assumption but a result we formally derive below.

Equilibrium Policies. The worker quits and the firm fires the worker at stopping times τ^h and τ^j, which denote the stochastic time at which productivity falls outside the continuation set of the worker and the firm, respectively. Thus, agents' optimal stopping times are given by

$$\tau^{j*}(w,z) = \inf\left\{t \geq 0 : z_t \in \mathcal{C}^j(w)^c, z_0 = z\right\},$$
$$\tau^{h*}(w,z) = \inf\left\{t \geq 0 : z_t \in \mathcal{C}^h(w)^c, z_0 = z\right\}.$$

The optimality of workers' search decisions implies that the competitive entry wage satisfies

$$w^*(z) = \arg\max_w \theta(z;w)^{1-\alpha} \left[h(z;w) - u(z)\right]. \tag{19}$$

The following lemma shows that these conditions are necessary and sufficient to characterize a BRE. We relegate all proofs to the Online Appendix.

Lemma 1. *The policy functions* $\{\tau^{h*}, \tau^{j*}, w^*(z)\}$ *and the value functions* $\{U(z), H(w, z, \vec{\tau}^m), J(w, z, \vec{\tau}^m)\}$ *given by* (1), (2) *and* (3) *and the market tightness function* $\theta(w,z)$ *form a BRE if and only if* $\{u(z), h(z;w), j(z;w)\}$ *satisfy equations* (7)–(19) *and*

$$\begin{aligned} u(z) &= U(z), \\ h(z;w) &= H(w, z, \tau^{h*}(w,z), \tau^{j*}(w,z), \tau^{\delta}), \\ j(z;w) &= J(w, z, \tau^{h*}(w,z), \tau^{j*}(w,z), \tau^{\delta}). \end{aligned}$$

Finding the State. To understand the dependence of equilibrium objects on state variables, we recast the model in terms of a reduced state space. Since the flow income of unemployed workers and firms' vacancy costs are both proportional to productivity, Z, it turns out that the relevant state variable for both workers and firms is the log-wage-to-productivity ratio, $\hat{w} := w - z$. This result allows us to express agents' values and policies as functions of the scalar $\hat{w}$ instead of the duplet (w, z). To simplify notation, we define the transformed drift $\hat{\gamma} := \gamma + \sigma^2$ and the transformed discount factor $\hat{\rho} := \rho - \gamma - \sigma^2/2$. The following Lemma characterizes the equilibrium.

Lemma 2. *Suppose that the functions* $(u(z), h(z;w), j(z;w), \theta(w,z))$ *satisfy the equilibrium conditions in* (7)–(14), *given the continuation sets* $\mathcal{C}^h(w)$ *and* $\mathcal{C}^j(w)$ *defined in* (17)–(18). *Then, the transformed value and market tightness functions given by*

$$\left(\hat{U}, \hat{J}(w-z), \hat{W}(w-z), \hat{\theta}(w-z)\right) = \left(\frac{u(z)}{e^z}, \frac{j(z;w)}{e^z}, \frac{h(z;w) - u(z)}{e^z}, \theta(w,z)\right)$$

equivalently characterize the equilibrium if the following conditions are satisfied:

1. *The transformed value function of an unemployed worker,* $\hat{U}$, *satisfies*

$$\hat{\rho}\hat{U} = \tilde{B} + \max_{\hat{w}} \hat{\theta}(\hat{w})^{1-\alpha} \hat{W}(\hat{w}), \tag{20}$$

where the optimal choice of submarket for an unemployed worker to search in is $\hat{w}^* = w^*(z) - z$.

2. *The lower bounds of the game's values for workers and firms are:* $\hat{W}(\hat{w}) \geq 0$ *and* $\hat{J}(\hat{w}) \geq 0$.

3. *The variational inequalities for workers and firms are satisfied: Given*

$$\hat{\mathcal{C}}^h := int\left\{\hat{w} \in \mathbb{R} : \hat{W}(\hat{w}) > 0 \text{ or } 0 < e^{\hat{w}} - \hat{\rho}\hat{U}\right\} \text{ and } \hat{\mathcal{C}}^j := int\left\{\hat{w} \in \mathbb{R} : \hat{J}(\hat{w}) > 0 \text{ or } 0 < 1 - e^{\hat{w}}\right\},$$

the transformed value function of an employed worker, $\hat{W}(\hat{w})$*, and that of a filled vacancy,* $\hat{J}(\hat{w})$*, satisfy*

$$(\hat{\rho} + \delta)\hat{W}(\hat{w}) = \max\left\{e^{\hat{w}} - \hat{\rho}\hat{U} - \hat{\gamma}\hat{W}'(\hat{w}) + \frac{\sigma^2}{2}\hat{W}''(\hat{w}),\, 0\right\},\ \forall \hat{w} \in \hat{\mathcal{C}}^j, \tag{21}$$

$$(\hat{\rho} + \delta)\hat{J}(\hat{w}) = \max\left\{1 - e^{\hat{w}} - \hat{\gamma}\hat{J}'(\hat{w}) + \frac{\sigma^2}{2}\hat{J}''(\hat{w}),\, 0\right\},\ \forall \hat{w} \in \hat{\mathcal{C}}^h, \tag{22}$$

with $\hat{W} \in \mathbb{C}^1(\hat{\mathcal{C}}^j(w)) \cap \mathbb{C}(\mathbb{R})$ *and* $\hat{J} \in \mathbb{C}^1(\hat{\mathcal{C}}^h(w)) \cap \mathbb{C}(\mathbb{R})$. *The optimal stopping times are given by* $\tau^{h*} = \inf\{t \geq 0 : \hat{w}_t \notin \hat{\mathcal{C}}^h, w_0 = \hat{w}^*\}$ *and* $\tau^{j*} = \inf\{t \geq 0 : \hat{w}_t \notin \hat{\mathcal{C}}^j, w_0 = \hat{w}^*\}$.

4. *The value matching conditions are satisfied:* $\hat{W}(\hat{w}) = 0 \quad \forall \hat{w} \in (\hat{\mathcal{C}}^j)^c$*, and* $\hat{J}(\hat{w}) = 0 \quad \forall \hat{w} \in (\hat{\mathcal{C}}^h)^c$.

5. *Free entry for* $\hat{\theta}(\hat{w})$ *is satisfied:* $\tilde{K} - \hat{\theta}(\hat{w})^{-\alpha}\hat{J}(\hat{w}) \geq 0$ *and* $\hat{\theta}(\hat{w}) \geq 0$*, with complementary slackness.*

The equilibrium conditions in Lemma 2 are transformed versions of those stated above and follow similar intuitions. Equation (20) of Part 1 gives the value of unemployment under the optimal log-wage-to-productivity search strategy. For the unemployed, the optimal wage w^* trades off the job-finding rate $\hat{\theta}(\hat{w}^*)^{1-\alpha}$ with the value of employment $\hat{W}(\hat{w}^*)$. Part 2 describes the lower bounds on agents' transformed values. From equations (21)–(22) of Part 3, we can infer the thresholds that render the worker's and the firm's transformed flow payoffs negative. If $e^{\hat{w}} < \hat{\rho}\hat{U}$, then the worker's wage is below the flow value of unemployment. Similarly, if $e^{\hat{w}} > 1$, then the firm's flow profits are negative. Part 4 states the transformed value matching conditions. Finally, Part 5 states the transformed free-entry condition.

Equilibrium Existence and Uniqueness. Equipped with the equilibrium conditions summarized in Lemma 2, we now state an important result.

Proposition 1. *There exists a unique BRE.*

While the result in Proposition 1 is essential for any model with non-Coasean labor contracts, it does not follow from previous work. Theorems for the existence of a BRE with exogenous job separations in discrete time rely on Schauder's fixed-point theorem (e.g., Menzio and Shi, 2010a,b; Schaal, 2017). Two conditions are critical for applying this fixed-point theorem: continuity in the value functions and continuity in the mapping between the value functions that characterize the BRE. In the discrete-time version of our model, idiosyncratic worker productivity shocks, wage rigidity, and two-sided lack of commitment jointly generate endogenous job separations, which violate the regularity conditions on which traditional arguments rely. In contrast, our continuous-time setup allows us to get around this technical challenge. One of our contributions is to leverage techniques from the literature on variational inequalities in continuous-time models to prove the *existence* and *uniqueness* of a BRE in our environment.

2.3 Understanding the Mechanisms

We now characterize the mechanisms that drive workers' and firms' equilibrium behavior.

Equilibrium Policies. Based on the transformed state variable $\hat{w}$ and the equilibrium conditions in Lemma 2, we can characterize agents' equilibrium policies. Recalling the definition of the transformed state variable $\hat{w} := w - z$, we postulate that there exist optimal policies $\hat{w}^- < \hat{w}^* < \hat{w}^+$, where $\hat{w}^-$ is the worker's optimal job-separation threshold, $\hat{w}^*$ is the optimal search strategy at match formation, and $\hat{w}^+$ is the firm's optimal job separation threshold. We define the transformed surplus of the match as $\hat{S}(\hat{w}) := \hat{J}(\hat{w}) + \hat{W}(\hat{w})$ and the worker's share of the transformed surplus as $\eta(\hat{w}) := \hat{W}(\hat{w})/\hat{S}(\hat{w})$. The following proposition characterizes the properties of the BRE in its transformed notation.

Proposition 2. *The BRE has the following properties:*

1. *The joint match surplus satisfies*

$$\hat{S}(\hat{w}) = (1 - \hat{\rho}\hat{U})\mathcal{T}(\hat{w}, \hat{\rho}), \tag{23}$$

   where

$$\mathcal{T}(\hat{w}, \hat{\rho}) := \mathbb{E}\left[\int_0^{\tau^{m*}} e^{-\hat{\rho}t}\,\mathrm{d}t | \hat{w}_0 = \hat{w}\right] \tag{24}$$

   is the expected discounted match duration and $1 > \hat{\rho}\hat{U} > \tilde{B}$.

2. *The competitive entry wage* $\hat{w}^*$ *coincides with the Nash bargaining solution with worker's weight* α*:*

$$\hat{w}^* = \arg\max_{\hat{w}} \left\{\hat{W}(\hat{w})^\alpha \hat{J}(\hat{w})^{1-\alpha}\right\} = \arg\max_{\hat{w}} \left\{\eta(\hat{w})^\alpha (1 - \eta(\hat{w}))^{1-\alpha} \mathcal{T}(\hat{w}, \hat{\rho})\right\}, \tag{25}$$

with optimality condition

$$\underbrace{\eta'(\hat{w}^*)\left(\frac{\alpha}{\eta(\hat{w}^*)} - \frac{1-\alpha}{1-\eta(\hat{w}^*)}\right)}_{\text{Share channel}} = -\underbrace{\frac{\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{\rho})}{\mathcal{T}(\hat{w}^*,\hat{\rho})}}_{\text{Surplus channel}}. \tag{26}$$

3. *Given $\eta(\hat{w}^*)$ and $\mathcal{T}(\hat{w}^*,\hat{\rho})$, the equilibrium job-finding rate $f(\hat{w}^*)$ and the flow opportunity cost of employment $\hat{\rho}\hat{U}$ are given by*

$$f(\hat{w}^*) = \left[(1-\eta(\hat{w}^*))(1-\hat{\rho}\hat{U})\mathcal{T}(\hat{w}^*,\hat{\rho})/\tilde{K}\right]^{\frac{1-\alpha}{\alpha}}, \tag{27}$$

$$\hat{\rho}\hat{U} = \tilde{B} + \left(\tilde{K}^{\alpha-1}\left(1-\eta(\hat{w})\right)^{1-\alpha}\eta(\hat{w})^{\alpha}\left(1-\hat{\rho}\hat{U}\right)\mathcal{T}(\hat{w}^*,\hat{\rho})\right)^{\frac{1}{\alpha}}. \tag{28}$$

4. *Assume $\gamma \neq 0$ or $\sigma \neq 0$. Given $\hat{U}$, the worker's and the firm's continuation sets are connected, and the game's continuation set is bounded; i.e.,*

$$\hat{\mathcal{C}}^h = \{\hat{w} : \hat{w} > \hat{w}^-\} \quad \textit{and} \quad \hat{\mathcal{C}}^j = \{\hat{w} : \hat{w} < \hat{w}^+\}, \tag{29}$$

with $-\infty < \hat{w}^- \leq \log(\hat{\rho}\hat{U}) < 0 \leq \hat{w}^+ < \infty$. The worker's and firm's value functions satisfy smooth pasting conditions at $\hat{w}^-$ and $\hat{w}^+$, respectively: $\hat{W}'(\hat{w}^-) = \hat{J}'(\hat{w}^+) = 0$.

Starting with Part 1 of Proposition 2, equation (23) states that the match surplus equals the product of the transformed flow surplus $1-\hat{\rho}\hat{U}$ and the expected discounted match duration $\mathcal{T}(\hat{w},\hat{\rho})$ defined in equation (24), which depends on the entry wage $\hat{w}^*$ and the width of the match's continuation set $(\hat{w}^-,\hat{w}^+)$. Also, the flow opportunity cost of employment $\hat{\rho}\hat{U}$ is bounded between one (i.e., the transformed value of flow output in the match) and $\tilde{B}$ (i.e., the transformed value of home production). Since $1 > \hat{\rho}\hat{U}$, the joint match surplus is always strictly positive—thus all endogenous job separations are inefficient.

Equations (25)–(26) of Part 2 show that the competitive entry wage $\hat{w}^*$ balances a *share channel* and a *surplus channel*. Unemployed workers search for wages that are competitively set in a way that coincides with the Nash bargaining solution with worker's weight α, thereby satisfying the well-known condition of Hosios (1990). This result obtains due to the free-entry condition, which implies that a worker's job-finding rate is proportional to the value of a firm. A larger initial wage increases the worker's share by $\eta'(\hat{w}^*)\alpha/\eta(\hat{w}^*)$ but at the same time reduces the job-finding probability by $\eta'(\hat{w}^*)(1-\alpha)/(1-\eta(\hat{w}^*))$. This trade-off is reflected in the share channel and is standard in models with directed search.

With allocative wages, a novel *surplus channel* arises. Intuitively, the surplus channel captures the fact that the wage set at match formation affects the expected match duration, and thus the expected surplus.

The higher the entry wage, the sooner the firm will dissolve the match in expectation. Conversely, the lower the entry wage, the sooner the worker will dissolve the match in expectation. Only if $\mathcal{T}_{\hat{w}}(\hat{w}^*, \hat{\rho}) = 0$ will the worker's share of the surplus equal $\eta(\hat{w}^*) = \alpha$, as in efficient models with nonallocative wages. These considerations are unique to our environment.

Part 3 characterizes the unemployed worker's job-finding rate (27) and the flow opportunity cost of employment (28) as functions of the worker's surplus share and the expected discounted match duration.

Part 4 shows that the continuation set of the worker and that of the firm (29) follow threshold rules in the log-wage-to-productivity ratio $\hat{w}$. Workers do not quit as long as $\hat{w} > \hat{w}^-$, while firms refrain from firing the worker as long as $\hat{w} < \hat{w}^+$. Thus, the continuation set for the match is given by $\hat{\mathcal{C}}^h \cap \hat{\mathcal{C}}^j = (\hat{w}^-, \hat{w}^+)$. These thresholds satisfy $\hat{w}^- \leq \log(\hat{\rho}\hat{U})$ and $\hat{w}^+ \geq 0$, which reflects the fact that both parties are willing to accept flow payoffs below that from their respective outside option. Finally, the smooth pasting conditions apply at the worker's quit trigger $\hat{w}^-$ and at the firm's firing trigger $\hat{w}^+$, which reflects the optimality of agents' continuation thresholds.

Static Considerations. Before further characterizing the original dynamic problem, it is instructive to consider equilibrium policies when productivity is fixed—i.e., $\gamma = \sigma = 0$.[8] The following proposition characterizes the static considerations in this case.

Proposition 3. *Assume $\gamma = \sigma = 0$. Then, optimal policies are given by*

$$(\hat{w}^-, \hat{w}^*, \hat{w}^+) = \log(\hat{\rho}\hat{U}, \alpha + (1-\alpha)\hat{\rho}\hat{U}, 1),$$

with $\eta(\hat{w}^*) = \alpha$ and $\mathcal{T}(\hat{w}^*, \hat{\rho}) = 1/(\hat{\rho} + \delta)$.

Note that $\hat{w}^- < \hat{w}^* < \hat{w}^+$ and $\hat{w} = \hat{w}^*$ for the duration of the match, absent productivity fluctuations, so there are no endogenous job separations. From this result, we see that lack of commitment and wage rigidity by themselves do not generate any inefficient job separations. Absent productivity fluctuations, agents' behavior is privately efficient, in that it maximizes the joint match surplus.

In addition to the forces outlined in this static example, two important dynamic incentives guide workers' and firms' choices: the *option value effect* and the *anticipatory effect*.

Dynamic Considerations I: The Option Value Effect. To understand the role of productivity fluctuations in creating the option value effect, we assume away, for now, the drift of worker productivity—i.e., $\hat{\gamma} = 0$. The following proposition characterizes the option value effect in this case.

[8]Observe that if $\gamma = \sigma = 0$, then the smooth pasting conditions do not apply.

Proposition 4. *Assume $\hat{\gamma} = 0$ and $\alpha = 1/2$. Then, to a first-order approximation, the optimal entry wage is given by $\hat{w}^* = \log\left((1+\hat{\rho}\hat{U})/2\right)$ and job-separation thresholds satisfy $\hat{w}^{\pm} = \hat{w}^* \pm h(\varphi, \Phi)$ for some function $h(\varphi, \Phi)$ with $\varphi := \sqrt{2(\hat{\rho}+\delta)}/\sigma$ and $\Phi := (1-\hat{\rho}\hat{U})/(1+\hat{\rho}\hat{U})$. The following properties apply:*

1. *$h(\varphi, \Phi)$ is decreasing in φ and increasing in Φ.*

2. *$\lim_{\varphi\to 0} h(\varphi, \Phi) = 3\Phi$ and $\lim_{\varphi\to\infty} h(\varphi, \Phi) = \Phi$.*

3. *$\varphi h(\varphi, \Phi)$ is increasing in φ.*

Furthermore, the equilibrium surplus share is $\eta(\hat{w}) = \alpha = 1/2$ and the expected discounted match duration,

$$\mathcal{T}(\hat{w}^*, \hat{\rho}) = \frac{1 - 2\left(e^{\varphi h(\varphi,\Phi)} + e^{-\varphi h(\varphi,\Phi)}\right)^{-1}}{\hat{\rho}+\delta}, \tag{30}$$

is increasing in φ and Φ and satisfies $\mathcal{T}'_{\hat{w}}(\hat{w}^*, \hat{\rho}) = 0$.

Proposition 4 demonstrates that idiosyncratic volatility, by itself, does not affect the split of the match surplus between the worker and the firm. Such an economy is symmetric in the sense that $\mathcal{T}'_{\hat{w}}(\hat{w}^*, \rho) = 0$ and $\eta(\hat{w}) = \alpha$. Thus, a larger $\hat{w}^*$ reduces the match duration by increasing the likelihood of a layoff but increases the match duration by reducing the likelihood of a quit. Weighing both forces, $\mathcal{T}(\cdot, \rho)$ is maximized at $\hat{w}^* = (1+\hat{\rho}\hat{U})/2$ and $\eta(\hat{w}^*) = 1/2$.

This result provides a tight characterization of the worker's and the firm's optimal policy functions, which result in the continuation region of the match $(\hat{w}^-,\ \hat{w}^+)$ being symmetrically centered around the optimal entry wage $\hat{w}^*$. Second, the width of the continuation region is increasing in volatility σ and decreasing in $\hat{\rho}\hat{U}$ (Part 1). The width of the inaction region increases with σ due to the option value effect: Although the worker's productivity might be low today, the firm is willing to wait before firing the worker in case productivity improves in the future. The width of the inaction region decreases with $\hat{\rho}\hat{U}$ because a higher opportunity cost of employment makes it less attractive to delay job separations.

The option value effect naturally arises in models of inaction. However, our model features a departure from canonical models of inaction (e.g., Barro, 1972; Bernanke, 1983). In those models, the width of the continuation region typically grows unboundedly with the level of volatility σ. Instead, in our model, the width of the continuation region has an upper bound (Part 2). To see the intuition behind this result, consider the problem of a firm that finds itself in a match with negative flow profits. The marginal benefit from remaining in a currently unprofitable match is that, with some probability in the future, productivity increases enough to make the match profitable by rendering the wage-to-productivity ratio less than unity. At the same time, inaction on the part of the firm is risky: Productivity may increase by a large enough amount for the worker to choose to quit. Given the two job-separation thresholds, as the volatility goes to infinity, the probability of remaining in the profitable part of the inaction region approaches zero. Thus, the two-sided lack of commitment imposes an upper bound on the option value associated with remaining in a match with negative flow profits.

The inefficiency generated by the lack of commitment also manifests itself in the expected duration of the match given by equation (30). It is easy to see that a bounded option value effect (i.e., bounded separation thresholds), as indexed by $h(\varphi, \Phi)$, implies a lower expected duration as the volatility of productivity shocks increases (Part 3).

Dynamic Considerations II: The Anticipatory Effect. To understand the role of a nonzero productivity drift in generating the anticipatory effect, we assume away, for now, the volatility of worker productivity—i.e., $\sigma = 0$—and focus on the case with weakly positive drift—i.e., $\hat{\gamma} \geq 0$. The following proposition characterizes the anticipatory effect in this case.

Proposition 5. *Assume $\sigma = 0$ and $\hat{\gamma} \geq 0$. Then, $\hat{w}^- = \log\left(\hat{\rho}\hat{U}\right)$ and*

$$w^* = \hat{w}^- + \tilde{T}\left(\frac{\alpha + (1-\alpha)\hat{\rho}\hat{U}}{\hat{\rho}\hat{U}}, \frac{\hat{\rho}+\delta}{\gamma}, \frac{(1-\alpha)(1-\hat{\rho}\hat{U})}{\hat{\rho}\hat{U}}\right),$$

where $\tilde{T}(\cdot)$ is increasing in the first argument and decreasing in the second argument; see equation (B.36) in the Online Appendix for its definition. Moreover,

1. *If $\hat{\gamma} = 0$, then $\left(\tilde{T}(\cdot), \mathcal{T}(\hat{w}^*, \hat{\rho}), \eta(\hat{w}^*)\right) \to \left(\log\left(\frac{\alpha+(1-\alpha)\hat{\rho}\hat{U}}{\hat{\rho}\hat{U}}\right), \frac{1}{\hat{\rho}+\delta}, \alpha\right)$.*
2. *If $\hat{\gamma} \to \infty$, then $\tilde{T}(\cdot) \to \tilde{T}^{limit}$, $\mathcal{T}(\hat{w}^*, \hat{\rho}) \to 0$, and $\eta(\hat{w}^*) \to \eta^{limit}$, where $\tilde{T}^{limit}$ and η^{limit} are implicitly defined as*

$$\frac{\alpha + (1-\alpha)\hat{\rho}\hat{U}}{\hat{\rho}\hat{U}} = \frac{e^{\tilde{T}^{limit}} - 1 - \frac{(1-\alpha)(1-\hat{\rho}\hat{U})}{\hat{\rho}\hat{U}}\left(1 - \frac{\tilde{T}^{limit}}{e^{\tilde{T}^{limit}}-1}\right)}{\tilde{T}^{limit}},$$
$$\eta^{limit} = \alpha + \frac{1-\alpha}{\tilde{T}^{limit}} \frac{(1-\hat{\rho}\hat{U})\eta^{limit}}{\eta^{limit} + \hat{\rho}\hat{U}(1-\eta^{limit})}. \tag{31}$$

When productivity grows at a constant rate, the job-separation threshold $\hat{w}^-$ equals the static opportunity cost of employment, since there is no value for the worker to further delay the separation. A novel mechanism is embedded in the entry wage $\hat{w}^*$ and, therefore, in the function $\tilde{T}(\cdot)$. From Proposition 5, we can see that $\hat{w}^*$ is increasing in the Nash bargaining target and also in the drift. We refer to the latter as the anticipatory effect: Workers anticipate higher future productivity and modify their search strategy accordingly. Two limiting cases illustrate this point.

As $\gamma \to 0$ (Part 1), the equilibrium entry wage $\hat{w}^*$ is the same as in the case without drift; thus, $\eta(\hat{w}^*) = \alpha$. As the drift increases, workers optimally search for a job with a higher entry wage. Therefore, the average wage in the economy increases above the Nash bargaining target; recall that $\hat{w}^-$ remains fixed. This results from the worker internalizing the trade-off whereby a higher wage implies (i) a reduced job-finding rate and (ii) a lower frequency of inefficient job separations and, thus, a longer expected match duration. As the drift increases unboundedly (Part 2), the entry wage w^* becomes unresponsive to the drift because the job-finding rate becomes so small that it starts to dominate the trade-off. Finally, as seen in (31), the anticipatory effect causes the worker's share of the surplus to increase with the drift.

The worker's lack of commitment implies the invariance of $\hat{w}^-$ to the drift, which decreases the value of searching for a job. To see this, assume that the worker commits to some $\hat{w}^-$ and $\delta \to 0$. Under these assumptions, the job-separation rate is $s = \hat{\gamma}/(w^* - \hat{w}^-)$. Thus, the worker reduces the frequency of inefficient job separations by increasing $w^* - \hat{w}^-$, as captured by the surplus channel. At the same time, workers choose an entry wage that takes into account the trade-off captured by the share channel. For a given $\hat{w}^-$, the worker chooses only one scalar, namely $\hat{w}^*$, to balance two opposing objectives: increase $\hat{w}^*$ to avoid inefficient job separations (i.e., the surplus channel) or keep $\hat{w}^*$ close to the Nash bargaining target (i.e., the share channel). Thus, the lack of commitment distorts both the expected duration of the match and job-finding rates in equilibrium.

3 The Consequences of Monetary Shocks in Non-Coasean Labor Markets

How does the presence of inefficient job separations—due to the interaction between search frictions, productivity shocks, wage rigidity, and two-sided lack of commitment—matter for the transmission of monetary shocks? To answer this question, we introduce money as a numéraire in order to study the effects of monetary shocks on aggregate labor market outcomes and provide sufficient statistics for these effects based on readily available microdata.

3.1 A Monetary Economy

We modify the baseline model in four dimensions. First, we introduce preferences over real money:

$$\mathbb{E}_0 \left[\int_{t=0}^{\infty} e^{-\rho t} \left(C_{it} + \mu \log \left(\frac{\hat{M}_{it}}{P_t} \right) \right) \mathrm{d}t \right], \tag{32}$$

where $\hat{M}_{it}$ denotes the money holdings of worker i, P_t is the relative price of the consumption good in terms of money, and μ is a preference weight on real money holdings.

Second, workers face a budget constraint that reflects ownership of firms and access to complete financial markets. Given a history of labor market decisions regarding job search, job acceptance, and job dissolution, $lm_i^t := \{lm_{it'}\}_{t'=0}^t$, a worker's private income is $Y_t(lm_i^t)$, which equals the nominal value of the wage while employed and the nominal value of home production while unemployed. In addition, each worker receives transfers of T_{it} from the government and fully diversified claims on firms' profits. On the spending side, a worker pays for consumption expenditures P_tC_{it} and the opportunity cost of holding money $i_t\hat{M}_{it}$ at a given interest rate $i_t \geq 0$. Letting Q_t denote the time-0 Arrow-Debreu price under complete markets, the worker's budget constraint is

$$\mathbb{E}_0\left[\int_{t=0}^{\infty} Q_t\left(P_tC_{it} + i_t\hat{M}_{it} - Y_t(lm_i^t) - T_{it}\right)\mathrm{d}t\right] \leq M_{i0}. \tag{33}$$

The worker's problem is to choose a consumption stream $\{C_{it}\}_{t=0}^{\infty}$, labor market decisions $\{lm_{it}\}_{t=0}^{\infty}$, and money holdings $\{\hat{M}_{it}\}_{t=0}^{\infty}$ to maximize utility (32) subject to the budget constraint (33) at time 0.

Third, the economy is subject to shocks to the aggregate money supply M_t. We assume that the log of the aggregate money supply m_t follows a Brownian motion with drift π and volatility ζ:

$$\mathrm{d}m_t = \pi\,\mathrm{d}t + \zeta\,\mathrm{d}\mathcal{W}_t^m,$$

where $\mathcal{W}_t^m$ is a Wiener process.

Fourth and finally, we assume that the vacancy posting cost $K(Z_t)$ and the value of home production $B(Z_t)$ are both denominated in real terms.

Given these modifications, the market-clearing conditions for goods and money, respectively, are

$$\int_0^1 \left(C_{it} + \theta_{it}\mathbb{1}[E_{it} = u]K(Z_{it})\right)\mathrm{d}i = \int_0^1 \left(Z_{it}\mathbb{1}[E_{it} = h] + B(Z_{it})\mathbb{1}[E_{it} = u]\right)\mathrm{d}i, \tag{34}$$

$$\int_0^1 \hat{M}_{it}\,\mathrm{d}i = M_t, \tag{35}$$

where $\mathbb{1}[\cdot]$ is an indicator function that takes a logical expression as its argument. Equation (34) states that the sum of real consumption and recruiting expenses must equal the total market and home production of the good. Equation (35) states that the total demand of nominal money holdings across workers equals the aggregate money supply.

The following proposition characterizes the worker's problem in this monetary economy.

Proposition 6. *Let $Q_0 = 1$ be the numéraire and assume $\mu = \rho + \pi - \zeta^2/2$. Then, $P_t = M_t$ and the value of a worker at time 0 is*

$$V_0 = \max_{\{lm_{it}\}_{t=0}^{\infty}} \mathbb{E}_0 \left[\int_0^{\infty} e^{-\rho t} \frac{Y(lm_i^t)}{P_t} \, \mathrm{d}t \right] + k,$$

where k is a constant independent of the worker's choices, which captures the present discounted value of financial wealth.

Proposition 6 shows that the price level is equal to the aggregate money supply and that maximizing (32) subject to (33) is equivalent to maximizing expected discounted real income. The result relies on three assumptions: (i) markets are complete, (ii) workers have preferences that are quasi-linear in consumption, and (iii) the log of aggregate money supply follows a random walk with drift. The first two assumptions imply a constant marginal value of nominal wealth, which, combined with the last assumption, leads to a constant real interest rate and a one-for-one pass-through of money shocks to inflation.

Notation. The introduction of a monetary economy requires minor adjustments to our previous solution approach. Given fluctuations in the log price level p, the relevant state variable becomes the *real wage-to-productivity ratio* $\hat{w} := w - z - p$. All policies $(\hat{w}^+, \hat{w}^*, \hat{w}^-)$ are still expressed in real terms. In addition, it will be useful to keep track of the negative of a worker's cumulative shocks to *revenue productivity* $z + p$ since the beginning of the current employment or unemployment spell, which we denote by Δz. By definition, $\hat{w} = \hat{w}^* + \Delta z$ and the law of motion for Δz is

$$\mathrm{d}\Delta z = -(\gamma + \pi)\, \mathrm{d}t + \sigma \, \mathrm{d}\mathcal{W}_t^z + \zeta \, \mathrm{d}\mathcal{W}_t^m.$$

Let $G_h(\Delta z)$ and $g^h(\Delta z)$ denote the steady-state CDF and PDF of cumulative revenue productivity shocks z within job spells, respectively.[9] Note that the support of this distribution is given by $[-\Delta^-, \Delta^+]$, where $\Delta^- := \hat{w}^* - \hat{w}^-$ and $\Delta^+ := \hat{w}^+ - \hat{w}^*$. For any integer $k \in \mathbb{N}$, we define the moments of this distribution as $\mathbb{E}_h(\Delta z^k) := \int_{\Delta z} \Delta z^k \, \mathrm{d}G_h(\Delta z)$.

Our model implies a set of observable steady-state statistics. First, agents transition from employment to unemployment at rate s, from unemployment to employment at rate $f(\hat{w}^*)$, and total employment is $\mathcal{E}$. Second, the model implies a distribution of log nominal wage changes between consecutive job spells Δw and distributions of employment durations τ^m and unemployment durations τ^u. We use subscript $\mathcal{D}$ to denote moments of these distributions observed in the microdata—e.g., $\mathbb{E}_{\mathcal{D}}[\cdot]$ and $\mathbb{V}ar_{\mathcal{D}}[\cdot]$ denote the mean and the variance of this distribution, respectively. These moments will be useful to define sufficient

[9] In Online Appendix F, we show the Kolmogorov forward equations that characterize the steady-state distribution of cumulative productivity shocks z within employment and unemployment spells.

statistics for the effects of monetary shocks on aggregate labor market outcomes.

Discussion of the Monetary Environment. Previous literature has extensively studied the multifaceted effects nominal shocks have on the labor market. For example, monetary policy, through its effect on real interest rates, can affect match surplus and the overall labor market (Kehoe *et al.*, 2019, 2022). In this paper, we abstract from these important but well-understood considerations and focus on the effect of monetary policy "greasing the wheels of the labor market": By lowering real wages in the economy, an unexpected increase in the aggregate price level redistributes the surplus from workers to firms. Our goal is to understand how this redistribution affects the dynamics of wages, job creation, separations, and aggregate employment.

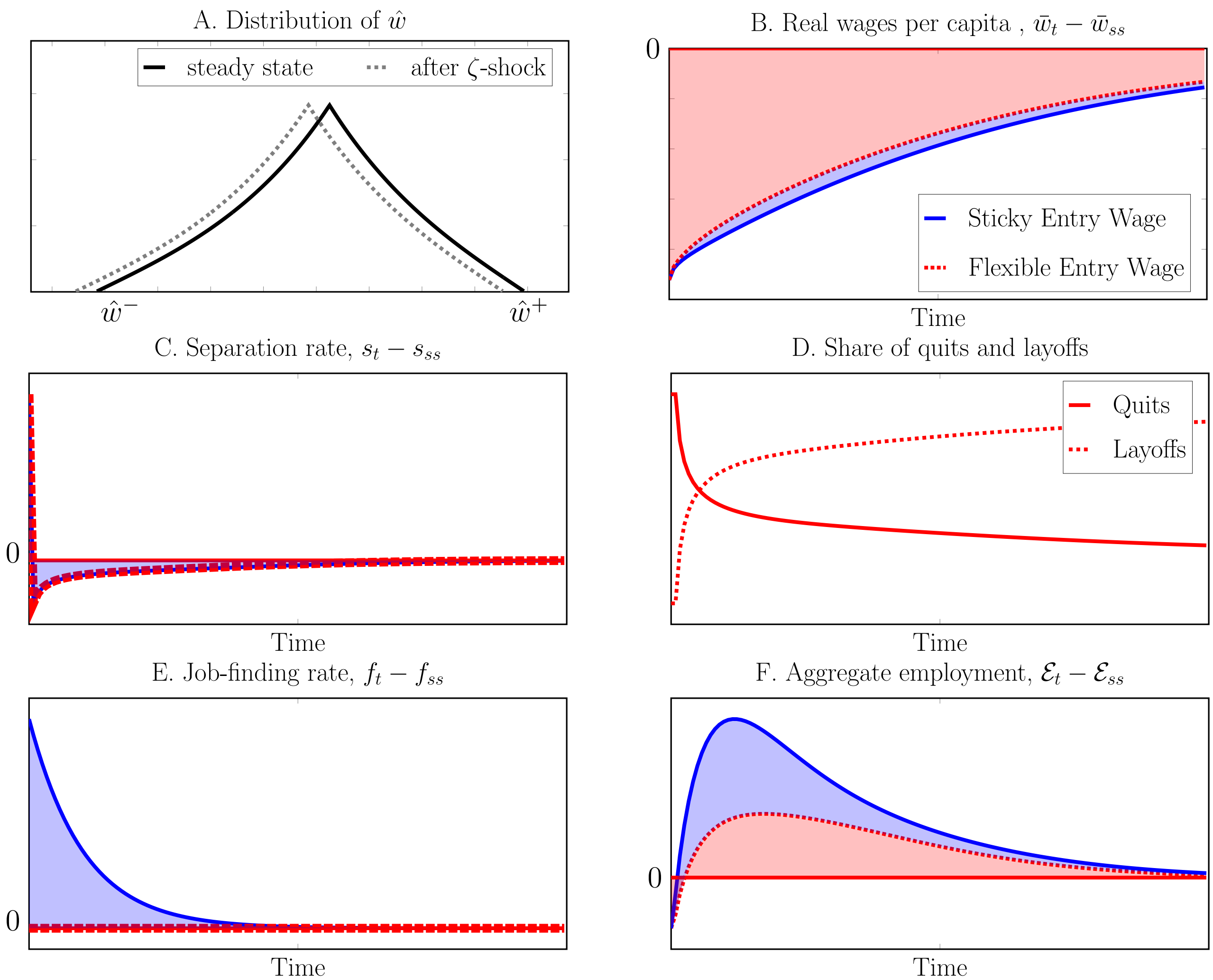
3.2 Monetary Multipliers for Aggregate Employment and Real Wages

Starting from the steady state without aggregate shocks, so that $\zeta = 0$, we consider a small and unanticipated shock $\zeta > 0$ to aggregate money supply at time $t = 0$—i.e., $\log(M_0) = \lim_{t \uparrow 0} \log(M_t) + \zeta$. This shock leads to a one-for-one increase in the price level. We are interested in the economy's *impulse response function (IRF)* and *cumulative impulse response (CIR)* of aggregate employment and real wages to such a monetary shock.[10]

An Illustration. Figure 2 shows the evolution of key variables after the sudden increase in the price level. The distribution of real wage-to-productivity ratios shifts to the left (Panel A), resulting in a lower average log real wage per capita $\bar{w}_t := \int_0^1 \mathbb{1}[E_{it} = h] w_{it} \, \mathrm{d}i$ (Panel B). In line with lower real wages among incumbents, the monetary shock affects the endogenous job-separation rate (Panel C), the relative importance of quits vs. layoffs (Panel D), and aggregate employment (Panel F). In addition, given that the wage level of new matches is a key determinant of the job-finding rate (Pissarides, 2009), we allow for two polar cases: flexible entry wages and sticky entry wages. With flexible entry wages, we assume that unemployed workers fully adjust their search behavior to incorporate the higher price level, so that $\hat{w}^*$ remains at its steady-state level. Therefore, aggregate log real wage is affected by the shock only because the nominal wages of incumbent workers are rigid. Since entry wages adjust one-for-one with the price level, the firm's real value of a filled vacancy is unaffected, so both vacancy-filling and job-finding rates remain at their steady-state levels (dashed line in Panel E). Thus, because the monetary shock changes

[10] By the certainty equivalence principle, the impulse response function following a money shock that departs from the steady state with steady-state policies is equivalent to the solution based on a first-order perturbation of the model with business cycle fluctuations.

FIGURE 2. IMPULSE RESPONSE FUNCTIONS OF LABOR MARKET VARIABLES



Notes: Panel A shows the distribution of real wage-to-productivity ratios $\hat{w} := \log(W_{it}/(Z_{it}P_t))$ in the steady state and after a monetary shock of size ζ. Panels B to F show the impulse response functions of the average log real per capita wage $\bar{w}_t$, the aggregate separation rates, the share of quits and layoffs in total separations, the job-finding rate $f_t - f_{ss}$, and aggregate employment $\mathcal{E}_t - \mathcal{E}_{ss}$, respectively.

the position of $\hat{w}$ in the inaction region, the employment response is determined by the effects of the aggregate shock on endogenous job separations.

In the sticky entry wage case, we allow the hiring margin to also be affected by assuming that unemployed workers do not adjust their search behavior to the higher price level. Instead, we assume that workers learn about the higher price level while employed, leading to a lower real entry wage until the worker's first job, after which it reverts to its steady-state level. As a consequence, temporarily lower entry wages induce firms to post more vacancies and the job-finding rate to increase (solid line in Panel E). Therefore, under sticky entry wages the employment dynamics are driven by both job-separation and job-finding rates. The assumption of sticky entry wages is motivated by the empirical evidence of Grigsby *et al.* (2021), who document that new-hire wages evolve similarly to incumbent workers within a firm at business cycle frequencies, and Hazell and Taska (2022), who show that wages for new hires rarely change between successive vacancies at the same job. Microfounding this assumption is outside the scope of this paper.[11] For notable models of rigid entry wages, see Fukui (2020) and Menzio (2022).

Defining IRFs and CIRs. Our goal is to characterize the effects of a monetary shock on aggregate employment $\mathcal{E}$ and aggregate real wages $\bar{w}$. To this end, we denote by $IRF_x(\zeta, t)$ the IRF for variable $x \in \{\mathcal{E}, \bar{w}\}$ at time t relative to its steady-state value, following a monetary shock ζ at time 0. The IRFs for aggregate employment and aggregate real wages are

$$IRF_{\mathcal{E}}(\zeta, t) = \mathcal{E}_t - \mathcal{E}_{ss}, \quad IRF_{\bar{w}}(\zeta, t) = \bar{w}_t - \bar{w}_{ss},$$

where $\mathcal{E}_{ss}$ and $\bar{w}_{ss}$ are the steady-state employment rate and average real per capita wage, respectively.

Following Alvarez *et al.* (2016), we define the CIR of a variable $x \in \{\mathcal{E}, \bar{w}\}$ to a monetary shock ζ as

$$CIR_x(\zeta) = \int_0^\infty IRF_x(\zeta, t)\, \mathrm{d}t,$$

which measures the area under the $IRF_x(\zeta, t)$ curve for $t > 0$. The CIR summarizes the on-impact response and the persistence of the response of the labor market to the monetary shock in a single scalar. Therefore, the CIR can be interpreted as a *monetary multiplier*. To illustrate the logic behind the CIR, suppose that there are no nominal rigidities so that the nominal wages of both newly hired and incumbent workers respond one-for-one to the price level. In this case, $IRF_x(\zeta, t) = 0$ for all t and thus $CIR_x(\zeta) = 0$ for $x \in \{\mathcal{E}, \bar{w}\}$, which reflects the fact that there are no real consequences of inflation. With nominal rigidities, an inflationary shock affects both employment and wages, the magnitude of which is given by the CIR.

Next, we characterize, up to a first-order approximation, the CIR of aggregate employment as a set of measurable objects in labor market microdata. The key insight is that the CIR of aggregate employment can be characterized only in terms of steady-state cross-sectional moments. Intuitively, changes in a

[11] Since the steady-state entry wage is optimal, any perturbation around that level has a second-order welfare effect on the worker. Thus, this assumption could be replaced by any first-order cost of entry wage adjustment. The literature offers a plethora of alternative microfoundations for these first-order adjustment costs based on imperfect knowledge about aggregate shocks. Examples include sticky information (Mankiw and Reis, 2002), rational inattention (Woodford, 2009; Maćkowiak and Wiederholt, 2009), dispersed knowledge (Hellwig *et al.*, 2014), and level-k reasoning (Farhi and Werning, 2019).

worker's productivity and changes in the aggregate price level affect the real wage-to-productivity ratio $W_{it}/(Z_{it}P_t)$ of a match in symmetric ways. Therefore, the response of a match to productivity changes in the steady state is informative of the aggregate effects of changes in the price level.

To convey the intuition more transparently, we assume $\pi = -\gamma$ for the remainder of this section. However, all propositions and proofs contained in the Online Appendix refer to the general case with $\pi \gtreqless -\gamma$. At the end of this section, we discuss the main differences with the general case.

CIR of Employment with Flexible Entry Wages. To facilitate exposition of the analysis, we first present the case with flexible entry wages. Proposition 7 characterizes the CIR up to a first order.[12]

Proposition 7. *Assume flexible entry wages. Up to first order, the CIR of employment is given by*

$$\frac{CIR_{\mathcal{E}}(\zeta)}{\zeta} = -(1-\mathcal{E}_{ss})\frac{\mathbb{E}_h[\Delta z]}{\sigma^2} + o(\zeta) \tag{36}$$

$$= \underbrace{\frac{1}{f(\hat{w}^*)}}_{\textit{avg. u dur.}} \underbrace{\frac{1}{\mathbb{V}ar_{\mathcal{D}}[\Delta w]}}_{\textit{dispersion}} \left[\underbrace{\frac{1}{3}\mathbb{E}_{\mathcal{D}}\left[\Delta w \frac{\Delta w^2}{\mathbb{E}_{\mathcal{D}}[\Delta w^2]}\right]}_{\textit{asymmetries}}\right] + o(\zeta). \tag{37}$$

To build the intuition behind this result, we first consider the implications of equation (36) in two cases in which the job-separation rate and aggregate employment do not respond to aggregate shocks. In the first case, all job separations are exogenous; therefore, the IRF of the job-separation rate is zero. In the second case, all job separations are endogenous, but the mass of additional workers who quit due to lower real wages is exactly compensated for by the mass of workers firms would have fired along the entire IRF (i.e., not just on impact) in the absence of the monetary shock. In both cases, the relevant sufficient statistic referenced by equation (36) is $\mathbb{E}_h[\Delta z] = 0$. Instead, $\mathbb{E}_h[\Delta z] < 0$ indicates a relatively larger incidence of layoffs than quits; therefore, a monetary policy-induced reduction in real wages reduces the separation rate, which increases aggregate employment along the IRF. Finally, notice that the CIR is scaled by the steady-state unemployment rate, $1 - \mathcal{E}_{ss}$, which is informative of workers' steady-state job-finding rate $f(\hat{w}^*)$ and their ability to quickly match with a firm again.

Next, we express $CIR_{\mathcal{E}}$ in terms of observable sufficient statistics of the distributions of wage changes and unemployment duration. Equation (37) shows that $CIR_{\mathcal{E}}$ is composed of three statistics: (i) the average duration of unemployment spells, (ii) the inverse of the dispersion of wage changes, and (iii) a measure of the asymmetry of the wage change distribution. Therefore, the relative mass of workers near the two job-separation triggers, which capture the immediate response of quits and layoffs, is not a

[12]That is, $CIR_x(\zeta) = CIR_x(0) + (CIR_x)'(0)\zeta + o(\zeta^2)$, where $CIR_x(0) = 0$.

sufficient statistic for characterizing the CIR of aggregate employment. Instead, the sufficient statistic is given by the product between the inverse of the job-finding rate, as explained above; the inverse of the dispersion of wage changes; and a measure of normalized skewness of the wage-change distribution. Intuitively, a larger dispersion of wage changes is indicative of a wider inaction region and the presence of more resilient matches to shocks. Thus, the larger this dispersion, the smaller the share of endogenous separations and the smaller the propagation of shocks. Finally, skewness measures how asymmetric the separation policies $\hat{w}^-$ and $\hat{w}^+$ are around the entry wage $\hat{w}^*$, which determines $\mathbb{E}_h[\Delta z]$. To illustrate, when the distribution of nominal wage changes is positively skewed, there is a large mass of workers who experience small wage cuts, which signals a relatively high layoff risk. Then, the higher price level reduces real wages and increases firms' incentives to keep their workers; as a result, aggregate employment increases. The opposite holds for a negatively skewed distribution of wage changes.

Proposition 7 also offers a new insight regarding the conventional wisdom whereby fluctuations in the job separation rate are not the primary driver of aggregate employment dynamics (e.g., Shimer, 2005b). In the context of a monetary shock, equation (36) points to conditions under which aggregate employment fluctuations due to endogenous job separations can be small. More importantly, it also highlights the conditions for these effects to be large,[13] and allows us to verify those conditions in the data. In light of this conventional wisdom, one might be tempted to conclude that sticky wages cannot lead to significant inefficiencies at the micro level. However, equation (36) allows for a small $CIR_{\mathcal{E}}$ despite the presence of inefficient separations at the micro level. Thus, aggregate time-series data cannot be used to discriminate between Coasean and non-Coasean theories of the labor market.

CIR of Employment with Sticky Entry Wages. Having characterized employment dynamics when entry wages are flexible, we now characterize the case with sticky entry wages.

Proposition 8. *Assume sticky entry wages. Up to first order, the CIR of employment is given by*

$$\frac{CIR_{\mathcal{E}}(\zeta)}{\zeta} = (1-\mathcal{E}_{ss})\left[-\frac{\mathbb{E}_h[\Delta z]}{\sigma^2} + \frac{1}{f(\hat{w}^*)+s}\left[\underbrace{\frac{1-\alpha}{\alpha}\left[\frac{\eta'(\hat{w}^*)}{(1-\eta(\hat{w}^*))} - \frac{\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{\rho})}{\mathcal{T}(\hat{w}^*,\hat{\rho})}\right]}_{\text{job-finding effect}} - \underbrace{\frac{\mathcal{T}'_{\hat{w}}(\hat{w}^*,0)}{\mathcal{T}(\hat{w}^*,0)}}_{\substack{\text{new-hires'}\\ \text{separation}\\ \text{effect}}}\right]\right] + o(\zeta) \tag{38}$$

[13] This sufficient statistic points to scenarios in which inefficient job separations matter for aggregate employment dynamics. For example, if trend inflation π is large in magnitude, then—all else equal—the rate of inefficient job separations will be more responsive to an inflationary shock. Alternatively, following a sequence of negative productivity shocks, an inflationary shock reduces the incidence of inefficient job separations due to firings (see Blanco *et al.*, 2022b, for empirical evidence consistent with this theoretical result).

$$= (1-\mathcal{E}_{ss})\left[-\frac{\mathbb{E}_h[\Delta z]}{\sigma^2}+\frac{1}{f(\hat{w}^*)+s}\left[\frac{\eta'(\hat{w}^*)}{\eta(\hat{w}^*)}+\frac{\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{\rho})}{\mathcal{T}(\hat{w}^*,\hat{\rho})}-\frac{\mathcal{T}'_{\hat{w}}(\hat{w}^*,0)}{\mathcal{T}(\hat{w}^*,0)}\right]\right]+o(\zeta). \tag{39}$$

Proposition 8 characterizes the new mechanisms that affect employment dynamics when entry wages are sticky. The elasticity of the firm's share of the surplus with respect to the entry wage (i.e., $\eta'(\hat{w}^*)/(1-\eta(\hat{w}^*)))$, together with the elasticity of the expected discounted duration of the match (i.e., $\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{\rho})/\mathcal{T}(\hat{w}^*,\hat{\rho})$), capture the effect of the increase in the job-finding probability following the decrease in the real entry wage. A drop in real entry wages increases the firm's share of the surplus and its incentive to post vacancies. In addition, a drop in the real entry wage could also change the expected duration and, therefore, the total surplus of the match, which also shapes the incentives to post vacancies for a given share. The last term captures the effect of a lower real entry wage on aggregate employment that arises from fluctuations in the job-separation rate of initially unemployed workers.

To further our understanding of the macroeconomic consequence of sticky entry wages, we combine (38) with the optimality condition for $\hat{w}^*$ in (26) to obtain (39). The goal of this step is to take advantage of the fact that workers internalize the effect of different entry wages on net job creation, which is the ultimate determinant of $CIR_{\mathcal{E}}$. To further characterize this expression, we show that the term $\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{\rho})/\mathcal{T}(\hat{w}^*,\hat{\rho})-\mathcal{T}'_{\hat{w}}(\hat{w}^*,0)/\mathcal{T}(\hat{w}^*,0)\approx 0$. While this property trivially holds when $\hat{\rho}\downarrow 0$, the following lemma shows that the elasticity of the expected duration to the entry wage is almost independent of the discount factor. We leave the full characterization of $\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{\rho})/\mathcal{T}(\hat{w}^*,\hat{\rho})$ to the Online Appendix.

Lemma 3. *Up to a second-order approximation of $\mathcal{T}(\hat{w},\hat{\rho})$ around $\hat{w}=\hat{w}^*$, and for all $\hat{\rho}$,*

$$\frac{\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{\rho})}{\mathcal{T}(\hat{w}^*,\hat{\rho})}=\frac{\Delta^+-\Delta^-}{\Delta^+\Delta^-}.$$

Thus, the key sufficient statistic for the effect of lower entry wages on job creation is $\eta'(\hat{w}^*)/\eta(\hat{w}^*)$. From this result, one may be inclined to conclude that the prevalence of inefficient separations cannot be an important determinant of aggregate job creation. However, we find the opposite result. The following proposition shows this by characterizing the elasticity of the worker's share to changes in the entry wage.

Proposition 9. *The following properties hold for* $\left.\frac{\mathrm{d}\log(\eta(\hat{w}))}{\mathrm{d}\hat{w}}\right|_{\hat{w}=\hat{w}^*}$*:*

1. *If $\Delta^+,\Delta^-\to\infty$, then*
$$\left.\frac{\mathrm{d}\log(\eta(\hat{w}))}{\mathrm{d}\hat{w}}\right|_{\hat{w}=\hat{w}^*}=\frac{\alpha+(1-\alpha)\hat{\rho}\hat{U}}{\alpha(1-\hat{\rho}\hat{U})}. \tag{40}$$

2. *Assume $\Delta^+=\Delta^-$, and Δ^+ is small enough, then*

$$\left.\frac{\mathrm{d}\log(\eta(\hat{w}))}{\mathrm{d}\hat{w}}\right|_{\hat{w}=\hat{w}^*}=\frac{\sqrt{s^{end}}}{2\alpha\sigma}. \tag{41}$$

We explain Proposition 9 with the help of Figure 3, which is constructed in two steps. First, we set $\delta = 0$ and calibrate the model to match the US economy's average job-finding and separation rates, together with a replacement ratio $\tilde{B}$ for newly employed workers of 0.46. We purposely choose α so that $\Delta^+ = \Delta^-$; thus, $\mathcal{T}'_{\hat{w}}(\hat{w}^*, \hat{\rho}) = 0$. Second, for different levels of the exogenous separation δ, we find the volatility of shocks σ that keeps the aggregate separation rate constant. The objective of this exercise is to change the fraction of endogenous separations s^{end}/s from 0 to 100 percent while keeping, by construction, the opportunity cost $\hat{\rho}\hat{U}$ and the aggregate separation rate fixed. Figure 3, Panels A and B show this "iso-separation rate curve" $\sigma(\delta)$ and the elasticity $\mathrm{d}\log(\eta(\hat{w}))/\,\mathrm{d}\hat{w}|_{\hat{w}=\hat{w}^*}$, respectively.

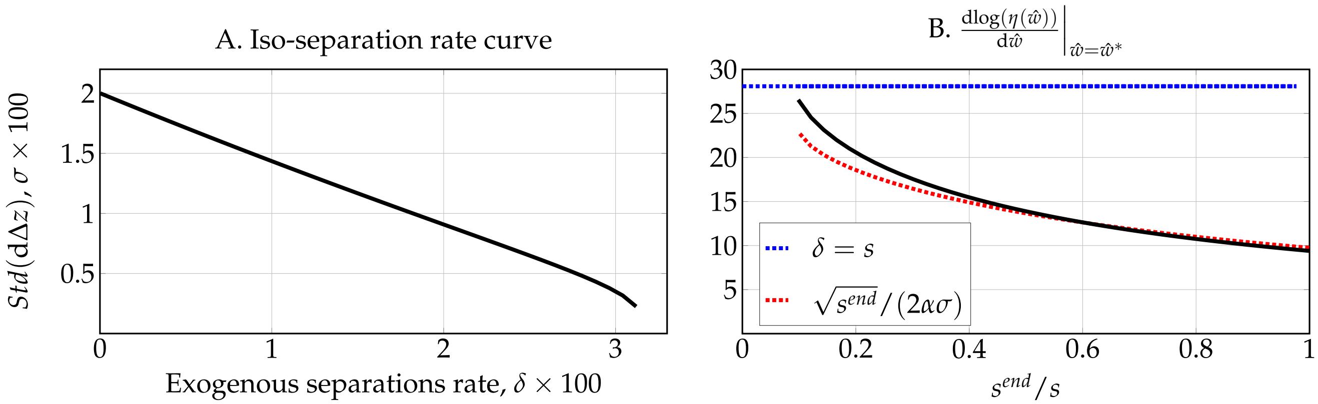
As a starting point, consider the limiting case with $\delta = s^{data}$ and $s^{end}/s = 0$. In this case, all separations are exogenous, and a marginal increase in the entry wage increases the worker's share of the surplus according to Equation (40). This reflects the well-known result whereby, in this limiting case, the elasticity of the share is proportional to the inverse of flow surplus $1 - \hat{\rho}\hat{U}$ (Shimer, 2005a). Figure 3, Panel B shows that when the share of inefficient separations increases, the elasticity of the worker's share to the entry wage decreases. In this case, a new mechanism that reduces the elasticity arises: With a higher entry wage, the probability that the worker gets fired increases and the probability that the worker chooses to quit decreases. By construction, the expected duration of the match does not change; thus, the match's joint surplus also does not change. In addition, because workers choose when to quit, the marginally lower probability of quitting does not affect their value (i.e., due to the envelope condition $\hat{W}'(\hat{w}^-) = 0$). Nevertheless, the marginal increase in the probability of being laid off reduces the worker's value, since the separation threshold is chosen by the firm. This renders the elasticity of the worker's share to the entry wage a decreasing function of the share of all separations that are endogenous. Equation (41) relates this elasticity to the prevalence of endogenous separations, which can be measured with labor market microdata as shown in the next section.

Discussion. We relegate several important results to Online Appendix E. There, we depart from the definition of the CIR and present an overview of the steps behind the proofs of Propositions 7 and 8. In addition, we extend the characterization of sufficient statistics to the general case with nonzero drift and trend inflation and present results for the CIR of the average real wage when entry wages are flexible.

4 Identifying the Model Based on Labor Market Microdata

This last section proves identification of the model in two steps. First, we show that the prevalence of inefficient job separations in our model critically depends on moments of the distribution of wage-to-

FIGURE 3. ELASTICITY OF WORKERS' SHARE FOR CONSTANT JOB-SEPARATION RATES AND DIFFERENT (δ, σ)



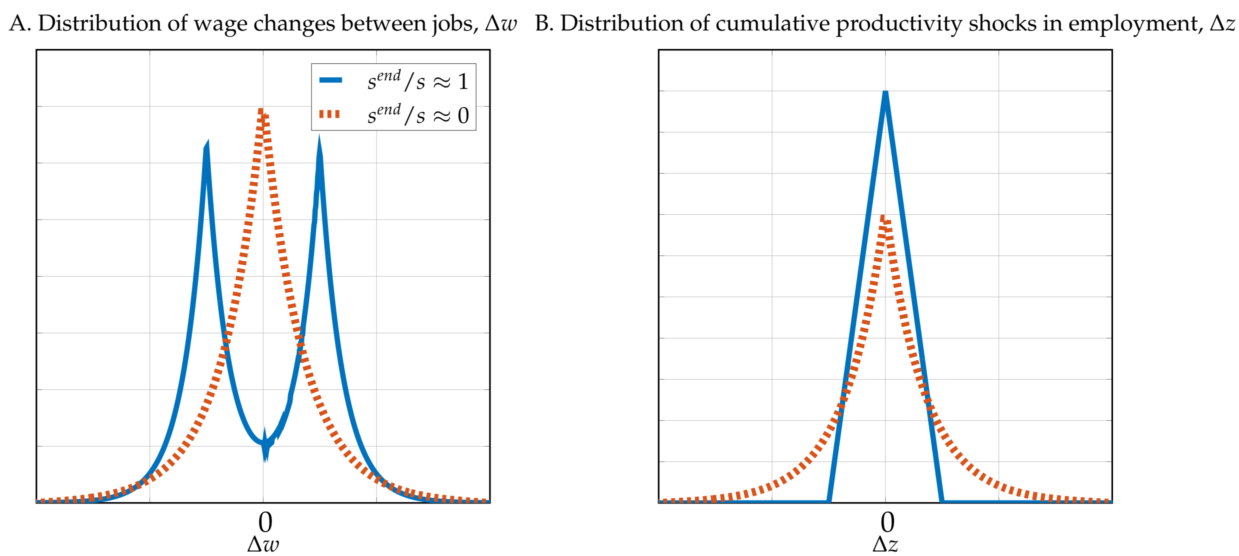
Notes: Panel A shows the level set of total job-separation rates for different values of (δ, σ). Panel B shows the elasticity of the worker's share with the entry wage and two theoretical limits when $\delta = s$ and $\delta = 0$, respectively. The parameter values for $\delta = 0$ are $(\gamma, \pi, \sigma, \rho, \alpha, \tilde{K}, \delta, \tilde{B}) = (0, 0, 0.02, 0.0033, 0.45, 2.2, 0, 0.45)$. The steady-state targets for this calibration are $(f(\hat{w}^*), s) = (0.45, 0.032)$ with a replacement ratio of 0.46 and $\Delta^+ = \Delta^- = 0$.

productivity ratios $\hat{w}$. Second, we demonstrate how to recover the unobserved distribution of wage-to-productivity ratios using microdata on wage changes and worker transitions between jobs. For clarity, here we focus on the case with $\gamma + \pi = 0$ and Online Appendix F presents the general results. We first provide the intuition of the identification strategy and then show the two necessary steps to recover $g^h(\Delta z)$ and inefficient separations.

Why are Wage Changes Informative about the Prevalence of Inefficient Separations? A critical insight is that the observed distribution of wage changes between jobs contains sufficient information to recover $g^h(\Delta z)$ and, therefore, the prevalence of endogenous job separations. We provide the intuition with the aid of Figure 4, which illustrates two possible marginal distributions of wage changes between jobs $l^w(\Delta w)$ in the left panel and the corresponding marginal distributions of cumulative productivity changes during employment $g^h(\Delta z)$ in the right panel. Each panel plots the distributions for two extreme calibrations: one that renders most job separations endogenous (solid blue line) and one that renders most job separations exogenous (dashed red line).

If, on the one hand, most job separations are due to endogenous quits and layoffs, then most separated workers must have experienced cumulative productivity shocks during employment of either $-\Delta^-$ or Δ^+. As a result, the probability mass associated with positive wage changes between jobs is concentrated around $-\Delta^-$, and the probability mass associated with negative wage changes is concentrated around Δ^+. This results in a bimodal distribution of wage changes between jobs, with additional dispersion around the two modes caused by cumulative productivity shocks during unemployment. If, on the other hand, most job separations are exogenous, then most separated workers experienced cumulative productivity shocks in employment close to zero. Because the job-finding probability is constant during unemployment, the distribution of wage changes between jobs mimics the distribution of cumulative productivity shocks in employment, which is symmetric and single peaked at zero. Therefore, after recovering the unobserved distribution $g^h(\Delta z)$, the mass of workers at the separation thresholds corresponds to the number of inefficient separations.

FIGURE 4. DISTRIBUTIONS OF WAGE CHANGES BETWEEN JOBS AND CUMULATIVE PRODUCTIVITY SHOCKS IN EMPLOYMENT



Notes: The figure plots the distribution of wage changes between jobs $l^w(\Delta w)$ and the distribution of cumulative worker shocks in employment $g^h(\Delta z)$ for two calibrations. In the first calibration, we set $(\Delta^-, \Delta^+, \gamma, \sigma, \delta, f(\hat{w}^*)) = (0.05, 0.05, 0, 0.02, 0, 0.5)$ so that $s^{end}/s \approx 1$ (blue solid line). In the second calibration, we set $(\Delta^-, \Delta^+, \gamma, \sigma, \delta, f(\hat{w}^*)) = (0.2, 0.2, 0, 0.1, 0.04, 0.05)$ so that $s^{end}/s \approx 0$ (red dashed line).

We formalize this identification argument in the next proposition.

Proposition 10. *The distribution of cumulative productivity shocks during employment $g^h(\Delta z)$ can be recovered in two steps:*

1. *The volatility of workers' productivity shocks is recovered from*

$$\sigma^2 = \frac{\mathbb{E}_{\mathcal{D}}[(\Delta w)^2]}{\mathbb{E}_{\mathcal{D}}[\tau^m + \tau^u]}. \tag{42}$$

2. *The distribution of workers' cumulative productivity shocks is recovered from*

$$g^h(\Delta z) = s\mathcal{E}\left[\int_{-\Delta^-}^{\Delta z} \frac{2(\Delta z - y)}{\sigma^2}\bar{g}^h(y)\,\mathrm{d}y + \bar{G}^h(-\Delta^-)\frac{2(\Delta z + \Delta^-)}{\sigma^2}\right], \tag{43}$$

where $\bar{G}^h(\Delta z)$ is the distribution of Δz conditional on a job separation. This distribution is given by

$$\bar{G}^h(\Delta z) = \frac{\sigma^2}{2f(\hat{w}^*)}\frac{\mathrm{d}l^w(-\Delta z)}{\mathrm{d}z} - \left[1 - L^w(-\Delta z)\right], \tag{44}$$

where $L^w(\Delta w)$ denotes the cumulative distribution function (CDF) corresponding to the observed distribution of wage changes between consecutive jobs $l^w(\Delta w)$.

Equation (42) shows that the volatility of productivity σ equals the dispersion of wage changes between jobs $\mathbb{E}_{\mathcal{D}}[(\Delta w)^2]$ divided by the average time elapsed between the starting dates of two consecutive jobs $\mathbb{E}_{\mathcal{D}}[\tau^m + \tau^u]$. To understand how to identify the distribution of Δz conditional on a job separation, we turn to the dynamics of h-to-u and u-to-h worker flows. Consider a worker who at time t_0 starts a job with wage w_{t_0}, at time $t_0 + \tau^m$ separates, and at time $t_0 + \tau^m + \tau^u$ finds a new job with wage $w_{t_0+\tau^m+\tau^u}$. This worker's wage change between jobs is given by

$$\Delta w = w_{t_0+\tau^m+\tau^u} - w_{t_0}, \tag{45}$$

$$= \underbrace{w_{t_0+\tau^m+\tau^u} - z_{t_0+\tau^m+\tau^u}}_{=\hat{w}^*} - \underbrace{(w_{t_0} - z_{t_0})}_{=\hat{w}^*} + \underbrace{z_{t_0+\tau^m+\tau^u} - z_{t_0},}_{=\Delta z \text{ after } h\text{-}u\text{-}h \text{ transition}} \tag{46}$$

$$= \underbrace{\hat{w}^* - \hat{w}^*}_{=0} + \underbrace{z_{t_0+\tau^m} - z_{t_0}}_{\Delta z | h\text{-}u \text{ transition starting from } z_{t_0}} + \underbrace{z_{t_0+\tau^m+\tau^u} - z_{t_0+\tau^m}.}_{\Delta z | u\text{-}h \text{ transition starting from } z_{t_0+\tau^m}} \tag{47}$$

Equation (45) starts from the definition of Δw. Then, equation (46) adds and subtracts $z_{t_0+\tau^h+\tau^u} - z_{t_0}$ before grouping terms into the wage-to-productivity ratio in the old job, the wage-to-productivity ratio in the new job, and the cumulative productivity shocks between the starting dates of the old and new jobs. Finally, equation (47) adds and subtracts $z_{t_0+\tau^m}$ before applying the definition of $\hat{w}^*$ and that of Δz. In summary, equations (45)–(47) show that the wage change across jobs is equal to the sum of three random variables: (i) the difference of entry wage-to-productivity ratios across jobs, which equals zero; (ii) Δz conditional on a job separation starting from productivity z_{t_0}; and (iii) Δz conditional on finding a new job, which is independent of productivity z_t for $t \in (t_0 + \tau^m, t_0 + \tau^m + \tau^u)$. Thus, by exploiting this independence, we can use data on Δw to infer the distribution of the second term, which is given by (44). Finally, the distribution of interest in (43) can be derived from (44) by exploiting the ergodicity of the productivity process (i.e., the cross-sectional distribution of cumulative shocks can be deduced from the distribution of shocks experienced during completed job spells).

We conclude this section with a brief discussion of the assumptions underlying the method described in Proposition 10. The first assumption is the threshold nature of job-separation policies, according to which the job-separation rate is equal to δ for $\Delta z \in (-\Delta^-, \Delta^+)$ and infinite for $\Delta z \in \{-\Delta^-, \Delta^+\}$. This assumption is not crucial and can be replaced with a general job-separation hazard as in Alvarez *et al.* (2020). The second assumption is the lack of other sources of wage adjustments, such as those arising from job-to-job transitions or wage adjustments within a job spell. This assumption could be relaxed following the methodology in Baley and Blanco (2022). Finally, while we assume a particular stochastic process for $d\Delta z_t$, this assumption can be empirically tested and adjusted if deemed necessary, as in Baley and Blanco (2021). For example, it would be straightforward to make the parameters of the productivity process depend on the worker's employment state. What is critical for the identification result in Proposition 10 to go through is that the data contain enough information to learn about the distribution of productivity changes during unemployment. In our model, the lack of selection in job finding and the preidentified stochastic process for Δz together yield this strong identification result.

5 Conclusion

There is mounting empirical evidence that wages are less than fully flexible. To understand the consequences of wage rigidity at the micro and macro levels, we developed a theory of non-Coasean labor markets. The realistic ingredients of this theory included fluctuations in individual output (i.e., productivity shocks), fixed pay within jobs (i.e., wage rigidity), and the possibility that workers can quit and firms can dissolve jobs at any time (i.e., two-sided lack of commitment). Our theory embedded these ingredients in an environment with search frictions, which are commonly viewed as central to the analysis of labor markets. We provide a first step in characterizing steady-state equilibrium policies and inefficient separations. Furthermore, we relate the labor market response to monetary shocks by providing sufficient statistics based on micro-data on wages and job flows.

While the parsimony of this framework is useful in delineating several novel theoretical insights, future work with quantitive objectives will need to incorporate extensions toward an empirically grounded framework. In our model, all endogenous separations are inefficient. Adding other sources of idiosyncratic risk (e.g., match-level shocks) or breaking the homotheticity assumptions (e.g., by including low-income workers with a significant relative value of home production) will yield efficient endogenous separations. In the current model, there is no opportunity for wage adjustments within the match. Adding asymmetric negotiation costs and on-the-job search would allow the model to match the asymmetric wage change distribution reported in the literature (Grigsby *et al.*, 2021; Blanco *et al.*, 2022a). Incorporating these features in our framework with empirical discipline will allow economists to assess the cost of non-Coasean labor markets and their business cycle and policy implications. Further examination of these extensions, specifically in an empirical context, is a crucial next step worthy of its own paper.

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A Theory of Non-Coasean Labor Markets

Online Appendix—Not for Publication

Andrés Blanco
University of Michigan

Andrés Drenik
University of Texas at Austin

Christian Moser
Columbia University and CEPR

Emilio Zaratiegui
Columbia University

Contents of the Online Appendix

A Auxiliary Theorems

A.1 Notation

We use the following mathematical notation throughout the appendices.

1. $H^l(\mathbb{R})$: Sobolev space; i.e., $H^l(\mathbb{R}) \subset L^2(\mathbb{R})$ and its weak derivatives up to order l have a finite L^p norm.
2. Characteristic operator $\mathcal{A}$: Given a function $f : \mathbb{R} \to \mathbb{R}$ and a diffusion process $\{x_t\}$, the characteristic operator of X is given by
$$\mathcal{A}f = \lim_{U \downarrow x} \frac{\mathbb{E}[f(X_{\tau_U}|x_0 = x] - f(x)]}{\mathbb{E}[\tau_U|x_0 = x]}$$
3. Let $u, v : \mathbb{R} \to \mathbb{R}$, $(u, v) = \int_{\mathbb{R}} u(x)v(x)\,\mathrm{d}x$ and $||u|| = \left(\int u(x)^2\,\mathrm{d}x\right)^{1/2}$.
4. $a(u, v)$ is a bilinear continuous form. We say $a(u, v)$ is coercive if $a(u, u) \geq \alpha||u||^2$.
5. We use $a \wedge b$ to denote the minimum between a and b.

A.2 Some Useful and Known Results

Proposition A.1. *Let $\mathcal{A}$ be the characteristic operator of $\{X_t\}$ with $X_t \in \mathbb{R}^n$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a twice differentiable function with compact (i.e., bounded and closed in $\mathbb{R}$) support $(support(f) = \{x : f(x) \neq 0\})$. If τ is a stopping time with $\mathbb{E}_x[\tau] < \infty$, then*

$$\mathbb{E}_x[f(x_\tau)] = f(x) + \mathbb{E}_x\left[\int_0^\tau \mathcal{A}f(X_t)\,\mathrm{d}t\right]. \tag{A.1}$$

Moreover, if τ is the first exit time of a bounded set, then (A.1) holds for any twice differentiable function.

Proof. This is Dynkin's formula, the proof of which can be found in Øksendal (2007). □

Proposition A.2. *Let x_t be a strong Markov process, τ be a stopping time measurable with the filtration generated by x_t, and τ^δ an exponential random variable independent of τ. Then*

$$\mathbb{E}\left[\int_0^{\tau\wedge\tau^\delta} e^{-\rho t} f(x_t)\,\mathrm{d}t + e^{-\rho(\tau\wedge\tau^\delta)} g(x_{\tau\wedge\tau^\delta}) \middle| x_0 = x\right] = \mathbb{E}\left[\int_0^\tau e^{-(\rho+\delta)t}[f(x_t) + \delta g(x_t)]\,\mathrm{d}t + e^{-(\rho+\delta)\tau} g(x_\tau) \middle| x_0 = x\right].$$

Proposition A.3. *Let V be a Hilbert space and P a closed convex cone of V satisfying $P = \{x \in V : (x, y) \geq 0\ \forall y \in P\}$. Let T be an increasing map from V to itself such that there exists a $\underline{x}, \overline{x} \in V$*

$$\underline{x} \leq \overline{x},\ \ \underline{x} \leq T(\underline{x}),\ \ T(\overline{x}) \leq \overline{x}.$$

Then, the subset of fixed points x^* of T satisfying $\underline{x} \leq x^* \leq \overline{x}$ is non-empty and has a larger and smallest element.

Proof. See the proof of Proposition 2 of Chapter 15 on page 539 of Aubin (2007). □

Proposition A.4. *Let V be a Hilbert space and P a closed convex set. Assume that $a(u, v)$ with $u, v \in V$ is a coercive bilinear continuous form. Then, there exists a unique solution to*

$$a(u, v - u) \geq (f, v - u), \forall v \in P, u \in P,$$

where f belongs to the dual of V.

Proof. See Lions and Stampacchia (1967). □

B Proofs for Section 2: A Model of Non-Coasean Labor Contracts

B.1 Proof of Lemma 1

To simplify the exposition, we divide the proof into a sequence of lemmas. Define the equilibrium conditions

$$\rho u(z) = \tilde{B}e^z + \gamma \frac{\partial u(z)}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 u(z)}{\partial z^2} + \max_w f(w,z)[h(z;w) - u(z)], \ \ \forall z \in \mathbb{R} \tag{B.1}$$

$$0 = \left[\tilde{K}e^z - q(w,z)j(z;w)\right]^+ \theta(w,z) \quad \forall (w,z) \in \mathbb{R}^2 \tag{B.2}$$

$$h(z;w) \geq u(z), \ \ \forall z \in \mathbb{R}, \tag{B.3}$$

$$j(z;w) \geq 0, \ \ \forall z \in \mathbb{R} \tag{B.4}$$

$$\text{If } z \in (\mathcal{C}^j(w))^c \Rightarrow h(z;w) = u(z), \tag{B.5}$$

$$\text{If } z \in (\mathcal{C}^h(w))^c \Rightarrow j(z;w) = 0, \tag{B.6}$$

$$0 = \max\{u(z) - h(z;w), \mathcal{A}^h h(z;w) + e^w\}, \ \forall z \in \mathcal{C}^j(w), h(\cdot;w) \in \mathbb{C}^1(\mathcal{C}^j(w)) \cap \mathbb{C}(\mathbb{R}), \tag{B.7}$$

$$0 = \max\{-j(z;w), \mathcal{A}^j j(z;w) + e^z - e^w\}, \ \forall z \in \mathcal{C}^h(w), j(\cdot;w) \in \mathbb{C}^1(\mathcal{C}^h(w)) \cap \mathbb{C}(\mathbb{R}), \tag{B.8}$$

$$\mathcal{C}^h(w) := int\left\{z \in \mathbb{R} : h(z;w) > u(z) \ or \ \mathcal{A}^h u(z) + e^w > 0\right\}, \tag{B.9}$$

$$\mathcal{C}^j(w) := int\left\{z \in \mathbb{R} : j(z;w) > 0 \ or \ e^z - e^w > 0\right\}, \tag{B.10}$$

$$\mathcal{A}^h(f(z)) := -\rho f + \delta(u(z) - f(z)) + \gamma \frac{\partial f(z)}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 f(z)}{\partial z^2}$$

$$\mathcal{A}^j(f(z)) := -\rho f + \delta(0 - f(z)) + \gamma \frac{\partial f(z)}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 f(z)}{\partial z^2}$$

Proposition B.1. *Let $x := (w,z)$. If there exist two functions $h(z;w)$ and $j(z;w)$ satisfying (B.3), (B.4), (B.6), (B.5), (B.7) and (B.8) given the continuation sets (B.9) and (B.10), then*

$$\tau^{h*}(x) = \inf\left\{t \geq 0 : z_t \notin \mathcal{C}^h(w)\right\}$$
$$\tau^{j*}(x) = \inf\left\{t \geq 0 : z_t \notin \mathcal{C}^j(w)\right\}$$

form a non-trivial Nash equilibrium and

$$h(z;w) = H(x, \tau^{h*}(x), \tau^{j*}(x), \tau^\delta), j(z;w) = J(x, \tau^{h*}(x), \tau^{j*}(x), \tau^\delta).$$

Moreover, if $(\tau^{h*}(x), \tau^{j*}(x))$ is a non-trivial Nash equilibrium, then

$$h(z;w) = H(x, \tau^{h*}(x), \tau^{j*}(x), \tau^\delta), j(z;w) = J(x, \tau^{h*}(x), \tau^{j*}(x), \tau^\delta).$$

satisfy (B.3) to (B.8).

Proof. **Quasi-variational inequalities as sufficient conditions.** First, we prove that if $h(z;w)$ and $j(z;w)$ satisfy (B.3) to (B.8), then

$$h(z;w) = H(x, \tau^{h*}(x), \tau^{j*}(x), \tau^\delta) \geq H(x, \tau^h(x), \tau^{j*}(x), \tau^\delta)$$

for any $\tau^h \in \mathcal{T}$. The proof of the statement

$$j(z;w) = J(x, \tau^{h*}(x), \tau^{j*}(x), \tau^{\delta}) \geq J(x, \tau^{h*}(x), \tau^{j}(x), \tau^{\delta}),$$

for any $\tau^j \in \mathcal{T}$, follows the same arguments.

Step 1: Here, we show that $h(z;w) \geq H(x, \tau^h(x), \tau^{j*}(x), \tau^{\delta})$. Let τ^h be any stopping time (not necessarily the optimal). Without loss of generality, we restrict the attention to $\tau^h \leq \tau_{(-\infty,a)}$, where $\tau_{(-\infty,a)} = \inf\{t > 0 : z_t \notin (-\infty, a)\}$. Intuitively, it is never optimal for the worker to stay in the job at wage w when productivity is sufficiently large. Let $U_k \subset \mathbb{R}$ be an increasing sequence of bounded sets s.t. $\cup_{k=1}^{\infty} U_k = \mathbb{R}$. Let $\tau_k = \inf\{t > 0 : z_t \notin U_k\}$. Since each U_k is bounded, we do not need to assume compact support of the function to apply Proposition A.1. Applying Dynkin's Lemma to the stopping time $\tau_k^h = \tau^h \wedge \tau^{j*} \wedge \tau^{\delta} \wedge \tau_k$,

$$\mathbb{E}[e^{-\rho\tau_k^h} h(x_{\tau_k^h})|z_0 = z] = h(z;w) + \mathbb{E}\left[\int_0^{\tau_k^h} \mathcal{A}^h h(z_t;w)\,\mathrm{d}t | z_0 = z\right].$$

Using condition (B.3), since $h(z;w) \geq u(z)$ for all z, we have that $\mathbb{E}[e^{-\rho\tau_k^h} h(z_{\tau_k^h};w)|z_0 = z] \geq \mathbb{E}[e^{-\rho\tau_k^h} u(z_{\tau_k^h})|z_0 = z]$. Thus,

$$\mathbb{E}[e^{-\rho\tau_k^h} u(z_{\tau_k^h})|z_0 = z] - \mathbb{E}\left[\int_0^{\tau_k^h} \mathcal{A}^h h(z_t;w)\,\mathrm{d}t | z_0 = z\right] \leq h(z;w).$$

From condition (B.7) , we have $\mathcal{A}^h h(z;w) + e^w \leq 0$ for all z. Thus,

$$\mathbb{E}\left[\int_0^{\tau_k^h} e^{-\rho t} e^w\,\mathrm{d}t | z_0 = z\right] \leq -\mathbb{E}\left[\int_0^{\tau_k^h} \mathcal{A}^h h(z;w)\,\mathrm{d}t | z_0 = z\right].$$

Using this result

$$\mathbb{E}\left[e^{-\rho\tau_k^h} u(z_{\tau_k^h}) + \int_0^{\tau_k^h} e^{-\rho t} e^w\,\mathrm{d}t | z_0 = z\right] \leq h(z;w)$$

Now, we take the limit $k \to \infty$. It is easy to see that $\int_0^{\tau^h\wedge\tau^{j*}\wedge\tau^{\delta}\wedge\tau_k} e^{-\rho t + w}\,\mathrm{d}t \leq \frac{1}{\rho} e^w$ a.e., so using the dominated convergence theorem $\lim_{k\to\infty} \mathbb{E}\left[\int_0^{\tau^h\wedge\tau^{j*}\wedge\tau^{\delta}\wedge\tau_k} e^{-\rho t + w}\,\mathrm{d}t | z_0 = z\right] = \mathbb{E}\left[\int_0^{\tau^h\wedge\tau^{j*}\wedge\tau^{\delta}} e^{-\rho t + w}\,\mathrm{d}t | z_0 = z\right]$.

As we show below, $u(z) \propto e^z$ and since $e^{z_t} \leq e^a$ for all $t \leq \tau^h \leq \tau_{(-\infty,a)}$, we have that $0 \leq e^{-\rho t} u(z_t) \leq e^a$. Applying the monotone convergence theorem again, we have that

$$\lim_{k\to\infty} \mathbb{E}\left[e^{-\rho(\tau^h\wedge\tau^{j*}\wedge\tau^{\delta}\wedge\tau_k)} u(z_{\tau^h\wedge\tau^{j*}\wedge\tau^{\delta}\wedge\tau_k})|z_0 = z\right] = \mathbb{E}\left[e^{-\rho(\tau^h\wedge\tau^{j*}\wedge\tau^{\delta})} u(z_{\tau^h\wedge\tau^{j*}\wedge\tau^{\delta}})|z_0 = z\right].$$

Therefore, taking the limit $k \to \infty$, we finally obtain

$$h(z;w) \geq H(x, \tau^h(x), \tau^{j*}(x), \tau^{\delta}).$$

Step 2: Now, we show that $h(z;w) = H(x, \tau^{h*}(x), \tau^{j*}(x), \tau^{\delta})$. Applying Proposition A.1 to the stopping time $\tau_k^{h*} = \tau^{h*} \wedge \tau^{j*} \wedge \tau_k \wedge \tau^{\delta}$ we obtain

$$\mathbb{E}[e^{-\rho\tau_k^{h*}} h(z_{\tau_k^{h*}};w)|z_0 = z] = h(z;w) + \mathbb{E}\left[\int_0^{\tau_k^{h*}} \mathcal{A}^h h(z_t;w)\,\mathrm{d}t | z_0 = z\right].$$

For all $t < \tau_k^{h*}$, we have that $u(z) < h(z;w)$. Therefore, by (B.7), $\mathcal{A}^h h(z;w) + e^w = 0$ for all z. Thus,

$$\mathbb{E}\left[e^{-\rho\tau_k^{h*}} h(z_{\tau_k^{h*}};w) + \int_0^{\tau_k^{h*}} e^{-\rho t} e^w \, \mathrm{d}t | z_0 = z\right] = h(z;w).$$

Taking the limit $k \to \infty$ and following similar arguments as above, we obtain

$$\mathbb{E}\left[e^{-\rho(\tau^{h*}\wedge\tau^{j*}\wedge\tau^{\delta})} h(z_{\tau^{h*}\wedge\tau^{j*}\wedge\tau^{\delta}};w) + \int_0^{\tau^{h*}\wedge\tau^{j*}\wedge\tau^{\delta}} e^{-\rho t} e^w \, \mathrm{d}t | z_0 = z\right] = h(z;w).$$

which, given Proposition A.2, is equivalent to

$$\mathbb{E}\left[e^{-(\rho+\delta)(\tau^{h*}\wedge\tau^{j*})} h(z_{\tau^{h*}\wedge\tau^{j*}};w) + \int_0^{\tau^{h*}\wedge\tau^{j*}} e^{-(\rho+\delta)t} (\delta u(z_t) + e^w) \, \mathrm{d}t | z_0 = z\right] = h(z;w).$$

Since $z_{\tau^{h*}\wedge\tau^{j*}} \in \partial(\mathcal{C}^h(w^*(z)) \cap \mathcal{C}^j(w^*(z)))$ and $h(\cdot;w)$ is continuous, we have that

$$\mathbb{E}\left[e^{-(\rho+\delta)(\tau^{h*}\wedge\tau^{j*})} u(z_{\tau^{h*}\wedge\tau^{j*}};w) + \int_0^{\tau^{h*}\wedge\tau^{j*}} e^{-(\rho+\delta)t} (\delta u(z_t) + e^w) \, \mathrm{d}t | z_0 = z\right] = h(z;w).$$

and $h(z;w) = H(x, \tau^{h*}(x), \tau^{j*}(x), \tau^{\delta})$.

Quasi-variational inequalities as necessary conditions. Now, we prove that if $\tau^{h*}(x)$ and $\tau^{j*}(x)$ is a non-trivial Nash equilibrium, then $h(z;w)$, $j(z;w)$ satisfy (B.3) to (B.10). It is easy to show that in a Nash equilibrium, the stopping time is Markovian—if any agent chooses to stop, then the game finishes. By definition, we have that

$$h(z;w) = \max_{\tau^h} \mathbb{E}\left[\int_0^{\tau^h\wedge\tau^{j*}\wedge\tau^{\delta}} e^{-\rho t + w} \, \mathrm{d}t + e^{-\rho(\tau^h\wedge\tau^{j*}\wedge\tau^{\delta})} u(z_{\tau^h\wedge\tau^{j*}\wedge\tau^{\delta}};w) \, \mathrm{d}t | z_0 = z\right]. \tag{B.11}$$

- Condition (B.3): We show it by contradiction. Assume that $h(z;w) < u(z)$. Then $\tau^h(x) = 0$, implies

  $$u(z) > h(z;w) \geq H(w, z, 0, \tau^{j*}(x), \tau^{\delta}) = \mathbb{E}_0\left[\int_0^0 e^{-\rho t} e^w \, \mathrm{d}t + e^{-\rho\tau^m} u(z_0) | z_0 = z\right] = u(z),$$

  so we have a contradiction.

- Condition (B.5): If $z \in (\mathcal{C}^j(w))^c$, then $\tau^{j*}(x) = 0$ and $Pr[\min\{\tau^{h*}(x), \tau^{j*}(x), \tau^{\delta}(x)\} \leq \tau^{j*}(x)] = 1$, and $h(z;w) = u(z)$.

- Condition (B.7): Observe that this condition is the best response of the worker, given that the firm continues. See Øksendal (2007) and Brekke and Øksendal (1990) for a discussion of the necessity of the smooth pasting condition.

- Condition (B.9): To show this, we need to characterize the continuation set in the Nash equilibrium that survives the iterated elimination of weakly dominated strategies. First, from the problem (B.11), if $\Pr(\tau^{j*}(x) > 0) = 1$, then $\Pr(\tau^{h*}(x) > 0) = 1$ if and only if

  $$z \in int\left\{z \in \mathbb{R} : h(z;w) > u(z)\right\}.$$

  Second, we show that staying in the match weakly dominates leaving it if

  $$0 < e^w + \mathcal{A}^h u(z), \tag{B.12}$$

  for all z. Take any stopping time τ such that $\Pr(\tau > 0 | z_0 = z) = 1$. Then, applying Dynkin's Lemma (and using similar arguments as in Step 1), we obtain

$$\mathbb{E}\left[e^{-\rho\tau}u(z_\tau)|z_0=z\right]=u(z)+\mathbb{E}\left[\int_0^\tau \mathcal{A}u(z_t)\,\mathrm{d}t|z_0=z\right].$$

Using the inequality in (B.12),

$$u(z)=\mathbb{E}\left[e^{-\rho\tau}u(z_\tau)|z_0=z\right]-\mathbb{E}\left[\int_0^\tau \mathcal{A}u(z_t)\,\mathrm{d}t|z_0=z\right]<\mathbb{E}\left[e^{-\rho\tau}u(z_\tau)|z_0=z\right]+\mathbb{E}\left[\int_0^\tau e^{-\rho t+w}\,\mathrm{d}t|z_0=z\right].$$

Thus, staying in the match strictly dominates dissolving the match.

□

Proposition B.2. *Define*

$$w^*(z)=\arg\max_w \theta(x)^{1-\alpha}(h(z;w)-u(z)).$$

and $\tau^{u*}=\inf\{t\geq 0:\Delta N_t^{f(w^*(z_t),z_t)}=1\}$ *where* $N_t^{f(w^*(z_t),z_t)}$ *is a Poisson counter with arrival rate* $f(w^*(z_t),z_t)$*. The function* $u(z)$ *satisfies* $u(z)\in\mathbb{C}^2(\mathbb{R})$ *and* (B.1) *if and only if*

$$u(z)=\max_{\{w_t\}_{t=0}^{\tau^u}}\mathbb{E}\left[\int_0^{\tau^u} e^{-\rho t}B(z_t)\,\mathrm{d}t+e^{-\rho\tau^u}h(z_{\tau^u};w)\right].$$

Proof. The proof is the standard optimality conditions in the HJB (see Øksendal, 2007). □

Lemma 1. *Assume* $u(z)$, $h(z;w)$, $j(z;w)$, $\theta(z;w)$ *satisfy* (B.1)—(B.8) *given the continuation sets* (B.9) *and* (B.10)*. Then* $\left\{\tau^{h*},\tau^{j*},\{w_t^*\}_{t=0}^{\tau^u}\right\}$ *constructed with*

$$\tau^{h*}(x)=\inf\left\{t\geq 0:z_t\notin\mathcal{C}^h(w)\right\}$$
$$\tau^{j*}(x)=\inf\left\{t\geq 0:z_t\notin\mathcal{C}^j(w)\right\}$$
$$w^*(z)=\arg\max_w \theta(x)^{1-\alpha}(h(z;w)-u(z)).$$

is a BRE with

$$h(z;w)=H(x,\tau^{h*}(x),\tau^{j*}(x),\tau^\delta),$$
$$j(z;w)=J(x,\tau^{h*}(x),\tau^{j*}(x),\tau^\delta),$$
$$u(z)=U(z).$$

If $\{H(w,z,\vec{\tau}^m),J(w,z,\vec{\tau}^m),U(z)\}$*, market tightness* $\theta(w,z)$*, and policy functions* $\left\{\tau^{h*}(w,z),\tau^{j*}(w,z),w^*(z_t)\right\}$ *is a BRE with*

$$h(z;w)=H(x,\tau^{h*}(x),\tau^{j*}(x),\tau^\delta),$$
$$j(z;w)=J(x,\tau^{h*}(x),\tau^{j*}(x),\tau^\delta),$$
$$u(z)=U(z).$$

then $u(z)$, $h(z;w)$, $j(z;w)$, $\theta(z;w)$ *satisfy* (B.1)—(B.8) *given the continuation sets* (B.9) *and* (B.10)*.*

Proof. The proof is a combination of Propositions B.1 and B.2. □

B.2 Proof of Lemma 2

For the next proof, it will be useful to define the normalized equilibrium conditions

$$\hat{\rho}\hat{U} = \tilde{B} + \max_{\hat{w}^*} \hat{\theta}(\hat{w}^*)^{1-\alpha}\hat{W}(\hat{w}^*), \tag{B.13}$$

$$0 = \left[\tilde{K} - \hat{\theta}(\hat{w})^{-\alpha}\hat{J}(\hat{w})\right]^+ \hat{\theta}(\hat{w}), \tag{B.14}$$

$$\hat{W}(\hat{w}) \geq 0, \tag{B.15}$$

$$\hat{J}(\hat{w}) \geq 0, \tag{B.16}$$

$$if\ \hat{w} \in (\hat{\mathcal{C}}^j)^c \Rightarrow \hat{W}(\hat{w}) = 0, \tag{B.17}$$

$$if\ \hat{w} \in (\hat{\mathcal{C}}^h)^c \Rightarrow \hat{J}(\hat{w}) = 0, \tag{B.18}$$

$$0 = \max\{-\hat{W}(\hat{w}), \hat{\mathcal{A}}\hat{W}(\hat{w}) + e^{\hat{w}} - \hat{\rho}\hat{U}\},\ \forall \hat{w} \in \hat{\mathcal{C}}^j, \hat{W} \in \mathbb{C}^1(\hat{\mathcal{C}}^j) \cap \mathbb{C}(\mathbb{R}) \tag{B.19}$$

$$0 = \max\{-\hat{J}(\hat{w}), \hat{\mathcal{A}}\hat{J}(\hat{w}) + 1 - e^{\hat{w}}\},\ \forall \hat{w} \in \hat{\mathcal{C}}^h, \hat{J} \in \mathbb{C}^1(\hat{\mathcal{C}}^h) \cap \mathbb{C}(\mathbb{R}) \tag{B.20}$$

$$\hat{\mathcal{C}}^h := int\left\{\hat{w} \in \mathbb{R} : \hat{W}(\hat{w}) > 0\ or\ (e^{\hat{w}} - \hat{\rho}\hat{U}) > 0\right\}, \tag{B.21}$$

$$\hat{\mathcal{C}}^j := int\left\{\hat{w} \in \mathbb{R} : \hat{J}(\hat{w}) > 0\ or\ \left(1 - e^{\hat{w}}\right) > 0\right\}, \tag{B.22}$$

$$\hat{\mathcal{A}}(f) := -(\hat{\rho} + \delta)f - \hat{\gamma}\frac{\partial f(\hat{w})}{\partial \hat{w}} + \frac{\sigma^2}{2}\frac{\partial^2 f(\hat{w})}{\partial \hat{w}^2},$$

where $\hat{w} = w - z$, $\hat{\rho} = \rho - \gamma - \sigma^2/2$ and $\hat{\gamma} = \gamma + \sigma^2$.

Lemma 2. *Assume that $(h(z;w), j(z;w), u(z), \theta(x), w^*(z))$ satisfy conditions (B.1) to (B.10), then*

$$\left(\hat{U}, \hat{J}(w-z), \hat{W}(w-z), \hat{\theta}(w-z), \hat{w}^*\right) = \left(\frac{u(z)}{e^z}, \frac{j(z;w)}{e^z}, \frac{h(z;w) - u(z)}{e^z}, \theta(w,z), w^*(z) - z\right).$$

satisfy (B.13) to (B.22). Moreover, if $\left(\hat{U}, \hat{J}(w-z), \hat{W}(w-z), \hat{\theta}(w-z)\right)$ satisfy (B.13) to (B.22), then

$$(u(z), j(z;w), h(z;w), \theta(w,z), w^*(z)) = \left(\hat{U}e^z, \hat{J}(w-z)e^z, (\hat{W}(w-z) + \hat{U})e^z, \hat{\theta}(w-z), \hat{w}^* + z\right)$$

satisfy (B.1) to (B.10).

Proof. The general idea for the proof is to use a guess-and-verify strategy for each equilibrium condition.

Condition (B.1) holds if and only if (B.13) is satisfied: Using $\hat{U} = \frac{u(z)}{e^z}$, we have that

$$\hat{U}e^z = u'(z) \text{ and } \hat{U}e^z = u''(z).$$

Using this result and the fact that $\theta(w,z) = \hat{\theta}(w-z)$, and $\hat{W}(w-z) = \frac{h(z;w)-u(z)}{e^z}$,

$$\rho u(z) = \tilde{B}e^z + \gamma u'(z) + \frac{\sigma^2}{2}u''(z) + \max_{w(z)} \theta(w,z)^{1-\alpha}[h(z;w) - u(z)] \iff$$

$$\rho\hat{U}e^z = \tilde{B}e^z + \gamma\hat{U}e^z + \frac{\sigma^2}{2}\hat{U}e^z + \max_{w(z)} \hat{\theta}\left(w-z\right)^{1-\alpha}\left[\hat{W}(w-z)e^z\right] \iff$$

$$\left(\rho - \gamma - \frac{\sigma^2}{2}\right)\hat{U}e^z = \tilde{B}e^z + e^z \max_{\hat{w}} \hat{\theta}\left(\hat{w}\right)^{1-\alpha}\hat{W}\left(\hat{w}\right) \iff \hat{\rho}\hat{U} = \tilde{B} + \max_{\hat{w}} \hat{\theta}\left(\hat{w}\right)^{1-\alpha}\hat{W}\left(\hat{w}\right).$$

Condition (B.2) holds if and only if (B.14) is satisfied: Using that $\hat{J}(w-z) = \frac{j(z;w)}{e^z}$, $\theta(w,z) = \hat{\theta}(w-z)$, and the assumption

$K(z) = \tilde{K}e^z$, we have that

$$[\tilde{K}e^z - q(\theta(w,z))j(z;w)]^+ \theta(w,z) = 0 \iff [\tilde{K}e^z - q(\theta(w,z))\hat{J}(w-z)e^z]^+ \theta(w,z) = 0$$
$$\iff [\tilde{K} - q(\hat{\theta}(\hat{w}))\hat{J}(\hat{w})]\,\hat{\theta}(\hat{w}) = 0.$$

Since $q(\theta(w,z)) = q(\hat{\theta}(\hat{w}))$, we have the result.

Condition (B.7) holds if and only if (B.19) is satisfied: Assume $h(z;w)$ satisfies (B.7). Then, for all $z \in \mathcal{C}^j(w)$

$$0 = \max\{u(z) - h(z;w), -\rho h(z;w) + \gamma\frac{\partial h(z;w)}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 h(z;w)}{\partial z^2} + \delta(u(z) - h(z;w)) + e^w\}.$$

Using that $\hat{U} = \frac{u(z)}{e^z}$ and $\hat{W}(w-z) = \frac{h(z;w)-u(z)}{e^z}$, we have that $h(z;w) = \hat{W}(w-z)e^z + \hat{U}e^z$, $\frac{\partial h(z;w)}{\partial z} = \hat{W}(w-z)e^z - \hat{W}'(w-z)e^z + \hat{U}e^z$, and $\frac{\partial^2 h(z;w)}{\partial z^2} = \hat{W}(w-z)e^z - 2\hat{W}'(w-z)e^z + \hat{W}''(w-z)e^z + \hat{U}e^z$. Thus, since $e^z > 0$

$$\begin{aligned}
0 &= \max\left\{u(z) - h(z;w), -\rho h(z;w) + \gamma\frac{\partial h(z;w)}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 h(z;w)}{\partial z^2} + \delta(u(z) - h(z;w)) + e^w\right\} \\
&= \max\{-\hat{W}(w-z)e^z, -\rho(\hat{W}(w-z)e^z + \hat{U}e^z) + \gamma(\hat{W}(w-z)e^z - \hat{W}'(w-z)e^z + \hat{U}e^z)\ldots \\
&+ \frac{\sigma^2}{2}(\hat{W}(w-z)e^z - 2\hat{W}'(w-z)e^z + \hat{W}''(w-z)e^z + \hat{U}e^z) - \delta\hat{W}(w-z)e^z + e^w\} \\
&= \max\{-\hat{W}(w-z), -(\rho - \gamma - \sigma^2/2 + \delta)\hat{W}(w-z) - (\gamma + \sigma^2)\hat{W}'(w-z) + \frac{\sigma^2}{2}\hat{W}''(w-z) - \left(\rho - \gamma - \sigma^2/2\right)\hat{U} + e^{w-z}\} \\
&= \max\left\{-\hat{W}(\hat{w}), -(\hat{\rho} + \delta)\hat{W}(\hat{w}) - \hat{\gamma}\hat{W}'(\hat{w}) + \frac{\sigma^2}{2}\hat{W}''(\hat{w}) - \hat{\rho}\hat{U} + e^{\hat{w}}\right\}.
\end{aligned}$$

The equivalence between (B.8) and (B.20) can be established following similar steps.

Condition (B.9) holds if and only if (B.21) is satisfied: Assume $z \in \mathcal{C}^h(w)$. Then,

$$h(z;w) > u(z) \text{ or } -\rho u(z) + \gamma u'(z) + \frac{\sigma^2}{2}u''(z) + e^w > 0$$

Using that $\hat{U} = \frac{u(z)}{e^z}$ and $\hat{W}(w-z) = \frac{h(z;w)-u(z)}{e^z}$, with $e^z > 0$

$$\begin{aligned}
&\hat{W}(w-z) > 0 \text{ or } -\rho\hat{U}e^z + \gamma\hat{U}e^z + \frac{\sigma^2}{2}\hat{U}e^z + e^w > 0 \iff \\
&\hat{W}(w-z) > 0 \text{ or } e^{w-z} - (\rho - \gamma - \sigma^2/2)\hat{U} > 0 \iff \\
&\quad \hat{W}(\hat{w}) > 0 \text{ or } e^{\hat{w}} - \hat{\rho}\hat{U} > 0
\end{aligned}$$

Thus, $z \in \mathcal{C}^h(w)$ if and only if $w - z \in \hat{\mathcal{C}}^h$. The equivalence between (B.10) and (B.22) can be established following similar steps.

Remaining conditions: The equivalence between equations (B.3), (B.4), (B.5), (B.6) and equations (B.15), (B.16), (B.17), and (B.18) is trivially established. □

B.3 Proof of Proposition 1

Proposition 1. *Let $\hat{W}(\hat{w}), \hat{J}(\hat{w}), \hat{\theta}(\hat{w})$ be bounded functions with compact support. Then, there exists a unique solution to*

$$\hat{\rho}\hat{U} = \tilde{B} + \max_{\hat{w}^*} \hat{\theta}(\hat{w}^*)^{1-\alpha}\hat{W}(\hat{w}^*),$$
$$0 = [\tilde{K} - \hat{\theta}(\hat{w})^{-\alpha}\hat{J}(\hat{w})]^+ \hat{\theta}(\hat{w}),$$

$$\hat{W}(\hat{w}) \geq 0, \tag{B.23}$$

$$\hat{J}(\hat{w}) \geq 0, \tag{B.24}$$

$$if\ \hat{w} \in (\hat{\mathcal{C}}^h)^c \Rightarrow \hat{J}(\hat{w}) = 0, \tag{B.25}$$

$$if\ \hat{w} \in (\hat{\mathcal{C}}^j)^c \Rightarrow \hat{W}(\hat{w}) = 0, \tag{B.26}$$

$$0 = \max\{-\rho\hat{W}(\hat{w}), \hat{\mathcal{A}}\hat{W}(\hat{w}) + e^{\hat{w}} - \hat{\rho}\hat{U}\},\ \forall \hat{w} \in \hat{\mathcal{C}}^j, \hat{W} \in \mathbb{C}^1(\hat{\mathcal{C}}^j) \cap \mathbb{C}(\mathbb{R}) \tag{B.27}$$

$$0 = \max\{-\rho\hat{J}(\hat{w}), \hat{\mathcal{A}}\hat{J}(\hat{w}) + 1 - e^{\hat{w}}\},\ \forall \hat{w} \in \hat{\mathcal{C}}^h, \hat{J} \in \mathbb{C}^1(\hat{\mathcal{C}}^h) \cap \mathbb{C}(\mathbb{R}) \tag{B.28}$$

$$\hat{\mathcal{C}}^h := int\left\{\hat{w} \in \mathbb{R} : \hat{W}(\hat{w}) > 0\ or\ (e^{\hat{w}} - \hat{\rho}\hat{U}) > 0\right\}, \tag{B.29}$$

$$\hat{\mathcal{C}}^j := int\left\{\hat{w} \in \mathbb{R} : \hat{J}(\hat{w}) > 0\ or\ \left(1 - e^{\hat{w}}\right) > 0\right\}, \tag{B.30}$$

$$\hat{\mathcal{A}}(f) := -(\hat{\rho} + \delta)f - \hat{\gamma}\frac{\partial f(\hat{w})}{\partial \hat{w}} + \frac{\sigma^2}{2}\frac{\partial^2 f(\hat{w})}{\partial \hat{w}^2},$$

The proof uses results from the mathematics literature that, generally, a well-trained economist has not used or seen before. For this reason, before presenting the proof, we provide some intuition about the steps we show below. In a nutshell, there are two steps in the proof. First, we need to show that, for a given value of unemployment $\hat{U}$, there is a unique non-trivial Nash equilibrium of the game played by the matched worker-firm pair. To understand the intuition behind this step, define $\hat{w}^+(\hat{w}^-; \rho\hat{U})$ as the best response function of the firm in terms of its layoff threshold, and $\hat{w}^-(\hat{w}^+; \rho\hat{U})$ as the best response function of the worker in terms of her quit threshold. It is easy to show that optimal policies are given by wage-to-productivity thresholds. $\hat{w}^+(\hat{w}^-; \rho\hat{U})$ is the solution to the differential equation

$$(\hat{\rho} + \delta)\hat{J}(\hat{w}) = 1 - e^{\hat{w}} - \hat{\gamma}\hat{J}'(\hat{w}) + \frac{\sigma^2}{2}\hat{J}''(\hat{w}),$$

with border conditions $\hat{J}(\hat{w}^+) = \hat{J}(\hat{w}^-) = \hat{J}'(\hat{w}^+) = 0$. In the same way, $\hat{w}^-(\hat{w}^+; \rho\hat{U})$ is the solution to the differential equation

$$(\hat{\rho} + \delta)\hat{W}(\hat{w}) = e^{\hat{w}} - \hat{\rho}\hat{U} - \hat{\gamma}\hat{W}'(\hat{w}) + \frac{\sigma^2}{2}\hat{W}''(\hat{w}),$$

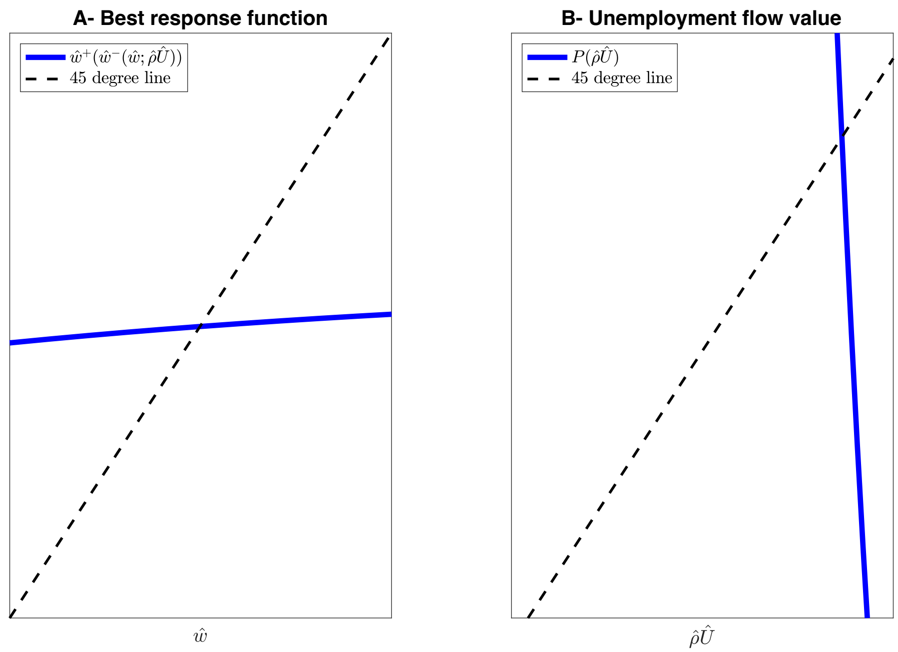
with border conditions $\hat{W}(\hat{w}^+) = \hat{W}(\hat{w}^-) = \hat{W}'(\hat{w}^-) = 0$. Let $\hat{W}(\hat{w}; \rho\hat{U})$ and $\hat{J}(\hat{w}; \rho\hat{U})$ be the values associated with the non-trivial equilibrium policies.

Second, we need to find the equilibrium value of unemployment. This value satisfies

$$\mathbb{P}(\hat{\rho}\hat{U}) = \tilde{B} + \max_{\hat{w}} \frac{1}{\tilde{K}^{1/\alpha}}\hat{J}(\hat{w}; \hat{\rho}\hat{U})^{\frac{1-\alpha}{\alpha}}\hat{W}(\hat{w}; \hat{\rho}\hat{U}).$$

Figure B1-Panel A shows the composition of $\mathbb{Q}(\hat{w}) := \hat{w}^+(\hat{w}^-(\hat{w}; \hat{\rho}\hat{U}))$ and Figure B1-Panel B shows $\mathbb{P}(\hat{\rho}\hat{U})$. As we can see in the figure, the composition of the best response functions satisfies two properties: (i) monotonicity (i.e., $\mathbb{Q}'(\hat{w}) > 0$) and (ii) concavity (i.e., $\mathbb{Q}''(\hat{w}) < 0$). Intuitively, the monotonicity property arises from the fact that if one agent prefers to stay in the match for longer, the other agent also prefers to stay longer. Concavity arises from the fact that there is a decreasing value of delaying the separation. As the figure clearly shows, a unique non-trivial Nash Equilibrium exists under these two properties. Equipped with the values from the non-trivial Nash Equilibrium as a function of $\hat{U}$, we can then characterize the decision problem of the unemployed worker. The mapping $\mathbb{P}(\hat{\rho}\hat{U})$ satisfies three properties: (i) $\mathbb{P}(\tilde{B}) > \tilde{B}$ with $\mathbb{P}(1) = \tilde{B}$, (ii) it is continuous and (iii) it is decreasing. Intuitively, if the flow value of unemployment is equal to $\tilde{B}$, then the surplus of the match is positive, and the unemployed worker obtains a positive continuation value from searching for a job. If, instead, the flow value of unemployment equals the value of (normalized) output, then the surplus is zero, and the unemployed worker does not benefit from finding a job. Also, the larger the unemployment value, the lower the value of the match, and, therefore, the value of searching for a job. As the figure clearly shows, a unique equilibrium exists under these three properties of $\mathbb{P}(\hat{\rho}\hat{U})$.

FIGURE B1. INTUITION



Notes: The figure illustrates the properties of the policy and value functions. Panel A shows the composition of $\mathbb{Q}(\hat{w}) := \hat{w}^+(\hat{w}^-(\hat{w};\hat{\rho}\hat{U}))$ and the 45 degree line. The non-trivial Nash Equilibrium is given by the intersection between these two lines. Panel B shows the composition of the individual best response functions and the fixed point in the equilibrium $\mathbb{P}(\hat{\rho}\hat{U})$.

Proof. We divide the proof into four steps.

Step 1 shows the existence of a non-trivial Nash equilibrium for a given $\hat{U}$. In this step, we show the existence of a solution to conditions (B.23) to (B.30). To simplify the exposure, we divide step 1 into three propositions. Proposition B.3 shows the equivalence between the equilibrium conditions and the quasi-variational inequalities. Intuitively, Lemma 1 established the equivalence between the sequential and the recursive formulations of the model represented by a set of differential equations. Here, we establish a second equivalence between these differential equations and variational inequalities, which is required to apply known fixed-point theorems. Proposition B.4 shows the existence and uniqueness of the agents' best responses. Proposition B.5 shows the existence of equilibrium by invoking Proposition A.3 (Tartar's fixed point theorem). Observe that we restrict the functions $\hat{W}(\hat{w})$ and $\hat{J}(\hat{w})$ to have bounded support. This restriction is without loss of generality since it is a result of Proposition 2—i.e., the match's continuation region is bounded.

Step 2 shows the uniqueness of the solution to conditions (B.23) to (B.30). We divide this proof into two propositions. Proposition B.6 shows that the operator defined in step 1 is strong order concave. Using concavity and techniques in the spirit of Marinacci and Montrucchio (2019) applied to our own problem, we show uniqueness in Proposition B.7.

Step 3 shows that value functions are continuous and decreasing. We divide this step into two propositions. First, we show in Proposition B.8 that the value associated with the worker's "best response" is continuous and decreasing in $\hat{U}$. Proposition B.9 shows these properties for the non-trivial Nash equilibrium. Finally, step 4 proves the uniqueness of the equilibrium by showing the existence of the unique fixed point in the unemployed worker's value $\hat{U}$.

Step 1. We start by defining a continuous bilinear form in a more general space of functions. Let $V = H_0^1(\mathbb{R})$—where $H_0^1(\mathbb{R})$ is a Sobolev space of order 1—be a Hilbert space and define the bilinear continuous form $a : V \times V \to \mathbb{R}$

$$a(v_1, v_2) := \frac{\sigma^2}{2} \int_{\mathbb{R}} \frac{\mathrm{d}v_1}{\mathrm{d}\hat{w}} \frac{\mathrm{d}v_2}{\mathrm{d}\hat{w}} \,\mathrm{d}\hat{w} + \hat{\gamma} \int_{\mathbb{R}} \frac{\mathrm{d}v_1}{\mathrm{d}\hat{w}} v_2(\hat{w}) \,\mathrm{d}\hat{w} + (\hat{\rho} + \delta) \int_{\mathbb{R}} v_1(\hat{w}) v_2(\hat{w}) d\hat{w}$$

Proposition B.3. *Define $K^h(\hat{J})$ and $K^j(\hat{W})$ as*

$$K^h(\hat{J}) := \left\{\hat{W} \in V : \hat{W}(\hat{w}) \geq 0 \;\&\; if\; \hat{J}(\hat{w}) = 0 \;and\; \hat{w} \geq 0 \Rightarrow \hat{W}(\hat{w}) = 0\right\},$$
$$K^j(\hat{W}) := \left\{\hat{J} \in V : \hat{J}(\hat{w}) \geq 0 \;\&\; if\; \hat{W}(\hat{w}) = 0 \;and\; \hat{w} \leq \log(\hat{\rho}\hat{U}) \Rightarrow \hat{J}(\hat{w}) = 0\right\}.$$

Assume $\hat{W}(\hat{w}) \in \mathbb{C}^1(\hat{\mathcal{C}}^j) \cap \mathbb{C}(\mathbb{R})$ and $\hat{J}(\hat{w}) \in \mathbb{C}^1(\hat{\mathcal{C}}^h) \cap \mathbb{C}(\mathbb{R})$ bounded with compact support, where $\hat{\mathcal{C}}^h$ and $\hat{\mathcal{C}}^j$ are constructed with $\hat{W}$ and $\hat{J}$ following (B.29) *and* (B.30)*. Then, $\hat{W}(\hat{w})$ and $\hat{J}(\hat{w})$ solve*

$$\begin{aligned}
&\hat{W} \in K^h(\hat{J}), \;\; \hat{J} \in K^j(\hat{W}) \\
&a(\hat{J}, v - \hat{J}) \geq \int_{\mathbb{R}} \left(1 - e^{\hat{w}}\right) \left(v - \hat{J}\right) \mathrm{d}\hat{w}, \;\; \forall\, v \in K^j(\hat{W}) \\
&a(\hat{W}, v - \hat{W}) \geq \int_{\mathbb{R}} (e^{\hat{w}} - \hat{\rho}\hat{U}) \left(v - \hat{W}\right) \mathrm{d}\hat{w}, \;\; \forall\, v \in K^h(\hat{J}).
\end{aligned}$$

if and only if $\hat{W}(\hat{w})$ and $\hat{J}(\hat{w})$ solve (B.23), (B.24), (B.25), (B.26), (B.27), *and* (B.28).

Proof of Step 1 - Proposition B.3. We verify conditions (B.23), (B.24), (B.25), (B.26), (B.27), and (B.28) focusing on the firm (the worker's conditions are verified following similar steps). It is easy to show the converse.

Conditions (B.23) and (B.24) are satisfied. Since $\hat{J} \in K^j(\hat{W})$, we have that $\hat{J}(\hat{w}) \geq 0$.

Conditions (B.25) and (B.26) are satisfied. Define $\hat{\mathcal{C}}^h$ with $\hat{W}$. Then, $(\hat{\mathcal{C}}^h)^c$ is equal to

$$(\hat{\mathcal{C}}^h)^c = cl\{\hat{w} \in \mathbb{R} : \hat{W}(\hat{w}) \leq 0 \;and\; (e^{\hat{w}} - \hat{\rho}\hat{U}) \leq 0\}.$$

Since $\hat{W}(\hat{w}) \geq 0$, we have that

$$(\hat{\mathcal{C}}^h)^c = cl\{\hat{w} \in \mathbb{R} : \hat{W}(\hat{w}) = 0 \;and\; \hat{w} \leq \log(\hat{\rho}\hat{U})\}.$$

Since $\hat{J} \in K^j(\hat{W})$, if $\hat{w} \in (\hat{\mathcal{C}}^h)^c$, then $\hat{J}(\hat{w}) = 0$.

Conditions (B.27) and (B.28) are satisfied. Take any $v \in K^j(\hat{W})$. Then, if $\hat{w} \in (\hat{\mathcal{C}}^h)^c$, we have that $\hat{J}(\hat{w}) = v(\hat{w}) = 0$. Therefore, we have that for every $v \in K^j(\hat{W})$

$$\begin{aligned}
&a(\hat{J}, v - \hat{J}) \geq \int_{\mathbb{R}} (1 - e^{\hat{w}}) \left(v - \hat{J}\right) \iff \\
&\frac{\sigma^2}{2} \int_{(\hat{\mathcal{C}}^h)^c} \frac{\mathrm{d}\hat{J}(\hat{w})}{\mathrm{d}\hat{w}} \frac{\mathrm{d}(v(\hat{w}) - \hat{J}(\hat{w}))}{\mathrm{d}\hat{w}} \,\mathrm{d}\hat{w} + \hat{\gamma} \int_{(\hat{\mathcal{C}}^h)^c} \frac{\mathrm{d}\hat{J}(\hat{w})}{\mathrm{d}\hat{w}} (v(\hat{w}) - \hat{J}(\hat{w})) \,\mathrm{d}\hat{w} + (\hat{\rho} + \delta) \int_{(\hat{\mathcal{C}}^h)^c} \hat{J}(\hat{w})(v(\hat{w}) - \hat{J}(\hat{w})) \,\mathrm{d}\hat{w} + \\
&\frac{\sigma^2}{2} \int_{\hat{\mathcal{C}}^h} \frac{\mathrm{d}\hat{J}(\hat{w})}{\mathrm{d}\hat{w}} \frac{\mathrm{d}(v(\hat{w}) - \hat{J}(\hat{w}))}{\mathrm{d}\hat{w}} \,\mathrm{d}\hat{w} + \hat{\gamma} \int_{\hat{\mathcal{C}}^h} \frac{\mathrm{d}\hat{J}(\hat{w})}{\mathrm{d}\hat{w}} (v(\hat{w}) - \hat{J}(\hat{w})) \,\mathrm{d}\hat{w} + (\hat{\rho} + \delta) \int_{\hat{\mathcal{C}}^h} \hat{J}(\hat{w})(v(\hat{w}) - \hat{J}(\hat{w})) \,\mathrm{d}\hat{w} \geq
\end{aligned}$$

$$\int_{\hat{\mathcal{C}}^h} (1-e^{\hat{w}})\left(v(\hat{w})-\hat{J}(\hat{w})\right)\mathrm{d}\hat{w} + \int_{(\hat{\mathcal{C}}^h)^c} (1-e^{\hat{w}})\left(v(\hat{w})-\hat{J}(\hat{w})\right)\mathrm{d}\hat{w} \iff$$
$$\frac{\sigma^2}{2}\int_{\hat{\mathcal{C}}^h} \frac{\mathrm{d}\hat{J}(\hat{w})}{\mathrm{d}\hat{w}}\frac{\mathrm{d}(v(\hat{w})-\hat{J}(\hat{w}))}{\mathrm{d}\hat{w}}\mathrm{d}\hat{w} + \hat{\gamma}\int_{\hat{\mathcal{C}}^h}\frac{\mathrm{d}\hat{J}(\hat{w})}{\mathrm{d}\hat{w}}(v(\hat{w})-\hat{J}(\hat{w}))\,\mathrm{d}\hat{w} + (\hat{\rho}+\delta)\int_{\hat{\mathcal{C}}^h}\hat{J}(\hat{w})(v(\hat{w})-\hat{J}(\hat{w}))\,\mathrm{d}\hat{w} \geq$$
$$\int_{\hat{\mathcal{C}}^h} (1-e^{\hat{w}})\left(v(\hat{w})-\hat{J}(\hat{w})\right)\mathrm{d}\hat{w}.$$

Using integration by parts, we obtain

$$\begin{aligned}&\frac{\sigma^2}{2}\int_{\hat{\mathcal{C}}^h} \frac{\mathrm{d}\hat{J}(\hat{w})}{\mathrm{d}\hat{w}}\frac{\mathrm{d}(v(\hat{w})-\hat{J}(\hat{w}))}{\mathrm{d}\hat{w}}\mathrm{d}\hat{w}\\ &=^{(1)} \underbrace{\frac{\sigma^2}{2}\frac{\mathrm{d}\hat{J}(\hat{w})}{\mathrm{d}\hat{w}}(v(\hat{w})-\hat{J}(\hat{w}))\Big|_{\hat{w}\in\partial\in\hat{\mathcal{C}}^h}}_{=0} - \frac{\sigma^2}{2}\int_{\hat{\mathcal{C}}^h}\frac{d^2\hat{J}(\hat{w})}{\mathrm{d}\hat{w}}(v(\hat{w})-\hat{J}(\hat{w}))\,\mathrm{d}\hat{w}.\end{aligned}$$

In (1), there could be two cases for the first term. The first case is a finite limit of integration (i.e., $\hat{\mathcal{C}}^h$ is bounded). In this case, we use continuity of the functions and the fact that if $\hat{w}\to\partial\hat{\mathcal{C}}^h$ ($\hat{\mathcal{C}}^h$ is open) , then $\hat{w}\to(\hat{\mathcal{C}}^h)^c$ and, therefore, $\hat{J}(\hat{w}) = v(\hat{w}) = 0$. The second case is an infinite limit of integration. In this case, the assumption of bounded support implies $\hat{J}(\hat{w}) = 0$ for sufficiently large or small $\hat{w}$, thus $\hat{J}'(\hat{w}) = 0$. In conclusion,

$$\int_{\hat{\mathcal{C}}^h}\left(\hat{\mathcal{A}}\hat{J}(\hat{w}) + (1-e^{\hat{w}})\right)(v(\hat{w})-\hat{J}(\hat{w}))\,\mathrm{d}\hat{w} \leq 0.$$

Before continuing, we remark that the previous equality holds for all $v(\hat{w})\in K^j(\hat{W})$. Let $\mathcal{O}$ be an open ball in $\hat{\mathcal{C}}^h$ that covers an arbitrary point $\hat{w}\in\hat{\mathcal{C}}^h$. Then, we can find a family of smooth functions indexed by n with $o_{\hat{w}}(n)\in[0,1]$, s.t. $o_{\hat{w}}(n) = 0$ outside $\hat{\mathcal{C}}^h$, $o_{\hat{w}}(n)\to 1$ in $\mathcal{O}$, and $o_{\hat{w}}(n)\to 0$ outside $\mathcal{O}$. Since $\hat{J}(\hat{w}) + o_{\hat{w}}(n)\geq 0$, $\hat{J}(\hat{w}) + o_{\hat{w}}(n)\in K^j(\hat{W})$ and

$$\int_{\mathcal{O}}\left(\hat{\mathcal{A}}\hat{J}(\hat{w}) + (1-e^{\hat{w}})\right)o_{\hat{w}}(n)\,\mathrm{d}\hat{w} + \int_{\hat{\mathcal{C}}^h/\mathcal{O}}\left(\hat{\mathcal{A}}\hat{J}(\hat{w}) + (1-e^{\hat{w}})\right)o_{\hat{w}}(n)\,\mathrm{d}\hat{w} \leq 0.$$

Taking the limit $n\to\infty$, we have that

$$\int_{\mathcal{O}}\left(\hat{\mathcal{A}}\hat{J}(\hat{w}) + (1-e^{\hat{w}})\right)\mathrm{d}\hat{w} \leq 0.$$

Since $\mathcal{O}$ is arbitrary, $\hat{\mathcal{A}}\hat{J}(\hat{w}) + 1 - e^{\hat{w}} \leq 0$ a.e. in $\hat{\mathcal{C}}^h$. Since $\hat{J}(\hat{w})\in\mathbb{C}^1(\hat{\mathcal{C}}^h)$, then $\hat{\mathcal{A}}\hat{J}(\hat{w}) + 1 - e^{\hat{w}} \leq 0$ for all $\hat{w}$ whenever the second derivative is defined. To obtain the other inequality, consider $\hat{J}(\hat{w})(1-o_{\hat{w}}(n)) + 0o_{\hat{w}}(n)\in K^j(\hat{W})$ and we have that

$$-\int_{\mathcal{O}}\left(\hat{\mathcal{A}}\hat{J}(\hat{w}) + (1-e^{\hat{w}})\right)\hat{J}(\hat{w})o_{\hat{w}}(n)\,\mathrm{d}\hat{w} - \int_{\hat{\mathcal{C}}^h/\mathcal{O}}\left(\hat{\mathcal{A}}\hat{J}(\hat{w}) + (1-e^{\hat{w}})\right)\hat{J}(\hat{w})o_{\hat{w}}(n)\,\mathrm{d}\hat{w} \leq 0$$

Taking the limit $n\to\infty$, we have that $\int_{\mathcal{O}}\left(\hat{\mathcal{A}}\hat{J}(\hat{w}) + (1-e^{\hat{w}})\right)(-\hat{J}(\hat{w}))\,\mathrm{d}\hat{w}\leq 0$ a.e.. Since $\hat{J}(\hat{w})\in\mathbb{C}^1(\hat{\mathcal{C}}^h)$, we have that for all $\hat{w}\in\hat{\mathcal{C}}^h$

$$\left(\hat{\mathcal{A}}\hat{J}(\hat{w}) + (1-e^{\hat{w}})\right)(-\hat{J}(\hat{w}))\leq 0.$$

Since $\hat{J}(\hat{w})\geq 0$ and $\left(\hat{\mathcal{A}}\hat{J}(\hat{w}) + (1-e^{\hat{w}})\right)\leq 0$, we have that $\left(\hat{\mathcal{A}}\hat{J}(\hat{w}) + (1-e^{\hat{w}})\right)(-\hat{J}(\hat{w}))\geq 0$. Thus, $\left(\hat{\mathcal{A}}\hat{J}(\hat{w}) + 1 - e^{\hat{w}}\right)(-\hat{J}(\hat{w})) =$ 0 or written more compactly

$$0 = \max\{-\hat{J}(\hat{w}), \hat{\mathcal{A}}\hat{J}(\hat{w}) + 1 - e^{\hat{w}}\},\ \forall\hat{w}\in\hat{\mathcal{C}}^h,$$

with $\hat{J}(\hat{w})\in\mathbb{C}^1(\hat{\mathcal{C}}^h)\cap\mathbb{C}(\mathbb{R})$. □

Proposition B.4. *Define the value functions that are obtained from the best responses as* $BR^h : H^1(\mathbb{R})\to H^1(\mathbb{R})$ *and* $BR^j : H^1_0(\mathbb{R})\to$

$H_0^1(\mathbb{R})$ *such that*

$$BR^h(\hat{J}) = \{\hat{W} \in H^1(\mathbb{R}) : a(\hat{W}, v - \hat{W}) \geq (e^{\hat{w}} - \hat{\rho}\hat{U}, v - \hat{W}),\ \forall\, v \in K^h(\hat{J}),\ \hat{W} \in K^h(\hat{J})\},$$
$$BR^j(\hat{W}) = \{\hat{J} \in H^1(\mathbb{R}) : a(\hat{J}, v - \hat{J}) \geq (1 - e^{\hat{w}}, v - \hat{J}),\ \forall\, v \in K^j(\hat{W}),\ \hat{J} \in K^j(\hat{W})\}.$$

Then, $BR^h(\hat{J})$ and $BR^j(\hat{W})$ exist and are unique.

Proof of Step 1 - Proposition B.4. Here, we show that the value functions that are obtained from the best responses are well-defined. For this, we need to verify the conditions in Proposition A.4. Basically, we need to show that $K^j(\hat{W})$ is closed and convex, and that $a(\cdot,\cdot)$ is coercive.

$\mathbf{K^j(\hat{W})}$ is closed and convex. First, we show that $K^j(\hat{W})$ is closed. Take a sequence $\hat{J}^n \in K^j(\hat{W})$ s.t. $\hat{J}^n$ converges to some $\hat{J}^*$. Since $\hat{J}^n \in K^j(\hat{W})$,

$$\hat{J}^n \geq 0, \text{ if } \hat{W}(\hat{w}) = 0 \textit{ and } \hat{w} \leq \log(\hat{\rho}\hat{U}), \text{ then } \hat{J}^n = 0$$

for all n. Taking the limit,

$$\hat{J}^* \geq 0, \text{ if } \hat{W}(\hat{w}) = 0 \textit{ and } \hat{w} \leq \log(\hat{\rho}\hat{U}), \text{ then } \hat{J}^* = 0$$

where we use the fixed domain in the second limit. Thus, $K^j(\hat{W})$ is closed.

To show that $K^j(\hat{W})$ is convex, take $\hat{J}^1, \hat{J}^2 \in K^j(\hat{W})$, then

$$\hat{J}^1 \geq 0, \text{ if } \hat{W}(\hat{w}) = 0 \textit{ and } \hat{w} \leq \log(\hat{\rho}\hat{U}), \text{ then } \hat{J}^1 = 0,$$
$$\hat{J}^2 \geq 0, \text{ if } \hat{W}(\hat{w}) = 0 \textit{ and } \hat{w} \leq \log(\hat{\rho}\hat{U}), \text{ then } \hat{J}^2 = 0.$$

Taking the convex combination with $\lambda \in [0,1]$

$$\lambda\hat{J}^1 + (1-\lambda)\hat{J}^2 \geq 0, \text{ if } \hat{W}(\hat{w}) = 0 \textit{ and } \hat{w} \geq 0, \text{ then } \lambda\hat{J}^1 + (1-\lambda)\hat{J}^2 = 0.$$

Thus, $K^j(\hat{W})$ is convex.

$\mathbf{a(u,v)}$ is coercive. Operating over the bilinear operator

$$\begin{aligned}
a(v,v) &= \frac{\sigma^2}{2}\int_{\mathbb{R}} \frac{\mathrm{d}v(\hat{w})}{\mathrm{d}\hat{w}}\frac{\mathrm{d}v(\hat{w})}{\mathrm{d}\hat{w}}\,\mathrm{d}\hat{w} + \hat{\gamma}\int_{\mathbb{R}} \frac{\mathrm{d}v(\hat{w})}{\mathrm{d}\hat{w}} v(\hat{w})\,\mathrm{d}\hat{w} + (\hat{\rho}+\delta)\int_{\mathbb{R}} v(\hat{w})^2 d\hat{w} \\
&=^{(1)} \underbrace{\frac{\sigma^2}{2}\int_{\mathbb{R}}\left(\frac{\mathrm{d}v(\hat{w})}{\mathrm{d}\hat{w}}\right)^2 \mathrm{d}\hat{w}}_{\geq 0} + \hat{\gamma}\underbrace{v(\hat{w})^2\Big|_{-\infty}^{\infty}}_{=0} + (\hat{\rho}+\delta)\int_{\mathbb{R}} v(\hat{w})^2 d\hat{w} \\
&\geq^{(2)} (\hat{\rho}+\delta)\int_{\mathbb{R}} v(\hat{w})^2 d\hat{w} \\
&= (\hat{\rho}+\delta)||v||^2
\end{aligned}$$

Step (1) integrates $\int_{\mathbb{R}} \frac{\mathrm{d}v(\hat{w})}{\mathrm{d}\hat{w}} v(\hat{w})\,\mathrm{d}\hat{w} = \frac{1}{2}v(\hat{w})^2\Big|_{-\infty}^{\infty}$ and uses compact support. Step (2) uses the non-negativity of the squared derivative term.

With the properties verified, we can apply Proposition A.4. Thus, the best response exists, and it is unique. □

Proposition B.5. *Define $Q(\hat{W}) = (BR^h \circ BR^j)(\hat{W})$, then there exists a fixed point $Q(\hat{W}^*) = \hat{W}^*$ and $\hat{J}^* = BR^j(\hat{W}^*)$. The set of fixed* *points is bounded above and below by*

$$0 \le \underline{\hat{W}} \le \hat{W}^* \le \overline{\hat{W}},$$
$$0 \le \underline{\hat{J}} \le \hat{J}^* \le \overline{\hat{J}},$$

where

$$a(\underline{\hat{W}}, v - \underline{\hat{W}}) \ge (e^{\hat{w}} - \hat{\rho}\hat{U}, \underline{\hat{W}}),\ \forall\, v \in K^{small}, \underline{\hat{W}} \in K^{small},$$
$$a(\underline{\hat{J}}, v - \underline{\hat{J}}) \ge (1 - e^{\hat{w}}, \underline{\hat{J}}),\ \forall\, v \in K^{small}, \underline{\hat{J}} \in K^{small},$$
$$a(\overline{\hat{W}}, v - \overline{\hat{W}}) \ge (e^{\hat{w}} - \hat{\rho}\hat{U}, \overline{\hat{W}}),\ \forall\, v \in K^{big}, \overline{\hat{W}} \in K^{big},$$
$$a(\overline{\hat{J}}, v - \overline{\hat{J}}) \ge (1 - e^{\hat{w}}, \overline{\hat{J}}),\ \forall\, v \in K^{big}, \overline{\hat{J}} \in K^{big},$$

with

$$K^{small} := \left\{ v \in V : v(\hat{w}) \ge 0 \;\&\; if\ \ \hat{w} \notin (\log(\hat{\rho}\hat{U}), 0) \Rightarrow v(\hat{w}) = 0 \right\},$$
$$K^{big} := \left\{ v \in V : v(\hat{w}) \ge 0 \right\},$$

with a maximum and minimum element.

Proof of Step 1 - Proposition B.5. The first step consists in showing that the function $Q(W)$ is monotonically increasing—i.e., if $\hat{W}_1 \ge \hat{W}_2$, then $Q(\hat{W}_1) \ge Q(\hat{W}_2)$. To show this result, first, we need to prove that $K^j(\hat{W})$ is increasing—i.e., if $\hat{W}_1 \ge \hat{W}_2$, then $K^j(\hat{W}_2) \subset K^j(\hat{W}_1)$. Take $\hat{J}_2 \in K^j(\hat{W}_2)$, then

$$\hat{J}_2 \ge 0,\ \&\ if\ \hat{W}_2(\hat{w}) = 0\ and\ \hat{w} \le \log(\hat{\rho}\hat{U}) \Rightarrow \hat{J}_2(\hat{w}) = 0.$$

Since $\hat{W}_2(\hat{w}) \ge 0$, we have that

$$\hat{J}_2 \ge 0,\ \&\ \hat{J}_2(\hat{w}) = 0\ \forall \hat{w} \in \{\hat{w} : \hat{W}_2(\hat{w}) \le 0 \;\&\; \hat{w} \le \log(\hat{\rho}\hat{U})\}.$$

Now, we show that $\{\hat{w} : \hat{W}_1(\hat{w}) \le 0 \;\&\; \hat{w} \le \log(\hat{\rho}\hat{U})\} \subset \{\hat{w} : \hat{W}_2(\hat{w}) \le 0 \;\&\; \hat{w} \le \log(\hat{\rho}\hat{U})\}$. Take $\hat{w} \in \{\hat{w} : \hat{W}_1(\hat{w}) \le 0 \;\&\; \hat{w} \le \log(\hat{\rho}\hat{U})\}$. Then, $\hat{W}_1(\hat{w}) \le 0$ and, since $\hat{W}_1(\hat{w}) \ge \hat{W}_2(\hat{w})$, we have that $\hat{W}_2(\hat{w}) \le 0$. Since $\{\hat{w} : \hat{W}_1(\hat{w}) \le 0 \;\&\; \hat{w} \le \log(\hat{\rho}\hat{U})\} \subset \{\hat{w} : \hat{W}_2(\hat{w}) \le 0 \;\&\; \hat{w} \le \log(\hat{\rho}\hat{U})\}$, the previous condition holds for the larger set, so it will also hold for the smaller set

$$\hat{J}_2 \ge 0,\ \&\ \hat{J}_2(\hat{w}) = 0, \forall \hat{w} \in \{\hat{w} : \hat{W}_1(\hat{w}) \le 0 \;\&\; \hat{w} \le \log(\hat{\rho}\hat{U})\}.$$

Thus, $\hat{J}_2 \in K^j(W_1)$ and $K^j(\hat{W}_2) \subset K^j(W_1)$.

Now, let $\hat{W}_1 \ge \hat{W}_2$. We need to show that $\hat{J}_1 = BR^j(\hat{W}_1) \ge BR^j(\hat{W}_2) = \hat{J}_2$. Since $K^j(\hat{W})$ is increasing—i.e., $K^j(\hat{W}_2) \subset K^j(\hat{W}_1)$—$\hat{J}_1, \hat{J}_2 \in K^j(\hat{W}_1)$ and the envelope $\max\{\hat{J}_1, \hat{J}_2\} \in K^j(\hat{W}_1)$. Now, we show that $\min\{\hat{J}_1, \hat{J}_2\} \in K^j(\hat{W}_2)$. Since $\hat{J}_1, \hat{J}_2 \ge 0$, we have that $\min\{\hat{J}_1, \hat{J}_2\} \ge 0$. Moreover, take a $\hat{w}$ s.t. $\hat{W}_2(\hat{w}) \le 0$ and $\hat{w} \le \log(\hat{\rho}\hat{U})$, then $0 = \hat{J}_2 = \min\{\hat{J}_2, \hat{J}_1\}$. Thus, $\min\{\hat{J}_1, \hat{J}_2\} \in K^j(\hat{W}_2)$. In conclusion, we can use $\max\{\hat{J}_1, \hat{J}_2\}$ as a test function for $K^j(\hat{W}_1)$ and $\min\{\hat{J}_1, \hat{J}_2\}$ as a test function for $K^j(\hat{W}_2)$:

$$\min\{\hat{J}_1, \hat{J}_2\} = \hat{J}_2 - \max\{\hat{J}_2 - \hat{J}_1, 0\} \text{ for test function for } K^j(\hat{W}_2)$$

$$\max\{\hat{J}_1,\hat{J}_2\} = \hat{J}_1 + \max\{\hat{J}_2 - \hat{J}_1, 0\} \text{ for test function for } K^j(\hat{W}_1)$$

Using the quasi-variational inequality

$$\begin{aligned}
a(\hat{J}_2, -\max\{\hat{J}_2 - \hat{J}_1, 0\}) &\geq (1 - e^{\hat{w}}, -\max\{\hat{J}_2 - \hat{J}_1, 0\}) \\
a(\hat{J}_1, \max\{\hat{J}_2 - \hat{J}_1, 0\}) &\geq (1 - e^{\hat{w}}, \max\{\hat{J}_2 - \hat{J}_1, 0\}).
\end{aligned}$$

Thus,

$$\begin{aligned}
-a(\hat{J}_2, \max\{\hat{J}_2 - \hat{J}_1, 0\}) &\geq -(1 - e^{\hat{w}}, \max\{\hat{J}_2 - \hat{J}_1, 0\}) \\
a(\hat{J}_1, \max\{\hat{J}_2 - \hat{J}_1, 0\}) &\geq (1 - e^{\hat{w}}, \max\{\hat{J}_2 - \hat{J}_1, 0\}).
\end{aligned}$$

Summing these two equalities, we obtain

$$a(\hat{J}_1, \max\{\hat{J}_2 - \hat{J}_1, 0\}) - a(\hat{J}_2, \max\{\hat{J}_2 - \hat{J}_1, 0\}) \geq 0$$

or equivalently,

$$a(\hat{J}_2, \max\{\hat{J}_2 - \hat{J}_1, 0\}) - a(\hat{J}_1, \max\{\hat{J}_2 - \hat{J}_1, 0\}) \leq 0.$$

Next, we show that the previous inequality implies $a(\max\{\hat{J}_2 - \hat{J}_1, 0\}, \max\{\hat{J}_2 - \hat{J}_1, 0\}) \leq 0$. Define the set $\mathbb{X} = \{x : \hat{J}_2 > \hat{J}_1\}$. Then,

$$\begin{aligned}
&a(\hat{J}_2, \max\{\hat{J}_2 - \hat{J}_1, 0\}) - a(\hat{J}_1, \max\{\hat{J}_2 - \hat{J}_1, 0\}) \\
&= \frac{\sigma^2}{2}\left(\int_{\mathbb{X}} \frac{\mathrm{d}\hat{J}_2(\hat{w})}{\mathrm{d}\hat{w}} \frac{\mathrm{d}(\hat{J}_2 - \hat{J}_1)}{\mathrm{d}\hat{w}}\,\mathrm{d}\hat{w} - \int_{\mathbb{X}} \frac{\mathrm{d}\hat{J}_1(\hat{w})}{\mathrm{d}\hat{w}} \frac{\mathrm{d}(\hat{J}_2 - \hat{J}_1)}{\mathrm{d}\hat{w}}\,\mathrm{d}\hat{w} + \int_{\mathbb{R}/\mathbb{X}} 0\,\mathrm{d}x\right) \\
&\cdots + \hat{\gamma}\left(\int_{\mathbb{X}} \frac{\mathrm{d}\hat{J}_2(\hat{w})}{\mathrm{d}\hat{w}}(\hat{J}_2 - \hat{J}_1)\,\mathrm{d}\hat{w} - \int_{\mathbb{X}} \frac{\mathrm{d}\hat{J}_1(\hat{w})}{\mathrm{d}\hat{w}}(\hat{J}_2 - \hat{J}_1)\,\mathrm{d}\hat{w} + \int_{\mathbb{R}/\mathbb{X}} 0\,\mathrm{d}x\right) \\
&\cdots + (\hat{\rho} + \delta)\left(\int_{\mathbb{X}} \hat{J}_2(\hat{J}_2 - \hat{J}_1)\,\mathrm{d}\hat{w} - \int_{\mathbb{X}} \hat{J}_1(\hat{J}_2 - \hat{J}_1)\,\mathrm{d}\hat{w} + \int_{\mathbb{R}/\mathbb{X}} 0\,\mathrm{d}\hat{w}\right) \\
&= \frac{\sigma^2}{2}\int_{\mathbb{X}} \left(\frac{\mathrm{d}(\hat{J}_2 - \hat{J}_1)}{\mathrm{d}\hat{w}}\right)^2 \mathrm{d}\hat{w} + \hat{\gamma}\int_{\mathbb{X}} \frac{\mathrm{d}(\hat{J}_2(\hat{w}) - \hat{J}_1)}{\mathrm{d}\hat{w}}(\hat{J}_2 - \hat{J}_1)\,\mathrm{d}\hat{w} + (\hat{\rho} + \delta)\left(\int_{\mathbb{X}} (\hat{J}_2 - \hat{J}_1)^2\,\mathrm{d}\hat{w}\right) \\
&= a(\max\{\hat{J}_2 - \hat{J}_1, 0\}, \max\{\hat{J}_2 - \hat{J}_1, 0\}).
\end{aligned}$$

In conclusion, since $a(\cdot,\cdot)$ is a coercive bilinear form, $0 \geq a(\max\{\hat{J}_2 - \hat{J}_1, 0\}, \max\{\hat{J}_2 - \hat{J}_1, 0\}) \geq K||\max\{\hat{J}_2 - \hat{J}_1, 0\}||^2$. Thus, $\hat{J}_1 \geq \hat{J}_2$ a.e., and by continuity $\hat{J}_1 \geq \hat{J}_2$ for all $\hat{w}$. Applying similar arguments to $BR^h(\hat{J})$, we have that if $\hat{W}_1 \geq \hat{W}_2$, then $Q(\hat{W}_1) \geq Q(\hat{W}_2)$, so by Proposition A.3, there exists a fixed point. Moreover, the set of fixed points has a maximum and a minimum—i.e.,

$$\{\hat{W} \in H_0^1(\mathbb{R}) : \hat{W} = Q(\hat{W})\}$$

has a $\hat{W}^{\min}$ and $\hat{W}^{\max}$ s.t. $\hat{W}^{\min} \leq \hat{W}^* \leq \hat{W}^{\max}$ for all $\hat{W}^* \in \{\hat{W} \in H_0^1(\mathbb{R}) : \hat{W} = Q(\hat{W})\}$. To find the upper and lower bound, observe that we can write the non-trivial Nash equilibrium policies as

$$\hat{J}^*(w) = \max_{\{\tau^j \in \mathcal{T} : \tau^j \leq \tau^{h*}\}} \mathbb{E}\left[\int_0^{\tau^j} e^{-(\hat{\rho}+\delta)t}(1 - e^{\hat{w}_t})\,\mathrm{d}t | \hat{w}_0 = \hat{w}\right].$$

Since $\infty > \tau^{h*} \geq \tau_{(\log(\hat{\rho}\hat{U},0))}$,[14] we have that

$$
\begin{aligned}
0 \leq \underline{\hat{J}} &= \max_{\{\tau^j \in \mathcal{T}: \tau^j \leq \tau_{(\log(\hat{\rho}\hat{U},0))}\}} \mathbb{E}\left[\int_0^{\tau^j} e^{-(\hat{\rho}+\delta)t}(1-e^{\hat{w}_t})\,\mathrm{d}t|\hat{w}_0 = \hat{w}\right] \\
&\leq \max_{\{\tau^j \in \mathcal{T}: \tau^j \leq \tau^{h*}\}} \mathbb{E}\left[\int_0^{\tau^j} e^{-(\hat{\rho}+\delta)t}(1-e^{\hat{w}_t})\,\mathrm{d}t|\hat{w}_0 = \hat{w}\right] \\
&= \hat{J}^*(w) \\
&\leq \max_{\{\tau^j \in \mathcal{T}\}} \mathbb{E}\left[\int_0^{\tau^j} e^{-(\hat{\rho}+\delta)t}(1-e^{\hat{w}_t})\,\mathrm{d}t|\hat{w}_0 = \hat{w}\right] \\
&= \overline{\hat{J}}.
\end{aligned}
$$

□

Step 2. This step proves the uniqueness of the fixed point. The first proposition shows that $Q : H_0^1(\mathbb{R}) \to H_0^1(\mathbb{R})$ is concave. Since the Q operator is only defined for non-negative functions, we assume that the domain is restricted to non-negative functions without loss of generality. Since the game's continuation region is bounded, flow payoffs are bounded. Therefore, the equilibrium value functions are also bounded. For these reasons, without loss of generality, we restrict the $Q : \mathcal{A} \to \mathcal{A}$ operator in

$$\mathcal{A} = \{v \in H_0^1(\mathbb{R}) : v(\hat{w}) \in [0, \overline{v}],\ \forall \hat{w}\}$$

Observe that $\mathcal{A}$ is order convex—i.e., if $a, b \in \mathcal{A}$ with $a \leq c \leq b$, then $c \in \mathcal{A}$.

Define the operator $\alpha : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$, where

$$\alpha(\hat{W}', \hat{W}'') = \alpha(\hat{w})\hat{W}'(\hat{w}) + (1-\alpha(\hat{w}))\hat{W}''(\hat{w}),$$

with $\alpha(\hat{w}) \in [0,1]$.

Proposition B.6. *$Q : \mathcal{A} \to \mathcal{A}$ is strongly order concave—i.e.,*

$$Q(\alpha(\hat{W}', \hat{W}'')) \geq \alpha(Q(\hat{W}'), Q(\hat{W}''))$$

for all $\hat{W}' \leq \hat{W}''$.

Proof of Step 2 - Proposition B.6. Take $\hat{W}' \leq \hat{W}''$. The proof has three arguments. First, we show that $K^j(\alpha(\hat{W}', \hat{W}'')) = K^j(\hat{W}'')$. Then, with this result in hand, we show that the $BR^j(\alpha(\hat{W}', \hat{W}'')) \geq \alpha(BR^j(\hat{W}'), BR^j(\hat{W}''))$. Finally, we show that $Q(\alpha(\hat{W}', \hat{W}'')) \geq \alpha(Q(\hat{W}'), Q(\hat{W}''))$.

To see that $K^j(\alpha(\hat{W}', \hat{W}'')) = K^j(\hat{W}'')$, observe that since $\alpha(\hat{W}', \hat{W}'') \leq \hat{W}''$ and since $K^j(\cdot)$ is increasing, we have that $K^j(\alpha(\hat{W}', \hat{W}'')) \subset K^j(\hat{W}'')$. Now, we show that $K^j(\hat{W}'') \subset K^j(\alpha(\hat{W}', \hat{W}''))$. Take any $\hat{J} \in K^j(\hat{W}'')$. Then,

$$\hat{J} \geq 0,\ \&\ if\ \hat{W}''(\hat{w}) = 0\ and\ \hat{w} \leq \log(\hat{\rho}\hat{U}) \Rightarrow \hat{J}(\hat{w}) = 0.$$

[14] $\tau_{(\log(\hat{\rho}\hat{U},0))} := \inf\{t \geq 0 : \hat{w}_t \notin (\log(\hat{\rho}\hat{U}, 0))\}$.

If $\hat{W}''(\hat{w}) = 0$, then $\hat{W}''(\hat{w}) \geq \hat{W}'(\hat{w}) = 0$, which is then also true for any convex combination. Thus, $\alpha(\hat{W}', \hat{W}'') \leq \hat{W}'' = 0$ and

$$\hat{J} \geq 0, \ \& \ if \ \alpha(\hat{W}', \hat{W}'') = 0 \ and \ \hat{w} \leq \log(\hat{\rho}\hat{U}) \Rightarrow \hat{J}(\hat{w}) = 0.$$

In conclusion, $\hat{J} \in K^j(\alpha(\hat{W}', \hat{W}''))$ and $K^j(\hat{W}'') \subset K^j(\alpha(\hat{W}', \hat{W}''))$. Therefore, we have that $K^j(\alpha(\hat{W}', \hat{W}'')) = K^j(\hat{W}'')$.

Since the constraint set—i.e., $\hat{W}$ and any test function v in $K^j(\cdot)$—is the same for $\alpha(\hat{W}', \hat{W}'')$ and $\hat{W}''$, we have that

$$\begin{aligned} BR^j(\alpha(\hat{W}', \hat{W}'')) &= BR^j(\hat{W}''), \\ &= \alpha(BR^j(\hat{W}''), BR^j(\hat{W}'')), \\ &\geq \alpha(BR^j(\hat{W}'), BR^j(\hat{W}'')), \end{aligned}$$

where we used the monotonicity of $BR^j(\hat{W})$ in the last inequality. A similar property holds for $BR^h(\hat{J})$. In conclusion, $BR^j(\hat{W})$ and $BR^h(\hat{J})$ are increasing and strongly order concave. Using this result, for $\hat{W}' \leq \hat{W}''$, we have

$$\begin{aligned} Q(\alpha(\hat{W}', \hat{W}'')) &= BR^h(BR^j(\alpha(\hat{W}', \hat{W}''))) \\ &\geq^{(1)} BR^h(\alpha(BR^j(\hat{W}'), BR^j(\hat{W}''))) \\ &\geq^{(2)} \alpha(BR^h(BR^j(\hat{W}')), BR^h(BR^j(\hat{W}''))) \\ &= \alpha(Q(\hat{W}'), Q(\hat{W}'')). \end{aligned}$$

Step (1) uses the monotonicity of $BR^h(\hat{J})$ and the strongly order concavity of $BR^j(\hat{W})$. Step (2) uses the strongly order concavity of $BR^h(\hat{J})$. $\square$

Proposition B.7. *$Q : \mathcal{A} \to A$ has a unique fixed point.*

Proof of Step 2 - Proposition B.7. We have shown that $Q(\hat{W})$ is monotone and order concave defined in an order convex set. Now, we prove the result by contradiction. Let $\underline{\hat{W}}$ be the minimum fixed point and let $\hat{W}^*$ be another fixed point with $\hat{W}^* > \underline{\hat{W}}$. Then, we can write $\underline{\hat{W}} = \alpha^*(0, \hat{W}^*)$ for some $\alpha^*(\hat{w})$ function, where zero is the lower bound in the domain. Importantly, it is easy to see that $\alpha^*(\hat{w}) \in (0,1)$ for all $\hat{w} \in (\log(\hat{\rho}\hat{U}, 0))$. Thus,

$$\begin{aligned} \underline{\hat{W}} &=^{(1)} Q(\underline{\hat{W}}) \\ &=^{(2)} Q(\alpha^*(0, \hat{W}^*)) \\ &\geq^{(3)} \alpha^*(Q(0), Q(\hat{W}^*)) \\ &=^{(4)} \alpha^*(Q(0), \hat{W}^*) \\ &>^{(5)} \alpha^*(0, \hat{W}^*) \\ &=^{(6)} \underline{\hat{W}} \end{aligned}$$

Step (1) uses the fact that $\underline{\hat{W}}$ is a fixed point and step (2) uses the fact that $\underline{\hat{W}} = \alpha^*(0, \hat{W}^*)$. Step (3) uses the strongly order concavity of Q. Step (4) uses the fact that $\hat{W}^*$ is a fixed point. Step (5) uses that $Q(0) > 0$ for all $\hat{w} \in (\log(\hat{\rho}\hat{U}), 0)$. Since it cannot be that $\underline{\hat{W}} > \underline{\hat{W}}$, we have a contradiction. $\square$

Step 3. Let $\hat{W}^*(\hat{w}; \hat{\rho}\hat{U})$ and $\hat{J}^*(\hat{w}; \hat{\rho}\hat{U})$ be the value functions from the unique non-trivial Nash equilibrium. We now show that they are continuous and decreasing in $\hat{U}$.

Proposition B.8. *Fix $\hat{J}$. Let $\hat{W}(\hat{w};\hat{\rho}\hat{U}) = BR^h(\hat{J};\hat{\rho}\hat{U})$ be the solution of*

$$a(\hat{W}, v - \hat{W}) \geq (1 - \hat{\rho}\hat{U}, v - \hat{W}),\ \forall v \in K^h(\hat{J}),\ \hat{W} \in K^h(\hat{J})$$

Then, $\hat{W}(\hat{w};\hat{\rho}\hat{U})$ is continuous and decreasing in $\hat{\rho}\hat{U}$.

Proof of Step 3 - Proposition B.8. First, we prove continuity. Take $\hat{U}_1$ and $\hat{U}_2$ and define $\hat{W}_1 = BR^h(\hat{J};\hat{\rho}\hat{U}_1)$ and $\hat{W}_2 = BR^h(\hat{J};\hat{\rho}\hat{U}_2)$. Then,

$$a(\hat{W}_1, v - \hat{W}_1) \geq (1 - \hat{\rho}\hat{U}_1, v - \hat{W}_1), \tag{B.31}$$

$$a(\hat{W}_2, v - \hat{W}_2) \geq (1 - \hat{\rho}\hat{U}_2, v - \hat{W}_2). \tag{B.32}$$

Let $\hat{W}_2$ be the test function for (B.31) and let $\hat{W}_1$ be the test function for (B.32). Summing both equations

$$a(\hat{W}_1, \hat{W}_2 - \hat{W}_1) + a(\hat{W}_2, \hat{W}_1 - \hat{W}_2) \geq (1 - \hat{\rho}\hat{U}_1, \hat{W}_2 - \hat{W}_1) + (1 - \hat{\rho}\hat{U}_2, \hat{W}_1 - \hat{W}_2)$$

or equivalently

$$a(\hat{W}_1 - \hat{W}_2, \hat{W}_2 - \hat{W}_1) \geq (\hat{\rho}(\hat{U}_2 - \hat{U}_1), \hat{W}_2 - \hat{W}_1).$$

Multiplying by -1 on both sides and under the observation that $(\hat{\rho}(\hat{U}_2 - \hat{U}_1), \hat{W}_2 - \hat{W}_1) = \hat{\rho}(\hat{U}_2 - \hat{U}_1)(1, \hat{W}_2 - \hat{W}_1)$, we obtain

$$a(\hat{W}_2 - \hat{W}_1, \hat{W}_2 - \hat{W}_1) \leq \hat{\rho}(\hat{U}_1 - \hat{U}_2)(1, \hat{W}_2 - \hat{W}_1).$$

Given that the operator is coercive and that

$$(1, \hat{W}_2 - \hat{W}_1) = \int_{\mathbb{R}} \left(\hat{W}(\hat{w};\hat{\rho}\hat{U}_2) - \hat{W}(\hat{w};\hat{\rho}\hat{U}_1)\right) \mathrm{d}\hat{w} \leq \left(\int_{\mathbb{R}} \left(\hat{W}(\hat{w};\hat{\rho}\hat{U}_2) - \hat{W}(\hat{w};\hat{\rho}\hat{U}_1)\right)^2 \mathrm{d}\hat{w}\right)^{1/2},$$

we have that

$$\beta||\hat{W}_2 - \hat{W}_1||^2 \leq a(\hat{W}_2 - \hat{W}_1, \hat{W}_2 - \hat{W}_1) \leq \hat{\rho}(\hat{U}_1 - \hat{U}_2)(1, \hat{W}_2 - \hat{W}_1) \leq \hat{\rho}|\hat{U}_1 - \hat{U}_2|||\hat{W}_2 - \hat{W}_1||$$

for some $\beta > 0$. Thus,

$$||\hat{W}_2 - \hat{W}_1|| \leq \frac{\hat{\rho}}{\beta}|\hat{U}_1 - \hat{U}_2|$$

With this inequality, we can verify the continuity of $\hat{W}(\hat{w};\hat{\rho}\hat{U})$. Let $\epsilon > 0$ and choose $|\hat{U}_1 - \hat{U}_2| < \epsilon\frac{\beta}{\hat{\rho}}$. Then

$$||\hat{W}_2 - \hat{W}_1|| < \epsilon.$$

Thus, $\hat{W}(\hat{w};\hat{\rho}\hat{U})$ is continuous.

Now, we prove that $\hat{W}(\hat{w};\hat{\rho}\hat{U})$ is decreasing in the second argument. Let $\hat{U}_1 > \hat{U}_2$ and define $\hat{W}_1 = BR^h(\hat{J};\hat{\rho}\hat{U}_1)$ and $\hat{W}_2 = BR^h(\hat{J};\hat{\rho}\hat{U}_2)$. Observe that $\hat{W}_1, \hat{W}_2 \in K^h(\hat{J})$. Thus, $\min\{\hat{W}_1, \hat{W}_2\}$ and $\max\{\hat{W}_1, \hat{W}_2\} \in K^h(\hat{J})$. Therefore, we can use $\min\{\hat{W}_1, \hat{W}_2\} = \hat{W}_1 - \max\{\hat{W}_1 - \hat{W}_2, 0\}$ as a test function with $\hat{U}_1$ and $\max\{\hat{W}_1, \hat{W}_2\} = \hat{W}_2 + \max\{\hat{W}_1 - \hat{W}_2, 0\}$ as a test function with $\hat{U}_2$. Therefore,

$$-a(\hat{W}_1, \max\{\hat{W}_1 - \hat{W}_2, 0\}) \geq -(1 - \hat{\rho}\hat{U}_1, \max\{\hat{W}_1 - \hat{W}_2, 0\}),$$
$$a(\hat{W}_2, \max\{\hat{W}_1 - \hat{W}_2, 0\}) \geq (1 - \hat{\rho}\hat{U}_2, \max\{\hat{W}_1 - \hat{W}_2, 0\}).$$

Adding both inequalities, we obtain

$$a(\hat{W}_2 - \hat{W}_1, \max\{\hat{W}_1 - \hat{W}_2, 0\}) \geq \hat{\rho}\left(\hat{U}_1 - \hat{U}_2\right)(1, \max\{\hat{W}_1 - \hat{W}_2, 0\}).$$

Multiplying by -1 and under the observation that $a(\hat{W}^1 - \hat{W}^2, \max\{\hat{W}^1 - \hat{W}^2, 0\}) = a(\max\{\hat{W}^1 - \hat{W}^2, 0\}, \max\{\hat{W}^1 - \hat{W}^2, 0\}) \geq \beta || \max\{\hat{W}^1 - \hat{W}^2, 0\}||^2$ for some $\beta > 0$, we have that

$$|| \max\{\hat{W}_1 - \hat{W}_2, 0\}||^2 \leq \frac{\hat{\rho}}{\beta}\left(\hat{U}_2 - \hat{U}_1\right)(1, \max\{\hat{W}_1 - \hat{W}_2, 0\}).$$

Since $\hat{U}_1 > \hat{U}_2$, we have that $\hat{U}_2 - \hat{U}_1 < 0$. Assume, by contradiction, that $\hat{W}_1 > \hat{W}_2$, then $(1, \max\{\hat{W}_1 - \hat{W}_2, 0\}) > 0$. Operating,

$$0 < || \max\{\hat{W}_1 - \hat{W}_2, 0\}||^2 \leq \frac{\hat{\rho}}{\beta}\left(\hat{U}_2 - \hat{U}_1\right)(1, \max\{\hat{W}_1 - \hat{W}_2, 0\}) < 0.$$

Thus, we have a contradiction. In conclusion, $\hat{W}(\hat{w}; \hat{\rho}\hat{U})$ is decreasing in the second argument. Observe that $\hat{J}(\hat{w}) = BR^j(\hat{W})$ is independent of $\hat{\rho}\hat{U}$. □

Proposition B.9. *Let $\hat{W}^*(\hat{w}; \hat{\rho}\hat{U})$ be the non-trivial Nash Equilibrium, then it is continuous and decreasing in the second argument.*

Proof of Step 3 - Proposition B.9. First, we show that the value function from the non-trivial Nash equilibrium is decreasing in $\hat{U}$. If $\hat{U}_1 > \hat{U}_2$, we have, by the previous step, that

$$Q(\hat{W}, \hat{\rho}\hat{U}_1) \leq Q(\hat{W}, \hat{\rho}\hat{U}_2).$$

Define recursively $Q^n(\hat{W}, \hat{\rho}\hat{U}_1) = Q \circ Q^{n-1}(\hat{W}, \hat{\rho}\hat{U}_1)$. By monotonicity,

$$Q^n(\hat{W}, \hat{\rho}\hat{U}_1) \leq Q^n(\hat{W}, \hat{\rho}\hat{U}_2)$$

also holds for all n. By Theorem 18 of Marinacci and Montrucchio (2019),

$$Q^n(\hat{W}, \hat{\rho}\hat{U}_1) \to \hat{W}^*(\hat{w}; \hat{\rho}\hat{U}_1) \text{ and } Q^n(\hat{W}, \hat{\rho}\hat{U}_2) \to \hat{W}^*(\hat{w}; \hat{\rho}\hat{U}_2).$$

Thus,

$$\hat{W}^*(\hat{w}; \hat{\rho}\hat{U}_1) \leq \hat{W}^*(\hat{w}; \hat{\rho}\hat{U}_2).$$

In conclusion, the non-trivial Nash equilibrium is decreasing in $\hat{U}$.

Now, we show continuity. Take $\hat{U}_n \uparrow \hat{U}^*$ (resp. $\hat{U}_n \downarrow \hat{U}^*$). Then, it is easy to see that $\hat{W}^*(\hat{w}; \hat{\rho}\hat{U}^n)$ is monotonic, and by completeness, it is easy to see that $\hat{W}^*(\hat{w}; \hat{\rho}\hat{U}^n)$ is a convergent series. Thus, $\hat{W}^*(\hat{w}; \hat{\rho}\hat{U})$ is continuous in the second element. □

Step 4. We now show the existence of the unique fixed point in $\hat{\rho}\hat{U}$. Using the free entry condition, we can define the value of the unemployed worker as

$$P(\hat{\rho}\hat{U}) := \tilde{B} + \max_{\hat{w}} \frac{1}{\tilde{K}^{1/\alpha}} \hat{J}(\hat{w};\hat{\rho}\hat{U})^{\frac{1-\alpha}{\alpha}} \hat{W}(\hat{w};\hat{\rho}\hat{U}).$$

We now show two propositions: (i) we show relevant properties of $P(\hat{\rho}\hat{U})$, (ii) we use these properties to show the existence of a unique fixed point $P(\hat{\rho}\hat{U}^*) = \hat{\rho}\hat{U}^*$.

Proposition B.10. *The following properties hold for $P(\hat{\rho}\hat{U})$:*

- *$P(\hat{\rho}\hat{U})$ exists and is unique.*
- *$P(\hat{\rho}\hat{U})$ is continuous.*
- *$P : [\tilde{B}, \bar{P}] \to [\tilde{B}, \bar{P}]$ and it is decreasing.*

Proof of Step 4 - Proposition B.10. From Proposition 2, we have that $\hat{\mathcal{C}}^h \cap \hat{\mathcal{C}}^j$ is bounded, thus

$$\max_{\hat{w}} \frac{1}{\tilde{K}^{1/\alpha}} \hat{J}(\hat{w};\hat{\rho}\hat{U})^{\frac{1-\alpha}{\alpha}} \hat{W}(\hat{w};\hat{\rho}\hat{U}) = \max_{\hat{w}\in cl\{\mathcal{C}^j\cap\mathcal{C}^h\}} \frac{1}{\tilde{K}^{1/\alpha}} \hat{J}(\hat{w};\hat{\rho}\hat{U})^{\frac{1-\alpha}{\alpha}} \hat{W}(\hat{w};\hat{\rho}\hat{U}).$$

Since $\hat{J}(\cdot;\hat{\rho}\hat{U})$ and $\hat{W}(\cdot;\hat{\rho}\hat{U})$ are continuous and the optimization is conducted over a compact support, by the extreme value theorem there exists a maximum and, clearly, is unique.

Since $\hat{J}(\hat{w};\hat{\rho}\hat{U})$ and $\hat{W}(\hat{w};\hat{\rho}\hat{U})$ are continuous in both arguments, by the maximum theorem, the maximal value is continuous. Let $\hat{w}^*(\hat{\rho}\hat{U})$ be the solution to the optimization problem. Then, if $\hat{U} < \hat{U}'$,

$$\begin{aligned}\frac{1}{\tilde{K}^{1/\alpha}} \hat{J}(\hat{w}^*(\hat{\rho}\hat{U});\hat{\rho}\hat{U})^{\frac{1-\alpha}{\alpha}} \hat{W}(\hat{w}^*(\hat{\rho}\hat{U});\hat{\rho}\hat{U}) &\geq^{(1)} \frac{1}{\tilde{K}^{1/\alpha}} \hat{J}(\hat{w}^*(\hat{\rho}\hat{U}');\hat{\rho}\hat{U})^{\frac{1-\alpha}{\alpha}} \hat{W}(\hat{w}^*(\hat{\rho}\hat{U}'),\hat{\rho}\hat{U}) \\ &\geq^{(2)} \frac{1}{\tilde{K}^{1/\alpha}} \hat{J}(\hat{w}^*(\hat{\rho}\hat{U}'),\hat{\rho}\hat{U}')^{\frac{1-\alpha}{\alpha}} \hat{W}(\hat{w}^*(\hat{\rho}\hat{U}'),\hat{\rho}\hat{U}').\end{aligned}$$

Step (1) uses the optimality of $\hat{w}^*(\hat{\rho}\hat{U})$ and step (2) uses the fact that $\hat{J}$ and $\hat{W}$ are decreasing in the second argument. Thus, $P(\hat{\rho}\hat{U})$ is decreasing. By Proposition 2, we have that $\bar{P} =: P(\tilde{B}) > \tilde{B}$. Since $P(\hat{\rho}\hat{U}) \geq \tilde{B}$ ($\hat{J}(\cdot)$ and $\hat{W}(\cdot)$ are non-negative), we have that $P : [\tilde{B}, \bar{P}] \to [\tilde{B}, \bar{P}]$. □

Proposition B.11. *$P(\hat{\rho}\hat{U})$ has a unique fixed point.*

Proof of Step 4 - Proposition B.11. The existence of the fixed point follows directly from Brouwer's fixed point theorem. To show uniqueness, observe that if there were two fixed points $\hat{U}_1 < \hat{U}_2$, by definition, we would have that $P(\hat{\rho}\hat{U}_1) = \hat{\rho}\hat{U}_1 < \hat{\rho}\hat{U}_2 = P(\hat{\rho}\hat{U}_2)$ and $P(\hat{\rho}\hat{U})$ would be strictly increasing. By Step 4-Proposition B.10, this is a contradiction. □

B.4 Proof of Proposition 2

Proposition 2. *The BRE has the following properties:*

1. *The joint match surplus satisfies*

$$\hat{S}(\hat{w}) = (1 - \hat{\rho}\hat{U})\mathcal{T}(\hat{w}, \hat{\rho}),$$

where

$$\mathcal{T}(\hat{w}, \hat{\rho}) := \mathbb{E}\left[\int_0^{\tau^{m*}} e^{-\hat{\rho}t}\, \mathrm{d}t | \hat{w}_0 = \hat{w}\right]$$

is the expected discounted match duration and $1 > \hat{\rho}\hat{U} > \tilde{B}$.

2. *The competitive entry wage* $\hat{w}^*$ *coincides with the Nash bargaining solution with worker's weight* α:

$$\hat{w}^* = \arg\max_{\hat{w}} \left\{ \hat{W}(\hat{w})^{\alpha} \hat{J}(\hat{w})^{1-\alpha} \right\} = \arg\max_{\hat{w}} \left\{ \eta(\hat{w})^{\alpha} (1-\eta(\hat{w}))^{1-\alpha} \mathcal{T}(\hat{w}, \hat{\rho}) \right\},$$

with optimality condition

$$\underbrace{\eta'(\hat{w}^*) \left(\frac{\alpha}{\eta(\hat{w}^*)} - \frac{1-\alpha}{1-\eta(\hat{w}^*)} \right)}_{\text{Share channel}} = - \underbrace{\frac{\mathcal{T}_{\hat{w}}(\hat{w}^*, \hat{\rho})}{\mathcal{T}(\hat{w}^*, \hat{\rho})}}_{\text{Surplus channel}}.$$

3. *Given* $\eta(\hat{w}^*)$ *and* $\mathcal{T}(\hat{w}^*, \hat{\rho})$, *the equilibrium job finding rate* $f(\hat{w}^*)$ *and the flow opportunity cost of employment* $\hat{\rho}\hat{U}$ *are given by*

$$f(\hat{w}^*) = \left[(1-\eta(\hat{w}^*))(1-\hat{\rho}\hat{U}) \mathcal{T}(\hat{w}^*, \hat{\rho}) / \tilde{K} \right]^{\frac{1-\alpha}{\alpha}},$$
$$\hat{\rho}\hat{U} = \tilde{B} + \left(\tilde{K}^{\alpha-1} \left(1-\eta(\hat{w})\right)^{1-\alpha} \eta(\hat{w})^{\alpha} \left(1-\hat{\rho}\hat{U}\right) \mathcal{T}(\hat{w}^*, \hat{\rho}) \right)^{\frac{1}{\alpha}}.$$

4. *Assume* $\gamma \neq 0$ *or* $\sigma \neq 0$. *Given* $\hat{U}$, *the worker's and the firm's continuation sets are connected, and the game's continuation set is bounded; i.e.*

$$\hat{\mathcal{C}}^h = \{\hat{w} : \hat{w} > \hat{w}^-\} \quad \text{and} \quad \hat{\mathcal{C}}^j = \{\hat{w} : \hat{w} < \hat{w}^+\},$$

with $-\infty < \hat{w}^- \leq \log(\hat{\rho}\hat{U}) < 0 \leq \hat{w}^+ < \infty$. *The worker's and firm's value functions satisfy* smooth pasting conditions *at* $\hat{w}^-$ *and* $\hat{w}^+$, *respectively:* $\hat{W}'(\hat{w}^-) = \hat{J}'(\hat{w}^+) = 0$.

Proof. Now, we prove each equilibrium property.

1. Using the free entry condition and worker optimality, we have that $\hat{\theta}(\hat{w}) \geq 0$ and $\hat{W}(\hat{w}) \geq 0$ for all $\hat{w}$; thus, the product is also non-negative at $\hat{w}^*$ and

$$\hat{\rho}\hat{U} = \tilde{B} + \max_{\hat{w}} \hat{\theta}(\hat{w})^{1-\alpha} \hat{W}(\hat{w}) \geq \hat{B} \quad \Longrightarrow \quad \hat{\rho}U \geq \tilde{B}$$

Using the recursive definition of the value function, we have

$$\hat{W}(\hat{w}) = \mathbb{E}_{\hat{w}} \left[\int_0^{\tau^{m*}} e^{-\hat{\rho}t} (e^{\hat{w}_t} - \hat{\rho}\hat{U}) \, \mathrm{d}t \right]$$
$$\hat{J}(\hat{w}) = \mathbb{E}_{\hat{w}} \left[\int_0^{\tau^{m*}} e^{-\hat{\rho}t} (1 - e^{\hat{w}_t}) \, \mathrm{d}t \right]$$

where τ^{m*} is the non-trivial Nash equilibrium of the game between the firm and the worker. Summing up the previous two equations, we have

$$\hat{S}(\hat{w}) = \hat{W}(\hat{w}) + \hat{J}(\hat{w}) = \mathbb{E}_{\hat{w}} \left[\int_0^{\tau^{m*}} e^{-\hat{\rho}t} (1-\hat{\rho}\hat{U}) \, \mathrm{d}t \right] = (1-\hat{\rho}\hat{U}) \mathbb{E}_{\hat{w}} \left[\int_0^{\tau^{m*}} e^{-\hat{\rho}t} \, \mathrm{d}t \right] = (1-\hat{\rho}\hat{U}) \mathcal{T}(\hat{w}, \hat{\rho}).$$

Since $\hat{W}(\hat{w}), \hat{J}(\hat{w}) \geq 0$, $\hat{S}(\hat{w}) \geq 0$ and

$$0 \leq \hat{S}(\hat{w}^*) = (1-\hat{\rho}\hat{U}) \underbrace{\mathcal{T}(\hat{w}^*, \hat{\rho})}_{>0} \iff 0 \leq 1 - \hat{\rho}\hat{U} \iff 1 \geq \hat{\rho}\hat{U}.$$

Therefore, $1 \geq \hat{\rho}\hat{U} \geq \tilde{B}$.

Now, we show the strict inequality by contradiction. Assume that $\hat{\rho}\hat{U} = \tilde{B} < 1$. Then, we have that $\max_{\hat{w}} \hat{\theta}(\hat{w})^{1-\alpha}\hat{W}(\hat{w}) = 0$ and, therefore, $\hat{\theta}(\hat{w})^{1-\alpha}\hat{W}(\hat{w}) = 0\ \forall \hat{w}$. By weakly dominated strategies, we have that $(\log(\hat{\rho}\hat{U}), 0) = (\log(\tilde{B}), 0) \subset \hat{C}^j \cap \hat{C}^h$. Thus, for any $\hat{w} \in (\log(\tilde{B}), 0)$, we have that $(\hat{J}(\hat{w}), \hat{W}(\hat{w})) > (0, 0)$ and using the free entry condition $\hat{\theta}(\hat{w})^{1-\alpha}\hat{W}(\hat{w}) > 0$. Thus, a contradiction. Assume instead that $\hat{\rho}\hat{U} = 1$. Then, $0 = \hat{S}(\hat{w}) \geq (\hat{J}(\hat{w}), \hat{W}(\hat{w})) \geq 0\ \forall \hat{w}$ and $\max_{\hat{w}^*} \hat{\theta}(\hat{w}^*)^{1-\alpha}\hat{W}(\hat{w}^*) = 0$. With these argument, we have that $1 = \tilde{B} + \max_{\hat{w}^*} \hat{\theta}(\hat{w}^*)^{1-\alpha}\hat{W}(\hat{w}^*) = \hat{B} < 1$, and we have the contradiction.

2. To show this property, first we show that $\hat{J}(\hat{w}) > 0$ for all $\hat{w} \in (\log(\hat{\rho}\hat{U}), 0)$. Define

$$\tau_{(w^-,0)} = \inf_t \{t : \hat{w}_t \notin (\log(\hat{\rho}\hat{U}), 0)\}.$$

By optimality of the firm,

$$\hat{J}(\hat{w}) = \mathbb{E}_{\hat{w}}\left[\int_0^{\tau^{m*}} e^{-\hat{\rho}t}(1 - e^{\hat{w}_t})\,\mathrm{d}t\right] \geq \mathbb{E}_{\hat{w}}\left[\int_0^{\min\{\tau_{(\log(\hat{\rho}\hat{U}),0)},\tau^{m*}\}} e^{-\hat{\rho}t}(1 - e^{\hat{w}_t})\,\mathrm{d}t\right] > 0.$$

Thus, there is an open set around the optimally chosen starting wage $\hat{w}$ that lies entirely within the continuation region s.t. $\hat{J}(\hat{w}) > 0$, $\hat{\theta}(\hat{w}) > 0$, and $\hat{J}(\hat{w}) - \hat{K}\hat{\theta}(\hat{w})^{\alpha} = 0$. Therefore,

$$\arg\max_{\hat{w}} \{f(\hat{\theta}(\hat{w}))\hat{W}(\hat{w})\} = \arg\max_{\hat{w}} \left\{\left(\frac{\hat{J}(\hat{w})}{\hat{K}}\right)^{\frac{1-\alpha}{\alpha}} \hat{W}(\hat{w})\right\} = \arg\max_{\hat{w}} \left\{\hat{J}(\hat{w})^{1-\alpha}\hat{W}(\hat{w})^{\alpha}\right\}.$$

Since $\hat{W}(\hat{w}) = \eta(\hat{w})\hat{S}(\hat{w})$ and $\hat{J}(\hat{w}) = (1 - \eta(\hat{w}))\hat{S}(\hat{w})$ and $\hat{S}(\hat{w}) = (1 - \hat{\rho}\hat{U})\mathcal{T}(\hat{w}, \hat{\rho})$,

$$\arg\max_{\hat{w}} \{f(\hat{\theta}(\hat{w}))\hat{W}(\hat{w})\} = \arg\max_{\hat{w}} \left\{\hat{J}(\hat{w})^{1-\alpha}\hat{W}(\hat{w})^{\alpha}\right\} = \arg\max_{\hat{w}} \left\{(1 - \eta(\hat{w}))^{1-\alpha}\eta(\hat{w})^{\alpha}\mathcal{T}(\hat{w}, \hat{\rho})\right\}.$$

Taking first order conditions

$$\eta'(\hat{w}^*)\left(\frac{\alpha}{\eta(\hat{w}^*)} - \frac{1-\alpha}{1 - \eta(\hat{w}^*)}\right) = -\frac{\mathcal{T}'_{\hat{w}}(\hat{w}^*, \hat{\rho})}{\mathcal{T}(\hat{w}^*, \hat{\rho})}.$$

3. This step follows directly from workers' and firms' optimality conditions.

4. To show that $\hat{C}^h$ and $\hat{C}^j$ are connected, assume they are not. Without loss of generality, assume that $\hat{C}^h = \{\hat{w} : \hat{w} > \hat{w}^-\} \cup (a, b)$ with $a < b < w^-$. Then, since $\hat{w}^- \leq \hat{\rho}\hat{U}$, it must be the case that for all $\hat{w} \in (a, b)$, we have $(e^{\hat{w}} - \hat{\rho}\hat{U}) < 0$ for all $\hat{w} \in (a, b)$, and $\hat{W}(\hat{w}) = \mathbb{E}_{\hat{w}}\left[\int_0^{\tau_{\hat{C}^h \cap \hat{C}^j}} e^{-(\hat{\rho}+\delta)t}(e^{w_t} - \hat{\rho}\hat{U})\,\mathrm{d}t\right] < 0$ for all $\hat{w} \in (a, b)$ due to continuity of Brownian motions. Since $\hat{W}(\hat{w}) \geq 0$, we have a contradiction. A similar argument holds for the firm's continuation set.

We prove that $-\infty < \hat{w}^-$ by contradiction. Assume that $-\infty = \hat{w}^-$, then

$$\hat{W}(\hat{w}, \hat{w}^+) := \mathbb{E}\left[\int_0^{\tau_{(-\infty,\hat{w}^+)} \wedge \tau^{\delta}} e^{-\hat{\rho}t}\left(e^{\hat{w}_t} - \hat{\rho}\hat{U}\right)\mathrm{d}t | \hat{w}_0 = \hat{w}\right].$$

Then, since $\hat{\rho}\hat{U} < e^{\hat{w}^+}$, it is easy to show

$$\begin{aligned}\hat{W}(\hat{w}, \hat{w}^+) &= \mathbb{E}\left[\int_0^{\tau_{(-\infty,\hat{w}^+)} \wedge \tau^{\delta}} e^{-\hat{\rho}t}\left(e^{\hat{w}_t} - \hat{\rho}\hat{U}\right)\mathrm{d}t | \hat{w}_0 = \hat{w}\right] \\ &\leq \mathbb{E}\left[\int_0^{\infty} e^{-(\hat{\rho}+\delta)t}\left(e^{\hat{w}_t} - \hat{\rho}\hat{U}\right)\mathrm{d}t | \hat{w}_0 = \hat{w}\right] \\ &= \frac{e^{w}}{\hat{\rho} + \delta + \hat{\gamma} - \sigma^2/2} - \frac{\hat{\rho}\hat{U}}{\hat{\rho} + \delta}\end{aligned}$$

$$= \frac{e^w}{\rho - \gamma - \sigma^2/2 + \delta + \gamma + \sigma^2 - \sigma^2/2} - \frac{\hat{\rho}\hat{U}}{\rho - \gamma - \sigma^2/2 + \delta}$$
$$= \frac{e^w}{\rho + \delta} - \frac{\hat{\rho}\hat{U}}{\rho - \gamma - \sigma^2/2 + \delta}$$

Thus, there exists a small enough $\hat{w}$ s.t. $\hat{W}(\hat{w}, \hat{w}^+) < 0$, and we have a contradiction. A similar argument holds for the firm's separation threshold.

Finally, the smooth pasting conditions are necessary and sufficient for the optimal stopping times (see Brekke and Øksendal, 1990). □

B.5 Proof of Propositions 3, 4, and 5

Define $\hat{C} = (\hat{w}^-, \hat{w}^+)$. From Proposition 2, we can work with the following HJB conditions

$$(\hat{\rho} + \delta)\hat{W}(\hat{w}) = e^{\hat{w}} - \hat{\rho}\hat{U} - \hat{\gamma}\hat{W}'(\hat{w}) + \frac{\sigma^2}{2}\hat{W}''(\hat{w}) \quad \forall \hat{w} \in \hat{C}^j \cap \hat{C}^h \tag{B.33}$$
$$(\hat{\rho} + \delta)\hat{J}(\hat{w}) = 1 - e^{\hat{w}} - \hat{\gamma}\hat{J}'(\hat{w}) + \frac{\sigma^2}{2}\hat{J}''(\hat{w}) \quad \forall \hat{w} \in \hat{C}^j \cap \hat{C}^h \tag{B.34}$$
$$\hat{\rho}\hat{U} = \tilde{B} + \tilde{K}^{1-\alpha^{-1}}\hat{J}(\hat{w}^*)^{\frac{1-\alpha}{\alpha}}\hat{W}(\hat{w}^*)$$
$$(1-\alpha)\frac{\mathrm{d}\log \hat{J}(\hat{w}^*)}{\mathrm{d}\hat{w}} = -\alpha\frac{\mathrm{d}\log \hat{W}(\hat{w}^*)}{\mathrm{d}\hat{w}},$$

with the value matching conditions

$$\hat{W}(\hat{w}^-) = \hat{J}(\hat{w}^-) = \hat{W}(\hat{w}^+) = \hat{J}(\hat{w}^+) = 0$$

and smooth pasting conditions

$$\hat{W}'(\hat{w}^-) = \hat{J}'(\hat{w}^+) = 0.$$

Proposition 3. *Assume $\hat{\gamma} = \sigma = 0$. Then, optimal policies are given by*

$$(\hat{w}^-,\ \hat{w}^*,\ \hat{w}^+) = \log(\hat{\rho}\hat{U}, \alpha + (1-\alpha)\hat{\rho}\hat{U}, 1),$$

with $\eta(\hat{w}^*) = \alpha$ and $\mathcal{T}(\hat{w}^*, \hat{\rho}) = 1/(\hat{\rho} + \delta)$.

Proof. If $\hat{\gamma} = \sigma = 0$, conditions (B.33) and (B.34) imply

$$\hat{W}(\hat{w}) = \frac{e^{\hat{w}} - e^{\hat{w}^-}}{\hat{\rho} + \delta} \quad ; \quad \hat{J}(\hat{w}) = \frac{e^{\hat{w}^+} - e^{\hat{w}}}{\hat{\rho} + \delta}.$$

The variation inequalities imply

$$(\hat{\rho} + \delta)\hat{W}(\hat{w}) = \max\{0, e^{\hat{w}} - \hat{\rho}\hat{U}\},\ \forall \hat{w} \in \hat{C}^j,$$
$$(\hat{\rho} + \delta)\hat{J}(\hat{w}) = \max\{0, 1 - e^{\hat{w}}\},\ \forall \hat{w} \in \hat{C}^h, .$$

Thus, $\hat{W}(\hat{w}^-) = \hat{J}(\hat{w}^+) = 0$ and

$$\hat{w}^+ = 0 \quad ; \quad \hat{w}^- = \log\left(\hat{\rho}\hat{U}\right).$$

Since

$$\mathcal{T}(\hat{w},\hat{\rho}) = \begin{cases} (\hat{\rho}+\delta)^{-1} & \text{if } \hat{w} \in [\hat{w}^-, \hat{w}^+] \\ 0 & \text{otherwise,} \end{cases}$$

and $\mathcal{T}_{\hat{w}}(\hat{w}^*,\hat{\rho}) = 0$, we have that the worker's share of the surplus $\eta(\hat{w}^*) = \alpha$. □

Proposition 4. *Assume $\hat{\gamma} = 0$, $\alpha = 1/2$ and a first-order approximation of the flow payoffs around $\hat{w}^*$. Then $\hat{w}^{\pm} = \hat{w}^* \pm h(\varphi, \Phi)$*

$$e^{\hat{w}^*} = \frac{1+\hat{\rho}\hat{U}}{2} \text{ and } h(\varphi,\Phi) = \omega(2\varphi\Phi)\Phi$$

with $\varphi = \sqrt{2(\hat{\rho}+\delta)}/\sigma$, $\Phi = \frac{1-\hat{\rho}\hat{U}}{1+\hat{\rho}\hat{U}}$. The following properties hold for ω: (i) $\omega(z)$ decreases for all $z \in (0,\infty)$, (ii) $\lim_{z\to 0}\omega(z) = 3$, (iii) $\lim_{z\to\infty}\omega(z) = 1$, (iv) $\omega(2\varphi\Phi)\Phi$ is increasing in Φ, and (v) $\varphi\omega(2\varphi\Phi)$ is increasing in φ. $\eta(\hat{w}) = \alpha$ and

$$\mathcal{T}(\hat{w}^*,\hat{\rho}) = \frac{1 - sech(\varphi h(\varphi,\Phi))}{\hat{\rho}+\delta}$$

increasing in φ and Φ.

Proof. Let us guess and verify the following solution $w^* = \log\left(\frac{1+\hat{\rho}\hat{U}}{2}\right)$ and $\hat{w}^- = \hat{w}^* - h$ and $\hat{w}^+ = \hat{w}^* + h$ for a given h. Using a Taylor approximation over the flow profits around w^*

$$\begin{aligned} e^{\hat{w}} - \hat{\rho}\hat{U} &\approx e^{\hat{w}^*}(1 + (\hat{w} - \hat{w}^*)) - \hat{\rho}\hat{U} = \frac{1-\hat{\rho}\hat{U}}{2} + e^{\hat{w}^*}(\hat{w} - \hat{w}^*), \\ 1 - e^{\hat{w}} &\approx 1 - e^{\hat{w}^*}(1 + (\hat{w} - \hat{w}^*)) = \frac{1-\hat{\rho}\hat{U}}{2} - e^{\hat{w}^*}(\hat{w} - \hat{w}^*). \end{aligned}$$

We can write the optimality conditions as

$$\begin{aligned} (\hat{\rho}+\delta)\hat{W}(\hat{w}) &= \frac{1-\hat{\rho}\hat{U}}{2} + e^{\hat{w}^*}(\hat{w} - \hat{w}^*) + \frac{\sigma^2}{2}\hat{W}''(\hat{w}), \quad \forall \hat{w} \in (w^* - h, w^* + h) \\ (\hat{\rho}+\delta)\hat{J}(\hat{w}) &= \frac{1-\hat{\rho}\hat{U}}{2} - e^{\hat{w}^*}(\hat{w} - \hat{w}^*) + \frac{\sigma^2}{2}\hat{J}''(\hat{w}), \quad \forall \hat{w} \in (w^* - h, w^* + h) \end{aligned}$$

with the border conditions

$$\begin{aligned} &\hat{W}(\hat{w}^* - h) = \hat{J}(\hat{w}^* - h) = \hat{W}(\hat{w}^* + h) = \hat{J}(\hat{w}^* + h) = 0, \\ &\hat{W}'(\hat{w}^* - h) = \hat{J}'(\hat{w}^* + h) = 0. \end{aligned}$$

Now, we show that we can transform $J(x) = \frac{\hat{J}(x+\hat{w}^*) - \frac{1-\hat{\rho}\hat{U}}{2(\hat{\rho}+\delta)}}{e^{w^*}}$. A similar argument applies to the value function of the worker. Making the following transformation $J(x) = \frac{\hat{J}(x+\hat{w}^*) - \frac{1-\hat{\rho}\hat{U}}{2(\hat{\rho}+\delta)}}{e^{w^*}}$, and using (B.34)

$$\begin{aligned} (\hat{\rho}+\delta)J(x) &= (\hat{\rho}+\delta)\left(\frac{\hat{J}(x+\hat{w}^*) - \frac{1-\hat{\rho}\hat{U}}{2(\hat{\rho}+\delta)}}{e^{w^*}}\right), \\ &= -x + \frac{\sigma^2}{2}\frac{1}{e^{w^*}}\hat{J}''(x+\hat{w}^*), \\ &= -x + \frac{\sigma^2}{2}J''(x). \end{aligned}$$

Thus,

$$(\hat{\rho}+\delta)W(x) = x + \frac{\sigma^2}{2}W''(x) \quad \forall x \in (-h,h)$$
$$(\hat{\rho}+\delta)J(x) = -x + \frac{\sigma^2}{2}J''(x) \quad \forall x \in (-h,h)$$

Defining

$$\Phi = \frac{\frac{1-\hat{\rho}\hat{U}}{2}}{e^{w^*}} = \frac{1-\hat{\rho}\hat{U}}{1+\hat{\rho}\hat{U}} > 0,$$

it is easy to show

$$W(h) = J(h) = W(-h) = J(-h) = -\frac{\Phi}{\hat{\rho}+\delta} \quad ; \quad W'(-h) = J'(h) = 0.$$

Thus, $W(x) = J(-x)$. Given that this problem is symmetric, we verify the guess of symmetry of the Ss bands and $\frac{1}{2}W'(0) = -\frac{1}{2}J'(-0)$. The latter property, implies that w^* satisfies the proposed Nash bargaining solution.

Now, we show that $h = \omega(\varphi)\Phi$ with $\varphi = \sqrt{2(\hat{\rho}+\delta)}/\sigma$. Note that $W(x) = J(-x)$. Thus, we can only focus on $W(x)$ using the smooth pasting condition evaluated at $-h$. The solution to this system of differential equations is given by

$$W(x) = Ae^{\varphi x} + Be^{-\varphi x} + \frac{x}{\hat{\rho}+\delta}$$
$$W(h) = W(-h) = -\frac{\Phi}{\hat{\rho}+\delta} \;\; and \;\; W'(-h) = 0$$

with $\varphi = \sqrt{2(\hat{\rho}+\delta)}/\sigma$. Writing the value matching conditions

$$Ae^{\varphi h} + Be^{-\varphi h} + \frac{h}{\hat{\rho}+\delta} = -\frac{\Phi}{\hat{\rho}+\delta}$$
$$Ae^{-\varphi h} + Be^{\varphi h} - \frac{h}{\hat{\rho}+\delta} = -\frac{\Phi}{\hat{\rho}+\delta}$$

Taking the difference and the sum

$$A\left(e^{\varphi h} + e^{-\varphi h}\right) + B\left(e^{-\varphi h} + e^{\varphi h}\right) = -2\frac{\Phi}{\hat{\rho}+\delta}$$
$$A\left(e^{\varphi h} - e^{-\varphi h}\right) + B\left(e^{-\varphi h} - e^{\varphi h}\right) = -2\frac{h}{\hat{\rho}+\delta}.$$

Solving for A and B,

$$A = \frac{-2\frac{\Phi}{\hat{\rho}+\delta}\left(e^{-\varphi h} - e^{\varphi h}\right) + 2\frac{h}{\hat{\rho}+\delta}\left(e^{-\varphi h} + e^{\varphi h}\right)}{\left(e^{\varphi h} + e^{-\varphi h}\right)\left(e^{-\varphi h} - e^{\varphi h}\right) - \left(e^{-\varphi h} + e^{\varphi h}\right)\left(e^{\varphi h} - e^{-\varphi h}\right)}$$
$$= \frac{e^{-\varphi h}\left(-\frac{\Phi}{\hat{\rho}+\delta} + \frac{h}{\hat{\rho}+\delta}\right) + e^{\varphi h}\left(\frac{h}{\hat{\rho}+\delta} + \frac{\Phi}{\hat{\rho}+\delta}\right)}{\left(e^{\varphi h} + e^{-\varphi h}\right)\left(e^{-\varphi h} - e^{\varphi h}\right)}$$
$$= -\frac{1}{\hat{\rho}+\delta}\frac{e^{-\varphi h}\left(-\Phi + h\right) + e^{\varphi h}\left(h + \Phi\right)}{e^{2\varphi h} - e^{-2\varphi h}}$$
$$B = \frac{-2\frac{h}{\hat{\rho}+\delta}\left(e^{\varphi h} + e^{-\varphi h}\right) + 2\frac{\Phi}{\hat{\rho}+\delta}\left(e^{\varphi h} - e^{-\varphi h}\right)}{\left(e^{\varphi h} + e^{-\varphi h}\right)\left(e^{-\varphi h} - e^{\varphi h}\right) - \left(e^{-\varphi h} + e^{\varphi h}\right)\left(e^{\varphi h} - e^{-\varphi h}\right)}$$

$$= -\frac{e^{\varphi h}\left(-\frac{\Phi}{\hat{\rho}+\delta}+\frac{h}{\hat{\rho}+\delta}\right)+e^{-\varphi h}\left(\frac{h}{\hat{\rho}+\delta}+\frac{\Phi}{\hat{\rho}+\delta}\right)}{\left(e^{\varphi h}+e^{-\varphi h}\right)\left(e^{-\varphi h}-e^{\varphi h}\right)}$$
$$= \frac{1}{\hat{\rho}+\delta}\frac{e^{\varphi h}\left(-\Phi+h\right)+e^{-\varphi h}\left(h+\Phi\right)}{e^{2\varphi h}-e^{-2\varphi h}}$$

Therefore
$$W(x) = -\frac{1}{\hat{\rho}+\delta}\frac{e^{-\varphi h}\left(-\Phi+h\right)+e^{\varphi h}\left(h+\Phi\right)}{e^{2\varphi h}-e^{-2\varphi h}}e^{\varphi x}+\frac{1}{\hat{\rho}+\delta}\frac{e^{\varphi h}\left(\Phi+h\right)+e^{-\varphi h}\left(h-\Phi\right)}{e^{2\varphi h}-e^{-2\varphi h}}e^{-\varphi x}+\frac{x}{\hat{\rho}+\delta}$$

Taking the derivative and evaluating in $x=-h$

$$W'(-h) = -\frac{1}{\hat{\rho}+\delta}\frac{e^{-\varphi h}\left(-\Phi+h\right)+e^{\varphi h}\left(h+\Phi\right)}{e^{2\varphi h}-e^{-2\varphi h}}\varphi e^{-\varphi h}-\frac{1}{\hat{\rho}+\delta}\frac{e^{\varphi h}\left(-\Phi+h\right)+e^{-\varphi h}\left(h+\Phi\right)}{e^{2\varphi h}-e^{-2\varphi h}}\varphi e^{\varphi h}+\frac{1}{\hat{\rho}+\delta}=0$$

or equivalently
$$-\Phi(e^{-2\varphi h}+e^{2\varphi h}-2)=\frac{1}{\varphi}(e^{2\varphi h}-e^{-2\varphi h})-\frac{1}{2\varphi}2\varphi h\left(e^{2\varphi h}+e^{-2\varphi h}+2\right). \tag{B.35}$$

It would be useful to express equation (B.35) using $\sinh(x)=\frac{e^x-e^{-x}}{2}$ and $\cosh(x)=\frac{e^x+e^{-x}}{2}$. Using the hyperbolic functions,

$$-\Phi 2\left(\cosh(2\varphi h)-1\right)=\frac{2\sinh(2\varphi h)}{\varphi}-2h\left(\cosh(2\varphi h)+1\right).$$

Multiplying both sides by φ

$$-\Phi 2\varphi\left(\cosh(2\varphi h)-1\right)=2\sinh(2\varphi h)-\varphi 2h\left(\cosh(2\varphi h)+1\right).$$

Next, we change variables with $x\equiv 2\varphi h$ and define x as the implicit solution of

$$-2\Phi\varphi\left(\cosh(x)-1\right)+x\left(\cosh(x)+1\right)=2\sinh(x).$$

Thus, $h=\frac{x(2\Phi\varphi)}{2\varphi}$. Let $b=2\Phi\varphi>0$, then we can express the function $x(\cdot)$ as the solution of

$$b=-\frac{2\sinh(x(b))-x(b)\left(\cosh(x(b))+1\right)}{\left(\cosh(x(b))-1\right)}.$$

Notice that if we define
$$f(x)=-\frac{2\sinh(x)-x\left(\cosh(x)+1\right)}{\left(\cosh(x)-1\right)},$$
the following properties about $f(x)$ hold:

1. $\lim_{x\to 0}f(x)=0$ and $\lim_{x\to\infty}f(x)=\infty$.
2. $f(x)$ is increasing and convex, with $\lim_{x\to 0}f'(x)=1/3$ and $\lim_{x\to\infty}f(x)=1$.
3. $\frac{\mathrm{dlog}(f(x))}{\mathrm{dlog}(x)}>1$.

Given these properties, we can write $h(\varphi,\Phi)=\frac{f^{-1}(2\varphi\Phi)}{2\varphi}$ and show the following properties of $h(\varphi,\Phi)$

1. $h(\varphi,\Phi)$ is increasing in Φ: Since $f^{-1}(\cdot)$ is increasing, we have the result.
2. $h(\varphi,\Phi)$ is decreasing in φ: Taking the derivative of $h(\varphi,\Phi)=\frac{f^{-1}(2\varphi\Phi)}{2\varphi}$ with respect to φ and operating

$$\frac{\partial h(\varphi,\Phi)}{\partial\varphi}=\left.\frac{\mathrm{d}f^{-1}(x)}{\mathrm{d}x}\right|_{x=2\varphi\Phi}\frac{2\Phi}{2\varphi}-\frac{f^{-1}(2\varphi\Phi)}{2\varphi^2}$$

$$= \frac{f^{-1}(2\varphi\Phi)}{2\varphi^2}\left[\left.\frac{\mathrm{d}f^{-1}(x)}{\mathrm{d}x}\right|_{x=2\varphi\Phi}\frac{2\varphi\Phi}{f^{-1}(2\varphi\Phi)}-1\right]$$
$$= \frac{f^{-1}(2\varphi\Phi)}{2\varphi^2}\left[\left.\frac{\mathrm{dlog}(x)}{\mathrm{dlog}(f(x))}\right|_{x=2\varphi\Phi}\frac{2\varphi\Phi}{f^{-1}(2\varphi\Phi)}-1\right]$$
$$< 0.$$

3. $\lim_{\varphi\downarrow 0} h(\varphi,\Phi) = 3\Phi$ and $\lim_{\varphi\to\infty} h(\varphi,\Phi) = \Phi$: Applying L'Hopital's rule and using properties of the derivative of the inverse,

$$\lim_{\varphi\to\infty} h(\varphi,\Phi) = \lim_{\varphi\to\infty}\frac{f^{-1}(2\varphi\Phi)}{2\varphi} = \lim_{\varphi\to\infty}\frac{1}{f'(2\varphi\Phi)}\Phi = \Phi$$
$$\lim_{\varphi\downarrow 0} h(\varphi,\Phi) = \lim_{\varphi\downarrow 0}\frac{f^{-1}(2\varphi\Phi)}{2\varphi} = \lim_{\varphi\downarrow 0}\frac{1}{f'(2\varphi\Phi)}\Phi = 3\Phi$$

4. $h(\varphi,\Phi) = \omega(2\varphi\Phi)\Phi$: Define $\omega(z) = \frac{f^{-1}(z)}{z}$, then it is easy to see that $h(\varphi,\Phi) = \omega(2\varphi\Phi)\Phi$. Moreover, from property 2 and 3, $\omega(z)$ is decreasing with $\lim_{z\downarrow 0}\omega(z) = 3$ and $\lim_{z\to\infty}\omega(z) = 1$. Moreover, it is easy to show with similar arguments that $\omega(2\varphi\Phi)\Phi$ is increasing in Φ and $\omega(2\varphi\Phi)\varphi$ is increasing in φ.

Now, we can compute $\eta(\hat{w}^*)$ and $\mathcal{T}(\hat{w}^*,\hat{\rho})$. Note that we can define $T(x) = \mathcal{T}(x+\hat{w}^*,\hat{\rho})$, which solves

$$(\hat{\rho}+\delta)T(x) = 1 + \frac{\sigma^2}{2}T''(x), \text{ with } T(\pm h(\varphi,\Phi)) = 0.$$

The solution to this differential equation is given by

$$T(x) = \frac{1-\frac{e^{\varphi x}+e^{-\varphi x}}{e^{\varphi h}+e^{-\varphi h}}}{\hat{\rho}+\delta}.$$

Thus, $T'(0) = 0$ and $\eta(\hat{w}^*) = \alpha$. Finally, using the property that $sech(x) = \frac{2}{e^x+e^{-x}}$, we have

$$\mathcal{T}(\hat{w}^*,\hat{\rho}) = \frac{1-sech(\varphi\omega(2\varphi\Phi)\Phi)}{\hat{\rho}+\delta}.$$

□

Proposition 5. *Assume $\sigma = 0$ and $\hat{\gamma} \geq 0$. Then $\hat{w}^- = \log\left(\hat{\rho}\hat{U}\right)$ and*

$$w^* = \hat{w}^- + \tilde{T}\left(\frac{\alpha+(1-\alpha)\hat{\rho}\hat{U}}{\hat{\rho}\hat{U}}, \frac{\hat{\rho}+\delta}{\hat{\gamma}}, \frac{(1-\alpha)(1-\hat{\rho}\hat{U})}{\hat{\rho}\hat{U}}\right).$$

$\tilde{T}(\cdot)$ is defined as

$$a = e^{\tilde{T}(a,b,c)}\frac{1-e^{-(1+b)\tilde{T}(a,b,c)}}{1-e^{-b\tilde{T}(a,b,c)}}\frac{b}{b+1} - cb\frac{e^{-b\tilde{T}(a,b,c)}}{1-e^{-b\tilde{T}(a,b,c)}}\left[1-\frac{b+1}{b}\frac{1-a^{-b\tilde{T}(a,b,c)}}{e^{\tilde{T}(a,b,c)}-e^{-b\tilde{T}(a,b,c)}}\right] \tag{B.36}$$

where $\tilde{T}(\cdot)$ is increasing in the first argument and decreasing in the second argument. The expected discounted duration and worker's share satisfies:

$$\mathcal{T}(\hat{w}^*,\hat{\rho}) = \frac{1-e^{-\frac{\hat{\rho}+\delta}{\hat{\gamma}}\tilde{T}(\cdot)}}{\hat{\rho}+\delta}$$

$$\eta(\hat{w}^*) = \frac{e^{\tilde{T}(\cdot)} \frac{1-e^{-\left(1+\frac{\hat{\rho}+\delta}{\hat{\gamma}}\right)\tilde{T}(\cdot)}}{1-e^{-\frac{\hat{\rho}+\delta}{\hat{\gamma}}\tilde{T}(\cdot)}} \frac{\hat{\rho}+\delta}{\hat{\rho}+\delta+\hat{\gamma}} - 1}{1-\hat{\rho}\hat{U}} \hat{\rho}\hat{U}$$

Moreover,

1. *If* $\hat{\gamma} = 0$, *then*

$$(\tilde{T}(\cdot), \mathcal{T}(\hat{w}^*, \hat{\rho}), \eta(\hat{w}^*)) \to \left(\log\left(\frac{\alpha + (1-\alpha)\hat{\rho}\hat{U}}{\hat{\rho}\hat{U}}\right), \frac{1}{\hat{\rho}+\delta}, \alpha\right).$$

2. *If* $\hat{\gamma} \to \infty$, *then* $\tilde{T}(\cdot) \to \tilde{T}^{limit}$ *where*

$$\frac{\alpha + (1-\alpha)\hat{\rho}\hat{U}}{\hat{\rho}\hat{U}} = \frac{e^{\tilde{T}^{limit}} - 1 - \frac{(1-\alpha)(1-\hat{\rho}\hat{U})}{\hat{\rho}\hat{U}}\left(1 - \frac{\tilde{T}^{limit}}{e^{\tilde{T}^{limit}}-1}\right)}{\tilde{T}^{limit}},$$

$\mathcal{T}(\hat{w}^*, \hat{\rho}) \to 0$ *and* $\eta(\hat{w}^*) \to \eta^{limit}$

$$\eta^{limit} = \alpha + \frac{1-\alpha}{\tilde{T}^{limit}} \frac{(1-\hat{\rho}\hat{U})\eta^{limit}}{\eta^{limit} + \hat{\rho}\hat{U}(1-\eta^{limit})}$$

Proof. Now, we take the limit $\sigma \downarrow 0$. The equilibrium conditions in this case are

$$(\hat{\rho}+\delta)\hat{W}(\hat{w}) = e^{\hat{w}} - \hat{\rho}\hat{U} - \hat{\gamma}\hat{W}'(\hat{w}) \quad \forall \hat{w} \in \hat{\mathcal{C}}^j \cap \hat{\mathcal{C}}^h$$
$$(\hat{\rho}+\delta)\hat{J}(\hat{w}) = 1 - e^{\hat{w}} - \hat{\gamma}\hat{J}'(\hat{w}) \quad \forall \hat{w} \in \hat{\mathcal{C}}^j \cap \hat{\mathcal{C}}^h$$
$$(1-\alpha)\frac{\mathrm{dlog}\,\hat{J}(\hat{w}^*)}{\mathrm{d}\hat{w}} = -\alpha\frac{\mathrm{dlog}\,\hat{W}(\hat{w}^*)}{\mathrm{d}\hat{w}}$$

with the value matching and smooth pasting conditions

$$\hat{W}(\hat{w}^-) = \hat{J}(\hat{w}^-) = \hat{W}(\hat{w}^+) = \hat{J}(\hat{w}^+) = 0 \quad ; \quad \hat{W}'(\hat{w}^-) = \hat{J}'(\hat{w}^+) = 0.$$

Without idiosyncratic shocks and $\gamma > 0$ the upper Ss band is not active. Thus, we discard the optimality condition for $\hat{w}^+$. In this case, the stopping time is a deterministic function; hence, it is easier to work with the sequential formulation.

$$\hat{W}(\hat{w}) = \max_T \int_0^T e^{-(\hat{\rho}+\delta)s}\left(e^{\hat{w}-\hat{\gamma}s} - \hat{\rho}\hat{U}\right)\mathrm{d}s \tag{B.37}$$
$$\hat{J}(\hat{w}) = \int_0^{T(\hat{w})} e^{-(\hat{\rho}+\delta)s}\left(1 - e^{\hat{w}-\hat{\gamma}s}\right)\mathrm{d}s. \tag{B.38}$$

In equation (B.38), $T(\hat{w})$ is the optimal policy of the worker. Taking the first order conditions with respect to $T(\hat{w})$

$$e^{\hat{w}-\hat{\gamma}T(\hat{w})} = \hat{\rho}\hat{U}.$$

Solving the previous equation,

$$T(\hat{w}) = \frac{\hat{w} - \log(\hat{\rho}\hat{U})}{\hat{\gamma}}.$$

Thus, if $\hat{w} = \hat{w}^*$, we have that $\hat{w}^- = \hat{w}^* - \hat{\gamma}T(\hat{w}^*)$ satisfies

$$\hat{w}^- = \log(\hat{\rho}\hat{U}).$$

Taking the derivatives of $\hat{W}(\hat{w})$ and $\hat{J}(\hat{w})$, and using the envelope condition for $\hat{W}'(\hat{w})$, we have that

$$\hat{W}'(\hat{w}) = \int_0^{T(w)} e^{-(\hat{\rho}+\delta)s} \left(e^{\hat{w}-\hat{\gamma}s} \right) \mathrm{d}s, \tag{B.39}$$

$$\hat{J}'(\hat{w}) = -\int_0^{T(w)} e^{-(\hat{\rho}+\delta)s} \left(e^{\hat{w}-\hat{\gamma}s} \right) \mathrm{d}s + e^{-(\hat{\rho}+\delta)T(\hat{w})} \left(1 - e^{\hat{w}-\hat{\gamma}T(\hat{w})} \right) \underbrace{T'(\hat{w})}_{=1/\hat{\gamma}}. \tag{B.40}$$

From equations (B.39) and (B.40), we get the Nash bargaining solution

$$-\alpha \frac{\int_0^{T^*} e^{-(\hat{\rho}+\delta)s} \left(e^{\hat{w}^*-\hat{\gamma}s} \right) \mathrm{d}s}{\int_0^{T^*} e^{-(\hat{\rho}+\delta)s} \left(e^{\hat{w}^*-\hat{\gamma}s} - \hat{\rho}\hat{U} \right) \mathrm{d}s} = (1-\alpha) \frac{\left[-\int_0^{T^*} e^{-(\hat{\rho}+\delta)s} \left(e^{\hat{w}^*-\hat{\gamma}s} \right) \mathrm{d}s + e^{-(\hat{\rho}+\delta)T^*} \frac{(1-\hat{\rho}\hat{U})}{\hat{\gamma}} \right]}{\int_0^{T^*} e^{-(\hat{\rho}+\delta)s} \left(1 - e^{\hat{w}^*-\hat{\gamma}s} \right) \mathrm{d}s} \tag{B.41}$$

Define

$$\Omega(a, T^*) := \frac{1 - e^{-aT^*}}{a}$$

$$\mathcal{Z} := \frac{e^{-(\hat{\rho}+\delta)T^*} (1 - \hat{\rho}\hat{U})}{\hat{\gamma} \int_0^{T^*} e^{-(\hat{\rho}+\delta)s} \left(e^{\hat{w}^*-\hat{\gamma}s} \right) \mathrm{d}s}$$

Operating

$$\alpha \int_0^{T^*} e^{-(\hat{\rho}+\delta)s} \left(1 - e^{\hat{w}^*-\hat{\gamma}s} \right) \mathrm{d}s = (1-\alpha) \int_0^{T^*} e^{-(\hat{\rho}+\delta)s} \left(e^{\hat{w}^*-\hat{\gamma}s} - \hat{\rho}\hat{U} \right) \mathrm{d}s \, [1 - \mathcal{Z}] \iff$$

$$\alpha \left[\Omega(\hat{\rho}+\delta, T^*) - e^{\hat{w}^*} \Omega(\hat{\rho}+\delta+\hat{\gamma}, T^*) \right] = (1-\alpha) \left[e^{\hat{w}^*} \Omega(\hat{\rho}+\delta+\hat{\gamma}, T^*) - \hat{\rho}\hat{U}\Omega(\hat{\rho}+\delta, T^*) \right] \times$$

$$\ldots \left[1 - \frac{e^{-(\hat{\rho}+\delta)T^*} (1 - \hat{\rho}\hat{U})}{\hat{\gamma} e^{\hat{w}^*} \Omega(\hat{\rho}+\delta+\hat{\gamma}, T^*)} \right] \iff$$

$$\left(\alpha + (1-\alpha)\hat{\rho}\hat{U} \right) \Omega(\hat{\rho}+\delta, T^*) = e^{\hat{w}^*} \Omega(\hat{\rho}+\delta+\hat{\gamma}, T^*) + \ldots$$

$$\ldots (1-\alpha) \left[e^{\hat{w}^*} \Omega(\hat{\rho}+\delta+\hat{\gamma}, T^*) - \hat{\rho}\hat{U}\Omega(\hat{\rho}+\delta, T^*) \right] \frac{e^{-(\hat{\rho}+\delta)T^*} (1 - \hat{\rho}\hat{U})}{\hat{\gamma} e^{\hat{w}^*} \Omega(\hat{\rho}+\delta+\hat{\gamma}, T^*)} \iff$$

$$\left(\alpha + (1-\alpha)\hat{\rho}\hat{U} \right) \Omega(\hat{\rho}+\delta, T^*) = e^{\hat{w}^*} \Omega(\hat{\rho}+\delta+\hat{\gamma}, T^*) - \frac{(1-\alpha) e^{-(\hat{\rho}+\delta)T^*} (1 - \hat{\rho}\hat{U})}{\hat{\gamma}} \left[1 - \hat{\rho}\hat{U} \frac{\Omega(\hat{\rho}+\delta, T^*)}{e^{\hat{w}^*} \Omega(\hat{\rho}+\delta+\hat{\gamma}, T^*)} \right]$$

The policy $(T^*, \hat{w}^*)$ solves

$$e^{\hat{w}^*-\hat{\gamma}T^*} = \hat{\rho}\hat{U}$$

$$\left(\alpha + (1-\alpha)\hat{\rho}\hat{U} \right) \Omega(\hat{\rho}+\delta, T^*) = e^{\hat{w}^*} \Omega(\hat{\rho}+\delta+\hat{\gamma}, T^*) - \frac{(1-\alpha) e^{-(\hat{\rho}+\delta)T^*} (1 - \hat{\rho}\hat{U})}{\hat{\gamma}} \left[1 - \hat{\rho}\hat{U} \frac{\Omega(\hat{\rho}+\delta, T^*)}{e^{\hat{w}^*} \Omega(\hat{\rho}+\delta+\hat{\gamma}, T^*)} \right].$$

Define $\tilde{T} = \hat{\gamma}T^*$ and $\Omega(a, T^*) := \frac{1-e^{-aT^*}}{a} = \hat{\gamma}^{-1}\Omega(\frac{a}{\hat{\gamma}}, \tilde{T})$. Then

$$e^{\hat{w}^*-\tilde{T}} = \hat{\rho}\hat{U}$$

$$\left(\alpha + (1-\alpha)\hat{\rho}\hat{U} \right) \hat{\gamma}^{-1} \Omega\left(\frac{\hat{\rho}+\delta}{\hat{\gamma}}, \tilde{T} \right) = e^{\hat{w}^*} \hat{\gamma}^{-1} \Omega\left(\frac{\hat{\rho}+\delta}{\hat{\gamma}} + 1, \tilde{T} \right) - \frac{(1-\alpha) e^{-\frac{\hat{\rho}+\delta}{\hat{\gamma}}\tilde{T}} (1 - \hat{\rho}\hat{U})}{\hat{\gamma}} \left[1 - \hat{\rho}\hat{U} \frac{\Omega\left(\frac{\hat{\rho}+\delta}{\hat{\gamma}}, \tilde{T} \right)}{e^{\hat{w}^*} \Omega\left(\frac{\hat{\rho}+\delta}{\hat{\gamma}} + 1, \tilde{T} \right)} \right].$$

Therefore, the optimal stopping is given by

$$\frac{\alpha+(1-\alpha)\hat{\rho}\hat{U}}{\hat{\rho}\hat{U}}\frac{\Omega\left(\frac{\hat{\rho}+\delta}{\hat{\gamma}},\tilde{T}\right)}{\Omega\left(\frac{\hat{\rho}+\delta}{\hat{\gamma}}+1,\tilde{T}\right)}=e^{\tilde{T}}-\frac{(1-\alpha)(1-\hat{\rho}\hat{U})\left[1-\frac{\hat{\rho}+\delta}{\hat{\gamma}}\Omega\left(\frac{\hat{\rho}+\delta}{\hat{\gamma}},\tilde{T}\right)\right]}{\hat{\rho}\hat{U}\Omega\left(\frac{\hat{\rho}+\delta}{\hat{\gamma}}+1,\tilde{T}\right)}\left[1-\frac{\Omega\left(\frac{\hat{\rho}+\delta}{\hat{\gamma}},\tilde{T}\right)}{e^{\tilde{T}}\Omega\left(\frac{\hat{\rho}+\delta}{\hat{\gamma}}+1,\tilde{T}\right)}\right]$$

or

$$\frac{\alpha+(1-\alpha)\hat{\rho}\hat{U}}{\hat{\rho}\hat{U}}=e^{\tilde{T}}\frac{\Omega\left(\frac{\hat{\rho}+\delta}{\hat{\gamma}}+1,\tilde{T}\right)}{\Omega\left(\frac{\hat{\rho}+\delta}{\hat{\gamma}},\tilde{T}\right)}-\frac{(1-\alpha)(1-\hat{\rho}\hat{U})\left[1-\frac{\hat{\rho}+\delta}{\hat{\gamma}}\Omega\left(\frac{\hat{\rho}+\delta}{\hat{\gamma}},\tilde{T}\right)\right]}{\hat{\rho}\hat{U}\Omega\left(\frac{\hat{\rho}+\delta}{\hat{\gamma}},\tilde{T}\right)}\left[1-\frac{\Omega\left(\frac{\hat{\rho}+\delta}{\hat{\gamma}},\tilde{T}\right)}{e^{\tilde{T}}\Omega\left(\frac{\hat{\rho}+\delta}{\hat{\gamma}}+1,\tilde{T}\right)}\right]$$

Now, we show the properties satisfied by $\tilde{T}\left(\frac{\alpha+(1-\alpha)\hat{\rho}\hat{U}}{\hat{\rho}\hat{U}},\frac{\hat{\rho}+\delta}{\hat{\gamma}},\frac{(1-\alpha)(1-\hat{\rho}\hat{U})}{\hat{\rho}\hat{U}}\right)$. Let us define the following function:

$$\begin{aligned}
f(a,b,c) &:= e^{a}\frac{1-e^{-(1+b)a}}{1-e^{-ba}}\frac{b}{b+1}-c\left[\frac{1-b\frac{1-e^{-ba}}{b}}{\frac{1-e^{-ba}}{b}}\right]\left[1-\frac{1-e^{-ba}}{1-e^{-(1+b)a}}\frac{b+1}{be^{a}}\right]\\
&= e^{a}\frac{1-e^{-(1+b)a}}{1-e^{-ba}}\frac{b}{b+1}-c\left[\frac{b}{1-e^{-ba}}-b\right]\left[\frac{b(e^{a}-e^{-ba})-(1-e^{-ba})(b+1)}{b(e^{a}-e^{-ba})}\right]\\
&= e^{a}\frac{1-e^{-(1+b)a}}{1-e^{-ba}}\frac{b}{b+1}-cb\frac{e^{-ba}}{1-e^{-ba}}\left[\frac{b(e^{a}-e^{-ba})-(b+1)+(1+b)e^{-ba}}{b(e^{a}-e^{-ba})}\right]\\
&= e^{a}\frac{1-e^{-(1+b)a}}{1-e^{-ba}}\frac{b}{b+1}-cb\frac{e^{-ba}}{1-e^{-ba}}\left[1-\frac{b+1}{b}\frac{1-e^{-ba}}{e^{a}-e^{-ba}}\right].
\end{aligned}$$

Observe that with this function:

$$\frac{\alpha+(1-\alpha)\hat{\rho}\hat{U}}{\hat{\rho}\hat{U}}=f\left(\tilde{T}\left(\frac{\alpha+(1-\alpha)\hat{\rho}\hat{U}}{\hat{\rho}\hat{U}},\frac{\hat{\rho}+\delta}{\hat{\gamma}},\frac{(1-\alpha)(1-\hat{\rho}\hat{U})}{\hat{\rho}\hat{U}}\right),\frac{\hat{\rho}+\delta}{\hat{\gamma}},\frac{(1-\alpha)(1-\hat{\rho}\hat{U})}{\hat{\rho}\hat{U}}\right).$$

The following properties are easy to show:

1. $f(a,b,c)$ is increasing in a.
2. If $a,c>0$, $b\to\infty$, then $f(a,b,c)\to e^{a}$: To see this property, taking the limit

$$\begin{aligned}
&\lim_{a>0,b\to\infty,c\propto b}\left[e^{a}\frac{1-e^{-(1+b)a}}{1-e^{-ba}}\frac{b}{b+1}-cb\frac{e^{-ba}}{1-e^{-ba}}\left[1-\frac{b+1}{b}\frac{1-e^{-ba}}{e^{a}-e^{-ba}}\right]\right]\\
&=e^{a}\underbrace{\lim_{a>0,b\to\infty}\frac{1-e^{-(1+b)a}}{1-e^{-ba}}}_{=1}\underbrace{\lim_{a>0,b\to\infty}\frac{b}{b+1}}_{=1}-\underbrace{\lim_{a>0,b\to\infty}cb\frac{e^{-ba}}{1-e^{-ba}}}_{=0}\left[1-\underbrace{\lim_{b\to\infty}\frac{b+1}{b}}_{=1}\underbrace{\lim_{a>0,b\to\infty}\frac{1-e^{-ba}}{e^{a}-e^{-ba}}}_{=e^{-a}}\right]\\
&=e^{a}.
\end{aligned}$$

3. If $a,c>0$ and $b\to 0$ then $f(a,b,c)\to\frac{e^{a}-1-c\left(1-\frac{a}{e^{a}-1}\right)}{a}$: To see this property, taking the limit

$$\begin{aligned}
&\lim_{a>0,b\to 0}\left[e^{a}\frac{1-e^{-(1+b)a}}{1-e^{-ba}}\frac{b}{b+1}-cb\frac{e^{-ba}}{1-e^{-ba}}\left[1-\frac{b+1}{b}\frac{1-e^{-ba}}{e^{a}-e^{-ba}}\right]\right]\\
&=e^{a}(1-e^{-a})\underbrace{\lim_{a>0,b\to 0}\frac{b}{1-e^{-ba}}}_{=1/a}-c\underbrace{\lim_{a>0,b\to 0}\frac{b}{1-e^{-ba}}}_{=1/a}\left[1-\frac{1}{e^{a}-1}\underbrace{\lim_{b\to\infty}\frac{1-e^{-ba}}{b}}_{=a}\right]
\end{aligned}$$

$$= \frac{e^a - 1 - c\left(1 - \frac{a}{e^a-1}\right)}{a}.$$

4. $e^a \geq f(a,b,c) \geq \frac{e^a-1-c\left(1-\frac{a}{e^a-1}\right)}{a}$ where the upper bound is reached when $b \to \infty$ and the lower bound when $b \downarrow 0$.
5. Duration of the match: It is easy to show that

$$\mathcal{T}(\hat{w}^*, \hat{\rho}) = \frac{1 - e^{-\frac{\hat{\rho}+\delta}{\hat{\gamma}}\tilde{T}(\cdot)}}{\hat{\rho}+\delta}.$$

6. The worker's share is given by

$$\begin{aligned}
\eta(\hat{w}^*) &= \frac{e^{\hat{\gamma}T^*(\cdot)+\log(\hat{\rho}\hat{U})}\int_0^{T^*} e^{-(\hat{\rho}+\delta+\hat{\gamma})t}\,\mathrm{d}t - \hat{\rho}\hat{U}\int_0^{T^*} e^{-(\hat{\rho}+\delta)t}\,\mathrm{d}t}{(1-\hat{\rho}\hat{U})\int_0^{T^*} e^{-(\hat{\rho}+\delta)t}\,\mathrm{d}t} \\
&= \frac{e^{\tilde{T}(\cdot)}\frac{1-e^{-\left(1+\frac{\hat{\rho}+\delta}{\hat{\gamma}}\right)\tilde{T}(\cdot)}}{1-e^{-\frac{\hat{\rho}+\delta}{\hat{\gamma}}\tilde{T}(\cdot)}}\frac{\hat{\rho}+\delta}{\hat{\rho}+\delta+\hat{\gamma}} - 1}{1-\hat{\rho}\hat{U}}\hat{\rho}\hat{U}
\end{aligned} \tag{B.42}$$

With these properties, we can characterize the equilibrium policies:

1. $\tilde{T}\left(\frac{\alpha+(1-\alpha)\hat{\rho}\hat{U}}{\hat{\rho}\hat{U}}, \frac{\hat{\rho}+\delta}{\hat{\gamma}}, \frac{(1-\alpha)(1-\hat{\rho}\hat{U})}{\hat{\gamma}\hat{\rho}\hat{U}}\right)$ is increasing in the first argument.
2. If $\hat{\gamma} \to 0$, then $\frac{\hat{\rho}+\delta}{\hat{\gamma}} \to \infty$

$$\lim_{(\hat{\rho}+\delta)/\hat{\gamma}\to\infty} \tilde{T}(\cdot) = \log\left(\frac{\alpha+(1-\alpha)\hat{\rho}\hat{U}}{\hat{\rho}\hat{U}}\right)$$

The expected discounted duration in the limit is equal to

$$\lim_{\hat{\gamma}\to 0} \mathcal{T}(\hat{w}^*, \hat{\rho}) = \frac{1}{\hat{\rho}+\delta}$$

The worker's share in the limit is equal to

$$\eta(\hat{w}^*) = \frac{e^{\tilde{T}(\cdot)}\frac{1-e^{-\left(1+\frac{\hat{\rho}+\delta}{\hat{\gamma}}\right)\tilde{T}(\cdot)}}{1-e^{-\frac{\hat{\rho}+\delta}{\hat{\gamma}}\tilde{T}(\cdot)}}\frac{\hat{\rho}+\delta}{\hat{\rho}+\delta+\hat{\gamma}} - 1}{1-\hat{\rho}\hat{U}}\hat{\rho}\hat{U} = \frac{e^{\tilde{T}(\cdot)}-1}{1-\hat{\rho}\hat{U}}\hat{\rho}\hat{U} = \frac{\frac{\alpha+(1-\alpha)\hat{\rho}\hat{U}}{\hat{\rho}\hat{U}} - 1}{1-\hat{\rho}\hat{U}}\hat{\rho}\hat{U} = \alpha$$

3. If $\hat{\gamma} \to \infty$, then $\frac{\hat{\rho}+\delta}{\hat{\gamma}} \to 0$, which provides the same $\tilde{T}(\cdot)$ as $\hat{\rho}+\delta \to 0$. As we have shown before, under this limit, $\tilde{T}(\cdot)$ converges to the implicit solution given by

$$\frac{\alpha+(1-\alpha)\hat{\rho}\hat{U}}{\hat{\rho}\hat{U}} = \frac{e^{\tilde{T}(\cdot)} - 1 - \frac{(1-\alpha)(1-\hat{\rho}\hat{U})}{\hat{\rho}\hat{U}}\left(1 - \frac{\tilde{T}(\cdot)}{e^{\tilde{T}(\cdot)}-1}\right)}{\tilde{T}(\cdot)}.$$

Given the convergence, we now show the limit for $\eta(\hat{w}^*)$ since clearly $\mathcal{T}(\hat{w}^*, \rho) \to 0$. Let us depart from equation (B.41)

$$-\alpha\frac{\int_0^{T^*} e^{-(\hat{\rho}+\delta)s}\left(e^{\hat{w}^*-\hat{\gamma}s}\right)\mathrm{d}s}{\int_0^{T^*} e^{-(\hat{\rho}+\delta)s}\left(e^{\hat{w}^*-\hat{\gamma}s} - \hat{\rho}\hat{U}\right)\mathrm{d}s} = (1-\alpha)\frac{\left[-\int_0^{T^*} e^{-(\hat{\rho}+\delta)s}\left(e^{\hat{w}^*-\hat{\gamma}s}\right)\mathrm{d}s + e^{-(\hat{\rho}+\delta)T^*}\frac{(1-\hat{\rho}\hat{U})}{\hat{\gamma}}\right]}{\int_0^{T^*} e^{-(\hat{\rho}+\delta)s}\left(1 - e^{\hat{w}^*-\hat{\gamma}s}\right)\mathrm{d}s}$$

Taking the limit as $\hat{\rho}+\delta \to 0$

$$\alpha\int_0^{T^*}\left(1-e^{w_t}\right)\mathrm{d}t = (1-\alpha)\int_0^{T^*}\left(e^{w_t} - \hat{\rho}\hat{U}\right)\mathrm{d}t - \frac{(1-\alpha)(1-\hat{\rho}\hat{U})}{\hat{\gamma}}\frac{\int_0^{T^*}\left(e^{w_t}-\hat{\rho}\hat{U}\right)\mathrm{d}t}{\int_0^{T^*} e^{w_t}\,\mathrm{d}t}.$$

Operating and using the occupancy measure

$$\alpha + (1-\alpha)\hat{\rho}\hat{U} + \frac{(1-\alpha)(1-\hat{\rho}\hat{U})}{\hat{\gamma}T^*} \frac{\frac{\int_0^{T^*} e^{w_t}\,\mathrm{d}t}{T^*} - \hat{\rho}\hat{U}}{\frac{\int_0^{T^*} e^{w_t}\,\mathrm{d}t}{T^*}} = \frac{\int_0^{T^*} e^{w_t}\,\mathrm{d}t}{T^*}$$

It is easy to check that

$$\alpha + (1-\alpha)\hat{\rho}\hat{U} + \frac{1-\alpha}{\hat{\gamma}T^*} \frac{\mathbb{E}[e^{\hat{w}}] - \hat{\rho}\hat{U}}{\mathbb{E}[e^{\hat{w}}]}(1-\hat{\rho}\hat{U}) = \mathbb{E}[e^{\hat{w}}].$$

From (B.42), since $\hat{\rho} + \delta \to 0$, we have that $\eta(\hat{w}^*) = \frac{\mathbb{E}[e^{\hat{w}}] - \hat{\rho}\hat{U}}{1-\hat{\rho}\hat{U}}$. Combining these results

$$\begin{aligned}
\alpha + (1-\alpha)\hat{\rho}\hat{U} + \frac{1-\alpha}{\hat{\gamma}T^*} \frac{\mathbb{E}[e^{\hat{w}}] - \hat{\rho}\hat{U}}{\mathbb{E}[e^{\hat{w}}]}(1-\hat{\rho}\hat{U}) = \mathbb{E}[e^{\hat{w}}] \iff \\
\alpha + (1-\alpha)\hat{\rho}\hat{U} + \frac{1-\alpha}{\hat{\gamma}T^*} \frac{(1-\hat{\rho}\hat{U})\eta(\hat{w}^*)}{(1-\hat{\rho}\hat{U})\eta(\hat{w}^*) + \hat{\rho}\hat{U}}(1-\hat{\rho}\hat{U}) = (1-\hat{\rho}\hat{U})\eta(\hat{w}^*) + \hat{\rho}\hat{U} \iff \\
\alpha(1-\hat{\rho}\hat{U}) + \frac{1-\alpha}{\hat{\gamma}T^*} \frac{(1-\hat{\rho}\hat{U})\eta(\hat{w}^*)}{(1-\hat{\rho}\hat{U})\eta(\hat{w}^*) + \hat{\rho}\hat{U}}(1-\hat{\rho}\hat{U}) = (1-\hat{\rho}\hat{U})\eta(\hat{w}^*) \\
\eta(\hat{w}^*) = \alpha + \frac{1-\alpha}{\tilde{T}} \frac{(1-\hat{\rho}\hat{U})\eta(\hat{w}^*)}{\eta(\hat{w}^*) + \hat{\rho}\hat{U}(1-\eta(\hat{w}^*))}.
\end{aligned}$$

□

C Extension with Staggered Wage Renegotiations

Overview. In the baseline model, wages are completely rigid within a worker-firm match. We now consider a generalization of the baseline model that adds staggered wage renegotiations à la Calvo (1983). The resulting insight is that all our key results can be extended to this more general environment.

In the model extension, renegotiations occur within a match according to a Poisson process with rate $\delta^r \geq 0$. The renegotiation hazard induces a stochastic duration τ^r between the last wage setting event (i.e., match formation or renegotiations) and a new round of renegotiations. We assume that renegotiations entail the wage within a worker-firm match being set according to a Nash bargain with worker bargaining weight α (i.e., the elasticity of the matching function with respect to unemployment) over the prevailing surplus at the time of renegotiations. As $\delta^r \to 0$, we recover the baseline model with fully rigid wages within a match. As $\delta^r \to \infty$, we obtain the polar opposite case with flexible wages that are continuously reset.

Notation. Let $N \in \mathbb{N}_+$ denote the number of segments with distinct wages realized during a job spell (i.e., $N-1$ is the number of renegotiations that occur) between match formation and match dissolution. We index all stopping times by the segment $n \in \{1, 2, \ldots, N\}$ during which they occur. By construction, wages are constant within but vary across segments.

For each segment $n \in \{1, 2, \ldots, N\}$, there are two cases. First, if $\min\{\tau^{h,n}, \tau^{j,n}, \tau^{\delta,n}\} < \tau^{r,n}$, then the job spell ends with segment $n = N$. Second, if $\min\{\tau^{h,n}, \tau^{j,n}, \tau^{\delta,n}\} > \tau^{r,n}$, then renegotiations occur and the match enters segment $n+1 \leq N$ of the job spell with a renegotiated wage.[15] Given a collection of stopping times associated with a job spell, $\vec{\tau}^m = \{\vec{\tau}^{m,n}\}_{n \in \{1,\ldots,N\}}$, where $\vec{\tau}^{m,n} = \{\tau^{h,n}, \tau^{j,n}, \tau^{\delta,n}, \tau^{r,n}\}$ for each segment $n \in \{1, \ldots, N\}$, we define $\tau^{m,n} := \min \vec{\tau}^{m,n}$ as the minimum stopping time of segment n and we define $\tau^m := \sum_{n=1}^{N} \tau^{m,n} = \sum_{n=1}^{N-1} \tau^{r,n} + \min\{\tau^{h,N}, \tau^{j,N}, \tau^{\delta,N}\}$ as the total match duration.

The wage prevailing during segment $n \in \{1, 2, \ldots, N\}$ of a job spell starting at time $t = 0$ is given by

$$w_t = \begin{cases} w^*(z_0) & t \in \left[0, \tau^{m,1}\right), \\ w^*(z_{\tau^{r,n-1}}) & t \in \left[\tau^{r,n-1}, \tau^{m,n}\right), \quad 1 < n \leq N. \end{cases}$$

Sequence Formulation. The value of an unemployed worker with productivity z is

$$U(z) = \max_{\{w_t\}_{t=0}^{\tau^u}} \mathbb{E}_0 \left[\int_0^{\tau^u} e^{-\rho t} B(e^{z_t}) \, \mathrm{d}t + e^{-\rho \tau^u} H(w_{\tau^u}, z_{\tau_u}, \vec{\tau}^m(w_{\tau^u}, z_{\tau^u})) \right].$$

The value of a worker employed at wage w with productivity z is

$$H(w, z, \vec{\tau}^m) = \mathbb{E}_0 \left[\int_0^{\tau^m} e^{-\rho t} e^{w_t} \, \mathrm{d}t + e^{-\rho \tau^m} U(z_{\tau^m}) \right].$$

That is, an employed worker consumes a stochastic wage stream w_t until time τ^m, when she either endogenously or exogenously transitions to unemployment. Similarly, the value of a firm matched with a worker with wage wage w and productivity z is

$$J(w, z, \vec{\tau}^m) = \mathbb{E}_0 \left[\int_0^{\tau^m} e^{-\rho t} \left[e^{z_t} - e^{w_t} \right] \mathrm{d}t \right].$$

[15] Note that $\mathbb{P}[\min\{\tau^{h,n}, \tau^{j,n}, \tau^{\delta,n}\} = \tau^{r,n}] = 0$ for all $n \in \{1, 2, \ldots, N\}$ since renegotiations occur independently from other separation events.

Recursive Formulation. The Hamilton-Jacobi-Bellman (HJB) equation of an unemployed worker is still

$$\rho u(z) = \tilde{B}e^z + \gamma \frac{\partial u(z)}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 u(z)}{\partial z^2} + \max_w f(w,z)[h(z;w) - u(z)].$$

The HJB equation of a worker employed at log wage w with log productivity $z \in \mathcal{C}^j(w)$, for which the firm prefers to continue, is now

$$\rho h(z;w) = \max\left\{e^w + \gamma\frac{\partial h(z;w)}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 h(z;w)}{\partial z^2} - \delta^r[h(z;w) - h(z;w^*(z))] - \delta[h(z;w) - u(z)]\,,\, \rho u(z)\right\}.$$

Similarly, the HJB equation of a firm employing a worker at log wage w with log productivity $z \in \mathcal{C}^h(w)$, for which the worker prefers to continue, is now

$$\rho j(z;w) = \max\left\{e^z - e^w + \gamma\frac{\partial j(z;w)}{\partial z} + \frac{\sigma^2}{2}\frac{\partial^2 j(z;w)}{\partial z^2} - \delta^r[j(z;w) - j(z;w^*(z))] - \delta j(z;w)\,,\, 0\right\}.$$

Here, $w^*(z)$ is the solution to a Nash bargain with worker bargaining weight α (i.e., the elasticity of the matching function with respect to unemployment), which satisfies the Hosios (1990) condition:

$$w^*(z) = \arg\max_w \left\{h(z;w)^\alpha j(z;w)^{1-\alpha}\right\}$$

Change of Notation. Using the change of notation from Lemma 2, $\hat{w} := w - z$, $\hat{\rho} := \rho - \gamma - \sigma^2/2$ and $\hat{\gamma} := \gamma + \sigma^2$, we define

$$(\hat{U}, \hat{J}(w-z), \hat{W}(w-z), \hat{\theta}(w-z)) = \left(\frac{u(z)}{e^z}, \frac{j(z;w)}{e^z}, \frac{h(z;w) - u(z)}{e^z}, \theta(w,z)\right).$$

Rewriting the HJB equations using this change of notation, we get

$$\hat{\rho}\hat{U} = \tilde{B} + \max_{\hat{w}} \hat{\theta}(\hat{w})^{1-\alpha}\hat{W}(\hat{w})$$

$$(\hat{\rho} + \delta + \delta^r)\hat{W}(\hat{w}) = \max\left\{e^{\hat{w}} + \delta^r\hat{W}(\hat{w}^*) - \hat{\rho}\hat{U} - \hat{\gamma}\hat{W}'(\hat{w}) + \frac{\sigma^2}{2}\hat{W}''(\hat{w})\,,\, 0\right\}$$

$$(\hat{\rho} + \delta + \delta^r)\hat{J}(\hat{w}) = \max\left\{1 - e^{\hat{w}} + \delta^r\hat{J}(\hat{w}^*) - \hat{\gamma}\hat{J}'(\hat{w}) + \frac{\sigma^2}{2}\hat{J}''(\hat{w})\,,\, 0\right\}.$$

Here, $\delta^r\hat{W}(\hat{w}^*)$ and $\delta^r\hat{J}(\hat{w}^*)$ are just constants, since the reset wage $\hat{w}^*$ does not depend on the current value of $\hat{w}$. Therefore, the problem is identical to that in the baseline model with completely rigid wages (i.e., with $\delta^r = 0$), except for three features. First, the effective discount rate of workers and firms is now $\hat{\rho} + \delta + \delta^r$ instead of the previous expression $\hat{\rho} + \delta$. Second, the flow value of the worker is now $e^{\hat{w}} + \delta^r\hat{W}(\hat{w}^*) - \hat{\rho}\hat{U}$ instead of the previous expression $e^{\hat{w}} - \hat{\rho}\hat{U}$. Third, the flow value of the firm is now $1 - e^{\hat{w}} + \delta^r\hat{J}(\hat{w}^*)$ instead of the previous expression $1 - e^{\hat{w}}$. Note that these expressions simplify to those from the baseline model as $\delta^r \to 0$.

Results. We now restate without proof some of the key results from the baseline model to the extended environment with staggered wage renegotiations.

Proposition C.1. *There exists a unique block recursive equilibrium (BRE).*

Proposition C.2. *With wage renegotiations à la Calvo (1983), the BRE has the following properties:*

1. *The joint match surplus satisfies*

$$\hat{S}(\hat{w}) = \frac{1-\hat{\rho}\hat{U}}{1-\delta^r \mathcal{T}(\hat{w}^*,\hat{\rho})}\mathcal{T}(\hat{w},\hat{\rho}+\delta^r),$$

where

$$\mathcal{T}(\hat{w},\hat{\rho}+\delta^r) := \mathbb{E}_{\hat{w}}\left[\int_0^{\tau^{m*}} e^{-(\hat{\rho}+\delta^r)t}\,\mathrm{d}t\right]$$

is the expected discounted duration of current wages and $1 > \hat{\rho}\hat{U} > \tilde{B}$.

2. *The competitive entry wage* $\hat{w}^*$ *coincides with the Nash bargaining solution with worker's weight* α*:*

$$\hat{w}^* = \arg\max_{\hat{w}}\left\{\hat{W}(\hat{w})^{\alpha}\hat{J}(\hat{w})^{1-\alpha}\right\} = \arg\max_{\hat{w}}\left\{\eta(\hat{w})^{\alpha}(1-\eta(\hat{w}))^{1-\alpha}\mathcal{T}(\hat{w},\hat{\rho}+\delta^r)\right\},$$

with optimality condition

$$\underbrace{\eta'(\hat{w}^*)\left(\frac{\alpha}{\eta(\hat{w}^*)}-\frac{1-\alpha}{1-\eta(\hat{w}^*)}\right)}_{\text{Share channel}} = -\underbrace{\frac{\mathcal{T}_{\hat{w}}(\hat{w}^*,\hat{\rho}+\delta^r)}{\mathcal{T}(\hat{w}^*,\hat{\rho}+\delta^r)}}_{\text{Surplus channel}}.$$

3. *Given* $\eta(\hat{w}^*)$ *and* $\mathcal{T}(\hat{w}^*,\hat{\rho}+\delta^r)$*, the equilibrium job finding rate* $f(\hat{w}^*)$ *and the flow opportunity cost of employment* $\hat{\rho}\hat{U}$ *are given by*

$$f(\hat{w}^*) = \left[(1-\eta(\hat{w}^*))(1+\delta^r\hat{S}(\hat{w}^*)-\hat{\rho}\hat{U})\mathcal{T}(\hat{w}^*,\hat{\rho}+\delta^r)/\tilde{K}\right]^{\frac{1-\alpha}{\alpha}},$$
$$\hat{\rho}\hat{U} = \tilde{B} + \left(\tilde{K}^{\alpha-1}(1-\eta(\hat{w}))^{1-\alpha}\eta(\hat{w})^{\alpha}(1+\delta^r\hat{S}(\hat{w}^*)-\hat{\rho}\hat{U})\mathcal{T}(\hat{w}^*,\hat{\rho}+\delta^r)\right)^{\frac{1}{\alpha}}.$$

4. *Assume* $\gamma \neq 0$ *or* $\sigma \neq 0$*. Given* $\hat{U}$*, the worker's and the firm's continuation sets are connected, and the game's continuation set is bounded; i.e.,*

$$\hat{\mathcal{C}}^h = \{\hat{w} : \hat{w} > \hat{w}^-\} \quad \textit{and } \hat{\mathcal{C}}^j = \{\hat{w} : \hat{w} < \hat{w}^+\}.$$

The worker's and firm's value functions satisfy smooth pasting conditions *at* $\hat{w}^-$ *and* $\hat{w}^+$*, respectively:* $\hat{W}'(\hat{w}^-) = \hat{J}'(\hat{w}^+) = 0$.

Proof. Now, we prove each equilibrium property.

1. The fact that $\hat{\rho}U \geq \tilde{B}$ follows from the same argument as before. Combining the sequence and recursive formulations of the value functions, we have

$$\hat{W}(\hat{w}) = \mathbb{E}_{\hat{w}}\left[\int_0^{\tau^{m*}} e^{-(\hat{\rho}+\delta^r)t}(e^{\hat{w}_t}+\delta^r\hat{W}(\hat{w}^*)-\hat{\rho}\hat{U})\,\mathrm{d}t\right]$$
$$\hat{J}(\hat{w}) = \mathbb{E}_{\hat{w}}\left[\int_0^{\tau^{m*}} e^{-(\hat{\rho}+\delta^r)t}(1-e^{\hat{w}_t}+\delta^r\hat{J}(\hat{w}^*))\,\mathrm{d}t\right]$$

where τ^{m*} is the optimal stopping time. Summing up the previous two equations, we have

$$\hat{S}(\hat{w}) = \hat{W}(\hat{w})+\hat{J}(\hat{w}) = \mathbb{E}_{\hat{w}}\left[\int_0^{\tau^{m*}} e^{-(\hat{\rho}+\delta^r)t}(1+\delta^r\hat{S}(\hat{w}^*)-\hat{\rho}\hat{U})\,\mathrm{d}t\right]$$
$$= (1+\delta^r\hat{S}(\hat{w}^*)-\hat{\rho}\hat{U})\mathbb{E}_{\hat{w}}\left[\int_0^{\tau^{m*}} e^{-(\hat{\rho}+\delta^r)t}\,\mathrm{d}t\right]$$

$$= (1 + \delta^r \hat{S}(\hat{w}^*) - \hat{\rho}\hat{U})\mathcal{T}(\hat{w}, \hat{\rho} + \delta^r).$$

Note that, in the limit as $\delta^r \to \infty$, we get

$$\begin{aligned}
\lim_{\delta^r \to \infty} \mathcal{T}(\hat{w}, \hat{\rho} + \delta^r) &= \lim_{\delta^r \to \infty} \mathbb{E}_{\hat{w}} \left[\int_0^{\tau^{m*}} e^{-(\hat{\rho}+\delta^r)t} \, \mathrm{d}t \right] \\
&= \lim_{\delta^r \to \infty} \mathbb{E}_{\hat{w}} \left[-\frac{1}{\hat{\rho} + \delta^r} \left[e^{-(\hat{\rho}+\delta^r)t} \right]_{t=0}^{\tau^m} \right] \\
&= \lim_{\delta^r \to \infty} -\frac{1}{\hat{\rho} + \delta^r} \mathbb{E}_{\hat{w}} \left[e^{-(\hat{\rho}+\delta^r)\tau^m} - 1 \right] = 0
\end{aligned}$$

Evaluating the expression for match surplus $\hat{S}(\hat{w})$ at $\hat{w}^*$, we get

$$\hat{S}(\hat{w}^*) = (1 + \delta^r \hat{S}(\hat{w}^*) - \hat{\rho}\hat{U})\mathcal{T}(\hat{w}^*, \hat{\rho} + \delta^r)$$

and thus

$$\hat{S}(\hat{w}^*) = \frac{1 - \hat{\rho}\hat{U}}{1 - \delta^r \mathcal{T}(\hat{w}^*, \hat{\rho} + \delta^r)} \mathcal{T}(\hat{w}^*, \hat{\rho} + \delta^r).$$

Plugging this back into the above, we get

$$\hat{S}(\hat{w}) = \frac{1 - \hat{\rho}\hat{U}}{1 - \delta^r \mathcal{T}(\hat{w}^*, \hat{\rho} + \delta^r)} \mathcal{T}(\hat{w}, \hat{\rho} + \delta^r),$$

which is an expression for $\hat{S}(\hat{w})$ that depends only on $\hat{U}$, $\mathcal{T}(\hat{w}^*, \hat{\rho} + \delta^r)$, and $\mathcal{T}(\hat{w}, \hat{\rho} + \delta^r)$ but not on $\hat{S}(\hat{w}^*)$. Since $\hat{W}(\hat{w}), \hat{J}(\hat{w}) \geq 0$, then $\hat{S}(\hat{w}) \geq 0$ and thus

$$0 \leq \hat{S}(\hat{w}^*) = (1 - \hat{\rho}\hat{U}) \underbrace{\frac{\mathcal{T}(\hat{w}^*, \hat{\rho} + \delta^r)}{1 - \delta^r \mathcal{T}(\hat{w}^*, \hat{\rho} + \delta^r)}}_{>0} \iff 0 \leq 1 - \hat{\rho}\hat{U} \iff 1 \geq \hat{\rho}\hat{U}.$$

Therefore, $1 \geq \hat{\rho}\hat{U} \geq \tilde{B}$. To go from weak to strict inequalities, we follow the same logic as previously.

Notice that in the limit as $\delta^r \to \infty$, $\frac{\mathcal{T}(\hat{w}^*, \hat{\rho}+\delta^r)}{1-\delta^r \mathcal{T}(\hat{w}^*, \hat{\rho}+\delta^r)} \to \frac{1}{\hat{\rho}+\delta}$, thus the model converges to the model with flexible wages as in Moen (1997).

2. An analogous proof goes through.
3. The same optimality conditions apply.
4. Once more, an analogous proof goes through.

□

Interpretation. The introduction of staggered wage renegotiations à la Calvo (1983) changes the match surplus in a single way: It changes the expected discounted match duration $\mathcal{T}(\hat{w}^*, \hat{\rho})$ to $\mathcal{T}(\hat{w}^*, \hat{\rho} + \delta^r)/(1 - \delta^r \mathcal{T}(\hat{w}^*, \hat{\rho} + \delta^r)) > 1$, which reflects the extended match duration due to future renegotiations.

D Proofs for Section 3: The Consequences of Monetary Shocks in Non-Coasean Labor Markets

D.1 Proof of Proposition 6

Proposition 6. *Let $Q_0 = 1$ be the numéraire and assume $\mu = \rho + \pi - \zeta^2/2$. Then, $P_t = M_t$ and the value of a worker at time 0 is*

$$V_0 = \max_{\{lm_{it}\}_{t=0}^{\infty}} \mathbb{E}_0 \left[\int_0^{\infty} e^{-\rho t} \frac{Y(lm_i^t)}{P_t} \, \mathrm{d}t \right] + k,$$

where k is a constant independent of the worker's choices and the present discounted value of financial wealth.

Proof. Let V_0 be the present discounted value of the optimal plan. The worker's value is given by

$$V_0 = \max_{\{C_{it}, \hat{M}_{it}, lm_{it}\}_{t=0}^{\infty}} \mathbb{E}_0 \left[\int_{t=0}^{\infty} e^{-\rho t} \left(C_{it} + \mu \log \left(\frac{\hat{M}_{it}}{P_t} \right) \right) \mathrm{d}t \right],$$

subject to

$$\mathbb{E}_0 \left[\int_{t=0}^{\infty} Q_t \left(P_t C_{it} + i_t \hat{M}_{it} - Y(lm_i^t) - T_{it} \right) \mathrm{d}t \right] \leq M_{i0}. \tag{D.1}$$

The first-order conditions for consumption and money holdings, combined with the definition of the nominal interest rate, are given by

$$e^{-\rho t} = \Lambda_i Q_t P_t, \tag{D.2}$$

$$\mu \frac{e^{-\rho t}}{\hat{M}_{it}} = \Lambda_i Q_t i_t, \tag{D.3}$$

$$\mathbb{E}[\mathrm{d}Q_t] = -i_t Q_t \, \mathrm{d}t. \tag{D.4}$$

Here, Λ_i is the Lagrange multiplier of (D.1) for each worker. Equation (D.2) shows that $\Lambda_i = \Lambda$ for all i. Taking integrals over (D.3), we can replace $\hat{M}_{ti} = M_t$. With these results, we guess and verify the following equilibrium outcomes

$$\begin{aligned} P_t &= A^p M_t, \\ i_t &= A^i, \\ Q_t &= \frac{A^Q e^{-\rho t}}{M_t}. \end{aligned} \tag{D.5}$$

given a set of constants A^p, A^i, and A^Q. Using the guess in (D.2) and (D.3)

$$1 = \Lambda A^Q A^P, \tag{D.6}$$

$$\mu = \Lambda A^Q A^i. \tag{D.7}$$

Equations (D.6) and (D.7) provide the equilibrium values for A^Q and A^P given A^i. Applying Ito's lemma and using the guess over (D.4)

$$\mathrm{d}Q_t = A^Q \, \mathrm{d} \left(\frac{e^{-\rho t}}{e^{\log(M_t)}} \right),$$

$$
\begin{aligned}
&= -\rho A^Q \left(\frac{e^{-\rho t}}{e^{\log(M_t)}}\right) \mathrm{d}t - A^Q \frac{e^{-\rho t}}{e^{\log(M_t)}} \mathrm{dlog}(M_t) + A^Q \frac{e^{-\rho t}}{2e^{\log(M_t)}} (\mathrm{dlog}(M_t))^2, \\
&= -\rho Q_t \, \mathrm{d}t - \pi Q_t \, \mathrm{d}t - \zeta Q_t \, \mathrm{d}\mathcal{W}_t^m + \frac{\zeta^2}{2} Q_t \, \mathrm{d}t.
\end{aligned}
$$

Thus, using the guess (D.5) and $\mathbb{E}[\mathrm{d}\mathcal{W}_t^m] = 0$

$$
\mathbb{E}[\mathrm{d}Q_t] = -\underbrace{\left(\rho + \pi - \frac{\zeta^2}{2}\right)}_{=A^i} Q_t \, \mathrm{d}t.
$$

If we take as numéraire $Q_0 = 1$, then we verify the guess with $\mu = \rho + \pi - \frac{\zeta^2}{2}$:

$$
\begin{aligned}
A^Q &= M_0, \\
A^i &= \rho + \pi - \frac{\zeta^2}{2} = \mu, \\
\Lambda &= \frac{\mu}{M_0(\rho + \pi - \zeta^2/2)} = \frac{1}{M_0}, \\
A^P &= \frac{\rho + \pi - \zeta^2/2}{\mu} = 1.
\end{aligned}
$$

Using the budget constraint (D.1)

$$
\begin{aligned}
\mathbb{E}_0 \left[\int_0^\infty Q_t \left(P_t C_{it} + i_t \hat{M}_{it} - Y(lm_i^t) - T_{it}\right) \mathrm{d}t\right] = M_{i0} &\iff \\
\mathbb{E}_0 \left[\int_0^\infty \frac{M_0 e^{-\rho t}}{M_t} \left(M_t C_{it} + \mu M_t - Y(lm_i^t) - T_{it}\right) \mathrm{d}t\right] = M_{i0} &\iff \\
M_0 \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} C_{it} \, \mathrm{d}t\right] = M_{i0} + M_0 \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \frac{Y(lm_i^t)}{M_t}\right] + M_0 \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \frac{T_{it}}{M_t} \, \mathrm{d}t\right] - \frac{M_0}{\rho}\mu &\iff \\
\mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} C_{it} \, \mathrm{d}t\right] = \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \frac{Y(lm_i^t)}{M_t}\right] + k_i&,
\end{aligned}
$$

where k_i is a constant independent of the worker's policies. Thus,

$$
\begin{aligned}
V_0 &= \max_{\{C_{it}, \hat{M}_{it}, lm_{it}\}_{t=0}^\infty} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \left(C_{it} + \mu \log\left(\frac{\hat{M}_{it}}{P_t}\right)\right) \mathrm{d}t\right], \\
&= \max_{\{C_{it}, lm_{it}\}_{t=0}^\infty} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \left(C_{it} + \mu \log\left(\frac{\mu}{\rho + \pi - \zeta^2/2}\right)\right) \mathrm{d}t\right], \\
&= \max_{\{C_{it}, lm_{it}\}_{t=0}^\infty} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} C_{it} \, \mathrm{d}t\right], \\
&= \max_{\{lm_{it}\}_{t=0}^\infty} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \frac{Y(lm_i^t)}{M_t}\right] + k_i.
\end{aligned}
$$

□

D.2 Proof of Proposition 7: CIR of employment with Flexible Entry Wage

We divide the proof of Proposition 7 into three propositions. Proposition D.1 relates the CIR to a perturbation of two Bellman equations describing future employment fluctuations for initially employed and unemployed workers. This proposition covers both the case with flexible and sticky entry wages. Proposition D.2 relates steady-state moments of the perturbed Bellman equations to steady-state moments of the distribution of Δz. Finally, Proposition D.3 related the steady-state moments of Δz to observable moments in the steady-state.

Let $g^h(\Delta z)$ and $g^u(\Delta z)$ be the distributions of Δz across employed and unemployed workers, respectively. The support of $g^h(\Delta z)$ is given by $[-\Delta^-, \Delta^+]$, where $\Delta^- := \hat{w}^* - \hat{w}^-$ and $\Delta^+ := \hat{w}^+ - \hat{w}^*$. We denote by $\mathbb{E}_h[\cdot]$ and $\mathbb{E}_u[\cdot]$ the expectation operators under the distributions $g^h(\Delta z)$ and $g^u(\Delta z)$, respectively.

Proposition D.1. *Given steady-state policies $(\hat{w}^-, \hat{w}^*, \hat{w}^+)$ and distributions $(g^h(\Delta z), g^u(\Delta z))$, the CIR is given by*

$$CIR_{\mathcal{E}}(\zeta) = \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z) g^h(\Delta z + \zeta)\, \mathrm{d}\Delta z + \int_{-\infty}^{\infty} m_{\mathcal{E},u}(\Delta z, \zeta) g^u(\Delta z + \zeta)\, \mathrm{d}\Delta z,$$

where the value functions $m_{\mathcal{E},h}(\Delta z)$ and $m_{\mathcal{E},u}(\Delta z, \zeta)$ are defined as:

$$m_{\mathcal{E},h}(\Delta z) = \mathbb{E}\left[\int_0^{\tau^m} (1 - \mathcal{E}_{ss})\, \mathrm{d}t + m_{\mathcal{E},u}(0,0) \,\middle|\, \Delta z_0 = \Delta z \right], \tag{D.8}$$

$$m_{\mathcal{E},u}(\Delta z, \zeta) = \mathbb{E}\left[\int_0^{\tau^u(\zeta)} (-\mathcal{E}_{ss})\, \mathrm{d}t + m_{\mathcal{E},h}(-\zeta) \,\middle|\, \Delta z_0 = \Delta z \right]. \tag{D.9}$$

$$0 = \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z) g^h(\Delta z)\, \mathrm{d}\Delta z + \int_{-\infty}^{\infty} m_{\mathcal{E},u}(\Delta z, 0) g^u(\Delta z)\, \mathrm{d}\Delta z.$$

with $\tau^u(\zeta)$ being distributed according to a Poisson process with arrival rate $f(\hat{w}^* - \zeta)$.

Proof. We define the cumulative impulse response of aggregate employment to a monetary shock as

$$\text{CIR}_{\mathcal{E}}(\zeta) = \int_0^{\infty} \int_{-\infty}^{\infty} \left(g^h(\Delta z, \zeta, t) - g^h(\Delta z) \right) \mathrm{d}\Delta z\, \mathrm{d}t$$

Note that $\mathcal{E}_t = \int_{-\infty}^{\infty} g^h(\Delta z, \zeta, t)\, \mathrm{d}\Delta z$ is a function of ζ since aggregate shocks affect net flows into employment. The proof proceeds in three steps. Step 1 rewrites the CIR as the integral over time of two value functions, one for employed and unemployed workers, up to a finite time $\mathcal{T}$. Step 2 expresses the CIR as $\mathcal{T} \to \infty$. Step 3 uses the equivalence of the combined Dirichlet-Poisson problem (i.e., the mapping between the sequential problem and the corresponding HJB equations and boundary conditions).

Step 1.

Here, we follow a recursive representation for the CIR. The CIR satisfies

$$\text{CIR}_{\mathcal{E}}(\zeta) = \int_{-\infty}^{\infty} \lim_{\mathcal{T} \to \infty} \left[m_{\mathcal{E},h}(\Delta z, \mathcal{T}) g^h(\Delta z + \zeta) + m_{\mathcal{E},u}(\Delta z, \mathcal{T}) g^u(\Delta z + \zeta) \right] \mathrm{d}\Delta z$$

where we defined

$$m_{\mathcal{E},h}(\Delta z_0, \mathcal{T}) := \int_0^{\mathcal{T}} \left[\int_{-\infty}^{\infty} \left[(1 - \mathcal{E}_{ss})\, g^h(\Delta z, t | \Delta z_0, h) + (-\mathcal{E}_{ss}) g^u(\Delta z, t | \Delta z_0, h) \right] \mathrm{d}\Delta z\, \mathrm{d}t \right] \tag{D.10}$$

$$m_{\mathcal{E},u}(\Delta z_0, \zeta, \mathcal{T}) := \int_0^{\mathcal{T}} \left[\int_{-\infty}^{\infty} \left[(1 - \mathcal{E}_{ss})\, g^h(\Delta z, \zeta, t | \Delta z_0, u) + (-\mathcal{E}_{ss}) g^u(\Delta z, \zeta, t | \Delta z_0, u) \right] \mathrm{d}\Delta z\, \mathrm{d}t \right] \tag{D.11}$$

Proof of Step 1. Starting from the definition of the CIR, (1) adds and subtracts employment in t, $\mathcal{E}_t$; (2) operates over the integral;

(3) and (4) use the fact that the integral operator is a linear operator; (5) applies the definition of a conditional expectation, where $g^h(\Delta z, t|\Delta z_0, h)\,\mathrm{d}\Delta z$ is the probability of a worker being in the state Δz at time t when the initial productivity is Δz_0 and the initial employment state is h (mutatis mutandis if the initial employment state is u); (6) uses the fact that conditional on being initially employed, the transition probabilities are independent of ζ; (6) and (7) apply Fubini's theorem and the definition of the limit of an integral; (8) relabels the resulting terms.

$$
\begin{aligned}
\mathrm{CIR}_{\mathcal{E}}(\zeta) &= \int_0^\infty \int_{-\infty}^\infty \Big(g^h(\Delta z,\zeta,t) - g^h(\Delta z) \Big)\,\mathrm{d}\Delta z\,\mathrm{d}t \\
&=^{(1)} \int_0^\infty \int_{-\infty}^\infty \Big(g^h(\Delta z,\zeta,t) - g^h(\Delta z)(\mathcal{E}_t + 1 - \mathcal{E}_t) \Big)\,\mathrm{d}\Delta z\,\mathrm{d}t \\
&=^{(2)} \int_0^\infty \left[\int_{-\infty}^\infty g^h(\Delta z,\zeta,t)\,\mathrm{d}\Delta z - \mathcal{E}_{ss}\left(\int_{-\infty}^\infty g^h(\Delta z,\zeta,t)\,\mathrm{d}\Delta z + \int_{-\infty}^\infty g^u(\Delta z,\zeta,t)\,\mathrm{d}\Delta z \right) \right] \mathrm{d}t \\
&=^{(3)} \int_0^\infty \int_{-\infty}^\infty (1-\mathcal{E}_{ss})\, g^h(\Delta z,\zeta,t)\,\mathrm{d}\Delta z\,\mathrm{d}t + \int_0^\infty \int_{-\infty}^\infty (-\mathcal{E}_{ss}) g^u(\Delta z,\zeta,t)\,\mathrm{d}\Delta z\,\mathrm{d}t \\
&=^{(4)} \int_0^\infty \int_{-\infty}^\infty \Big[(1-\mathcal{E}_{ss})\, g^h(\Delta z,\zeta,t) + (-\mathcal{E}_{ss}) g^u(\Delta z,\zeta,t) \Big]\,\mathrm{d}\Delta z\,\mathrm{d}t \\
&=^{(5)} \int_0^\infty \int_{-\infty}^\infty \left[\int_{-\infty}^\infty \Big[(1-\mathcal{E}_{ss})\, g^h(\Delta z,t|\Delta z_0,h) + (-\mathcal{E}_{ss}) g^u(\Delta z,t|\Delta z_0,h) \Big] g^h(\Delta z_0,0)\,\mathrm{d}\Delta z_0\,\mathrm{d}\Delta z\,\mathrm{d}t \right] \dots \\
&\quad \dots + \int_0^\infty \int_{-\infty}^\infty \left[\int_{-\infty}^\infty \Big[(1-\mathcal{E}_{ss})\, g^h(\Delta z,\zeta,t|\Delta z_0,u) + (-\mathcal{E}_{ss}) g^u(\Delta z,\zeta,t|\Delta z_0,u) \Big] g^u(\Delta z_0,0)\,\mathrm{d}\Delta z_0\,\mathrm{d}\Delta z\,\mathrm{d}t \right] \\
&=^{(6)} \int_{-\infty}^\infty \int_0^\infty \left[\int_{-\infty}^\infty \Big[(1-\mathcal{E}_{ss})\, g^h(\Delta z,t|\Delta z_0,h) + (-\mathcal{E}_{ss}) g^u(\Delta z,t|\Delta z_0,h) \Big]\,\mathrm{d}\Delta z\,\mathrm{d}t \right] g^h(\Delta z_0+\zeta)\,\mathrm{d}\Delta z_0 \dots \\
&\quad \dots + \int_{-\infty}^\infty \int_0^\infty \left[\int_{-\infty}^\infty \Big[(1-\mathcal{E}_{ss})\, g^h(\Delta z,\zeta,t|\Delta z_0,u) + (-\mathcal{E}_{ss}) g^u(\Delta z,\zeta,t|\Delta z_0,u) \Big]\,\mathrm{d}\Delta z\,\mathrm{d}t \right] g^u(\Delta z_0+\zeta)\,\mathrm{d}\Delta z_0 \\
&=^{(7)} \int_{-\infty}^\infty \lim_{\mathcal{T}\to\infty} \underbrace{\int_0^{\mathcal{T}} \left[\int_{-\infty}^\infty \Big[(1-\mathcal{E}_{ss})\, g^h(\Delta z,t|\Delta z_0,h) + (-\mathcal{E}_{ss}) g^u(\Delta z,t|\Delta z_0,h) \Big]\,\mathrm{d}\Delta z\,\mathrm{d}t \right]}_{m_{\mathcal{E},h}(\Delta z_0,\mathcal{T})} g^h(\Delta z_0+\zeta)\,\mathrm{d}\Delta z_0 \dots \\
&\quad \dots + \int_{-\infty}^\infty \lim_{\mathcal{T}\to\infty} \underbrace{\int_0^{\mathcal{T}} \left[\int_{-\infty}^\infty \Big[(1-\mathcal{E}_{ss})\, g^h(\Delta z,\zeta,t|\Delta z_0,u) + (-\mathcal{E}_{ss}) g^u(\Delta z,\zeta,t|\Delta z_0,u) \Big]\,\mathrm{d}\Delta z\,\mathrm{d}t \right]}_{m_{\mathcal{E},u}(\Delta z_0,\zeta,\mathcal{T})} g^u(\Delta z_0+\zeta)\,\mathrm{d}\Delta z_0 \\
&=^{(8)} \int_{-\infty}^\infty \lim_{\mathcal{T}\to\infty} m_{\mathcal{E},h}(\Delta z,\mathcal{T}) g^h(\Delta z+\zeta)\,\mathrm{d}\Delta z + \int_{-\infty}^\infty \lim_{\mathcal{T}\to\infty} m_{\mathcal{E},u}(\Delta z,\zeta,\mathcal{T}) g^u(\Delta z+\zeta)\,\mathrm{d}\Delta z
\end{aligned} \tag{D.12}
$$

where we define

$$
\begin{aligned}
m_{\mathcal{E},h}(\Delta z_0,\mathcal{T}) &\equiv \int_0^{\mathcal{T}} \left[\int_{-\infty}^\infty \Big[(1-\mathcal{E}_{ss})\, g^h(\Delta z,t|\Delta z_0,h) + (-\mathcal{E}_{ss}) g^u(\Delta z,t|\Delta z_0,h) \Big]\,\mathrm{d}\Delta z\,\mathrm{d}t \right] \\
m_{\mathcal{E},u}(\Delta z_0,\zeta,\mathcal{T}) &\equiv \int_0^{\mathcal{T}} \left[\int_{-\infty}^\infty \Big[(1-\mathcal{E}_{ss})\, g^h(\Delta z,\zeta,t|\Delta z_0,u) + (-\mathcal{E}_{ss}) g^u(\Delta z,\zeta,t|\Delta z_0,u) \Big]\,\mathrm{d}\Delta z\,\mathrm{d}t \right].
\end{aligned}
$$

Step 2. The CIR satisfies

$$
\mathrm{CIR}_{\mathcal{E}}(\zeta) = \int_{-\infty}^\infty m_{\mathcal{E},h}(\Delta z) g^h(\Delta z+\zeta)\,\mathrm{d}\Delta z + \int_{-\infty}^\infty m_{\mathcal{E},u}(\Delta z,\zeta) g^u(\Delta z+\zeta)\,\mathrm{d}\Delta z
$$

and the value functions $m_{\mathcal{E},h}(\Delta z_0)$ and $m_{\mathcal{E},u}(\Delta z_0,\zeta)$ satisfy the following HJB and border conditions:

$$
0 = 1 - \mathcal{E}_{ss} - (\gamma+\pi)\frac{\mathrm{d}m_{\mathcal{E},h}(\Delta z)}{\mathrm{d}\Delta z} + \frac{\sigma^2}{2}\frac{d^2 m_{\mathcal{E},h}(\Delta z)}{\mathrm{d}\Delta z^2} + \delta(m_{\mathcal{E},u}(0,0) - m_{\mathcal{E},h}(\Delta z)), \tag{D.13}
$$

$$0 = -\mathcal{E}_{ss} - (\gamma+\pi)\frac{\mathrm{d}m_{\mathcal{E},u}(\Delta z,\zeta)}{\mathrm{d}\Delta z} + \frac{\sigma^2}{2}\frac{d^2 m_{\mathcal{E},u}(\Delta z,\zeta)}{\mathrm{d}\Delta z^2} + f(\hat{w}^* - \zeta)(m_{\mathcal{E},h}(-\zeta) - m_{\mathcal{E},u}(\Delta z,\zeta)) \tag{D.14}$$

$$m_{\mathcal{E},u}(0,0) = m_{\mathcal{E},h}(\Delta z), \text{ for all } \Delta z \notin (-\Delta^-,\Delta^+) \tag{D.15}$$

$$0 = \lim_{\Delta z\to-\infty}\frac{\mathrm{d}m_{\mathcal{E},u}(\Delta z,\zeta)}{\mathrm{d}\Delta z} = \lim_{\Delta z\to\infty}\frac{\mathrm{d}m_{\mathcal{E},u}(\Delta z,\zeta)}{\mathrm{d}\Delta z} \tag{D.16}$$

$$0 = \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z)g^h(\Delta z)\,\mathrm{d}\Delta z + \int_{-\infty}^{\infty} m_{\mathcal{E},u}(\Delta z,0)g^u(\Delta z)\,\mathrm{d}\Delta z. \tag{D.17}$$

Proof of Step 2. We divide this proof in steps *a* to *d*.

a. We show that $\lim_{\mathcal{T}\to\infty} m_{\mathcal{E},h}(\Delta z,\mathcal{T}) = m_{\mathcal{E},h}(\Delta z)$ and $\lim_{\mathcal{T}\to\infty} m_{\mathcal{E},u}(\Delta z,\zeta,\mathcal{T}) = m_{\mathcal{E},u}(\Delta z,\zeta)$: This property holds due to the convergence of the distribution of Δz over time to its ergodic distribution for any initial condition (Stokey, 1989).

b. We show that $0 = \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z,\mathcal{T})g^h(\Delta z)\,\mathrm{d}\Delta z + \int_{-\infty}^{\infty} m_{\mathcal{E},u}(\Delta z,0,\mathcal{T})g^u(\Delta z)\,\mathrm{d}\Delta z$: The logic of the proof is to repeat the steps behind (D.12) in the reverse order. Departing from the definition,

$$\begin{aligned}
&\int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z_0,\mathcal{T})g^h(\Delta z_0)\,\mathrm{d}\Delta z_0 + \int_{-\infty}^{\infty} m_{\mathcal{E},u}(\Delta z_0,0,\mathcal{T})g^u(\Delta z_0)\,\mathrm{d}\Delta z_0\\
&=^{(1)} \int_{-\infty}^{\infty}\int_0^{\mathcal{T}}\left[\int_{-\infty}^{\infty}\left[(1-\mathcal{E}_{ss})\,g^h(\Delta z,t|\Delta z_0,h) + (-\mathcal{E}_{ss})g^u(\Delta z,t|\Delta z_0,h)\right]\mathrm{d}\Delta z\,\mathrm{d}t\right]g^h(\Delta z_0)\,\mathrm{d}\Delta z_0\\
&\cdots + \int_{-\infty}^{\infty}\int_0^{\mathcal{T}}\left[\int_{-\infty}^{\infty}\left[(1-\mathcal{E}_{ss})\,g^h(\Delta z,0,t|\Delta z_0,u) + (-\mathcal{E}_{ss})g^u(\Delta z,0,t|\Delta z_0,u)\right]\mathrm{d}\Delta z\,\mathrm{d}t\right]g^u(\Delta z_0)\,\mathrm{d}\Delta z_0\\
&=^{(2)} \int_0^{\mathcal{T}}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\left[(1-\mathcal{E}_{ss})\,g^h(\Delta z,t|\Delta z_0,h) + (-\mathcal{E}_{ss})g^u(\Delta z,t|\Delta z_0,h)\right]g^h(\Delta z_0)\,\mathrm{d}z_0\,\mathrm{d}\Delta z\right]\mathrm{d}t\\
&\cdots + \int_0^{\mathcal{T}}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}\left[(1-\mathcal{E}_{ss})\,g^h(\Delta z,0,t|\Delta z_0,u) + (-\mathcal{E}_{ss})g^u(\Delta z,0,t|\Delta z_0,u)\right]g^u(\Delta z_0)\;\mathrm{d}\Delta z_0\,\mathrm{d}\Delta z\right]\mathrm{d}t\\
&=^{(3)} \int_0^{\mathcal{T}}\int_{-\infty}^{\infty}\left[(1-\mathcal{E}_{ss})\underbrace{\int_{-\infty}^{\infty} g^h(\Delta z,0,t|\Delta z_0)g(\Delta z_0)\,\mathrm{d}\Delta z_0}_{=g^h(\Delta z)}\mathrm{d}\Delta z + (-\mathcal{E}_{ss})\underbrace{\int_{-\infty}^{\infty} g^u(\Delta z,0,t|\Delta z_0)g(\Delta z_0)\,\mathrm{d}\Delta z_0}_{=g^u(\Delta z)}\mathrm{d}\Delta z\right]\mathrm{d}t\\
&=^{(4)} \int_0^{\mathcal{T}}(1-\mathcal{E}_{ss})\,\mathcal{E}_{ss}\,\mathrm{d}t + \int_0^{\mathcal{T}}(-\mathcal{E}_{ss})\,(1-\mathcal{E}_{ss})\,\mathrm{d}t\\
&= 0
\end{aligned}$$

In (1), we apply the definitions (D.10) and (D.11); (2) applies Fubini's theorem; (3) uses the steady-state conditions for $g^h(\cdot)$ and $g^u(\cdot)$, and the definition $g(\Delta z) = g^h(\Delta z) + g^u(\Delta z)$; and (4) computes the integral using the definitions of aggregate employment and unemployment.

c. We show that $0 = \int_{-\Delta^-}^{\Delta^+} m_{\mathcal{E},h}(\Delta z)g^h(\Delta z)\,\mathrm{d}\Delta z + \int_{-\Delta^-}^{\Delta^+} m_{\mathcal{E},u}(\Delta z,0)g^u(\Delta z)\,\mathrm{d}\Delta z$: See Baley and Blanco (2022).

d. We show that the CIR satisfies (D.12) with $m_{\mathcal{E},h}(\Delta z_0)$ and $m_{\mathcal{E},u}(\Delta z_0,\zeta)$ satisfying (D.13)–(D.17): Writing the HJB for $m_{\mathcal{E},h}(\Delta z_0,\mathcal{T})$ and $m_{\mathcal{E},u}(\Delta z_0,\zeta,\mathcal{T})$, we have that

$$\begin{aligned}
0 &= 1-\mathcal{E}_{ss} - \frac{\mathrm{d}m_{\mathcal{E},h}(\Delta z,\mathcal{T})}{\mathrm{d}\mathcal{T}} - (\gamma+\pi)\frac{\mathrm{d}m_{\mathcal{E},h}(\Delta z,\mathcal{T})}{\mathrm{d}\Delta z} + \frac{\sigma^2}{2}\frac{d^2 m_{\mathcal{E},h}(\Delta z,\mathcal{T})}{\mathrm{d}\Delta z^2} + \delta(m_{\mathcal{E},u}(0,0,\mathcal{T}) - m_{\mathcal{E},h}(\Delta z,\mathcal{T})),\\
0 &= -\mathcal{E}_{ss} - \frac{\mathrm{d}m_{\mathcal{E},u}(\Delta z,\zeta,\mathcal{T})}{\mathrm{d}\mathcal{T}} - (\gamma+\pi)\frac{\mathrm{d}m_{\mathcal{E},u}(\Delta z,\zeta,\mathcal{T})}{\mathrm{d}\Delta z} + \frac{\sigma^2}{2}\frac{d^2 m_{\mathcal{E},u}(\Delta z,\zeta,\mathcal{T})}{\mathrm{d}\Delta z^2} + f(\hat{w}^* - \zeta)(m_{\mathcal{E},h}(-\zeta,\mathcal{T}) - m_{\mathcal{E},u}(\Delta z,\zeta,\mathcal{T}))
\end{aligned}$$

$$m_{\mathcal{E},u}(0,0,\mathcal{T}) = m_{\mathcal{E},h}(\Delta z,\mathcal{T}),\text{ for all } \Delta z \notin (-\Delta^-,\Delta^+)$$
$$0 = \lim_{\Delta z\to-\infty} \frac{\mathrm{d}m_{\mathcal{E},u}(\Delta z,\zeta,\mathcal{T})}{\mathrm{d}\Delta z} = \lim_{\Delta z\to\infty} \frac{\mathrm{d}m_{\mathcal{E},u}(\Delta z,\zeta,\mathcal{T})}{\mathrm{d}\Delta z}$$
$$0 = \int_{-\Delta^-}^{\Delta^+} m_{\mathcal{E},h}(\Delta z,\mathcal{T})g^h(\Delta z)\,\mathrm{d}\Delta z + \int_{-\Delta^-}^{\Delta^+} m_{\mathcal{E},u}(\Delta z,0,\mathcal{T})g^u(\Delta z)\,\mathrm{d}\Delta z.$$

The border condition for $m_{\mathcal{E},u}(\Delta z,\zeta,\mathcal{T})$ is implied from the fact that the job finding rate $f(\hat{w}^*)$ is independent of Δz, so the function $m_{\mathcal{E},u}(\Delta z,\zeta,\mathcal{T})$ is constant in the entire domain. Taking the limit $\mathcal{T}\to\infty$ and using point-wise convergence of $m_{\mathcal{E},h}(\Delta z_0,\mathcal{T})$ and $m_{\mathcal{E},u}(\Delta z_0,\zeta,\mathcal{T})$, we have the desired result.

Step 3. The solution of the differential equations (D.13) to (D.16) satisfy (D.8) and (D.9).

Proof of Step 3. This is just an application of Øksendal (2007), Chapter 9.

□

Before starting the next step of the proof, we summarize the conditions that characterize the distributions of Δz.

Steady-State Cross-Sectional Distribution Δz. Below, we describe the Kolmogorov Forward Equations (KFE) for $g^h(\Delta z)$ and $g^u(\Delta z)$.

$$\delta g^h(\Delta z) = (\gamma+\pi)(g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z) \quad \text{for all } \Delta z \in (-\Delta^-,\Delta^+)/\{0\}, \tag{D.18}$$
$$f(\hat{w}^*)g^u(\Delta z) = (\gamma+\pi)(g^u)'(\Delta z) + \frac{\sigma^2}{2}(g^u)''(\Delta z) \quad \text{for all } \Delta z \in (-\infty,\infty)/\{0\}. \tag{D.19}$$
$$g^h(\Delta z) = 0,\text{ for all } \Delta z \notin (-\Delta^-,\Delta^+) \tag{D.20}$$
$$\lim_{\Delta z\to-\infty} g^u(\Delta z) = \lim_{\Delta z\to\infty} g^u(\Delta z) = 0. \tag{D.21}$$
$$1 = \int_{-\infty}^{\infty} g^u(\Delta z)\,\mathrm{d}\Delta z + \int_{-\Delta^-}^{\Delta^+} g^h(\Delta z)\,\mathrm{d}\Delta z, \tag{D.22}$$
$$f(\hat{w}^*)(1-\mathcal{E}) = \delta\mathcal{E} + \frac{\sigma^2}{2}\left[\lim_{\Delta z\downarrow-\Delta^-}(g^h)'(\Delta z) - \lim_{\Delta z\uparrow\Delta^+}(g^h)'(\Delta z)\right], \tag{D.23}$$
$$g^h(\Delta z), g^u(\Delta z) \in \mathbb{C}, \mathbb{C}^1(\{0\}), \mathbb{C}^2(\{0\})$$

Proposition D.2. *Assume flexible entry wages. Up to first order, the CIR of employment is given by:*

$$\frac{CIR_{\mathcal{E}}(\zeta)}{\zeta} = -(1-\mathcal{E}_{ss})\frac{(\gamma+\pi)\mathbb{E}_h[a] + \mathbb{E}_h[\Delta z]}{\sigma^2} + o(\zeta).$$

Proof. The proof proceeds in three steps. Step 1 computes the value function for an unemployed worker $m_{\mathcal{E},u}(\Delta z)$ (when entry wages are flexible, the job-finding rate and this value function are independent of the shock ζ, so we omit this argument). Step 2 computes the value for the employed worker at $\Delta z = 0$—i.e., $m_{\mathcal{E},h}(0)$. Step 3 characterizes the CIR as a function of steady-state aggregate variables and moments.

Step 1. The CIR is given by

$$\text{CIR}_{\mathcal{E}}(\zeta) = \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z) g^h(\Delta z + \zeta)\, \mathrm{d}\Delta z + \left(-\frac{\mathcal{E}_{ss}}{f(\hat{w}^*)} + m_{\mathcal{E},h}(0)\right)(1-\mathcal{E}_{ss}),$$

with

$$0 = 1 - \mathcal{E}_{ss} - (\gamma+\pi)\frac{\mathrm{d}m_{\mathcal{E},h}(\Delta z)}{\mathrm{d}\Delta z} + \frac{\sigma^2}{2}\frac{d^2 m_{\mathcal{E},h}(\Delta z)}{\mathrm{d}\Delta z^2} + \delta\left(-\frac{\mathcal{E}_{ss}}{f(\hat{w}^*)} + m_{\mathcal{E},h}(0) - m_{\mathcal{E},h}(\Delta z)\right),$$

$$-\frac{\mathcal{E}_{ss}}{f(\hat{w}^*)} + m_{\mathcal{E},h}(0) = m_{\mathcal{E},h}(\Delta z), \text{ for all } \Delta z \notin (-\Delta^-, \Delta^+)$$

$$0 = \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z) g^h(\Delta z)\, \mathrm{d}\Delta z + \left(-\frac{\mathcal{E}_{ss}}{f(\hat{w}^*)} + m_{\mathcal{E},h}(0)\right)(1-\mathcal{E}_{ss}). \tag{D.24}$$

Proof of Step 1. To show this result, observe that the solution to (D.14) and (D.16) is

$$m_{\mathcal{E},u}(\Delta z) = m_{\mathcal{E},u}(0), \text{ for all } \Delta z.$$

Thus,

$$0 = -\mathcal{E}_{ss} + f(\hat{w}^*)(m_{\mathcal{E},h}(0) - m_{\mathcal{E},u}(0)) \iff m_{\mathcal{E},u}(0) = -\frac{\mathcal{E}_{ss}}{f(\hat{w}^*)} + m_{\mathcal{E},h}(0). \tag{D.25}$$

Replacing (D.25) into the CIR, we have the result.

Step 2. We show that $m_{\mathcal{E},h}(0) = \frac{\mathcal{E}_{ss}}{f(\hat{w}^*)} - (1-\mathcal{E}_{ss})\mathbb{E}_h[a]$, where $\mathbb{E}_h[a]$ is the cross-sectional expected age of the match or the worker's tenure at the current match.

Proof of Step 2. Observe that $m_{\mathcal{E},h}(\Delta z)$ satisfies the following recursive representation

$$m_{\mathcal{E},h}(\Delta z) = \mathbb{E}\left[\int_0^{\tau^m}(1-\mathcal{E}_{ss})\,\mathrm{d}t + \left(-\frac{\mathcal{E}_{ss}}{f(\hat{w}^*)} + m_{\mathcal{E},h}(0)\right)\middle|\, \Delta z_0 = \Delta z\right]. \tag{D.26}$$

Define the following auxiliary function

$$\Psi(\Delta z|\varphi) = \mathbb{E}\left[\int_0^{\tau^m} e^{\varphi t}(1-\mathcal{E}_{ss})\,\mathrm{d}t + e^{\varphi\tau^m}\left(-\frac{\mathcal{E}_{ss}}{f(\hat{w}^*)} + m_{\mathcal{E},h}(0)\right)\middle|\, \Delta z_0 = \Delta z\right]. \tag{D.27}$$

and note that $\Psi(\Delta z|0) = m_{\mathcal{E},h}(\Delta z)$. The auxiliary function $\Psi(\Delta z|\varphi)$ satisfies the following HJB and border conditions:

$$-\varphi\Psi(\Delta z|\varphi) + \delta\left(\Psi(\Delta z|\varphi) - \left(-\frac{\mathcal{E}_{ss}}{f(\hat{w}^*)} + m_{\mathcal{E},h}(0)\right)\right) = (1-\mathcal{E}_{ss}) - (\gamma+\pi)\frac{\partial\Psi(\Delta z|\varphi)}{\partial\Delta z} + \frac{\sigma^2}{2}\frac{\partial^2\Psi(\Delta z|\varphi)}{\partial\Delta z^2}, \tag{D.28}$$

$$\Psi(\Delta z,\varphi) = \left(-\frac{\mathcal{E}_{ss}}{f(\hat{w}^*)} + m_{\mathcal{E},h}(0)\right) \text{ for all } \Delta z \notin (-\Delta^-, \Delta^+).$$

Taking the derivative with respect to φ in (D.28), we have that

$$(\delta-\varphi)\frac{\partial\Psi(\Delta z|\varphi)}{\partial\varphi} - \Psi(\Delta z|\varphi) = -(\gamma+\pi)\frac{\partial^2\Psi(\Delta z,\varphi)}{\partial\Delta z\partial\varphi} + \frac{\sigma^2}{2}\frac{\partial^3\Psi(\Delta z|\varphi)}{\partial\Delta z^2\partial\varphi},$$

$$\frac{\partial\Psi(\Delta z|\varphi)}{\partial\varphi} = 0 \text{ for all } \Delta z \notin (-\Delta^-, \Delta^+).$$

Using the Schwarz theorem to exchange partial derivatives, evaluating at $\varphi = 0$, and using $\Psi(\Delta z|0) = m_{\mathcal{E},h}(\Delta z)$, we obtain

$$\delta \frac{\partial \Psi(\Delta z|0)}{\partial \varphi} - m_{\mathcal{E},h}(\Delta z) = -(\gamma+\pi)\frac{\partial}{\partial \Delta z}\left(\frac{\partial \Psi(\Delta z|0)}{\partial \varphi}\right) + \frac{\sigma^2}{2}\frac{\partial^2}{\partial \Delta z^2}\left(\frac{\partial \Psi(\Delta z|0)}{\partial \varphi}\right), \tag{D.29}$$

$$\frac{\partial \Psi(-\Delta^-|0)}{\partial \varphi} = \frac{\partial \Psi(\Delta^+|0)}{\partial \varphi} = 0. \tag{D.30}$$

Equations (D.29) and (D.30) correspond to the HJB and border conditions of the function $\frac{\partial \Psi(\Delta z|0)}{\partial \varphi} = \mathbb{E}\left[\int_0^{\tau^m} m_{\mathcal{E},h}(\Delta z_t)\,\mathrm{d}t \middle| \Delta z_0 = \Delta z\right]$. Evaluating $\frac{\partial \Psi(\Delta z|0)}{\partial \varphi}$ at $\Delta z = 0$, using the occupancy measure and result (D.24), we write the previous equation as:

$$\begin{aligned}
\frac{\partial \Psi(0|0)}{\partial \varphi} &= \mathbb{E}\left[\int_0^{\tau^m} m_{\mathcal{E},h}(\Delta z_t)\,\mathrm{d}t \middle| \Delta z_0 = 0\right] \\
&= \mathbb{E}_{\mathcal{D}}[\tau^m]\frac{\int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z) g^h(\Delta z)\,\mathrm{d}\Delta z}{\mathcal{E}_{ss}} \\
&= \mathbb{E}_{\mathcal{D}}[\tau^m]\left(\frac{\mathcal{E}_{ss}}{f(\hat{w}^*)} - m_{\mathcal{E},h}(0)\right)\frac{(1-\mathcal{E}_{ss})}{\mathcal{E}_{ss}},
\end{aligned} \tag{D.31}$$

where $\mathbb{E}_{\mathcal{D}}[\cdot]$ is the mean duration of completed employment spells (the subscript highlights that the moment can be easily computed from the data). From (D.27), we also have that

$$\begin{aligned}
\frac{\partial \Psi(0|0)}{\partial \varphi} &= \mathbb{E}\left[\int_0^{\tau^m} s\,(1-\mathcal{E}_{ss})\,\mathrm{d}s + \tau^m\left(-\frac{\mathcal{E}_{ss}}{f(\hat{w}^*)} + m_{\mathcal{E},h}(0)\right) \middle| \Delta z_0 = 0\right] \\
&= \mathbb{E}_{\mathcal{D}}\left[\tau^m\right]\left[(1-\mathcal{E}_{ss})\frac{\mathbb{E}_h[a]}{\mathcal{E}_{ss}} + \left(-\frac{\mathcal{E}_{ss}}{f(\hat{w}^*)} + m_{\mathcal{E},h}(0)\right)\right],
\end{aligned} \tag{D.32}$$

Combining (D.31) and (D.32), and solving for $m_{\mathcal{E},h}(0)$ we obtain:

$$m_{\mathcal{E},h}(0) = \frac{\mathcal{E}_{ss}}{f(\hat{w}^*)} - (1-\mathcal{E}_{ss})\mathbb{E}_h[a]$$

Step 3. Up to a first-order approximation, the CIR is given by:

$$\mathrm{CIR}_{\mathcal{E}}(\zeta) = -(1-\mathcal{E}_{ss})\frac{(\gamma+\pi)\mathbb{E}_h[a] + \mathbb{E}_h[\Delta z]}{\sigma^2}\zeta + o(\zeta^2).$$

Proof of Step 3. To help the reader, we summarize below the conditions used in this step of the proof.

$$\mathrm{CIR}_{\mathcal{E}}(\zeta) = \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z) g^h(\Delta z + \zeta)\,\mathrm{d}\Delta z + \left(-\frac{\mathcal{E}_{ss}}{f(\theta^*)} + m_{\mathcal{E},h}(0)\right)(1-\mathcal{E}_{ss}) \tag{D.33}$$

with

$$\delta m_{\mathcal{E},h}(\Delta z) = 1 - \mathcal{E}_{ss} - (\gamma+\pi)\frac{\mathrm{d}m_{\mathcal{E},h}(\Delta z)}{\mathrm{d}\Delta z} + \frac{\sigma^2}{2}\frac{d^2 m_{\mathcal{E},h}(\Delta z)}{\mathrm{d}\Delta z^2} + \delta m_{\mathcal{E},u}(0), \tag{D.34}$$

$$m_{\mathcal{E},u}(0) = m_{\mathcal{E},h}(\Delta z) \text{ for all } \Delta z \notin (-\Delta^-, \Delta^+) \tag{D.35}$$

$$0 = \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z) g^h(\Delta z)\,\mathrm{d}\Delta z + m_{\mathcal{E},u}(0)(1-\mathcal{E}_{ss}). \tag{D.36}$$

1. **Zero-order:** If $\zeta = 0$, condition (D.36) implies

$$\mathrm{CIR}_{\mathcal{E}}(0) = \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z) g^h(\Delta z)\,\mathrm{d}\Delta z + \left(-\frac{\mathcal{E}_{ss}}{f(\theta^*)} + m_{\mathcal{E},h}(0)\right)(1-\mathcal{E}_{ss}) = 0.$$

2. **First-order:** Taking the derivative of (D.33) we obtain

$$\text{CIR}'_{\mathcal{E}}(\zeta) = \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z + \zeta)\,\mathrm{d}\Delta z,$$

which evaluated at $\zeta = 0$ becomes

$$\text{CIR}'_{\mathcal{E}}(0) = \int_{-\Delta^-}^{\Delta^+} m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\,\mathrm{d}\Delta z.$$

Using condition (D.18) to replace $\delta = \frac{(\gamma+\pi)(g^h)'(\Delta z)+\frac{\sigma^2}{2}(g^h)''(\Delta z)}{g^h(\Delta z)}$ into equation (D.34), we obtain

$$\frac{(\gamma+\pi)(g^h)'(\Delta z)+\frac{\sigma^2}{2}(g^h)''(\Delta z)}{g^h(\Delta z)} m_{\mathcal{E},h}(\Delta z) = 1 - \mathcal{E}_{ss} - (\gamma+\pi)m'_{\mathcal{E},h}(\Delta z) + \frac{\sigma^2}{2}m''_{\mathcal{E},h}(\Delta z)$$
$$+ \frac{(\gamma+\pi)g'(\Delta z)+\frac{\sigma^2}{2}g''(\Delta z)}{g(\Delta z)} m_{\mathcal{E},u}(0).$$

Multiplying both sides by $g^h(\Delta z)\Delta z$ and integrating between $-\Delta^-$ and Δ^+,

$$0 = (1-\mathcal{E}_{ss})\mathbb{E}_h[\Delta z] - (\gamma+\pi)T_1 + \frac{\sigma^2}{2}T_2 + m_{\mathcal{E},u}(0)T_3 \tag{D.37}$$
$$T_1 = \int_{-\Delta^-}^{\Delta^+} \Delta z \left[m'_{\mathcal{E},h}(\Delta z)g^h(\Delta z) + m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\right]\mathrm{d}\Delta z$$
$$T_2 = \int_{-\Delta^-}^{\Delta^+} \Delta z \left[m''_{\mathcal{E},h}(\Delta z)g^h(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z)\right]\mathrm{d}\Delta z$$
$$T_3 = \int_{-\Delta^-}^{\Delta^+} \Delta z \left((\gamma+\pi)(g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z)\right)\mathrm{d}\Delta z.$$

Next, we operate on the terms T_1, T_2, and T_3. The term T_1 is equal to

$$T_1 = \int_{-\Delta^-}^{\Delta^+} \Delta z \left[m'_{\mathcal{E},h}(\Delta z)g^h(\Delta z) + m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\right]\mathrm{d}\Delta z \tag{D.38}$$
$$=^{(1)} \int_{-\Delta^-}^{0} \Delta z \left[m'_{\mathcal{E},h}(\Delta z)g^h(\Delta z) + m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\right]\mathrm{d}\Delta z + \int_{0}^{\Delta^+} \Delta z \left[m'_{\mathcal{E},h}(\Delta z)g^h(\Delta z) + m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\right]\mathrm{d}\Delta z$$
$$=^{(2)} \int_{-\Delta^-}^{0} \Delta z \frac{\mathrm{d}\left(m_{\mathcal{E},h}(\Delta z)g^h(\Delta z)\right)}{\mathrm{d}\Delta z}\mathrm{d}\Delta z + \int_{0}^{\Delta^+} \Delta z \frac{\mathrm{d}\left(m_{\mathcal{E},h}(\Delta z)g^h(\Delta z)\right)}{\mathrm{d}\Delta z}\mathrm{d}\Delta z$$
$$=^{(3)} \underbrace{\Delta z m_{\mathcal{E},h}(\Delta z)g^h(\Delta z)\Big|_{-\Delta^-}^{0} + \Delta z m_{\mathcal{E},h}(\Delta z)g^h(\Delta z)\Big|_{0}^{\Delta^+}}_{=0}$$
$$\cdots - \left[\int_{-\Delta^-}^{0} m_{\mathcal{E},h}(\Delta z)g^h(\Delta z)\,\mathrm{d}\Delta z + \int_{0}^{\Delta^+} m_{\mathcal{E},h}(\Delta z)g^h(\Delta z)\,\mathrm{d}\Delta z\right]$$
$$=^{(4)} -\int_{-\Delta^-}^{\Delta^+} m_{\mathcal{E},h}(\Delta z)g^h(\Delta z)\,\mathrm{d}\Delta z$$
$$=^{(5)} m_{\mathcal{E},u}(0)(1-\mathcal{E}_{ss}).$$

Here, (1) divides the integral at the discontinuity point of $g^h(\Delta z)$; (2) uses the property of the derivative of a product of functions; (3) integrates and uses the border conditions for $g^h(\Delta z)$; (4) uses the continuity of $m_{\mathcal{E},h}(\Delta z)g^h(\Delta z)$; and (5) uses (D.36).

The term T_2 satisfies

$$
\begin{aligned}
T_2 &= \int_{-\Delta^-}^{\Delta^+} \Delta z \left[m''_{\mathcal{E},h}(\Delta z) g^h(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z) \right] \mathrm{d}\Delta z \qquad \text{(D.39)} \\
&=^{(1)} \int_{-\Delta^-}^{0} \Delta z \left[m''_{\mathcal{E},h}(\Delta z) g^h(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z) \right] \mathrm{d}\Delta z + \int_{0}^{\Delta^+} \Delta z \left[m''_{\mathcal{E},h}(\Delta z) g(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z) \right] \mathrm{d}\Delta z \\
&=^{(2)} \Delta z \left[m'_{\mathcal{E},h}(\Delta z) g^h(\Delta) - m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) \right] \Big|_{-\Delta^-}^{0} + \Delta z \left[m'_{\mathcal{E},h}(\Delta z) g^h(\Delta) - m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) \right] \Big|_{0}^{\Delta^+} \\
&\cdots - \left[\int_{-\Delta^-}^{0} \left[m'_{\mathcal{E},h}(\Delta z) g^h(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) \right] \mathrm{d}\Delta z + \int_{0}^{\Delta^+} \left[m'_{\mathcal{E},h}(\Delta z) g^h(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) \right] \mathrm{d}\Delta z \right] \\
&=^{(3)} \underbrace{\Delta z \left[m'_{\mathcal{E},h}(\Delta z) g^h(\Delta) - m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) \right] \Big|_{\Delta^-}^{\Delta^+}}_{= -m_{\mathcal{E},u}(0)\, \Delta z (g^h)'(\Delta z)|_{\Delta^-}^{\Delta^+}} \\
&\cdots - \left[\int_{-\Delta^-}^{0} \left[m'_{\mathcal{E},h}(\Delta z) g^h(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) \right] \mathrm{d}\Delta z + \int_{0}^{\Delta^+} \left[m'_{\mathcal{E},h}(\Delta z) g^h(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) \right] \mathrm{d}\Delta z \right] \\
&=^{(4)} -m_{\mathcal{E},u}(0)\ \Delta z (g^h)'(\Delta z) \Big|_{\Delta^-}^{\Delta^+} - \int_{\Delta^-}^{\Delta^+} m'_{\mathcal{E},h}(\Delta z) g^h(\Delta z)\, \mathrm{d}\Delta z + \int_{\Delta^-}^{\Delta^+} m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\, \mathrm{d}\Delta z \\
&=^{(5)} -m_{\mathcal{E},u}(0)\ \Delta z (g^h)'(\Delta z) \Big|_{\Delta^-}^{\Delta^+} - \left[\underbrace{m_{\mathcal{E},h}(\Delta z) g^h(\Delta z) \Big|_{\Delta^-}^{\Delta^+}}_{=0} - \int_{\Delta^-}^{\Delta^+} m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\, \mathrm{d}\Delta z \right] + \int_{\Delta^-}^{\Delta^+} m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\, \mathrm{d}\Delta z \\
&= -m_{\mathcal{E},u}(0)\ \Delta z (g^h)'(\Delta z) \Big|_{\Delta^-}^{\Delta^+} + 2 \int_{\Delta^-}^{\Delta^+} m_{\mathcal{E},h}(\Delta z) g'(\Delta z)\, \mathrm{d}\Delta z.
\end{aligned}
$$

Here, (1) divides the integral at the discontinuity point of $g^h(\Delta z)$; (2) uses the equality $m''_{\mathcal{E},h}(\Delta z)g^h(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z) = \frac{\mathrm{d}\left[m'_{\mathcal{E},h}(\Delta z)g^h(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\right]}{\mathrm{d}\Delta z}$ and integrates by parts; (3) uses conditions (D.35) and the border conditions of $g^h(\Delta z)$; and (4)-(5) integrate by parts and operate.

Finally, the term T_3 is equal to

$$
\begin{aligned}
T_3 &= \int_{-\Delta-}^{\Delta^+} \Delta z \left((\gamma + \pi)(g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z) \right) \mathrm{d}\Delta z \qquad \text{(D.40)} \\
&=^{(1)} (\gamma+\pi) \left[\int_{-\Delta^-}^{0} \Delta z (g^h)'(\Delta z)\, \mathrm{d}\Delta z + \int_{0}^{\Delta^+} \Delta z (g^h)'(\Delta z)\, \mathrm{d}\Delta z \right] + \frac{\sigma^2}{2} \left[\int_{\Delta^-}^{0} \Delta z (g^h)''(\Delta z)\, \mathrm{d}\Delta z + \int_{0}^{\Delta^+} \Delta z (g^h)''(\Delta z)\, \mathrm{d}\Delta z \right] \\
&=^{(2)} (\gamma+\pi) \left[\underbrace{\Delta z g^h(\Delta z) \Big|_{-\Delta^-}^{0} + \Delta z g^h(\Delta z) \Big|_{0}^{\Delta^+}}_{=0} - \underbrace{\int_{-\Delta^-}^{\Delta^+} g^h(\Delta z)\, \mathrm{d}\Delta z}_{=\mathcal{E}_{ss}} \right] \\
&\cdots + \frac{\sigma^2}{2} \left[\Delta z (g^h)'(\Delta z) \Big|_{-\Delta^-}^{0} + \Delta z (g^h)'(\Delta z) \Big|_{0}^{\Delta^+} - \int_{-\Delta^-}^{\Delta^+} (g^h)'(\Delta z)\, \mathrm{d}\Delta z \right] \\
&=^{(3)} -(\gamma+\pi)\mathcal{E}_{ss} + \frac{\sigma^2}{2} \left[\Delta z (g^h)'(\Delta z) \Big|_{\Delta^-}^{\Delta^+} - \underbrace{g^h(\Delta z) \Big|_{\Delta^-}^{\Delta^+}}_{=0} \right] \\
&=^{(4)} -(\gamma+\pi)\mathcal{E}_{ss} + \frac{\sigma^2}{2} \left[\Delta z (g^h)'(\Delta z) \Big|_{\Delta^-}^{\Delta^+} \right]
\end{aligned}
$$

Here, (1) divides the integral at the discontinuity point of $g^h(\Delta z)$; (2) integrates by parts; and (3) and (4) use the border conditions of $g^h(\Delta z)$.

Combining results (D.37), (D.38), (D.39), (D.40) and those in Step 2, we obtain

$$
\begin{aligned}
0 &= (1-\mathcal{E}_{ss})\mathbb{E}_h[\Delta z] - (\gamma+\pi)T_1 + \frac{\sigma^2}{2}T_2 + m_{\mathcal{E},u}(0)T_3 \\
&= (1-\mathcal{E}_{ss})\mathbb{E}_h[\Delta z] - (\gamma+\pi)m_{\mathcal{E},u}(0)(1-\mathcal{E}_{ss}) + \frac{\sigma^2}{2}\left[-m_{\mathcal{E},u}(0)\ \Delta z (g^h)'(\Delta z)\Big|_{-\Delta^-}^{\Delta^+} + 2\int_{-\Delta^-}^{\Delta^+} m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\,\mathrm{d}\Delta z\right] \\
&\cdots + m_{\mathcal{E},u}(0)\left[-(\gamma+\pi)\mathcal{E}_{ss} + \frac{\sigma^2}{2}\left[\Delta z (g^h)'(\Delta z)\Big|_{-\Delta^-}^{\Delta^+}\right]\right] \\
&= (1-\mathcal{E}_{ss})\mathbb{E}_h[\Delta z] - (\gamma+\pi)m_{\mathcal{E},u}(0) + \sigma^2\int_{-\Delta^-}^{\Delta^+} m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\,\mathrm{d}\Delta z,
\end{aligned}
$$

which implies

$$
\begin{aligned}
\int_{-\Delta^-}^{\Delta^+} m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\,\mathrm{d}\Delta z &= \frac{(\gamma+\pi)\left(-\frac{\mathcal{E}_{ss}}{f(\theta^*)} + \frac{\mathcal{E}_{ss}}{f(\theta^*)} - (1-\mathcal{E}_{ss})\mathbb{E}_h[a]\right) - (1-\mathcal{E}_{ss})\mathbb{E}_h[\Delta z]}{\sigma^2}, \\
&= -(1-\mathcal{E}_{ss})\frac{[(\gamma+\pi)\mathbb{E}_h[a] + \mathbb{E}_h[\Delta z]]}{\sigma^2}.
\end{aligned}
$$

□

Proposition D.3. *Up to first order, the $CIR_{\mathcal{E}}(\zeta)$ can be expressed in terms of data moments as follows:*

(i) If $(\gamma+\pi)=0$,

$$
\frac{CIR_{\mathcal{E}}(\zeta)}{\zeta} = \underbrace{\frac{1}{f(\hat{w}^*)}}_{\text{avg. u dur.}}\ \underbrace{\frac{1}{\mathbb{V}ar_{\mathcal{D}}[\Delta w]}}_{\text{dispersion}}\left[\underbrace{\frac{1}{3}\mathbb{E}_{\mathcal{D}}\left[\Delta w\frac{\Delta w^2}{\mathbb{E}_{\mathcal{D}}[\Delta w^2]}\right]}_{\text{asymmetries}}\right] + o(\zeta).
$$

(ii) If $(\gamma+\pi)\neq 0$,

$$
\begin{aligned}
\frac{CIR_{\mathcal{E}}(\zeta)}{\zeta} &= \underbrace{\frac{1}{f(\hat{w}^*)}}_{\text{avg. u dur.}}\ \underbrace{\frac{1}{\mathbb{V}ar_{\mathcal{D}}\left[\left(\Delta w - \frac{\mathbb{E}_{\mathcal{D}}[\Delta w]}{\mathbb{E}_{\mathcal{D}}[\tau]}\tau\right)\right]}}_{\text{dispersion}} \\
&\times\left[\underbrace{\mathbb{E}_{\mathcal{D}}[\Delta w]\left(\mathcal{E}_{ss}\left(\mathbb{C}ov_{\mathcal{D}}\left[\widetilde{\Delta w},\widetilde{\Delta w}-\tilde{\tau}\right] + \frac{\mathbb{V}ar_{\mathcal{D}}[\tilde{\tau}] - \mathcal{E}_{ss}\mathbb{V}ar_{\mathcal{D}}\left[\widetilde{\tau^m}\right]}{2}\right) + (1-\mathcal{E}_{ss})\left(\frac{\mathbb{V}ar_{\mathcal{D}}\left[\widetilde{\Delta w}\right]-1}{2}\right)\right)}_{\text{asymmetries}}\right] + o(\zeta).
\end{aligned}
$$

Proof. The goal is to express the sufficient statistics of the CIR, $\mathbb{E}_h[a]$ and $\mathbb{E}_h[\Delta z]$, in terms of moments of the distribution of Δw and (τ^u, τ^m). We do so separately for the case with $(\gamma+\pi)=0$ and $(\gamma+\pi)\neq 0$. Let $\tilde{x}\equiv x/\mathbb{E}_{\mathcal{D}}[x]$ denote random variable x relative to its mean in the data.

Proposition G.5 expresses moments of the wage distribution as a linear combination of moments of the distribution of productivity changes among completed employment and unemployment spells:

$$
\begin{aligned}
\mathbb{E}_{\mathcal{D}}[\Delta w] &= -\left[\bar{\mathbb{E}}_u[\Delta z] + \bar{\mathbb{E}}_h[\Delta z]\right] \\
\mathbb{E}_{\mathcal{D}}\left[\Delta w^2\right] &= \left[\bar{\mathbb{E}}_u\left[\Delta z^2\right] + 2\bar{\mathbb{E}}_h[\Delta z]\,\bar{\mathbb{E}}_u[\Delta z] + \bar{\mathbb{E}}_h\left[\Delta z^2\right]\right]
\end{aligned}
$$

$$\mathbb{E}_{\mathcal{D}}\left[\Delta w^3\right] = -\left[\bar{\mathbb{E}}_u\left[\Delta z^3\right] + 3\bar{\mathbb{E}}_h\left[\Delta z\right]\bar{\mathbb{E}}_u\left[\Delta z^2\right] + 3\bar{\mathbb{E}}_h\left[\Delta z^2\right]\bar{\mathbb{E}}_u\left[\Delta z\right] + \bar{\mathbb{E}}_h\left[\Delta z^3\right]\right],$$

where $\bar{\mathbb{E}}_h[\cdot]$ and $\bar{\mathbb{E}}_u[\cdot]$ denote the expectation operators under the distributions $\bar{g}^h(\Delta z)$ and $\bar{g}^u(\Delta z)$, respectively. Using results from the same Proposition, we can express the moments of productivity changes for completed unemployment spells in terms of model parameters:

$$\begin{aligned}
\bar{\mathbb{E}}_u\left[\Delta z\right] &= \frac{\left(\mathcal{L}_2^{-1} - \mathcal{L}_2\right)}{\mathcal{L}_1} \\
\bar{\mathbb{E}}_u\left[\Delta z^2\right] &= \frac{2\left(\mathcal{L}_2^{-2} + \mathcal{L}_2^{2} - 1\right)}{\mathcal{L}_1^2} \\
\bar{\mathbb{E}}_u\left[\Delta z^3\right] &= \frac{6\left(-\mathcal{L}_2^{3} + \mathcal{L}_2 - \mathcal{L}_2^{-1} + \mathcal{L}_2^{-3}\right)}{\mathcal{L}_1^3},
\end{aligned}$$

where

$$\mathcal{L}_1 = \sqrt{\frac{2f(\hat{w}^*)}{\sigma^2}} \text{ and } \mathcal{L}_2 = \sqrt{\frac{(\gamma+\pi) + \sqrt{(\gamma+\pi)^2 + 2\sigma^2 f(\hat{w}^*)}}{-(\gamma+\pi) + \sqrt{(\gamma+\pi)^2 + 2\sigma^2 f(\hat{w}^*)}}}.$$

From these two sets of equations, we solve for the moments of productivity changes for completed employment spells and obtain

$$\begin{aligned}
\bar{\mathbb{E}}_h\left[\Delta z\right] &= -\left(\frac{\left(\mathcal{L}_2^{-1} - \mathcal{L}_2\right)}{\mathcal{L}_1}\right) - \mathbb{E}_{\mathcal{D}}\left[\Delta w\right] \\
\bar{\mathbb{E}}_h\left[\Delta z^2\right] &= \mathbb{E}_{\mathcal{D}}\left[\Delta w^2\right] + 2\mathbb{E}_{\mathcal{D}}\left[\Delta w\right]\left(\frac{\left(\mathcal{L}_2^{-1} - \mathcal{L}_2\right)}{\mathcal{L}_1}\right) - \frac{2}{\mathcal{L}_1^2} \\
\bar{\mathbb{E}}_h\left[\Delta z^3\right] &= -\mathbb{E}_{\mathcal{D}}\left[\Delta w^3\right] - 3\mathbb{E}_{\mathcal{D}}\left[\Delta w^2\right]\left(\frac{\left(\mathcal{L}_2^{-1} - \mathcal{L}_2\right)}{\mathcal{L}_1}\right) + \frac{6}{\mathcal{L}_1^2}\mathbb{E}_{\mathcal{D}}\left[\Delta w\right].
\end{aligned}$$

The remaining steps are case-specific.

Case I: $(\gamma + \pi) = 0$. To obtain $\mathbb{E}_h[\Delta z]$, evaluate (G.10) at $m = 1$, use the fact that $\mathcal{L}_2 = 1$, $\mathbb{E}_{\mathcal{D}}\left[\Delta w\right] = 0$ and $\frac{\mathbb{E}_{\mathcal{D}}[\tau^u]}{\mathbb{E}_{\mathcal{D}}[\tau]} = \mathcal{E}_{ss}$, and substitute σ^2 from Lemma F.1:

$$\begin{aligned}
\mathbb{E}_h\left[\Delta z\right] &= \frac{\mathcal{E}_{ss}}{3}\frac{\bar{\mathbb{E}}_h\left[\Delta z^3\right]}{\bar{\mathbb{E}}_h\left[\Delta z^2\right]} \\
&= \frac{\mathcal{E}_{ss}}{3}\left(\frac{-\mathbb{E}_{\mathcal{D}}\left[\Delta w^3\right] + \mathbb{E}_{\mathcal{D}}\left[\Delta w\right]\frac{6}{\mathcal{L}_1^2}}{\mathbb{E}_{\mathcal{D}}\left[\Delta w^2\right] - \frac{2}{\mathcal{L}_1^2}}\right) \\
&= -\frac{\mathcal{E}_{ss}}{3}\left(\frac{\mathbb{E}_{\mathcal{D}}\left[\Delta w^3\right]}{\mathbb{E}_{\mathcal{D}}\left[\Delta w^2\right] - \sigma^2\mathbb{E}_{\mathcal{D}}[\tau^u]}\right) \\
&= -\frac{\mathbb{E}_{\mathcal{D}}\left[\Delta w^3\right]}{3\mathbb{E}_{\mathcal{D}}\left[\Delta w^2\right]}.
\end{aligned}$$

Finally, replace this expression into (36):

$$\frac{CIR_{\mathcal{E}}(\zeta)}{\zeta} = -\left(1 - \mathcal{E}_{ss}\right)\frac{\mathbb{E}_h\left[\Delta z\right]}{\sigma^2}$$

$$= (1-\mathcal{E}_{ss}) \frac{\frac{\mathbb{E}_{\mathcal{D}}[\Delta w^3]}{3\mathbb{E}_{\mathcal{D}}[\Delta w^2]}}{\frac{\mathbb{E}_{\mathcal{D}}[\Delta w^2]}{\mathbb{E}_{\mathcal{D}}[\tau]}}$$

$$= \frac{1}{f(\hat{w}^*)} \frac{\mathbb{E}_{\mathcal{D}}[\Delta w^3]}{3\mathbb{E}_{\mathcal{D}}[\Delta w^2]^2}$$

$$= \frac{1}{f(\hat{w}^*)} \frac{1}{\mathbb{V}ar_{\mathcal{D}}[\Delta w^2]} \frac{1}{3}\mathbb{E}_{\mathcal{D}}\left[\Delta w \frac{\Delta w^2}{\mathbb{E}_{\mathcal{D}}[\Delta w^2]}\right].$$

Case II: $(\gamma+\pi) \neq 0$**.** To obtain $\mathbb{E}_h[\Delta z]$, evaluate (G.11) at $m=1$, use the result that $\left(\mathcal{L}_2^{-1} - \mathcal{L}_2\right)/\mathcal{L}_1 = -(\gamma+\pi)/f(\hat{w}^*)$ and $\frac{\mathbb{E}_{\mathcal{D}}[\tau^u]}{\mathbb{E}_{\mathcal{D}}[\tau]} = \mathcal{E}_{ss}$, and substitute $(\gamma+\pi)$ and σ^2 from Lemma F.1:

$$\mathbb{E}_h[\Delta z] = \frac{\mathcal{E}_{ss}}{2} \frac{\bar{\mathbb{E}}_h[\Delta z^2]}{\bar{\mathbb{E}}_h[\Delta z]} - \frac{\sigma^2}{2(\gamma+\pi)}$$

$$= \frac{\mathcal{E}_{ss}}{2} \left(\frac{\mathbb{E}_{\mathcal{D}}[\Delta w^2] - 2\frac{\mathbb{E}_{\mathcal{D}}[\Delta w]^2 \mathbb{E}_{\mathcal{D}}[\tau^u]}{\mathbb{E}_{\mathcal{D}}[\tau]} - \sigma^2 \mathbb{E}_{\mathcal{D}}[\tau^u]}{-\mathbb{E}_{\mathcal{D}}[\Delta w]\left(1 - \frac{\mathbb{E}_{\mathcal{D}}[\tau^u]}{\mathbb{E}_{\mathcal{D}}[\tau]}\right)} \right) - \frac{\sigma^2}{2(\gamma+\pi)}$$

$$= -\frac{1}{2}\mathbb{E}_{\mathcal{D}}[\Delta w] \left(\mathbb{E}_{\mathcal{D}}\left[\widetilde{\Delta w}^2\right] - 2\frac{\mathbb{E}_{\mathcal{D}}[\tau^u]}{\mathbb{E}_{\mathcal{D}}[\tau]} \right) + \frac{1}{2} \frac{\sigma^2}{\mathbb{E}_{\mathcal{D}}[\Delta w]} \left(\mathbb{E}_{\mathcal{D}}[\tau^u] - \mathbb{E}_{\mathcal{D}}[\tau]\right)$$

$$= -\frac{1}{2}\mathbb{E}_{\mathcal{D}}[\Delta w] \left(\mathbb{E}_{\mathcal{D}}\left[\widetilde{\Delta w}^2\right] - 2\frac{\mathbb{E}_{\mathcal{D}}[\tau^u]}{\mathbb{E}_{\mathcal{D}}[\tau]} \right) + \frac{1}{2} \frac{\mathbb{E}_{\mathcal{D}}[(\Delta w - (\gamma+\pi)\tau)^2]}{\mathbb{E}_{\mathcal{D}}[\tau]\mathbb{E}_{\mathcal{D}}[\Delta w]} \left(\mathbb{E}_{\mathcal{D}}[\tau^u] - \mathbb{E}_{\mathcal{D}}[\tau]\right)$$

$$= -\mathbb{E}_{\mathcal{D}}[\Delta w] \left(\frac{1}{2} \left(\mathbb{E}_{\mathcal{D}}\left[\widetilde{\Delta w}^2\right] - 2\frac{\mathbb{E}_{\mathcal{D}}[\tau^u]}{\mathbb{E}_{\mathcal{D}}[\tau]} \right) + \frac{1}{2}\mathbb{E}_{\mathcal{D}}[(\widetilde{\Delta w} - \tilde{\tau})^2] \left(1 - \frac{\mathbb{E}_{\mathcal{D}}[\tau^u]}{\mathbb{E}_{\mathcal{D}}[\tau]}\right) \right)$$

$$= -\mathbb{E}_{\mathcal{D}}[\Delta w] \left(\frac{1}{2} \left(var_{\mathcal{D}}\left[\widetilde{\Delta w}\right] - 1 \right) + \mathcal{E}_{ss}\left(1 + \frac{1}{2} var_{\mathcal{D}}[(\widetilde{\Delta w} - \tilde{\tau})]\right) \right).$$

The average cross-sectional age of a job spell is obtained from the occupancy measure:

$$\mathbb{E}_h[a] = \mathcal{E}_{ss} \frac{\mathbb{E}[\int_0^{\tau^m} t\, dt | \hat{w}_0 = \hat{w}^*]}{\mathbb{E}[\tau^m | \hat{w}_0 = \hat{w}^*]} = \frac{\mathcal{E}_{ss}}{2} \frac{\mathbb{E}_{\mathcal{D}}[\tau^{m2}]}{\mathbb{E}_{\mathcal{D}}[\tau^m]},$$

where we solve the Reimann integral.

Finally, we replace these expressions into (36):

$$\frac{CIR_{\mathcal{E}}(\zeta)}{\zeta} = -(1-\mathcal{E}_{ss}) \frac{\mathbb{E}_h[\Delta z] + (\gamma+\pi)\mathbb{E}[a]}{\sigma^2}$$

$$= -\frac{\mathbb{E}_{\mathcal{D}}[\tau^u]}{\mathbb{E}_{\mathcal{D}}[\tau]} \frac{\mathbb{E}_{\mathcal{D}}[\tau]}{\mathbb{E}_{\mathcal{D}}\left[\left(\Delta w - \frac{\mathbb{E}_{\mathcal{D}}[\Delta w]}{\mathbb{E}_{\mathcal{D}}[\tau]}\tau\right)^2\right]} \left(-\mathbb{E}_{\mathcal{D}}[\Delta w] \left(\frac{1}{2}\left(\mathbb{V}ar_{\mathcal{D}}\left[\widetilde{\Delta w}\right] - 1\right) + \mathcal{E}_{ss}\left(1 + \frac{1}{2}\left(\mathbb{V}ar_{\mathcal{D}}[\widetilde{\Delta w} - \tilde{\tau}]\right)\right)\right) + (\gamma+\pi)\mathbb{E}[a] \right)$$

$$= \frac{\mathbb{E}_{\mathcal{D}}[\Delta w]}{f(\hat{w}^*)\,\mathbb{V}ar_{\mathcal{D}}\left[\left(\Delta w - \frac{\mathbb{E}_{\mathcal{D}}[\Delta w]}{\mathbb{E}_{\mathcal{D}}[\tau]}\tau\right)\right]} \left(\frac{1}{2}\left(\mathbb{V}ar_{\mathcal{D}}\left[\widetilde{\Delta w}\right] - 1\right) + \mathcal{E}_{ss}\left(1 + \frac{1}{2}\left(\mathbb{V}ar_{\mathcal{D}}[\widetilde{\Delta w} - \tilde{\tau}]\right)\right) - \frac{1}{\mathbb{E}_{\mathcal{D}}[\tau]}\mathbb{E}[a] \right)$$

$$= \frac{\mathbb{E}_{\mathcal{D}}[\Delta w]}{f(\hat{w}^*)\,\mathbb{V}ar_{\mathcal{D}}\left[\left(\Delta w - \frac{\mathbb{E}_{\mathcal{D}}[\Delta w]}{\mathbb{E}_{\mathcal{D}}[\tau]}\tau\right)\right]} \left(\frac{1}{2}\left(\mathbb{V}ar_{\mathcal{D}}\left[\widetilde{\Delta w}\right] - 1\right)(1 - \mathcal{E}_{ss} + \mathcal{E}_{ss}) + \mathcal{E}_{ss}\left(1 + \frac{1}{2}\left(\mathbb{V}ar_{\mathcal{D}}[\widetilde{\Delta w} - \tilde{\tau}]\right)\right) - \frac{1}{\mathbb{E}_{\mathcal{D}}[\tau]}\mathbb{E}[a] \right)$$

$$= \frac{\mathbb{E}_{\mathcal{D}}[\Delta w]}{2f(\hat{w}^*)\,\mathbb{V}ar_{\mathcal{D}}\left[\left(\Delta w - \frac{\mathbb{E}_{\mathcal{D}}[\Delta w]}{\mathbb{E}_{\mathcal{D}}[\tau]}\tau\right)\right]} \left(\left(\mathbb{V}ar_{\mathcal{D}}\left[\widetilde{\Delta w}\right] - 1\right)(1 - \mathcal{E}_{ss}) + \mathcal{E}_{ss}\left(1 + \left(\mathbb{V}ar_{\mathcal{D}}[\widetilde{\Delta w} - \tilde{\tau}]\right) + \mathbb{V}ar_{\mathcal{D}}\left[\widetilde{\Delta w}\right]\right) - \frac{2}{\mathbb{E}_{\mathcal{D}}[\tau]}\mathbb{E}[a] \right)$$

$$= \frac{\mathbb{E}_{\mathcal{D}}[\Delta w]}{2f(\hat{w}^*)\,\mathbb{V}ar_{\mathcal{D}}\left[\left(\Delta w - \frac{\mathbb{E}_{\mathcal{D}}[\Delta w]}{\mathbb{E}_{\mathcal{D}}[\tau]}\tau\right)\right]} \left(\left(\mathbb{V}ar_{\mathcal{D}}\left[\widetilde{\Delta w}\right] - 1\right)(1 - \mathcal{E}_{ss}) + \mathcal{E}_{ss}\left(1 + \left(\mathbb{V}ar_{\mathcal{D}}[\widetilde{\Delta w} - \tilde{\tau}]\right) + \mathbb{V}ar_{\mathcal{D}}\left[\widetilde{\Delta w}\right]\right) - \mathcal{E}_{ss}\frac{\mathbb{E}_{\mathcal{D}}[\tau^m]}{\mathbb{E}_{\mathcal{D}}[\tau]}\mathbb{E}_{\mathcal{D}}[\widetilde{\tau^m}^2] \right)$$

$$= \frac{\mathbb{E}_{\mathcal{D}}[\Delta w]}{2f(\hat{w}^*)\,\mathbb{V}ar_{\mathcal{D}}\left[\left(\Delta w - \frac{\mathbb{E}_{\mathcal{D}}[\Delta w]}{\mathbb{E}_{\mathcal{D}}[\tau]}\tau\right)\right]} \left(\left(\mathbb{V}ar_{\mathcal{D}}\left[\widetilde{\Delta w}\right] - 1\right)(1 - \mathcal{E}_{ss}) + \mathcal{E}_{ss}\left(1 + \left(\mathbb{V}ar_{\mathcal{D}}[\widetilde{\Delta w} - \tilde{\tau}]\right) + \mathbb{V}ar_{\mathcal{D}}\left[\widetilde{\Delta w}\right] - \mathcal{E}_{ss}\mathbb{E}_{\mathcal{D}}[\widetilde{\tau^m}^2]\right) \right)$$

$$= \frac{\mathbb{E}_{\mathcal{D}}[\Delta w]}{f(\hat{w}^*)\,\mathbb{V}ar_{\mathcal{D}}\left[\left(\Delta w - \frac{\mathbb{E}_{\mathcal{D}}[\Delta w]}{\mathbb{E}_{\mathcal{D}}[\tau]}\tau\right)\right]} \left(\frac{\left(\mathbb{V}ar_{\mathcal{D}}\left[\widetilde{\Delta w}\right] - 1\right)}{2}(1-\mathcal{E}_{ss}) + \mathcal{E}_{ss}\left(\mathbb{C}ov_{\mathcal{D}}\left[\widetilde{\Delta w}, \widetilde{\Delta w} - \tilde{\tau}\right] + \frac{\mathbb{V}ar_{\mathcal{D}}[\tilde{\tau}] - \mathcal{E}_{ss}\mathbb{V}ar_{\mathcal{D}}[\widetilde{\tau^m}^2]}{2}\right)\right).$$

□

D.3 Proof of Proposition 8: CIR of Employment with Sticky Entry Wage

Proposition 8. *Assume sticky entry wages. Up to first order, the CIR of employment is given by:*

$$\frac{CIR_{\mathcal{E}}(\zeta)}{\zeta} = (1-\mathcal{E}_{ss})\left[-\frac{[\gamma\mathbb{E}_h[a] + \mathbb{E}_h[\Delta z]]}{\sigma^2} + \frac{1}{f(\hat{w}^*)+s}\left[\frac{\eta'(\hat{w}^*)}{\eta(\hat{w}^*)} + \frac{\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{\rho})}{\mathcal{T}(\hat{w}^*,\hat{\rho})} - \frac{\mathcal{T}'_{\hat{w}}(\hat{w}^*,0)}{\mathcal{T}(\hat{w}^*,0)}\right]\right] + o(\zeta). \tag{D.41}$$

Proof. We divide the proof in two steps. Step 1 characterizes $m_{\mathcal{E},u}(\Delta z,\zeta)$. Steps 2 uses the equilibrium conditions to show (D.41). The starting point is the CIR for employment, which is given by

$$\text{CIR}_{\mathcal{E}}(\zeta) = \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z) g^h(\Delta z+\zeta)\,\text{d}\Delta z + \int_{-\infty}^{\infty} m_{\mathcal{E},u}(\Delta z,\zeta) g^u(\Delta z+\zeta)\,\text{d}\Delta z, \tag{D.42}$$

with

$$0 = 1-\mathcal{E}_{ss} - \gamma\frac{\text{d}m_{\mathcal{E},h}(\Delta z)}{\text{d}\Delta z} + \frac{\sigma^2}{2}\frac{d^2 m_{\mathcal{E},h}(\Delta z)}{\text{d}\Delta z^2} + \delta(m_{\mathcal{E},u}(0,0) - m_{\mathcal{E},h}(\Delta z)),\ \text{for all } \Delta z \in (-\Delta^-,\Delta^+) \tag{D.43}$$

$$0 = -\mathcal{E}_{ss} - \gamma\frac{\text{d}m_{\mathcal{E},u}(\Delta z,\zeta)}{\text{d}\Delta z} + \frac{\sigma^2}{2}\frac{d^2 m_{\mathcal{E},u}(\Delta z,\zeta)}{\text{d}\Delta z^2} + f(\hat{w}^*-\zeta)(m_{\mathcal{E},h}(-\zeta) - m_{\mathcal{E},u}(\Delta z,\zeta)) \tag{D.44}$$

$$m_{\mathcal{E},u}(0,0) = m_{\mathcal{E},h}(\Delta z),\ \text{for all } \Delta z \notin (-\Delta^-,\Delta^+) \tag{D.45}$$

$$0 = \lim_{\Delta z\to-\infty}\frac{\text{d}m_{\mathcal{E},u}(\Delta z,\zeta)}{\text{d}\Delta z} = \lim_{\Delta z\to\infty}\frac{\text{d}m_{\mathcal{E},u}(\Delta z,\zeta)}{\text{d}\Delta z} \tag{D.46}$$

$$0 = \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z) g^h(\Delta z)\,\text{d}\Delta z + \int_{-\infty}^{\infty} m_{\mathcal{E},u}(\Delta z) g^u(\Delta z)\,\text{d}\Delta z \tag{D.47}$$

The key differences between the CIR with flexible wages and the CIR with sticky wages are found in the HJB equation at the moment of the shock. With sticky entry wages, the job-finding probability is given by $f(\hat{w}^*-\zeta)$, since now the real entry wage is lower. As a consequence, we need to evaluate $m_{\mathcal{E},h}(\Delta z)$ at $\Delta z = -\zeta$ because conditional on finding a job, the real entry wage is lower. Observe that following the first job separation, the monetary shock is fully absorbed (see the term $m_{\mathcal{E},u}(0,0)$ in equation (D.43)).

Step 1. The value function $m_{\mathcal{E},u}(\Delta z,\zeta)$ is independent of Δz and satisfies

$$m_{\mathcal{E},u}(\Delta z,\zeta) = -\frac{\mathcal{E}_{ss}}{f(\hat{w}^*-\zeta)} + m_{\mathcal{E},h}(-\zeta).$$

Proof of Step 1. We guess and verify that $m_{\mathcal{E},u}(\Delta z,\zeta) = m_{\mathcal{E},u}(0,\zeta)$ for all Δz. From the equilibrium conditions (D.44) and (D.46),

$$0 = -\mathcal{E}_{ss} + f(\hat{w}^*-\zeta)(m_{\mathcal{E},h}(-\zeta) - m_{\mathcal{E},u}(0,\zeta)).$$

Thus,

$$m_{\mathcal{E},u}(0,\zeta) = m_{\mathcal{E},u}(\Delta z,\zeta) = -\frac{\mathcal{E}_{ss}}{f(\hat{w}^*-\zeta)} + m_{\mathcal{E},h}(-\zeta).$$

Step 2. Up to a first-order approximation, the CIR is given by:

$$\text{CIR}_{\mathcal{E}}(\zeta) = -(1-\mathcal{E}_{ss})\frac{\gamma\mathbb{E}_h[a]+\mathbb{E}_h[\Delta z]}{\sigma^2}\zeta + \frac{(1-\mathcal{E}_{ss})}{f(\hat{w}^*)+s}\left(\frac{\eta'(\hat{w}^*)}{\eta(\hat{w}^*)}+\frac{\mathcal{T}'(\hat{w}^*,\hat{\rho})}{\mathcal{T}(\hat{w}^*,\hat{\rho})}-\frac{\mathcal{T}'(\hat{w}^*,0)}{\mathcal{T}(\hat{w}^*,0)}\right)\zeta + o(\zeta^2).$$

Proof of Step 2. From Step 1, we have that

$$\text{CIR}'_{\mathcal{E}}(0) = \int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\,\mathrm{d}\Delta z + \left(-\frac{\mathcal{E}_{ss}f'(\hat{w}^*)}{f(\hat{w}^*)^2} - m'_{\mathcal{E},h}(0)\right)(1-\mathcal{E}_{ss}).$$

Since $\int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\,\mathrm{d}\Delta z$ satisfies the same system of functional equations as the CIR with flexible entry wages characterized in Online Appendix D.2,

$$\int_{-\infty}^{\infty} m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\,\mathrm{d}\Delta z = -(1-\mathcal{E}_{ss})\frac{\gamma\mathbb{E}_h[a]+\mathbb{E}_h[\Delta z]}{\sigma^2}. \tag{D.48}$$

Observe that we can write

$$\begin{aligned} m_{\mathcal{E},h}(\Delta z) &= \mathbb{E}\left[\int_0^{\tau^m}(1-\mathcal{E}_{ss})\,\mathrm{d}t + m_{\mathcal{E},u}(\Delta z,0)\,\middle|\,\Delta z_0=\Delta z\right], \\ &= (1-\mathcal{E}_{ss})\mathcal{T}(\hat{w}^*+\Delta z,0) - \frac{\mathcal{E}_{ss}}{f(\hat{w}^*)} + m_{\mathcal{E},h}(0). \end{aligned}$$

Taking the derivative with respect to Δz, evaluating it at $\Delta z = 0$, and using $s = 1/\mathcal{T}(\hat{w}^*,0)$ from the Renewal Principle, we have that

$$m'_{\mathcal{E},h}(0) = (1-\mathcal{E}_{ss})\mathcal{T}'_{\hat{w}}(\hat{w}^*,0) = \frac{s}{f(\hat{w}^*)+s}\mathcal{T}'_{\hat{w}}(\hat{w}^*,0) = \frac{1}{f(\hat{w}^*)+s}\frac{\mathcal{T}'(\hat{w}^*,0)}{\mathcal{T}(\hat{w}^*,0)}. \tag{D.49}$$

From the free entry condition

$$f(\hat{w}^*) = \left(\frac{\hat{J}(\hat{w}^*)}{\tilde{K}}\right)^{\frac{1-\alpha}{\alpha}},$$

and the definition $(1-\eta(\hat{w}^*)) = \hat{J}(\hat{w}^*)/\hat{S}(\hat{w}^*)$, we can compute the elasticity of the job finding rate with respect to the entry wage:

$$\begin{aligned} \frac{f'(\hat{w}^*)}{f(\hat{w}^*)} &= \frac{\frac{1-\alpha}{\alpha}\left(\frac{\hat{J}(\hat{w}^*)}{\tilde{K}}\right)^{\frac{1-\alpha}{\alpha}-1}\frac{\hat{J}'(\hat{w}^*)}{\tilde{K}}}{\left(\frac{\hat{J}(\hat{w}^*)}{\tilde{K}}\right)^{\frac{1-\alpha}{\alpha}}}, \\ &= \frac{1-\alpha}{\alpha}\frac{\hat{J}'(\hat{w}^*)}{\hat{J}(\hat{w}^*)}, \\ &= \frac{1-\alpha}{\alpha}\left[-\frac{\eta'(\hat{w}^*)}{(1-\eta(\hat{w}^*))}+\frac{\mathcal{T}'(\hat{w}^*,\hat{\rho})}{\mathcal{T}(\hat{w}^*,\hat{\rho})}\right]. \end{aligned}$$

Finally, combining this result with the fact that $\mathcal{E}_{ss} = \frac{f(\hat{w}^*)}{f(\hat{w}^*)+s}$, $s = \frac{1}{\mathcal{T}(\hat{w}^*,0)}$, $\eta'(\hat{w}^*)\left(\frac{\alpha}{\eta(\hat{w}^*)} - \frac{1-\alpha}{1-\eta(\hat{w}^*)}\right) = -\frac{\mathcal{T}'(\hat{w}^*,\hat{\rho})}{\mathcal{T}(\hat{w}^*,\hat{\rho})}$, and operating, we obtain

$$\begin{aligned} -\frac{\mathcal{E}_{ss}f'(\hat{w}^*)}{f(\hat{w}^*)^2} &= -\frac{\mathcal{E}_{ss}}{f(\hat{w}^*)}\frac{f'(\hat{w}^*)}{f(\hat{w}^*)} \\ &= \frac{1}{f(\hat{w}^*)+s}\left[-\frac{1-\alpha}{\alpha}\left[-\frac{\eta'(\hat{w}^*)}{(1-\eta(\hat{w}^*))}+\frac{\mathcal{T}'(\hat{w}^*,\hat{\rho})}{\mathcal{T}(\hat{w}^*,\hat{\rho})}\right]\right] \\ &= \frac{1}{f(\hat{w}^*)+s}\left[-\frac{1}{\alpha}\left[-\frac{\eta'(\hat{w}^*)(1-\alpha)}{(1-\eta(\hat{w}^*))}+(1-\alpha)\frac{\mathcal{T}'(\hat{w}^*,\hat{\rho})}{\mathcal{T}(\hat{w}^*,\hat{\rho})}\right]\right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{f(\hat{w}^*)+s}\left[-\frac{1}{\alpha}\left[-\frac{\eta'(\hat{w}^*)\alpha}{\eta(\hat{w}^*)} - \alpha\frac{\mathcal{T}'(\hat{w}^*,\hat{\rho})}{\mathcal{T}(\hat{w}^*,\hat{\rho})}\right]\right] \\
&= \frac{1}{f(\hat{w}^*)+s}\left[\frac{\eta'(\hat{w}^*)}{\eta(\hat{w}^*)} + \frac{\mathcal{T}'(\hat{w}^*,\hat{\rho})}{\mathcal{T}(\hat{w}^*,\hat{\rho})}\right].
\end{aligned} \tag{D.50}
$$

Combining results in equations (D.48), (D.49), and (D.50), we obtain the desired result:

$$
\text{CIR}'_{\mathcal{E}}(0) = -(1-\mathcal{E}_{ss})\frac{[\gamma\mathbb{E}_h[a]+\mathbb{E}_h[\Delta z]]}{\sigma^2} + \frac{1-\mathcal{E}_{ss}}{f(\hat{w}^*)+s}\left[\frac{\eta'(\hat{w}^*)}{\eta(\hat{w}^*)} + \frac{\mathcal{T}'(\hat{w}^*,\hat{\rho})}{\mathcal{T}(\hat{w}^*,\hat{\rho})} - \frac{\mathcal{T}'(\hat{w}^*,0)}{\mathcal{T}(\hat{w}^*,0)}\right].
$$

□

D.4 Proof of Lemma 3

The following proposition provides a broad characterization of $\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{\rho})/\mathcal{T}(\hat{w}^*,\hat{\rho})$, which includes the result in Lemma 3.

Proposition D.4. *The following properties hold for* $\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{\rho})/\mathcal{T}(\hat{w}^*,\hat{\rho})$*:*

a. Assume that $\Delta^- = \Delta^+$ *and* $\gamma + \pi = 0$*. Then,* $\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{\rho}) = 0$ *and, up to a 3rd order approximation of* $\mathcal{T}(\hat{w},\hat{\rho})$ *around* $\hat{w} = \hat{w}^*$*,*

$$
\mathcal{T}(\hat{w}^*,\hat{\rho}) = \frac{1}{\hat{\rho}+\delta+(\sigma/\Delta^+)^2}.
$$

b. Up to a 2nd order approximation of $\mathcal{T}(\hat{w},\hat{\rho})$ *around* $\hat{w} = \hat{w}^*$*,*

$$
\frac{\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{\rho})}{\mathcal{T}(\hat{w}^*,\hat{\rho})} = \frac{\Delta^+ - \Delta^-}{\Delta^+\Delta^-}.
$$

c. If $\hat{\rho} = 0$*, then*

$$
\frac{\mathcal{T}'_{\hat{w}}(\hat{w}^*,0)}{\mathcal{T}(\hat{w}^*,0)} = \frac{1}{\sigma^2 g^h(0)}\left[s^{end}\left(\mathcal{E}_{ss} - 2G^h(0)\right) + \frac{\sigma^2}{2}\left(\lim_{\Delta z\uparrow\Delta^+}(g^h)'(\Delta z) + \lim_{\Delta z\downarrow-\Delta^-}(g^h)'(\Delta z)\right)\right].
$$

d. If $\hat{\rho} > 0$*, then*

$$
\frac{\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{\rho})}{\mathcal{T}(\hat{w}^*,\hat{\rho})} = \frac{\mathcal{T}(\hat{w}^*,0)}{\mathcal{T}(\hat{w}^*,\hat{\rho})\mathcal{E}_{ss}}\left[-\hat{\rho}\frac{(\gamma+\pi)\mathbb{E}_h[a]+\mathbb{E}_h[\Delta z]}{\sigma^2} + \frac{\sigma^2}{4}\left[\lim_{\Delta z\downarrow\Delta^-} - \lim_{\Delta z\uparrow\Delta^+}\right]\frac{\mathrm{d}^2\left[\mathcal{T}(\hat{w}^*+\Delta z,\hat{\rho})g^h(\Delta z)\right]}{\mathrm{d}\Delta z^2}\right] + o(\hat{\rho}^2),
$$

where $\mathcal{T}(\hat{w}^*+\Delta z,\hat{\rho})$ *solves, up to a first-order approximation around* $\hat{\rho}=0$*, the 2-step procedure given by*

$$
\mathcal{T}(\hat{w}^*+\Delta z,\hat{\rho}) = \frac{g^h(\Delta z)}{g^h(0)}\left[\mathcal{T}(\hat{w}^*,\hat{\rho})e^{\frac{(\gamma+\pi)2}{\sigma^2}\Delta z} + \frac{2g^h(0)}{\sigma^2}\begin{cases}\int_{\Delta z}^0 e^{\frac{(\gamma+\pi)2}{\sigma^2}(\Delta z-s)}\int_{-\Delta^-}^s \frac{(1+\hat{\rho}\mathcal{T}(\hat{w}^*+x,0))g^h(x)}{g^h(s)^2}\,\mathrm{d}x\,\mathrm{d}s & \text{if } \Delta z<0 \\ \int_0^{\Delta z} e^{\frac{(\gamma+\pi)2}{\sigma^2}(\Delta z-s)}\int_s^{\Delta^+} \frac{(1+\hat{\rho}\mathcal{T}(\hat{w}^*+x,0))g^h(x)}{g^h(s)^2}\,\mathrm{d}x\,\mathrm{d}s & \text{if } \Delta z>0\end{cases}\right].
$$

with

$$
\lim_{\Delta z\downarrow\Delta^-}\frac{\mathrm{d}^2\left[\mathcal{T}(\hat{w}^*+\Delta z,\hat{\rho})g^h(\Delta z)\right]}{\mathrm{d}\Delta z^2} = \lim_{\Delta z\downarrow-\Delta^-} 2\frac{(g^h)'(\Delta z)^2}{g^h(0)}\left[\tilde{\mathcal{T}}(0,\hat{\rho})e^{-\frac{(\gamma+\pi)2}{\sigma^2}\Delta^-} + \frac{2g^h(0)}{\sigma^2}\int_{-\Delta^-}^0 e^{\frac{(\gamma+\pi)2}{\sigma^2}(-\Delta^- - s)}\int_{-\Delta^-}^s \frac{(1+\hat{\rho}\mathcal{T}(\hat{w}^*+x,0))g^h(x)}{g^h(s)^2}\,\mathrm{d}
$$

$$
\lim_{\Delta z\uparrow\Delta^+}\frac{\mathrm{d}^2\left[\mathcal{T}(\hat{w}^*+\Delta z,\hat{\rho})g^h(\Delta z)\right]}{\mathrm{d}\Delta z^2} = \lim_{\Delta z\uparrow\Delta^+} 2\frac{(g^h)'(\Delta z)^2}{g^h(0)}\left[\tilde{\mathcal{T}}(0,\hat{\rho})e^{\frac{(\gamma+\pi)2}{\sigma^2}\Delta^+} + \frac{2g^h(0)}{\sigma^2}\int_0^{\Delta^+} e^{\frac{(\gamma+\pi)2}{\sigma^2}(\Delta^+ - s)}\int_s^{\Delta^+} \frac{(1+\hat{\rho}\mathcal{T}(\hat{w}^*+x,0))g^h(x)}{g^h(s)^2}\,\mathrm{d}x\,\mathrm{d}s\right]
$$

Proof. We proceed to prove items *a-d* of the Proposition. To show these properties, it would be useful to change the state variable in $\mathcal{T}(\hat{w},\hat{\rho})$ from $\hat{w}$ to Δz. Define $\tilde{\mathcal{T}}(\Delta z,\hat{\rho}) := \mathcal{T}(\hat{w}^* + \Delta z,\hat{\rho})$. Then, applying Itô's Lemma, we obtain

$$\delta\tilde{\mathcal{T}}(\Delta z,\hat{\rho}) = 1 - \hat{\rho}\tilde{\mathcal{T}}(\Delta z,\hat{\rho}) - (\gamma+\pi)\tilde{\mathcal{T}}'_{\Delta z}(\Delta z,\hat{\rho}) + \frac{\sigma^2}{2}\tilde{\mathcal{T}}''_{\Delta z^2}(\Delta z,\hat{\rho}) \;\forall\; \Delta z \in (-\Delta^-,\Delta^+), \tag{D.51}$$

$$\tilde{\mathcal{T}}(\Delta z,\hat{\rho}) = 0 \;\forall\; \Delta z \notin (-\Delta^-,\Delta^+). \tag{D.52}$$

a. Assume that $\Delta^- = \Delta^+$ and $(\gamma+\pi) = 0$. Then, it is easy to show that $\tilde{\mathcal{T}}(\Delta z,\hat{\rho}) = \tilde{\mathcal{T}}(-\Delta z,\hat{\rho})$, and by definition of a derivative,

$$\tilde{\mathcal{T}}'_{\Delta z}(0,\hat{\rho}) = \lim_{\epsilon\downarrow 0}\underbrace{\frac{\tilde{\mathcal{T}}(\epsilon,\hat{\rho}) - \tilde{\mathcal{T}}(-\epsilon,\hat{\rho})}{2\epsilon}}_{=0} = 0.$$

A similar argument applies to $\tilde{\mathcal{T}}'''_{\Delta z^3}(0,\hat{\rho})$. Thus, $\tilde{\mathcal{T}}'_{\Delta z}(0,\hat{\rho}) = \tilde{\mathcal{T}}'''_{\Delta z^3}(0,\hat{\rho}) = 0$. Applying a third-order Taylor approximation to $\tilde{\mathcal{T}}(\Delta z,\hat{\rho})$ around $\Delta z = 0$,

$$\tilde{\mathcal{T}}(\Delta z,\hat{\rho}) = \tilde{\mathcal{T}}(0,\hat{\rho}) + \frac{1}{2}\tilde{\mathcal{T}}''_{\Delta z^2}(0,\hat{\rho})\Delta z^2 + O(\Delta z^4).$$

From the HJB equation in (D.51),

$$\tilde{\mathcal{T}}(0,\hat{\rho}) = \frac{1 + \frac{\sigma^2}{2}\tilde{\mathcal{T}}''_{\Delta z^2}(0,\hat{\rho})}{\hat{\rho}+\delta}.$$

Combining the Taylor approximation with the border conditions in (D.52), we obtain (we omit the term $O(\Delta z^4)$ to save on notation)

$$\tilde{\mathcal{T}}(0,\hat{\rho}) = -\frac{1}{2}\tilde{\mathcal{T}}''_{\Delta z^2}(0,\hat{\rho})(\Delta^+)^2.$$

Combining the last two equations, we have that

$$-\frac{1}{2}\tilde{\mathcal{T}}''_{\Delta z^2}(0,\hat{\rho})(\Delta^+)^2 = \frac{1 + \frac{\sigma^2}{2}\tilde{\mathcal{T}}''_{\Delta z^2}(0,\hat{\rho})}{\hat{\rho}+\delta} \iff \tilde{\mathcal{T}}''_{\Delta z^2}(0,\hat{\rho}) = -\frac{1}{\hat{\rho}+\delta}\left(\frac{(\Delta^+)^2}{2} + \frac{\sigma^2}{2(\hat{\rho}+\delta)}\right)^{-1}$$

and

$$\tilde{\mathcal{T}}(0,\hat{\rho}) = \mathcal{T}(\hat{w}^*,\hat{\rho}) = \frac{1}{\hat{\rho}+\delta+\left(\frac{\sigma}{\Delta^+}\right)^2} \qquad ; \qquad \tilde{\mathcal{T}}'_{\Delta z}(0,\hat{\rho}) = \mathcal{T}'_{\hat{w}^*}(\hat{w}^*,\hat{\rho}) = 0.$$

b. Now, we let $(\gamma+\pi) \neq 0$ and $\Delta^+ \neq \Delta^-$. In this case, we proceed with a second-order Taylor approximation of $\tilde{\mathcal{T}}(\Delta z,\hat{\rho})$ around $\Delta z = 0$,

$$\tilde{\mathcal{T}}(\Delta z,\hat{\rho}) = \tilde{\mathcal{T}}(0,\hat{\rho}) + \tilde{\mathcal{T}}'_{\Delta z}(0,\hat{\rho})\Delta z + \frac{1}{2}\tilde{\mathcal{T}}''_{\Delta z^2}(0,\hat{\rho})\Delta z^2 + O(\Delta z^3).$$

From the border conditions in (D.52), we obtain (we omit the term $O(\Delta z^3)$ to save on notation)

$$\begin{aligned}\tilde{\mathcal{T}}(0,\hat{\rho}) + \tilde{\mathcal{T}}'_{\Delta z}(0,\hat{\rho})\Delta^+ + \frac{1}{2}\tilde{\mathcal{T}}''_{\Delta z^2}(0,\hat{\rho})(\Delta^+)^2 &= 0,\\ \tilde{\mathcal{T}}(0,\hat{\rho}) + \tilde{\mathcal{T}}'_{\Delta z}(0,\hat{\rho})(-\Delta^-) + \frac{1}{2}\tilde{\mathcal{T}}''_{\Delta z^2}(0,\hat{\rho})(\Delta^-)^2 &= 0.\end{aligned} \tag{D.53}$$

Taking the difference

$$\tilde{\mathcal{T}}'_{\Delta z}(0,\hat{\rho})(\Delta^+ + \Delta^-) = -\frac{1}{2}\tilde{\mathcal{T}}''_{\Delta z^2}(0,\hat{\rho})\left((\Delta^+)^2 - (\Delta^-)^2\right) \iff \tilde{\mathcal{T}}'_{\Delta z}(0,\hat{\rho}) = -\frac{1}{2}\tilde{\mathcal{T}}''_{\Delta z^2}(0,\hat{\rho})\left(\Delta^+ - \Delta^-\right).$$

Replacing this last equation into the HJB equation in (D.51) evaluated at $\Delta z = 0$ and into (D.53), we obtain

$$\tilde{\mathcal{T}}(0,\hat{\rho}) = \frac{1 + \left(\frac{\sigma^2+(\gamma+\pi)(\Delta^+-\Delta^-)}{2}\right)\tilde{\mathcal{T}}''_{\Delta z^2}(0,\hat{\rho})}{\hat{\rho}+\delta}$$
$$\tilde{\mathcal{T}}(0,\hat{\rho}) = -\frac{1}{2}\tilde{\mathcal{T}}''_{\Delta z^2}(0,\hat{\rho})\left((\Delta^+)^2 - \Delta^+\left(\Delta^+ - \Delta^-\right)\right).$$

Combining these equations and solving for $\tilde{\mathcal{T}}(0,\hat{\rho})$, we have

$$\tilde{\mathcal{T}}(0,\hat{\rho}) = \frac{1}{\hat{\rho}+\delta+\frac{\sigma^2+(\gamma+\pi)(\Delta^+-\Delta^-)}{(\Delta^+)^2-\Delta^+(\Delta^+-\Delta^-)}},$$
$$\tilde{\mathcal{T}}'_{\Delta z}(0,\hat{\rho}) = \frac{(\Delta^+-\Delta^-)}{(\hat{\rho}+\delta)\left((\Delta^+)^2-\Delta^+(\Delta^+-\Delta^-)\right)+\sigma^2+(\gamma+\pi)(\Delta^+-\Delta^-)},$$
$$\tilde{\mathcal{T}}''_{\Delta z^2}(0,\hat{\rho}) = -\frac{2}{(\hat{\rho}+\delta)\left((\Delta^+)^2-\Delta^+(\Delta^+-\Delta^-)\right)+\sigma^2+(\gamma+\pi)(\Delta^+-\Delta^-)}.$$

and
$$\frac{\tilde{\mathcal{T}}'_{\Delta z}(0,\hat{\rho})}{\tilde{\mathcal{T}}(0,\hat{\rho})} = \frac{\mathcal{T}'_{\hat{w}^*}(\hat{w}^*,\hat{\rho})}{\mathcal{T}(\hat{w}^*,\hat{\rho})} = \frac{\Delta^+-\Delta^-}{\Delta^+\Delta^-}.$$

c. Set $\hat{\rho} = 0$. Combining (D.18), (D.51), and (D.52)

$$0 = g^h(\Delta z) - (\gamma+\pi)\left((g^h)'(\Delta z)\tilde{\mathcal{T}}(\Delta z,0) + g^h(\Delta z)\tilde{\mathcal{T}}'_{\Delta z}(\Delta z,0)\right) + \frac{\sigma^2}{2}\left(\tilde{\mathcal{T}}''_{\Delta z^2}(\Delta z,0)g^h(\Delta z) - \tilde{\mathcal{T}}(\Delta z,0)(g^h)''(\Delta z)\right)$$

Integrating between $-\Delta^-$ and $\Delta z < 0$,

$$\begin{aligned}
0 &= G^h(\Delta z) - (\gamma+\pi)\int_{-\Delta^-}^{\Delta z}\left((g^h)'(x)\tilde{\mathcal{T}}(x,0) + g^h(x)\tilde{\mathcal{T}}'_{\Delta z}(x,0)\right)\mathrm{d}x + \frac{\sigma^2}{2}\int_{-\Delta^-}^{\Delta z}\left(\tilde{\mathcal{T}}''_{\Delta z^2}(x,0)g^h(x) - \tilde{\mathcal{T}}(x,0)(g^h)''(x)\right)\mathrm{d}x \\
&= G^h(\Delta z) - (\gamma+\pi)\int_{-\Delta^-}^{\Delta z}\frac{\mathrm{d}\left(g^h(x)\tilde{\mathcal{T}}(x,0)\right)}{\mathrm{d}x}\mathrm{d}x + \frac{\sigma^2}{2}\int_{-\Delta^-}^{\Delta z}\frac{\mathrm{d}\left(\tilde{\mathcal{T}}'_{\Delta z}(x,0)g^h(x) - \tilde{\mathcal{T}}(x,0)(g^h)'(x)\right)}{\mathrm{d}x}\mathrm{d}x, \\
&= G^h(\Delta z) - (\gamma+\pi)g^h(\Delta z)\tilde{\mathcal{T}}(\Delta z,0) + \frac{\sigma^2}{2}(\tilde{\mathcal{T}}'_{\Delta z}(\Delta z,0)g^h(\Delta z) - \tilde{\mathcal{T}}(\Delta z,0)(g^h)'(\Delta z)).
\end{aligned}$$

In the last step, we use the fact that $\lim_{\Delta z\downarrow -\Delta^-} g^h(\Delta z) = \lim_{\Delta z\downarrow -\Delta^-}\tilde{\mathcal{T}}(\Delta z,0) = 0$. Applying similar steps to integrate from $\Delta z > 0$ to Δ^+, we have that

$$0 = \mathcal{E}_{ss} - G^h(\Delta z) + (\gamma+\pi)g^h(\Delta z)\tilde{\mathcal{T}}(\Delta z,0) - \frac{\sigma^2}{2}(\tilde{\mathcal{T}}'_{\Delta z}(\Delta z,0)g^h(\Delta z) - \tilde{\mathcal{T}}(\Delta z,0)(g^h)'(\Delta z)).$$

Thus, $\tilde{\mathcal{T}}(\Delta z,0)$ satisfies the following first order differential equation, once we write it as a function of $g^h(\Delta z)$:

$$\tilde{\mathcal{T}}'_{\Delta z}(\Delta z,0) = \begin{cases} -\frac{2}{\sigma^2}\frac{G^h(\Delta z)}{g^h(\Delta z)} + \left(\frac{(g^h)'(\Delta z)}{g^h(\Delta z)} + \frac{2(\gamma+\pi)}{\sigma^2}\right)\tilde{\mathcal{T}}(\Delta z,0) & \text{if } \Delta z \in (-\Delta^-,0) \\ \frac{2}{\sigma^2}\frac{\mathcal{E}_{ss}-G^h(\Delta z)}{g^h(\Delta z)} + \left(\frac{(g^h)'(\Delta z)}{g^h(\Delta z)} + \frac{2(\gamma+\pi)}{\sigma^2}\right)\tilde{\mathcal{T}}(\Delta z,0) & \text{if } \Delta z \in (0,\Delta^+) \end{cases} \tag{D.54}$$

Integrating the Kolmogorov forward equation (D.18) from 0 to Δ^+ and from $-\Delta^-$ to 0, we obtain

$$-\frac{2}{\sigma^2}\left[\delta\left(\mathcal{E}_{ss} - G^h(0)\right) - \frac{\sigma^2}{2}\lim_{\Delta z\uparrow\Delta^+}(g^h)'(\Delta z) + (\gamma+\pi)g^h(0)\right] = \lim_{\Delta z\downarrow 0}(g^h)'(\Delta z), \tag{D.55}$$
$$\frac{2}{\sigma^2}\left[\delta G^h(0) + \frac{\sigma^2}{2}\lim_{\Delta z\downarrow -\Delta^-}(g^h)'(\Delta z) - (\gamma+\pi)g^h(0)\right] = \lim_{\Delta z\uparrow 0}(g^h)'(\Delta z), \tag{D.56}$$

respectively. Next, we sum the limits of (D.54) as $\Delta z \to 0$ from the left and right, use the continuity of $G^h(\Delta)$, $g^h(\Delta)$, $\tilde{\mathcal{T}}'(\Delta z, 0)$ together with (D.55) and (D.56) to obtain

$$2\tilde{\mathcal{T}}'_{\Delta z}(0,0) = \frac{2}{\sigma^2}\frac{\mathcal{E}_{ss} - 2G^h(0)}{g^h(0)} + \left(\frac{\lim_{\Delta z \downarrow 0}(g^h)'(\Delta z) + \lim_{\Delta z \uparrow 0}(g^h)'(\Delta z)}{g^h(0)} + \frac{4(\gamma+\pi)}{\sigma^2}\right)\tilde{\mathcal{T}}(0,0),$$
$$= \frac{2}{\sigma^2}\frac{\mathcal{E}_{ss} - 2G^h(0)}{g^h(0)}\left[1 - \delta\tilde{\mathcal{T}}(0,0)\right] + \frac{\lim_{\Delta z \downarrow -\Delta^-}(g^h)'(\Delta z) + \lim_{\Delta z \uparrow \Delta^+}(g^h)'(\Delta z)}{g^h(0)}\tilde{\mathcal{T}}(0,0),$$
$$= \frac{2}{\sigma^2}\frac{\mathcal{E}_{ss} - 2G^h(0)}{g^h(0)}\underbrace{\left[1 - \frac{s^{exo}}{s}\right]}_{= s^{end}/s} + \frac{\lim_{\Delta z \downarrow -\Delta^-}(g^h)'(\Delta z) + \lim_{\Delta z \uparrow \Delta^+}(g^h)'(\Delta z)}{g^h(0)}\tilde{\mathcal{T}}(0,0),$$

where the last equation uses $s = 1/\tilde{\mathcal{T}}(0,0)$. Operating the last expression, we obtain

$$\frac{\tilde{\mathcal{T}}'_{\Delta z}(0,0)}{\tilde{\mathcal{T}}(0,0)} = \frac{\mathcal{T}'_{\hat{w}^*}(\hat{w}^*,0)}{\mathcal{T}(\hat{w}^*,0)} = \frac{1}{\sigma^2}\left[s^{end}\frac{\mathcal{E}_{ss} - 2G^h(0)}{g^h(0)} + \frac{\sigma^2}{2}\frac{\lim_{\Delta z \uparrow \Delta^+}(g^h)'(\Delta z) + \lim_{\Delta z \downarrow -\Delta^-}(g^h)'(\Delta z)}{g^h(0)}\right].$$

d. Now, we study the case with $\hat{\rho} > 0$. Let $\Psi(\Delta z, \hat{\rho}) := \tilde{\mathcal{T}}'_{\Delta z}(\Delta z, \hat{\rho})$. Differentiating the HJB in (D.51), we obtain a new HJB

$$\delta\Psi(\Delta z, \hat{\rho}) = -\hat{\rho}\tilde{\mathcal{T}}'_{\Delta z}(\Delta z, \hat{\rho}) - (\gamma+\pi)\Psi'_{\Delta z}(\Delta z, \hat{\rho}) + \frac{\sigma^2}{2}\Psi''_{\Delta z^2}(\Delta z, \hat{\rho})\ \forall\ \ \Delta z \in (-\Delta^-, \Delta^+),$$

with new border conditions for $\Psi(\Delta z, \hat{\rho})$

$$\tilde{\mathcal{T}}'_{\Delta z}(-\Delta^-, \hat{\rho}) = \Psi(-\Delta^-, \hat{\rho})\ \ ; \tilde{\mathcal{T}}'_{\Delta z}(\Delta^+, \hat{\rho}) = \Psi(\Delta^+, \hat{\rho}).$$

Thus,

$$\Psi(\Delta z, \hat{\rho}) = \mathbb{E}\left[\int_0^{\tau^m} -\hat{\rho}\tilde{\mathcal{T}}'_{\Delta z}(\Delta z_t, \hat{\rho})\,dt + \tilde{\mathcal{T}}'_{\Delta z}(\Delta z_{\tau^m}, \hat{\rho})\mathbb{1}[\Delta z_{\tau^m} = \Delta^+ \text{ or } \Delta z_{\tau^m} = -\Delta^-]|\Delta z_0 = \Delta z\right].$$

Evaluating at zero and using the occupancy measure,

$$\Psi(0, \hat{\rho}) = -\hat{\rho}\frac{\mathbb{E}_h[\tilde{\mathcal{T}}'_{\Delta z}(\Delta z, \hat{\rho})]}{\mathcal{E}_{ss}}\tilde{\mathcal{T}}(0,0) + \frac{\sigma^2}{2}\left[\lim_{\Delta z \downarrow -\Delta^-}\tilde{\mathcal{T}}'_{\Delta z}(\Delta z, \hat{\rho})\frac{(g^h)'(\Delta z)}{s\mathcal{E}_{ss}} - \lim_{\Delta z \uparrow \Delta^+}\tilde{\mathcal{T}}'_{\Delta z}(\Delta z, \hat{\rho})\frac{(g^h)'(\Delta z)}{s\mathcal{E}_{ss}}\right].$$

Using the fact that $s = \frac{1}{\tilde{\mathcal{T}}(0,0)}$, we have that

$$\frac{\tilde{\mathcal{T}}'_{\Delta z}(0,\hat{\rho})}{\tilde{\mathcal{T}}(0,\hat{\rho})} = \frac{\tilde{\mathcal{T}}(0,0)}{\tilde{\mathcal{T}}(0,\hat{\rho})\mathcal{E}_{ss}}\left[-\hat{\rho}\mathbb{E}_h[\tilde{\mathcal{T}}'_{\Delta z}(\Delta z, \hat{\rho})] + \frac{\sigma^2}{2}\left[\lim_{\Delta z \downarrow -\Delta^-}\tilde{\mathcal{T}}'_{\Delta z}(\Delta z, \hat{\rho})(g^h)'(\Delta z) - \lim_{\Delta z \uparrow \Delta^+}\tilde{\mathcal{T}}'_{\Delta z}(\Delta z, \hat{\rho})(g^h)'(\Delta z)\right]\right].$$

Notice that for small $\hat{\rho}$, we can apply a first-order approximation to $\hat{\rho}\mathbb{E}_h[\tilde{\mathcal{T}}'_{\Delta z}(\Delta z, \hat{\rho})]$ around $\hat{\rho} = 0$:

$$\hat{\rho}\mathbb{E}_h[\tilde{\mathcal{T}}'_{\Delta z}(\Delta z, \hat{\rho})] = \left(\mathbb{E}_h[\tilde{\mathcal{T}}'_{\Delta z}(\Delta z, 0)] + \underbrace{\hat{\rho}\mathbb{E}_h\left[\frac{\partial\tilde{\mathcal{T}}'_{\Delta z}}{\partial\hat{\rho}}(\Delta z, 0)\right]}_{=0}\right)\hat{\rho} + o(\hat{\rho}^2) = \mathbb{E}_h[\tilde{\mathcal{T}}'_{\Delta z}(\Delta z, 0)]\hat{\rho} + O(\hat{\rho}^2).$$

Thus,

$$\frac{\tilde{\mathcal{T}}'_{\Delta z}(0,\hat{\rho})}{\tilde{\mathcal{T}}(0,\hat{\rho})} = \frac{\tilde{\mathcal{T}}(0,0)}{\tilde{\mathcal{T}}(0,\hat{\rho})\mathcal{E}_{ss}}\left[-\hat{\rho}\mathbb{E}_h[\tilde{\mathcal{T}}'_{\Delta z}(\Delta z, 0)] + \frac{\sigma^2}{2}\left[\lim_{\Delta z \downarrow -\Delta^-}\tilde{\mathcal{T}}'_{\Delta z}(\Delta z, \hat{\rho})(g^h)'(\Delta z) - \lim_{\Delta z \uparrow \Delta^+}\tilde{\mathcal{T}}'_{\Delta z}(\Delta z, \hat{\rho})(g^h)'(\Delta z)\right]\right] + O(\hat{\rho}^2).$$

Next, we show that $\mathbb{E}_h[\tilde{\mathcal{T}}'_{\Delta z}(\Delta z,0)] = \frac{(\gamma+\pi)\mathbb{E}_h[a]+\mathbb{E}_h[\Delta z]}{\sigma^2}$:

$$\begin{aligned}
\mathbb{E}_h[\tilde{\mathcal{T}}'_{\Delta z}(\Delta z,0)] &= \int_{-\Delta^-}^{\Delta^+} \tilde{\mathcal{T}}'_{\Delta z}(\Delta z,0) g^h(\Delta z)\,\mathrm{d}\Delta z,\\
&=^{(1)} \mathcal{T}(\Delta z,0)g^h(\Delta z)\Big|_{-\Delta^-}^{\Delta^+} - \int_{-\Delta^-}^{\Delta^+} \mathcal{T}(\Delta z,0)(g^h)'(\Delta z)\,\mathrm{d}\Delta z,\\
&=^{(2)} -\int_{-\Delta^-}^{\Delta^+} \left[\frac{m_{\mathcal{E},h}(\Delta z)}{1-\mathcal{E}_{ss}} - \frac{m_{\mathcal{E},u}(0,0)}{1-\mathcal{E}_{ss}}\right](g^h)'(\Delta z)\,\mathrm{d}\Delta z,\\
&=^{(3)} \frac{m_{\mathcal{E},u}(0)}{1-\mathcal{E}_{ss}}\, g^h(\Delta z)\Big|_{-\Delta^-}^{\Delta^+} - \int_{-\Delta^-}^{\Delta^+} \frac{m_{\mathcal{E},h}(\Delta z)}{1-\mathcal{E}_{ss}}(g^h)'(\Delta z)\,\mathrm{d}\Delta z,\\
&=^{(4)} \frac{[(\gamma+\pi)\mathbb{E}_h[a]+\mathbb{E}_h[\Delta z]]}{\sigma^2}.
\end{aligned}$$

Step (1) applies integration by parts; step (2) uses the border conditions for $g^h(\Delta z)$; step (3) uses the recursive definition of $m_{\mathcal{E},h}(\Delta z) = (1-\mathcal{E}_{ss})\tilde{\mathcal{T}}(\Delta z) + m_{\mathcal{E},u}(0,0)$; and step (4) uses the results in Subsection D.2. Thus,

$$\frac{\tilde{\mathcal{T}}'_{\Delta z}(0,\hat{\rho})}{\tilde{\mathcal{T}}(0,\hat{\rho})} = \frac{\tilde{\mathcal{T}}(0,0)}{\tilde{\mathcal{T}}(0,\hat{\rho})\mathcal{E}_{ss}}\left[-\hat{\rho}\frac{[(\gamma+\pi)\mathbb{E}_h[a]+\mathbb{E}_h[\Delta z]]}{\sigma^2} + \frac{\sigma^2}{2}\left[\lim_{\Delta z\downarrow -\Delta^-}\tilde{\mathcal{T}}'_{\Delta z}(\Delta z,\hat{\rho})(g^h)'(\Delta z) - \lim_{\Delta z\uparrow \Delta^+}\tilde{\mathcal{T}}'_{\Delta z}(\Delta z,\hat{\rho})(g^h)'(\Delta z)\right]\right] + o(\hat{\rho}^2). \tag{D.57}$$

To further operate on the limits, observe that

$$\frac{\mathrm{d}^2(\tilde{\mathcal{T}}(\Delta z,\hat{\rho})(g^h)(\Delta z))}{\mathrm{d}\Delta z^2} = \tilde{\mathcal{T}}''_{\Delta z^2}(\Delta z,\hat{\rho})g^h(\Delta z) + 2\tilde{\mathcal{T}}'_{\Delta z}(\Delta z,\hat{\rho})(g^h)'(\Delta z) + \tilde{\mathcal{T}}(\Delta z,\hat{\rho})(g^h)''(\Delta z).$$

The limits of this expression as $\Delta z \downarrow -\Delta^-$ and $\Delta z \downarrow \Delta^+$ are given by

$$\left[\lim_{\Delta z\downarrow -\Delta^-} - \lim_{\Delta z\uparrow \Delta^+}\right]\frac{\mathrm{d}^2(\tilde{\mathcal{T}}(\Delta z,\hat{\rho})(g^h)(\Delta z))}{\mathrm{d}\Delta z^2} = \left[\lim_{\Delta z\downarrow -\Delta^-} - \lim_{\Delta z\uparrow \Delta^+}\right] 2\tilde{\mathcal{T}}'_{\Delta z}(\Delta z,\hat{\rho})(g^h)'(\Delta z).$$

Replacing this expression into (D.57), we obtain

$$\frac{\mathcal{T}'_{\hat{w}^*}(\hat{w}^*,\hat{\rho})}{\mathcal{T}(\hat{w}^*,\hat{\rho})} = \frac{\mathcal{T}(\hat{w}^*,0)}{\mathcal{T}(\hat{w}^*,\hat{\rho})\mathcal{E}_{ss}}\left[-\hat{\rho}\frac{(\gamma+\pi)\mathbb{E}_h[a]+\mathbb{E}_h[\Delta z]}{\sigma^2} + \frac{\sigma^2}{4}\left[\lim_{\Delta z\downarrow -\Delta^-} - \lim_{\Delta z\uparrow \Delta^+}\right]\frac{\mathrm{d}^2\left[\mathcal{T}(\hat{w}^*+\Delta z,\hat{\rho})g^h(\Delta z)\right]}{\mathrm{d}\Delta z^2}\right] + O(\hat{\rho}^2).$$

Finally, we characterize the marginal duration at the separation triggers as a function of $g^h(\Delta z)$. Using (D.18), (D.51), and (D.52)

$$\begin{aligned}
0 = {} & g^h(\Delta z) - \hat{\rho}\tilde{\mathcal{T}}(\Delta z,\hat{\rho})g^h(\Delta z) - (\gamma+\pi)\left((g^h)'(\Delta z)\tilde{\mathcal{T}}(\Delta z,\hat{\rho}) + g^h(\Delta z)\tilde{\mathcal{T}}'_{\Delta z}(\Delta z,\hat{\rho})\right)\\
&+ \frac{\sigma^2}{2}\left(\tilde{\mathcal{T}}''_{\Delta z^2}(\Delta z,\hat{\rho})g^h(\Delta z) - \tilde{\mathcal{T}}(\Delta z,\hat{\rho})(g^h)''(\Delta z)\right).
\end{aligned}$$

After applying a similar first-order approximation to $\hat{\rho}\tilde{\mathcal{T}}(\Delta z,\hat{\rho})g^h(\Delta z)$ around $\hat{\rho}=0$, we obtain

$$\begin{aligned}
0 = {} & g^h(\Delta z) - \hat{\rho}\tilde{\mathcal{T}}(\Delta z,0)g^h(\Delta z) - (\gamma+\pi)\left((g^h)'(\Delta z)\tilde{\mathcal{T}}(\Delta z,\hat{\rho}) + g^h(\Delta z)\tilde{\mathcal{T}}'_{\Delta z}(\Delta z,\hat{\rho})\right)\\
&+ \frac{\sigma^2}{2}\left(\tilde{\mathcal{T}}''_{\Delta z^2}(\Delta z,\hat{\rho})g^h(\Delta z) - \tilde{\mathcal{T}}(\Delta z,\hat{\rho})(g^h)''(\Delta z)\right) + O(\hat{\rho}^2).
\end{aligned}$$

To save on notation, we omit the term $O(\hat{\rho}^2)$. Define

$$\Phi^-(\Delta z) = \int_{-\Delta^-}^{\Delta z} (1 - \hat{\rho}\tilde{\mathcal{T}}(x,0))g^h(x)\,\mathrm{d}x$$
$$\Phi^+(\Delta z) = \int_{\Delta z}^{\Delta^+} (1 - \hat{\rho}\tilde{\mathcal{T}}(x,0))g^h(x)\,\mathrm{d}x.$$

Then, after applying similar steps as in item c, $\tilde{\mathcal{T}}(\Delta z, \hat{\rho})$ satisfies a first order differential equation, once we write it as a function of $g^h(\Delta z)$:

$$\tilde{\mathcal{T}}'_{\Delta z}(\Delta z, 0) = \begin{cases} -\frac{2}{\sigma^2}\frac{\Phi^-(\Delta z)}{g^h(\Delta z)} + \left(\frac{(g^h)'(\Delta z)}{g^h(\Delta z)} + \frac{2(\gamma+\pi)}{\sigma^2}\right)\tilde{\mathcal{T}}(\Delta z, 0) & \text{if } \Delta z \in (-\Delta^-, 0) \\ \frac{2}{\sigma^2}\frac{\Phi^+(\Delta z)}{g^h(\Delta z)} + \left(\frac{(g^h)'(\Delta z)}{g^h(\Delta z)} + \frac{2(\gamma+\pi)}{\sigma^2}\right)\tilde{\mathcal{T}}(\Delta z, 0) & \text{if } \Delta z \in (0, \Delta^+) \end{cases}$$

Guessing and verifying the solution, it is easy to see that the solution is given by

$$\begin{aligned} \tilde{\mathcal{T}}(\Delta z, \hat{\rho}) &= \tilde{\mathcal{T}}(0,\hat{\rho})\frac{g^h(\Delta z)}{g^h(0)}e^{\frac{2(\gamma+\pi)}{\sigma^2}\Delta z} + \begin{cases} \frac{2g(\Delta z)}{\sigma^2}\int_{\Delta z}^0 e^{\frac{2(\gamma+\pi)}{\sigma^2}(\Delta z - s)}\frac{\Phi^-(s)}{g^h(s)^2}\,\mathrm{d}s & \text{if } \Delta z < 0 \\ \frac{2g(\Delta z)}{\sigma^2}\int_0^{\Delta z} e^{\frac{2(\gamma+\pi)}{\sigma^2}(\Delta z - s)}\frac{\Phi^+(s)}{g^h(s)^2}\,\mathrm{d}s & \text{if } \Delta z > 0 \end{cases} \\ &= \frac{g^h(\Delta z)}{g^h(0)}\left[\tilde{\mathcal{T}}(0,\hat{\rho})e^{\frac{2(\gamma+\pi)}{\sigma^2}\Delta z} + \frac{2g(0)}{\sigma^2}\begin{cases} \int_{\Delta z}^0 e^{\frac{2(\gamma+\pi)}{\sigma^2}(\Delta z - s)}\frac{\Phi^-(s)}{g^h(s)^2}\,\mathrm{d}s & \text{if } \Delta z < 0 \\ \int_0^{\Delta z} e^{\frac{2(\gamma+\pi)}{\sigma^2}(\Delta z - s)}\frac{\Phi^+(s)}{g^h(s)^2}\,\mathrm{d}s & \text{if } \Delta z > 0 \end{cases}\right] \end{aligned}$$

Next, we characterize the marginal duration. Taking the derivative of the solution, and using L'Hopital's rule, $\lim_{\Delta\downarrow -\Delta^-}\left(\frac{G^h(\Delta z)}{g^h(\Delta z)}, \frac{G^h(\Delta z)}{g^h(\Delta z)^2}\right) = (0,1)$, when $\Delta z \downarrow -\Delta^-$

$$\frac{\tilde{\mathcal{T}}(0,\hat{\rho})}{g(0)}e^{\frac{2(\gamma+\pi)}{\sigma^2}\Delta z}\left[g'(\Delta z) + \frac{2(\gamma+\pi)}{\sigma^2}g(\Delta z)\right] \to \frac{\tilde{\mathcal{T}}(0,\hat{\rho})}{g(0)}e^{-\frac{2(\gamma+\pi)}{\sigma^2}\Delta^-}g'(-\Delta^-),$$
$$\frac{2g'(\Delta z)}{\sigma^2}\int_{\Delta z}^0 e^{\frac{2(\gamma+\pi)}{\sigma^2}(\Delta z - s)}\frac{\Phi^-(s)}{g^h(s)^2}\,\mathrm{d}s \to \frac{2g'(-\Delta^-)}{\sigma^2}\int_{-\Delta^-}^0 e^{\frac{2(\gamma+\pi)}{\sigma^2}(\Delta z - s)}\frac{\Phi^-(s)}{g^h(s)^2}\,\mathrm{d}s,$$
$$\frac{2g(\Delta z)}{\sigma^2}\frac{2(\gamma+\pi)}{\sigma^2}\int_{\Delta z}^0 e^{\frac{2(\gamma+\pi)}{\sigma^2}(\Delta z - s)}\frac{g^h(s)}{g^h(s)^2}\,\mathrm{d}s + \frac{2g(\Delta z)}{\sigma^2}\frac{2(\gamma+\pi)}{\sigma^2}\frac{G^h(\Delta z)}{g^h(\Delta z)^2}\,\mathrm{d}s \to 0.$$

Combining these results, we obtain

$$\lim_{\Delta z\downarrow -\Delta^-}\tilde{\mathcal{T}}'_{\Delta z}(\Delta z, \hat{\rho}) = \lim_{\Delta z\downarrow -\Delta^-}\frac{(g^h)'(\Delta z)}{g^h(0)}\left[\tilde{\mathcal{T}}(0,\hat{\rho})e^{-\frac{2(\gamma+\pi)}{\sigma^2}\Delta^-} + \frac{2g^h(0)}{\sigma^2}\int_{-\Delta^-}^0 e^{\frac{2(\gamma+\pi)}{\sigma^2}(-\Delta^- - s)}\frac{\Phi^-(\Delta z)}{g^h(s)^2}\,\mathrm{d}s\right]$$

and

$$\lim_{\Delta z\uparrow \Delta^+}\tilde{\mathcal{T}}'_{\Delta z}(\Delta z, \hat{\rho}) = \lim_{\Delta z\uparrow \Delta^+}\frac{(g^h)'(\Delta z)}{g(0)}\left[\tilde{\mathcal{T}}(0,\hat{\rho})e^{\frac{2(\gamma+\pi)}{\sigma^2}\Delta^+} + \frac{2g^h(0)}{\sigma^2}\int_0^{\Delta^+} e^{\frac{2(\gamma+\pi)}{\sigma^2}(\Delta^+ - s)}\frac{\Phi^+(\Delta z)}{g^h(s)^2}\,\mathrm{d}s\right].$$

Therefore,

$$\lim_{\Delta z\downarrow \Delta^-}\frac{\mathrm{d}^2\left[\mathcal{T}(\hat{w}^* + \Delta z, \hat{\rho})g^h(\Delta z)\right]}{\mathrm{d}\Delta z^2} = \lim_{\Delta z\downarrow -\Delta^-}2\frac{(g^h)'(\Delta z)^2}{g^h(0)}\left[\tilde{\mathcal{T}}(0,\hat{\rho})e^{-\frac{2(\gamma+\pi)}{\sigma^2}\Delta^-} + \frac{2g^h(0)}{\sigma^2}\int_{-\Delta^-}^0 e^{\frac{2(\gamma+\pi)}{\sigma^2}(-\Delta^- - s)}\frac{\Phi^-(\Delta z)}{g^h(s)^2}\,\mathrm{d}s\right]$$
$$\lim_{\Delta z\uparrow \Delta^+}\frac{\mathrm{d}^2\left[\mathcal{T}(\hat{w}^* + \Delta z, \hat{\rho})g^h(\Delta z)\right]}{\mathrm{d}\Delta z^2} = \lim_{\Delta z\uparrow \Delta^+}2\frac{(g^h)'(\Delta z)^2}{g^h(0)}\left[\tilde{\mathcal{T}}(0,\hat{\rho})e^{\frac{2(\gamma+\pi)}{\sigma^2}\Delta^+} + \frac{2g^h(0)}{\sigma^2}\int_0^{\Delta^+} e^{\frac{2(\gamma+\pi)}{\sigma^2}(\Delta^+ - s)}\frac{\Phi^+(\Delta z)}{g^h(s)^2}\,\mathrm{d}s,\right]$$

which proves the result. □

D.5 Proof of Proposition 9

We divide Proposition 9 into two propositions. Proposition D.5 "re-scales the speed of time" to provide a recursive representation of $\eta(\hat{w})$.

Proposition D.5. *Define*

$$\tau^{end} = \inf\{t \geq 0 : \Gamma_t \notin (\hat{w}^-, \hat{w}^+)\}$$

where $(\hat{w}^-, \hat{w}^+)$ is a Nash equilibrium. Then, the worker's share $\eta(\hat{w})$ satisfies the following Bellman equation

$$\eta(\hat{w}) = \mathbb{E}\left[\int_0^{\tau^{end}} e^{-(\hat{\rho}+\delta)t}(\hat{\rho}+\delta)\frac{e^{\Gamma_t} - \hat{\rho}\hat{U}}{1-\hat{\rho}\hat{U}}\, \mathrm{d}t + e^{-(\hat{\rho}+\delta)\tau^{end}}\mathbb{1}[\Delta z_{\tau^{end}} = \Delta^+] | \Gamma_0 = \hat{w}\right]$$

with

$$\mathrm{d}\Gamma_t = (\hat{\rho}+\delta)(-\hat{\gamma}\mathcal{T}(\Gamma_t, \hat{\rho}) + \sigma^2\mathcal{T}'_{\hat{w}}(\Gamma_t, \hat{\rho}))\, \mathrm{d}t + \sigma\sqrt{\mathcal{T}(\Gamma_t, \hat{\rho})(\hat{\rho}+\delta)}\, \mathrm{d}\mathcal{W}^z_t.$$

Proof. The HJB equations for the worker's value and the surplus of the match are

$$(\hat{\rho}+\delta)\hat{W}(\hat{w}) = e^{\hat{w}} - \hat{\rho}\hat{U} - \hat{\gamma}\hat{W}'(\hat{w}) + \frac{\sigma^2}{2}\hat{W}''(\hat{w}) \quad \forall \hat{w} \in (\hat{w}^-, \hat{w}^+)$$
$$(\hat{\rho}+\delta)\hat{S}(\hat{w}) = 1 - \hat{\rho}\hat{U} - \hat{\gamma}\hat{S}'(\hat{w}) + \frac{\sigma^2}{2}\hat{S}''(\hat{w}) \quad \forall \hat{w} \in (\hat{w}^-, \hat{w}^+),$$

respectively. Replacing the definition of the worker's share $\eta(\hat{w}) = \hat{W}(\hat{w})/\hat{S}(\hat{w})$ into the worker's value function, we obtain

$$(\hat{\rho}+\delta)(\eta(\hat{w})\hat{S}(\hat{w})) = e^{\hat{w}} - \hat{\rho}\hat{U} - \hat{\gamma}\left(\eta(\hat{w})\hat{S}'(\hat{w}) + \eta'(\hat{w})\hat{S}(\hat{w})\right) + \frac{\sigma^2}{2}\left(\eta(\hat{w})\hat{S}''(\hat{w}) + 2\eta'(\hat{w})\hat{S}'(\hat{w}) + \eta''(\hat{w})\hat{S}(\hat{w})\right) \forall \hat{w} \in (\hat{w}^-, \hat{w}^+).$$

Using the HJB equation of the surplus to replace $(\hat{\rho}+\delta)\hat{S}(\hat{w})$ on the left hand side,

$$(1-\hat{\rho}\hat{U})\eta(\hat{w}) = e^{\hat{w}} - \hat{\rho}\hat{U} + \eta'(\hat{w})(-\hat{\gamma}\hat{S}(\hat{w}) + \sigma^2\hat{S}'(\hat{w})) + \eta''(\hat{w})\frac{\sigma^2}{2}\hat{S}(\hat{w}) \quad \forall \hat{w} \in (\hat{w}^-, \hat{w}^+).$$

Since $\hat{S}(\hat{w}) = (1-\hat{\rho}\hat{U})\mathcal{T}(\hat{w}, \hat{\rho})$,

$$\eta(\hat{w}) = \frac{e^{\hat{w}} - \hat{\rho}\hat{U}}{1-\hat{\rho}\hat{U}} + \eta'(\hat{w})(-\hat{\gamma}\mathcal{T}(\hat{w}, \hat{\rho}) + \sigma^2\mathcal{T}'_{\hat{w}}(\hat{w}, \hat{\rho})) + \eta''(\hat{w})\frac{\sigma^2}{2}\mathcal{T}(\hat{w}, \hat{\rho}) \quad \forall \hat{w} \in (\hat{w}^-, \hat{w}^+).$$

Multiplying by $(\hat{\rho}+\delta)$, we arrive at

$$(\hat{\rho}+\delta)\eta(\hat{w}) = (\hat{\rho}+\delta)\frac{e^{\hat{w}} - \hat{\rho}\hat{U}}{1-\hat{\rho}\hat{U}} + \eta'(\hat{w})(\hat{\rho}+\delta)(-\hat{\gamma}\mathcal{T}(\hat{w}, \hat{\rho}) + \sigma^2\mathcal{T}'_{\hat{w}}(\hat{w}, \hat{\rho})) + \eta''(\hat{w})\frac{\sigma^2}{2}(\hat{\rho}+\delta)\mathcal{T}(\hat{w}, \hat{\rho}) \quad \forall \hat{w} \in (\hat{w}^-, \hat{w}^+).$$

Finally, recall the value-matching conditions

$$\hat{W}(\hat{w}^-) = \hat{J}(\hat{w}^-) = \hat{W}(\hat{w}^+) = \hat{J}(\hat{w}^+) = 0,$$

and the smooth pasting conditions

$$\hat{W}'(-\Delta^-) = \hat{J}'(\Delta^+) = 0.$$

The L'Hôpital's rule implies

$$\lim_{\hat{w}\downarrow\hat{w}^-} \eta(\hat{w}) = \lim_{\hat{w}\downarrow\hat{w}^-} \frac{\hat{W}(\hat{w})}{\hat{S}(\hat{w})} = \lim_{\hat{w}\downarrow\hat{w}^-} \frac{\hat{W}'(\hat{w})}{\hat{J}'(\hat{w})} = 0$$
$$\lim_{\hat{w}\uparrow\hat{w}^+} \eta(\hat{w}) = \lim_{\hat{w}\uparrow\hat{w}^+} \frac{\hat{W}(\hat{w})}{\hat{S}(\hat{w})} = \lim_{\hat{w}\uparrow\hat{w}^+} \frac{\hat{W}'(\hat{w})}{\hat{W}'(\hat{w})} = 1,$$

which are the boundary values for the worker's share at the separation triggers.

Finally, the equivalence of the combined Dirichlet-Poisson problem (i.e., the mapping from the corresponding HJB equations and boundary conditions of $\eta(\hat{w})$ to the sequential formulation) gives us the following Bellman equation

$$\eta(\hat{w}) = \mathbb{E}\left[\int_0^{\tau^{end}} e^{-(\hat{\rho}+\delta)t}(\hat{\rho}+\delta)\frac{e^{\Gamma_t} - \hat{\rho}\hat{U}}{1-\hat{\rho}\hat{U}}\,\mathrm{d}t + e^{-(\hat{\rho}+\delta)\tau^{end}}\mathbb{1}[\Delta z_{\tau^{end}} = \Delta^+]|\Gamma_0 = \hat{w}\right],$$

where

$$\tau^{end} = \inf\{t \geq 0 : \Gamma_t \notin (\hat{w}^-, \hat{w}^+)\}$$

and

$$\mathrm{d}\Gamma_t = (\hat{\rho}+\delta)(-\hat{\gamma}\mathcal{T}(\Gamma_t,\hat{\rho}) + \sigma^2\mathcal{T}'_{\hat{w}}(\Gamma_t,\hat{\rho}))\,\mathrm{d}t + \sigma\sqrt{\mathcal{T}(\Gamma_t,\hat{\rho})(\hat{\rho}+\delta)}\,\mathrm{d}\mathcal{W}^z_t.$$

□

Proposition 9. *The following properties hold for* $\left.\frac{\mathrm{d}\log(\eta(\hat{w}))}{\mathrm{d}\hat{w}}\right|_{\hat{w}=\hat{w}^*}$*:*

a. If $\Delta^+, \Delta^- \to \infty$*, then*

$$\left.\frac{\mathrm{d}\log(\eta(\hat{w}))}{\mathrm{d}\hat{w}}\right|_{\hat{w}=\hat{w}^*} = \frac{\left[\alpha + (1-\alpha)\hat{\rho}\hat{U}\right]}{\alpha(1-\hat{\rho}\hat{U})}.$$

b. Assume $\gamma + \pi = 0$*,* $\Delta^+ = \Delta^-$*, and* Δ^+ *small enough, then*

$$\left.\frac{\mathrm{d}\log(\eta(\hat{w}))}{\mathrm{d}\hat{w}}\right|_{\hat{w}=\hat{w}^*} = \frac{\sqrt{s^{end}}}{2\alpha\sigma}.$$

Proof. Below, we prove each property.

1. If $\Delta^+, \Delta^- \to \infty$, then $\mathcal{T}(\hat{w},\hat{\rho}) = \int_0^\infty e^{-(\hat{\rho}+\delta)t}\,\mathrm{d}t = \frac{1}{\hat{\rho}+\delta}$. The optimality condition for $\hat{w}^*$ implies

$$0 = -\frac{\mathcal{T}'(\hat{w}^*,\hat{\rho})}{\mathcal{T}(\hat{w}^*,\hat{\rho})} = \eta'(\hat{w}^*)\left(\frac{\alpha}{\eta(\hat{w}^*)} - \frac{1-\alpha}{1-\eta(\hat{w}^*)}\right) \iff \alpha = \eta(\hat{w}^*).$$

Therefore, by the definition of $\eta(\hat{w})$,

$$\alpha = \eta(\hat{w}^*) = \frac{\mathbb{E}\left[\int_0^{\tau^m} e^{-\hat{\rho}t+\hat{w}_t}\,\mathrm{d}t|\hat{w}_0 = \hat{w}^*\right] - \hat{\rho}\hat{U}\mathcal{T}(\hat{w},\hat{\rho})}{(1-\hat{\rho}\hat{U})\mathcal{T}(\hat{w},\hat{\rho})} \iff \left[\alpha + (1-\alpha)\hat{\rho}\hat{U}\right]\mathcal{T}(\hat{w},\hat{\rho}) = \mathbb{E}\left[\int_0^{\tau^m} e^{-\hat{\rho}t+\hat{w}_t}\,\mathrm{d}t|\hat{w}_0 = \hat{w}^*\right].$$

Since $\mathcal{T}(\hat{w},\hat{\rho})$ is constant, the HJB equation of the worker's share $\eta(\hat{w})$ is given by

$$(\hat{\rho}+\delta)\eta(\hat{w}) = (\hat{\rho}+\delta)\frac{e^{\hat{w}} - \hat{\rho}\hat{U}}{1-\hat{\rho}\hat{U}} - \hat{\gamma}\eta'(\hat{w}) + \eta''(\hat{w})\frac{\sigma^2}{2} \quad \forall\hat{w} \in (-\infty,\infty). \tag{D.58}$$

Taking the derivative of (D.58) with respect to $\hat{w}$ yields

$$(\hat{\rho}+\delta)\eta'(\hat{w}) = (\hat{\rho}+\delta)\frac{e^{\hat{w}}}{1-\hat{\rho}\hat{U}} - \hat{\gamma}\eta''(\hat{w}) + \eta'''(\hat{w})\frac{\sigma^2}{2} \;\; \forall \hat{w} \in (-\infty, \infty).$$

This expression corresponds to the HJB of the function $\eta'(\hat{w})$, which can be expressed as

$$\eta'(\hat{w}^*) = (\hat{\rho}+\delta)\frac{\mathbb{E}\left[\int_0^{\tau^m} e^{-\hat{\rho}t+\hat{w}_t}\,\mathrm{d}t|\hat{w}_0 = \hat{w}^*\right]}{1-\hat{\rho}\hat{U}}$$

Combining all these results, we finally obtain

$$\frac{\eta'(\hat{w}^*)}{\eta(\hat{w}^*)} = \frac{\eta'(\hat{w}^*)}{\alpha} = (\hat{\rho}+\delta)\frac{\mathbb{E}\left[\int_0^{\tau^m} e^{-\hat{\rho}t+\hat{w}_t}\,\mathrm{d}t|\hat{w}_0 = \hat{w}^*\right]}{\alpha(1-\hat{\rho}\hat{U})} = (\hat{\rho}+\delta)\frac{\left[\alpha+(1-\alpha)\hat{\rho}\hat{U}\right]\mathcal{T}(\hat{w},\hat{\rho})}{\alpha(1-\hat{\rho}\hat{U})} = \frac{\left[\alpha+(1-\alpha)\hat{\rho}\hat{U}\right]}{\alpha(1-\hat{\rho}\hat{U})}.$$

2. If $\gamma+\pi = 0$ and $\Delta^+ = \Delta^-$, then $\mathcal{T}'_{\hat{w}}(\hat{w}^*,\hat{\rho}) = 0$ and $\eta(\hat{w}^*) = \alpha$ (see the proof of Proposition D.4, item *a*). If $(\Delta^+ + \Delta^-)$ is small enough, then we can use a second-order approximation of $\eta(\hat{w})$ around $\hat{w} = \hat{w}^*$ to characterize $\eta'(\hat{w}^*)$ *only* using the border conditions. The approximation is given by

$$\eta(\hat{w}) = \eta(\hat{w}^*) + \eta'(\hat{w}^*)(\hat{w}-\hat{w}^*) + \frac{1}{2}\eta''(\hat{w}^*)(\hat{w}-\hat{w}^*)^2 + O((\hat{w}-\hat{w}^*)^3).$$

Evaluating this expression at $\hat{w}^-$ and $\hat{w}^+$, and omiting any terms of the order $O((\hat{w}-\hat{w}^*)^3)$, we obtain

$$\eta(\hat{w}^*) + \eta'(\hat{w}^*)(\hat{w}^- - \hat{w}^*) + \frac{1}{2}\eta''(\hat{w}^*)(\hat{w}^- - \hat{w}^*)^2 = 0,$$
$$\eta(\hat{w}^*) + \eta'(\hat{w}^*)(\hat{w}^+ - \hat{w}^*) + \frac{1}{2}\eta''(\hat{w}^*)(\hat{w}^+ - \hat{w}^*)^2 = 1,$$

respectively. The difference between both equations is given by

$$\eta'(\hat{w}^*) = \frac{1}{\Delta^+ + \Delta^-}.$$

From the proof of Proposition D.4 item *b*, we know that $\tilde{\mathcal{T}}(0,0) = 1/s = 1/(\delta + (\sigma/\Delta^+)^2) \Rightarrow s^{end} = (\sigma/\Delta^+)^2$. Replacing this result in the previous equation, we obtain,

$$\frac{\eta'(\hat{w}^*)}{\eta(\hat{w}^*)} = \frac{1}{\alpha}\frac{1}{\Delta^+ + \Delta^-} = \frac{\sqrt{s^{end}}}{2\alpha\sigma}.$$

□

E Additional Results for Section 3: The Consequences of Monetary Shocks in Non-Coasean Labor Markets

E.1 Characterization of the CIR of employment as a function of the CIR of the job-separation and job-finding rates

Proposition E.1. *Define*

$$IRF_{\mathcal{E}}(\zeta,t) = \mathcal{E}_t(\zeta) - \mathcal{E}_{ss}, \;\; IRF_s(\zeta,t) = s_t(\zeta) - s_{ss}, \;\; IRF_f(\zeta,t) = f_t(\zeta) - f_{ss}$$

and

$$CIR_x(\zeta) = \int_0^\infty IRF_x(\zeta,t)\,\mathrm{d}t, \text{ with } x \in \{\mathcal{E}, s, f\}.$$

Assume that $\frac{\mathrm{d}\mathcal{E}_0(0)}{\mathrm{d}\zeta} = 0$ *(i.e., there is no first order change in employment on impact) and* $\lim_{t\to\infty} \mathcal{E}_t(\zeta) = \mathcal{E}_{ss}$*. Then*

$$\frac{CIR_{\mathcal{E}}(\zeta)}{\zeta} = (1-\mathcal{E}_{ss})\mathcal{E}_{ss}\left(f_{ss}\frac{CIR_f(0)}{\mathrm{d}\zeta} - s_{ss}\frac{CIR_s(0)}{\mathrm{d}\zeta}\right) + o(\zeta).$$

Proof. Since $(\mathcal{E}_t(0), p_t(0), s_t(0)) = (\mathcal{E}_{ss}, p_{ss}, s_{ss})$, a first order Taylor approximation with respect to ζ around $\zeta = 0$ yields

$$\begin{aligned}
\mathcal{E}_t(\zeta) &= \mathcal{E}_{ss} + \frac{\mathrm{d}\mathcal{E}_t(0)}{\mathrm{d}\zeta}\zeta + o_t(\zeta^2),\\
f_t(\zeta) &= f_{ss} + \frac{\mathrm{d}f_t(0)}{\mathrm{d}\zeta}\zeta + o_t(\zeta^2),\\
s_t(\zeta) &= s_{ss} + \frac{\mathrm{d}s_t(0)}{\mathrm{d}\zeta}\zeta + o_t(\zeta^2).
\end{aligned}$$

and the law of motion

$$\mathrm{d}\mathcal{E}_t = (-s_t\mathcal{E}_t + f_t(1-\mathcal{E}_t))\,\mathrm{d}t \quad , \quad \mathcal{E}_0(\zeta) = \mathcal{E}_{ss} + o(\zeta^2).$$

Replacing the first order Taylor approximations in the law of motion of employment, we obtain

$$\begin{aligned}
\mathrm{d}\mathcal{E}_t &= (-s_t\mathcal{E}_t + f_t(1-\mathcal{E}_t))\,\mathrm{d}t,\\
&\approx \left(\underbrace{-s_{ss}\mathcal{E}_{ss} + f_{ss}(1-\mathcal{E}_{ss})}_{=0} + (1-\mathcal{E}_{ss})\frac{\mathrm{d}f_t(0)}{\mathrm{d}\zeta} - \mathcal{E}_{ss}\frac{\mathrm{d}s_t(0)}{\mathrm{d}\zeta} - (f_{ss}+s_{ss})\frac{\mathrm{d}\mathcal{E}_t(0)}{\mathrm{d}\zeta}\right)\mathrm{d}t\zeta.
\end{aligned}$$

At the same time, we have

$$\begin{aligned}
\mathrm{d}\mathcal{E}_t &= \mathrm{d}(\mathcal{E}_{ss} + \frac{\mathrm{d}\mathcal{E}_t(0)}{\mathrm{d}\zeta}\zeta) + o_t(\zeta^2),\\
&\approx \mathrm{d}\left(\frac{\mathrm{d}\mathcal{E}_t(0)}{\mathrm{d}\zeta}\right)\zeta
\end{aligned}$$

Combining both expressions and cancelling ζ from both sides we obtain

$$\mathrm{d}\left(\frac{\mathrm{d}\mathcal{E}_t(0)}{\mathrm{d}\zeta}\right) = \left((1-\mathcal{E}_{ss})\frac{\mathrm{d}f_t(0)}{\mathrm{d}\zeta} - \mathcal{E}_{ss}\frac{\mathrm{d}s_t(0)}{\mathrm{d}\zeta} - (f_{ss}+s_{ss})\frac{\mathrm{d}\mathcal{E}_t(0)}{\mathrm{d}\zeta}\right)\mathrm{d}t.$$

Taking the integral between 0 and T,

$$\int_0^T \mathrm{d}\left(\frac{\mathrm{d}\mathcal{E}_t(0)}{\mathrm{d}\zeta}\right) = (1-\mathcal{E}_{ss})\int_0^T \frac{\mathrm{d}f_t(0)}{\mathrm{d}\zeta}\,\mathrm{d}t - \mathcal{E}_{ss}\int_0^T \frac{\mathrm{d}s_t(0)}{\mathrm{d}\zeta}\,\mathrm{d}t - (f_{ss}+s_{ss})\int_0^T \frac{\mathrm{d}\mathcal{E}_t(0)}{\mathrm{d}\zeta}\,\mathrm{d}t.$$

Since $\int_0^T \mathrm{d}\left(\frac{\mathrm{d}\mathcal{E}_t(0)}{\mathrm{d}\zeta}\right) = \frac{\mathrm{d}\mathcal{E}_T(0)}{\mathrm{d}\zeta} - \frac{\mathrm{d}\mathcal{E}_0(0)}{\mathrm{d}\zeta} = \frac{\mathrm{d}\mathcal{E}_T(0)}{\mathrm{d}\zeta}$, we have that

$$\frac{\mathrm{d}\mathcal{E}_t(0)}{\mathrm{d}\zeta} = (1-\mathcal{E}_{ss})\int_0^T \frac{\mathrm{d}f_t(0)}{\mathrm{d}\zeta}\,\mathrm{d}t - \mathcal{E}_{ss}\int_0^T \frac{\mathrm{d}s_t(0)}{\mathrm{d}\zeta}\,\mathrm{d}t - (f_{ss}+s_{ss})\int_0^T \frac{\mathrm{d}\mathcal{E}_t(0)}{\mathrm{d}\zeta}\,\mathrm{d}t.$$

Taking the limit, since $\lim_{t\to\infty}\mathcal{E}_t(\zeta) = \mathcal{E}_{ss}$, we have that $\lim_{T\to\infty}\frac{\mathrm{d}\mathcal{E}_t(0)}{\mathrm{d}\zeta} = 0$ and

$$\int_0^\infty \frac{\mathrm{d}\mathcal{E}_t(0)}{\mathrm{d}\zeta}\,\mathrm{d}t = \frac{(1-\mathcal{E}_{ss})}{(f_{ss}+s_{ss})}\int_0^\infty \frac{\mathrm{d}f_t(0)}{\mathrm{d}\zeta}\,\mathrm{d}t - \frac{\mathcal{E}_{ss}}{(f_{ss}+s_{ss})}\int_0^\infty \frac{\mathrm{d}s_t(0)}{\mathrm{d}\zeta}\,\mathrm{d}t.$$

Since

$$\begin{aligned}
\frac{\mathrm{dCIR}_{\mathcal{E}}(0)}{\mathrm{d}\zeta} &= \int_0^\infty \frac{\mathrm{d}\mathcal{E}_t(0)}{\mathrm{d}\zeta}\,\mathrm{d}t,\\
\frac{\mathrm{dCIR}_{f}(0)}{\mathrm{d}\zeta} &= \int_0^\infty \frac{\mathrm{d}f_t(0)}{\mathrm{d}\zeta}\,\mathrm{d}t,\\
\frac{\mathrm{dCIR}_{s}(0)}{\mathrm{d}\zeta} &= \int_0^\infty \frac{\mathrm{d}s_t(0)}{\mathrm{d}\zeta}\,\mathrm{d}t,
\end{aligned}$$

we can combine the previous result and $\mathcal{E}_{ss} = \frac{f_{ss}}{s_{ss}+f_{ss}}$, to obtain

$$\frac{\mathrm{dCIR}_{\mathcal{E}}(0)}{\mathrm{d}\zeta} = (1-\mathcal{E}_{ss})\mathcal{E}_{ss}\left(f_{ss}^{-1}\frac{\mathrm{dCIR}_f(0)}{\mathrm{d}\zeta} - s_{ss}^{-1}\frac{\mathrm{dCIR}_s(0)}{\mathrm{d}\zeta}\right).$$

Since $\mathrm{CIR}_x(\zeta) = \mathrm{CIR}_x(0) + \frac{\mathrm{dCIR}_x(0)}{\mathrm{d}\zeta}\zeta + o(\zeta^2)$ and $\mathrm{CIR}_x(0) = 0$, we have

$$\frac{\mathrm{CIR}_{\mathcal{E}}(\zeta)}{\zeta} = (1-\mathcal{E}_{ss})\mathcal{E}_{ss}\left(f_{ss}\frac{\mathrm{CIR}_f(0)}{\zeta} - s_{ss}\frac{\mathrm{CIR}_s(0)}{\zeta}\right) + o(\zeta)$$

□

E.2 Second-Order Approximation of the CIR of Employment with Flexible Wages

Proposition E.2. *Assume flexible entry wages. Up to a second-order approximation,*

$$\begin{aligned}
CIR''_{\mathcal{E}}(0) = {} & \frac{(1-\mathcal{E}_{ss})\mathcal{E}_{ss} - \gamma(CIR_{\mathcal{E}})'(0)}{\sigma^2} - \frac{1}{2}\left(\Delta z m'_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\right)\Big|_{-\Delta^-}^{\Delta^+}\\
& + \frac{1}{2}\left((g^h)'(0_-) - (g^h)'(0_+)\right)\left(m_{\mathcal{E},u}(0) - m_{\mathcal{E},h}(0)\right).
\end{aligned}$$

Proof. Taking the second derivative of (D.33), we obtain

$$\begin{aligned}
\mathrm{CIR}''_{\mathcal{E}}(\zeta) = {} & -\lim_{\Delta z\to -\zeta} m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z+\zeta) + \lim_{\Delta z\to -\Delta^- -\zeta} m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z+\zeta) + \int_{-\Delta^- -\zeta}^{-\zeta} m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z+\zeta)\,\mathrm{d}\Delta z\\
& - \lim_{\Delta z\to \Delta^+ -\zeta} m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z+\zeta) + \lim_{\Delta z\to -\zeta} m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z+\zeta) + \int_{-\zeta}^{\Delta^+ -\zeta} m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z+\zeta)\,\mathrm{d}\Delta z,
\end{aligned}$$

which evaluated at $\zeta = 0$ becomes

$$\text{CIR}''_{\mathcal{E}}(0) = -\ m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\ |^{0}_{-\Delta^-} + \int_{-\Delta^-}^{0} m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z) d\Delta z.$$
$$-\ m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\ |^{\Delta^+}_{0} + \int_{0}^{\Delta^+} m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z) d\Delta z.$$

Differentiating condition (G.1) to replace $\delta = \frac{\gamma(g^h)''(\Delta z)+\frac{\sigma^2}{2}(g^h)'''(\Delta z)}{(g^h)'(\Delta z)}$ into equation (D.34) we obtain

$$\frac{\gamma(g^h)''(\Delta z)+\frac{\sigma^2}{2}(g^h)'''(\Delta z)}{(g^h)'(\Delta z)} m_{\mathcal{E},h}(\Delta z) = 1 - \mathcal{E}_{ss} - \gamma m'_{\mathcal{E},h}(\Delta z) + \frac{\sigma^2}{2} m''_{\mathcal{E},h}(\Delta z) + \frac{\gamma g''(\Delta z)+\frac{\sigma^2}{2} g'''(\Delta z)}{(g^h)'(\Delta z)} m_{\mathcal{E},u}(0).$$

Multiplying by $(g^h)'(\Delta z)\Delta z$ and integrating between $-\Delta^-$ and Δ^+

$$0 = (1-\mathcal{E}_{ss})\, T_1 - \gamma T_2 + \frac{\sigma^2}{2} T_3 + m_{\mathcal{E},u}(0) T_4 \tag{E.1}$$
$$T_1 = \int_{-\Delta^-}^{\Delta^+} \Delta z (g^h)'(\Delta z)\, \mathrm{d}\ \Delta z$$
$$T_2 = \int_{-\Delta^-}^{\Delta^+} \Delta z \left[m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z) + m'_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\right] \mathrm{d}\ \Delta z$$
$$T_3 = \int_{-\Delta^-}^{\Delta^+} \Delta z \left[m''_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)'''(\Delta z)\right] \mathrm{d}\ \Delta z$$
$$T_4 = \int_{-\Delta^-}^{\Delta^+} \Delta z \left(\gamma (g^h)''(\Delta z) + \frac{\sigma^2}{2}(g^h)'''(\Delta z)\right) \mathrm{d}\ \Delta z.$$

T_1 is equal to

$$\begin{aligned} T_1 &= \int_{-\Delta-}^{\Delta^+} \Delta z (g^h)'(\Delta z)\, \mathrm{d}\ \Delta z \\ &= \int_{-\Delta-}^{0} \Delta z (g^h)'(\Delta z)\, \mathrm{d}\ \Delta z + \int_{0}^{\Delta^+} \Delta z (g^h)'(\Delta z)\, \mathrm{d}\ \Delta z \\ &= \underbrace{\Delta z g^h(\Delta z)\Big|_{-\Delta^-}^{0} + \Delta z g^h(\Delta z)\Big|_{0}^{\Delta^+}}_{=0} - \underbrace{\int_{-\Delta^-}^{\Delta^+} g^h(\Delta z)\, \mathrm{d}\ \Delta z}_{=\mathcal{E}_{ss}} \\ &= -\mathcal{E}_{ss}. \end{aligned} \tag{E.2}$$

T_2 satisfies

$$\begin{aligned} T_2 &= \int_{-\Delta^-}^{\Delta^+} \Delta z \left[m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z) - m'_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z)\right] \mathrm{d}\ \Delta z \\ &=^{(1)} \int_{-\Delta^-}^{0} \Delta z \left[m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z) - m'_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\right] \mathrm{d}\ \Delta z + \int_{0}^{\Delta^+} \Delta z \left[m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z) - m'_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\right] \mathrm{d}\ \Delta z \\ &=^{(2)}\ \Delta z m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\Big|_{-\Delta^-}^{0} + \Delta z m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\Big|_{0}^{\Delta^+} \\ &\cdots - \left[\int_{-\Delta^-}^{0} \left[m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\right] \mathrm{d}\ \Delta z + \int_{0}^{\Delta^+} \left[m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\right] \mathrm{d}\ \Delta z \right] \\ &=^{(3)}\ m_{\mathcal{E},u}(0)\ \Delta z (g^h)'(\Delta z)\Big|_{\Delta^-}^{\Delta^+} - \int_{\Delta^-}^{\Delta^+} m_{\mathcal{E},h}(\Delta z) g'(\Delta z)\, \mathrm{d}\ \Delta z \end{aligned} \tag{E.3}$$

$$=^{(4)} m_{\mathcal{E},u}(0)\ \Delta z (g^h)'(\Delta z)\Big|_{\Delta^-}^{\Delta^+} - CIR'_{\mathcal{E}}(0).$$

Here, step (1) divides the integral at the discontinuity point; (2) uses the equality $m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z) + m'_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) = \frac{\mathrm{d}\left[m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\right]}{\mathrm{d}\ \Delta z}$ and integrates by parts; (3) uses conditions (D.35) and (E.7); and (4) uses the definition of $CIR'_{\mathcal{E}}(0)$. T_3 satisfies

$$T_3 = \int_{-\Delta^-}^{\Delta^+} \Delta z \left[m''_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)'''(\Delta z)\right] \mathrm{d}\ \Delta z \tag{E.4}$$

$$=^{(1)} \int_{-\Delta^-}^{0} \Delta z \left[m''_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)'''(\Delta z)\right] \mathrm{d}\ \Delta z + \int_{0}^{\Delta^+} \Delta z \left[m''_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)'''(\Delta z)\right] \mathrm{d}\ \Delta z$$

$$=^{(2)} \Delta z \left(m'_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z)\right)\Big|_{-\Delta^-}^{\Delta^+}$$

$$\cdots - \left[\int_{-\Delta^-}^{0} \left[m'_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z)\right] \mathrm{d}\ \Delta z + \int_{0}^{\Delta^+} \left[m'_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z)\right] \mathrm{d}\ \Delta z\right]$$

$$=^{(3)} \Delta z \left(m'_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) - m_{\mathcal{E},u}(0)(g^h)''(\Delta z)\right)\Big|_{-\Delta^-}^{\Delta^+}$$

$$\cdots - \left[m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\Big|_{-\Delta^-}^{0} - 2\int_{-\Delta^-}^{0} \left[m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z)\right] \mathrm{d}\ \Delta z\right]$$

$$\cdots - \left[m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z)\Big|_{0}^{\Delta^+} - 2\int_{0}^{\Delta^+} \left[m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z)\right] \mathrm{d}\ \Delta z\right].$$

Here, (1) divides the integral at the discontinuity point; (2) uses the equality $m''_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)'''(\Delta z) = \frac{\mathrm{d}\left[m'_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) - m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z)\right]}{\mathrm{d}\ \Delta z}$ and integrates by parts; and (3) uses conditions (D.35) and integrates by parts.

Finally, for T_4

$$T_4 = \int_{-\Delta-}^{\Delta^+} \Delta z \left(\gamma (g^h)''(\Delta z) + \frac{\sigma^2}{2}(g^h)'''(\Delta z)\right) \mathrm{d}\ \Delta z \tag{E.5}$$

$$=^{(1)} \int_{-\Delta-}^{0} \Delta z \left(\gamma (g^h)''(\Delta z) + \frac{\sigma^2}{2}(g^h)'''(\Delta z)\right) \mathrm{d}\ \Delta z + \int_{0}^{\Delta^+} \Delta z \left(\gamma (g^h)''(\Delta z) + \frac{\sigma^2}{2}(g^h)'''(\Delta z)\right) \mathrm{d}\ \Delta z$$

$$=^{(2)} \Delta z \left[\gamma (g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z)\right]\Big|_{-\Delta^-}^{0} + \Delta z \left[\gamma (g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z)\right]\Big|_{0}^{\Delta^+}$$

$$\cdots - \int_{-\Delta^-}^{\Delta^+} \gamma (g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z)\,\mathrm{d}\ \Delta z$$

$$=^{(3)} \Delta z \left[\gamma (g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z)\right]\Big|_{-\Delta^-}^{\Delta^+} - \gamma \underbrace{\left[g^h(\Delta z)\Big|_{-\Delta^-}^{0} + g^h(\Delta z)\Big|_{0}^{\Delta^+}\right]}_{=0}$$

$$\cdots - \frac{\sigma^2}{2}\left[(g^h)'(\Delta z)\Big|_{-\Delta^-}^{0} + (g^h)'(\Delta z)\Big|_{0}^{\Delta^+}\right]$$

$$= \Delta z \left[\gamma (g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z)\right]\Big|_{-\Delta^-}^{\Delta^+} - \frac{\sigma^2}{2}\left[(g^h)'(\Delta z)\Big|_{-\Delta^-}^{0} + (g^h)'(\Delta z)\Big|_{0}^{\Delta^+}\right].$$

Here, (1) divides the integral at the discontinuity point; (2) integrates by parts; (3) operates the integral and uses the border conditions.

Combining results (E.1), (E.2), (E.3), (E.4), (E.5), we obtain

$$0 = (1 - \mathcal{E}_{ss})\, T_1 - \gamma T_2 + \frac{\sigma^2}{2} T_3 + m_{\mathcal{E},u}(0) T_4$$

$$
\begin{aligned}
0 = & -(1-\mathcal{E}_{ss})\mathcal{E}_{ss} - \gamma \left(m_{\mathcal{E},u}(0)\ \Delta z (g^h)'(\Delta z) \Big|_{\Delta^-}^{\Delta^+} - CIR'_{\mathcal{E}}(0) \right) \\
& \cdots + \frac{\sigma^2}{2} \Bigg[\Delta z \left(m'_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) - m_{\mathcal{E},u}(0)(g^h)''(\Delta z) \right) \Big|_{-\Delta^-}^{\Delta^+} \\
& \cdots - \left[m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) \Big|_{-\Delta^-}^{0} - 2\int_{-\Delta^-}^{0} \left[m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z) \right] \mathrm{d}\,\Delta z \right] \\
& \cdots - \left[m_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) \Big|_{0}^{\Delta^+} - 2\int_{0}^{\Delta^+} \left[m_{\mathcal{E},h}(\Delta z)(g^h)''(\Delta z) \right] \mathrm{d}\,\Delta z \right] \Bigg] \\
& \cdots + m_{\mathcal{E},u}(0) \left(\Delta z \left[\gamma (g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z) \right] \Big|_{-\Delta^-}^{\Delta^+} - \frac{\sigma^2}{2} \left[(g^h)'(\Delta z) \Big|_{-\Delta^-}^{0} + (g^h)'(\Delta z) \Big|_{0}^{\Delta^+} \right] \right),
\end{aligned}
$$

which implies

$$
\begin{aligned}
CIR''_{\mathcal{E}}(0) = & \frac{(1-\mathcal{E}_{ss})\mathcal{E}_{ss} - \gamma (CIR_{\mathcal{E}})'(0)}{\sigma^2} - \frac{1}{2}\left(\Delta z m'_{\mathcal{E},h}(\Delta z)(g^h)'(\Delta z) \right) \Big|_{-\Delta^-}^{\Delta^+} \\
& + \frac{1}{2}\left((g^h)'(0_-) - (g^h)'(0_+) \right) \left(m_{\mathcal{E},u}(0) - m_{\mathcal{E},h}(0) \right).
\end{aligned}
$$

□

E.3 Characterizing the CIR for real wages with flexible entry wages

We define the CIR of the average wage per capita to a monetary shock as

$$
\mathrm{CIR}_w(\zeta) = \int_0^\infty \int_{\hat{w}^-}^{\hat{w}^+} \hat{w} \left(g^h(\hat{w}, t) - g(\hat{w}) \right) \mathrm{d}\hat{w}\, \mathrm{d}t.
$$

The strategy to find the sufficient statistic is similar to the strategy used for $\mathrm{CIR}_{\mathcal{E}}(\zeta)$, with few differences in the implementation of the steps. For this reason, we skip the proof of some of the similar steps. The main difference is the associated Bellman equation that describes the total sum of the differences between the average wage in period t and the steady-state average wage.

Step 1. The CIR satisfies

$$
\mathrm{CIR}_w(\zeta) = \int_{-\infty}^{\infty} m_{w,h}(\Delta z) g^h(\Delta z + \zeta)\, \mathrm{d}\Delta z + \int_{-\infty}^{\infty} m_{w,u}(\Delta z) g^u(\Delta z + \zeta)\, \mathrm{d}\Delta z
$$

with

$$
\begin{aligned}
m_{w,h}(\Delta z) &\equiv \mathbb{E}\left[\int_0^{\tau^m} [\Delta z_t - \mathbb{E}_h[\Delta z]]\, \mathrm{d}t + m_{w,u}(0) | \Delta z_0 = \Delta z \right] \\
m_{w,u}(\Delta z) &\equiv \mathbb{E}\left[m_{w,h}(0) | \Delta z_0 = \Delta z \right] \\
0 &= \int_{-\Delta^-}^{\Delta^+} m_{w,h}(\Delta z) g^h(\Delta z)\, \mathrm{d}\Delta z + \int_{-\Delta^-}^{\Delta^+} m_{w,u}(\Delta z) g^u(\Delta z)\, \mathrm{d}\Delta z.
\end{aligned}
$$

Step 2. Up to first order, the $CIR_w(\zeta)$ is the solution of

$$
\mathrm{CIR}_w(\zeta) = \int_{-\infty}^{\infty} m_{w,h}(\Delta z) g^h(\Delta z + \zeta)\, \mathrm{d}\Delta z + m_{w,h}(0)(1 - \mathcal{E}_{ss}),
$$

where

$$0 = \Delta z - \mathbb{E}_h[\Delta z] - \gamma (m_{w,h})'(\Delta z) + \frac{\sigma^2}{2}(m_{w,h})''(\Delta z) + \delta(m_{w,h}(0) - (m_{w,h})'(\Delta z)) \tag{E.6}$$

$$\begin{aligned}
m_{w,h}(0) &= m_{w,h}(-\Delta^-) = m_{w,h}(\Delta^+), \\
0 &= \int_{-\Delta^-}^{\Delta^+} m_{w,h}(\Delta z) g^h(\Delta z)\,\mathrm{d}\Delta z + m_{w,h}(0)(1-\mathcal{E}_{ss}), \\
\delta g^h(\Delta z) &= \gamma (g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z) \quad \text{for all } \Delta z \in (-\Delta^-, \Delta^+)/\{0\}, \\
g^h(-\Delta^-) &= g^h(\Delta^+) = 0, \\
\mathcal{E}_{ss} &= \int_{-\Delta^-}^{\Delta_+} g^h(\Delta z)\,\mathrm{d}\Delta z, \\
g^h(\Delta z) &\in \mathbb{C}, \mathbb{C}^1(\{0\}), \mathbb{C}^2(\{0\}).
\end{aligned} \tag{E.7}$$

Step 3. We show that $m_{w,h}(0) = \frac{\mathbb{C}ov_h[a,\Delta z]}{1-\mathcal{E}_{ss}}$.

Proof of Step 3. Define $f(\Delta z_t) = \Delta z_t - \mathbb{E}_h[\Delta z]$. Observe that $m_{w,h}(\Delta z)$ satisfies the following recursive representation

$$m_{w,h}(\Delta z) \;=\; \mathbb{E}\left[\int_0^{\tau^m} f(\Delta z_t)\,\mathrm{d}t + m_{w,h}(0) \middle| \Delta z_0 = \Delta z\right].$$

Define the following auxiliary function

$$\Psi(\Delta z|\varphi) \;=\; \mathbb{E}\left[\int_0^{\tau^m} e^{\varphi t} f(\Delta z_t)\,\mathrm{d}t + e^{\varphi \tau^m} m_{w,h}(0) \middle| \Delta z_0 = \Delta z\right]. \tag{E.8}$$

Then, following similar steps, we obtain

$$\begin{aligned}
\frac{\partial \Psi(0|0)}{\partial \varphi} &= \mathbb{E}\left[\int_0^{\tau^m} m_{w,h}(\Delta z_t)\,\mathrm{d}t \middle| \Delta z_0 = 0\right] \\
&= \mathbb{E}[\tau^m] \frac{\int_{-\Delta^-}^{\Delta^+} m_{w,h}(\Delta z) g^h(\Delta z)\,\mathrm{d}\Delta z}{\mathcal{E}_{ss}} \\
&= -\mathbb{E}[\tau^m] m_{w,h}(0) \frac{(1-\mathcal{E}_{ss})}{\mathcal{E}_{ss}}.
\end{aligned} \tag{E.9}$$

From (E.8), we have that

$$\begin{aligned}
\frac{\partial \Psi(0|0)}{\partial \varphi} &= \mathbb{E}\left[\int_0^{\tau^m} s f(\Delta z_s)\,\mathrm{d}s + \tau^m m_{w,h}(0) \middle| \Delta z_0 = 0\right] \\
&= \overline{\mathbb{E}}\left[\tau^m\right] \left[\frac{\mathbb{E}_h[a f(\Delta z_s)]}{\mathcal{E}_{ss}} + m_{w,h}(0)\right],
\end{aligned} \tag{E.10}$$

Combining (E.9) and (E.10), and solving for $m_{w,h}(0)$ we obtain:

$$m_{w,h}(0) = \frac{\mathbb{C}ov_h[a,\Delta z]}{1-\mathcal{E}_{ss}}$$

Step 4. Up to a first-order approximation, the CIR is given by:

$$\text{CIR}_w(\zeta) = -\frac{\mathbb{C}ov_h[\Delta z + \gamma a, \Delta z]}{\sigma^2} + o(\zeta^2).$$

Proof of Step 4. To help the reader, we summarize below the conditions we use in this proof.

$$\text{CIR}_w(\zeta) = \int_{-\infty}^{\infty} m_{w,h}(\Delta z) g^h(\Delta z + \zeta)\, \mathrm{d}\Delta z + m_{w,h}(0)(1 - \mathcal{E}_{ss}), \tag{E.11}$$

where

$$0 = f(\Delta z) - \gamma (m_{w,h})'(\Delta z) + \frac{\sigma^2}{2}(m_{w,h})''(\Delta z) + \delta(m_{w,h}(0) - m_{w,h}(\Delta z))$$

$$m_{w,h}(0) = m_{w,h}(-\Delta^-) = m_{w,h}(\Delta^+), \tag{E.12}$$

$$0 = \int_{-\Delta^-}^{\Delta^+} m_{w,h}(\Delta z) g^h(\Delta z)\, \mathrm{d}\Delta z + m_{w,h}(0)(1 - \mathcal{E}_{ss}). \tag{E.13}$$

1. **Zero-order:** If $\zeta = 0$, condition (E.13) implies

$$\text{CIR}_w(0) = \int_{-\infty}^{\infty} m_{w,h}(\Delta z) g^h(\Delta z)\, \mathrm{d}\Delta z + m_{w,h}(0)(1 - \mathcal{E}_{ss}) = 0.$$

2. **First-order:** Taking the derivative of (E.11) we obtain

$$\text{CIR}_w'(\zeta) = \int_{-\infty}^{\infty} m_{w,h}(\Delta z)(g^h)'(\Delta z + \zeta)\, \mathrm{d}\Delta z,$$

which evaluated at $\zeta = 0$ becomes

$$\text{CIR}_w'(0) = \int_{-\Delta^-}^{\Delta^+} m_{w,h}(\Delta z)(g^h)'(\Delta z)\, \mathrm{d}\Delta z.$$

Using condition (G.1) to replace $\delta = \frac{\gamma (g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z)}{g^h(\Delta z)}$ into the HJB equation (E.6), we obtain

$$\frac{\gamma (g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z)}{g^h(\Delta z)} m_{w,h}(\Delta z) = f(\Delta z) - \gamma (m_{w,h})'(\Delta z) + \frac{\sigma^2}{2}(m_{w,h})''(\Delta z) + \frac{\gamma (g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z)}{g^h(\Delta z)} m_{w,h}(0).$$

Multiplying by $g^h(\Delta z)\Delta z$ and taking the integral between $-\Delta^-$ and Δ^+

$$0 = \mathbb{V}ar_h[\Delta z] - \gamma T_1 + \frac{\sigma^2}{2} T_2 + m_{w,h}(0) T_3$$

$$T_1 = \int_{-\Delta^-}^{\Delta^+} \Delta z \left[(m_{w,h})'(\Delta z) g^h(\Delta z) + m_{w,h}(\Delta z)(g^h)'(\Delta z) \right] \mathrm{d}\,\Delta z$$

$$T_2 = \int_{-\Delta^-}^{\Delta^+} \Delta z \left[(m_{w,h})''(\Delta z) g^h(\Delta z) - m_{w,h}(\Delta z)(g^h)''(\Delta z) \right] \mathrm{d}\,\Delta z$$

$$T_3 = \int_{-\Delta^-}^{\Delta^+} \Delta z \left(\gamma (g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z) \right) \mathrm{d}\,\Delta z.$$

T_1 is equal to

$$T_1 = \int_{-\Delta^-}^{\Delta^+} \Delta z \left[(m_{w,h})'(\Delta z) g^h(\Delta z) + m_{w,h}(\Delta z)(g^h)'(\Delta z) \right] \mathrm{d}\,\Delta z$$

$$
\begin{aligned}
&=^{(1)} \int_{-\Delta^-}^{0} \Delta z \left[(m_{w,h})'(\Delta z) g^h(\Delta z) + m_{w,h}(\Delta z)(g^h)'(\Delta z) \right] \mathrm{d}\,\Delta z \\
&+ \int_{0}^{\Delta^+} \Delta z \left[(m_{w,h})'(\Delta z) g^h(\Delta z) + m_{w,h}(\Delta z)(g^h)'(\Delta z) \right] \mathrm{d}\,\Delta z \\
&=^{(2)} \int_{-\Delta^-}^{0} \Delta z \frac{d\left(m_{w,h}(\Delta z) g^h(\Delta z)\right)}{\mathrm{d}\,\Delta z} \mathrm{d}\,\Delta z + \int_{0}^{\Delta^+} \Delta z \frac{d\left(m_{w,h}(\Delta z) g^h(\Delta z)\right)}{\mathrm{d}\,\Delta z} \mathrm{d}\,\Delta z \\
&=^{(3)} \underbrace{\Delta z m_{w,h}(\Delta z) g^h(\Delta z)\Big|_{-\Delta^-}^{0} + \Delta z m_{w,h}(\Delta z) g^h(\Delta z)\Big|_{0}^{\Delta^+}}_{=0} \\
&\cdots - \left[\int_{-\Delta^-}^{0} m_{w,h}(\Delta z) g^h(\Delta z)\,\mathrm{d}\,\Delta z + \int_{0}^{\Delta^+} m_{w,h}(\Delta z) g^h(\Delta z)\,\mathrm{d}\,\Delta z \right] \\
&=^{(4)} - \int_{-\Delta^-}^{\Delta^+} m_{w,h}(\Delta z) g^h(\Delta z)\,\mathrm{d}\,\Delta z \\
&=^{(5)} m_{w,h}(0)(1 - \mathcal{E}_{ss})
\end{aligned}
$$

Here, (1) divides the integral at the discontinuity point of $g^h(\Delta z)$; (2) uses the property of the derivative of a product of functions; (3) integrates and uses the border conditions (E.7); (4) uses continuity of $m_{w,h}(\Delta z) g^h(\Delta z)$; and (6) uses (E.13). T_2 satisfies

$$
\begin{aligned}
T_2 &= \int_{-\Delta^-}^{\Delta^+} \Delta z \left[(m_{w,h})''(\Delta z) g^h(\Delta z) - m_{w,h}(\Delta z)(g^h)''(\Delta z) \right] \mathrm{d}\,\Delta z \\
&=^{(1)} \int_{-\Delta^-}^{0} \Delta z \left[(m_{w,h})''(\Delta z) g^h(\Delta z) - m_{w,h}(\Delta z)(g^h)''(\Delta z) \right] \mathrm{d}\,\Delta z \\
&+ \int_{0}^{\Delta^+} \Delta z \left[(m_{w,h})''(\Delta z) g^h(\Delta z) - m_{w,h}(\Delta z)(g^h)''(\Delta z) \right] \mathrm{d}\,\Delta z \\
&=^{(2)} \Delta z \left[(m_{w,h})'(\Delta z) g^h(\Delta z) - m_{w,h}(\Delta z)(g^h)'(\Delta z) \right]\Big|_{-\Delta^-}^{0} + \Delta z \left[(m_{w,h})'(\Delta z) g^h(\Delta z) - m_{w,h}(\Delta z)(g^h)'(\Delta z) \right]\Big|_{0}^{\Delta^+} \\
&\cdots - \left[\int_{-\Delta^-}^{0} \left[(m_{w,h})'(\Delta z) g^h(\Delta z) - m_{w,h}(\Delta z)(g^h)'(\Delta z) \right] \mathrm{d}\,\Delta z + \int_{0}^{\Delta^+} \left[(m_{w,h})'(\Delta z) g^h(\Delta z) - m_{w,h}(\Delta z)(g^h)'(\Delta z) \right] \mathrm{d}\,\Delta z \right] \\
&=^{(3)} \underbrace{\Delta z \left[(m_{w,h})'(\Delta z) g^h(\Delta z) - m_{w,h}(\Delta z)(g^h)'(\Delta z) \right]\Big|_{\Delta^-}^{\Delta^+}}_{= -m_{w,h}(0)\,\Delta z (g^h)'(\Delta z)|_{\Delta^-}^{\Delta^+}} \\
&\cdots - \left[\int_{-\Delta^-}^{0} \left[(m_{w,h})'(\Delta z) g^h(\Delta z) - m_{w,h}(\Delta z)(g^h)'(\Delta z) \right] \mathrm{d}\,\Delta z + \int_{0}^{\Delta^+} \left[(m_{w,h})'(\Delta z) g^h(\Delta z) - m_{w,h}(\Delta z)(g^h)'(\Delta z) \right] \mathrm{d}\,\Delta z \right] \\
&=^{(4)} -m_{w,h}(0)\ \Delta z (g^h)'(\Delta z)\Big|_{\Delta^-}^{\Delta^+} - \int_{\Delta^-}^{\Delta^+} (m_{w,h})'(\Delta z) g^h(\Delta z)\,\mathrm{d}\,\Delta z + \int_{\Delta^-}^{\Delta^+} m_{w,h}(\Delta z)(g^h)'(\Delta z)\,\mathrm{d}\,\Delta z \\
&=^{(5)} -m_{w,h}(0)\ \Delta z (g^h)'(\Delta z)\Big|_{\Delta^-}^{\Delta^+} - \left[\underbrace{m_{w,h}(\Delta z) g^h(\Delta z)\Big|_{\Delta^-}^{\Delta^+}}_{=0} - \int_{\Delta^-}^{\Delta^+} m_{w,h}(\Delta z)(g^h)'(\Delta z)\,\mathrm{d}\,\Delta z \right] \\
&+ \int_{\Delta^-}^{\Delta^+} m_{w,h}(\Delta z)(g^h)'(\Delta z)\,\mathrm{d}\,\Delta z \\
&= -m_{w,h}(0)\ \Delta z (g^h)'(\Delta z)\Big|_{\Delta^-}^{\Delta^+} + 2 \int_{\Delta^-}^{\Delta^+} m_{w,h}(\Delta z)(g^h)'(\Delta z)\,\mathrm{d}\,\Delta z.
\end{aligned}
$$

Here, (1) divides the integral at the discontinuity point; (2) uses the equality $(m_{w,h})''(\Delta z) g^h(\Delta z) - m_{w,h}(\Delta z)(g^h)''(\Delta z) =$

$\frac{\mathrm{d}\left[(m_{w,h})'(\Delta z) g^h(\Delta z) - m_{w,h}(\Delta z)(g^h)'(\Delta z)\right]}{\mathrm{d}\,\Delta z}$ and integrates by parts; (3) uses conditions (E.12) and (E.7); and (4)-(5) integrate by parts and operate.

Finally, for T_3

$$
\begin{aligned}
T_3 &= \int_{-\Delta-}^{\Delta^+} \Delta z \left(\gamma (g^h)'(\Delta z) + \frac{\sigma^2}{2} (g^h)''(\Delta z) \right) \mathrm{d}\,\Delta z \\
&= \gamma \left[\int_{-\Delta^-}^{0} \Delta z (g^h)'(\Delta z)\, \mathrm{d}\,\Delta z + \int_{0}^{\Delta^+} \Delta z (g^h)'(\Delta z)\, \mathrm{d}\,\Delta z \right] + \frac{\sigma^2}{2} \left[\int_{\Delta^-}^{0} \Delta z (g^h)''(\Delta z)\, \mathrm{d}\,\Delta z + \int_{0}^{\Delta^+} \Delta z (g^h)''(\Delta z)\, \mathrm{d}\,\Delta z \right] \\
&= \gamma \left[\underbrace{\Delta z g^h(\Delta z)\Big|_{-\Delta^-}^{0} + \Delta z g^h(\Delta z)\Big|_{0}^{\Delta^+}}_{=0} - \underbrace{\int_{-\Delta^-}^{\Delta^+} g^h(\Delta z)\, \mathrm{d}\,\Delta z}_{=\mathcal{E}_{ss}} \right] \\
&\cdots + \frac{\sigma^2}{2} \left[\Delta z (g^h)'(\Delta z)\Big|_{-\Delta^-}^{0} + \Delta z (g^h)'(\Delta z)\Big|_{0}^{\Delta^+} - \int_{-\Delta^-}^{\Delta^+} (g^h)'(\Delta z)\, \mathrm{d}\,\Delta z \right] \\
&= -\gamma \mathcal{E}_{ss} + \frac{\sigma^2}{2} \left[\Delta z (g^h)'(\Delta z)\Big|_{\Delta^-}^{\Delta^+} - \underbrace{g^h(\Delta z)\Big|_{\Delta^-}^{\Delta^+}}_{=0} \right] \\
&= -\gamma \mathcal{E}_{ss} + \frac{\sigma^2}{2} \left[\Delta z (g^h)'(\Delta z)\Big|_{\Delta^-}^{\Delta^+} \right].
\end{aligned}
$$

Combining all these results

$$
\begin{aligned}
0 &= \mathbb{V}ar_h[\Delta z] - \gamma T_1 + \frac{\sigma^2}{2} T_2 + m_{w,h}(0) T_3 \\
0 &= \mathbb{V}ar_h[\Delta z] - \gamma m_{w,h}(0)(1 - \mathcal{E}_{ss}) \\
&\quad + \frac{\sigma^2}{2} \left(-m_{w,h}(0)\ \Delta z (g^h)'(\Delta z)\Big|_{\Delta^-}^{\Delta^+} + 2 \int_{\Delta^-}^{\Delta^+} m_{w,h}(\Delta z)(g^h)'(\Delta z)\, \mathrm{d}\,\Delta z \right) \\
&\quad + m_{w,h}(0) \left(-\gamma \mathcal{E}_{ss} + \frac{\sigma^2}{2} \left[\Delta z (g^h)'(\Delta z)\Big|_{\Delta^-}^{\Delta^+} \right] \right) \\
0 &= \mathbb{V}ar_h[\Delta z] - \gamma m_{w,h}(0) + \sigma^2 \int_{\Delta^-}^{\Delta^+} m_{w,h}(\Delta z)(g^h)'(\Delta z)\, \mathrm{d}\,\Delta z,
\end{aligned}
$$

which implies

$$
\begin{aligned}
\int_{-\Delta^-}^{\Delta^+} m_{w,h}(\Delta z)(g^h)'(\Delta z) &= \frac{\gamma m_{w,h}(0) - \mathbb{V}ar_h[\Delta z]}{\sigma^2} \\
&= -\frac{\mathbb{C}ov_h[\Delta z + \gamma a, \Delta z]}{\sigma^2}.
\end{aligned}
$$

F Proofs for Section 4: Identifying the Model Based on Labor Market Microdata

Characterizing the Equilibrium Distributions of Cumulative Productivity Shocks $g^h(\Delta z)$ and $g^u(\Delta z)$. The equilibrium policies $(\hat{w}^-, \hat{w}^*, \hat{w}^+)$ together with the stochastic process guiding Δz and the exogenous job separation rate, determine the equilibrium distributions of cumulative productivity shocks $g^h(\Delta z)$ and $g^u(\Delta z)$. Due to the law of motion for Δz being independent of the worker's employment state, the Kolmogorov forward equations (KFEs) for employed and unemployed workers are

$$\delta g^h(\Delta z) = (\gamma+\pi)(g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z) \quad \forall \Delta z \in (-\Delta^-, \Delta^+)\backslash\{0\}, \tag{F.1}$$

$$f(\hat{w}^*)g^u(\Delta z) = (\gamma+\pi)(g^u)'(\Delta z) + \frac{\sigma^2}{2}(g^u)''(\Delta z) \quad \forall \Delta z \in \mathbb{R}\backslash\{0\}. \tag{F.2}$$

Here, δ is the exogenous exit rate of employed workers, and $f(\hat{w}^*)$ is the job finding rate of unemployed workers. Since the entry state for a newly employed or unemployed worker is $\Delta z = 0$, the KFEs (F.1)–(F.2) do not hold at this point, but $g^h(\cdot)$ and $g^u(\cdot)$ must be continuous there.

The boundary conditions impose a zero measure of workers at the borders of the support,

$$g^h(-\Delta^-) = g^h(\Delta^+) = 0,$$
$$\lim_{\Delta z \to -\infty} g^u(\Delta z) = \lim_{\Delta z \to \infty} g^u(\Delta z) = 0.$$

These distributions must also be consistent with (i) a unit measure of workers and (ii) a flow balance equation implying constant steady-state employment:

$$1 = \int_{-\infty}^{\infty} g^u(\Delta z)\, \mathrm{d}\Delta z + \int_{-\Delta^-}^{\Delta^+} g^h(\Delta z)\, \mathrm{d}\Delta z, \tag{F.3}$$

$$\underbrace{f(\hat{w}^*)(1-\mathcal{E})}_{u\text{-to-}h \text{ flows}} = \underbrace{\delta\mathcal{E} + \frac{\sigma^2}{2}\left[\lim_{\Delta z \downarrow -\Delta^-}(g^h)'(\Delta z) - \lim_{\Delta z \uparrow \Delta^+}(g^h)'(\Delta z)\right]}_{h\text{-to-}u \text{ flows}}. \tag{F.4}$$

In equation (F.3), the unit measure of workers is composed of $\int_{-\infty}^{\infty} g^u(\Delta z)\,\mathrm{d}\Delta z = 1-\mathcal{E}$ unemployed and $\int_{-\Delta^-}^{\Delta^+} g^h(\Delta z)\,\mathrm{d}\Delta z = \mathcal{E}$ employed workers. In equation (F.4), the mass of u-to$h$ flows is $f(\hat{w}^*)(1-\mathcal{E})$, while the mass of h-to-u flows is $\delta\mathcal{E} + \frac{\sigma^2}{2}[\lim_{\Delta z \downarrow -\Delta^-}(g^h)'(\Delta z) - \lim_{\Delta z \uparrow \Delta^+}(g^h)'(\Delta z)]$—i.e., the sum of exogenous and endogenous job separations.

To summarize, equations (F.1)–(F.4), together with the continuity of $g^u(\Delta z)$ and $g^h(\Delta z)$ at $\Delta z = 0$, constitute the equilibrium conditions for the steady-state distributions of cumulative productivity shocks.

F.1 Proof of Proposition 10

We divide the proof of Proposition 10 into three steps. Proposition F.1 identifies the parameters of the stochastic process of the wage-to-revenue productivity ratio Δz. Proposition F.2 recovers the distribution of cumulative productivity shocks conditional on job transitions $\bar{G}^h(\Delta z)$. Finally, Proposition F.3 recovers the distribution of Δz among employed workers $G^h(\Delta z)$.

Proposition F.1. *The drift $(\gamma+\pi)$ and volatility σ of the stochastic process guiding cumulative productivity shocks can be recovered from* *the data with*

$$\gamma + \pi = \frac{\mathbb{E}_{\mathcal{D}}[\Delta w]}{\mathbb{E}_{\mathcal{D}}[\tau]},$$
$$\sigma^2 = \frac{\mathbb{E}_{\mathcal{D}}[(\Delta w - (\gamma+\pi)\tau)^2]}{\mathbb{E}_{\mathcal{D}}[\tau]}$$

where $\tau := \tau^m + \tau^u$.

Proof. From the law of motion $\mathrm{d}z_t = (\gamma+\pi)\,\mathrm{d}t + \sigma\,\mathrm{d}\mathcal{W}_t^z$ and the fact that at the beginning of a new job spell $w_{t_0} - z_{t_0} = \hat{w}^*$, we have that

$$\Delta w = -\Delta z_\tau = (\gamma+\pi)\tau + \sigma \mathcal{W}_\tau^z. \tag{F.5}$$

Drift: Taking expectation on both sides conditional on a h-to-u-to-h transition, we have that $\sigma\mathbb{E}[\mathcal{W}_\tau^z] = \mathbb{E}_{\mathcal{D}}[\Delta w] - (\gamma+\pi)\mathbb{E}_{\mathcal{D}}[\tau]$. Since $\mathcal{W}_t^z$ is a martingale, by Doob's Optional Stopping Theorem (OST) $\mathcal{W}_\tau^z$ is also a martingale, and $E[\mathcal{W}_\tau^z] = \mathbb{E}[\mathcal{W}_0^z] = 0$. Thus,

$$(\gamma+\pi) = \frac{\mathbb{E}_{\mathcal{D}}[\Delta w]}{\mathbb{E}_{\mathcal{D}}[\tau]}.$$

Idiosyncratic volatility: Let us define $Y_t = (\Delta z_t + (\gamma+\pi)t)^2$. We apply Itô's Lemma to Y_t and obtain

$$\mathrm{d}Y_t = 2\left(\Delta z_t + (\gamma+\pi)t\right)\left(\mathrm{d}\Delta z_t + (\gamma+\pi)\,\mathrm{d}t\right) + \frac{1}{2}2(\mathrm{d}\Delta z_t)^2 = 2\sigma\left(\Delta z_t + (\gamma+\pi)t\right)\mathrm{d}\mathcal{W}_t^z + \sigma^2\,\mathrm{d}t$$

Integrating the previous equation between 0 and τ and using condition (F.5), we obtain

$$(\Delta w - (\gamma+\pi)\tau)^2 = 2\sigma \int_0^\tau \left(\Delta z_t + (\gamma+\pi)t\right)\mathrm{d}\mathcal{W}_t^z + \sigma^2\tau.$$

Since $\int_0^t \left(\Delta z_t + (\gamma+\pi)t\right)\mathrm{d}\mathcal{W}_t^z$ is a martingale, by the OST, $\int_0^\tau \left(\Delta z_t + (\gamma+\pi)t\right)\mathrm{d}\mathcal{W}_t^z$ is a martingale and $\mathbb{E}[\int_0^\tau \left(\Delta z_t + (\gamma+\pi)t\right)\mathrm{d}\mathcal{W}_t^z] = 0$. Thus,

$$\mathbb{E}_{\mathcal{D}}[(\Delta w - (\gamma+\pi)\tau)^2] = 2\sigma\mathbb{E}\left[\int_0^\tau \left(\Delta z_t + (\gamma+\pi)t\right)\mathrm{d}\mathcal{W}_t^z,\right] + \sigma^2\mathbb{E}_{\mathcal{D}}[\tau] = \sigma^2\mathbb{E}_{\mathcal{D}}[\tau]$$

which completes the proof of Proposition F.1. □

Proposition F.2. *The cumulative distribution of* Δz *conditional on a job separation is given by*

$$\bar{G}^h(\Delta z) = \frac{\sigma^2}{2f(\hat{w}^*)}\frac{\mathrm{d}l^w(-\Delta z)}{\mathrm{d}z} - \frac{(\gamma+\pi)}{f(\hat{w}^*)}l^w(-\Delta z) - \left[1 - L^w(-\Delta z)\right]. \tag{F.6}$$

where $L^w(\Delta w)$ *denotes the cumulative distribution function (CDF) corresponding to the marginal distribution* $l^w(\Delta w)$.

Proof. The objective in this proof is to use the non-differentiability of the distribution of $\bar{g}_s(\Delta z)$ for $s = \{h, u\}$ at $\Delta z = 0$ to express the distribution of Δz conditional on a separation. Observe that

$$\begin{aligned} L^w(a) &= Pr^{l^w}(\Delta w \le a) \\ &=^{(1)} Pr^{\bar{G}^h,\bar{G}^u}(-(\Delta z^h + \Delta z^u) \le a) \\ &=^{(2)} Pr^{\bar{G}^h,\bar{G}^u}(\Delta z^h + \Delta z^u \ge -a) \\ &=^{(3)} 1 - Pr(\Delta z^h + \Delta z^u \le -a) \end{aligned}$$

$$=^{(4)} 1-\int_{-\infty}^{\infty} \bar{G}^h(-a-y)\bar{g}^u(y)\,\mathrm{d}y.$$

Here, in step (1) we use the definition of Δw, and steps (2) to (4) operate and use the independence of $\bar{G}^h(\cdot)$ and $\bar{g}^u(\cdot)$. From Proposition G.1, we have

$$\bar{g}^u(\Delta z) = \mathcal{G}_u \begin{cases} e^{\beta_2(f(\hat{w}^*))\Delta z} & \text{if } \Delta z \in (-\infty, 0] \\ e^{\beta_1(f(\hat{w}^*))\Delta z} & \text{if } \Delta z \in [0, \infty) \end{cases}$$

and

$$L^w(\Delta w) = 1 - C_1(\Delta w) - C_2(\Delta w), \tag{F.7}$$

where

$$C_1(\Delta w) = \mathcal{G}_u \int_0^{\infty} \bar{G}^h(-\Delta w - u)e^{\beta_1(f(\hat{w}^*))u}\,\mathrm{d}u,$$
$$C_2(\Delta w) = \mathcal{G}_u \int_{-\infty}^{0} \bar{G}^h(-\Delta w - u)e^{\beta_2(f(\hat{w}^*))u}\,\mathrm{d}u.$$

Departing from $L^w(\Delta w) = 1 - \int_{-\infty}^{\infty} \bar{G}^h(-\Delta w - y)\bar{g}^u(y)\,\mathrm{d}y$ and doing the change of variable $x = -\Delta w - y$ with $dx = -dy$, we obtain

$$L^w(\Delta w) = 1 - \int_{-\infty}^{\infty} \bar{G}^h(x)\bar{g}^u(-\Delta w - x)\,\mathrm{d}x.$$

Taking the derivative on both sides with respect to Δw we obtain

$$l^w(\Delta w) = \int_{-\infty}^{\infty} \bar{G}^h(x)\left(\bar{g}^u\right)'(-\Delta w - x)\,\mathrm{d}x.$$

Reverting the change of variables and using the fact that $\bar{g}^u(-\Delta w - x)$ is non-differentiable at 0, we obtain

$$\begin{aligned} l^w(\Delta w) &= \int_{-\infty}^{\infty} \bar{G}^h(-\Delta w - u)(\bar{g}^u)'(u)\,\mathrm{d}u \\ &= \int_{-\infty}^{0} \bar{G}^h(-\Delta w - u)\mathcal{G}_u\beta_2(f(\hat{w}^*))e^{\beta_2(f(\hat{w}^*))u}\,\mathrm{d}u + \int_0^{\infty} \bar{G}^h(-\Delta w - u)\mathcal{G}_u\beta_1(f(\hat{w}^*))e^{\beta_1(f(\hat{w}^*))u}\,\mathrm{d}u \\ &= \beta_1(f(\hat{w}^*))C_1(\Delta w) + \beta_2(f(\hat{w}^*))C_2(\Delta w). \end{aligned}$$

Thus,

$$l^w(\Delta w) = \beta_1(f(\hat{w}^*))C_1(\Delta w) + \beta_2(f(\hat{w}^*))C_2(\Delta w). \tag{F.8}$$

To obtain the last condition, observe that

$$\begin{aligned} C_1(\Delta w) &= \int_0^{\infty} \bar{G}^h(-\Delta w - u)\mathcal{G}_u e^{\beta_1(f(\hat{w}^*))u}\,\mathrm{d}u, \\ &= -\mathcal{G}_u \int_{-\Delta w}^{-\infty} \bar{G}^h(y)e^{\beta_1(f(\hat{w}^*))(-\Delta w - y)}\,\mathrm{d}y, \\ &= \mathcal{G}_u \int_{-\infty}^{-\Delta w} \bar{G}^h(y)e^{\beta_1(f(\hat{w}^*))(-\Delta w - y)}\,\mathrm{d}y. \end{aligned}$$

and

$$C_2(\Delta w) = \int_{-\infty}^{0} \bar{G}^h(-\Delta w - u)\mathcal{G}_u e^{\beta_2(f(\hat{w}^*))u}\,\mathrm{d}u.$$

$$= -\mathcal{G}_u \int_{\infty}^{-\Delta w} \bar{G}^h(y) e^{\beta_2(f(\hat{w}^*))(-\Delta w - y)} \, \mathrm{d}y,$$
$$= \mathcal{G}_u \int_{-\Delta w}^{\infty} \bar{G}^h(y) e^{\beta_2(f(\hat{w}^*))(-\Delta w - y)} \, \mathrm{d}y.$$

Taking the derivative with respect to Δw and using the Leibniz rule, we obtain

$$C_1'(\Delta w) = -\mathcal{G}_u \bar{G}^h(-\Delta w) - \beta_1(f(\hat{w}^*))\mathcal{G}_u \int_{-\infty}^{-\Delta w} \bar{G}^h(y) e^{\beta_1(f(\hat{w}^*))(-\Delta w - y)} \, \mathrm{d}y,$$
$$= -\mathcal{G}_u \bar{G}^h(-\Delta w) - \beta_1(f(\hat{w}^*)) C_1(\Delta w) \tag{F.9}$$
$$C_2'(\Delta w) = \mathcal{G}_u \bar{G}^h(-\Delta w) - \beta_2(f(\hat{w}^*)) C_2(\Delta w). \tag{F.10}$$

Taking derivative of (F.8),

$$(l^w)'(\Delta w) = \beta_1(f(\hat{w}^*)) C_1'(\Delta w) + \beta_2(f(\hat{w}^*)) C_2'(\Delta w)$$

and using conditions (F.9) and (F.10),

$$(l^w)'(\Delta w) = \bar{G}^h(-\Delta w)\mathcal{G}_u[\beta_2(f(\hat{w}^*)) - \beta_1(f(\hat{w}^*))] - \beta_1(f(\hat{w}^*))^2 C_1(\Delta w) - \beta_2(f(\hat{w}^*))^2 C_2(\Delta w). \tag{F.11}$$

Equations (F.7), (F.8), and (F.11) provide the following system of three functional equations with three unknowns

$$1 - L^w(\Delta w) = C_1(\Delta w) + C_2(\Delta w),$$
$$l^w(\Delta w) = \beta_1(f(\hat{w}^*)) C_1(\Delta w) + \beta_2(f(\hat{w}^*)) C_2(\Delta w),$$
$$(l^w)'(\Delta w) = \bar{G}^h(-\Delta w)\mathcal{G}_u[\beta_2(f(\hat{w}^*)) - \beta_1(f(\hat{w}^*))] - \beta_1(f(\hat{w}^*))^2 C_1(\Delta w) - \beta_2(f(\hat{w}^*))^2 C_2(\Delta w).$$

Operating over the system of functional equations

$$(l^w)'(\Delta w) + [\beta_2(f(\hat{w}^*)) + \beta_1(f(\hat{w}^*))]\, l^w(\Delta w) + \beta_1(f(\hat{w}^*))\beta_2(f(\hat{w}^*))\, [1 - L^w(\Delta w)] = \bar{G}^h(-\Delta w)\mathcal{G}_u[\beta_2(f(\hat{w}^*)) - \beta_1(f(\hat{w}^*))],$$

with

$$\mathcal{G}_u = \left(\beta_2(f(\hat{w}^*))^{-1} - \beta_1(f(\hat{w}^*))^{-1}\right)^{-1}$$
$$\beta_1(f(\hat{w}^*)) = \frac{-(\gamma + \pi) - \sqrt{(\gamma + \pi)^2 + 2\sigma^2 f(\hat{w}^*)}}{\sigma^2},$$
$$\beta_2(f(\hat{w}^*)) = \frac{-(\gamma + \pi) + \sqrt{(\gamma + \pi)^2 + 2\sigma^2 f(\hat{w}^*)}}{\sigma^2}.$$

Then,

$$\mathcal{G}_u[\beta_2(f(\hat{w}^*)) - \beta_1(f(\hat{w}^*))] = \frac{\beta_2(f(\hat{w}^*)) - \beta_1(f(\hat{w}^*))}{\beta_2(f(\hat{w}^*))^{-1} - \beta_1(f(\hat{w}^*))^{-1}},$$
$$= -\beta_1(f(\hat{w}^*))\beta_2(f(\hat{w}^*)),$$
$$= -\left(\frac{-(\gamma + \pi) - \sqrt{(\gamma + \pi)^2 + 2\sigma^2 f(\hat{w}^*)}}{\sigma^2}\right)\left(\frac{-(\gamma + \pi) + \sqrt{(\gamma + \pi)^2 + 2\sigma^2 f(\hat{w}^*)}}{\sigma^2}\right)$$
$$= \frac{2f(\hat{w}^*)}{\sigma^2}$$

$$
\begin{aligned}
\frac{\beta_2(f(\hat{w}^*))+\beta_1(f(\hat{w}^*))}{\mathcal{G}_u[\beta_2(f(\hat{w}^*))-\beta_1(f(\hat{w}^*))]} &= \frac{\beta_2(f(\hat{w}^*))+\beta_1(f(\hat{w}^*))}{\beta_1(f(\hat{w}^*))\beta_2(f(\hat{w}^*))}, \\
&= \frac{\left(\frac{-(\gamma+\pi)-\sqrt{(\gamma+\pi)^2+2\sigma^2 f(\hat{w}^*)}-(\gamma+\pi)+\sqrt{(\gamma+\pi)^2+2\sigma^2 f(\hat{w}^*)}}{\sigma^2}\right)}{\frac{2f(\hat{w}^*)}{\sigma^2}} \\
&= -\frac{(\gamma+\pi)}{f(\hat{w}^*)}. \\
\frac{\beta_1(f(\hat{w}^*))\beta_2(f(\hat{w}^*))}{\mathcal{G}_u[\beta_2(f(\hat{w}^*))-\beta_1(f(\hat{w}^*))]} &= -1.
\end{aligned}
$$

Therefore, the differential equation is given by (F.6).

□

Proposition F.3. *If $(\gamma+\pi)=0$, the distribution of cumulative productivity shocks $g^h(\Delta z)$ is given by*

$$
g^h(\Delta z) = s\mathcal{E}\left[\int_{-\Delta^-}^{\Delta z} \frac{2(\Delta z - y)}{\sigma^2}\bar{g}^h(y)\,\mathrm{d}y + \bar{G}^h(-\Delta^-)\frac{2(\Delta z+\Delta^-)}{\sigma^2}\right].
$$

If $(\gamma+\pi)\neq 0$, the distribution of cumulative productivity shocks $g^h(\Delta z)$ is given by

$$
g^h(\Delta z) = \frac{s\mathcal{E}}{(\gamma+\pi)}\left[\int_{-\Delta^-}^{\Delta z}\left(1-e^{\frac{2(\gamma+\pi)}{\sigma^2}(y-\Delta z)}\right)\bar{g}^h(y)\,\mathrm{d}y + \bar{G}^h(-\Delta^-)\left[1-e^{-\frac{2(\gamma+\pi)}{\sigma^2}(\Delta z+\Delta^-)}\right]\right].
$$

Proof. During employment, the distribution of cumulative productivity shocks satisfies the following KFE and the boundary conditions

$$
\begin{aligned}
\delta g^h(\Delta z) &= (\gamma+\pi)(g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z) \quad \forall \Delta z \in (-\Delta^-,\Delta^+)/\{0\} \\
g^h(-\Delta^-) &= g^h(\Delta^+) = 0, G^h(\Delta^+) = \mathcal{E}, \\
g^h(\Delta z) &\in \mathbb{C}.
\end{aligned}
$$

The distribution of cumulative productivity shocks conditional on a job separation satisfies

$$
\bar{G}^h(\Delta z) = \begin{cases} 1 & \text{if } \Delta z \in [\Delta_+,\infty) \\ \frac{1}{s\mathcal{E}}\left[\frac{\sigma^2}{2}\lim_{\Delta z \downarrow -\Delta^-}(g^h)'(\Delta z) + \delta\int_{-\Delta^-}^{\Delta z} g^h(x)\,\mathrm{d}x\right] & \text{if } \Delta z \in [-\Delta^-,\Delta^+) \\ 0 & \text{if } \Delta z \in (-\infty,-\Delta^-). \end{cases}
$$

Combining these two conditions, we obtain

$$
\begin{aligned}
s\mathcal{E}\bar{g}^h(\Delta z) &= (\gamma+\pi)(g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z) \quad \forall \Delta z \in (-\Delta^-,\Delta^+)/\{0\} \\
g^h(-\Delta^-) &= g^h(\Delta^+) = 0, G^h(\Delta^+) = \mathcal{E}.
\end{aligned}
$$

Multiplying both sides of the first equation by $e^{\frac{2(\gamma+\pi)}{\sigma^2}\Delta z}$ we get

$$
\begin{aligned}
s\mathcal{E}e^{\frac{2(\gamma+\pi)}{\sigma^2}\Delta z}\bar{g}^h(\Delta z) &= (\gamma+\pi)e^{\frac{2(\gamma+\pi)}{\sigma^2}\Delta z}(g^h)'(\Delta z) + \frac{\sigma^2}{2}e^{\frac{2(\gamma+\pi)}{\sigma^2}\Delta z}(g^h)''(\Delta z) \quad \forall \Delta z \in (-\Delta^-,\Delta^+)/\{0\} \\
&= \frac{\sigma^2}{2}\frac{\mathrm{d}(e^{\frac{2(\gamma+\pi)}{\sigma^2}\Delta z}(g^h)'(\Delta z))}{\mathrm{d}\Delta z}.
\end{aligned}
$$

Integrating both sides from $-\Delta^-$ to Δz, we obtain

$$s\mathcal{E}\int_{-\Delta^-}^{\Delta z} e^{\frac{2(\gamma+\pi)}{\sigma^2}x}\bar{g}^h(x)\,\mathrm{d}x = \frac{\sigma^2}{2}\left[e^{\frac{2(\gamma+\pi)}{\sigma^2}\Delta z}(g^h)'(\Delta z) - \lim_{x\downarrow -\Delta^-} e^{\frac{2(\gamma+\pi)}{\sigma^2}x}(g^h)'(x)\right],$$
$$= \frac{\sigma^2}{2}e^{\frac{2(\gamma+\pi)}{\sigma^2}\Delta z}(g^h)'(\Delta z) - s\mathcal{E}e^{-\frac{2(\gamma+\pi)}{\sigma^2}\Delta^-}\bar{G}^h(-\Delta^-),$$

where the last equation uses the value of $\bar{G}^h(\Delta z)$ evaluated at $\Delta z = -\Delta^-$. Solving for $(g^h)'(\Delta z)$,

$$\frac{2s\mathcal{E}}{\sigma^2}\left[\int_{-\Delta^-}^{\Delta z} e^{\frac{2(\gamma+\pi)}{\sigma^2}(x-\Delta z)}\bar{g}^h(x)\,\mathrm{d}x + e^{-\frac{2(\gamma+\pi)}{\sigma^2}(\Delta^-+\Delta z)}\bar{G}^h(-\Delta^-)\right] = (g^h)'(\Delta z).$$

Integrating this equation from $-\Delta^-$ to Δz, we obtain

$$\begin{aligned}
\int_{-\Delta^-}^{\Delta z}(g^h)'(x)\,\mathrm{d}x &= g^h(\Delta z) - \underbrace{g^h(-\Delta^-)}_{=0} \\
&= \frac{2s\mathcal{E}}{\sigma^2}\int_{-\Delta^-}^{\Delta z}\left[\int_{-\Delta^-}^{x} e^{\frac{2(\gamma+\pi)}{\sigma^2}(y-x)}\bar{g}^h(y)\,\mathrm{d}y + e^{-\frac{2(\gamma+\pi)}{\sigma^2}(\Delta^-+x)}\bar{G}^h(-\Delta^-)\right]\mathrm{d}x \\
&= \frac{2s\mathcal{E}}{\sigma^2}\left[\int_{-\Delta^-}^{\Delta z}\int_{-\Delta^-}^{x} e^{\frac{2(\gamma+\pi)}{\sigma^2}(y-x)}\bar{g}^h(y)\,\mathrm{d}y\,\mathrm{d}x + \frac{\sigma^2}{2(\gamma+\pi)}\bar{G}^h(-\Delta^-)\left[1-e^{-\frac{2(\gamma+\pi)}{\sigma^2}(\Delta z+\Delta^-)}\right]\right] \\
&= \frac{2s\mathcal{E}}{\sigma^2}\left[\int_{-\Delta^-}^{\Delta z}\int_{y}^{\Delta z} e^{\frac{2(\gamma+\pi)}{\sigma^2}(y-x)}\bar{g}^h(y)\,\mathrm{d}x\,\mathrm{d}y + \frac{\sigma^2}{2(\gamma+\pi)}\bar{G}^h(-\Delta^-)\left[1-e^{-\frac{2(\gamma+\pi)}{\sigma^2}(\Delta z+\Delta^-)}\right]\right] \\
&= \frac{2s\mathcal{E}}{\sigma^2}\left[\int_{-\Delta^-}^{\Delta z}\underbrace{\left[\int_{y}^{\Delta z} e^{\frac{2(\gamma+\pi)}{\sigma^2}(y-x)}\,\mathrm{d}x\right]}_{=\frac{\sigma^2}{2(\gamma+\pi)}\left(1-e^{\frac{2(\gamma+\pi)}{\sigma^2}(y-\Delta z)}\right)}\bar{g}^h(y)\,\mathrm{d}y + \frac{\sigma^2}{2(\gamma+\pi)}\bar{G}^h(-\Delta^-)\left[1-e^{-\frac{2(\gamma+\pi)}{\sigma^2}(\Delta z+\Delta^-)}\right]\right] \\
&= \frac{s\mathcal{E}}{(\gamma+\pi)}\left[\int_{-\Delta^-}^{\Delta z}\left(1-e^{\frac{2(\gamma+\pi)}{\sigma^2}(y-\Delta z)}\right)\bar{g}^h(y)\,\mathrm{d}y + \bar{G}^h(-\Delta^-)\left[1-e^{-\frac{2(\gamma+\pi)}{\sigma^2}(\Delta z+\Delta^-)}\right]\right].
\end{aligned}$$

Taking the limit as $(\gamma+\pi)\downarrow 0$

$$\begin{aligned}
g^h(\Delta z) &= \lim_{(\gamma+\pi)\to 0}\frac{s\mathcal{E}}{(\gamma+\pi)}\left[\int_{-\Delta^-}^{\Delta z}\left(1-e^{\frac{2(\gamma+\pi)}{\sigma^2}(y-\Delta z)}\right)\bar{g}^h(y)\,\mathrm{d}y + \bar{G}^h(-\Delta^-)\left[1-e^{-\frac{2(\gamma+\pi)}{\sigma^2}(\Delta z+\Delta^-)}\right]\right] \\
&= s\mathcal{E}\left[\int_{-\Delta^-}^{\Delta z}\lim_{(\gamma+\pi)\to 0}\frac{1-e^{\frac{2(\gamma+\pi)}{\sigma^2}(y-\Delta z)}}{(\gamma+\pi)}\bar{g}^h(y)\,\mathrm{d}y + \bar{G}^h(-\Delta^-)\lim_{(\gamma+\pi)\to 0}\frac{1-e^{-\frac{2(\gamma+\pi)}{\sigma^2}(\Delta z+\Delta^-)}}{(\gamma+\pi)}\right] \\
&= s\mathcal{E}\left[\int_{-\Delta^-}^{\Delta z}\frac{2(\Delta z-y)}{\sigma^2}\bar{g}^h(y)\,\mathrm{d}y + \bar{G}^h(-\Delta^-)\frac{2(\Delta z+\Delta^-)}{\sigma^2}\right].
\end{aligned}$$

□

G Additional Results for Section 4: Identifying the Model Based on Labor Market Microdata

G.1 Characterization of $g^h(\Delta z)$ and $g^u(\Delta z)$

Proposition G.1. *Assume $\delta > 0$. Then, $g^h(\Delta z)$ and $g^u(\Delta z)$ are given by*

$$g^h(\Delta z) = \mathcal{E}\mathcal{G}_h \begin{cases} \frac{e^{\beta_1(\delta)(\Delta z+\Delta^-)} - e^{\beta_2(\delta)(\Delta z+\Delta^-)}}{e^{\beta_1(\delta)\Delta^-} - e^{\beta_2(\delta)\Delta^-}} & \text{if } \Delta z \in (-\Delta^-, 0] \\ \frac{e^{\beta_1(\delta)(\Delta z-\Delta^+)} - e^{\beta_2(\delta)(\Delta z-\Delta^+)}}{e^{-\beta_1(\delta)\Delta^+} - e^{-\beta_2(\delta)\Delta^+}} & \text{if } \Delta z \in [0, \Delta^+) \end{cases}$$

$$g^u(\Delta z) = (1-\mathcal{E})\mathcal{G}_u \begin{cases} e^{\beta_2(f(\hat{w}^*))\Delta z} & \text{if } \Delta z \in (-\infty, 0] \\ e^{\beta_1(f(\hat{w}^*))\Delta z} & \text{if } \Delta z \in [0, \infty) \end{cases}$$

where

$$\beta_1(x) = \frac{-\gamma - \sqrt{\gamma^2 + 2\sigma^2 x}}{\sigma^2}, \beta_2(x) = \frac{-\gamma + \sqrt{\gamma^2 + 2\sigma^2 x}}{\sigma^2},$$

$$\mathcal{E} = \frac{f(\hat{w}^*)}{f(\hat{w}^*) + \delta + \frac{\sigma^2}{2}\mathcal{G}_h \left[\frac{\beta_1(\delta)-\beta_2(\delta)}{e^{\beta_1(\delta)\Delta^-} - e^{\beta_2(\delta)\Delta^-}} - \frac{\beta_1(\delta)-\beta_2(\delta)}{e^{-\beta_1(\delta)\Delta^+} - e^{-\beta_2(\delta)\Delta^+}} \right]},$$

$$\mathcal{G}_h = \left[\frac{\frac{e^{\beta_1(\delta)\Delta^-}-1}{\beta_1(\delta)} - \frac{e^{\beta_2(\delta)\Delta^-}-1}{\beta_2(\delta)}}{e^{\beta_1(\delta)\Delta^-} - e^{\beta_2(\delta)\Delta^-}} + \frac{\frac{1-e^{-\beta_1\Delta^+}}{\beta_1(\delta)} - \frac{1-e^{-\beta_2\Delta^+}}{\beta_2(\delta)}}{e^{-\beta_1(\delta)\Delta^+} - e^{-\beta_2(\delta)\Delta^+}} \right]^{-1},$$

$$\mathcal{G}_u = \left[-\beta_1(f(\hat{w}^*))^{-1} + \beta_2(f(\hat{w}^*))^{-1} \right]^{-1}.$$

Proof. Let us write the KFE and border conditions:

$$\delta g^h(\Delta z) = \gamma (g^h)'(\Delta z) + \frac{\sigma^2}{2}(g^h)''(\Delta z) \quad \forall \Delta z \in (-\Delta^-, \Delta^+)/\{0\} \tag{G.1}$$

$$g^h(-\Delta^-) = g^h(\Delta^+) = 0, \tag{G.2}$$

$$f(\hat{w}^*) g^u(\Delta z) = \gamma (g^u)'(\Delta z) + \frac{\sigma^2}{2}(g^u)''(\Delta z) \quad \forall \Delta z \in (-\infty, \infty)/\{0\}, \tag{G.3}$$

$$\lim_{\Delta z \to -\infty} g^u(\Delta z) = \lim_{\Delta z \to \infty} g^u(\Delta z) = 0, \tag{G.4}$$

$$1 = \int_{-\infty}^{\infty} g^u(\Delta z)\, d\Delta z + \int_{-\Delta^-}^{\Delta^+} g^h(\Delta z)\, d\Delta z, \tag{G.5}$$

$$f(\hat{w}^*)(1-\mathcal{E}) = \delta \mathcal{E} + \frac{\sigma^2}{2}\left[\lim_{\Delta z \downarrow -\Delta^-} (g^h)'(\Delta z) - \lim_{\Delta z \uparrow \Delta^+} (g^h)'(\Delta z) \right], \tag{G.6}$$

$$g^h(\Delta z), g^u(\Delta z) \in \mathbb{C}.$$

We guess and verify the proposed solution. Substituting the guess for $g^h(\Delta z)$ in (G.1) for $\Delta z < 0$, we have

$$0 = -\delta \mathcal{E}\mathcal{G}_h \frac{e^{\beta_1(\delta)(\Delta z+\Delta^-)}}{e^{\beta_1(\delta)\Delta^-} - e^{\beta_2(\delta)\Delta^-}} + \gamma\beta_1(\delta)\mathcal{E}\mathcal{G}_h \frac{e^{\beta_1(\delta)(\Delta z+\Delta^-)}}{e^{\beta_1(\delta)\Delta^-} - e^{\beta_2(\delta)\Delta^-}} + \mathcal{E}\mathcal{G}_h \frac{\sigma^2}{2}\beta_1(\delta)^2 \frac{e^{\beta_1(\delta)(\Delta z+\Delta^-)}}{e^{\beta_1(\delta)\Delta^-} - e^{\beta_2(\delta)\Delta^-}} \iff$$

$$0 = -\delta + \gamma\beta_1(\delta) + \frac{\sigma^2}{2}\beta_1(\delta)^2,$$

mutatis mutandis for the terms that include $\beta_2(\delta)$. Given the definition of $\beta_1(\delta)$, the guess satisfies (G.1). A similar argument applies when (G.1) is evaluated at $\Delta z > 0$. It is easy to verify that the boundary conditions (G.2) are satisfied and that $g^h(\Delta z)$ is continuous at $\Delta z = 0$. Following the same steps for $g^u(\Delta z)$, we verify conditions (G.3) and (G.4). Next, we verify condition (G.5):

$$
\begin{aligned}
&\int_{-\infty}^{\infty} g^u(\Delta z)\,\mathrm{d}\Delta z + \int_{-\Delta^-}^{\Delta^+} g^h(\Delta z)\,\mathrm{d}\Delta z \\
&= (1-\mathcal{E})\mathcal{G}_u \left[\int_{-\infty}^{0} e^{\beta_2(f(\hat{w}^*))\Delta z}\,\mathrm{d}\Delta z + \int_{0}^{\infty} e^{\beta_1(f(\hat{w}^*))\Delta z}\,\mathrm{d}\Delta z \right] + \dots \\
&\dots \mathcal{E}\mathcal{G}_h \left[\int_{-\Delta^-}^{0} \frac{e^{\beta_1(\delta)(\Delta z+\Delta^-)} - e^{\beta_2(\delta)(\Delta z+\Delta^-)}}{e^{\beta_1(\delta)\Delta^-} - e^{\beta_2(\delta)\Delta^-}}\,\mathrm{d}\Delta z + \int_{0}^{\Delta^+} \frac{e^{\beta_1(\delta)(\Delta z-\Delta^+)} - e^{\beta_2(\delta)(\Delta z-\Delta^+)}}{e^{-\beta_1(\delta)\Delta^+} - e^{-\beta_2(\delta)\Delta^+}}\,\mathrm{d}\Delta z \right] \\
&= (1-\mathcal{E})\mathcal{G}_u \left[\frac{1 - \lim_{\Delta z \to -\infty} e^{\beta_2(f(\hat{w}^*))\Delta z}}{\beta_2(f(\hat{w}^*))} + \frac{\lim_{\Delta z \to \infty} e^{\beta_1(f(\hat{w}^*))\Delta z} - 1}{\beta_1(f(\hat{w}^*))} \right] + \dots \\
&\dots \mathcal{E}\mathcal{G}_h \left[\frac{\frac{e^{\beta_1(\delta)\Delta^-}-1}{\beta_1(\delta)} - \frac{e^{\beta_2(\delta)\Delta^-}-1}{\beta_2(\delta)}}{e^{\beta_1(\delta)\Delta^-} - e^{\beta_2(\delta)\Delta^-}} + \frac{\frac{1-e^{-\beta_1\Delta^+}}{\beta_1(\delta)} - \frac{1-e^{-\beta_2\Delta^+}}{\beta_2(\delta)}}{e^{-\beta_1(\delta)\Delta^+} - e^{-\beta_2(\delta)\Delta^+}} \right] \\
&= (1-\mathcal{E}) + \mathcal{E} = 1.
\end{aligned}
$$

Finally, combining condition (G.6) with the definition of $g^h(\Delta z)$, the employment rate is

$$
\begin{aligned}
\mathcal{E} &= \frac{f(\hat{w}^*)}{f(\hat{w}^*) + \delta + \frac{\sigma^2}{2\mathcal{E}}\left[\lim_{\Delta z \downarrow -\Delta^-} (g^h)'(\Delta z) - \lim_{\Delta z \uparrow \Delta^+} (g^h)'(\Delta z)\right]}, \\
&= \frac{f(\hat{w}^*)}{f(\hat{w}^*) + \delta + \frac{\sigma^2}{2}\mathcal{G}_h\left[\frac{\beta_1(\delta)-\beta_2(\delta)}{e^{\beta_1(\delta)\Delta^-}-e^{\beta_2(\delta)\Delta^-}} - \frac{\beta_1(\delta)-\beta_2(\delta)}{e^{-\beta_1(\delta)\Delta^+}-e^{-\beta_2(\delta)\Delta^+}}\right]}.
\end{aligned}
$$

□

G.2 Characterization of the job finding rate $f(\hat{w}^*)$ and job separation rate s

Proposition G.2. *The job finding rate $f(\hat{w}^*)$ and the job separation rate s are given by*

$$
\begin{aligned}
f(\hat{w}^*) &= \frac{\sigma^2}{2(1-\mathcal{E})}\left[\lim_{\Delta z \uparrow 0} (g^h)'(\Delta z) - \lim_{\Delta z \downarrow 0} (g^h)'(\Delta z)\right], \\
s &= \frac{\sigma^2}{2\mathcal{E}}\left[\lim_{\Delta z \uparrow 0} (g^u)'(\Delta z) - \lim_{\Delta z \downarrow 0} (g^u)'(\Delta z)\right].
\end{aligned}
$$

The ratio of endogenous s^{end} to total job separations s is given by

$$
\frac{s^{end}}{s} = \frac{\frac{\sigma^2}{2\mathcal{E}}\left[\lim_{\Delta z \downarrow -\Delta^-} (g^h)'(\Delta z) - \lim_{\Delta z \uparrow \Delta^+} (g^h)'(\Delta z)\right]}{s}
$$

and $\delta = s - s^{end}$.

Proof. The proof comes directly from the equilibrium conditions described in Online Appendix F.

□

G.3 Characterization of $l^u(\tau^u)$ and $l^m(\tau^m)$

In the proposition below, γ should be interpred as the sum of the productivity and inflation drift. The symbol π instead represents the irrational number 3.14....

Proposition G.3. *Let $\Delta_{+-} := \Delta^+ + \Delta^-$. The distributions of τ^u and τ^m are given by*

$$\begin{aligned} l^u(\tau^u) &= f(\hat{w}^*)e^{-f(\hat{w}^*)\tau^u}, \\ l^m(\tau^m) &= e^{-\delta\tau^m}\left[\delta + \sum_{n=1}^{\infty}\mathcal{A}(n)\left(e^{-\mathcal{B}(n)\tau^m} - \frac{\delta}{\mathcal{B}(n)}\left(1-e^{-\mathcal{B}(n)\tau^m}\right)\right)\right]. \end{aligned} \tag{G.7}$$

with

$$\begin{aligned} \mathcal{A}(n) &= \frac{\pi\sigma^2 n(-1)^{n-1}}{\Delta_{+-}^2}\left[\sin\left(\pi n\frac{\Delta^-}{\Delta_{+-}}\right)e^{-\frac{\gamma}{\sigma^2}\Delta^+} + \sin\left(\frac{\pi n\Delta^+}{\Delta_{+-}}\right)e^{\frac{\gamma}{\sigma^2}\Delta^-}\right], \\ \mathcal{B}(n) &= \frac{1}{2}\left(\frac{n^2\pi^2\sigma^2}{\Delta_{+-}^2} + \frac{\gamma^2}{\sigma^2}\right). \end{aligned}$$

Proof. The distribution $l^u(\tau^u)$ is the exponential distribution that arises from a Poisson process with arrival rate $f(\hat{w}^*)$. Next, we derive the formula for $l^m(\tau^m)$. Since $\tau^m = \min\{\tau^\delta, \tau^{h*}, \tau^{j*}\}$, we have that

$$\begin{aligned} \mathbb{P}r(\tau^m \le T) &= \mathbb{P}r(\min\{\tau^{h*}, \tau^{j*}\} \le T|\tau^\delta \ge T)\mathbb{P}r(\tau^\delta \ge T) + \mathbb{P}r(\tau^\delta \le T) \\ &= \mathbb{P}r(\min\{\tau^{h*}, \tau^{j*}\} \le T|\tau^\delta \ge T)[1-\mathbb{P}r(\tau^\delta \le T)] + \mathbb{P}r(\tau^\delta \le T) \\ &= \mathbb{P}r(\min\{\tau^{h*}, \tau^{j*}\} \le T)[e^{-\delta T}] + [1-e^{-\delta T}], \end{aligned}$$

where the last equation uses the fact that τ^δ is distributed according to an exponential distribution with rate δ. Taking the derivative with respect to T, we obtain

$$l^m(T) = e^{-\delta T}\left[\frac{\mathrm{d}\mathbb{P}r(\min\{\tau^{h*}, \tau^{j*}\} \le T)}{\mathrm{d}T} - \delta\mathbb{P}r(\min\{\tau^{h*}, \tau^{j*}\} \le T) + \delta\right]. \tag{G.8}$$

From Kolkiewicz (2002), we have that

$$\begin{aligned} &\frac{\mathrm{d}\mathbb{P}r(\min\{\tau^{h*}, \tau^{j*}\} \le T)}{\mathrm{d}T} \\ &= \frac{\pi\sigma^2}{(\hat{w}^+ - \hat{w}^-)^2}\sum_{n=1}^{\infty}n(-1)^{n-1}e^{-\frac{n^2\pi^2\sigma^2}{2(\hat{w}^+-\hat{w}^-)^2}T}\left[\sin\left(\pi n\frac{\hat{w}^* - \hat{w}^-}{\hat{w}^+ - \hat{w}^-}\right)e^{-\frac{\gamma}{2\sigma^2}(2(\hat{w}^+-\hat{w}^*)+\gamma T)} + \sin\left(\pi n\frac{\hat{w}^+ - \hat{w}^*}{\hat{w}^+ - \hat{w}^-}\right)e^{-\frac{\gamma}{2\sigma^2}(2(\hat{w}^- - \hat{w}^*)+\gamma T)}\right] \\ &= \frac{\pi\sigma^2}{\Delta_{+-}^2}\sum_{n=1}^{\infty}n(-1)^{n-1}e^{-\left(\frac{n^2\pi^2\sigma^2}{2\Delta_{+-}^2}+\frac{\gamma^2}{2\sigma^2}\right)T}\left[\sin\left(\frac{\pi n\Delta^-}{\Delta_{+-}}\right)e^{-\frac{\gamma}{\sigma^2}\Delta^+} + \sin\left(\frac{\pi n\Delta^+}{\Delta_{+-}}\right)e^{\frac{\gamma}{\sigma^2}\Delta^-}\right] \\ &= \sum_{n=1}^{\infty}\mathcal{A}(n)e^{-\mathcal{B}(n)T}. \end{aligned} \tag{G.9}$$

Combining (G.8) and (G.9), we obtain (G.7). □

G.4 Characterization of $l^w(\Delta w)$

Proposition G.4. *The distribution of log nominal wage changes satisfies*

$$l^w(\Delta w) = \mathcal{G}_u\left[\beta_2(f(\hat{w}^*))e^{-\beta_2(f(\hat{w}^*))\Delta w}\Gamma_2(\Delta w) + \beta_1(f(\hat{w}^*))e^{-\beta_1(f(\hat{w}^*))\Delta w}\Gamma_1(\Delta w)\right]$$

with

$$(\Gamma_1(c), \Gamma_2(c)) = \left(\int_{-\infty}^{-c} e^{-\beta_1(f(\hat{w}^*)))x}\bar{G}^h(x)\,\mathrm{d}x, \int_{-c}^{\infty} e^{-\beta_2(f(\hat{w}^*))x}\bar{G}^h(x)\,\mathrm{d}x\right).$$

Proof. Fix a date t_0 and focus on a newly hired worker. Then, the distribution of wage changes between two new jobs is given by

$$\begin{aligned}
Pr(\Delta w \leq c) &= Pr(w_{t_0+\tau^m+\tau^u} - w_{t_0} \leq c) \\
&=^{(1)} Pr(w_{t_0+\tau^m+\tau^u} - z_{t_0+\tau^m+\tau^u} - (w_{t_0} - z_{t_0}) + (z_{t_0+\tau^m+\tau^u} - z_{t_0}) \leq c) \\
&=^{(2)} Pr(\hat{w}^* - \hat{w}^* + (z_{t_0+\tau^m+\tau^u} - z_{t_0}) \leq c) \\
&=^{(3)} Pr(-(\Delta z^h + \Delta z^u) \leq c),
\end{aligned}$$

where Δz^h and Δz^u denote cumulative productivity shocks during completed employment and unemployment spells, respectively. Here, step (1) adds and subtracts productivity at the beginning of both job spells. In step (2), we use the result that $\hat{w}^*$ is constant across jobs. Step 3 uses the facts that τ^u and the Brownian motion increments are independent of the filtration $\mathcal{F}_{\tau_u}$. Therefore, the distributions of cumulative productivity shocks for completed employment and unemployment spells are given by

$$\begin{aligned}
\bar{G}^h(\Delta z) &= \begin{cases} 1 & \text{if } \Delta z \in [\Delta^+, \infty) \\ \frac{1}{s\mathcal{E}}\left[\frac{\sigma^2}{2}\lim_{\Delta z \downarrow -\Delta^-}(g^h)'(\Delta z) + \delta\int_{-\Delta^-}^{\Delta z} g^h(x)\,\mathrm{d}x\right] & \text{if } \Delta z \in [-\Delta^-, \Delta^+) \\ 0 & \text{if } \Delta z \in (-\infty, -\Delta^-) \end{cases} \\
\bar{g}^u(\Delta z) &= \mathcal{G}_u \begin{cases} e^{\beta_2(f(\hat{w}^*))\Delta z} & \text{if } \Delta z \in (-\infty, 0] \\ e^{\beta_1(f(\hat{w}^*))\Delta z} & \text{if } \Delta z \in [0, \infty) \end{cases}
\end{aligned}$$

Thus,

$$\begin{aligned}
&\Pr(\Delta w \leq c) \\
&= \Pr(-(\Delta z^u + \Delta z^h) \leq c) \\
&= 1 - \Pr(\Delta z^u + \Delta z^h \leq -c) \\
&=^{(1)} 1 - \int_{-\infty}^{\infty} \bar{G}^h(-(c+\Delta z))\bar{g}^u(\Delta z)\,\mathrm{d}\Delta z \\
&=^{(2)} 1 - \mathcal{G}_u\left[\int_{-\infty}^{0} e^{\beta_2(f(\hat{w}^*))\Delta z}\bar{G}^h(-(c+\Delta z))\,\mathrm{d}\Delta z + \int_{0}^{\infty} e^{\beta_1(f(\hat{w}^*))\Delta z}\bar{G}^h(-(c+\Delta z))\,\mathrm{d}\Delta z\right] \\
&=^{(3)} 1 + \mathcal{G}_u\left[\int_{\infty}^{-c} e^{-\beta_2(f(\hat{w}^*))(c+x)}\bar{G}^h(x)\,\mathrm{d}x + \int_{-c}^{-\infty} e^{-\beta_1(f(\hat{w}^*))(c+x)}\bar{G}^h(x)\,\mathrm{d}x\right] \\
&=^{(4)} 1 - \mathcal{G}_u\left[e^{-\beta_2(f(\hat{w}^*))c}\int_{-c}^{\infty} e^{-\beta_2(f(\hat{w}^*))x}\bar{G}^h(x)\,\mathrm{d}x + e^{-\beta_1(f(\hat{w}^*))c}\int_{-\infty}^{-c} e^{-\beta_1(f(\hat{w}^*))x}\bar{G}^h(x)\,\mathrm{d}x\right].
\end{aligned}$$

In step (1), we use the independence of Δz^u and Δz^h. In step (2), we use the definition of $\bar{g}^u(\Delta z)$. In step (3), we integrate by substituting $x = -c - \Delta z$, and in step (4), we use the properties of an integral. The last step involves defining

$$(\Gamma_1(c), \Gamma_2(c)) = \left(\int_{-\infty}^{-c} e^{-\beta_1(f(\hat{w}^*))x} \bar{G}^h(x)\,\mathrm{d}x, \int_{-c}^{\infty} e^{-\beta_2(f(\hat{w}^*))x} \bar{G}^h(x)\,\mathrm{d}x \right).$$

□

G.5 Characterization of $\mathbb{E}_h[\Delta z^n]$

We denote by $\bar{\mathbb{E}}_h[\cdot]$ and $\bar{\mathbb{E}}_u[\cdot]$ the expectation operators under the distributions $\bar{g}^h(\Delta z)$ and $\bar{g}^u(\Delta z)$, respectively.

Proposition G.5. *Define the weights* $\omega^{hn}(\Delta z) = \frac{\Delta z^n}{\bar{\mathbb{E}}_h[\Delta z^n]}$ *with the property that*

$$\bar{\mathbb{E}}_h \left[\omega^{hn}(\Delta z) \right] = 1.$$

If $\gamma = 0$*, then* $\mathbb{E}_h[(\Delta z)^n]$ *can be recovered from*

$$\mathbb{E}_h[(\Delta z)^n] = \frac{2\mathcal{E}}{(n+1)(n+2)} \bar{\mathbb{E}}_h \left[(\Delta z)^n \omega^{h2}(\Delta z) \right]. \tag{G.10}$$

If $\gamma \neq 0$*, then* $\mathbb{E}_h[(\Delta z)^n]$ *can be recovered recursively from*

$$\mathbb{E}_h[(\Delta z)^n] = \frac{\mathcal{E}}{n+1} \bar{\mathbb{E}}_h[(\Delta z)^n \omega^{h1}(\Delta z)] + \frac{\sigma^2 n}{2\gamma} \mathbb{E}_h[(\Delta z)^{n-1}]. \tag{G.11}$$

The moments $\bar{\mathbb{E}}_h \left[(\Delta z)^n \omega^{hk}(\Delta z) \right] = \frac{\bar{\mathbb{E}}_h\left[(\Delta z)^{n+k}\right]}{\bar{\mathbb{E}}_h[(\Delta z)^k]}$ *can be recovered from the following linear system of equations:*

$$\mathbb{E}_{\mathcal{D}}[\Delta w^n] = (-1)^n \sum_{i=0}^{n} \binom{n}{i} \bar{\mathbb{E}}_h[\Delta z^i] \bar{\mathbb{E}}_u[\Delta z^{n-i}],$$
$$\bar{\mathbb{E}}_u[(\Delta z)^{n-i}] = \frac{(n-i)!}{\mathcal{L}_1^{n-i}\left(\mathcal{L}_2 + \mathcal{L}_2^{-1}\right)} \left(\mathcal{L}_2^{-(n-i+1)} - (-\mathcal{L}_2)^{(n-i+1)} \right),$$

where

$$\mathcal{L}_1 = \sqrt{\frac{2f(\hat{w}^*)}{\sigma^2}} \text{ and } \mathcal{L}_2 = \sqrt{\frac{\gamma + \sqrt{\gamma^2 + 2\sigma^2 f(\hat{w}^*)}}{-\gamma + \sqrt{\gamma^2 + 2\sigma^2 f(\hat{w}^*)}}}.$$

Proof. We divide the proof into 3 steps.

Step 1. We first show that

$$\mathbb{E}_h[(\Delta z)^n] = \frac{\mathcal{E}}{n+1} \bar{\mathbb{E}}_h[(\Delta z)^n \omega^{h1}(\Delta z)] - \frac{\sigma^2 n}{2\gamma} \mathbb{E}_h[(\Delta z)^{n-1}].$$

when $\gamma \neq 0$. For the case with $\gamma = 0$, see Baley and Blanco (2021).

Let us define $Y_t = (\Delta z_t)^n$. The law of motion for Δz_t is given by $\mathrm{d}\Delta z_t = -\gamma\,\mathrm{d}t + \sigma\,\mathrm{d}\mathcal{W}_t^z$. Applying Itô's Lemma, we obtain

$$\begin{aligned} \mathrm{d}Y_t &= n(\Delta z_t)^{n-1}\,\mathrm{d}\Delta z_t + \frac{1}{2} n(n-1)(\Delta z_t)^{n-2}(\mathrm{d}\Delta z_t)^2 \\ &= \left[-\gamma n(\Delta z_t)^{n-1} + \frac{\sigma^2}{2} n(n-1)(\Delta z_t)^{n-2} \right] \mathrm{d}t + n\sigma(\Delta z_t)^{n-1}\,\mathrm{d}\mathcal{W}_t^z \end{aligned}$$

Thus,

$$(\Delta z_{\tau^m})^n = -\gamma n \int_0^{\tau^m} (\Delta z_t)^{n-1}\,\mathrm{d}t + \frac{\sigma^2}{2} n(n-1) \int_0^{\tau^m} (\Delta z_t)^{n-2}\,\mathrm{d}t + n \int_0^{\tau^m} (\Delta z_t)^{n-1} \sigma\,\mathrm{d}\mathcal{W}_t^z.$$

Following the same arguments as in the proof of Proposition F.1 and using the Renewal Principle to have $\mathbb{E}_{\mathcal{D}}[\tau^m] = 1/s$, we obtain

$$\bar{\mathbb{E}}_h[(\Delta z)^n] = -\gamma n \mathbb{E}_{\mathcal{D}}[\tau^m] \frac{\mathbb{E}_h[(\Delta z)^{n-1}]}{\mathcal{E}} + \frac{\sigma^2 n(n-1)}{2s} \frac{\mathbb{E}_h[(\Delta z)^{n-2}]}{\mathcal{E}}$$

or equivalently

$$\mathbb{E}_h[(\Delta z)^n] = -\frac{\mathcal{E}}{\gamma \mathbb{E}_{\mathcal{D}}[\tau^m]} \frac{\bar{\mathbb{E}}_h[(\Delta z)^{n+1}]}{n+1} + \frac{\sigma^2 n}{2\gamma} \mathbb{E}_h[(\Delta z)^{n-1}].$$

From Propostion F.1, we have that $\gamma \mathbb{E}_{\mathcal{D}}[\tau^m] = -\bar{\mathbb{E}}_h[(\Delta z)]$ and $\frac{\bar{\mathbb{E}}_h[(\Delta z)^{n+1}]}{\bar{\mathbb{E}}_h[(\Delta z)]} = \bar{\mathbb{E}}_h[(\Delta z)^n \omega^{h1}(\Delta z)]$. Thus,

$$\mathbb{E}_h[(\Delta z)^n] = \frac{\mathcal{E}}{n+1} \bar{\mathbb{E}}_h[(\Delta z)^n \omega^{h1}(\Delta z)] + \frac{\sigma^2 n}{2\gamma} \mathbb{E}_h[(\Delta z)^{n-1}].$$

Step 2. Here we show that

$$\mathbb{E}_{\mathcal{D}}[\Delta w^n] = (-1)^n \sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} \bar{\mathbb{E}}_h[\Delta z^n] \bar{\mathbb{E}}_u[\Delta z^{n-i}].$$

Using the independence of cumulative productivity shocks during employment and unemployment, we obtain

$$\begin{aligned}
\mathbb{E}_{\mathcal{D}}[\Delta w^n] &= \bar{\mathbb{E}}[(-\Delta z^h - \Delta z^u)^n], \\
&= \sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} \bar{\mathbb{E}}[(-\Delta z^h)^i (-\Delta u^u)^{n-i}], \\
&= \sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} \bar{\mathbb{E}}_h[(-\Delta z)^i] \bar{\mathbb{E}}_u[(-\Delta z)^{n-i}], \\
&= (-1)^n \sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} \bar{\mathbb{E}}_h[\Delta z^i] \bar{\mathbb{E}}_u[\Delta z^{n-i}],
\end{aligned}$$

Step 3. Here we show that

$$\bar{\mathbb{E}}_u[(\Delta z)^{n-i}] = \frac{(n-i)!}{\mathcal{L}_1^{n-i}\left(\mathcal{L}_2 + \mathcal{L}_2^{-1}\right)} \left(\mathcal{L}_2^{-(n-i+1)} - (-\mathcal{L}_2)^{(n-i+1)}\right).$$

Let us depart from the definition of $\bar{g}^u(\Delta z)$, which is given by

$$\bar{g}^u(\Delta z) = \left[-\beta_1(f(\hat{w}^*))^{-1} + \beta_2(f(\hat{w}^*))^{-1}\right]^{-1} \begin{cases} e^{\beta_2(f(\hat{w}^*))\Delta z} & \text{if } \Delta z \in (-\infty, 0] \\ e^{\beta_1(f(\hat{w}^*))\Delta z} & \text{if } \Delta z \in [0, \infty) \end{cases}$$

where $\beta_1(x) = \frac{-\gamma - \sqrt{\gamma^2 + 2\sigma^2 x}}{\sigma^2}$ and $\beta_2(x) = \frac{-\gamma + \sqrt{\gamma^2 + 2\sigma^2 x}}{\sigma^2}$. This step consist of showing that $\bar{g}^u(\Delta z)$ is an asymmetric Laplace distribution with parameters

$$\mathcal{L}_1 = \sqrt{\frac{2f(\hat{w}^*)}{\sigma^2}} \text{ and } \mathcal{L}_2 = \sqrt{\frac{\gamma + \sqrt{\gamma^2 + 2\sigma^2 f(\hat{w}^*)}}{-\gamma + \sqrt{\gamma^2 + 2\sigma^2 f(\hat{w}^*)}}}$$

The ratio between $\mathcal{L}_1$ and $\mathcal{L}_2$ is

$$\begin{aligned}
\frac{\mathcal{L}_1}{\mathcal{L}_2} &= \sqrt{\frac{2f(\hat{w}^*)}{\sigma^2}\frac{-\gamma+\sqrt{\gamma^2+2\sigma^2 f(\hat{w}^*)}}{\gamma+\sqrt{\gamma^2+2\sigma^2 f(\hat{w}^*)}}} \\
&= \sqrt{\frac{2f(\hat{w}^*)}{\sigma^2}(-\gamma+\sqrt{\gamma^2+2\sigma^2 f(\hat{w}^*)})\frac{-\gamma+\sqrt{\gamma^2+2\sigma^2 f(\hat{w}^*)}}{\left(\sqrt{\gamma^2+2\sigma^2 f(\hat{w}^*)}\right)^2-\gamma^2}} \\
&= (-\gamma+\sqrt{\gamma^2+2\sigma^2 f(\hat{w}^*)})\sqrt{\frac{2f(\hat{w}^*)}{\sigma^2}\frac{1}{2\sigma^2 f(\hat{w}^*)}} \\
&= \frac{-\gamma+\sqrt{\gamma^2+2\sigma^2 f(\hat{w}^*)}}{\sigma^2} = \beta_2(f(\hat{w}^*)).
\end{aligned}$$

The product between $\mathcal{L}_1$ and $\mathcal{L}_2$ is

$$\begin{aligned}
-\mathcal{L}_1\mathcal{L}_2 &= -\sqrt{\frac{2f(\hat{w}^*)}{\sigma^2}\frac{\gamma+\sqrt{\gamma^2+2\sigma^2 f(\hat{w}^*)}}{-\gamma+\sqrt{\gamma^2+2\sigma^2 f(\hat{w}^*)}}} \\
&= -\sqrt{\frac{2f(\hat{w}^*)}{\sigma^2}(\gamma+\sqrt{\gamma^2+2\sigma^2 f(\hat{w}^*)})\frac{\gamma+\sqrt{\gamma^2+2\sigma^2 f(\hat{w}^*)}}{-\gamma^2+\left(\sqrt{\gamma^2+2\sigma^2 f(\hat{w}^*)}\right)^2}} \\
&= -(\gamma+\sqrt{\gamma^2+2\sigma^2 f(\hat{w}^*)})\sqrt{\frac{2f(\hat{w}^*)}{\sigma^2}\frac{1}{2\sigma^2 f(\hat{w}^*)}} \\
&= -\frac{\gamma+\sqrt{\gamma^2+2\sigma^2 f(\hat{w}^*)}}{\sigma^2} = \beta_1(f(\hat{w}^*)).
\end{aligned}$$

Therefore, we can write $\bar{g}^u(\Delta z)$

$$\bar{g}^u(\Delta z) = \frac{\mathcal{L}_1}{\mathcal{L}_2+\mathcal{L}_2^{-1}}\begin{cases} e^{\frac{\mathcal{L}_1}{\mathcal{L}_2}\Delta z} & \text{if } \Delta z \in (-\infty, 0] \\ e^{-\mathcal{L}_1\mathcal{L}_2\Delta z} & \text{if } \Delta z \in [0,\infty), \end{cases}$$

which is the probability distribution function of an asymmetric Laplace distribution. It is a standard result that the n-th moment for an asymmetric Laplace distribution is given by

$$\bar{\mathbb{E}}_u[(\Delta z)^n] = \frac{n!}{\mathcal{L}_1^n\left(\mathcal{L}_2+\mathcal{L}_2^{-1}\right)}\left(\mathcal{L}_2^{-(n+1)}-(-\mathcal{L}_2)^{(n+1)}\right).$$

□

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