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Abstract

We reexamine the tests for dynamic inefficiency in productive overlapping-generations economies with stochastic growth. The size of real, long-term, safe interest rates relative to average GDP growth is an inconclusive test for dynamic inefficiency. A more accurate test should take into account the correlation between growth and the marginal utility of wealth. This typically restricts the room for inefficiency and welfare-improving policies. We also distinguish capital overaccumulation from an inefficient distribution of consumption risk. The refined test for capital overaccumulation is rather stringent: capital is not overaccumulated if the net dividend remains positive with some probability, as opposed to always, as in the original Abel et al. (1989)'s formulation.

JEL Classification: D60, G1, E21, E62, H2, H21

Keywords: Interest rates

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LOW SAFE INTEREST RATES: A CASE FOR DYNAMIC INEFFICIENCY?

GAETANO BLOISE AND PIETRO REICHLIN

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Abstract

We reexamine the tests for dynamic inefficiency in productive overlapping-generations economies with stochastic growth. Contrary to certain recent claims in the recent literature, we argue that the size of real, long-term, safe interest rates relative to average GDP growth is an inconclusive test for dynamic inefficiency. A more accurate test should take into account the correlation between growth and the marginal utility of wealth. We propose a necessary and sufficient criterion based on the spectral radius of the stochastic discount factor commonly used in macroeconomic finance theory. We also distinguish capital overaccumulation from an inefficient distribution of consumption risk. The refined test for capital overaccumulation is rather stringent: Capital is not overaccumulated if the net dividend remains positive *with some probability*, as opposed to *always*, as in the original Abel et al. [1]'s formulation.

JEL Classification Numbers: D60, G1, E21, E62, H2, H21.

1. INTRODUCTION

Real yields on safe bonds have been persistently low relative to GDP growth for most of the last seventy years, and especially since the late 1980s. According to the evidence documented by Blanchard [7], the 10-year rate on US T-bills has averaged 5.6%, while nominal GDP growth has averaged 6.3% from 1950 onward (see also Del Negro et al. [14] and Rogoff et al. [22]). This phenomenon has generated an intense debate about *dynamic inefficiency* and the social benefits of government debt rollover. Using Blanchard [7]'s own words, 'the signal sent by low rates is not only that debt may not have a substantial fiscal cost, but also that it may have limited welfare costs'. In a deterministic environment, the assessment reduces to a straight comparison of the safe interest rate with the rate of growth of the econony. In an uncertain world, however, a welfare evaluation is more controversial because returns and growth rates are time-varying, affected by uncertain events, and historical averages are only unreliable statistics. The purpose of this paper is to clarify which relation between safe rates, risky returns and GDP growth rates is relevant for assessing dynamic inefficiency in an economy with truly stochastic growth.

A renewed interest for dynamic inefficiency under low interest rates is testified by a flourishing recent literature on stochastic overlapping-generations economies with capital

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accumulation. Abel and Panageas [2] claim that a welfare-improving debt rollover is feasible even in a dynamically efficient economy whenever the *safe* rate falls short the growth rate of the economy. Hellwig [17] vigorously submits that the assessment of dynamic inefficiency must consider the *safe* rate of return and that '[c]ontrary claims in the literature are based on misunderstandings'. Kocherlakota [18] asserts that equilibrium is dynamically inefficient when the yield of a long-term discount bond (a sort of long-term *safe* rate) is dominated by the population growth rate, even when the short-term safe rate itself exceeds growth. Altogether these findings advocate a determinant role of safe rates for dynamic inefficiency. Instead, more in agreement with Abel et al. [1]'s influential paper, we will argue that a comparison of safe rates with expected (or average) growth provides only a misguided intuition when growth is properly stochastic.[1]

We study a conventional overlapping-generations economy with capital accumulation where uncertainty derives from productivity shocks affecting the GDP growth rate.[2] We distinguish between two potential sources of inefficiency: conditional Pareto inefficiency, defined as the occurrence of feasible Pareto improvements conditional on the state at which generations are born, and capital overaccumulation, defined as the possibility of increasing aggregate consumption at all contingencies through progressive capital reductions. It is known that, under uncertainty, conditional Pareto inefficiency might occur without capital overaccumulation, due to a misallocation of consumption risk (Barbie et al. [6]). The distinction is not a merely scholastic exercise, and it is relevant both for the assessment and for the implied policy prescriptions.

To ascertain conditional Pareto inefficiency involves judgements on individuals' preferences for intertemporal substitution and attitudes towards consumption risk, whereas capital overaccumulation is exhaustively reflected by capital returns relative to growth rates, independently of individuals' preferences. Comparatively more information is to be extracted from market prices in order to establish conditional Pareto inefficiency and, as we shall argue extensively in this paper, a persistently low safe interest rate might well be a misguided sufficient statistic. Furthermore, the schemes of transfers correcting conditional Pareto inefficiency are in general highly state-dependent, require calibrated compensations across generations, and do not reduce to a straight reallocation from young to old individuals.

[1] Abel et al. [1, Section III] ostensively clarify that a low safe rate can coexist with a permanently high rate of profit in a dynamically efficient economy. It is rather unfortunate that they illustrate their point with a simple example involving an infinitely representative-individual, as opposed as within the framework of an overlapping-generations economy with production. Furthermore, dynamic inefficiency might ambiguously refer to a failure of conditional (or *interim*) Pareto efficiency or alternatively to capital overaccumulation. Both facts might have concurred in generating a certain misalignment of the criteria appearing in the old and the recent literature.

[2] The nature of uncertainty is not relevant for our analysis, and we can straightforwardly encompass stochastic population growth. We take expositional advantage from the fact the growth rates are exogenously determined. Otherwise our approach would preliminarily need the identification of a maximum sustainable growth path for the economy, and then an adjustment all the arguments consistently.

Drawing on the established literature, we present exhaustive and operational criteria both for conditional Pareto inefficiency and for capital overaccumulation. For heuristic purposes, inspired by Kocherlakota [18], we preliminarily consider a simplified framework in which a stochastically growing endowment can be stored yielding an uncertain return. We assume that both the growth of endowments and the rate of return on storage follow a Markov chain and that utility is homothetic, so that competitive equilibrium inherits the Markov property. In such an environment, we argue that conditional Pareto inefficiency is fully characterized by the *dominant root* of the matrix of *growth-adjusted state prices*, extending Aiyagari and Peled [3].[3] This allows us to draw certain implications of low interest rates for dynamic inefficiency.

The necessity of a dominant root approach is a natural consequence of a time-varying environment. Interest rates need be compounded over time, so as to estimate the welfare-effects of consumption reallocations propagating across periods, and the dominant root serves to extrapolate long-term tendencies. Understated by the previous literature was the role of stochastic growth rates. A straight comparison of long-term interest rate with the average growth is highly deceptive. Indeed, the same physical transfer entails different implications for social welfare when growth is low rather than high. The interest rate reflects the first-order effect for a single individual, but the need of intergenerational compensations imposes further discipline in terms of feasibility of a perpetual scheme of transfers. As a consequence, *safe* interest rates are not really *safe* when growth is stochastic and have to be upward corrected by the negative correlation between growth and the marginal utility of wealth: low interest rates might well be consistent with conditional Pareto efficiency of a competitive equilibrium. In an hypothetical world in which all variables are identically and independently distributed, a test for conditional Pareto efficiency would reduce to

$$r > \mathbb{E}g + (1 + r) \operatorname{cov}(g, m),$$

where r is the safe rate, g is the growth rate and m is the marginal utility of wealth, which is typically *negatively* correlated with capital returns and output growth.

We further argue that, in line with the previous literature (*e.g.*, Barbie et al. [6]), conditional Pareto inefficiency might occur even absent capital overaccumulation. A criterion for capital overaccumulation necessarily requires marginal productivity of capital falling short growth in all states of the world and in all periods over the infinite horizon. This is substantially more demanding that Abel et al. [1]'s net dividend criterion established in the theoretical and empirical literature. The amended criterion might help to dissipate certain empirical ambiguities: Abel et al. [1] claim that the net dividend criterion has been historically verified in the US economy, whereas Geerolf [15] documents a failure of the criterion for a variety of advanced economies.

[3] A growth-adjustment in a stochastic environment is also studied by Kocherlakota [19, Section 4].

Unfortunately, competitive equilibria are only fortuitously Markovian in an overlapping-generations economy with capital accumulation, and this complicates our analysis substantially. However, our theory extends, and all intuitions remain, by means of a more sophisticated approach. The method is based on locating the *spectral radius* of a sort of valuation operator commonly used in macroeconomic finance theory, an extension of the dominant root approach for finite Markov chains. As the method is grounded on the conventional stochastic discount factor, it is operationally computable and suitable of an empirical application, given the recent progresses on long-term risk and asset prices pioneered by Alvarez and Jermann [5], Christensen [13] and Hansen and Scheinkman [16]. In fact, in the context of a nonparametric recursive utility model of risk, Christensen [13] provides empirical estimates of the spectral radius and of the related long-term yield for the US economy.

The spectral radius of the valuation operator is related to the yield of long-term discount bonds, as previously noticed by Bloise et al. [8, Appendix C] and recently advocated by Kocherlakota [18, 19]. Kocherlakota [18] argues that the long-term yield, relative to growth, is the relevant statistic for dynamic inefficiency. We find that, under stochastic growth, this characterization requires a major amendment, as the long-term yield might fall short growth even when the economy is dynamically efficient. The relevant statistic is the yield of an *hypothetical* long-term discount bond *indexed* to long-term growth. Therefore, the historical observation of low long yields on government bonds, documented in Blanchard [7], cannot be taken as a persuasive evidence of dynamic inefficiency.

Turning to a comparison with Abel and Panageas [2] and Hellwig [17], we notice that they both argue that conditional Pareto inefficiency arises if the safe rate falls short the growth rate of the economy.[4] Differently from our findings, these assessments are not related to the returns on risky assets. As our characterization of dynamic efficiency is (almost) exhaustive, this discrepancy can only be justified by a misalignment of the assumptions on the primitives of the economy. In fact, their characterizations crucially rely on the absence of a proper stochastic growth.

2. Less recent literature

Capital overaccumulation was initially studied by Cass [10] in a deterministic environment. He proved that dynamic inefficiency can be characterized in terms of the sequence of gross interest rates which, absent uncertainty, are equal to the gross marginal products of capital. Too much capital implies a small marginal productivity and small interest rates.

[4]Literally, Abel and Panageas [2] assert that a welfare-improving debt rollover is feasible in a dynamically efficient economy. However, dynamic efficiency in their analysis is to be understood as the absence of capital overaccumulation, as implied by their explicit reference to Zilcha [23]'s criterion. Thus, for our purposes of comparison, they prove that equilibrium is conditionally Pareto inefficient when the safe rate is dominated by the growth rate.

The Cass Criterion for testing dynamic inefficiency asserts that the sum of future values of capital units over the infinite time, net of population or GDP growth, diverges.

Similar characterizations were obtained by Peled [21] and Manuelli [20] under stationary uncertainty, whereas Chattopadhyay and Gottardi [12] developed a more general Cass Criterion in a pure endowment economy based on the convergence of the weighted sum of the reciprocals of present value prices. Assuming stationary uncertainty, Zilcha [23] provided a test for dynamic inefficiency based on the expected value of the log of a function representing the asymptotic value of compounded marginal products of capital. The intuition is that, under uncertainty, a relatively large marginal productivity of capital does not necessarily imply a large value of the long-run safe rate (as it would be the case in a deterministic setting).

In an influential paper, Abel et al. [1] provided a sufficient condition guaranteeing conditional Pareto optimality, and absence of capital overaccumulation, as well as a sufficient condition for absence of conditional Pareto optimality, and for the overaccumulation of capital. These conditions are known as the *net dividend criterion*, since they are obtained by comparing capital income and aggregate investment. If capital income is greater than investment at all date-events, there is no way to reallocate resources so as to increase aggregate consumption and individuals' welfare at any date-event. If, on the other hand, the market is always investing more than it is getting from capital income, there exists a reallocation of resources characterized by a progressive reduction of investment achieving higher aggregate consumption and a higher per generation utility at all date-events. The net dividend criterion is inconclusive when capital income exceeds investment or falls short of it some of the time.

3. An illustration

3.1. **A Markov setting.** To illustrate our criterion, we consider a simple overlapping-generations economy of two-period lived individuals born at all $t = 0, 1, 2, \ldots$. We denote by y_t the aggregate endowment of the unique consumption good available at period t, and we suppose that the good can be stored for one period yielding an uncertain return R. The aggregate endowment y_t grows at rate g. We assume that both g and R follow a simple Markov stochastic process. Namely, we let (g_{ij}, R_{ij}) be the realizations of capital return R and growth rate g from state i to state j in some finite state space, with $\mu_{ij} > 0$ being the transition probability.

The utility of a young agent born at time t is

$$U_t\left(c_t^y, c_{t+1}^o\right) = u\left(c_t^y\right) + \mathbb{E}_t u\left(c_{t+1}^o\right),$$

where c_t^y and c_{t+1}^o are the young and old age consumption and utility exhibits constant elasticity of substitution, namely, marginal utility satisfies $u'(c) = c^{-\sigma}$ for some $\sigma > 0$.

Each individual maximizes this lifetime utility subject to budget constraints,

$$c_t^y \leq y_t - k_t \text{ and } c_{t+1}^o \leq R_{t+1} k_t,$$

where k_t is the capital stored at time t. At a competitive equilibrium, the first-order condition for the optimality of capital investment requires

$$u'\left(c_t^y\right) = \mathbb{E}_t u'\left(c_{t+1}^o\right) R_{t+1}.$$

By constant elasticity of substitution, along with the simple demographic and productive structure, a competitive equilibrium is fully determined by $c_t^y = \phi_i y_t$, $k_t = (1 - \phi_i) y_t$ and $c_{t+1}^o = R_{ij} (1 - \phi_i) y_t$ for some constant $0 < \phi_i < 1$, where i is the state when young and j is the state when old. Importantly, equilibrium marginal rate of substitution depends only on current and future states,

$$m_{ij} = \left(\frac{u'\left(c_{t+1}^o\right)}{u'\left(c_t^y\right)}\right) = \left(\frac{c_t^y}{c_{t+1}^o}\right)^\sigma = \left(\frac{\phi_i}{R_{ij}\left(1 - \phi_i\right)}\right)^\sigma.$$

As usual, m_{ij} can be interpreted as the traditional stochastic discount factor in asset pricing theory.

3.2. **Pareto optimality.** We argue that the test for conditional Pareto optimality reduces to locating the dominant root ρ of the positive matrix Q of implicit *growth-adjusted state prices* given as

$$q_{ij} = \left(1 + g_{ij}\right) \mu_{ij} m_{ij},$$

where q_{ij} is the implicit price in state i of a share of output in the next period, conditional on state j, in terms of an equal share of current output. Competitive equilibrium is conditionally Pareto efficient when $\rho < 1$, and inefficient when $\rho > 1$. This criterion is perfectly consistent with the dominant root characterization provided by Aiyagari and Peled [3] for a non-growing economy. It is worth noticing that, when returns and growth rates are *identically and independently distributed* over time, conditional Pareto efficiency obtains if

$$\rho = \mathbb{E}\left(1 + g\right) m = \frac{1 + \mathbb{E}g}{1 + r} + \operatorname{cov}\left(g, m\right) < 1,$$

where the safe interest rate satisfies $r = \left(\mathbb{E}m\right)^{-1} - 1$. Therefore, at an efficient competitive equilibrium, the safe rate can be substantially smaller than the expected growth rate of output, provided that the latter is sufficiently positively correlated with the rate of return on capital (and, hence, negatively correlated with the stochastic discount factor).

By Perron-Frobenius Theorem, the positive matrix Q admits a strictly positive dominant eigenvector v satisfying the eigenvalue equation

$$\rho v = Qv.$$

With some abuse of notation, we denote with v itself a stochastic process taking value v_i when i is the current state.[5] In this alternative notation, the eigenvalue equation becomes

$$\rho v_t = \mathbb{E}_t \left(1 + g_{t+1}\right) m_{t+1} v_{t+1},$$

which corresponds to $\rho v_i = (Qv)_i$ where i is the current state. We will use the dominant eigenvector process to estimate directions of welfare-improving changes in consumptions relative to aggregate endowment.

It is immediate to verify that, when $\rho > 1$, a competitive equilibrium is conditionally Pareto inefficient. Consider a small reallocation of consumptions given by

$$\hat{c}_t^y = c_t^y - \epsilon v_t y_t \qquad \text{and} \qquad \hat{c}_{t+1}^o = c_{t+1}^o + \epsilon v_{t+1} y_{t+1}.$$

Evaluating the welfare impact for a sufficiently small $\epsilon > 0$, we obtain

$$\begin{aligned} \Delta U_t &\approx \epsilon \left(-u'\left(c_t^y\right) v_t + \mathbb{E}_t u'\left(c_{t+1}^o\right)\left(1 + g_{t+1}\right) v_{t+1}\right) y_t \\ &= \epsilon u'\left(c_t^y\right)\left(\mathbb{E}_t\left(1 + g_{t+1}\right) m_{t+1} v_{t+1} - v_t\right) y_t \\ &= \epsilon u'\left(c_t^y\right)\left(\rho - 1\right) v_t y_t > 0. \end{aligned}$$

As consumption of the initial old generation increases, this reallocation is Pareto improving, thus confirming our claim.

We now turn to the sufficient condition for conditional Pareto efficiency. To this end, suppose that $\rho < 1$ and assume that a planner is able to Pareto improve upon equilibrium by reallocating consumptions and capital over time. Furthermore, define

$$\begin{aligned} \tilde{c}_t^y - c_t^y &= -\tau_t y_t + \left(k_t - \tilde{k}_t\right), \\ \tilde{c}_{t+1}^o - c_{t+1}^o &= \tau_{t+1} y_{t+1} - R_{t+1}\left(k_t - \tilde{k}_t\right), \end{aligned}$$

where we conveniently measure the implicit transfer from young to old individuals, net of the readjustment in capital investment, as a share of current endowment. We assume that $\tau_t \leq 1$ and, at no loss of generality, we postulate that $\tau_0 > 0$, as the consumption of the initial old individual cannot decrease in a Pareto improving reallocation and the initial stock of capital is inherited from the past. Estimating the impact on welfare, by decreasing marginal utility, we obtain

$$\begin{aligned} \Delta U_t &\leq u'\left(c_t^y\right)\left(\tilde{c}_t^y - c_t^y\right) + \mathbb{E}_t u'\left(c_{t+1}^o\right)\left(\tilde{c}_{t+1}^o - c_{t+1}^o\right) \\ &= \left(\mathbb{E}_t\left[u'\left(c_{t+1}^o\right)\left(1 + g_{t+1}\right)\tau_{t+1}\right] - u'\left(c_t^y\right)\tau_t\right) y_t \\ &\quad + \left(\mathbb{E}_t\left[u'\left(c_{t+1}^o\right) R_{t+1}\right] - u'\left(c_t^y\right)\right)\left(\tilde{k}_t - k_t\right). \end{aligned}$$

[5] More precisely, given a history of states, $v_t = v_t\left(i_0, \ldots, i_{t-1}, i_t\right) = v_{i_t}$.

The last term vanishes because capital investment fulfills individual first-order conditions. As $\Delta U_t \geq 0$ by Pareto dominance, rearranging terms leads to

$$\tau_t^+ \leq \mathbb{E}_t \left(1 + g_{t+1}\right) m_{t+1} \tau_{t+1}^+,$$

where $\tau^+ = \max\{\tau, 0\}$. Since the eigenvector is determined up to a scalar factor, we can assume that $\tau_{t+1}^+ \leq v_{t+1}$, so obtaining

$$\tau_t^+ \leq \mathbb{E}_t \left(1 + g_{t+1}\right) m_{t+1} \tau_{t+1}^+ \leq \mathbb{E}_t \left(1 + g_{t+1}\right) m_{t+1} v_{t+1} = \rho v_t.$$

As $\tau_t^+ \leq \rho v_t$, reproducing the same logic leads to

$$\tau_{t-1}^+ \leq \mathbb{E}_{t-1} \left(1 + g_t\right) m_t \tau_t^+ \leq \mathbb{E}_{t-1} \left(1 + g_t\right) m_t \rho v_t = \rho^2 v_{t-1}.$$

Proceeding by backward induction, we finally obtain $\tau_0^+ \leq \rho^t v_0$ which, as $\lim_{t\to\infty} \rho^t = 0$, reveals that the redistribution was never initiated, thus contradicting conditional Pareto inefficiency. Intuitively, a Pareto improving reallocation would be exploding along some path of states, thus violating feasibility.

3.3. **Capital inefficiency.** In a stochastic environment conditional Pareto optimality might fail without implying any capital overaccumulation. Capital is overaccumulated whenever aggregate consumption can be increased in some period without requiring any contraction in future periods. In our maintained example, letting

$$\gamma = \max_i \min_j \frac{1 + g_{ij}}{R_{ij}},$$

capital is not overaccumulated if $\gamma < 1$. The argument is almost immediate: as capital is sufficiently productive with some probability, any reallocation preserving aggregate consumption would require an increasing contraction of the capital stock to compensate for the output losses, eventually violating feasibility.

More formally, to the purpose of contradiction, assume that aggregate consumption can be increased. By feasibility, current capital contraction needs to exceed output losses due to previous capital decumulation. Therefore, capital readjustments necessarily satisfy

$$R_{t+1} \left(k_t - \hat{k}_t\right) \leq \left(k_{t+1} - \hat{k}_{t+1}\right),$$

where we assume that capital is reduced over time, so that $k_t - \hat{k}_t \geq 0$. Evaluating relative to output, this yields

$$\left(\frac{R_{t+1}}{1 + g_{t+1}}\right) \epsilon_t \leq \epsilon_{t+1},$$

where $k_t - \hat{k}_t = \epsilon_t y_t$. Along a path in which the ratio of growth into returns is bounded by $\gamma < 1$, which occurs with positive probability over any horizon of length t, feasibility implies $\gamma^{-t} \epsilon_0 \leq \epsilon_t$. As $\gamma < 1$, the dynamics is explosive, violating feasibility, $\epsilon_t \leq 1$.

3.4. **Why does growth adjustment matter?** The upshot of the above discussion is that a low interest rate might not be a symptom of inefficiency, as the pattern of output growth should be taken into account. We provide here a heuristic explanation of this result and comment on a possible relevant implication for policies. To this end, we consider a situation in which safe interest rate r is constant.

In the absence of uncertainty, the gap between r and g is the relevant sufficient statistic for dynamic efficiency and the failure of efficiency implies that positive transfers from young to old (a sort of *social security policy*) would Pareto improve upon a competitive equilibrium. Both of these properties fail in a stochastic environment. To see this, suppose that $r < \mathbb{E}g$ and evaluate the effect of transferring a certain amount $\epsilon_t > 0$ from young to old individuals at time t, with transfers growing at the average rate, $\epsilon_{t+1} = (1 + \mathbb{E}g)\,\epsilon_t$. The first-order effect of this consumption adjustment on the young individual is always welfare improving, as

$$\Delta U_t \approx -u'\left(c_t^y\right)\epsilon_t + \mathbb{E}_t\left[u'\left(c_{t+1}^o\right)\epsilon_{t+1}\right] = u'\left(c_t^y\right)\left(\frac{\mathbb{E}g - r}{1+r}\right)\epsilon_t > 0.$$

This first-order approximation, however, is only a misguided intuition.

The transfers are intergenerational and, in order to be welfare-improving, they have to satisfy, in all periods and for all realizations of uncertainty, the condition

$$U_t\left(c_t^y - \epsilon_t, c_{t+1}^o + \epsilon_{t+1}\right) \geq U_t\left(c_t^y, c_{t+1}^o\right).$$

Considering a log-utility, simple manipulations yield

$$\sum_j \mu_{ij} \log\left(\frac{R_{ij} + (1+\mathbb{E}g)\left(2\epsilon_t/y_t\right)}{R_{ij}}\right) \geq \log\left(\frac{1}{1-\left(2\epsilon_t/y_t\right)}\right).$$

The left hand-side is the benefit from additional consumption in the old-age, whereas the right is the utility loss due to less consumption when young. Using Jensen's inequality, a Pareto improvement requires

$$\frac{\epsilon_t}{y_t} = \frac{(1+\mathbb{E}g)^t\epsilon_0}{y_t} \leq \frac{\mathbb{E}g - r}{2(1+\mathbb{E}g)}.$$

And, evaluating along a path with low growth rate $g < \mathbb{E}g$,

$$\left(\frac{1+\mathbb{E}g}{1+g}\right)^t \frac{\epsilon_0}{y_0} \leq \frac{\mathbb{E}g - r}{2(1+\mathbb{E}g)},$$

a condition that cannot be verified for t large enough. The transfer scheme entails a perpetual commitment to compensate old individuals at the average growth rate $\mathbb{E}g$ for the consumption contraction when young, but this might turn unsustainable because output might grown at lower rate $g < \mathbb{E}g$ for a prolonged phase with some small probability. Along this path, the transfer ϵ_t grows faster than income y_t, thus becoming large relative to *status quo* consumption, and the first-order effect loses any informative content: the transfer might be actually welfare depressing. In other terms, the scheme of transfers cannot be implemented without compromising the welfare of certain future generations with small (however positive) probability.

In an economy without growth, in order to ensure the feasibility of uncontingent transfers, Hellwig [17] assumes that each generation receives an additional small income $\epsilon > 0$. This might admittedly seem an innocuous assumption. In an economy with stochastic growth, however, an analogous experiment would require an additional income growing according to

$$\epsilon_{t+1} = (1 + \mathbb{E}g)\,\epsilon_t.$$

This is instead a dramatic alteration of the growth pattern of the economy: the added resources would become dominant along some paths and the overall income would be eventually growing at least at the previous average growth rate $\mathbb{E}g$ of the economy. Under stochastic growth, the infeasibility of uncontingent transfers is the natural implication of uncertain growth prospects.

We conclude with the observation that the described scheme of transfers might not be welfare improving even though competitive equilibrium is conditional Pareto inefficient, with $\rho > 1$. Assuming shocks are identically and independently distributed, and considering a $\log$-utility for computations,

$$\left(\frac{1}{1+r}\right) = \mathbb{E}\left(\frac{1}{R}\right) \text{ and } \rho = \mathbb{E}\left(\frac{1+g}{R}\right).$$

By direct inspection, it is immediate to verify that these conditions are certainly consistent with $r < \mathbb{E}g$ and $\rho\,(T) > 1$.

4. Competitive equilibrium

We study a canonical overlapping-generations economy with capital accumulation. To simplify our analysis, we assume that growth is only determined by an exogenous technological progress. All other elements are conventional, and shared with Abel et al. [1] and the related literature. Unconventional is the spectral radius condition we propose to assess dynamic inefficiency.

We assume that uncertainty is governed by an irreducible Markov transition $\mu : S \to \Delta(S)$, where S is a finite state space and $\Delta(S)$ is the space of probability measures on S. This process will be affecting productivity and technological progress. For notational parsimony, we descrive all relevant variables as stochastic processes. In particular, we let $\mathcal{L}$ be the space of stochastic processes with values in $\mathbb{R}$, that is, an element f of $\mathcal{L}$ is a sequence $(f_t)_{t\in\mathbb{T}}$ of $\mathcal{F}_t$-measurable random variables $f_t : \Omega \to \mathbb{R}$, where $\mathcal{F}_t$ is the algebra generated by partial histories of Markov states in S and $\mathbb{T} = \{0, 1, \ldots, t, \ldots\}$ is the infinite sequence of periods. All our statements will be understood as relative to histories occurring with positive probability.

Production is described by a smooth, concave, strictly increasing, bounded (reduced-form) production function $f : \mathbb{R}^+ \to \mathbb{R}^+$. Production is subject to a stochastic process a in $\mathcal{L}^+$ affecting productivity exogenously. Thus, given capital stock k_t in $\mathbb{R}^+$, the output in the next period is $y_{t+1} = a_{t+1} f(k_t)$ in $\mathbb{R}^+$. We assume that the technology exhibits constant returns to scale and factor prices are determined by competitive markets. Consistently, capital return $R_{t,t+1}$ in $\mathbb{R}^+$ and wage w_t in $\mathbb{R}^+$ satisfy

$$R_{t,t+1} = a_{t+1} f'(k_t) \text{ and } w_t = a_t f(k_{t-1}) - a_t f'(k_{t-1}) k_{t-1}.$$

In particular, capital return equates the marginal product of capital.

The utility of a young agent is

$$U_t\left(c_t^y, c_{t+1}^o\right) = u\left(c_t^y\right) + \delta \mathbb{E}_t u\left(c_{t+1}^o\right),$$

where $u : \mathbb{R}^+ \to \mathbb{R}$ is a smooth, strictly increasing, concave Bernoulli utility and δ in $\mathbb{R}^{++}$ is a discount factor. Notice that this utility is evaluated *interim*, that is, conditional on information available at the birth of the generation. This lifetime utility is maximized subject to budget constraints

$$k_t + c_t^y \leq w_t \text{ and } c_{t+1}^o \leq R_{t,t+1} k_t,$$

where c_t^y and c_{t+1}^o in $\mathbb{R}^+$ are consumption when young and when old, and k_t in $\mathbb{R}^+$ is the investment in capital, the only asset available to transfer wealth over time.[6] At an interior optimal plan, the first-order condition imposes

$$u'\left(c_t^y\right) = \delta \mathbb{E}_t R_{t,t+1} u'\left(c_{t+1}^o\right).$$

Adhering to a common practice in this literature, we interpret marginal rates of substitution as *state prices*, or as a *stochastic discount factor*, that is,

$$m_{t,t+1} = \delta \frac{u'\left(c_{t+1}^o\right)}{u'\left(c_t^y\right)}.$$

In other terms, $m_{t,t+1}$ in $\mathbb{R}^+$ is the price at time t in $\mathbb{T}$ of one unit of output to be delivered in the next period conditional on the occurrence of some Markov state.

Given an initial stock of capital k_{-1} in $\mathbb{R}^+$, a *competitive equilibrium* is defined by a capital accumulation path k in $\mathcal{L}^+$, consumption plans (c^y, c^o) in $\mathcal{L}^+ \times \mathcal{L}^+$ and factor prices (R, w) in $\mathcal{L}^+ \times \mathcal{L}^+$ such that (a) the consumption plan maximizes lifetime utility subject to budget constraints of each generation, given prices, and (b) all markets clear, that is, at every t in $\mathbb{T}$,

$$k_t + c_t^o + c_t^y = a_t f(k_{t-1}).$$

We impose certain regularity conditions on the competitive equilibrium. These restrictions are minimal and will be maintained throughout the analysis.

[6] Completing the asset market with a full set of elementary Arrow securities would be immaterial in this framework, because young and old individuals cannot share risk due to the simple demographic structure.

First, we assume that consumption is always strictly positive, so that the first-order condition applies. Second, we postulate that expected output growth rate is bounded uniformly, that is, for some sufficiently large λ in $\mathbb{R}^+$, at every t in $\mathbb{T}$,

$$\mathbb{E}_t a_{t+1} \leq \lambda a_t.$$

The technological progress might sustain any arbitrarily large, though bounded, growth rate. Third, the marginal rate of substitution is also bounded uniformly, that is, for some sufficiently large λ in $\mathbb{R}^+$,

$$\delta u' \left(c_{t+1}^o \right) \leq \lambda u' \left(c_t^y \right).$$

Fourth, we assume that the marginal product of capital is uniformly strictly positive, relative to growth, that is, for some sufficiently large λ in $\mathbb{R}^+$,

$$a_{t+1} \leq \lambda a_t R_{t,t+1}.$$

These conditions will allow us to assess dynamic inefficiency by only evaluating first-order effects.

An allocation is *feasible* if, given initial capital stock k_{-1} in $\mathbb{R}^+$, for every t in $\mathbb{T}$,

$$\tilde{k}_t + \tilde{c}_t^o + \tilde{c}_t^y = a_t f \left(\tilde{k}_{t-1} \right).$$

A competitive equilibrium is conditional Pareto inefficient if there exists an alternative feasible allocation such that $\tilde{c}_0^o > c_0^o$ and, for every t in $\mathbb{T}$, $U_t \left(\tilde{c}_t^y, \tilde{c}_{t+1}^o \right) \geq U_t \left(c_t^y, c_{t+1}^o \right)$. It is only at no loss of generality that we assume that the welfare of the initial old generation increases (if not, we could reinitiate the economy at some future contingency when the first change takes place). Notice that, in evaluating inefficiency, the planner is entitled to modify the path of capital accumulation *and* lifetime consumption of all generations. Therefore, we do not distinguish inefficiencies arising from an overaccumulation of capital from those arising because of a misallocation of consumption.

5. Dynamic inefficiencies

5.1. **Overview.** In order to assess conditional Pareto inefficiency, we introduce a spectral radius approach. This method is a generalization of the dominant root criterion in the established literature (*e.g.*, Aiyagari and Peled [3]), and it has recently been applied by Christensen [13] and Hansen and Scheinkman [16] to long-run risk. The spectral radius is related to the yield of an hypothetical long-run discount bond delivering consumption indexed to growth. A competitive equilibrium is Pareto efficient when this yield is positive, and Pareto inefficient when negative. We also clarify that Pareto inefficiency might occur in the absence of capital overaccumulation, and propose an amended spectral radius approach to overaccumulation. We finally compare our criteria with the traditional net dividend criterion proposed by Abel et al. [1].

5.2. **Spectral radius.** To verify conditional Pareto efficiency of a competitive equilibrium, we introduce

$$\mathcal{L}(a) = \left\{ v \in \mathcal{L} : |v_t| \leq \lambda a_t \text{ for some } \lambda \in \mathbb{R}^+ \right\}.$$

This is the space of stochastic processes growing no faster than output over time. We then consider the operator $T : \mathcal{L}(a) \to \mathcal{L}(a)$ given by

$$(Tv)_t = \mathbb{E}_t m_{t,t+1} v_{t+1} = \delta \mathbb{E}_t \left(\frac{u'\left(c^o_{t+1}\right)}{u'\left(c^y_t\right)} \right) v_{t+1}.$$

This operator is well-defined because of the maintained assumptions on equilibrium. As in [5, 13, 16], we introduce the *spectral radius* defined as

$$\rho(T) = \lim_{n \to \infty} \sqrt[n]{\|T^n\|},$$

where the underlying supremum norm is

$$\|v\| = \inf \left\{ \lambda \in \mathbb{R}^+ : |v| \leq \lambda a \right\}.$$

The spectral radius coincides with the Perron-Frobenius dominant root when a corresponding eigenprocess v in the interior of $\mathcal{L}^+(a)$ exists, that is, $\rho(T) v_t = (Tv)_t$ at every t in $\mathbb{T}$. To illustrate the nature of the spectral radius, we provide examples and a heuristic interpretation. As the spectral radius method is not fully conventional, we present a basic theory in a dedicated Appendix A.

Example 5.1 (Markov framework)**.** Assume the stochastic discount factor is Markov with respect to finite state space S. In other terms, $q_{ij} = \mu_{ij} m_{ij}$ in $\mathbb{R}^+$ is the price in state i in S of one unit of output delivered in the next period conditional on the occurrence of state j in S. We can arrange all such state prices in a positive matrix Q in $\mathbb{R}^{S \times S}$. By Perron-Frobenius Theorem there exists a positive vector v in $\mathbb{R}^S$ such that, for some ρ in $\mathbb{R}^+$,

$$\rho v = Qv.$$

Alternatively,

$$\rho v_i = (Tv)_i = \sum_{j \in S} q_{ij} v_j.$$

By a well-know theorem of analysis, this is the spectral radius (see Appendix A).

Example 5.2 (Abel et al. [1])**.** This example is taken from Abel et al. [1, Section III]. The exogenous stochastic process a in $\mathcal{L}^+$ satisfies

$$a_{t+1} = (1 + g + \nu_{t+1}) a_t,$$

where g lies in $\mathbb{R}^{++}$ and ν in $\mathbb{R}$ is an identically and independently distributed shock with $1 + g + \nu > 0$, $\mathbb{E}\nu = 0$, and utility takes the log-form. The stochastic discount factor is

$$m_{t,t+1} = \frac{\delta}{1 + g + \nu_{t+1}}.$$

By direct computation,

$$(Ta)_t = \mathbb{E}_t m_{t,t+1} a_{t+1} = \delta \mathbb{E}_t \frac{1}{1+g+\nu_{t+1}} (1+g+\nu_{t+1}) a_t = \delta a_t.$$

Therefore, δ in $(0,1) \subset R^+$ is an eigenvalue, $\delta a = (Ta)$, and, as a lies in the interior of $\mathcal{L}^+(a)$, it is the spectral radius $\rho(T) = \delta$ (see Claim A.3 in Appendix A).

Remark 5.1 (Interpretation)**.** In an hypothetical steady-state, the spectral radius is

$$\rho = \left(\frac{1+g}{1+r}\right),$$

where g in $(-1,\infty) \subset \mathbb{R}$ is the growth rate and r in $(-1,\infty) \subset \mathbb{R}$ is the safe interest rate. Thus, interest rate exceeds growth when $\rho < 1$, and falls below growth when $\rho > 1$. In a stochastic environment, the assessment is less immediate. A straight comparison of average interest rate and average growth would not be meaningful, as interest rate and growth rate are compounded over time. The spectral radius extrapolates long-term tendencies. One interpretation is as the rate of growth of an investment fund relative to output. Consider a fund evolving according to

$$\mathbb{E}_t m_{t,t+1} w_{t+1} = w_t.$$

An initial wealth w_0 in $\mathbb{R}^+$ is invested in available securities and the proceedings of this investment are rolled over forever. To estimate the rate of growth of the fund relative to output, we consider the least ρ in $\mathbb{R}^+$ such that, for some sufficiently small λ in $\mathbb{R}^{++}$,

$$\rho^t w_t \geq \lambda a_t.$$

Thus, the ρ^{-1}-detrended value of the fund remains a share of output forever over time. Letting $v_t = \rho^t w_t$, we notice that

$$\mathbb{E}_t m_{t,t+1} v_{t+1} = \rho v_t \text{ or, equivalently, } (Tv) = \rho v.$$

By Claim A.3 in Appendix A, under further mild restrictions, we conclude that ρ in $\mathbb{R}^+$ is the spectral radius. Therefore, a well-managed fund will be growing, relative to output, at rate $\rho(T)^{-1} - 1$ over time.

Remark 5.2 (Long-term yield)**.** As remarked by Bloise et al. [8, Appendix C] and Kocherlakota [18, 19] among others, the spectral radius can be interpreted as the limit of the yield of a long-term discount bond. Indeed, by no arbitrage, the price of a bond paying off a share of the output after n periods, in terms of current output, is

$$q_t^n = \frac{1}{a_t} \mathbb{E}_t m_{t,t+n} a_{t+n} = \frac{1}{a_t} (T^n a)_t,$$

where the stochastic discount factor is compounded according to

$$m_{t,t+n} = m_{t,t+1} m_{t+1,t+2} \cdots m_{t+n-1,t+n}.$$

The yield to maturity r_t^∞ in $(-1, \infty) \subset \mathbb{R}$ of an hypothetical infinite-maturity discount bond, indexed to growth, is

$$\left(\frac{1}{1 + r_t^\infty}\right) = \lim_{n \to \infty} \sqrt[n]{q_t^n}.$$

Under certain regularity conditions (*e.g.*, [5, Assumption 1] and [19, Assumption 1]), this yield is precisely $r^\infty = \rho(T)^{-1} - 1$.

Remark 5.3 (Long-term *versus* short-term rate)**.** Kocherlakota [19, Proposition 4] argues that long yield falls short the expected short yield. More precisely, in our notation, he proves that

$$\mathbb{E} \log(1 + r) \geq -\log \rho,$$

where the unconditional expectation is taken with respect to the invariant measure. As a consequence, in an economy without growth, equilibrium is dynamically inefficient ($\rho > 1$) whenever the expected short-term interest rate is negative. This feature is illustrated by Example 5.3. Despite first appearance, this is not inconsistent with our previous claims. In our streamlined example, the short-term interest rate is constant and, thus, coincides with the long-term interest rate. The spectral radius, instead, is given as

$$\rho = \mathbb{E}m(1 + g) = \left(\frac{1 + \mathbb{E}g}{1 + r}\right) + \text{cov}(m, g).$$

Equilibrium might be dynamically efficient ($\rho < 1$) even though the short-term interest rate falls short expected growth.

Example 5.3 (Long-term *versus* short-term rate)**.** Consider a simple Markov setting with two states and deterministic cyclic transitions ($\mu_{ii} = 0$). There is no growth of the endowment. Let $R_1 > 0$ and $R_2 > 0$ be distinct safe returns in the two states, and assume asset-pricing under risk-neutrality ($m_{ij} = \mu_{ij} R_i^{-1}$). We argue that

$$\rho = \sqrt{\frac{1}{R_1 R_2}}.$$

Indeed, the eigenvalue equation writes as

$$\rho v_i = \frac{1}{R_i} v_j,$$

where the deterministic transition is from state i to the other state j. Setting $v_i = \sqrt{R_i^{-1}}$, straightforward calculations confirm our claim. We compare this dominant root with the average short-term rate in this simple setting, where the only invariant measure is $\mu = (1/2, 1/2)$. As shown in Figure 1, a positive expected short yield is consistent with dynamic inefficiency.

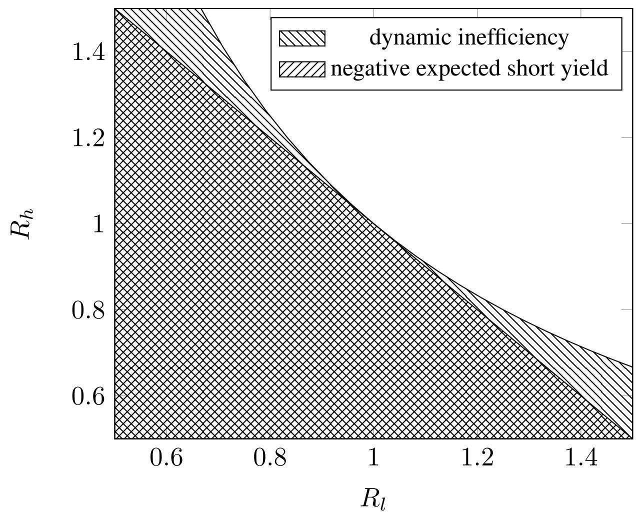

FIGURE 1. Short *versus* long yield

5.3. **Pareto inefficiency.** We are now ready to present a sufficient condition for Pareto efficiency. In particular, efficiency occurs when long-term interest rate exceeds growth, as precisely expressed by the condition $\rho(T) < 1$.

Proposition 5.1 (Sufficiency)**.** *A competitive equilibrium is efficient if* $\rho(T) < 1$.

Proof. Consider a planner improving upon the equilibrium allocation. Feasibility imposes

$$\hat{k}_t + \hat{c}_t^y + \hat{c}_t^o = a_t f\left(\hat{k}_{t-1}\right),$$

given the initial stock of capital k_{-1} in $\mathbb{R}^+$. The competitive equilibrium instead satisfies

$$k_t + c_t^y + c_t^o = a_t f\left(k_{t-1}\right).$$

We define

$$\tau_t = \tilde{c}_t^o - c_t^o - a_t\left(f\left(\tilde{k}_{t-1}\right) - f\left(k_{t-1}\right)\right),$$

and notice that by feasibility

$$\tilde{c}_t^y - c_t^y = -\tau_t - \left(\tilde{k}_t - k_t\right).$$

It worth remarking that $\tau_0 > 0$ at no loss of generality, as welfare increases for initial old individual and the initial capital stock is inherited from the past. We now derive the implication in terms of operator $T : \mathcal{L}^+(a) \to \mathcal{L}^+(a)$.

Concavity of the utility implies

$$0 \leq U_t\left(\tilde{c}_t^y, \tilde{c}_{t+1}^o\right) - U_t\left(c_t^y, c_{t+1}^o\right) \leq u'\left(c_t^y\right)\left(\tilde{c}_t^y - c_t^y\right) + \delta \mathbb{E}_t u'\left(c_{t+1}^o\right)\left(\tilde{c}_{t+1}^o - c_{t+1}^o\right).$$

Using the definition of process τ in $\mathcal{L}(a)$,

$$\begin{aligned} u'(c_t^y)\tau_t &\leq \delta\mathbb{E}_t u'(c_{t+1}^o)\tau_{t+1} \\ &\quad -u'(c_t^y)\left(\tilde{k}_t - k_t\right) + \delta\mathbb{E}_t u'(c_{t+1}^o)a_{t+1}\left(f\left(\tilde{k}_t, s_{t+1}\right) - f(k_t, s_{t+1})\right). \end{aligned}$$

By concavity of the production function, we finally obtain

$$\begin{aligned} u'(c_t^y)\tau_t &\leq \delta\mathbb{E}_t u'(c_{t+1}^o)\tau_{t+1} \\ &\quad -u'(c_t^y)\left(\tilde{k}_t - k_t\right) + \delta\mathbb{E}_t u'(c_{t+1}^o)R_{t.t+1}\left(\tilde{k}_t - k_t\right). \end{aligned}$$

As capital is optimally chosen by the young individual at equilibrium, the last term vanishes, so establishing that $\tau \leq (T\tau)$. Finally, possibly replacing it with $\max\{0, \tau_t\}$, we can assume that process τ lies in $\mathcal{L}^+(a)$. We now derive a contradiction.

At no loss of generality, we can assume that $\tau \leq a$. Notice that, for every n in $\mathbb{N}$, as linear operator $T : \mathcal{L}^+(a) \to \mathcal{L}^+(a)$ is monotone,

$$\tau \leq (T^n\tau) \leq (T^n a) \leq \rho^n a,$$

where ρ lies in $(\rho(T), 1) \subseteq \mathbb{R}^+$ and n in $\mathbb{N}$ is chosen sufficiently large. The middle inequality is due to monotonicity. To prove the extreme right hand-side inequality, we use the fact that, for any sufficiently large n in $\mathbb{N}$, $\sqrt[n]{\|T^n\|} \leq \rho$, so that

$$\|(T^n a)\| \leq \|T^n\| \leq \rho^n.$$

Moreover, by the supremum norm, $(T^n a) \leq \|(T^n a)\| a$. Hence, combining the last two inequalities, $(T^n a) \leq \rho^n a$. Finally, going to the limit,

$$0 < \tau \leq \lim_{n\to\infty} \rho^n a = 0,$$

a contradiction. □

We now turn to dynamic inefficiency, thus proving that equilibrium is inefficient whenever $\rho(T) > 1$. This involves two complications of independent nature. First, as the spectral radius capture first-order effects on welfare, some restrictions are needed to ensure that first-order welfare increases translate into actual welfare increases. This is not obvious because the economy involves *infinitely many* individuals. Second, the spectral radius might not be an eigenvalue of the operator and, thus, no eigenprocess can be associated to the spectral radius. As the eigenprocess identifies the direction of consumption changes yielding the Pareto improvement, the approach might fail. We repair to these potential issues by adding further assumptions on equilibrium, and on the induced operator $T : \mathcal{L}^+(a) \to \mathcal{L}^+(e)$. The first group of restrictions are shared with the established literature (see, for instance, the curvature conditions in Chattopadhyay and Gottardi [12]). The existence of an eigenprocess is specific to our approach, and can be weakened by adding restrictions on equilibrium processes, so as to ensure the applicability of a general Perron-Frobenius Theorem (as in Christensen [13] and Hansen and Scheinkman [16]).[7]

Non-vanishing consumption. *For some λ in $\mathbb{R}^{++}$, $c_t^y \geq \lambda a_t$ for every t in $\mathbb{T}$.*

Uniformly smooth preferences. *Given any process v in $\mathcal{L}^+(a)$, and given any η in $(0,1) \subset \mathbb{R}^+$,*

$$u'(c_t^y)\, v_t \leq \eta \delta \mathbb{E}_t u'\left(c_{t+1}^o\right) v_{t+1}$$

implies, for some sufficiently small ϵ in $\mathbb{R}^{++}$, uniformly over all periods t in $\mathbb{T}$,

$$U_t\left(c_t^y - \epsilon v_t, c_{t+1}^o + \epsilon v_{t+1}\right) \geq U_t\left(c_t^y, c_{t+1}^o\right).$$

The nature of this last condition is to prevent the insurgence of kinks in the limit indifference curves. For finitely many generations, this restriction is always satisfied: the first-order utility increase translates into an actual utility increase in an open neighborhood. With infinitely many generations, this is not granted anymore without further assumptions ensuring some uniformity of the open neighborhood.

Existence of an eigenprocess. *A dominant eigenprocess exists, that is, there exists v in $\mathcal{L}^+(a)$ such that*

$$\rho(T)\, v = (Tv).$$

Proposition 5.2 (Necessity)**.** *A competitive equilibrium is inefficient if $\rho(T) > 1$, provided that all the previous assumptions are satisfied.*

Proof. For a given η in $\left(\rho(T)^{-1}, 1\right) \subset \mathbb{R}^+$, we have $v \leq \eta(Tv)$. Consider the reallocation of consumption only given by $\tilde{c}_t^y = c_t^y - v_t$ and $\tilde{c}_t^o + v_t$. Up to an innocuous rescaling, because consumption of the young individual is not vanishing relative to output, this is feasible. Welfare increases for all generations by the assumption of uniformly smooth preferences. □

We add two remarks: first, on the gap between necessary and sufficient conditions; second, on the separation of consumption from production inefficiency. We do not assess efficiency when $\rho(T) = 1$. This is a situation in which the long-term interest rate exactly balances growth. In such a situation, the first-order effect is ambiguous and an evaluation of second-order effects becomes necessary. Under appropriate curvature assumptions, in line with the established literature (*e.g.*, Chattopadhyay and Gottardi [12]), a competitive equilibrium is efficient whenever $\rho(T) = 1$. We omit the proof because it would be laborious, reproducing steps in the previous literature and without adding any further insight. Therefore, up to technicalities, and subject to some appropriate curvature restrictions, we submit an educated conjecture.

[7]Assumption 1 in Kocherlakota [19] implies the existence of some sort of eigenprocess.

Conjecture 5.1 (Necessity and sufficiency)**.** *A competitive equilibrium is Pareto efficient if and only if $\rho(T) \leq 1$, provided that appropriate curvature conditions are satisfied in addition to the maintained assumptions.*

As a last observation, it is worth remarking that, whenever $\rho(T) > 1$, a planner is able to induce a welfare improvement by a mere reallocation of consumption, without any alteration of capital accumulation. This, of course, does not imply that a readjustment of production plans would not permit further welfare gains. Rather, it implies that a failure of Pareto efficiency is always revealed by a misallocation of consumption, and a supplemental investigation of the production side of the economy is unnecessary.

Conjecture 5.2 (Consumption inefficiency)**.** *A competitive equilibrium is Pareto inefficient if and only if a welfare improvement is feasible by a mere reallocation of consumptions, provided that appropriate curvature conditions are satisfied in addition to the maintained assumptions.*

5.4. **Capital overaccumulation.** Pareto efficiency depends of individual preferences, and such preferences might not be observable through prices when markets are incomplete. Mostly for this reason, the previous literature studied *capital overaccumulation*. This is a situation in which aggregate consumption might be increased in some period without being decreased in any other period. To ascertain overaccumulation of capital requires no evaluation of trade-offs and, hence, no knowledge of individual preferences. We now identify conditions ensuring the absence of capital overaccumulation.

Define the operator $D : \mathcal{L}(a) \to \mathcal{L}(a)$ as

$$(Dv)_t = \sup z_t \text{ subject to } R_{t,t+1} z_t \leq v_{t+1} \ \mathcal{F}_t\text{-almost surely},$$

where $\mathcal{F}_t$ is the information available at t in $\mathbb{T}$. This operator is monotone superlinear. As in our previous analysis, we define the spectral radius $\gamma(D)$ in $\mathbb{R}^+$. The (reciprocal of) the spectral radius might be interpreted as an estimation of the long-term return to capital along the most optimistic path. We provide an explicit computation in a simple Markov framework (Example 5.4).

Proposition 5.3 (Capital overaccumulation)**.** *Capital is not overaccumulated if $\gamma(D) < 1$.*

Proof. Supposing not, there exists an alternative capital accumulation path such that

$$\tilde{k}_t - k_t \leq a_t \left(f\left(\tilde{k}_{t-1}\right) - f(k_{t-1}) \right).$$

Indeed, this condition ensures that aggregate consumption does not decrease at any contingency. Exploiting concavity of the utility function, we obtain

$$R_{t-1,t}\left(k_{t-1} - \tilde{k}_{t-1}\right) \leq k_t - \tilde{k}_t.$$

Setting $v_t = \tilde{k}_t - k_t$, this implies

$$R_{t,t+1} v_t \leq v_{t+1} \text{ if and only if } v \leq (Dv) .$$

We can assume that v lies in $\mathcal{L}^+ (a)$ at no loss of generality, and argue as in the proof of our first proposition. □

Example 5.4 (Capital overaccumulation)**.** How to interpret the spectral radius condition for the absence of capital overaccumulation? It is worth considering a simplified framework in which the return to capital is governed by a Markov transition with strictly positive probabilities on a finite state space S. Consistently, we let R_{ij} in $\mathbb{R}^+$ be the return to capital invested in previous state i in S when state j in S occurs. We claim that

$$\gamma (D) \leq \gamma = \max_{i \in S} \min_{j \in S} \frac{1}{R_{ij}}.$$

Indeed, letting $\mathbf{1}$ in $\mathbb{R}^S$ be the unit vector, we see that

$$(D\mathbf{1})_i = \min_{j \in S} \frac{1}{R_{ij}} \leq \gamma \mathbf{1}_i,$$

thus implying that $\|D\| \leq \gamma$. It is then immediate to conclude that

$$\gamma (D) = \lim_{n \to \mathbb{N}} \sqrt[n]{\|D^n\|} = \inf_{n \in \mathbb{N}} \sqrt[n]{\|D^n\|} \leq \|D\| \leq \gamma.$$

In other terms, with no output growth, as long as the net return to capital remains strictly positive in every state with some probability, capital is not overaccumulated.

5.5. **A comparison.** We compare our characterization with the traditional net dividend criterion proposed by Abel et al. [1]. In particular, we relate that criterion to conditional Pareto efficiency and capital overaccumulation, and argue that our refinements permit an assessment even when the net dividend criterion remains ambiguous.[8]

The *net divided criterion* for Pareto efficiency is the requirement that, for some ϵ in $\mathbb{R}^{++}$,

$$\epsilon v_t \leq d_t,$$

where d in $\mathcal{L}^+$ is the net dividend and v in $\mathcal{L}^+$ is the value of the market portfolio. Identifying terms in our (as well as in Abel et al. [1]'s) framework, the value of the market portfolio is $v_t = k_t$, and the dividend is determined as

$$d_{t+1} = a_{t+1} f' (k_t) k_t - k_{t+1}.$$

In this notation, consumptions of young and old individuals are

$$c_t^y = w_t - v_t \text{ and } c_{t+1}^o = v_{t+1} + d_{t+1},$$

[8] Abel et al. [1]'s notion of dynamic efficiency coincides *stricto sensu* with conditional Pareto efficiency. However, their narrative unfolds along the idea of a socially inefficient stock of capital. The intuition provided after their Proposition 1 is based on a comparison between the total return on the aggregate stock of capital and the new investment and, as a matter of fact, their proof of inefficiency mirrors Cass [10]'s argument for capital overaccumulation.

exactly as in Abel et al. [1, Equations (1.2)-(1.3)]. As noticed by Chattopadhyay [11], the net dividend criterion is incomplete, and its amendment requires that the value of the market portfolio does not vanish relative to output. For this reason, we also postulate that, for some λ in $\mathbb{R}^{++}$, $v_t \geq \lambda a_t$ uniformly over all periods t in $\mathbb{T}$.

We argue that our criterion for conditional Pareto efficiency is in fact a refinement of Abel et al. [1]'s net dividend criterion.

Proposition 5.4 (Net dividend, I)**.** *The net dividend criterion is satisfied only if* $\rho(T) < 1$.

Proof. By no arbitrage, a necessary condition at a competitive equilibrium,

$$v_t = \mathbb{E}_t m_{t,t+1}(v_{t+1} + d_{t+1}).$$

Invoking the net dividend criterion,

$$v_t \geq (1+\epsilon)\,\mathbb{E}_t m_{t,t+1} v_{t+1}.$$

Therefore, setting $\rho(1+\epsilon) = 1$, we obtain $\rho v \geq (Tv)$, thus delivering $\rho(T) < 1$ (see Claim A.2 in Appendix A). □

Turning to the overaccumulation of capital, we prove that a substantially weaker criterion rules out this inefficiency, as illustrated by Example 5.5. To the purpose of comparison, we say that the net dividend criterion is satisfied *with some probability* if

$$\mu(\{\epsilon v_{t+1} \leq d_{t+1}\} \,|\mathcal{F}_t) > 0,$$

where $\mathcal{F}_t$ denotes the information available at t in $\mathbb{T}$. In fact, our spectral radius characterization implies a probabilistic version of the net dividend criterion.

Proposition 5.5 (Net dividend, II)**.** *The net divided criterion for Pareto efficiency is satisfied* with some probability *only if* $\gamma(D) < 1$.

Proof. The probabilistic net dividend criterion imposes, for some γ in $(0,1) \subset \mathbb{R}^{+}$,

$$k_{t+1} \leq \gamma R_{t,t+1} k_t \text{ with positive } \mathcal{F}_t\text{-conditional probability.}$$

By definition of $D : \mathcal{L}^{+}(a) \to \mathcal{L}^{+}(a)$, there exists z_t in $\mathbb{R}^{+}$ such that

$$(Dk)_t = z_t \text{ and } R_{t,t+1} z_t \leq k_{t+1}\ \mathcal{F}_t\text{-almost surely.}$$

Comparing with the net dividend criterion, we obtain

$$(Dk)_t = z_t \leq \gamma k_t.$$

As $k \geq \lambda a$ for some λ in $\mathbb{R}^{++}$, and $(Dk) \leq \gamma k$, this immediately implies $\gamma(D) < 1$ □

Example 5.5 (Net dividend)**.** Here is a simple example in which our criterion is satisfied, whereas the net dividend criterion fails. States $S = \{l, h\}$ can occur with equal probability in each period. Assume that capital stock is constant at level k in $\mathbb{R}^{++}$. Capital returns satisfy $R_l < 1 < R_h$. Hence, the net dividend criterion cannot be satisfied. By the characterization in Example 5.4, $\gamma(D) < 1$.

6. Conclusion

We have provided a characterization of dynamic inefficiency in overlapping generations economy with stochastic growth. Our major insight is that a meaningful assessment of dynamic efficiency cannot be based on a comparison between interest rates and average growth rate. In fact, low interest rates are not necessarily a symptom of dynamic inefficiency. Furthermore, even when equilibrium is inefficient, they do not imply that a social security scheme will be welfare improving.

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APPENDIX A. SPECTRAL RADIUS

A mathematical treatment of the spectral radius of positive linear operators can be found in Aliprantis and Border [4, Chapter 20]. For completeness we present a short treatment of the relevant theory for the linear monotone operator $T : \mathcal{L}(a) \to \mathcal{L}(a)$ used throughout our analysis.

The *spectral radius* is given by

$$\rho(T) = \lim_{n \in \mathbb{N}} \sqrt[n]{\|T^n\|},$$

where the operator norm is as usual defined as

$$\|T^n\| = \sup_{v \in \mathcal{L}(a)} \left\{ \|(T^n v)\| : \|v\| \leq 1 \right\}.$$

We preliminarily clarify that, by monotonicity, the operator norm is easily determined.

Claim A.1 (Operator norm)**.** *For every n in $\mathbb{N}$,*

$$\|T^n\| = \|(T^n a)\|.$$

Proof. The operator norm is given by

$$\|T^n\| = \sup_{v \in [-a,a]} \|(T^n v)\|.$$

Notice that, by monotonicity,

$$|(T^n v)| \leq (T^n |v|).$$

Indeed,

$$-(T^n |v|) \leq (T^n (-|v|)) \leq (T^n v) \leq (T^n |v|).$$

It follows that

$$\|(T^n a)\| \leq \|T^n\| \leq \sup_{|v| \leq a} \|(T^n |v|)\| = \|(T^n a)\|,$$

so proving the claim. □

We next provide a characterization of the spectral radius.

Claim A.2 (Spectral radius)**.** *Spectral radius $\rho(T)$ lies in $(0,1) \subset \mathbb{R}^+$ if and only if there exists ρ in $(0,1) \subset \mathbb{R}^+$ such that, for some f in the interior of $\mathcal{L}^+(a)$,*

(*) $$\rho f \geq (Tf).$$

Proof of necessity. As f lies in the interior of $\mathcal{L}^+(a)$, there exist λ_h and λ_l in $\mathbb{R}^{++}$ such that $\lambda_h a \geq f \geq \lambda_l a$. When condition (*) is satisfied, by iterating, we obtain

$$\rho^n f \geq \Pi^n(f).$$

Using the bounds, and exploiting monotonicity, this yields

$$\rho^n \left(\frac{\lambda_h}{\lambda_l}\right) a \geq (T^n a),$$

which in turn implies

$$\rho^n \left(\frac{\lambda_h}{\lambda_l}\right) \geq \|T^n\|.$$

Taking the root,

$$\rho \sqrt[n]{\frac{\lambda_h}{\lambda_l}} \geq \sqrt[n]{\|T^n\|},$$

thus proving that $\rho \geq \rho(T)$. □

Proof of sufficiency. Pick any ρ in $(\rho(T), 1) \subset \mathbb{R}^+$. For any sufficiently large n in $\mathbb{N}$,

(**) $$(T^n a) \leq \|T^n\| a \leq \rho^n a.$$

Defining

$$f^n = a + (Ta) + \cdots + (T^n a),$$

notice that, by linearity,

$$a + (Tf^n) = f^{n+1}.$$

The process monotonically (pointwise) converges to f in $\mathcal{L}^+(a)$ because condition (**) provides a geometric upper bound eventually. We so obtain that, in the limit,

$$a + (Tf) = f.$$

Furthermore, for some sufficiently large ρ in $(0, 1) \subset \mathbb{R}^+$,

$$(1 - \rho) f \leq a.$$

We so conclude that

$$(1 - \rho) f + (Tf) \leq f \text{ if and only if } (Tf) \leq \rho f.$$

This proves our claim. □

We conclude with a sort of Perron-Frobenius Theorem

Claim A.3 (Dominant root)**.** *If root ρ in $\mathbb{R}^+$ satisfies the eigenvalue equation $\rho f = (Tf)$ for some eigenprocess f in the interior of $\mathcal{L}^+(a)$, then it is the spectral radius, $\rho = \rho(T)$.*

Proof. At no loss of generality, we can assume that $\|f\| = \|a\| = 1$. By linearity, iterating the eigenvalue equation implies $\rho^n f = (T^n f)$. Using the definition of the operator norm,

$$\|T^n\| \geq \|(T^n f)\| = \|\rho^n f\| = \rho^n \|f\| = \rho^n.$$

This yields $\rho(T) \geq \rho$. Let λ in $\mathbb{R}^{++}$ be such that $f \leq a \leq \lambda f$. This is possible because f lies in the interior of $\mathcal{L}^+(f)$. By monotonicity, we obtain

$$(T^n a) \leq \lambda (T^n f) \leq \lambda \rho^n f \leq \lambda \rho^n a,$$

thus implying $\|T^n\| \leq \lambda \rho^n$ and, hence, $\sqrt[n]{\|T^n\|} \leq \sqrt[n]{\lambda}\rho$. As $\lim_{n \to \infty} \sqrt[n]{\lambda} = 1$, this proves the claim. □