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JEL Classification: C11, C32, D81, E32

Keywords: Identification, Heteroskedasticity, Sign restrictions

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Blended Identification in Structural VARs

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October 2022

Abstract

We propose a blended approach which combines identification via heteroskedasticity with the widely used methods of sign restrictions, narrative restrictions, and external instruments. Since heteroskedasticity in the reduced form can be exploited to point identify a set of orthogonal shocks, its use results in a sharp reduction of the potentially large identified sets stemming from the typical approaches. Conversely, the identifying information in the form of sign and narrative restrictions or external instruments can prove necessary when the conditions for point identification through heteroskedasticity are not met and offers a natural solution to the labeling problem inherent in purely statistical identification strategies. As a result, we argue that blending these methods together resolves their respective key issues and leverages their advantages, which allows to sharpen identification at virtually no cost. We illustrate the blending approach using several examples taken from recent and influential literature. Specifically, we consider labour market shocks, oil market shocks, monetary and fiscal policy shocks, and find that their effects can be rather different from what previously obtained with simpler identification strategies.

Keywords: SVAR, Identification, Heteroskedasticity, Sign restrictions, Proxy variables.
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1 Introduction

Since the seminal contribution of Sims (1980), the empirical analysis of shocks propagation is most often carried out using Vector Autoregressive (VAR) models. VARs can offer a very good description of the dynamics of macroeconomic and financial data, and - when coupled with an appropriate prior - have shown a remarkably good performance in out of sample forecasting.[1] If one wants to use VARs to answer policy questions, to analyze the effects of shocks, or more generally to give a causal interpretation to such class of models, it becomes necessary to map the VAR into a system of simultaneous equations.

As is well known in the macroeconometrics literature, the main difficulty of this task comes from the fact that such a mapping is not unique, and one has to rely on assumptions derived from economic theory or on external information to restrict the set of admissible structural models that share the same reduced form representation. The fundamental problem is that the data are only informative about the joint dynamics of macroeconomic and financial variables, but are silent about the contemporaneous causal links in economic variables unless further assumptions are made.

Traditionally, these additional assumptions took the form of zero restrictions on the matrix that defines the contemporaneous relations across the variables, or the matrix measuring the effect of a structural shock on a certain variable, either on impact or in the long run. However, the assumption that a shock does not affect at all certain macroeconomic variables is too restrictive in many cases. Moreover, the number of zero restrictions required increases rapidly with the size of the model, making restrictions even harder to choose and to defend. For this reason researchers have endeavored to devise alternative identification strategies able to narrow down the set of structural models in which a VAR can be mapped (see Kilian and Lutkepohl (2017) for a thorough treatment). Particularly successful strategies entail the use of sign restrictions, narrative restrictions, and external instruments (proxies).

The sign restriction approach, formulated by Faust (1998), Canova and De Nicolò (2002), and Uhlig (2005), is based on the conjecture that the effect of a shock on a given variable, at some horizon, has a pre-specified sign. More recently, this idea has been revisited by Baumeister

[1] See e.g. Litterman (1986), Banbura et al. (2010), Koop and Korobilis (2013) and Carriero et al. (2019).

and Hamilton (2015), who impose sign restrictions on the structural contemporaneous relationships between variables on the basis of widely accepted theoretical postulates. The remarkable merit of this class of identification strategies is that it replaces the strong assumptions on which zero restrictions are based with arguably milder conjectures about the sign of macroeconomic relations. The price for being able to impose these milder conditions is potentially high: sign restrictions provide set identification, as opposed to point identification. This means that, contrary to what is usually achieved imposing zeros restrictions, the resulting set of admissible structural models associated with the reduced form VAR is not singleton, and inference must be drawn on the basis of the whole set. [2]

Antolin-Diaz and Rubio-Ramirez (2018) propose to further restrict the set of admissible models by exploiting generally acceptable accounts of historical events. The authors call these additional constraints 'narrative restrictions' because they confine the set of admissible models to those that produce structural innovations in line with the chronicle of specific events. For instance, narrative restrictions may require that a certain structural shock takes large positive or negative values at specific dates, or that the shock is the main contributor to the change in a certain variable observed on a specific day. Similar restrictions were introduced by Ludvigson et al. (2021) in a frequentist setting to identify the effects of macroeconomic and financial uncertainty on the real economy. In many relevant cases, narrative restrictions are able to reduce the uncertainty about structural parameters, but, in general, they do not provide point identification.

On the other hand, the external instruments approach, pioneered by Stock (2008), Stock and Watson (2012) and Mertens and Ravn (2013), exploits the availability of variables that are believed to be correlated with some structural shocks of interest, but uncorrelated with all the other shocks. Also this approach allows to avoid the imposition of excessively strong restrictions on the model and is being widely and successfully implemented. However, also this strategy can (and does) in some cases run into problems. In particular, when more than one proxy variable is available for more than one shock, also this approach can only yield set identification, and some researchers in some instances had to complement this identification

[2] Bayesian methods for set identified model have been proposed by Rubio-Ramirez et al. (2010), Arias et al. (2018), Giacomini and Kitagawa (2021) and Inoue and Kilian (2021a), while frequentist methods are the focus of Granziera et al. (2018).

strategy with the reintroduction of some additional zero restrictions (e.g. Mertens and Ravn (2013), Lakdawala (2019)).

Parallel to these approaches (which are used widely), a separate strand of the recent literature has pointed towards exploiting heteroskedasticity for identification. Notable examples are Sentana and Fiorentini (2001), Rigobon (2003), Rigobon and Sack (2004), Lanne et al. (2010) and Wright (2012), who show that, if changes in the variances of structural shocks are heterogeneous enough, reduced form information is sufficient to pin down the parameters of the structural model. Lewis (2020, 2021) extends these ideas to more general and unknown variance processes, showing in a non-parametric setup that identification can be achieved even in the presence of misspecification of the variance process. However, the condition that changes in the variance of structural shocks are heterogeneous enough is not always supported by the data. If some of the variances change proportionally throughout the sample, Bacchiocchi et al. (2022) have shown that also this strategy only yields set identification. Moreover, even if the volatility processes of different shocks are not exactly proportional, the identifying information provided by the heteroskedasticity in the data may still be weak due to limited heterogeneity or limited samples.[3]

Moreover, another important limitation that has prevented identification through heteroskedasticity from being widely used by empirical researchers is that this method allows to pin down a set of orthogonal shocks, but does not offer a clear way to attach economic meaning to such shocks. In this sense identification trough heteroskedasticity offers statistical identification, not economic identification, an issue referred to as the "labelling" problem. In fact, the presence of heteroskedasticity allows the econometrician to determine uniquely the structural shocks that drive economic fluctuations, but it does not convey information about what the nature of the identified innovations is; one can only attach an ex-post label to shocks based on the effects it produces on the variables of the system, as done, for instance, by Lanne and Luoto (2020) and Brunnermeier et al. (2021).

In this paper we argue that the set identification strategies and the heteroskedasticity based strategies can be fruitfully used together, thereby resolving their respective key issues. We sup-

[3]See Stock and Watson (2016) and Lewis (2022) for an instrumental variable interpretation of identification through heteroskedasticity and a discussion of the weak identification problem.

port a view that researchers should consider blending together these alternative identification strategies as the blend contains more identifying information than the parts. Specifically, the identifying information embedded in heteroskedasticity can be used to sharpen the identification schemes based on sign restrictions, narrative restrictions, and external instruments. When the conditions for point identification through heteroskedasticity are met, the likelihood of the data becomes perfectly informative about structural parameters, making the model immune to the problem of never-updated priors intrinsic in set-identified models. We show how heteroskedasticity can be routinely and effectively used to reduce the identified set of VARs identified via sign restrictions, narrative restrictions, and external-instruments approaches. Similarly, we show that the information contained in sign restrictions, narrative restrictions, and external instruments schemes can help overcoming the key issues inherent in heteroskedasticity-based strategies, i.e. the lack of labeling and the possibly weak identification power.

Following different Bayesian and frequentist strategies respectively, Braun (2021), and Drautzburg and Wright (2021) also aim at refining set identified models by considering features of the data that are usually overlooked. Both these papers exploit non-Gaussianity to further restrict the set of admissible structural parametrization but consider different restrictions on the distribution of the structural shocks. Different from our case, Braun's (2021) method rules out the presence of cross-sectional volatility clustering, while Drautzburg and Wright (2021) grant more flexibility in the number of moments conditions imposed, adapting their method to allow for co-kurtosis. Furtehrmore, both the contributions only consider combining the identifying information coming from non-Gaussianity with sign restrictions, while we also extend our blended approach to the case of set-identifying information in the form of narrative restrictions and instrumental variables. In a related series of papers including Lutkepohl and Netsunajev (2014), Lutkepohl and Wozniak (2020), and Schlaak et al. (2021), heteroskedasticity is exploited to develop tests of over-identifying assumptions in the form of zero restrictions and sign restrictions, or to test the validity of external instrument in a Proxy-SVAR. Differently from these contributions, we propose to exploit heteroskedasticity in conjunction with trusted incomplete identifying assumptions, maintaining both types of restrictions at the same time. The difference in the objectives of the two strategies is therefore substantial: while they exploit statistical identification to verify the validity of economically motivated restrictions, our goal is to combine both types of information in order to compensate their respective weaknesses and maximize the extent to which we can learn from the data.

We develop three algortihms for estimation of SVARs under blended identification. The first algorithm applies to SVARs in which sign restrictions are imposed trough a truncation in the prior on the parameters describing the contemporaneous relations in the structural form as in Baumeister and Hamilton (2015). The second algorithm applies to SVARs in which sign restrictions are imposed on the impact matrix trough the specification of a rotation matrix with a uniform prior distribution, as in Uhlig's (2005) and Rubio-Ramirez et al. (2010), and combined with narrative restrictions as in Antolin-Diaz and Rubio-Ramirez (2018). The third algorithm applies to Proxy-SVARs along the lines of Stock (2008), Stock and Watson (2012) and Mertens and Ravn (2013).

We illustrate the convenience of the blended approach by revisiting empirical results presented in seminal papers based on either sign restrictions, narrative restrictions, or proxy variables. With reference to sign restrictions, we consider both Baumeister and Hamilton's approach and the conventional approach. In particular, we revisit the applications in Baumeister and Hamilton (2015) and in Kilian and Murphy (2012), analyzing the effects of labour market and oil market shocks, respectively. With reference to the narrative restrictions, we revisit the monetary policy application in Antolin-Diaz and Rubio-Ramirez (2018), which is based on the seminal Uhlig's (2005) sign-restricted SVAR. Finally, with reference to the external instruments approach, we reproduce the Proxy-SVAR model of Mertens and Ravn (2013), whose aim is to assess the effects of APITR (Average Personal Income Tax Rate) and ACITR (Average Corporate Income Tax Rate) shocks, and we do this by relaxing the zero restrictions that allow them to achieve point identification.

Our empirical applications show that the introduction of identifying information coming from heteroskedasticity can generate substantial changes in the results. For example, in the labor market application, blending leads to a significantly steeper aggregate labour supply and a demand that is almost flat. The responses of employment to demand and supply shocks are importantly revised, with wages that are significantly more responsive to demand shocks, when compared to the model that does not exploit changes in variances. In the application on oil, our results show that oil-specific demand shocks affect global economic activity only marginally, but they represent the main driver of oil price changes in the short run. In the monetary policy application, a monetary contraction has deeper recessive consequences once our blended identification is applied, and the ability to curb inflation at short horizons is called into question. In the fiscal policy application, both types of fiscal contractions have deep negative effects on output once heteroskedasticity is taken into consideration. Following a tax shock, the corresponding income tax base decreases. This mechanism, together with the negative effect on output, prevents government debt from diminishing significantly in response to the unexpected corporate tax increase. On the other hand, a personal income tax shock allows a reduction of the government debt, but only at short to medium horizons.

The paper is structured as follows: Section 2 describes the structural VAR (SVAR) model and discusses the identification problem. Section 3 reviews the use of sign restrictions in a SVAR, it illustrates the Bayesian algorithm we employ in order to draw from the posterior of a heteroskedastic version of the model (SVAR-H), and revisits some empirical results obtained by seminal papers in this literature, namely Baumeister and Hamilton (2015) and Kilian and Murphy (2012). Section 4 extends the idea to the case in which narrative restriction are available and shows how the results of the monetary application in Antolin-Diaz and Rubio-Ramirez (2018) are affected when our blended strategy is applied. Section 5 describes the Proxy-SVAR approach, adapts it to account for heteroskedasticity and re-examines the effects of personal and corporate tax surprises using data from Mertens and Ravn (2013). Section 6 discusses some robustness checks and section 7 summarizes and concludes.

2 Identification approaches in SVAR models

This section provides a brief overview of the alternative identification methods used in this paper. Specifically, we describe identification trough sign restrictions, narrative restrictions, external instruments, and heteroskedasticity. This is not intended as an exhaustive treatment, it rather aims at laying out the ground work and notation for the description of our blending approach discussed in Section 3, Section 4, and Section 5.

2.1 Baseline SVAR model and Cholesky identification

The joint dynamics of macroeconomic variables is often modeled as a VAR(p) process:

$$y_t = \Pi x_{t-1} + u_t, \tag{1}$$

where y_t is a $n \times 1$ vector of endogenous variables, x_{t-1} is the $k \times 1$ ($k = np + 1$) vector $\left[1, y'_{t-1}, \ldots, y'_{t-p}\right]'$, and Π is a $n \times k$ matrix of coefficients. The $n \times 1$ vector of prediction errors u_t is assumed to be normally distributed with zero mean and variance Ω, and it is a linear combination of structural shocks: $u_t = A^{-1}\Lambda^{0.5}\varepsilon_t$, $\varepsilon_t \sim N(0, I_n)$, with $\Lambda = diag[\lambda_1, ..., \lambda_n]$ and A a full matrix with ones on the main diagonal.[4]

Macroeconomists are interested in the propagation of structural innovations, which are considered the drivers of economic fluctuations. For this reason, policy analysis is usually carried out considering the structural form of the VAR model. This is obtained premultiplying both sides of equation (1) by A:

$$Ay_t = Bx_{t-1} + \Lambda^{0.5}\varepsilon_t, \tag{2}$$

where $B = A\Pi$.

The main difficulty when moving from the reduced form in (1) to the structural representation in (2) is that this mapping is not unique, unless more information is introduced in the model. This problem is known as the identification problem in the SVAR literature and it arises because the data are only informative about the parameters (Π, Ω), but they cannot distinguish between alternative structural specifications, (A, B, Λ), that give rise to the same reduced form model. To see this, note that the following must hold:

$$\Omega = A^{-1}\Lambda\left(A^{-1}\right)'. \tag{3}$$

Since the reduced form variance Ω is symmetric, (3) provides a system of $n(n+1)/2$ equations in which the left hand side can be inferred from the data, but the right hand side contains

[4] In some applications, it may be more convenient to impose unit diagonal elements in A_0^{-1} rather than in A_0. This is the case, for example, in the application described in Section 5 and leaves our reasoning unaffected.

$n(n-1)+n$ unknown elements. As a result, (3) does not have a unique solution and more information has to be introduced in order to obtain an economically meaningful parametrization of the SVAR (2).

A typical solution is to use a Cholesky decomposition, which amounts to put (linearly independent) zero restrictions on $n(n-1)/2$ coefficients in A^{-1} , or other types of restrictions implementing beliefs on the long run behavior of the system, or other choices of restrictions based on economic theory. All these approaches are valid but typically require more zero restrictions than researchers and macroeconomist are comfortable with.[5]This issue becomes worse as the size of the model increases, which is problematic since there is much theoretical and empirical support for using VARs with a rather large cross-sectional dimension in order to describe the data appropriately and avoid issues of nonfundamentalness.[6]

To avoid the problems associated with zero restrictions, researchers have developed alternative identification strategies, some of which are being increasingly used, namely sign restrictions, narrative restrictions, and external instruments. These strategies have the clear advantage of imposing a constraint on the model which is not only milder, but also in line with high level, broadly defendable, identifying assumption.

2.2 Sign restrictions, narrative restrictions, and external instruments

The use of sign restrictions consists in considering only the solutions of equation (3) that satisfy the assumption that certain elements of A^{-1}(or A in the case of Baumaister and Hamilton, 2015) have a specified sign. These assumptions are usually made on the basis of widely acceptable theoretical justifications, which explains why this identification strategy has become one of the most popular in the empirical literature. The set of admissible solutions can be further restricted considering only those structural parameterizations that produce structural shocks possessing characteristics in line with the narrative information at hand, as proposed by Antolin-Diaz and

[5]As discussed by Primiceri (2005), Carriero et al. (2019) and Arias et al. (2022), a separate issue arises at the estimation stage when using a Cholesky-type factorization of the error variance, if the priors are elicited on the individual elements of the Cholesky factor. In this case alternative orderings imply alternative priors on the error variance, and therefore different posterior distributions for the reduced form parameters.

Our methodology does not suffer from this issue because it allows for a full impact matrix A. The ordering invariance of point and density forecasts produced by a VAR identified through heteroskedasticity is in fact the focus of Chan et al. (2021).

[6]See e.g. Banbura et al. (2010).

Rubio-Ramirez (2018) and Ludvigson et al. (2021). Such mild assumptions, however, are often not enough to make the solution of (3) unique. For this reason, inference must be drawn from a set of equally likely parameterizations (A, B, Λ) with possibly vague interpretations.

External instruments (also referred to as proxy variables), when available, provide precious information that can be used to obtain identification. The special feature of proxy variables is that they are known to be correlated with the structural shocks of interest (*relevance condition*) but uncorrelated with all other shocks (*exogeneity condition*). If one proxy is available for the i-th shock, the parameters of the i-th equation in the SVAR (2) can be uniquely determined. However, in practice it is often the case that $m > 1$ instruments are correlated with m shocks of interest and uncorrelated with all the other $n-m$. In that case, the parameters of the m equations in (2) that are associated with the m innovations of interest are only set identified, in the sense that there exists a set of parameterizations that satisfy the conditions implied by the external instruments, but all the elements of this set are observationally equivalent. In order to reduce the set to a collection of economically meaningful models, researchers often impose additional zero restrictions, as in Mertens and Ravn (2013) and Lakdawala (2019), or sign restrictions, as in Piffer and Podstawski (2017) , Braun and Bruggemann (2022) and Arias et al. (2021).

In summary while sign restrictions, narrative restrictions, and external instruments do allow to obtain identification using just some mild high-level assumptions, this comes at the cost of the system being often only set-identified, with the identified set being potentially large. In the following subsection we discuss how this cost can be partially offset by exploiting the information contained in heteroskedasticity.

2.3 Identification through heteroskedasticity

Another identification strategy that has been put to the fore is based on heteroskedasticity. This is a purely statistical method to achieve a unique parametrization (A, B, Λ) and is based on results developed by Sentana and Fiorentini (2001), Rigobon (2003) and Lanne et al. (2010). The strategy is a typical application of the identification through stability restrictions discussed by Magnusson and Mavroedis (2014).

Intuitively, heteroskedasticity adds more equations than free parameters to the system in (3).

If one considers the case with S volatility-regimes, the mapping of the reduced form variance Ω_s, $s = 1,...,S$, into the structural parameters (A, Λ_s) becomes:

$$\Omega_s = A^{-1} \Lambda_s \left(A^{-1}\right)', \tag{4}$$

which is now a system of $S[n(n+1)/2]$ equations in $n(n-1) + Sn$ unknowns. Results in Lanne et al. (2010) and Lutkepohl and Wozniak (2020) prove that, as long as there exist at least two regimes, s_1 and s_2, for which $\lambda_{i,s_1}/\lambda_{j,s_1} \neq \lambda_{i,s_2}/\lambda_{j,s_2}$, $\forall j \neq i$ and $\forall i \in \{1,...,n\}$, then the parameters of the structural model (2) are uniquely identified up to a permutation of the columns of A^{-1}. In other words, if changes in the variance of shocks are not proportional across all regimes, heteroskedasticity allows to achieve point identification. This procedure is silent about the economic nature of ε_t, and any permutation of these shocks is observationally equivalent.

Differently from the approaches discussed above, as long as the heteroskedasticity process is not proportional across shocks, this approach does achieve a unique structural parameterization, and therefore it delivers point identification. However, this approach has a shortcoming insofar it is a purely statistical method which identifies a set of mutually orthogonal shocks, but gives no guide to the researcher as to what these shocks are representing in terms of relevant economic variables. This issue is known as the labeling problem. Furthermore, when the change in the variances of some structural shocks is close to proportional, the sampling uncertainty may significantly weaken the ability of disentangling the effects of those shocks.

There is a definite gain when introducing heteroskedasticity in SVARs identified via sign restrictions, narrative restrictions, or external instruments: the identifying information stemming from the variation in the variances will allow to dramatically reduce the identified set stemming from these approaches, in some cases reducing it to a point. In other words, heteroskedasticity in the reduced form can be used to pick one model out of all the ones in the identified set. One can also think of this procedure backwards: starting from a set of structural shocks (possibly weakly) identified via heteroskedasticity, and therefore un-labeled, one can use sign restrictions, narrative restrictions, and external instruments to label the shocks, and to help discerning the effects of those shocks whose volatility process cannot be clearly distinguished by the data alone. Since both these interpretations are valid we refer to our strategy as “blending” the two identification approaches.

The benefits of "blending" come at the cost of requiring a parametric specification for the time variation in the reduced form conditional variances, which we will detail below. As stressed by Montiel Olea et al. (2021), identification from heteroskedasticity does not simply exploit more information in the data than traditional SVAR methods, it also requires stronger assumptions on the shock processes than unconditional second-moment methods do.

We argue that this cost is largely mitigated by two considertations. First, the overwhelming evidence in favour of time varying volatility in macroeconomic data, documented in studies such as Clark (2011), Clark and Ravazzolo (2015) and Chan and Eisenstat (2018). The risk of misspecifying the specific law of motion of the volatilities is present, but results in Lewis (2021) and Sims (2020) show that even in presence of a misspecified process for the volatility regimes, the identification via heteroskedasticity results still hold.

Second, the cost emphasized by Montiel Olea et al. (2021) is already present - and to a larger extent - in some of the most popular methods in the literature. Specifically, the sign and narrative restriction approaches of Baumeister and Hamilton (2015) and Antolin-Diaz and Rubio-Ramirez (2018) both assume that the shocks are homoskedastic and unconditionally Gaussian. This assumption goes directly against the evidence mentioned above. Instead, our assumption that the shocks are heteroskedastic is milder and implies that the shocks are unconditionally a mixture of Gaussians. The homoskedastic Gaussian VARs of Baumeister and Hamilton (2015) and Antolin-Diaz and Rubio-Ramirez (2018) are nested in our more general specification, they can be obtained as a special case by imposing the (strong) restriction that the variance of the volatility states is zero.

We will assume for the rest of the paper that the variances of structural innovations evolve as a regime switching process, where the dates of regime changes are either known to the researcher or determined on the basis of a change-point specification. It is worth emphasizing that recognizing the exact date of regime changes is not necessary for consistency. Rigobon (2003) and Sims (2020) show that estimates of the structural parameters are still consistent if the windows of heteroskedasticity are moderately misspecified. The main assumption behind identification through heteroskedasticity is that the variation in the reduced form error variance

Ω_s is entirely driven by the variation in the structural shocks variance Λ_s while the contemporaneous impact of shocks on economic variables, defined by A^{-1}, is time invariant.[7]

3 Blending sign restrictions and heteroskedasticity

The traditional way of implementing sign restrictions is to estimate reduced form parameters (Π, Ω) and then to consider all the possible structural representations that are consistent with the desired signs for the impact response to an impulse. This is achieved by defining an orthonormal rotation $QQ' = I_n$ and the corresponding rotated shocks $Q\varepsilon_t$. The rotated shocks enter (2) in the place of ε_t and the resulting system is observationally equivalent to the original one, since equation (3) continues to hold:

$$\Omega = A^{-1}QQ'\Lambda Q\left(A^{-1}Q\right)' = \widetilde{A}^{-1}\widetilde{\Lambda}\left(\widetilde{A}^{-1}\right)'. \tag{5}$$

Finally, a uniform prior is specified on Q and draws are performed from this distribution, retaining only those draws yielding the desired sign in the impact matrix $\widetilde{A}^{-1}$. [8]

In an influential paper, Baumeister and Hamilton (2015, BH henceforth) point out some limitations of this approach. Their main argument is that the prior over the admissible SVAR parameterizations is never updated by the data, and apparently uninformative priors for the rotation matrix Q can translate into very informative and implausible priors for objects of interest, such as Impulse-Response Functions (IRF).

Their proposed method to overcome these limitations follows from the idea of Sims and Zha (1998) of eliciting informative priors on the structural form in (2) directly. Specifically, BH introduce sign restrictions in this setting by placing truncated student-t priors on specific elements of A. Although it is not an essential aspect of their suggested method, BH's preference for considering sign restrictions on the contemporaneous relations in A rather than on the impact matrix $A^{-1}$is based on a view that the former type of constraints are more easily elicited on the basis of economic theory. For example, in order to identify the simultaneous equations describ-

[7]Lutkepohl and Schlaak (2021) propose to exploit the over-identifying restrictions in a heteroskedastic Proxy-SVAR to test this assumption.

[8]Rubio-Ramirez, Waggoner and Zha (2010) develop an efficient algorithm for this approach.

ing demand and supply in SVAR models, the researcher can impose that price and quantity are negatively related in the former and positively related in the latter.

It must be noted that in the class of models proposed by BH, even though the posterior distribution is able to assign different probabilities to alternative structural models in the identified set, such distinction is entirely based on - albeit well motivated - prior information. Instead, the likelihood remains unable to distinguish between points in the structural parameter space associated with the same reduced form representation, i.e. it features some flat regions. The blended approach we suggest in this paper often allows to achieve point identification, because the mere introduction of heteroskedasticity in the structural shocks (when indeed present in the data) removes these flat regions from the likelihood.

As pointed out by Montiel Olea et al. (2021), traditional identification strategies do not necessarily rely on the assumption of homoskedastic shocks, because standard Quasi-ML estimators remain consistent even in the presence of heteroskedasticity. However, it is important to stress that the consistency argument made by Montiel Olea et al. (2021) does not directly apply to the BH model considered in this subsection, as this is a fully parametric homoskedastic Gaussian VAR, estimated with Bayesian methods.

In this section, we begin by showing how heteroskedasticity can be introduced in the SVAR to obtain what we label SVAR-H and make the data more informative about the structural parameters. In this way, as long as the conditions stated in Lanne et al. (2010) and Lutkepohl and Wozniak (2020) are satisfied, there will be no regions of the parameters space in which the likelihood is flat. The SVAR-H we present is a natural extension of the model proposed by BH as it directly elicits priors on structural parameters. However, following arguments put forward by Inoue and Kilian (2021a) and Arias et al. (2022), we remark that BH's method does not represent a definitive solution to the problem of never-updated priors in sign restricted SVAR, and researchers may still feel more comfortable eliciting Normal-Inverse-Wishart (N-IW) priors on the reduced form parameters and a uniform (U) prior over the set of admissible rotations Q. As a matter of fact, Arias et al. (2022) have proven that in order to specify a uniform joint prior for the vector of impulse response functions, it is both necessary and sufficient to assume a uniform prior over the set of orthonormal matrices Q. For this reason, we continue the section by showing how our blended identification approach can be easily implemented in the conventional U-N-IW setting thanks to the distributional equivalence pointed out by Arias et al. (2018).

Subsection 3.1 describes the general Bayesian algorithm we use to draw from the posterior distribution of the SVAR-H model, while in subsection 3.2.1 we revisit the empirical results in BH through the lenses of our heteroskedastic model. Subsection 3.2.2 demonstrates the ability of heteroskedasticity to sharpen structural inference in the oil market model of Kilian and Murphy (2012).

3.1 Structural VAR with heteroskedasticity (SVAR-H)

In this section we borrow notation from Lutkepohl and Wozniak (2020) and write $\lambda_{i,s} = \lambda_{i,1}\omega_{i,s}$, $s = 1,\ldots,S$, where it is understood that $\omega_{i,1} = 1$ for every $i \in \{1,\ldots,n\}$. Although this specification does not change any property of the model, it provides an immediate way to check that enough heterogeneity is present in the changes of variance regimes.

If we assume that the data are generated by the SVAR(p) model in (2) with time varying Λ_s, $s = 1,\ldots,S$, the likelihood is:

$$p\left(y_{1:T}|A,B,\Lambda_{1:S}\right) = (2\pi)^{-Tn/2}\left|\det\left(A\right)\right|^{T}\left(\prod_{i=1}^{n}\lambda_{i,1}^{-T/2}\right)\left(\prod_{i=1}^{n}\prod_{s=1}^{S}\omega_{i,s}^{-T_s/2}\right)\times$$

$$\exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\lambda_{i,1}^{-1}\left[\sum_{s=1}^{S}\omega_{i,s}^{-1}\left(A_i y^{(s)} - B_i x^{(\mathrm{s})}\right)\left(A_i y^{(s)} - B_i x^{(\mathrm{s})}\right)'\right]\right\}, \tag{6}$$

where T is the time series size of the sample and T_s is the length of regime s. The matrices $y^{(s)}$ and $x^{(s)}$ collect the values of y_t and x_{t-1} observed during regime s and have size $n \times T_s$ and $k \times T_s$ respectively, while A_i and B_i denote the i-th row of A and B.

Adopting the notation in Braun (2021), we write $A_i' = W_i\alpha_i + w_i$,[9] and assume independent priors for the free elements in A_i, $p\left(\alpha_i\right)$, conditionally normal priors with mean μ_{b_i} and variance V_{b_i} on the vectors of coefficients B_i, independent inverse-gamma priors with $d_{i,1}/2$ degrees of freedom and scale parameter $\zeta_{i,1}/2$ for the variances $\lambda_{i,1}$, inverse-gamma priors with $d_{i,s}/2$ degrees of freedom and scale parameter $\zeta_{i,s}/2$ for the ratios $\omega_{i,s}$, for $s = 2,\ldots,S$.

[9] W_i is a $n \times n_\alpha$ selection matrix and w_i is a $n \times 1$ vector fixes one of the elements of $A_i{}'$ to 1.

The free elements α_i have a conditional posterior distribution whose density is:

$$p(\alpha_i|Y,\Lambda_{1:S}) \propto p(\alpha_i)\left|\det(A)\right|^T \exp\left[-\frac{1}{2}(\alpha_i-\bar{\alpha}_i)'\bar{V}_{\alpha_i}^{-1}(\alpha_i-\bar{\alpha}_i)\right], \quad (7)$$

where:

$$\bar{V}_{\alpha_i}^{-1} = W_i'\left(\lambda_{i,1}^{-1}\sum_{s=1}^{S}\omega_{i,s}^{-1}y^{(s)}y^{(s)\prime} - \Upsilon'\bar{V}_{b_i}\Upsilon\right)W_i,$$

$$\bar{\alpha}_i = \bar{V}_{\alpha_i}W_i'\left[\Upsilon'\bar{V}_{b_i}\left(V_{b_i}^{-1}\mu_{b_i} + \lambda_{i,1}^{-1}\sum_{s=1}^{S}\omega_{i,s}^{-1}x^{(s)}\left(w_i'y^{(s)}\right)'\right) - \left(\lambda_{i,1}^{-1}\sum_{s=1}^{S}\omega_{i,s}^{-1}y^{(s)}y^{(s)\prime}w_i\right)\right],$$

with $\Upsilon = \lambda_{i,1}^{-1}\sum_{s=1}^{S}\omega_{i,s}^{-1}y^{(s)}x^{(s)\prime}$.[10] The conditional posterior distribution of B_i is shown in Appendix A to be normal with moments:

$$\bar{V}_{b_i} = \left[V_{b_i}^{-1} + \lambda_{i,1}^{-1}\sum_{s=1}^{S}\omega_{i,s}^{-1}x^{(s)}\left(x^{(s)}\right)'\right]^{-1},$$

$$\bar{\mu}_{b_i}' = \left[\mu_{b_i}'V_{b_i}^{-1} + \lambda_{i,1}^{-1}\sum_{s=1}^{S}\omega_{i,s}^{-1}A_iy^{(s)}\left(x^{(s)}\right)'\right]\bar{V}_{b_i}. \quad (8)$$

The variances $\lambda_{i,1}$, conditional on all other parameters, have inverse-gamma posteriors with $\frac{\bar{d}_{i,1}}{2} = \frac{d_{i,1}+T}{2}$ degrees of freedom and scale parameters

$$\frac{\bar{\zeta}_{i,1}}{2} = \frac{\zeta_{i,1} + \sum_{s=1}^{S}\omega_{i,s}^{-1}\left(A_iy^{(s)} - B_ix^{(\mathbf{s})}\right)\left(A_iy^{(s)} - B_ix^{(\mathbf{s})}\right)'}{2}. \quad (9)$$

Finally, the ratios $\omega_{i,s}$ have inverse-gamma conditional posteriors with $\frac{\bar{d}_{i,s}}{2} = \frac{d_{i,s}+T_s}{2}$ degrees of freedom and scale parameters

$$\frac{\bar{\zeta}_{i,s}}{2} = \frac{\zeta_{i,s} + \lambda_{i,1}^{-1}\left(A_iy^{(s)} - B_ix^{(\mathbf{s})}\right)\left(A_iy^{(s)} - B_ix^{(\mathbf{s})}\right)'}{2}. \quad (10)$$

The detection of the variance regimes is performed through a change-point specification of the type proposed by Chib (1998). Chib's method can be thought of as a standard Markov

[10]The derivation of the posterior distribution is illustrated in Appendix A.

Switching model with banded $S \times S$ transition probability matrix P, [11] so that once the model has entered a regime it can only move to the next one. Setting independent Dirichlet priors $D(\underline{\tau}_{s,s}, \underline{\tau}_{s,s+1})$, $s = 1, ..., S-1$, for the free elements in P, the conditional posteriors are conjugate: $D(\underline{\tau}_{s,s} + (T_s - 1), \underline{\tau}_{s,s+1} + 1)$. The states, $\delta_t \in \{1, ..., S\}$, $t = 1, ..., T$, can thus be smoothed using the Hamilton filter. In what follows, we set a priori the number of regimes $S = 3$; we replicated all the examples in the paper with $S = 2$, with almost no difference in the results.

Given these conditional densities, draws from the joint posterior of the parameters of the SVAR-H model can be obtained implementing a Markov Chain Monte Carlo (MCMC) algorithm that draws from each full conditional posterior M times, where M is a sufficiently large number of iterations.[12] Defining the set of SVAR parameters $\theta = \{B, A, \Lambda_{1:S}\}$, the steps of the algorithm can be summarized as follows:

[11] The only non-zero elements in P are those on the main diagonal and those on the one below, with the (S,S) element equal to 1

[12] In our empirical applications $M = 15,000$ with a burn-in of $M/3$

Algorithm 1 SVAR-H

1. Draw from $p\left(\theta^{(m)}, P^{(m)}|Y, \delta_{1:T}^{(m-1)}\right)$ by:

   (a) Drawing the SVAR parameters from $p\left(\theta^{(m)}|Y, \delta_{1:T}^{(m-1)}\right)$:

   i. Draw from $p\left(\Lambda_s|Y, A^{(m-1)}, B^{(m-1)}, \delta_{1:T}^{(m-1)}\right)$ for $s = 1, ..., S$, which is accomplished by:

   A. drawing from $p\left(\lambda_{i,1}|Y, A_i^{(m-1)}, B_i^{(m-1)}, \omega_{i,2:S}^{(m-1)}, \delta_{1:T}^{(m-1)}\right)$ for $i = 1, ..., n$;

   B. drawing from $p\left(\omega_{i,s}|Y, A_i^{(m-1)}, B_i^{(m-1)}, \lambda_{i,1}^{(m)}, \delta_{1:T}^{(m-1)}\right)$ for $i = 1, ..., n$ and for $s = 2, ..., S$;

   ii. Draw from $p\left(A, B|Y, \Lambda_{1:S}^{(m)}, \delta_{1:T}^{(m-1)}\right)$:

   A. Draw from $p\left(\alpha_i|Y, \Lambda_{1:S}^{(m)}, \delta_{1:T}^{(m-1)}\right)$, for $i = 1, ..., n$, using an Independence Metropolis-Hastings step, which:

   - draws a candidate $\alpha^\star$ using the likelihood as proposal density;
   - compute $\vartheta = \min\left\{\frac{p(\alpha^\star)}{p\left(\alpha^{(m-1)}\right)}, 1\right\}$, and set $\alpha^{(m)} = \alpha^\star$ with probability ϑ and $\alpha^{(m)} = \alpha^{(m-1)}$ with probability $(1-\vartheta)$;

   B. Draw from $p\left(B_i|Y, A_i^{(m)}, \lambda_{i,1:S}^{(m)}, \delta_{1:T}^{(m-1)}\right)$, for $i = 1, ..., n$;

   (b) Drawing the transition matrix from $p\left(P^{(m)}|Y, \delta_{1:T}^{(m-1)}\right)$ following Chib (1998);

2. Draw from $p\left(\delta_{1:T}^{(m)}|Y, \theta^{(m)}, P^{(m)}\right)$ through a Hamilton smoother.

Although we enforce sign restrictions through the exact same priors on α_i proposed by BH, it is important to stress a subtle difference between our approach and the one implemented in their original paper, in that for the coefficients B_i and Λ_s we use independent Normal-Inverse Gamma priors, while BH use a natural conjugate Normal-Inverse Gamma prior. The conjugate prior is computationally convenient but it implies that the prior variances of the coefficients in the various equations are proportional to the error variance of the system, and conjugacy requires the likelihood to be symmetric across equations.[13] Indeed, if heteroskedasticity is added to the model, the symmetry in the likelihood breaks down and the conjugate prior is no longer a viable option. As a result, one should keep in mind that the prior distributions we use for these parameters in the following subsection are broadly comparable but not exactly identical to the ones elicited in the original BH paper and there is, in principle, a possibility that

[13] See Sims and Zha (1998) and Carriero et al. (2019) for a discussion of this issue

some of the differences reported in the results are due to differences in the prior distributions. To check if this is the case, we also repeated the analysis using a version of the BH model that assumes the same independent Normal-Inverse Gamma prior we use in this paper. Results are reported in section B of the Supplemental materials available online and show that the effect of the differences in the form of the prior (natural conjugate vs independent) plays only a marginal role.

In recent contributions, Inoue and Kilian (2021a) and Arias et al. (2022) argue that BH's approach does not represent a uniformly superior method to implement sign restrictions in SVAR models. In particular, Arias et al. (2022) show that the conventional practice of specifying a uniform prior over the set of orthogonal matrices is the only way to reflect a joint uniform prior for the vector of impulse response functions. If this is the case, researchers may still want to work with the conventional Normal-Inverse Wishart prior distributions on the reduced form parameters coupled with a Uniform prior for the rotation matrix Q, as done by most of the literature ever since Uhlig (2005). Furthermore, Arias et al. (2018) show that this Uniform-Normal-Inverse Wishart (U-N-IW) prior specified on the reduced form parameters can be mapped into a Normal-Generalized Normal (N-GN) prior on the parameters of the structural form. In such case, given the conjugacy properties of N-GN priors, the Independence Metropolis step (1.a.ii.A) of the algorithm described in section 3.1 is no longer needed, and sign restrictions can be enforced by rejection sampling. In practice, this amounts to a truncation discarding the draws of α_i, $i = 1, \ldots, n$, that do not imply the desired signs in impulse response functions. In summary, the following algorithm can be used when working with the U-N-IW prior:

Algorithm 2 SVAR-H with Uniform-Normal-Inverse Wishart prior.

1. Draw from $p\left(\theta^{(m)}, P^{(m)}|Y, \delta_{1:T}^{(m-1)}\right)$ by:

   (a) Drawing the SVAR parameters from $p\left(\theta^{(m)}|Y, \delta_{1:T}^{(m-1)}\right)$:

      i. Draw from $p\left(\Lambda_s|Y, A^{(m-1)}, B^{(m-1)}, \delta_{1:T}^{(m-1)}\right)$ for $s = 1, ..., S$, which is accomplished by:

         A. drawing from $p\left(\lambda_{i,1}|Y, A_i^{(m-1)}, B_i^{(m-1)}, \omega_{i,2:S}^{(m-1)}, \delta_{1:T}^{(m-1)}\right)$ for $i = 1, ..., n$;

         B. drawing from $p\left(\omega_{i,s}|Y, A_i^{(m-1)}, B_i^{(m-1)}, \lambda_{i,1}^{(m)}, \delta_{1:T}^{(m-1)}\right)$ for $i = 1, ..., n$ and for $s = 2, ..., S$;

      ii. Draw from $p\left(A, B|Y, \Lambda_{1:S}^{(m)}, \delta_{1:T}^{(m-1)}\right)$:

         A. Draw from $p\left(A, B|Y, \Lambda_{1:S}^{(m)}, \delta_{1:T}^{(m-1)}\right)$, which is accomplished by:

            - drawing from $p\left(\alpha_i|Y, \Lambda_{1:S}^{(m)}, \delta_{1:T}^{(m-1)}\right)$, for $i = 1, ..., n$, implementing Waggoner and Zha's (2003) sampler;
            - drawing from $p\left(B_i|Y, A_i^{(m)}, \lambda_{i,1:S}^{(m)}, \delta_{1:T}^{(m-1)}\right)$, for $i = 1, ..., n$;

         B. Check sign restrictions: if they are satisfied, move to the next step, otherwise return to step (ii.A).

   (b) Drawing the transition matrix from $p\left(P^{(m)}|Y, \delta_{1:T}^{(m-1)}\right)$ following Chib (1998);

2. Draw from $p\left(\delta_{1:T}^{(m)}|Y, \theta^{(m)}, P^{(m)}\right)$ through a Hamilton smoother.

3.2 Empirical applications

This section illustrates the blended approach by revisiting two widely known empirical applications on the effects of shocks to the labor market and oil price.

3.2.1 Identifying labour market shocks [Baumeister and Hamilton (2015)]

The empirical example used by BH to illustrate their method is a bivariate SVAR describing aggregate labour demand and supply in the U.S. The model takes the form in (2) with:

$$A = \begin{bmatrix} -\beta^d & 1 \\ -\alpha^s & 1 \end{bmatrix}, \tag{11}$$

and $y_t = [\Delta w_t, \Delta n_t]$, where w_t and n_t are 100 times the logarithm of the wage level and employment level, while Δ denotes the quarterly difference. [14] The free elements of A, $\beta^d$and α^s, are interpreted as the short run demand and supply elasticity respectively. The authors place on β^d a student-t prior with 3 degrees of freedom, location parameter $c_\beta = -0.6$ and scale parameter $\sigma_\beta = 0.6$, and they set for α^s the same type of prior with $c_\alpha = 0.6$ and $\sigma_\alpha = 0.6$. Furthermore, in accordance with any plausible theoretical model, BH truncate the priors for these elasticities to have respectively only negative and positive support.

In our specification of the model, we set the exact same prior on the free elements of A and use an implied Minnesota type conditional prior for the structural parameters B_i: $\mu_i = A_i\Pi_0$, with Π_0 an $n \times k$ matrix of zeros.[15]In addition, we follow the original paper and introduce the non-dogmatic restriction that the long run effect of a labour demand shock on employment is zero. We do this by adding an independent normal prior for the sum of the coefficients in B_2 associated with lagged values of Δw_t.[16] The prior distributions for the variances $\lambda_{i,1}$ are inverse-gamma with degrees of freedom $d_{i,1} = 2$ and scale $\zeta_{i,1} = \left(d_{i,1}\hat{A}\Sigma^{(1)}\hat{A}'\right)_{i,i}$, where $\hat{A}$ is the matrix that maximizes Baumeister and Hamilton's posterior and Σ is the variance/covariance matrix of AR(1) residuals computed for the first regime. Finally, the variance ratios $\omega_{i,s}$, $s = 2,...,S$, have also inverse-gamma prior with $d_{i,s} = 2$ and scale $\zeta_{i,s} = d_{i,s}\frac{\left(\hat{A}\Sigma^{(s)}\hat{A}'\right)_{i,i}}{\left(\hat{A}\Sigma^{(1)}\hat{A}'\right)_{i,i}}$.

For this application, we use the same data analyzed in the original paper, which are quarterly and cover the period 1970:Q1-2014:Q2. The two change-points are detected by the model between 1983:Q2 and 1984:Q4 and around the beginning of the Great Financial Crisis respectively.

Figure 1 illustrates the contribution provided by heteroskedasticity to the identification of the structural parameters $\beta^d$and α^s. The histograms show the marginal posterior distributions of the demand elasticity (left panel) and the supply elasticity (right panel) obtained from the homoskedastic SVAR (BH) and from the SVAR-H model. The continuous red lines depict the prior distributions of the two parameters. The figure shows a considerable difference between the results of the two models, SVAR-H portrays a significantly steeper aggregate labour supply

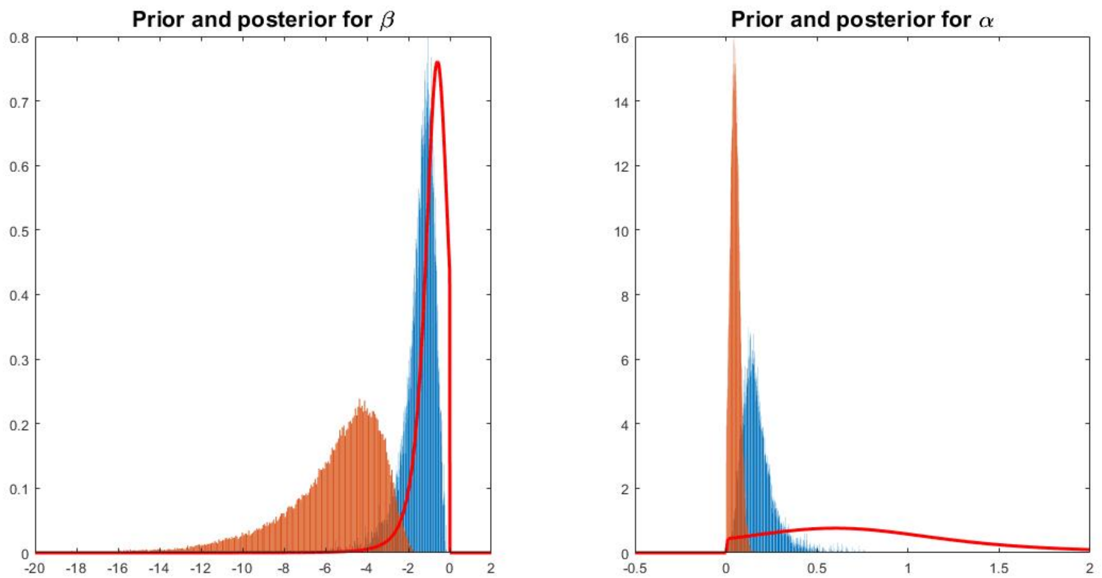
[14]Wage level is measured as the seasonally adjusted real compensation per hour and employment level is the seasonally adjusted number of people on non-farm payrolls.

[15]The variance V_i is a diagonal $k \times k$ matrix with the top left element equal to δ_0, and j-th main diagonal element equal to $\frac{\delta_1 \delta_2 \left(1_{j \neq i}\right)}{l^{\delta_3}} \frac{\sigma_i}{\sigma_j}$, where l is the lag of the associated regressor, σ_i and σ_j are the residual variances of an AR(1) estimated via OLS for the i-th and j-th variable of the system. In our application, δ_0=100, $\delta_1 = 0.2$, δ_2=1, $\delta_3 = 1$.

[16]This prior is centered at $-\alpha^s$ and has variance $R = 0.1\sigma_i$.

and a demand that is almost flat. Although the demand elasticity depicted may seem extreme compared to the consensus in the microeconometric literature, Akerlof and Dickens (2007) provide theoretical reasons explaining why large demand elasticities are plausible in macroeconomic models. Interstingly, a large demand elasticity is also estimated by Doppelt (2021) using the same model but exploiting seasonality for identification.

Figure 1: Revisiting Baumeister and Hamilton (2015): Posteriors under alternative identification strategies

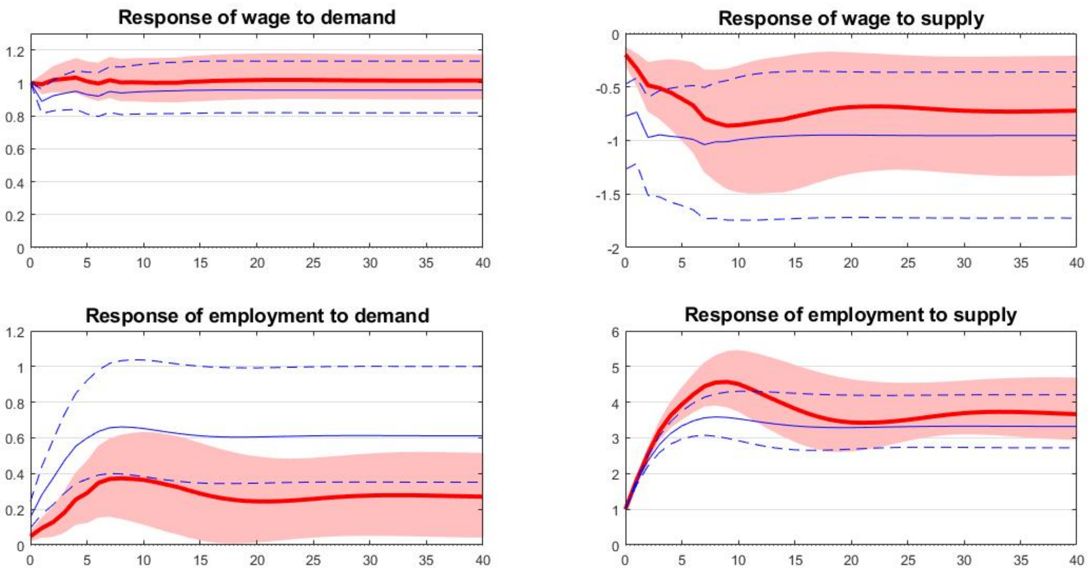


Notes: BH (blue histogram) denotes the homoskedastic SVAR with natural conjugate priors, SVAR-H (red histogram) indicates the heteroskedastic model. The red line is the prior density, which is identical in both specifications.

The divergent implications of the two specifications are reflected in the cumulated IRFs depicted in Figure 2. [17] In the figure, the size of the demand shock has been normalized to have a unit impact on wages, while the size of the supply shocks has been normalized to have a unit impact on employment. As a consequence of the different elasticities, the responses of employment to demand and supply shocks are substantially revised with the introduction of heteroskedasticity. Due to the steepness of the supply curve, employment is less responsive to demand shocks on impact when compared to the case with constant variances, and the cu-

[17] In order to facilitate comparisons with the original applications, throughout the paper we do not adopt Inoue and Kilian's (2021b) Bayes estimator for IRFs, but we report point-wise medians and credible bands. Point-wise medians are shown by Baumeister and Hamilton (2018) to be indeed optimal Bayes estimates when the researcher's loss function takes a linear-absolute form.

Figure 2: Revisiting Baumeister and Hamilton (2015): Impulse responses.



Notes: BH (blue) denotes the homoskedastic SVAR with natural conjugate priors, SVAR-H (red) indicates the heteroskedastic model. Solid lines indicate point-wise median responses while shaded areas and dashed lines represent point-wise 68% credible bands.

mulated effect remains lower at all horizons. On the other hand, the different supply-demand schedules also imply that a supply shock increasing employment by 1% produces a significantly smaller response of wages on impact in the model that exploits heteroskedasticity for identification. This is in line with the anecdotical evidence collected by Friedberg and Hunt (1995), who notice that large increases of the supply of labour due to exogenous immigration waves generated only limited effects on salaries in a number of historical episodes.

Although point identification is not necessarily the objective of our blended approach, it may be of interest to know whether the presence of heteroskedasticty is actually supported by the data and if the non-proportionality conditions are satisfied. For this reason, we report in Table 1 the Log- Bayes factors in favour of the homoskedastic model ($\omega_{i,1} = \omega_{i,2} = 1,\ i = 1,\dots,n$) and a model with proportional changes in the shocks variances ($\omega_{is}/\omega_{js} = 1,\ s = 2,3$) for all the applications considered in the paper. The Bayes factors are computed as Savage-Dickey Density Ratios (SDDR) as suggested by Lutkepohl and Wozniak (2020). Based on the scale recommended by Kass and Raftery (1995), the authors consider SDDR with an absolute value larger than five as providing strong evidence against the restricted model. The table shows that, for BH's model, the data do support the presence of regimes changes in the variance of both structural shocks, and there is no sign of proportional changes across regimes.[18]

Table 1: Log-SDDR

		Restriction: $\omega_{i,2} = \omega_{i,3} = 1$	Restriction: $\omega_{i,1}/\omega_{j,1} = \omega_{i,2}/\omega_{j,2} = 1$					
			$j = 2$	$j = 3$	$j = 4$	$j = 5$	$j = 6$	$j = 7$
BH15	$i = 1$	**−8.1970**	**−18.7085**					
	$i = 2$	**−14.2412**						
KM12	$i = 1$	**−93.5873**	**−126.3354**	**−107.3837**				
	$i = 2$	**−172.3215**		**−20.4317**				
	$i = 3$	**−6.4990**						
AR18	$i = MP$	$-\infty$	**−27.5061**	**−92.8301**	**−89.8227**	−1.9582	**−227.4372**	
MR13	$i = APITR$	**−7.8672**	**−35.6227**	**−7.0247**	−17.5384	**−10.6323**	**−7.3824**	**−7.1596**
	$i = ACITR$	**−25.7048**		**−27.1317**	0.0964	**−5.3968**	**−25.4468**	−4.9284

Notes: Log-SDDR comparing SVAR-H with a model with homoskedastic shocks ($\omega_{i,2} = \omega_{i,3} = 1$), and with a model with proportional changes in shocks variances ($\omega_{i,1}/\omega_{j,1} = \omega_{i,2}/\omega_{j,2} = 1$). BH15: Baumeister and Hamilton (2015); KM12: Kilian and Murphy (2012); AR18: Antolin-Diaz and Rubio-Ramirez (2018); MR13: Mertens and Ravn (2013). SDDRs with absolute value greater than five are in bold.

3.2.2 Identifying oil market shocks [Kilian and Murphy (2012)]

In order to illustrate the use of our blended approach in the context of conventional sign restrictions, we revisit the seminal oil market model developed by Kilian and Murphy (2012, KM henceforth). KM's model is a SVAR(24) with monthly endogenous variables $y_t = [\Delta q_t, y_t, p_t]'$, where Δq_t is the growth rate of global oil production, y_t is a measure of real global economic activity, and p_t is the logarithm of the real price of oil. The authors identify three structural shocks that they label 'oil supply', 'aggregate demand', and 'oil-specific demand', through the sign restrictions on A^{-1} illustrated in Table 2. To further reduce the set of admissible models, on the basis of previous empirical studies and historical evidence, KM also impose an upper bound of 0.0258 to the short-run price elasticity of oil supply.[19][20]

[18]Log-SDDR considering regime specific homoskedasticity or proportionality are reported in the online Appendix E.

[19]In the terminology used in oil market modeling, supply (demand) elasticity is represented by the ratio between the response of Δq_t and the response of p_t to a demand (supply) shock. See Kilian (2022) for a detailed discussion.

[20]The elasticity bound and other features of KM's approach have been superseded on various grounds by subsequent contributions (see e.g. Kilian and Murphy (2014), Baumeister and Hamilton [2019] and Caldara et al. [2019]), nevertheless the model represents an ideal example to clarify the implementation and usefulness of our blended approach due to its simplicity and popularity.

Table 2: Sign Restrictions in Kilian and Murphy (2012)

Variable\Shock	Oil Supply	Aggregate Demand	Oil-specific Demand
Δq_t	$-$	$+$	$+$
y_t	$-$	$+$	$-$
p_t	$+$	$+$	$+$

The original analysis was performed by KM in a Bayesian setting with flat priors, thus, in order to maximize comparability, we elicit the prior $p(A,B) \propto |\det(A)|^{-n}$, which is equivalent to the Jeffreys' (flat) prior on the reduced-form parameters. Furthermore, to facilitate the comparison between priors, we do not impose the unit normalization of specific elements of A as done in the previous section, but we instead normalize the structural shocks to have unit variance in the first regime, $\lambda_{i,1} = 1$ for every i. For the ratios $\omega_{i,s}$, $i = 1,...,n$ and $s = 2,3$, we set conservative Inverse Gamma priors with n degrees of freedom and scale $n-1$, which imply prior means equal to 1. The conditional posterior distribution of α_i, $i = 1,...,n$, is therefore Generalized Normal:

$$p(\alpha_i|Y,\Lambda_{1:S}) \propto |\det(A)|^{T-n} \exp\left[-\frac{1}{2}\alpha_i{}'\bar{V}_{\alpha_i}^{-1}\alpha_i\right] \tag{12}$$

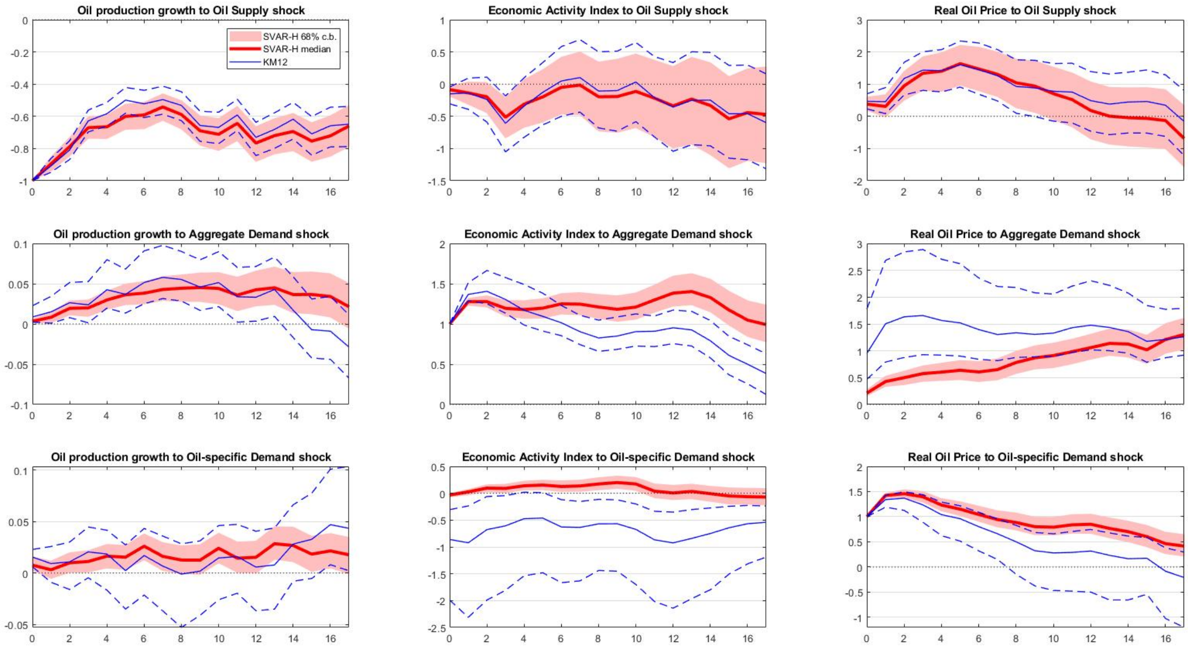
where $\bar{V}_{\alpha_i}^{-1} = \sum_{s=1}^{S} \omega_{i,s}^{-1} y^{(s)} y^{(s)\prime} - \Upsilon' \bar{V}_{b_i} \Upsilon$, and $\Upsilon = \sum_{s=1}^{S} \omega_{i,s}^{-1} y^{(s)} x^{(s)\prime}$.

The rows B_i have a normal conditional posterior with variance $\bar{V}_{b_i} = \left[\sum_{s=1}^{S} \omega_{i,s}^{-1} x^{(s)} \left(x^{(s)}\right)'\right]^{-1}$ and mean $\bar{\mu}_{b_i}' = \left[\sum_{s=1}^{S} \omega_{i,s}^{-1} A_i y^{(s)} \left(x^{(s)}\right)'\right] \bar{V}_{b_i}$[21]. The posterior moments for all other parameters are identical to those derived in section 3.1. We estimate both KM original model and our SVAR-H version based on the sample, extended by Antolin-Diaz and Rubio-Ramirez (2018), that covers the period from January 1971 to December 2015.

Figure 3 compares the impulse response functions obtained from KM's model, identified through sign-restrictions and elasticity bounds only, with those obtained from the same model that also accounts for regimes changes in the variances of structural shocks. Form the figure it is clear that while KM's sign-restrictions are extremely informative in identifying an oil supply shock, they are less effective in pinning down the two demand shocks in their system. As shown by the differences in both shape and width, the responses to demand shocks (both aggregate and

[21] These posterior moments can be obtained by substituting $W_i = I$, $w_i = 0$, and $V_{b_i}^{-1} = 0$ in the derivations of Appendix A.

Figure 3: Revisiting Kilian and Murphy's (2012): Impulse responses.

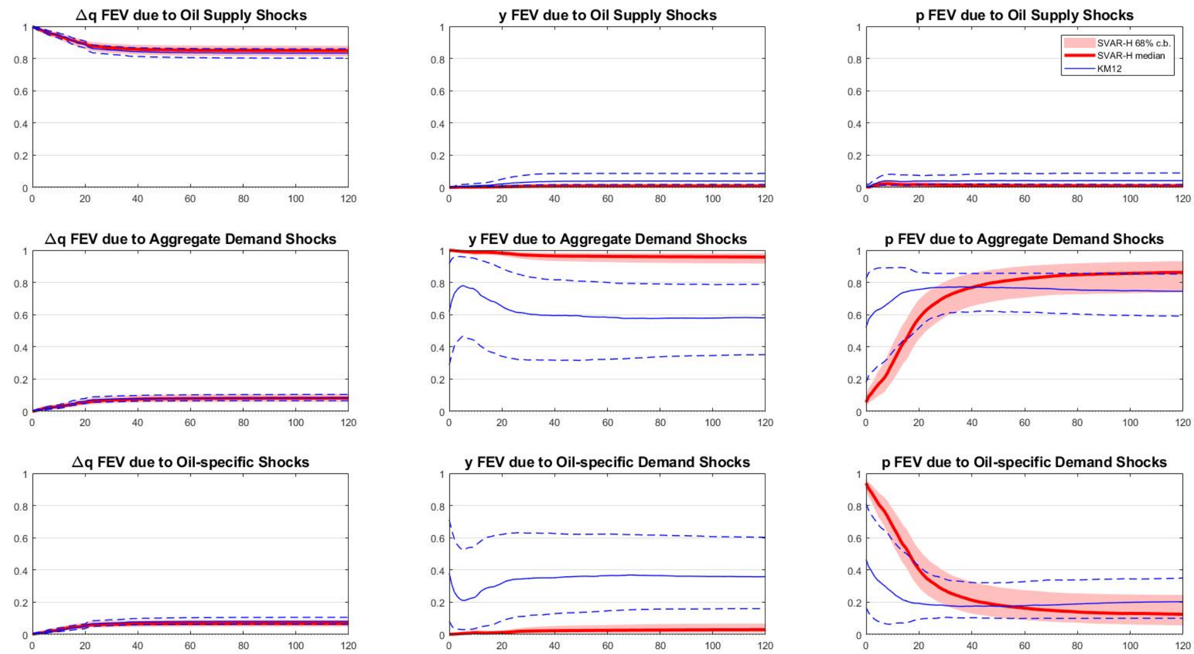


Notes: KM12 (blue) denotes Kilian and Murphy's (2012) SVAR, SVAR-H (red) indicates the heteroskedastic model. Solid lines indicate median responses while shaded areas and dashed lines represent 68% credible bands. To aid comparisons, the figure normalizes the shocks to have a unit impact on the target variable.

oil-specific) are largely affected by the additional information introduced by heteroskedasticity. The effects of an aggregate demand shock appear less pronounced on impact but tend to be higher and more persistent at longer horizons. More specifically, the response of the economic activity index never becomes lower than its impact level for the first 16 months according to the SVAR-H, while, in the sign-restricted only model, it starts to dissipate already after one quarter. A similar pattern is shown by the response of real oil price. In the SVAR-H specification, the impact effect is significantly lower and it tends to strengthen over time, whereas, according to the alternative model, the response is large on impact and slightly decreases over the horizon. Contrarily, the effect of an oil-specific demand shock on the real price of oil appears more persistent when heteroskedasticity is exploited, and its repercussion on economic activity is significantly downsized and even sign-reverted at medium horizons.

It is worth noting at this point that the revisions in IRFs shown in Figure 3 are strikingly similar to those obtained by Antolin-Diaz and Rubio-Ramirez (2018) through a completely different method, i.e. the addition of narrative restrictions. Besides supporting the robustness of our results, this fact points out the effectiveness of both these methods (narrative restrictions and heteroskedasticity) in sharpening identification. There is further scope to fruitfully blend these two approaches, as we shall discuss in the next section. The way in which the introduction of heteroskedasticity sharpens identification is also consistent with the results obtained by KM adding further economically motivated restrictions to the baseline model described. In particular, to reduce the identified set, KM also impose a bound on the response of global economic activity to oil-specific demand shock finding results largely in line with ours.

Figure 4: Revisiting Kilian and Murphy's (2012): Forecast error variance decompositions.



Notes: KM12 (blue) denotes Kilian and Murphy's (2012) SVAR, SVAR-H (red) indicates the heteroskedastic model. Solid lines indicate median FEVDs while shaded areas and dashed lines represent 68% credible bands. For the heteroskedastic specification, the depicted FEVD are computed by setting the variances of structural shocks to their arithmetic average across regimes.

Besides IRFs, other quantities that are of particular interest in the oil market modeling literature are the Forecast Error Variance Decompositions (FEVDs) of oil production and price. The FEVD represents a measure of the relative importance of each shock for the unexpected change in the endogenous variables, and it has been the object of strong debate in the recent literature.[22] One particular question at the core of the debate is whether changes in oil price are

[22] See e.g Baumeister and Hamilton (2019), Caldara et al .(2019) and Kilian (2021) and references therein.

mainly driven by demand or supply shocks.

Figure 4 compares the variance decompositions obtained by the original Kilian and Murphy's (2012) model with those obtained by the SVAR-H specification.[23] The answer provided by the two models to the crucial question is similar: the bulk of oil price variation is due to demand shocks. However, the SVAR-H results show that oil-specific demand is the most important factor at shorter horizons, while aggregate demand becomes the dominant driver of oil price in the long run. The latter mechanism is absent in the model identified only through sign-restrictions and elasticity bounds due to the large uncertainty about the parameters in the matrix Q, but it also becomes evident once the identified set is reduced exploiting narrative information as done by Antolin-Diaz and Rubio-Ramirez (2018). Moreover, the SVAR-H also sheds light on the fact that only a negligible share of global economic fluctuations is associated with oil market shocks, whereas KM's model attributes between 20% and 60% of the economic activity forecast variance to oil-specific demand shocks.

Finally, as in the previous section, we compute the Log-SDDR in favour of the model with time invariant or proportional volatility changes. The density ratios are reported in Table 1 and show strong evidence against homoskedastic shocks or proportional shifts in their variances.

4 Blending narrative restrictions and heteroskedasticity

In many relevant instances, the information a researcher has before seeing the data is not only in the form of sign restrictions. Sometimes a reliable account of some shocks that materialized in association with specific events may be available. For example the econometrician may be reasonably sure that the shock of interest was positive at a specific point in time ("shock sign restrictions" in the terminology of Giacomini et al., 2021), or that it was the most significant contributor to the change in an observable variable ('shock rank restrictions' in the terminology of Giacomini et al., 2021). The idea of exploiting this narrative information for identification purposes has been only recently formalized by Antolin-Diaz and Rubio-Ramirez (2018) and Ludvigson et al. (2021). Despite the similarity with sign restrictions, narrative restrictions bear

[23] Because of the presence of heteroskedasticity, the FEVD is time-varying in the SVAR-H specification. However, to maximize comparability, we consider the FEVD computed by setting the variances of structural shocks to their arithmetic average across regimes.

some peculiarities that are worth emphasizing. In particular, they do not restrict the support of the prior distribution, they rather truncate the likelihood by restricting the set of admissible realizations of the structural shocks ε_t.[24] The likelihood in (6) becomes:

$$p\left(y_{1:T}|A,B,\Lambda_{1:S}\right) = (2\pi)^{-Tn/2}\left|\det\left(A\right)\right|^{T}\left(\prod_{i=1}^{n}\lambda_{i,1}^{-T/2}\right)\left(\prod_{i=1}^{n}\prod_{s=1}^{S}\omega_{i,s}^{-T_s/2}\right)\times$$

$$\exp\left\{-\frac{1}{2}\sum_{i=1}^{n}\lambda_{i,1}^{-1}\left[\sum_{s=1}^{S}\omega_{i,s}^{-1}\left(A_i y^{(s)} - B_i x^{(\mathbf{s})}\right)\left(A_i y^{(s)} - B_i x^{(\mathbf{s})}\right)'\right]\right\}\times\mathscr{D}_{NR}\left(\theta,y\right), \tag{13}$$

where $\mathscr{D}_{NR}\left(\theta,y\right)$ is the Dirac delta function that takes value 1 when the narrative restrictions are satisfied, and 0 otherwise.

Narrative restrictions are usually implemented in conjunction with sign restrictions to reduce the set of SVAR parameterizations associated with a given reduced form model. Inside this smaller set, however, the likelihood remains flat, allowing for infinitely many equally likely parameter points. The structural model is therefore still set identified, and there is scope for heteroskedasticity to be exploited in order to make the likelihood more informative. This is achieved by truncating the posterior via rejection sampling. In particular, this is achieved by appropriately specifying the truncation set in step (1.a.ii) of Algorithm 2.

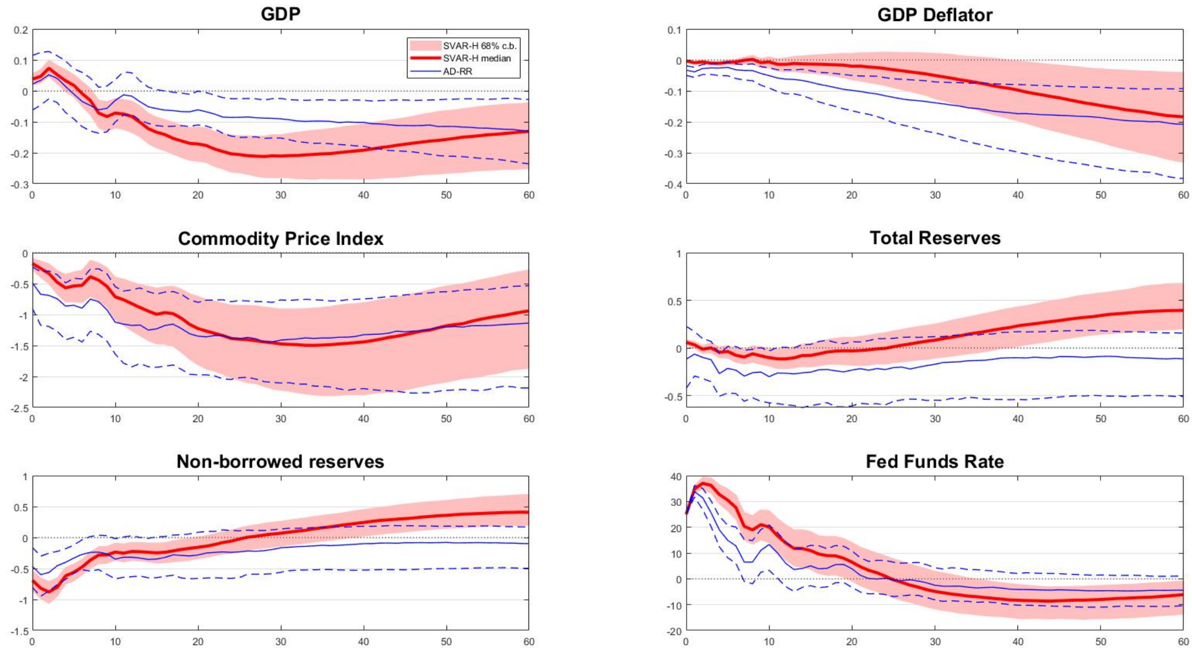
4.1 Monetary policy shocks [Antolin-Diaz and Rubio-Ramirez (2018)]

Antolin-Diaz and Rubio-Ramirez (2018, AR henceforth) consider the six-variable monthly SVAR(12) designed by Uhlig (2005) to study the propagation of monetary policy shocks. The endogenous variables in the model are: real output, the GDP deflator, a price index for commodities, total reserves, non-borrowed reserves, and the fed funds rate. The monetary policy shock is identified by Uhlig (2005) by assuming that the response, up to a five-month horizon, of the fed funds rate to such shock is positive, while it is negative for the GDP deflator, the commodity price index and non-borrowed reserves. In this setting, AR show that the identified set can be narrowed down significantly by exploiting the sole uncontroversial fact that in Octo-

[24] See Giacomini et al. (2021) for a formal discussion of the differences between sign restrictions and narrative restrictions.

ber 1979 the monetary policy was contractionary, and it was the main driver of the unexpected change in the fed funds rate.[25]

Figure 5: Revisiting Antolin-Diaz and Rubio-Ramirez (2018): Responses to a monetary policy shock



Notes: AD-RR (blue) denotes Antolin-Diaz and Rubio-Ramirez (2012) SVAR, SVAR-H (red) indicates the heteroskedastic model. Solid lines indicate median responses while shaded areas and dashed lines represent 68% credible bands. To aid comparisons, the figure normalizes the shocks to 25 basis points impact on the fed funds rate.

Figure 5 shows that the presence of heteroskedasticity in the structural shocks can be exploited for identification also in this context. The red impulse responses are derived from the posterior distribution of the model that includes the same sign and narrative restrictions considered by AR and also accounts for heteroskedasticity, while the blue IRFs are generated imposing only AR's sign and narrative restrictions. Both results are based on the sample period January 1965 to November 2007.[26] Also in this case, we take advantage of the distributional equivalence derived by Arias et al. (2018) and consider a prior distribution for the structural parameters that translates into the Jeffreys (flat) prior on the reduced-form parameters, $p(\Pi,\Omega) \propto |\det(\Omega)|^{-(n+1+k)/2}$, and a uniform prior over the space of admissible rotations

[25]In October 1979 the Fed Chairman Paul Volcker announced new operating procedures to curb inflation, which comprised large increases of short-term interest rates. See e.g. Lindsey, Orphanides and Rasche (2005).

[26]The dataset is available for download from Juan Antolin-Diaz website.

Q.

The figure shows that, once again, the introduction of the identifying information coming from heteroskedasticity exerts a remarkable contribution in reducing uncertainty about IRFs, especially at shorter horizons. Moreover, the implications of a monetary policy shock are revised significantly in several directions. In particular, when compared to the original model, the SVAR-H highlights a reduced ability of contractionary monetary policy surprises to reduce inflation in the short-run. In line with the evidence provided by Nakamura and Steinsson (2018), the response of the general inflation measure is virtually null for the first three years from the shock. At the same time, the recessionary effects brought in by the monetary contraction appear larger and more significant, although they are positive in the first few months after the shock. Finally, the mean reversion of the fed funds rate is slower once changes in the variance of structural shocks are considered, presumably reflecting a stronger preference of the Fed for interest rates smoothing.

Table 1 shows that the data strongly favour the model with regime shifting in the variance of structural shocks. However, the Log-SDDR also show that the evidence against a proportional change between the volatility of the monetary policy shock and that of all other shocks is not strong. In particular, one of the remaining (unlabelled) structural innovation seems to display a volatility process that cannot be clearly distinguished from that of the monetary policy shock. This demonstrates that the convenience of our blending approach is not limited to the solution of the labelling problem. As shown by Bacchiocchi et al (2022), if we were to identify the monetary policy shock based on heteroskedasticity alone, we would likely be left with a set identified model. The combination of heteroskedasticity with nattarive and sign restrictions, however, is able to dramatically reduce the size of the set of admissible models. This is even more important when inference is performed through Monte Carlo algorithms, in which label switching can happen across posterior draws.

5 Blending proxy-SVARs and heteroskedasticity

A Proxy-SVAR is a Structural VAR identified by means of external instruments that are correlated with the shocks of interest but uncorrelated with all other shocks. That is, given the model in equation (2), a $r \times 1$ vector of variables, $\mathbf{z}_t$, is available, and evolves according to the following process:

$$\mathbf{z}_t = \Gamma_c + \Gamma_x x_{t-1} + \Gamma_{z,1}\mathbf{z}_{t-1} + \ldots + \Gamma_{z,p}\mathbf{z}_{t-p} + \Phi\Lambda^{0.5}\varepsilon_t + \Lambda_e^{0.5} e_t \tag{14}$$

where Γ_c is a $r \times 1$ vector of intercepts, Γ_x has dimension $r \times k$, and $\Gamma_{z,l}$, $l = 1, \ldots, p$, are $r \times r$ matrices of lagged coefficients. The matrix Φ links the external instruments to the identified structural shocks of the SVAR and can be partitioned as $\Phi = \begin{bmatrix} \Phi_r & \mathbf{0}_{n-r} \end{bmatrix}$, in which Φ_r is a full rank $r \times r$ matrix and $\mathbf{0}_{n-r}$ is a $r \times (n-r)$ matrix of zeros. This partition makes clear why the external instruments are able to distinguish the first r shocks from all the remaining innovations. Finally, e_t is a $r \times 1$ i.i.d. vector of measurement errors that are normally distributed with mean zero and variance Σ_e, and independent from the structural shocks ε_t.

In most practical cases, and in our application in section 5.2, instruments are such that Γ_c, Γ_x and $\Gamma_{z,l}$ are all zeros, therefore equation (14) simplifies to: $\mathbf{z}_t = \Phi\varepsilon_t + e_t$. The Proxy-SVAR can then be written compactly in reduced form as:

$$\underbrace{\begin{bmatrix} \mathbf{z}_t \\ y_t \end{bmatrix}}_{\widetilde{y}_t} = \underbrace{\begin{bmatrix} 0 \\ \Pi \end{bmatrix}}_{\widetilde{\Pi}} \underbrace{x_{t-1}}_{x_{t-1}} + \underbrace{\begin{bmatrix} \Omega_e & \Phi \\ 0 & A^{-1} \end{bmatrix}}_{\widetilde{\mathbf{A}}^{-1}} \underbrace{\begin{bmatrix} \Lambda_e^{0.5} & 0 \\ 0 & \Lambda^{0.5} \end{bmatrix}}_{\widetilde{\Lambda}} \underbrace{\begin{bmatrix} e \\ \varepsilon_t \end{bmatrix}}_{\widetilde{\varepsilon}_t} \tag{15}$$

where $\widetilde{\varepsilon}_t$ is a multivariate standard Gaussian and Ω_e is a potentially full matrix with ones on the main diagonal, such that $\Sigma_e = \Omega_e \Lambda_e \Omega_e'$.

When r is greater than one, the Proxy-SVAR is only set identified, unless more constraints are introduced either on Φ or A^{-1}, or on both. Such constraints come often in the form of zero restrictions or sign restrictions. In the latter case, however, the identified set is only made smaller, but it is usually not reduced to a single parameter point.

In what follows we provide a method to point identify a Proxy-SVAR with $r > 1$ instruments and r shocks of interest, without imposing dubious restrictions on Φ or A^{-1}. Following our discussion of heteroskedasticity as a complementary identification strategy, we propose to reduce the identified set in Proxy-SVARs taking advantage of the time variation in the variances of structural shocks.

5.1 Proxy-SVAR-H

Extending the reduced form Proxy-SVAR in equation (15) to account for heteroskedasticity is straightforward, it suffices to consider ε_t as generated by a normal distribution with mean zero and variance Λ_s, for $s = 1, ..., S$, where S continues to indicate the number of regimes. Without loss of generality, we also set the diagonal elements of A^{-1} to 1 to fix the scale.

The likelihood of the resulting Proxy-SVAR-H is therefore:

$$p\left(y_{1:T}|\widetilde{\mathbf{A}}^{-1}, \widetilde{\Pi}, \widetilde{\Lambda}_{1:S}\right) = (2\pi)^{-Tn/2}\left|\det\left(\widetilde{\mathbf{A}}^{-1}\right)\right|^{-T}\left(\prod_{j=1}^{r}\lambda_{e,j}^{-T/2}\right)\left(\prod_{i=1}^{n}\lambda_{i,1}^{-T/2}\right)\left(\prod_{i=r+1}^{r+n}\prod_{s=1}^{S}\widetilde{\omega}_{i,s}^{-T_s/2}\right)\times$$

$$\exp\left\{-\frac{1}{2}\left[\sum_{s=1}^{S}\left(vec\left(\widetilde{y}_t^{(s)'}\right)-\left(I_n\otimes x^{(s)'}\right)\widetilde{\pi}\right)'\left(\left(\widetilde{\mathbf{A}}^{-1}\widetilde{\Lambda}_s\left(\widetilde{\mathbf{A}}^{-1}\right)'\right)\otimes I_{T_s}\right)^{-1}\left(vec\left(\widetilde{y}_t^{(s)'}\right)-\left(I_n\otimes x^{(s)'}\right)\widetilde{\pi}\right)\right]\right\},$$

where $\widetilde{\pi} = vec\left(\widetilde{\Pi}\right)$ is a vector of reduced form VAR coefficients.

Assuming an independent Gaussian prior, $N(\widetilde{\pi}_0, V)$, conditional on $\widetilde{\mathbf{A}}^{-1}$ and $\widetilde{\Lambda}_{1:S}$, the posterior is Gaussian with variance and mean:

$$\bar{V} = \left[V^{-1} + \sum_{s=1}^{S}\left(\widetilde{\mathbf{A}}\otimes x^{(s)'}\right)'\left(\widetilde{\Lambda}_s\otimes I_{T_s}\right)^{-1}\left(\widetilde{\mathbf{A}}\otimes x^{(s)'}\right)\right]^{-1},$$

$$\bar{\pi} = \bar{V}^{-1}\left[V^{-1}\widetilde{\pi}_0 + \sum_{s=1}^{S}\left(\widetilde{\mathbf{A}}\otimes x^{(s)'}\right)'\left(\widetilde{\Lambda}_s\otimes I_{T_s}\right)^{-1} vec\left(\widetilde{y}_t^{(s)'}\widetilde{\mathbf{A}}'\right)\right]. \tag{16}$$

Furthermore, conditional on $\widetilde{\mathbf{A}}$, the equations of the SVAR are independent. Accordingly, given an inverse gamma prior, $IG(d_{i,1}, \zeta_{i,1})$, $i = 1, ..., r+n$, for each element of $\widetilde{\Lambda}_1$, the posteriors are conjugate with $\bar{d}_{i,1} = d_{i,1} + T$ degrees of freedom, and scale parameters:

$$\bar{\zeta}_{i,1} = \zeta_{i,1} + \sum_{s=1}^{S}\omega_{i,s}^{-1}\left[\widetilde{\mathrm{A}}_i\left(y^{(s)} - \widetilde{\Pi}x^{(\mathbf{s})}\right)\left(y^{(s)} - \widetilde{\Pi}x^{(\mathbf{s})}\right)'\widetilde{\mathrm{A}}_i'\right]. \tag{17}$$

Since in many applications we do not have reasons to believe that the measurement error in proxies is heteroskedastic, we focus on the case in which the first r diagonal elements of $\widetilde{\Lambda}_s$ are time invariant, that is, we set $\widetilde{\omega}_{i,s} = 1$ for $i = 1,...,r$ and $s = 2,...,S$. As a result, we consider regime-switching variance ratios only for the structural shocks ε_t. For given priors, $\widetilde{\omega}_{i,s} \sim IG(d_{i,s}/2, \zeta_{i,s}/2)$, $i = r+1,...,r+n$ and $s = 2,...,S$, the posteriors are then also Inverse Gamma with $\frac{\bar{d}_{i,s}}{2} = \frac{d_{i,s}}{2} + \frac{T_s}{2}$ degrees of freedom and scale parameters

$$\frac{\bar{\zeta}_{i,s}}{2} = \frac{\zeta_{i,s}}{2} + \frac{\widetilde{\lambda}_{i,1}^{-1}}{2}\left[\widetilde{\mathrm{A}}_i\left(y^{(s)} - \widetilde{\Pi}x^{(\mathbf{s})}\right)\left(y^{(s)} - \widetilde{\Pi}x^{(\mathbf{s})}\right)'\widetilde{\mathrm{A}}_i'\right]. \tag{18}$$

Finally, the conditional posterior of $\widetilde{\mathbf{A}}^{-1}$ is not of a standard form because of the term $\left|\det\left(\widetilde{\mathbf{A}}\right)\right|^T$ appearing in the likelihood. Provided a prior for the n_α free elements in $\widetilde{\mathbf{A}}^{-1}$, $p(\alpha)$, we can thus write its posterior distribution as:

$$p\left(\alpha|Y,\widetilde{\Pi},\widetilde{\Lambda}_{1:S}\right) \propto p(\alpha)\left|\det\left(\widetilde{\mathbf{A}}^{-1}\right)\right|^{-T} \times$$

$$\exp\left\{-\frac{1}{2}\left[\sum_{s=1}^{S}\left(vec\left(\widetilde{y}_t^{(s)'}\right) - \left(I_n \otimes x^{(s)'}\right)\widetilde{\pi}\right)'\left(\left(\widetilde{\mathbf{A}}^{-1}\widetilde{\Lambda}_s\widetilde{\mathbf{A}}^{-1\prime}\right)\otimes I_{T_s}\right)^{-1}\left(vec\left(\widetilde{y}_t^{(s)'}\right) - \left(I_n \otimes x^{(s)'}\right)\widetilde{\pi}\right)\right]\right\},$$

which can be used as a target density in a Metropolis-Hastings step.

As in the previous sections, the detection of change-points is performed by means of Chib's (1998) method. Define $\theta = \{\widetilde{\Pi}, \widetilde{\mathbf{A}}^{-1}, \widetilde{\Lambda}_{1:S}\}$. The following MCMC algorithm draws from the posterior of the Proxy-SVAR-H parameters, the states and the transition probabilities:

Algorithm 3 Proxy-SVAR-H

1. Draw from $p\left(\theta^{(m)}, P^{(m)} | Y, \delta_{1:T}^{(m-1)}\right)$ by:

   (a) Drawing the SVAR parameters from $p\left(\theta^{(m)} | Y, \delta_{1:T}^{(m-1)}\right)$:

   i. Draw from $p\left(\widetilde{\Lambda}_s | Y, \left(\widetilde{\mathbf{A}}^{-1}\right)^{(m-1)}, \widetilde{\Pi}^{(m-1)}, \delta_{1:T}^{(m-1)}\right)$ for $s = 1, ..., S$, which is accomplished by:

   A. drawing from $p\left(\widetilde{\lambda}_{i,1} | Y, \left(\widetilde{\mathbf{A}}^{-1}\right)^{(m-1)}, \widetilde{\Pi}^{(m-1)}, \widetilde{\omega}_{i,2:S}^{(m-1)}, \delta_{1:T}^{(m-1)}\right)$ for $i = 1, ..., r+n$;

   B. drawing from $p\left(\widetilde{\omega}_{i,s} | Y, \left(\widetilde{\mathbf{A}}^{-1}\right)^{(m-1)}, \widetilde{\Pi}^{(m-1)}, \widetilde{\lambda}_{i,1}^{(m)}, \delta_{1:T}^{(m-1)}\right)$ for $i = r+1, ..., n$ and for $s = 2, ..., S$;

   ii. Draw from $p\left(\alpha | Y, \widetilde{\Pi}^{(m-1)}, \widetilde{\Lambda}_{1:S}^{(m)}, \delta_{1:T}^{(m-1)}\right)$ using a Random Walk Metropolis-Hastings step, which:

   A. draws a candidate $\alpha^{\star} = \alpha^{(m-1)} + \xi \nu$, where $\nu \sim t(2)$ and has dimension $n_{\alpha} \times 1$, while ξ is a tuning parameter that guarantees an acceptance rate between 30% and 40%;

   B. compute $\vartheta = \min \left\{ \frac{p\left(\alpha^{\star} | Y, \widetilde{\Pi}^{(m-1)}, \widetilde{\Lambda}_{1:S}^{(m)}\right)}{p\left(\alpha^{(m-1)} | Y, \widetilde{\Pi}^{(m-1)}, \widetilde{\Lambda}_{1:S}^{(m)}\right)}, 1 \right\}$, and set $\alpha^{(m)} = \alpha^{\star}$ with probability ϑ and $\alpha^{(m)} = \alpha^{(m-1)}$ with probability $(1 - \vartheta)$;

   iii. Draw from $p\left(\widetilde{\Pi}^{(m-1)} | Y, \left(\widetilde{\mathbf{A}}^{-1}\right)^{(m)}, \widetilde{\Lambda}_{1:S}^{(m)}, \delta_{1:T}^{(m-1)}\right)$;

   (b) Drawing the transition matrix from $p\left(P^{(m)} | Y, \delta_{1:T}^{(m-1)}\right)$ following Chib (1998);

2. Draw from $p\left(\delta_{1:T}^{(m)} | Y, \theta^{(m)}, P^{(m)}\right)$ through a Hamilton smoother.

The sampler is iterated for $m = 1, ..., M$, where M is a sufficiently large number, discarding the first M_0 draws.[27]

Notice that, since the coefficients in the first r rows of $\widetilde{\Pi}$ are all zeros and the last n reduced form errors are independent from the measurement errors, the last step of the sampler can be performed by focusing only on the last n equations of the augmented VAR.

[27] In our empirical application $M = 2,000,000$ and $M_0 = 1,000,000$.

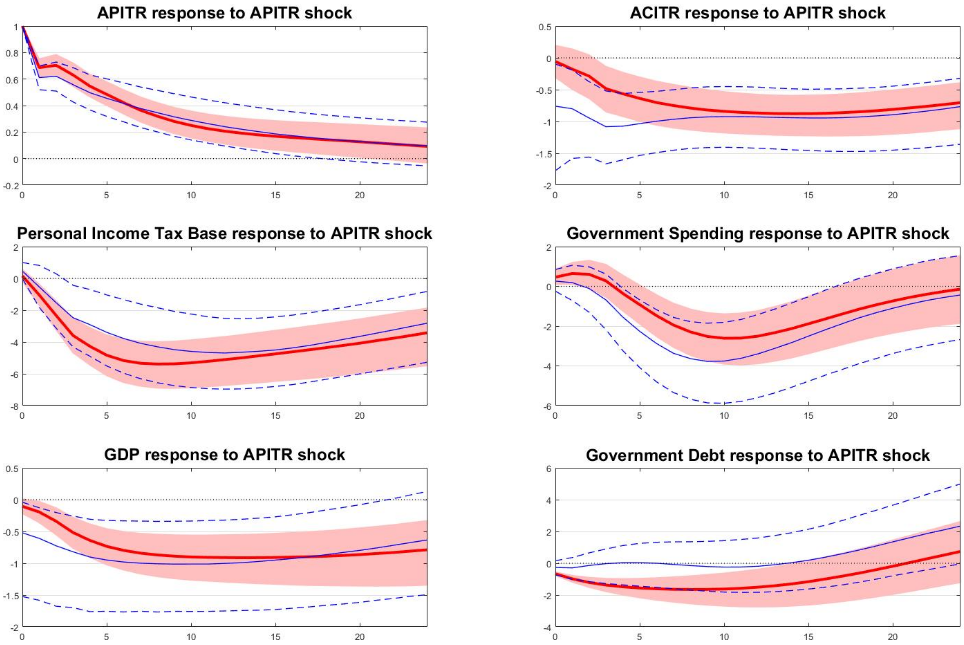
5.2 Identifying tax shocks [Mertens and Ravn (2013)]

One of the most important studies exploiting the availability of external instruments in a SVAR framework is Mertens and Ravn (2013, MR henceforth), subsequently replicated and extended by Jentsch and Lunsford (2019), Mertens and Ravn (2019) and Giacomini et al. (2020). The authors construct two narrative measures correlated with personal income tax (APITR) and corporate income tax (ACITR) shocks for the U.S.. They then acknowledge the fact that these proxies contain measurement errors and therefore rely on a Proxy-SVAR analysis. This is a typical example of the situation in which $r > 1$ ($r = 2$ in this case) instruments are available to identify $r > 1$ shocks. As a result, the structural model is only set-identified. MR obtain point identification by imposing that the matrix Φ_r is lower triangular, ordering either the APITR or the ACITR instrument first. In other words, the authors assume that the instrument ordered first only contains information about one shock, while the other is correlated with both fiscal shocks.

As argued by Giacomini et al. (2020), this additional restriction may be undesirable because it introduces a significant degree of arbitrariness in the identification scheme. To avoid this problem, Giacomini et al. (2020) propose to relax the zero restriction and to base inference on the whole identified set, i.e. rely on set identification. Our blended identification strategy instead allows to relax the zero restriction and obtain point identification using heteroskedasticity. In fact, the Log-SDDR reported in Table 1 show strong evidence against homoskedasticity or proportional variance processes, at least for one of the shocks of interest, namely the APITR shock. It can be noticed that, despite one of the shocks seems to experience variance changes that cannot be clearly distinguished by the change-point specification, the model is still point identified by the presence of proxy variables. Therefore, the application represents another example in which identification through heteroskedasticity can be fruitfully combined with set identifying information, not just for labelling purposes, but also to compensate the respective weaknesses. The only additional condition we need is a *"relevance condition"*, which imposes that the first shock explains a larger portion of the first instrument's variance, and it is required to avoid column switchings in $\widetilde{\mathbf{A}}^{-1}$.

We now revisit MR's results through the lenses of a Proxy-SVAR-H and show that the information brought in by heteroskedasticity can be substantial. The SVAR in the original paper

Figure 6: Revisiting Mertens and Ravn (2013): responses to personal income tax shock



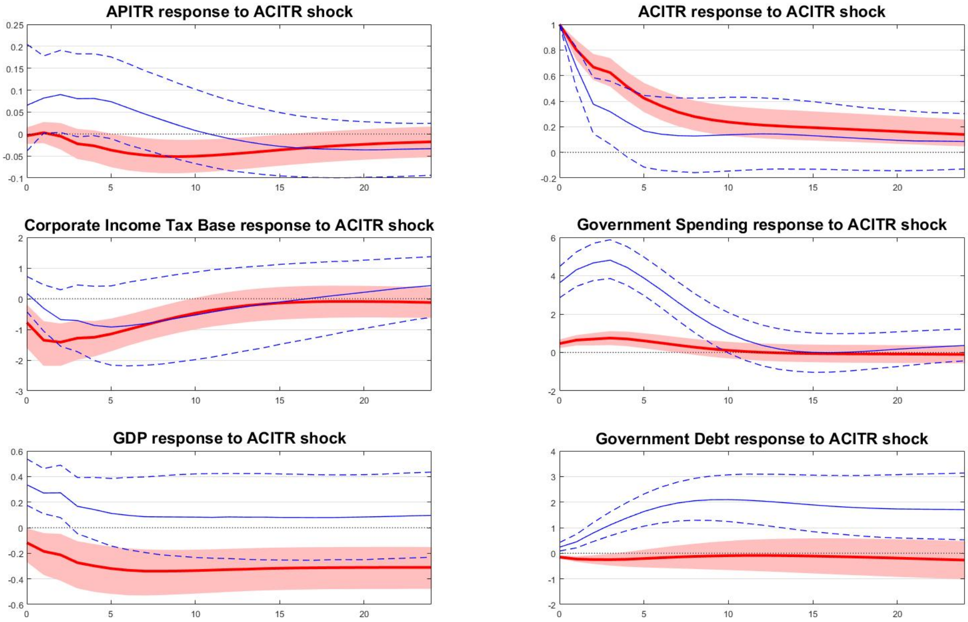
Notes: Proxy-SVAR (blue) denotes the homoskedastic SVAR with independent priors, Proxy-SVAR-H (red) indicates the heteroskedastic model. Solid lines indicate median responses while shaded areas and dashed lines represent 68% credible bands.

has 4 lags and a $n \times 1$ vector of endogenous variables, $y_t = [tp_t, tc_t, bp_t, bc_t, g_t, y_t, d_t]'$, where tp_t and tc_t are respectively the real personal and corporate average income tax rate, bp_t and bc_t are the real personal and corporate income tax base per capita, g_t measures government purchases of final goods per capita, y_t is the real GDP per capita, and d_t is the real Federal Government debt per capita. Data are quarterly and cover the period 1951:Q1-2006:Q4. The model defines a first variance regime that lasts until 1963:Q3, and selects 1985 as a further break point. The prior we set for the lagged coefficients, $\widetilde{\Pi}$, are Gaussian with an identity matrix as mean and Minnesota-type variances.[28] The inverse gamma priors for the diagonal elements of $\widetilde{\Lambda}_1$ and for the variance ratios $\omega_{i,s}$ have two degrees of freedom and scale parameters computed as in section 3.2.1. Finally, all the free elements in $\widetilde{\mathbf{A}}^{-1}$ have independent student-t prior distributions with three degrees of freedom, scale 0.5 and location at 0.

Results are presented in Figures 6 and 7, which depict the responses to one-unit tax shock for both the homoskedastic and the heteroskedastic version of the Proxy-SVAR. The IRFs in

[28] Variances are $\frac{\delta_1 \delta_2 (1_{j \neq i})}{l^{\delta_3}} \frac{\sigma_i}{\sigma_j}$, with δ_0=100, $\delta_1 = 0.5$, δ_2=1, $\delta_3 = 2$.

Figure 7: Revisiting Mertens and Ravn (2013): responses to corporate income tax shock



Notes: Proxy-SVAR (blue) denotes the homoskedastic SVAR with independent priors, Proxy-SVAR-H (red) indicates the heteroskedastic model. Solid lines indicate median responses while shaded areas and dashed lines represent 68% credible bands.

Figure 6 are generated by a fiscal surprise that rises APITR by 1 percentage point, while Figure 7 plots the responses to a 1% unexpected increase in ACITR. It is clear that heteroskedasticity adds important information in many directions. The fact that credible bands around responses are generally narrower especially at shorter horizons, once heteroskedasticity is considered, signals that the additional information introduced is able to reduce uncertainty about the free elements in the impact matrix $\widetilde{\mathbf{A}}^{-1}$. The Proxy-SVAR-H model makes evident that both types of fiscal contractions have deep and persistent negative effects on output. Another important feature of the heteroskedastic model is that following both types of shock the associated tax base decreases. Furthermore, while a personal income tax shock allows a reduction of the government debt (as one would expect), the effect of a corporate income tax surprise on debt is weak and negative only at shorter horizon. Finally, government purchases tend to increase after a corporate tax surprise and then tends to be reabsorbed within one year from the shock, while government spending drops following an unexpected personal tax increase.

6 Robustness

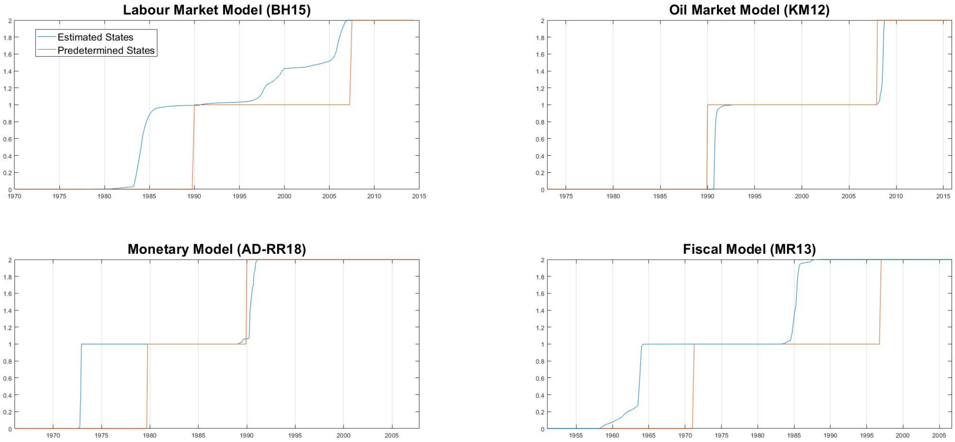
To check the robustness of our results, we re-estimated all of the models changing the number of states (S) from three to two. This did not produce any relevant difference in the results. We have also considered the possibility of four regimes (results available upon request) and for the first three applications the impulse responses are very similar. With S=4 in the Proxy-SVAR the responses to personal tax (APIRT) shocks do not change while the responses of Government spending and debt to corporate income (ACIRT) tax shocks change. Yet, in this model, due to the small sample available, the estimation algorithm does not manage to properly identify each of the four regimes, and it assigns very few observations to some of them.

In addition, we replicated the analysis fixing change-points to predetermined dates defined following Brunnermeier et al. (2021) or based on external information. Figure 8 compares the posterior mean of the states, $\delta_{1:T}$, obtained through the change-point specification of Chib (1998) with the ones built on the basis of predetermined regimes. It can be seen that the two series are fairly different. Nevertheless, the two models produce very few differences in the IRFs. This is not surprising in light of theoretical results pointed out by Rigobon (2003), Lewis (2021), and Sims (2020), according to which misspecification of the variance regimes does not undermine the consistency of the estimated structural parameters, as long as the variance of the shocks is sufficiently different across the misspecified regimes. To further support this interpretation we report in section C of the Supplemental materials results obtained fixing the variance regimes to the predetermined dates.

7 Conclusions

In this paper we proposed a blended approach to identification. Specifically, the approach consists in exploiting heteroskedasticity to sharpen identification in models with sign restrictions, narrative restrictions, and external instruments. The basic mechanism at work is one in which the introduction of regime shifts in the reduced form variances allows the data to be more informative about which of the structural representations is compatible with sign and narrative restrictions or which conditions coming from external instruments are admissible.

Figure 8: Estimated states and predetermined regimes



Notes: Posterior mean of states (blue) and predetermined states (red). For the first three applications, regimes are defined following Brunnermeier et al. (2021), for the fiscal application the first variance regime that goes from 1951:Q1 to 1971:Q1 and can be called the“Bretton Woods” regime, the second change point is in 1996:Q4 and separates a period of deficit increase from the subsequent period of deficit cuts.

We developed the estimation algorithms necessary to implement such blended approach. We presented a view that researchers should consider this approach as the blend contains more identifying information than the parts, while the cost is minimal. We illustrated this point by revisiting some key influential papers using sign restrictions, narrative restrictions, or proxy variables. In particular, we revisited the applications in Baumeister and Hamilton (2015), Kilian and Murphy (2012) and Antolin-Diaz and Rubio Ramirez (2018), and the Proxy-SVAR model in Mertens and Ravn (2013). In all these instances we found that combining the information introduced by heteroskedasticity with incomplete restrictions does sharpen identification, and leads to economically relevant changes in the effects of the identified structural shocks.

Appendix A. Conditional Posterior Distributions

A.1 SVAR Parameters

Recall that the likelihood of the SVAR-H model of section 3.1 is given by:

$$p\left(Y|A,B,\Lambda_{1:S}\right) = (2\pi)^{-\frac{Tn}{2}} \left|\det\left(A\right)\right|^{T} \left(\prod_{i=1}^{n} \lambda_{i,1}^{-T/2}\right) \left(\prod_{i=1}^{n} \omega_{i,s}^{-T_s/2}\right) \times$$

$$\exp\left\{-\frac{1}{2}\sum_{i=1}^{n} \lambda_{i,1}^{-1} \left[\sum_{s=1}^{S} \omega_{i,s}^{-1} \left(A_i y^{(s)} - B_i x^{(s)}\right) \left(A_i y^{(s)} - B_i x^{(s)}\right)'\right]\right\}$$

A.1.1 Conditional posterior distribution of (A,B) We elicit normal prior for the structural parameter B_i, $p(B_i) \sim N\left(\mu_{b_i}, V_{b_i}\right)$, for $i = 1,...,n$. Furthermore, we write $A_i = W_i\alpha_i + w_i$ and set priors $p\left(\alpha_i\right)$ for the free elements of A_i. Applying Bayes theorem, we obtain the posterior:

$$p(\alpha_i, B_i|Y,\Lambda_{1:S}) \propto p\left(Y|A,B,\Lambda_{1:S}\right) \times p\left(B_i\right) \times p\left(\alpha_i\right)$$

$$\propto \left|\det\left(A\right)\right|^{T} \exp\left\{-\frac{1}{2}\lambda_{i,1}^{-1} \left[\sum_{s=1}^{S} \omega_{i,s}^{-1} \left(\left(W_i\alpha_i + w_i\right) y^{(s)} - B_i x^{(s)}\right) \left(\left(W_i\alpha_i + w_i\right) y^{(s)} - B_i x^{(s)}\right)'\right]\right\} \times$$

$$\exp\left[-\frac{1}{2}\left(B_i - \mu_{b_i}\right) V_{b_i}^{-1} \left(B_i - \mu_{b_i}\right)'\right] \times p\left(\alpha_i\right).$$

Completing squares and rearranging terms, the density above can be written as:

$$p(\alpha_i, B_i|Y,\Lambda_{1:S}) \propto \left|\det\left(A\right)\right|^{T} \exp\left[-\frac{1}{2}\left(\alpha_i - \bar{\alpha}_i\right)' V_{\alpha_i}^{-1} \left(\alpha_i - \bar{\alpha}_i\right)\right] \times p\left(\alpha_i\right) \times$$

$$\exp\left[-\frac{1}{2}\left(B_i - \bar{\mu}_{b_i}\right) \bar{V}_{b_i}^{-1} \left(B_i - \bar{\mu}_{b_i}\right)'\right],$$

where the first line is the posterior density of α_i conditional on $\Lambda_{1:S}$ only, $p(\alpha_i|Y,\Lambda_{1:S})$, and the second line is the Gaussian posterior density of B_i, conditional on both $\Lambda_{1:S}$ and A_i, $p(B|Y,\alpha_i,\Lambda_{1:S}) \sim N\left(\bar{\mu}_{b_i}, \bar{V}_{b_i}\right)$, with $\bar{V}_{b_i} = \left[V_{b_i}^{-1} + \lambda_{i,1}^{-1}\sum_{s=1}^{S}\omega_{i,s}^{-1}x^{(s)}\left(x^{(s)}\right)'\right]^{-1}$ and $\bar{\mu}'_{b_i} = \left[\mu'_{b_i}V_{b_i}^{-1} + \lambda_{i,1}^{-1}\sum_{s=1}^{S}\omega_{i,s}^{-1}A_i y^{(s)}\left(x^{(s)}\right)'\right]\bar{V}_{b_i}$. Furthermore, it is straightforward to show that $\bar{V}_{\alpha_i}^{-1} = W_i'\left(\lambda_{i,1}^{-1}\sum_{s=1}^{S}\omega_{i,s}^{-1}y^{(s)}y^{(s)\prime} - \Upsilon'\bar{V}_{b_i}\Upsilon\right)W_i$ and $\bar{\alpha}_i = \bar{V}_{\alpha_i}W_i'\left[\Upsilon'\bar{V}_{b_i}\left(V_{b_i}^{-1}\mu_{b_i} + \lambda_{i,1}^{-1}\sum_{s=1}^{S}\omega_{i,s}^{-1}x^{(s)}\left(w_i'y^{(s)}\right)'\right) - \left(\lambda_{i,1}^{-1}\sum_{s=1}^{S}\omega_{i,s}^{-1}y^{(s)}y^{(s)\prime}w_i\right)\right]$, where $\Upsilon = \lambda_{i,1}^{-1} \sum_{s=1}^{S} \omega_{i,s}^{-1} y^{(s)} x^{(s)\prime}$.

If the priors $p(\alpha_i)$ are uniform or belong to the generalized normal family, draws from $p(\alpha_i|Y, \Lambda_{1:S})$ are readily available using an extension of Waggoner and Zha's (2003) sampler developed by Villani (2009) and Chan et al. (2021). For any arbitrary prior $p(\alpha_i)$, Waggoner and Zha's (2003) method can be still implemented to draw a candidate in the Independence Metropolis Hastings algorithm described in section **3.1**.

A.1.2 Posterior distribution of $\lambda_{1,i}$ Given the structural parameters A and B. If we focus on regime $s = 1$ and set an inverse gamma prior for $\lambda_{i,1}$, $\lambda_{i,1} \sim IG\left(d_{i,1}/2, \zeta_{i,1}/2\right)$, the posterior distribution is:

$$p(\lambda_{i,1}|Y, A, B, \omega_{i,s}) \propto p(Y|A, B, \Lambda_{1:S}) \times p(\lambda_{i,1})$$

$$\propto (\lambda_{i,1})^{-\left(\frac{T+d_{i,1}+2}{2}\right)} \exp\left\{-\frac{1}{2\lambda_{i,1}}\left[\zeta_{i,1} + \sum_{s=1}^{S} \omega_{i,s}^{-1}\left(A_i y^{(s)} - B_i x^{(\mathbf{s})}\right)\left(A_i y^{(s)} - B_i x^{(\mathbf{s})}\right)'\right]\right\}$$

which is the kernel of an Inverse Gamma pdf with $\frac{T+d_{i,1}}{2}$ degrees of freedom and scale equal to $\bar{\zeta}_{i,1}/2 = \left[\zeta_{i,1} + \sum_{s=1}^{S} \omega_{i,s}^{-1}\left(A_i y^{(s)} - B_i x^{(\mathbf{s})}\right)\left(A_i y^{(s)} - B_i x^{(\mathbf{s})}\right)'\right]/2$.

A.1.4 Posterior distribution of ω_i Given the prior distribution $\omega_{i,s} \sim IG\left(d_{i,s}/2, \zeta_{i,s}/2\right)$, the posterior for $\omega_{i,s}$ is:

$$p(\omega_{i,s}|Y, A, B, \lambda_{i,1}) \propto p(Y|A, B, \Lambda_{1:S}) \times p(\omega_{i,s})$$

$$\propto (\omega_{i,s})^{-\left(\frac{T_s+d_{i,s}+2}{2}\right)} \exp\left\{-\frac{1}{2\omega_{i,s}}\left[\zeta_{i,s} + \lambda_{i,1}^{-1}\left(A_i y^{(s)} - B_i x^{(\mathbf{s})}\right)\left(A_i y^{(s)} - B_i x^{(\mathbf{s})}\right)'\right]\right\}$$

which is again an Inverse Gamma kernel with $\frac{T_s+d_{i,s}}{2}$ degrees of freedom and scale equal to $\bar{\zeta}_{i,s}/2 = \left[\zeta_{i,s} + \lambda_{i,1}^{-1}\left(A_i y^{(s)} - B_i x^{(\mathbf{s})}\right)\left(A_i y^{(s)} - B_i x^{(\mathbf{s})}\right)'\right]/2$.

endappendices

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Supplemental Materials

(intended for online publication)

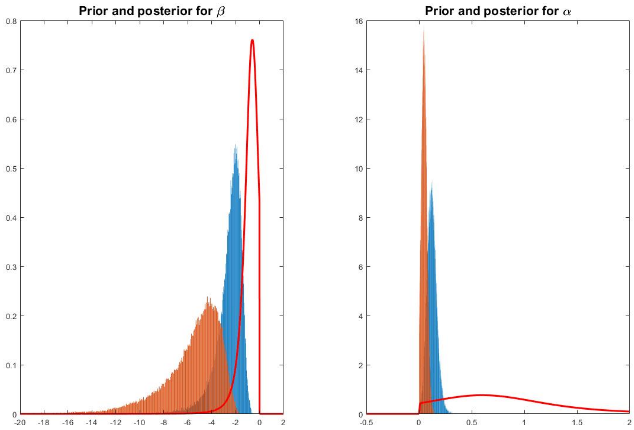
"Blended Identification in Structural VARs"

Andrea Carriero, Massimiliano Marcellino, and Tommaso Tornese

Appendix B. Baumeister and Hamilton (2015) with Independent Priors

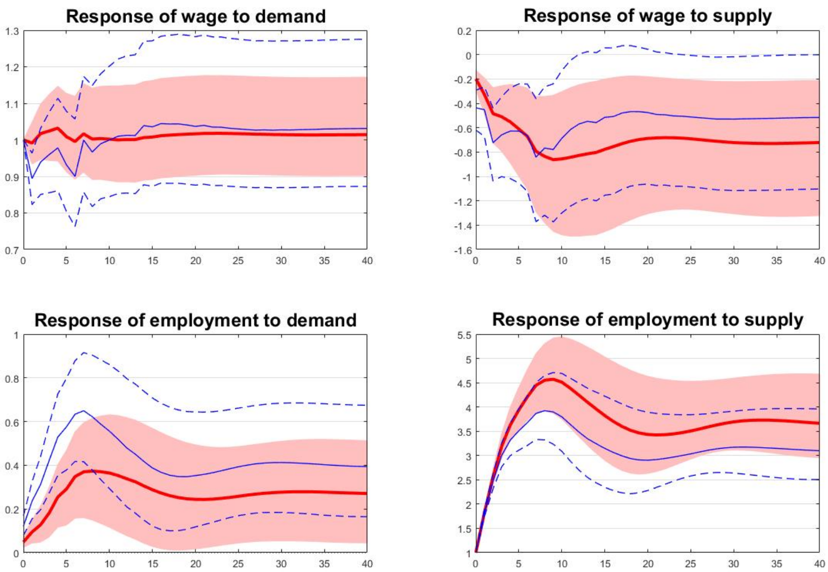
In this section we compare the results for generated by the SVAR-H model with those generated by a homoskedastic version of the same specification, obtained setting $\omega_{i,s} = 1$ for all i and s. Therefore, the differences shown in the figures can be only driven by information conveyed by the data, and cannot be affected by differences in parameters priors or in the sampling algorithm.

Figure A1: Prior vs Posteriors.



Notes: BH (blue histogram) denotes the homoskedastic SVAR with independent priors, SVAR-H (red histogram) indicates the heteroskedastic model. The red line is the prior density, which is identical in both specifications.

Figure A2: Cumulated IRFs.



Notes: BH (blue) denotes the homoskedastic SVAR with independent priors, SVAR-H (red) indicates the heteroskedastic model. Solid lines indicate median responses while shaded areas and dashed lines represent 68% credible bands.

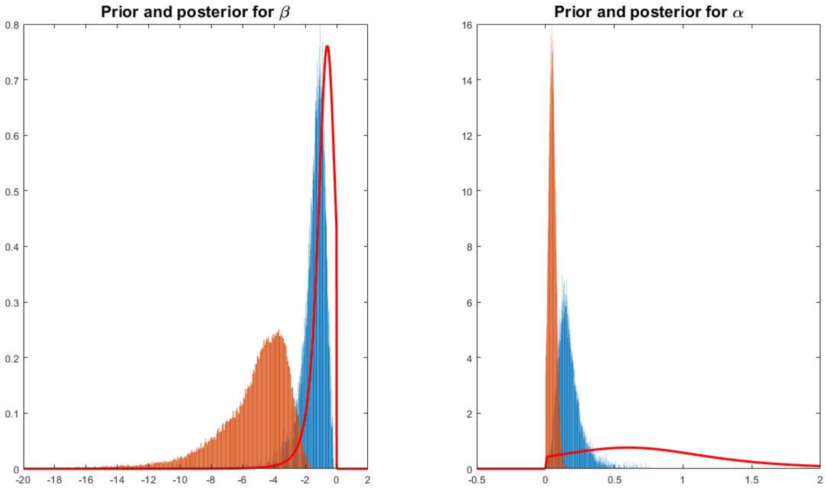
Appendix C. Model With Predetermined Regimes

In this section we compare the posterior distributions of the key structural parameters obtained using the change-point specification of Chib (1998) with those obtained fixing the variance regimes to the predetermined dates.

The fixed dates are defined following Brunnermeier et al. (2021) for the first three applications and a historical account of the U.S. budget policy for the last application.

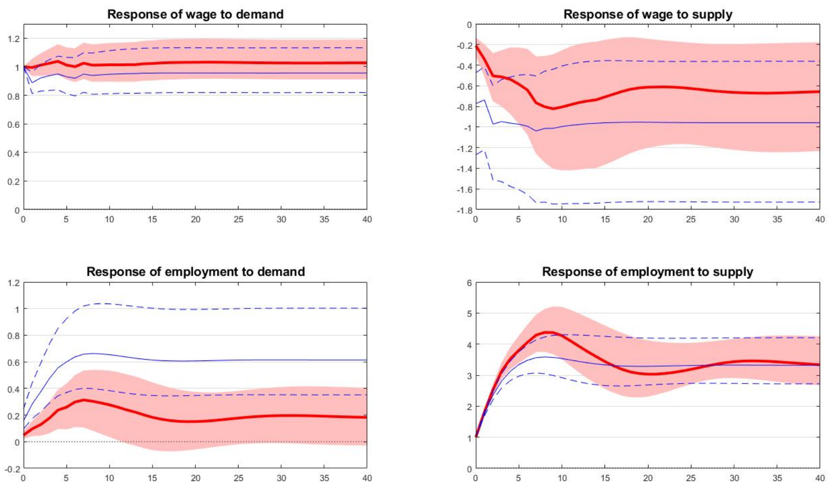
C.1. Labour Market Model [Baumeister and Hamilton (2015)]

Figure A3: Prior vs Posteriors.



Notes: BH (blue histogram) denotes the homoskedastic SVAR with natural conjugate priors, SVAR-H (red histogram) indicates the heteroskedastic model with predetermined regimes. The red line is the prior density, which is identical in both specifications. Predetermined regimes are: 1970:Q1-1989:Q4, 1990:Q1-2007:Q4, 2008:Q1-2014:Q4.

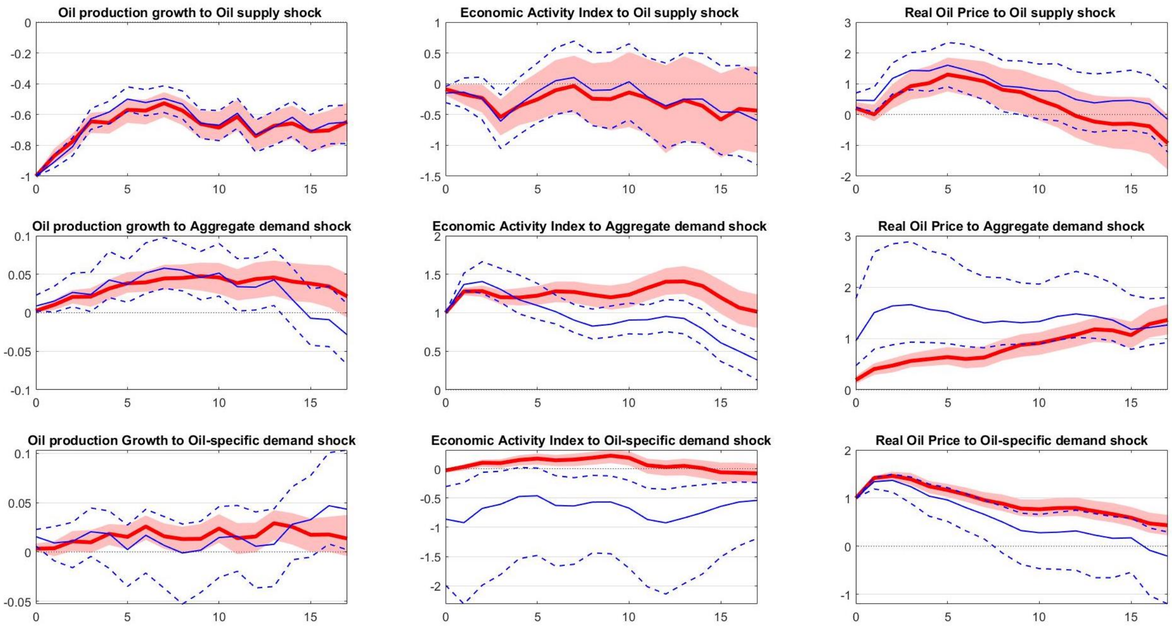
Figure A4: Cumulated IRFs.



Notes: BH (blue) denotes the homoskedastic SVAR with natural conjugate priors, SVAR-H (red) indicates the heteroskedastic model with predetermined regimes. Solid lines indicate pointwise median responses while shaded areas and dashed lines represent pointwise 68% credible bands. Predetermined regimes are: 1970:Q1-1989:Q4, 1990:Q1-2007:Q4, 2008:Q1-2014:Q4.

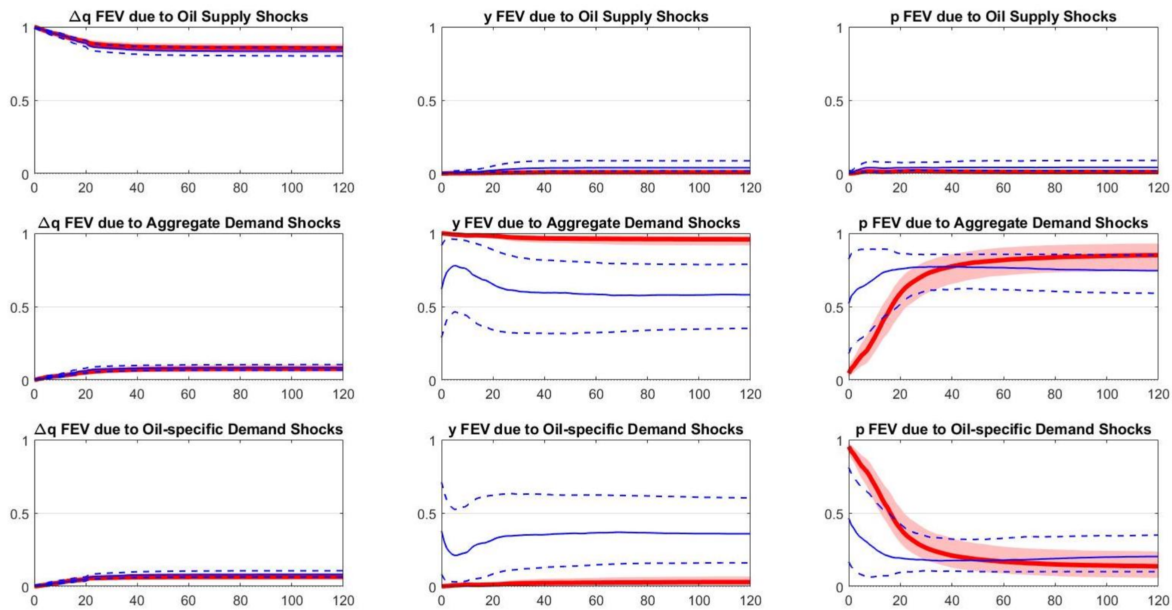
C.2. Oil Market Model [Kilian and Murphy (2012)]

Figure A5: IRFs.



Notes: KM12 (blue) denotes the sign-restricted only SVAR, SVAR-H (red) indicates the heteroskedastic model with predetermined regimes. Solid lines indicate pointwise median responses while shaded areas and dashed lines represent pointwise 68% credible bands. Predetermined regimes are: 1971:M1-1989:M12, 1990:M1-2007:M12, 2008:M1-2015:M12.

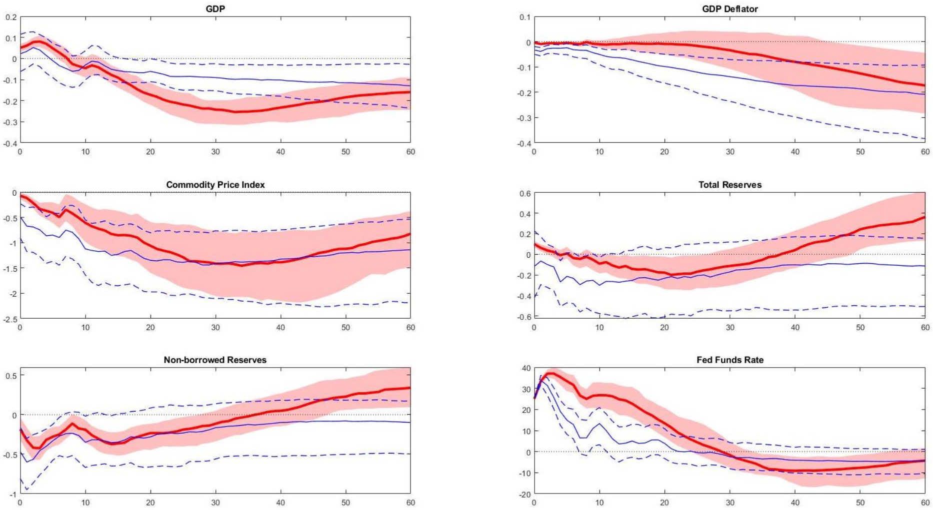
Figure A6: FEVD.



Notes: KM12 (blue) denotes the sign-restricted only SVAR, SVAR-H (red) indicates the heteroskedastic model with predetermined regimes. Solid lines indicate pointwise median FEVD while shaded areas and dashed lines represent pointwise 68% credible bands. Predetermined regimes are: 1971:M1-1989:M12, 1990:M1-2007:M12, 2008:M1-2015:M12.

C.3. Monetary Model [Antolin-Diaz and Rubio-Ramirez (2018)]

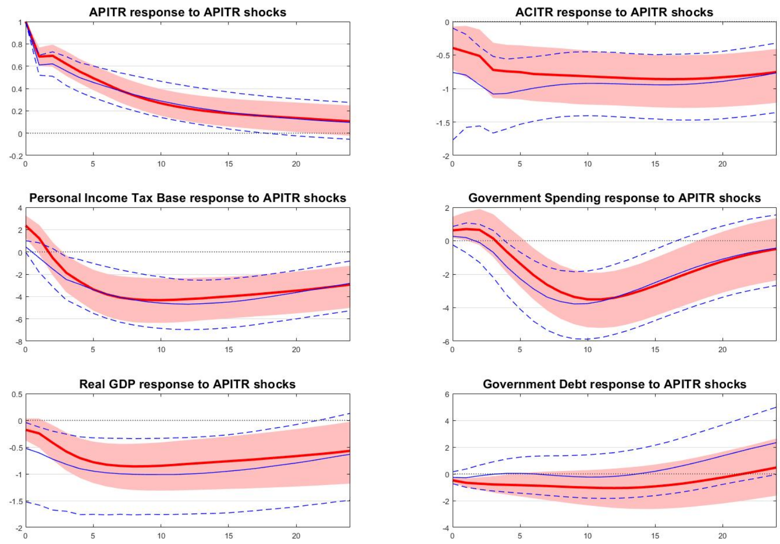
Figure A7: IRFs.



Notes: AD-RR18 (blue) denotes the sign/narrative-restricted only SVAR. SVAR-H (red) indicates the heteroskedastic model with predetermined regimes. Solid lines indicate median responses while shaded areas and dashed lines represent 68% credible bands. Predetermined regimes are: 1972:Q1-1989:Q4, 1990:Q1-2007:Q4, 2008:Q1-2019:Q4.

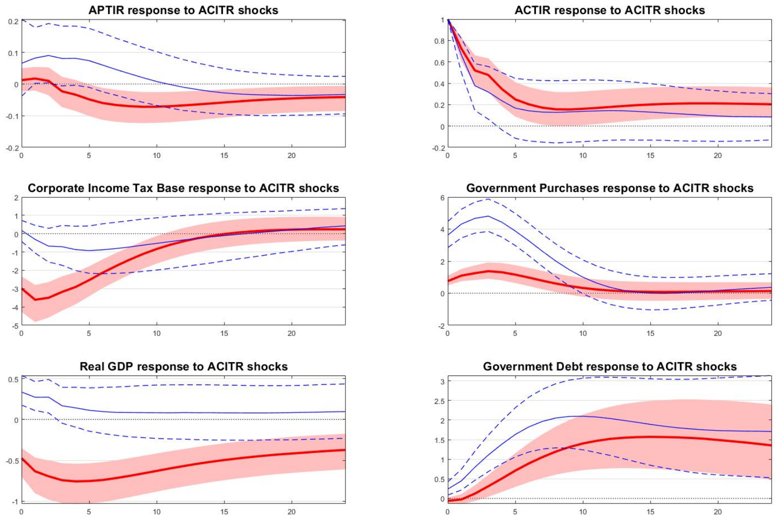
C.4. Fiscal Policy Model [Mertens and Ravn (2013)]

Figure A8: IRFs to APTIR shocks.



Notes: Proxy-SVAR (blue) denotes the homoskedastic SVAR with independent priors, Proxy-SVAR-H (red) indicates the heteroskedastic model. Solid lines indicate median responses while shaded areas and dashed lines represent 68% credible bands. Predetermined regimes are: 1951:Q1-1971:Q1 (Bretton Woods), 1971:Q2-1996:Q4 (deficit increase), 1997:Q1-2006:Q4 (deficit cuts).

Figure A9: IRFs to ACTIR shocks.



Notes: Proxy-SVAR (blue) denotes the homoskedastic SVAR with independent priors, Proxy-SVAR-H (red) indicates the heteroskedastic model. Solid lines indicate median responses while shaded areas and dashed lines represent 68% credible bands. Predetermined regimes are: 1951:Q1-1971:Q1 (Bretton Woods), 1971:Q2-1996:Q4 (deficit increase), 1997:Q1-2006:Q4 (deficit cuts).

Appendix D. Variance Ratios Between Regimes

In this section we report the 68% credible intervals obtained for the variance ratios, $\omega_{i,s}$, in the four applications analyzed in the paper. The tables show that heteroskedasticity is a feature of the data, and that regime changes are heterogeneous.

D.1. Labour Market Model [Baumeister and Hamilton (2015)]

Table A1: Variance Ratios

	$s=1$	$s=2$	$s=3$
$\omega_{1,s}$	1	$[1.0032, 1.9796]$	$[2.7348, 5.5208]$
$\omega_{2,s}$	1	$[0.1851, 0.3052$	$[0.2793, 0.5711]$

Notes: The table shows the 68% credible intervals for the variace ratios $\omega_{i,s} = {}^{\lambda_{i,s}}/_{\lambda_{i,1}}$.

D.2. Oil Market Model [Kilian and Murphy (2012)]

Table A2: Variance Ratios

	$s=1$	$s=2$	$s=3$
$\omega_{1,s}$	1	$[0.1889, 0.2500]$	$[0.1334, 0.2031]$
$\omega_{2,s}$	1	$[0.7276, 0.9726]$	$[7.5255, 10.9030]$
$\omega_{3,s}$	1	$[1.3688, 1.8332]$	$[2.0231, 3.0166]$

Notes: The table shows the 68% credible intervals for the variace ratios $\omega_{i,s} = {}^{\lambda_{i,s}}/_{\lambda_{i,1}}$.

D.3. Monetary Model [Antolin-Diaz and Rubio-Ramirez (2018)]

Table A3: Variance Ratios

	$s=1$	$s=2$	$s=3$
$\omega_{mp,s}$	1	$[10.8775, 15.0954]$	$[0.2889, 0.4246]$
$\omega_{2,s}$	1	$[11.4725, 16.4732]$	$[7.1076, 10.3021]$
$\omega_{3,s}$	1	$[1.5768, 2.9079]$	$[0.8366, 2.1579]$
$\omega_{4,s}$	1	$[1.4515, 2.8020]$	$[0.7571, 2.0046]$
$\omega_{5,s}$	1	$[8.6978, 12.2856]$	$[0.4705, 0.6747]$
$\omega_{6,s}$	1	$[2.5761, 3.9236]$	$[35.9409, 51.8108]$

Notes: The table shows the 68% credible intervals for the variace ratios $\omega_{i,s} = {}^{\lambda_{i,s}}/_{\lambda_{i,1}}$. The subscript mp indicates the monetary policy shock.

D.4. [Mertens and Ravn (2013)]

Table A4: Variance Ratios

	$s=1$	$s=2$	$s=3$
$\omega_{1,s}$	1	$[6.2132, 12.3128]$	$[3.4723, 6.9967]$
$\omega_{2,s}$	1	$[0.2945, 0.5278]$	$[0.0855, 0.1543]$

Notes: The table shows the 68% credible intervals for the variace ratios $\omega_{i,s} = {}^{\lambda_{i,s}}/_{\lambda_{i,1}}$.

Appendix E. Bayes Factors

In this section we report the logarithmic Bayes factors in favour of our identified model against the restricted alternatives with homoskedastic shocks or with proportional changes in the variances of shocks. Since the models characterized by shocks with constant variances or with variances that change proportionally across regimes are nested in the SVAR-H specification, Bayes factors can be computed as Savage-Dickey Density Ratios (SDDR) as suggested by Lutkepohl and Wozniak (2020). Based on the scale discussed by Kass and Raftery (1995), Lutkepohl and Wozniak (2020) argue that a log-SDDR with absolute value larger than 5 should be considered as strong evidence against the restricted models.

E.1. Labour Market Model [Baumeister and Hamilton (2015)]

Table A5: Log-SDDR against homoskedasticity

	Restriction: $\omega_{j,2} = 1$	Restriction: $\omega_{j,3} = 1$
$j = 1$	0.5570	**−6.8053**
$j = 2$	−4.1586	−2.0581

Notes: Log-SDDR comparing SVAR-H with a model with homoskedastic shocks.

Table A6: Log-SDDR against proportional changes in variances of shocks

	$s = 2$	$s = 3$
Restriction: $\omega_{1,s}/\omega_{2,s} = 1$	−3.7048	**−11.3288**

Notes: Log-SDDR comparing SVAR-H with a model with proportional changes in shocks variances.

E.2. Oil Market Model [Kilian and Murphy (2012)]

Table A7: Log-SDDR against homoskedasticity

	Restriction: $\omega_{j,2} = 1$	Restriction: $\omega_{j,3} = 1$
$j = 1$	**−60.0116**	**−33.4291**
$j = 2$	1.3080	**−163.3203**
$j = 3$	−2.4388	**−8.2182**

Notes: Log-SDDR comparing SVAR-H with a model with homoskedastic shocks.

Table A8: Log-SDDR against proportional changes in variances of shocks: $\omega_{i,s}/\omega_{j,s} = 1$

	$s = 2$		$s = 3$	
	$j = 2$	$j = 3$	$j = 2$	$j = 3$
$i = 1$	**−26.3433**	**−57.9711**	**−104.3174**	**−42.6160**
$i = 2$		−2.4626		**−10.0056**

Notes: Log-SDDR comparing SVAR-H with a model with proportional changes in shocks variances.

E.3. Monetary Model [Antolin-Diaz and Rubio-Ramirez (2018)]

Table A9: Log-SDDR against homoskedasticity

	$s = 1$	$s = 2$
Restriction: $\omega_{mp,s} = 1$	**−617.5537**	**−302.2689**

Notes: Log-SDDR comparing SVAR-H with a model with homoskedastic shocks.

Table A10: Log-SDDR against proportional changes in variances of shocks: ${}^{\omega_{i,s}}/_{\omega_{j,s}} = 1$

	$s=1$				
	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$
Restriction: ${}^{\omega_{mp,s}}/_{\omega_{j,s}} = 1$	1.1233	$\mathbf{-26.4753}$	$\mathbf{-32.9548}$	0.9047	$\mathbf{-11.4614}$
	$s=2$				
	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$
Restriction: ${}^{\omega_{mp,s}}/_{\omega_{j,s}} = 1$	$\mathbf{-9.9081}$	-3.3393	$\mathbf{-5.2298}$	-0.5995	$\mathbf{-101.6933}$

Notes: Log-SDDR comparing SVAR-H with a model with proportional changes in shocks variances.

E.4. [Mertens and Ravn (2013)]

Table A11: Log-SDDR against homoskedasticity

	$s=1$	$s=2$
Restriction: $\omega_{apitr,s} = 1$	$\mathbf{-9.4389}$	$\mathbf{-6.6678}$
Restriction: $\omega_{acitr,s} = 1$	-4.6126	$\mathbf{-27.2904}$

Notes: Log-SDDR comparing SVAR-H with a model with homoskedastic shocks.

Table A12: Log-SDDR against homoskedasticity

	$s=1$					
	$j=acitr$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$
Restriction: ${}^{\omega_{apitr,s}}/_{\omega_{j,s}} = 1$	$\mathbf{-12.9349}$	$\mathbf{-5.9923}$	$\mathbf{-7.0207}$	$\mathbf{-11.6530}$	$\mathbf{-7.3421}$	-4.2139
Restriction: ${}^{\omega_{acitr,s}}/_{\omega_{j,s}} = 1$		-1.2835	-1.3635	1.2729	-0.9778	-2.2920
	$s=2$					
	$j=acitr$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$
Restriction: ${}^{\omega_{apitr,s}}/_{\omega_{j,s}} = 1$	$\mathbf{-32.3109}$	-0.6315	$\mathbf{-19.3867}$	$\mathbf{-8.6897}$	-1.3102	$\mathbf{-7.3570}$
Restriction: ${}^{\omega_{acitr,s}}/_{\omega_{j,s}} = 1$		$\mathbf{-27.4489}$	-0.0844	$\mathbf{-5.2758}$	$\mathbf{-26.6766}$	$\mathbf{-5.3718}$

Notes: Log-SDDR comparing SVAR-H with a model with proportional changes in shocks variances.