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Abstract

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ZOOMERS AND BOOMERS: ASSET PRICES AND INTERGENERATIONAL INEQUALITY

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We construct a perpetual youth DSGE model with aggregate uncertainty in which there are dynamically complete markets and agents have Epstein-Zin preferences. We prove that, when endowments have a realistic hump-shaped age-profile, our model has three steady-state equilibria. One of these equilibria is dynamically inefficient and displays real price indeterminacy. We estimate the parameters of our model and we find that a fourth-order approximation around the indeterminate steady-state provides the best fit to U.S. data. Our work interprets the large and persistent generational inequality that has been observed in western economies over the past century as the result of uninsurable income shocks to birth cohorts.

KEYWORDS: Perpetual Youth, Asset Pricing, Indeterminacy.

1. INTRODUCTION

The use of dynamic stochastic general equilibrium (DSGE) models to understand the macroeconomy began in the 1980s with the Real Business Cycle (RBC) model of Kydland and Prescott (1982) and Long and Plosser (1983). Although the initial version of the RBC model contained a representative agent, more recent heterogeneous agent new-Keynesian (HANK) models built around an RBC core, contain multiple agents and in-

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complete asset markets (Kaplan et al., 2018, Auclert et al., 2020). This paper presents an alternative way – *the perpetual youth model* – of introducing heterogeneous agents into a DSGE model.

A perfect foresight version of the perpetual youth model in continuous time was introduced by Blanchard (1985) and was extended to discrete time with aggregate shocks by Farmer (1990) and to include multiple types of agents in Farmer (2018). The perpetual youth model is a variant of the overlapping generations (OLG) model of Allais (1947) and Samuelson (1958) and, as such, it has very different properties from the class of infinite-lived agent models of which the RBC model is a special case.

In the perpetual youth model there are always at least two steady state equilibria. Fiscal and monetary policy work very differently from the way they operate in representative agent models and active fiscal and monetary policy may fail to uniquely determine either the price level or the real interest rate (Farmer and Zabczyk, 2022). As in the OLG model, stationary equilibria may be indeterminate of arbitrary degree (Kehoe and Levine, 1985) and, as a consequence, monetary policy may have real effects in the short-run (Farmer, 1991) even in the absence of menu-costs or other forms of price-rigidity.

We focus in this paper on two sets of facts which, we argue, are connected. The first set concerns asset prices, interest rates and growth. The second set concerns the low frequency behavior of inter-generational inequality.

Our first set of facts concerns the behavior of the safe nominal interest rate, the nominal return on a risky portfolio, and the growth rate of nominal GDP. In data the safe nominal interest rate has been less than the nominal growth rate for long periods of time whereas the ex-post return to the aggregate stock market (as proxied by the CRSP value-weighted market portfolio) has been consistently higher than the safe rate by about 6% and higher also than the GDP growth rate. The risky rate is also much more volatile than the safe rate at both high and low frequencies. These facts are illustrated in Figure 1.

Next we turn our attention to generational inequality and its connection to asset market volatility. Figure 2 plots the cyclically adjusted price earnings ratio from Shiller (2014), measured on the right axis, alongside a measure of relative median lifetime income, measured on the left axis. This latter data-series measures the median lifetime income of the generation that attains the age of 25 in the year measured on the horizontal axis, relative to the generation that attained age 25 in 1957. We want to draw attention to two features

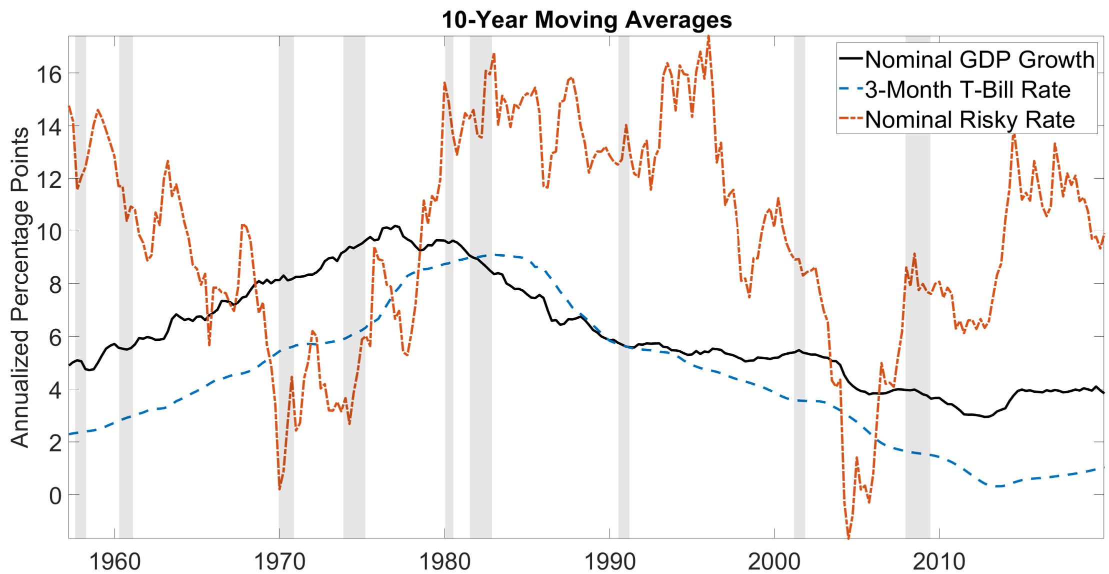

Figure 1.: Asset Returns and GDP growth

Note: This figure plots ten-year moving averages of three different quarterly U.S. time series from 1957-2019: 1) the nominal GDP growth rate (solid black), 2) the 3-month Treasury Bill rate (dashed blue), and 3) the return on the CRSP value-weighted market portfolio (dotted-dashed red). Each series is reported in annualized percentage points.

of these data. First, lifetime median income varies considerably by cohort and those that got their first job in the late 1960s and early 1970s were roughly 12% better off than both earlier and later generations. Second, lifetime earnings prospects are highly correlated with the state of the stock market at the time of attaining adulthood. In this paper, we connect these two sets of facts and we offer an explanation which ties together a theory of excess asset price volatility with a theory of generational inequality.

To explain the connection between our two sets of facts we construct a stochastic perpetual youth model of an exchange economy in which agents have Epstein Zin preferences (Epstein and Zin, 1989, 1991) and we estimate the parameters of our model using simulated method of moments. Our model explains excess asset price volatility as a sunspot equilibrium in a low safe-rate equilibrium and it connects asset pricing evidence with generational inequality using the fact that agents cannot insure against the state of the world they are born into.

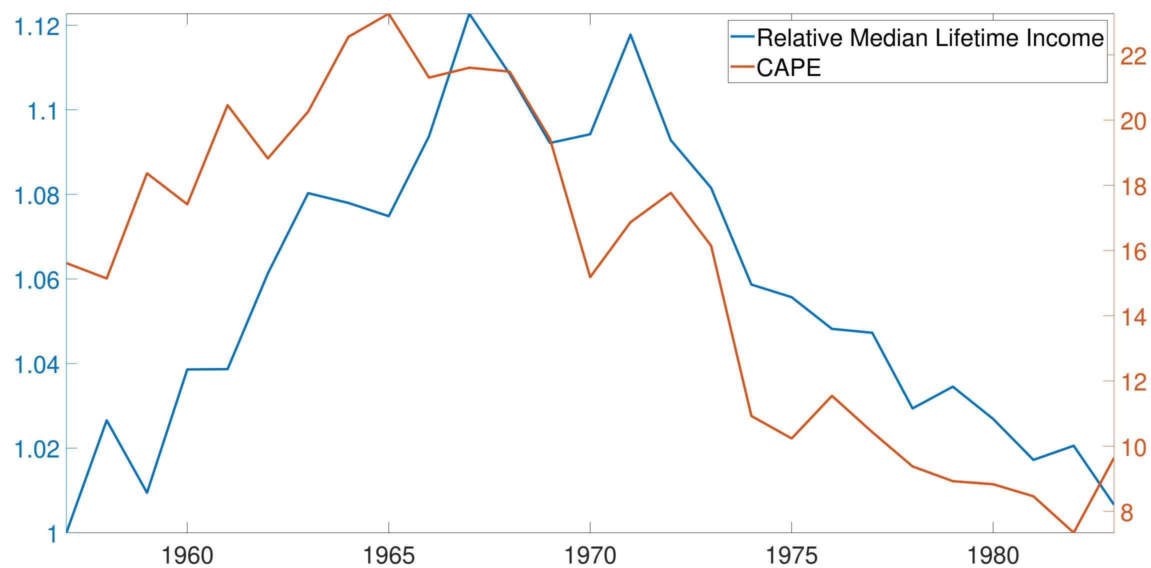

Figure 2.: Asset Prices and Generational Inequality

Note: This figure plots the cyclically adjusted price earnings ratio (CAPE) for the S&P 500 in red on the right axis and a measure of relative median lifetime income for males in the U.S. in blue on the left axis at an annual frequency. The CAPE is taken from Robert Shiller's website http://www.econ.yale.edu/ shiller/data.htm and the measure of median liftetime income is taken from Guvenen et al. (2022). Median lifetime income is normalized to 1 in 1957 and all subsequent observations are relative to 1957.

2. LITERATURE REVIEW

Our work is related to a number of previous papers that deal with asset pricing in an exchange economy. Much of this literature deals with a continuous time model with endowment shocks that follow a diffusion process. Papers in this literature include Gârleanu and Panageas (2015, 2021, 2022), Schmidt (2022), and Gomez (2022). Typically, there is no government sector and inflation is assumed to be exogenous.[1] In contrast, we build a model in which government intervenes in the goods and asset markets through fiscal and monetary policy and we assume that time is discrete. This places our work at the intersection of the representative agent macro models surveyed in Leeper and Leith (2016) and the asset pricing papers cited above.

[1] An important exception to this is Swanson (2021), who shows that a discrete-time representative agent New-Keynesian model with Epstein-Zin preferences is consistent with a number of asset pricing facts.

An emerging literature extends Representative Agent New-Keynesian (RANK) models to allow for uninsurable income risk by adding multiple agents and incomplete markets. These models come in two-agent varieties – TANK models – of the kind studied by Bilbiie (2008, 2020), and – HANK models – as in the work of Kaplan et al. (2018) and Auclert et al. (2020).[2] HANK models are more general than TANK models but they must carry around the wealth distribution as a state variable. Since the wealth distribution is an infinite dimensional object, solving and estimating HANK models is a challenging, but not insurmountable, problem. Techniques to solve and estimate HANK models, building on insights from Krusell and Smith (1998), have been developed by Reiter (2009), Winberry (2018), Auclert et al. (2021) and Bilal (2021).

Our work is complementary to the HANK literature, but we approach the issue of heterogeneity in a different way. In contrast to the literature reviewed in Kaplan and Violante (2018), where wealth inequality arises from uninsurable idiosyncratic income risk, we follow Campbell and Nosbusch (2007) by assuming that wealth inequality is caused by uninsurable aggregate risks to newborn generations who cannot insure across the state of the world they are born into. Unlike Campbell and Nosbusch (2007) who calibrate a perpetual youth model with logarithmic preferences, our agents have Epstein-Zin preferences and we estimate the parameters of our model on U.S. data. The extension to a more general preference specification is key to our results which exploit the existence of multiple autarkic steady-state equilibria when agents have a hump-shaped income profile and a low intertemporal elasticity of substitution.

The first DSGE perpetual youth model in discrete time, of which we are aware, is the paper by Farmer (1990) who builds a DSGE perpetual youth model using a special case of Epstein-Zin preferences. We generalize Farmer (1990) to the case of general Epstein-Zin (1989) preferences and we allow for a hump-shaped endowment process.[3] Both of these features are key to the ability of our model to fit asset pricing facts in U.S. data.

[2]TANK is an acronym for Two Agent New Keynesian and HANK stands for Heterogeneous Agent New Keynesian.

[3]In related developments in the DSGE perpetual youth literature, Farmer et al. (2011) show how to construct the pricing kernel in a discrete time DSGE perpetual youth model with complete markets and Farmer (2018) uses their result to construct a model with two types of agents who have Von-Neumann Morgenstern preferences. Gârleanu and Panageas (2015, 2021, 2022) present a series of results for the Epstein and Duffie (1992) continuous time case.

Our solution to the individual's problem is related to the results in Toda (2014) and Flynn et al. (2023) who study the solution to a problem in which agents have access to a limited set of assets. In contrast, our assumption that markets are dynamically complete allows us to aggregate individual decision rules and to generate a set of low dimensional aggregate equations that characterize equilibrium and facilitates our empirical application. Our ability to accommodate heterogeneous agents in a tractable way distinguishes our empirical work from DSGE models that solve and estimate Epstein-Zin models with a representative consumer (Epstein and Zin, 1991, van Binsbergen et al., 2008).

An important contribution of our paper is our proof that a hump-shaped endowment pattern interacts with a low intertemporal elasticity of substitution to generate multiple autarkic steady-state equilibria in a perpetual youth model. Our current paper extends results in Farmer and Zabczyk (2022) for a 62-period model to the perpetual youth model with aggregate shocks and estimates the parameters of the model using real-world U.S. data.

We are not the only authors to point to dynamic indeterminacy as a potential explanation for features of the asset markets. Brunnermeier et al. (2022b,a) study the existence of bubbles in infinite horizon models in both continuous and discrete time and Aguiar et al. (2021) study Pareto improving policies in a model with idiosyncratic income risk. Reis (2021) explicitly studies the role of liquidity effects in a model with aggregate shocks in which the interest rate is less than the growth rate and Miao and Su (2021) study the emergence of debt as a bubble in a Keynesian model with production. Unlike the papers cited here, our model has no frictions and dynamically complete markets and we estimate the parameters of our model on U.S. data.

Our theoretical model contains multiple steady-state equilibria each of which may be locally indeterminate under some combinations of monetary and fiscal regimes. In our empirical work we estimate both determinate and indeterminate versions of our model as in the work of Lubik and Schorfheide (2004), Aruoba et al. (2018) and Farmer and Nicolò (2018).[4] In our estimation strategy we first choose the dimension of the state and, for each choice of this dimension, we approximate the solution to the model by a fourth order approximation using Matlab code from Levintal (2017).

[4]Lubik and Schorfheide (2003) were the first to develop a method to estimate indeterminate models. Their approach was refined by Farmer et al. (2015) and Bianchi and Nicolò (2021).

3. THE AGENTS' PROBLEM

We construct a perpetual youth model in which agents die with probability $1-\pi$ and in which a member of the cohort born at date j is endowed with a before-tax fraction y_t^j of aggregate real GDP which we refer to with the symbol Y_t. Define

$$y_t^j = \frac{1}{1-\pi}\left(\kappa_1 \lambda_1^{t-j} + \kappa_2 \lambda_2^{t-j}\right), \tag{1}$$

for $(\kappa_1, \kappa_2) \in \mathbb{R}^2$ and $(\lambda_1, \lambda_2) \in [0,1]^2$. We choose the parameters $\kappa_1, \kappa_2, \lambda_1$ and λ_2 to match the U.S. income profile as in Gârleanu and Panageas (2015). Figure 3 shows the age-profile of individual before-tax income shares for our choice of parameters.

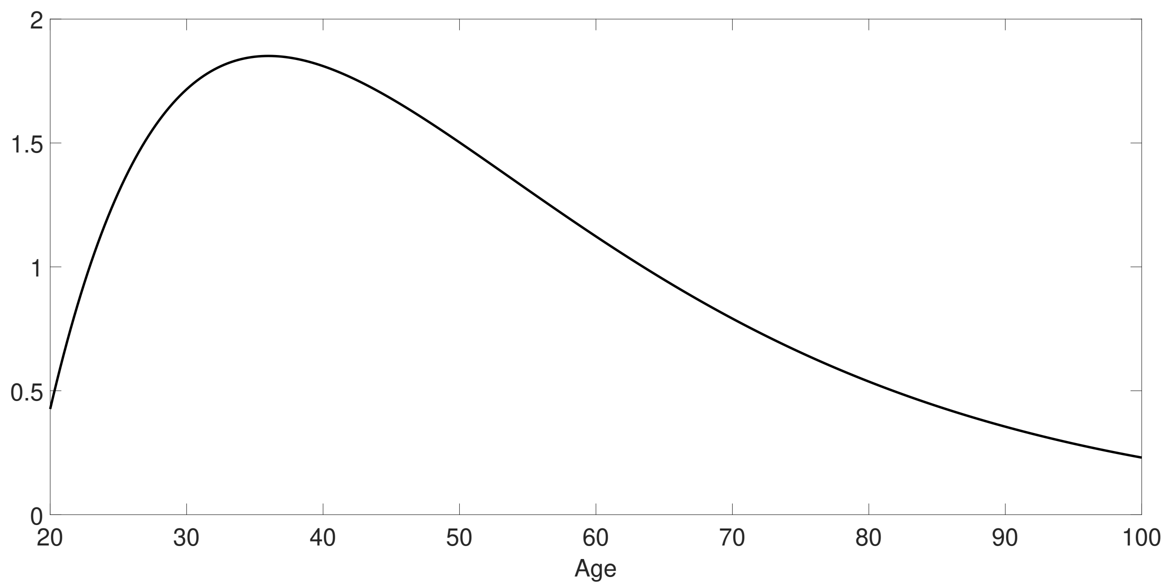

Figure 3.: Individual Before-Tax Income Share

Note: This figure plots the individual before-tax income share of an agent conditional on surviving. The x-axis is the age of the agent in years, assuming the agent "begins life" at age 20.

GDP is generated by the process $\frac{Y_t}{Y_{t-1}} = \gamma_t$, where γ_t is a non-negative random variable whose dynamics follow an autoregressive process in logs. Define $\tilde{\gamma}_t \equiv \log(\gamma_t)$. Let $0 < \rho_\gamma < 1$ be the persistence of $\tilde{\gamma}_t$, let $\bar{\gamma}$ be the steady-state growth rate of GDP, and let ε_γ be an i.i.d. random variable with mean 0 and variance σ_γ^2. The dynamics of $\tilde{\gamma}_t$ are given by

$$\tilde{\gamma}_{t+1} = (1-\rho_\gamma)\log(\bar{\gamma}) + \rho_\gamma \tilde{\gamma}_t + \varepsilon_{\gamma,t+1}. \tag{2}$$

The variables of our model are elements of a vector of random variables $X_t \in \mathcal{X} \subset \mathbb{R}_+^n$ which we partition into two subsets $X = \{S, T\}, \quad S \in \mathcal{X}_S \subset \mathbb{R}_+^{n_1}, \quad T \in \mathcal{X}_T \subset \mathbb{R}_+^{n_2}, \quad n = n_1 + n_2$. We refer to S as *states* and T as *co-states*. To keep the notation concise, in the remainder of the paper we refer to variables $x_t \in X_t$ and $x_{t+1} \in X_{t+1}$ with the notation x and x' where x here refers to a generic element of X.

Private agents maximize the discounted expected value of an Epstein-Zin recursive utility function. The problem of a member of cohort j is defined by the value function, v^j, that solves Problem 1.

PROBLEM 1:

$$v^j\left(A^j\right) = \max_{A^{j\prime}} \left[\left(C^j\right)^\rho + \beta\pi\left(m^j\right)^\rho\right]^{\frac{1}{\rho}}, \tag{3}$$

$$m^j = \left\{\mathbb{E}\left[v^{j\prime}\left(A^{j\prime}\right)^{\rho\theta}\right]\right\}^{\frac{1}{\rho\theta}},$$

$$C^j + \pi\mathbb{E}\left[Q' A^{j\prime}\right] = A^j + y^j(1-\tau)Y, \tag{4}$$

with initial condition $A^j(S_j) = 0$ and where τ is the tax rate.

PROPOSITION 1—Solution to the Consumers' Problem: *The value function and the policy function that solve Problem 1 are given by $C^j = \psi^{-1} W^j$ and $v^j = \psi^{\frac{1-\rho}{\rho}} W^j$. The variable ψ is defined recursively as, $\psi = 1 + \pi\beta^{\frac{1}{1-\rho}}\left(\mathbb{E}\left[\psi'^{\frac{(1-\rho)\theta}{(1-\rho\theta)}} Q'^{\frac{\rho\theta}{\rho\theta-1}}\right]\right)^{\frac{1-\rho\theta}{(1-\rho)\theta}}$, where W^j is the sum of three components. $W^j = H_1^j + H_2^j + A^j$, and H_1 and H_2 represent the discounted present values of the two components of the after-tax income shares from the right-hand-side of Eq. (1). A^j is the value of financial assets owned by a member of generation j in state S_t.*

PROOF: For a proof of Proposition 1, see Appendix A. *Q.E.D.*

The parameters ρ and θ are related to the intertemporal elasticity of substitution, ies, and the coefficient of relative risk aversion, rra, by the identities[5]

$$ies \equiv \frac{1}{1-\rho}, \quad rra \equiv 1 - \rho\theta. \tag{5}$$

$Q' \equiv \tilde{Q}'/\chi(S')$ is the pricing kernel, $\tilde{Q}'$ is the price at date t of a claim to one unit of the commodity in state S' and $\chi(S')$ is the date t conditional probability that state S' occurs. A^j is the value of state S dependent Arrow securities that were accumulated at date $t-1$ by generation j.

The term π that appears in equations (3) and (4) serves two roles. In Eq. (3) it is the probability that a person survives into period $t+1$. In Eq. (4) it is the price of a security that insures the life of the agent. This security sells for price π when there is free entry to the financial services industry.[6]

4. GOVERNMENT POLICY

In this section we discuss fiscal and monetary policy.

4.1. *Fiscal Policy*

The government purchases g goods as a fraction of nominal GDP which it pays for by raising a proportional income tax at rate τ and by issuing nominal debt with a maturity structure parameterized by δ. The government budget equation is given by the expression,

$$B^{\delta\prime} p^{\delta} = \left(1 + \delta p^{\delta}\right) B^{\delta} + PY\left(g - \tau\right). \tag{6}$$

[5]It is more usual to parameterize Epstein-Zin preference by a parameter ρ and a parameter α, where in our notation, $\alpha = \rho\theta$. Our alternative parameterization permits us to study the special case of $ies = 1$ by taking the limit as $\rho \to 0$. The more familiar parameterization using ρ and α leads to numerical instability in our empirical estimates for values of ρ close to 0. For the special case when $\theta = 1$, agents have Von-Neumann Morgenstern preferences on the space of lotteries over intertemporal consumption sequences.

[6]If the agent is a debtor, the contract pays her debts in the event that she dies. If she is a creditor, the security represents an annuity that gives the agent an additional stream of payments while she is alive and that returns her assets to the financial institution that issued the security in the event of her death. Because there is a continuum of agents in each generation, there is no aggregate risk to issuing these securities.

Here, P is the dollar price of commodities and the nominal bond $B^{\delta\prime}$ is a promise to repay $\$1$ plus $\delta B^{\delta\prime}$ nominal bonds in period $t+1$. By choosing $\delta \in [0,1]$ we can mimic the maturity structure of public debt in U.S. data.

$B^{\delta\prime}$ sells for price p^{δ} in period t and it follows from the assumption of no riskless arbitrage that p^{δ} is given by the expression

$$p^{\delta} = \mathbb{E}\left[\frac{Q'}{\Pi'}\left(1+\delta p^{\delta\prime}\right)\right]. \tag{7}$$

Using the definition $b^{\delta} = \frac{B^{\delta}}{P_L Y_L}$, where P_L and Y_L are the lagged dollar price of commodities and lagged GDP respectively, we can rewrite Eq. (6) in terms of ratios to nominal GDP,

$$p^{\delta} b^{\delta\prime} = \frac{b^{\delta}\left(1+\delta p^{\delta}\right)}{\Pi\gamma} + g - \tau, \tag{8}$$

where Π is the gross inflation rate between periods $t-1$ and t.

We model government purchases with the assumption that a transformation of government purchases is determined by an autoregressive process. Define $\tilde{g} \equiv \log\left(\frac{1}{1-g}\right)$, let $\bar{g}$ denote the steady-state government spending-GDP ratio, and define the persistence of $\tilde{g}$ by the parameter $0 < \rho_g < 1$. We assume that

$$\tilde{g}' = (1-\rho_g)\log\left(\frac{1}{1-\bar{g}}\right) + \rho_g \tilde{g} + \varepsilon_g', \tag{9}$$

where ε_g' is a zero mean random variable with standard deviation σ_g.

We assume further that the government follows a fiscal rule of the form

$$\tau = \bar{\tau} + \phi_{\tau}\left[\frac{b^{\delta}(1+\delta p^{\delta})}{\Pi\gamma} - \Phi\right],$$

where ϕ_τ is a fiscal response coefficient. By setting a positive response coefficient, ϕ_τ, our model can capture a passive fiscal policy in which the government actively stabilizes the economy at a given debt-GDP ratio, represented here by the symbol Φ.[7]

The target value of the debt-GDP ratio must be consistent with its steady-state value, $\Phi = \frac{\bar{b}^\delta(1+\bar{p}^\delta)}{\bar{\Pi}\bar{\gamma}}$. Because there may be multiple steady-states and the steady-state value of the debt-GDP ratio $\frac{\bar{b}^\delta(1+\bar{p}^\delta)}{\bar{\Pi}\bar{\gamma}}$ is different in each of them, Φ cannot be chosen independently; it is a function of $\bar{\tau}$ and ϕ_τ, as well of all of the other parameters of the model which contribute to the determination of the steady state values of $\bar{b}^\delta$, $\bar{p}^\delta$, and $\bar{\Pi}$.

4.2. *Monetary Policy*

The central bank sets the gross nominal interest rate R as a function of the date $t-1$ interest rate R_L, the gross inflation rate Π, and the gross real GDP growth rate γ, according to the Taylor Rule (Taylor, 1999),

$$R = {R_L}^{\phi_R} \left[\left(\frac{\Pi}{\Pi^*} \right)^{\phi_\pi} \left(\frac{\gamma}{\gamma^*} \right)^{\phi_\gamma} \left(\frac{\Pi^*}{Q^*} \right) \right]^{1-\phi_R} \exp(\varepsilon_R), \tag{10}$$

where ε_R is a policy shock, generated by an i.i.d. stochastic process with mean 0 and variance given by σ_R^2.

Π^* is the gross inflation target, γ^* is the target GDP growth rate, and Q^* is the target steady-state value of the pricing kernel. The parameters ϕ_R, ϕ_π, and ϕ_γ capture the interest rate smoothing motive, the inflation response, and the output growth response of the Taylor Rule. The central bank is free to choose any value for Π^*, γ^* and Q^*; but in order to hit the growth and inflation targets they must choose values that are consistent with equilibrium. In our estimation we approximate a solution to the model around a steady-state. We choose $\gamma^* = \bar{\gamma}$ and we pick values of $\Pi^* = \bar{\Pi}$ and $Q^* = \bar{Q}$ that are consistent with their target non-stochastic steady-state.

[7] A fiscal policy in which the government adjusts taxes and spending to maintain budget balance is referred to as *passive*. A fiscal policy in which the government sets a deficit rule that is independent of the debt-GDP ratio is said to be *active*. This definition originates in an attempt to provide a unified theory of fiscal and monetary interactions (Leeper, 1991). A government that *actively* adjusts its fiscal rule is said to follow a *passive* fiscal policy. We retain the definition here for consistency with previous literature.

5. DEFINITIONS OF THE VARIABLES

In this section we construct a set of aggregate variables and a set of equations that connect these variables at consecutive dates and in consecutive states. We divide the variables that are growing through time by GDP to create a set of stationary variables and we assemble the equations of the model into a function that defines equilibrium.

5.1. *The State Variables of the Model*

We discuss two representations of the model, one in which all of the state variables are fundamental, and one in which the state includes a non-fundamental variable driven by sunspot shocks. We refer to the fundamental representation of the state as S and to the non-fundamental representation of the state as $\tilde{S}$.

S includes the variables γ and g which we model as first order auto-correlated processes in the transformed variables $\log(\gamma)$ and $\log\left(\frac{1}{1-g}\right)$, b^{δ} which is related to the real value of the debt-GDP ratio, and ε_R, the monetary policy shock. S also includes R_L and c_L from the date $t-1$ information set and z_L, where z is the following function of the moments of ψ' and Q',

$$z = \mathbb{E}\left[\psi'^{\frac{(1-\rho)\theta}{1-\rho\theta}} Q'^{\frac{\rho\theta}{1-\rho\theta}}\right].$$

We include z_L and c_L in the state because the pricing kernel at date t holds in all pairs of consecutive states $\{S_{t-1}, S_t\}$ and, as we show in Proposition 2, Q is a function of c_L and z_L.

This discussion leads to the following vector of fundamental state variables, $S \equiv \{c_L, z_L, R_L, b^{\delta}, \gamma, g, \varepsilon_R\}$. Most existing DSGE models are estimated under the assumption that *all* of the states are fundamental. However, the perpetual youth model is not restricted to purely fundamental equilibria and for some parameterizations of the model we find that there exists an indeterminate steady-state. For these parameterizations we define an asset that we refer to as equity. Equity issued at date j is a claim to the income stream $\lambda^{t-j}Y_t$ for all $t \geq j$ and its price, P^E is determined by the recursion[8]

$$P^E = \mathbb{E}\left[Q'\left(\lambda P^{E'} + Y'\right)\right].$$

The *price-dividend ratio* is determined by the expression

$$p^E = \mathbb{E}\left[\gamma' Q'\left(\lambda p^{E'} + 1\right)\right].$$

Equity is a redundant asset and in a model with a unique determinate steady-state it would appear as a co-state variable. In contrast, in a model with an indeterminate steady-state, there are stationary equilibria driven purely by sunspots. In these equilibria, there are insufficient initial conditions to uniquely determine all of the variables and there may exist sunspot equilibria in which non-fundamental shocks influence prices and allocations (Azariadis, 1981, Cass and Shell, 1983).

In our empirical work, we explore the properties of sunspot equilibria close to an indeterminate steady-state by choosing p^E to be an additional state variable. In the non-fundamental version of the model we define the augmented state vector $\tilde{S} \equiv \{S, p^E\}$.

There is no unique way to choose an additional state variable although Farmer et al. (2015) show, in the context of a linear model, that if one allows for an arbitrary variance-covariance between the sunspot shock and the fundamental shocks, all choices of the additional state are observationally equivalent. The sunspot moves the economy to a point on the stable manifold of a locally indeterminate steady state. By specifying a variance-covariance structure for *all* the shocks one arrives at an empirically testable model that can be compared with non sunspot theories. In this paper we show that a sunspot-driven equilibrium provides a much better fit to U.S. data than fundamental equilibria of the same model.

[8] λ represents the decay rate of the claim and it is not identified independently of the volatility of the innovation to the sunspot shock. For existence of equilibrium, it must satisfy the inequality $\lambda\bar{\gamma}\bar{Q} < 1$ in a steady state parameterized by $\bar{Q}$. In our estimation we set $\lambda = \pi\lambda_1$ which guarantees that this inequality is satisfied in any equilibrium in which human wealth is well defined.

5.2. *The Co-state Variables of the Model*

The co-state vector T includes z, ψ, Q and c. It also includes the variables R, τ, Π, p^δ and two stationary variables, h_1 and h_2 which represent the net present values of the two components of the income streams in Eq. (1) added up over all living agents and expressed as ratios to GDP. These variables are defined recursively,[9] $h_1 = \alpha(1-\tau) + \pi\lambda_1\mathbb{E}\left[\gamma' Q' h_1'\right]$, and $h_2 = (1-\alpha)(1-\tau) + \pi\lambda_2\mathbb{E}\left[\gamma' Q' h_2'\right]$. For the specification of the state in which all states are fundamental, T also includes the price-dividend ratio p^E.

This discussion leads to the following vectors of co-state variables for the fundamental and non-fundamental versions of the model, $\tilde{T} = \{z, \psi, h_1, h_2, Q, c, R, \tau, \Pi, p^\delta\}$, and $T = \{p^E, \tilde{T}\}$.

6. COMPETITIVE EQUILIBRIUM

In this section we introduce the dynamic equations that link the variables through time and we define the concepts of a *competitive equilibrium*, a *steady-state equilibrium*, and a *balanced-budget steady-state*.

DEFINITION 1—Competitive Equilibrium: *A 'competitive equilibrium' is a stochastic sequence of prices and allocations such that markets clear at every period and allocations solve the households' utility maximization problems at every date and in every state.*

PROPOSITION 2—Characterization of Equilibrium:
Define a vector $X \in \mathcal{X} \subset \mathbb{R}^n_+$, where

$$X \equiv \{c_L, z_L, R_L, b^\delta, \gamma, g, \varepsilon_R, p^E, z, \psi, h_1, h_2, Q, c, R, \tau, \Pi, p^\delta\}$$

[9] See Appendix D.

and functions $F : \mathcal{X}^2 \to \mathbb{R}^n_+$ and $\phi : \mathcal{X}^2 \to \mathbb{R}_+$, where

$$F(X,X') \equiv \begin{bmatrix} c - {c_L}' \\ z - {z_L}' \\ R - {R_L}' \\ \psi c - h_1 - h_2 - \frac{b^\delta(1+\delta p^\delta)}{\Pi\gamma} \\ \tilde{\gamma}' - (1-\rho_\gamma)\log(\bar{\gamma}) - \rho_\gamma\tilde{\gamma} - \varepsilon'_\gamma \\ \tilde{g}' - (1-\rho_g)\log\left(\frac{1}{1-\bar{g}}\right) - \rho_g\tilde{g} - \varepsilon'_g \\ \varepsilon'_R \\ p^E - \gamma' Q'(\lambda p^{E'} + 1) \\ z - {\psi'}^{\frac{(1-\rho)\theta}{1-\rho\theta}} {Q'}^{\frac{\rho\theta}{\rho\theta-1}} \\ \psi - 1 - \pi\beta^{\frac{1}{1-\rho}} z^{\frac{1-\rho\theta}{(1-\rho)\theta}} \\ h_1 - \alpha(1-\tau) - \pi\lambda_1\gamma' Q' h'_1 \\ h_2 - (1-\alpha)(1-\tau) - \pi\lambda_2\gamma' Q' h'_2 \\ Q - \phi(X,X') \\ c + g - 1 \\ R - \frac{Q'}{\Pi'} \\ \tau - \bar{\tau} - \phi_\tau\left[\frac{b^\delta(1+\delta p^\delta)}{\Pi\gamma} - \Phi\right] \\ R - \left[R_L\right]^{\phi_R}\left[\left(\frac{\Pi}{\Pi^*}\right)^{\phi_\pi}\left(\frac{\gamma}{\gamma^*}\right)^{\phi_\gamma}\left(\frac{\Pi^*}{Q^*}\right)\right]^{1-\phi_R} \\ p^\delta - \frac{Q'}{\Pi'}\left(1 + \delta {p^\delta}'\right) \end{bmatrix},$$

and

$$\phi(X,X') \equiv \left(\frac{\pi\beta^{\frac{1}{1-\rho}} c_L}{\gamma\left(c - \psi^{-1}\left[(1-\lambda_1\pi)h_1 + (1-\lambda_2\pi)h_2\right]\right) z_L^{\frac{\theta-1}{(1-\rho)\theta}} \psi^{\frac{1-\theta}{1-\rho\theta}}}\right)^{1-\rho\theta}.$$

A competitive equilibrium is characterized by a bounded stationary stochastic process $\{X_t\}_{t=1}^\infty$ that satisfies the functional equation $\mathbb{E}\left[F(X,X')\right] = 0$, with boundary conditions $\psi_1^{-1}\left(h_{1,1} + h_{2,1} + \frac{b_1^\delta(1+\delta p_1^\delta)}{\Pi_1\gamma_1}\right) = 1 - g_1$, $R_L = R_{L,1}$, $c_L = c_{L,1}$, and $z_L = z_{L,1}$, where $\frac{b_1^\delta(1+\delta p_1^\delta)}{\Pi_1\gamma_1}$, $h_{1,1}$, and $h_{2,1}$ are the debt-GDP ratio and the two components of the *human wealth-GDP ratio in period 1, g_1 is the period 1 government spending-GDP ratio, and ψ_1 is the initial inverse savings propensity.*

A steady-state is a vector $\bar{X} \in \mathcal{X}$ that satisfies the equation

$$F(\bar{X}, \bar{X}) = 0. \tag{11}$$

The proof of Proposition 2 is found in Appendix D.

DEFINITION 2—Steady-State Equilibrium: *A 'steady-state equilibrium', or more compactly a 'steady-state', is a competitive equilibrium in which the variables of the model are non-stochastic and time invariant. A steady-state equilibrium is 'non-trivial' if the steady-state pricing kernel, $\bar{Q}$, is strictly positive. A 'balanced budget steady-state' is a steady-state equilibrium of the model in which the government follows the balanced budget policy $\bar{g} = \bar{\tau}$.*

Proposition 3 characterizes the properties of balanced-budget steady-states.

PROPOSITION 3—Multiplicity of Balanced-Budget Steady-States: *The model has at least two balanced-budget steady-states. In one of these steady-states $\bar{Q}_{gr} = \frac{1}{\bar{\gamma}}$. We refer to this as the golden rule and we index the elements of $\bar{X}$ in the golden-rule steady state with the subscript gr. In the one-commodity model, the golden-rule is unique. In addition to the golden rule, there is at least one other steady-state in which $\bar{b}_{au_i} = 0$. We refer to these steady states as a 'generationally autarkic' or more compactly as 'autarkic' and we index the elements of $\bar{X}$ in the i-th autarkic steady state with the subscript au_i.*

PROOF: Using equations (7) and (8) and exploiting the balanced budget assumption leads to the steady-state expression, $\bar{b}^{\delta}(\bar{Q})\left(1 - \frac{1}{\bar{Q}\bar{\gamma}}\right) = 0$, from which it follows that either $\bar{b}^{\delta}(\bar{Q}) = 0, \quad \text{or} \quad \bar{Q} = \frac{1}{\bar{\gamma}}$. This established Proposition 3. *Q.E.D.*

PROPOSITION 4—Multiplicity of Autarkic Steady-States: *Define the following compound parameters, $\delta_1 \equiv \frac{1}{1-\pi\lambda_1\bar{\gamma}}, \delta_2 \equiv \frac{1}{1-\pi\lambda_2\bar{\gamma}}, \delta_b \equiv \frac{1}{1-\pi\beta^{\frac{1}{1-\rho}}}, \Delta \equiv \delta_1 - \delta_2, \rho_c \equiv -\frac{\log(\pi\beta)}{\log(\lambda_1\bar{\gamma})}$, and the following inequalities*

$$\alpha > 1, \qquad \delta_b > \delta_1 > \delta_2, \qquad \delta_1 - \Delta(1-\alpha) > \delta_b, \qquad \rho < \rho_c < 0. \tag{12}$$

When the parameters satisfy the inequalities in (12), there is a trivial autarkic steady-state and two non-trivial autarkic steady-states. The steady-state pricing kernel in these steady-states are solutions to the equation,

$$\frac{h(Q)}{1-\bar{g}} \equiv \left[\frac{\alpha}{1-\pi\lambda_1\bar{\gamma}Q} + \frac{(1-\alpha)}{1-\pi\lambda_2\bar{\gamma}Q}\right] = \left[\frac{1}{1-\pi\beta^{\frac{1}{1-\rho}}Q^{\frac{\rho}{\rho-1}}}\right] \equiv \frac{\psi(Q)}{1-\bar{\tau}}, \tag{13}$$

where $h(Q) \equiv h_1(Q) + h_2(Q)$ is the aggregate human wealth to GDP ratio. We refer to the values of the non-trivial steady-state pricing kernel in these two steady-states as $\bar{Q}_{au_1}$ and $\bar{Q}_{au_2}$.

The steady state indexed by au_1 is dynamically efficient and the steady state pricing kernel $\bar{Q}_{au_1}$ satisfies the inequality $\bar{Q}_{au_1} < \bar{Q}_{gr} = \frac{1}{\bar{\gamma}}$. The steady state indexed by au_2 is dynamically inefficient and the steady state pricing kernel $\bar{Q}_{au_2}$ satisfies the inequality $\bar{Q}_{au_2} > \bar{Q}_{gr} = \frac{1}{\bar{\gamma}}$.

The parameter ρ is related to the intertemporal elasticity of substitution by the identity $ies \equiv \frac{1}{1-\rho}$, and Proposition 4 implies that, when the parameters of the model satisfy inequalities (12), there exists a critical value, $ies_c = \frac{1}{1-\rho_c}$, such that for all values of $ies < ies_c$ there exist multiple autarkic steady-states. In our empirical work we calibrate the parameters π, λ_1, and $\bar{\gamma}$ and estimate the parameter β. For our parameterization, this critical value is $ies = 0.453$. For an explanation and a proof of Proposition 4 see Appendix E.

7. THE DETERMINACY PROPERTIES OF THE STEADY-STATES

In this section we discuss the concept of local determinacy of equilibrium and we explain the solution and estimation strategy that we use to compare the model with data.

7.1. *The Definition of Local Determinacy*

A steady-state, $\bar{X}$ is said to be locally determinate if, in the absence of shocks, and for initial values of the state variables in the neighborhood of $\bar{X}$, there is a unique value for the co-state variables such that equilibrium sequences $\{X_t\}_{t\geq 0}$ converge to $\bar{X}$. We elaborate on this definition below.

Define the matrices $A_{eq} \equiv F_X|_{eq}$ and $B_{eq} \equiv F_{X'}|_{eq}$, where A_{eq} and B_{eq} represents the Jacobians of the function $F(X, X')$ with respect to the vectors X and X' evaluated at a steady state $eq \in \{gr, au_1, au_2\}$. Consider the following linear approximation of Eq. (11) $A_{eq}\tilde{X} + B_{eq}\tilde{X}' = 0$, where the tilde signifies deviations from the steady state.

Let $\sigma_{eq} \in \mathbb{C}^n$ denote the spectrum of the matrix pencil $(A_{eq} - \sigma_{eq} B_{eq})$ and let m_{eq} denote the number of elements of σ_{eq} inside the unit circle.[10] Let d_{eq} denote the *degree of indeterminacy* of the steady state. It follows from the Blanchard Kahn conditions (Blanchard and Kahn, 1980) that $d_{eq} = m_{eq} - n + n_1$, where n_1 is the number of fundamental state variables and n is the dimension of X.

For a simple version of our model with a balanced budget and monetary and fiscal policies that are *both* active, we computed the spectra at the three steady states for values of $ies \in [0.05, ies_c]$. For all values of ies in this range we found that $d_{au_1} = -1, d_{gr} = 0$ and $d_{au_2} = 1$. These results imply that the efficient autarkic steady state is explosive and would never be reached if monetary and fiscal policy were both active. The golden-rule steady state is locally determinate and, in the vicinity of the golden-rule, there exists a unique equilibrium that is a function only of fundamentals. In contrast, we found that the inefficient autarkic steady-state equilibrium displays one degree of indeterminacy even when both monetary and fiscal policy are active. This is in marked contrast to results from the representative agent model in which equilibrium, under an active monetary policy and an active fiscal policy, does not exist (Leeper and Leith, 2016). The indeterminacy that occurs at the inefficient autarkic steady-state is *real* as opposed to nominal and it leads to the possibility of a volatile pricing kernel, driven by sunspot fluctuations in non-fundamentals.

7.2. *Excess Volatility and the Equity Premium*

The fact that the overlapping generations model has an indeterminate dynamically inefficient steady-state equilibrium was established in Samuelson's seminal (1958) paper. In two-generation one-commodity models, the existence of an indeterminate steady-state equilibrium occurs only if debt is denominated in dollars. In models with three or more generations, that qualification is unnecessary and we have examples of multi-generation models that display indeterminacy of relative prices and real interest rates (Kehoe and Levine,

[10]The $\sigma_i(eq)$ are solutions to the polynomial equation: $\det(A_{eq} - \sigma_{eq} B_{eq}) = 0$.

1983, 1985, Farmer and Zabczyk, 2022). Our paper provides a further example of this phenomenon.

The existence of an indeterminate dynamically inefficient steady-state equilibrium is interesting because it offers the potential to understand three asset market facts that are otherwise difficult to explain. The first fact is that asset prices are far more volatile than can easily be explained by fluctuations in fundamentals (Shiller, 1981, Leroy and Porter, 1981). The second fact is that the return to government debt has been lower than the growth rate of GDP for long periods of time (Blanchard, 2019). And the third fact is that the average rate of return to the stock market has been two to three percentage points higher than the growth rate of GDP in a century of U.S. data (Mehra and Prescott, 1985).

For any risky asset with return R_r', the no-arbitrage condition in the asset markets implies that $\mathbb{E}[R_r'] = \frac{1-\mathrm{Cov}(R_r',Q')}{\mathbb{E}[Q']} > \frac{1}{\mathbb{E}[Q']} \equiv R_s'$, where R_s' is the return on a risk-free bond and the inequality follows if $\mathrm{Cov}(R_r', Q') < 0$. By choosing p^E as a state variable, we ensure that fluctuations in ε_s' cause excess volatility in the pricing kernel, Q', and conditional on a realization of γ', they induce a negative covariance between sunspot fluctuations in the pricing kernel and the return to a risky asset.

8. SOLUTION AND ESTIMATION STRATEGY

We parameterize the model by a finite vector of parameters $\boldsymbol{\vartheta} \in \Theta \subset \mathbb{R}^\ell$ and using the partition, $X \equiv \{S, T\}$, we define the function $G : \mathcal{X}_S^2 \times \mathcal{X}_T^2 \to \mathbb{R}^n$, where $G(S, S', T, T'; \boldsymbol{\vartheta}) \equiv F(X, X')$.

Define a vector shocks $\varepsilon \in \mathcal{E} \subset \mathbb{R}_+^k$. A *solution* to the model is pair of functions $f : \mathcal{X}_S \times \mathcal{E} \to \mathcal{X}_S$ and $g : \mathcal{X}_S \to \mathcal{X}_T$, where $S' = f(S, \varepsilon')$ and $T = g(S)$, where the functions f and g satisfy the functional equation, $\mathbb{E}\left[G\left(S, f[S,\varepsilon'], g[S], g[f(S,\varepsilon')]; \boldsymbol{\vartheta}\right)\right] \equiv 0$. For the fundamental version of the model we choose $S \equiv \{c_L, z_L, R_L, b^\delta, \gamma, g, \bar{\varepsilon}_R\}$, and we define three fundamental shocks, ε_γ, ε_g, and ε_R. In this representation of the model, $k = 3$ and we specify AR(1) processes for $\tilde{\gamma} = \log(\gamma)$ and $\tilde{g} = \log\left(\frac{1}{1-g}\right)$ and a zero mean i.i.d. process for ε_R,

$$\tilde{\gamma}' = (1-\rho_\gamma)\log(\bar{\gamma}) + \rho_\gamma \tilde{\gamma} + {\varepsilon_\gamma}',$$

$$\tilde{g}' = (1-\rho_g)\log\left(\frac{1}{1-\bar{g}}\right) + \rho_g \tilde{g} + {\varepsilon_g}',$$

$$\varepsilon'_R \sim i.i.d.(0, \sigma_R^2).$$

In our estimation strategy we further assume that the elements of ε are uncorrelated and we parameterize their standard deviations by σ_γ, σ_g, and σ_R. For the non-fundamental version of the model we choose $\tilde{S} = \{S, p^E\}$, and we add a non-fundamental shock ε_s. In this specification, $k = 4$, and the states γ, g, ε_R, and p^E follow the processes

$$\begin{aligned}
\tilde{\gamma}' &= (1-\rho_\gamma)\log(\bar{\gamma}) + \rho_\gamma \tilde{\gamma} + \varepsilon_\gamma{}', \\
\tilde{g}' &= (1-\rho_g)\log\left(\frac{1}{1-\bar{g}}\right) + \rho_g \tilde{g} + \varepsilon_g{}', \\
\varepsilon'_R &\sim i.i.d.(0, \sigma_R^2), \\
p^{E\prime} &= \mathbb{E}\left[p^{E\prime}\right] \exp(\varepsilon_s{}').
\end{aligned}$$

In the non-fundamental model there is an additional i.i.d. shock $\varepsilon_s{}'$ with mean 0 and standard deviation σ_s.

9. DATA SOURCES AND MOMENT MATCHING

This section describes data sources and partitions the parameter space into a subset of parameters that we calibrated, or estimated by OLS, and a subset that we estimated by simulated method-of-moments.

For the risky asset, we used data on the value-weighted market portfolio from the Center for Research in Security Prices (CRSP). The price-dividend ratio was computed as the price of the value-weighted market portfolio divided by a 12-month moving sum of daily dividends (as in Welch and Goyal (2008)). For the risk-free 1-period asset, we used the effective federal funds rate from FRED.[11] For inflation, we used Consumer Price Index (CPI) inflation. For the government debt-to-GDP ratio we used total public debt as a percentage of GDP from FRED. All data are quarterly and the sample period is 1990Q1-2019Q4.

The model has 23 parameters which we collect into the vector $\boldsymbol{\vartheta} \in \Theta$. We calibrated 11 of these parameters to match various observable features of the data and we refer to

[11]Federal Reserve Bank of St Louis Economic Database.

the subset of calibrated parameters as $\boldsymbol{\vartheta}_C$. The remaining 12 parameters, collected into the vector $\boldsymbol{\vartheta}_E$, were estimated by simulated method-of-moments, $\boldsymbol{\vartheta} \equiv \left[\boldsymbol{\vartheta}_C', \boldsymbol{\vartheta}_E'\right]'$.

9.1. *Parameters Calibrated or Estimated by Least-Squares*

Table I displays the values of $\boldsymbol{\vartheta}_C$. We chose the survival probability π to match an average life expectancy of 50 years. Agents are assumed to begin life as working-age adults, so if an agent enters the economy at age 20, they would live on average until they are 70.

We chose the parameters $\lambda_1, \lambda_2,$ and α to match the U.S. income profile as shown in Figure 3. These parameters are taken from Gârleanu and Panageas (2015) who use least-squares to fit a doubly exponential process to the age profile of U.S. cohort data.

We chose AR(1) processes for output growth and government spending from univariate first-order auto-regressions of the logs of real GDP growth and a transformation of the government spending-GDP ratio in U.S. data. The estimated parameters for output growth imply an annualized real GDP growth rate of 2.43% and an annualized unconditional standard deviation of 1.14%. The estimated parameters for government spending imply a mean real government spending-GDP ratio of 20.85% and unconditional standard deviation of 1.41%.

Finally, we chosen δ to match the average maturity of government debt in our data set which we estimate to be 5 years.

9.2. *Parameters Estimated by Method of Moments*

We collect the estimated parameters into a vector $\boldsymbol{\vartheta}_E = [\beta, \rho, \theta, \overline{\tau}, \phi_\tau, \rho_R, \overline{\pi}, \phi_\pi, \phi_\gamma, \kappa, \sigma_R, \sigma_s]^\top$ The parameter β is the discount factor of the household. The parameters ρ and θ are the functions of the intertemporal elasticity of substitution and the coefficient of relative risk aversion defined in Eq. (5); these are the only three estimated private-sector parameters. $\bar{\tau}$ and ϕ_τ parameterize the fiscal rule, ϕ_R, ϕ_π, and ϕ_γ parameterize the monetary rule and σ_R and σ_s are standard deviations of the monetary shock and the sunspot shock.

In order to match the equity premium and the Sharpe ratio in U.S. data we introduce the parameter κ which represents the fraction of a firm financed by debt. This parameter captures leverage and it allows us to increase our estimate of the equity premium and simultaneously increase the standard deviation of the return on the risky asset. The risk-return

Parameter	Calibrated Value
Survival Probability	
π	0.995
Endowment Profile	
λ_1	0.987
λ_2	0.985
α	8.522
Output Growth	
$100\log(\bar{\gamma})$	0.608
ρ_γ	0.402
$100\sigma_\gamma$	0.529
Government Spending	
$\bar{g}$	0.209
ρ_g	0.991
$100\sigma_g$	0.230
Government Debt	
δ	0.950

Note: We chose the survival probability to match an average working-age life-span of 50 years. We chose the endowment profile parameters to match estimates in Gârleanu and Panageas (2015). We estimate the output growth and government spending parameters by OLS using data from FRED. Finally, we chose the decay rate of government bonds to match an average maturity of 5 years.

TABLE I: Calibrated Parameters

trade-off to a leveraged asset is a direct application of the Modigliani-Miller theorem in a model with dynamically compete markets.

Let $R_{r,\ell}$, $R_{r,u}$ and R_s denote the gross real return on a levered risky asset, the gross real return on an unlevered risky asset and the gross real return on a riskless bond. It follows from the assumption of complete asset markets that $R_{r,\ell} - R_s = \frac{1}{1-\kappa}\left(R_{r,u} - R_s\right)$. When we report statistics related to the risky return we use $R_{r,\ell}$.

For some parameterizations, our model has three steady-state equilibria and for others it has only two. Our estimation strategy allows for both possibilities. First, we chose the state vector to be S, and we searched over the parameter space Θ_E for the minimum distance between the model and data moments. Our estimation procedure computes the steady states associated with any given vector and it rejects a steady state if it does not satisfy the Blanchard-Kahn conditions. This implies that, *for a given definition of the state* our equilibrium is determinate by construction.

We did not impose any assumptions, in advance, about whether fiscal and/or monetary policy are active or passive. Instead, we allowed the stance of policy to be chosen to achieve the best fit. In a model with a unique steady-state, our approach would require that either fiscal policy is active and monetary policy is passive, or monetary policy is active and fiscal policy is passive. In our model, in contrast, there are always at least two steady states and, for low values of the ies, there are three. This fact allows us to construct a determinate equilibria at any one of the three steady states by picking appropriate combinations of policy activism.

The novel aspect of our work, is that we are not restricted to the choice of S as the state vector. In our empirical work we repeated the estimation exercise using p^E as an additional state. We refer to the augmented state vector as $\tilde{S} = \{S, p^E\}$. The augmented model has one additional state variable and one additional non-fundamental shock that we assumed to be uncorrelated with the fundamental shocks. We parameterized the volatility of the non-fundamental shock by σ_s. This discussion implies that the standard model has 11 estimated parameters while the augmented model has 12. We refer to the standard and augmented models as models S and $\tilde{S}$ respectively.

We searched over all determinate equilibria under both definitions of the state and we compared the best fit for the two alternative specifications, where by best fit, we mean the model that most closely matches the following fifteen macro and financial moments:

- $\mu_{r_r^n}$: mean of the nominal risky rate
- $\sigma^2_{r_r^n}$: variance of the nominal risky rate
- $\mu_{r_f^n}$: mean of the nominal risk-free rate
- $\sigma^2_{r_f^n}$: variance of the nominal risk-free rate
- $\rho_{r_f^n}$: auto-correlation of the nominal risk-free rate
- μ_{pd}: mean of the log price-dividend ratio

- σ^2_{pd}: variance of the log price-dividend ratio
- ρ_{pd}: auto-correlation of the log price-dividend ratio
- μ_π: mean of inflation
- σ^2_π: variance of inflation
- μ_b: mean of the debt-to-GDP ratio
- σ^2_b: variance of the debt-to-GDP ratio
- $\sigma_{r^n_f,\pi}$: covariance between the nominal risk-free rate and inflation
- $\sigma_{r^n_f,\gamma}$: covariance between the nominal risk-free rate and real GDP growth
- $\sigma_{\pi,\gamma}$: covariance between inflation and real GDP growth

We estimated $\boldsymbol{\vartheta}_E$ using two-step simulated method of moments. For a given parameter vector, we solved the model using a fourth-order perturbation approximation with code from Levintal (2017). We simulated 5,000 periods of burn-in and we kept the subsequent 100,000 draws to compute moments.

9.3. *Model Fit*

We found that the data favor model $\tilde{S}$ in which the sunspot shock plays an important role. Table II compares the fit of models $\tilde{S}$ and S to the targeted moments. We report estimated parameter values and 95% bootstrapped confidence intervals for Model $\tilde{S}$ in Table III. We begin by discussing the results for Model $\tilde{S}$.

With a couple of exceptions, the moments of Model $\tilde{S}$ are close to their data analogues with a typical percentage difference of less than 10%. The two exceptions to the close fit are the mean and persistence of the price-dividend ratio. The mean of the log price-dividend ratio is 3.11 compared to 3.92 and its persistence, measured by ρ_{pd}, is estimated to be 0.99 compared to 0.75 in data.[12]

Model S exhibits major shortcomings relative to Model $\tilde{S}$ when it comes to fitting the targeted moments. The main issue is that Model S is incapable of producing enough volatility in interest rates and asset prices relative to the data. Model S produces a high equity premium of 7.45% using financial leverage but only produces a risky rate volatility of 3.03%

[12] We suspect that this aspect of our model could be improved by exploring alternative specifications for the additional state variable that allow the price-dividend ratio to respond to lagged and contemporaneous values of shocks to other variables in the model. For example, we have not allowed for the possibility that the stock market is too volatile because it over-reacts to fundamentals. Instead, we modeled *all* excess volatility as exogenous.

Moments	Data	Model $\tilde{S}$	Model S
Risky Rate			
$\mu_{r_r^n}$	9.23	9.34	9.86
$\sigma_{r_r^n}$	16.21	16.24	3.03
Risk-free Rate			
$\mu_{r_f^n}$	2.66	2.82	2.40
$\sigma_{r_f^n}$	1.10	1.02	0.73
$\rho_{r_f^n}$	0.85	0.87	0.74
Log Price-Dividend Ratio			
μ_{pd}	3.92	3.11	3.03
σ_{pd}	0.26	0.29	0.03
ρ_{pd}	0.75	0.99	0.94
Inflation			
μ_{π}	2.38	2.40	2.37
σ_{π}	1.18	1.26	1.21
Government Debt			
μ_b	74.02	75.28	73.33
σ_b	18.90	18.23	19.47
Correlations			
$\rho_{r_f^n,\gamma}$	0.20	0.22	0.33
$\rho_{r_f^n,\pi}$	0.32	0.42	0.55
$\rho_{\gamma,\pi}$	0.25	0.22	0.23

Note: We annualize all moments except correlations. Specifically, we multiply means by 4 and standard deviations are multiplied by 2, and we report both quantities in percentage points. We raise auto-correlations to the power 4. We compute moments from the model as the average of 105,000 simulated draws where the first 5,000 draws are discarded as burn-in.

TABLE II: Targeted Moments Fit

compared to 16.21% in the data. This leads to an annualized Sharpe ratio 2.53 which is significantly larger than the annualized Sharpe ratio of 0.41 in the data.

Similarly, the log price-dividend ratio is too low on average with a mean of 3.03 compared to 3.92 in the data and exhibits significantly less volatility, 0.03 compared to 0.26. Model S also produces a much stronger correlation between the nominal risk-free rate and inflation than in the data, 0.55 compared to 0.32.

9.4. *Parameter Estimates*

Our point estimate of the intertemporal elasticity of substitution, ies, is equal to 0.41. This estimate is less than ies_c implying a parameterization with three steady states. We found that the data favors an approximation around the dynamically inefficient steady-state, allowing the model to capture the fact that, in the U.S. data, the safe interest rate has been lower than the growth rate in much of the post-war period.

Our estimate of the ies is consistent with empirical studies using micro-level data to estimate Euler equations. A consistent finding of that literature is that the ies of poorer households tends to be small and close to zero, while the ies of richer households is substantially larger although often less than 1.[13]

Our point estimate of the coefficient of relative risk aversion, rra, is 17.37. In a model with constant-relative-risk-aversion (CRRA) preferences, a value for rra of 17.37 would imply a value for the ies of 0.06 which is well outside the 5% confidence bound of 0.37 for our estimate of that parameter. Similarly, the estimated value of the ies would, under CRRA preferences, imply a coefficient of relative risk-aversion of 2.48. This, once again, is below the 5% confidence bound of our estimate of this parameter which is equal to 14.87. We conclude that our estimates allow us to reject the hypothesis of Von-Neumann Morgenstern CRRA preferences in favor of Epstein-Zin.

Next, we turn to the fiscal rule parameters. We estimated a steady state tax-to-GDP ratio of 17.55% which implies a steady state deficit-to-GDP ratio of 3.33%. The debt stabilization parameter $\phi_\tau = 3.63 \times 10^{-6}$ implies a weak fiscal response of taxes to deviations of debt from its steady state. This response accounts implies a nearly constant tax rate as a fraction of GDP and is too small to act as an independent stabilization mechanism. We conclude that our estimates imply that fiscal policy during our sample period was active.

[13] See e.g. Zeldes (1989), Lawrance (1991), and Jorgenson (2002).

Parameter	**Estimate**	**95% Bootstrap CI**
Preferences		
$\beta\pi$	0.992	[0.991, 0.993]
ies	0.404	[0.370, 0.466]
rra	17.373	[14.874, 21.232]
Fiscal Rule		
$\overline{\tau}$	0.176	[0.156, 0.189]
ϕ_τ	3.63×10^{-6}	[3.24×10^{-6}, 3.97×10^{-6}]
Taylor Rule		
ρ_R	0.914	[0.884, 0.967]
$100 \log(\Pi^*)$	-0.398	[-0.447, -0.363]
ϕ_π	2.137	[1.765, 2.390]
ϕ_γ	0.939	[0.817, 1.143]
Leverage		
κ	0.812	[0.729, 0.860]
Exogenous Shocks		
$100\sigma_R$	7.69×10^{-4}	[6.80×10^{-4}, 8.46×10^{-4}]
$100\sigma_s$	1.443	[1.029, 1.560]

Note: We estimate parameters using the simulated method of moments (SMM). For each parameter value, we solve the model using fourth-order perturbation around all steady states for which a solution exists. We use the solution associated with the lowest value of the objective function. We simulated the model using a common set of random numbers for 105,000 draws. We discard the first 5,000 draws as burn-in and use the subsequent 100,000 to compute moments. We report bootstrapped 95% confidence intervals in brackets and account for data moment variability by using a block bootstrap with optimal block length chosen according to Politis and White (2004).

TABLE III: Estimated Parameters

For the monetary policy rule, we estimated a response coefficient to inflation of $\phi_\pi = 2.14$ and a response coefficient to real GDP growth of $\phi_\gamma = 0.94$. These estimates imply that monetary policy was active and are within the range estimated in previous literature. The 95% confidence intervals for these parameters significantly overlap with confidence intervals reported in other estimated DSGE models. For example, Gust et al. (2017), report point estimates of $\phi_\pi = 1.67$ and $\phi_\gamma = 0.73$ and corresponding 95% percent credible sets of $[1.21, 2.14]$ and $[0.39, 1.07]$, albeit in a richer model which includes production and a zero lower bound on nominal interest rates.

We estimated the leverage ratio, κ, to be 0.81 which implies a debt-to-equity ratio of approximately 4. This is higher than the value of 2 used in Bansal and Yaron (2004), like the monetary policy parameters ϕ_π, and ϕ_γ, it is relatively imprecisely estimated with a 95% confidence interval of $[2.69, 6.15]$.

In our preferred specification the state is $\tilde{S}$ and there are four shocks, $\varepsilon_\gamma, \varepsilon_g, \varepsilon_R$ and ε_s. The standard deviations of ε_γ and ε_g were recovered from least-squares regressions of univariate AR processes and our point estimates are $\sigma_\gamma = 5.3 \times 10^{-3}$ and $\sigma_g = 2.3 \times 10^{-3}$. Our estimate of the standard deviation of ε_R is $\sigma_R = 0.008 \times 10^{-3}$. These are all small numbers relative to the main driver of fluctuations in our model, the sunspot shock ε_s, which has an estimated standard deviation of $\sigma_s = 14 \times 10^{-3}$, three times larger than the growth shock. We conclude from these estimates that the hump-shaped income profile, in conjunction with sunspot shocks and a low ies are critical features of our explanation of the data that are inconsistent with a steady-state driven by fundamentals.

10. EXPLAINING TWO SETS OF FACTS

In the introduction to this paper, we drew attention to two sets of facts. The first set of facts concerned comparisons of the returns to safe and risky assets with GDP growth and the second set of facts linked the asset price data with generational inequality.

10.1. *Replicating the Facts*

In the top panel of Figure 4 we reproduce Figure 1 and in the bottom panel we graph a single simulation of our model using the estimated values of the parameters from the US data. This figure illustrates, visually, the ability of our perpetual youth model to replicate the three features of the asset pricing and growth data that we drew attention to in the introduction. Our model is able to replicate the fact that the safe rate of interest is lower than the GDP growth rate whilst the risky rate is more volatile, consistently above the safe rate and higher than the GDP growth rate.

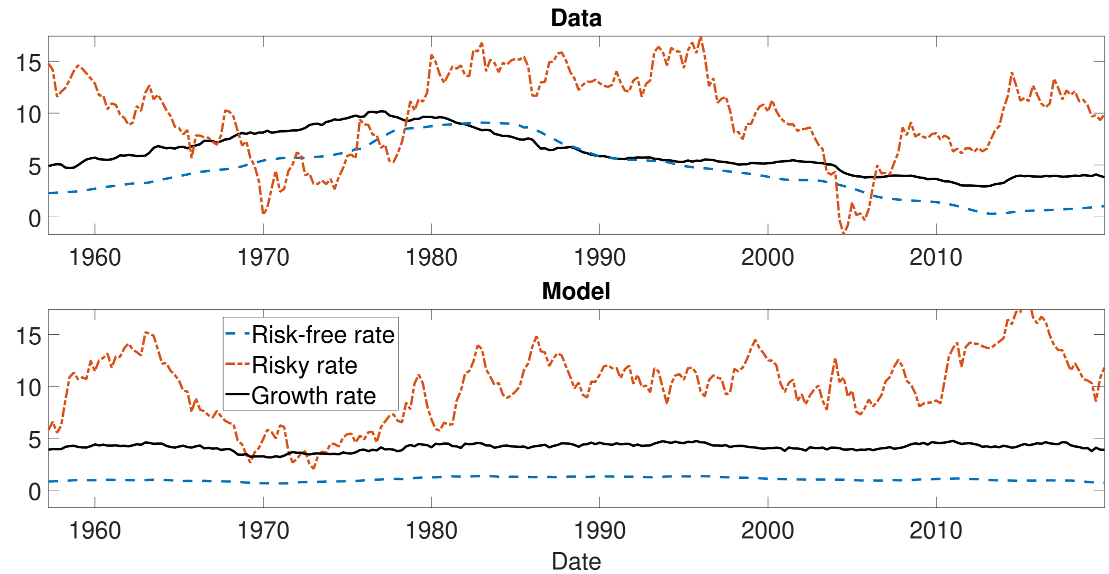

Figure 4.: Asset Returns and GDP Growth: Data and Model

Note: This figure plots ten-year moving averages of nominal output growth, the risk-free return, and the risky return in both U.S. data and simulated data from our model. The top panel replicates Figure 1 while the bottom panel plots one simulated draw of our model with the sample size as the U.S. data.

Next we turn to the implications of our model for the temporal structure of wealth and consumption inequality across generations. Although we assume the existence of dynamically complete markets over aggregate uncertainty, the generational structure of our model does not allow agents to insure over the state of the world they are born into. When this assumption is combined with large persistent fluctuations in non-fundamental uncertainty it implies that cohorts born at different points in time have vastly different lifetime earnings prospects.

The top panel of Figure 5 reproduces Figure 2 and the lower panel plots data for the price-dividend ratio against a measure of human wealth by cohort for our artificial economy, relative to the cohort in the initial period. In the model, state dependent consumption depends only on date of birth. Once a cohort has been born, its entire lifecycle consumption profile, state by state, is completely determined. This figure illustrates that in the model, as in the data, life-cycle consumption prospects are closely correlated with the state of the asset markets at the date that the agent enters adulthood.

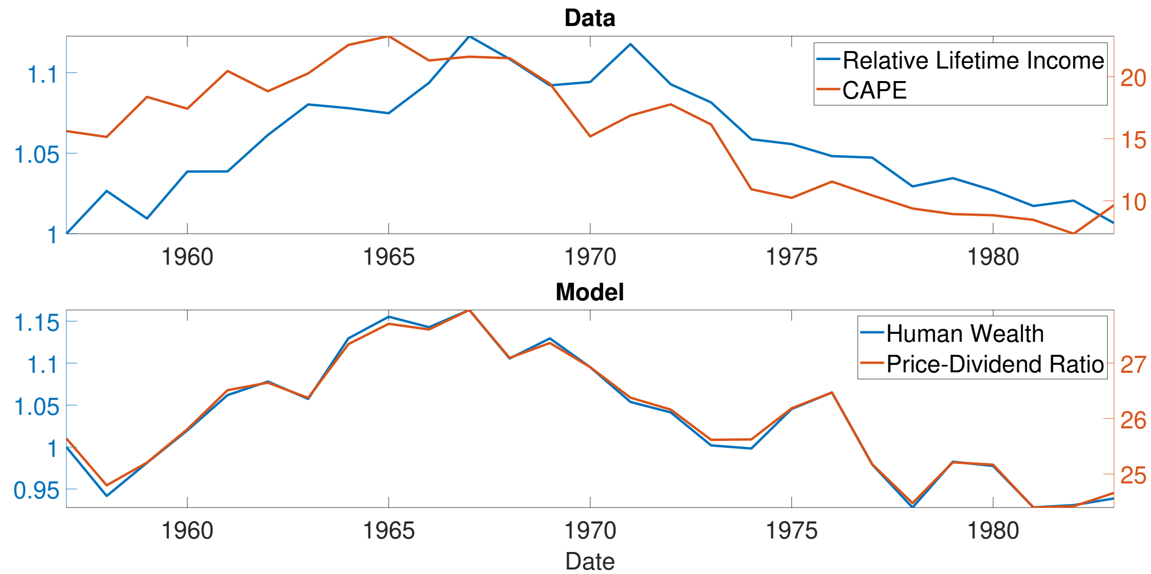

Figure 5.: Asset Prices and Generational Inequality: Data and Model

Note: The top panel of this figure reproduces Figure 2, where the blue line on the left axis is relative median lifetime income and the red line on the right axis is the CAPE. The bottom panel plots annual averages of aggregate human wealth relative to the first period in blue on the left axis and the price-dividend ratio in red on the right axis from one simulated draw of our model with the same sample size as the U.S. data.

To give a measure of how likely is any particular draw, Figure 6 plots 5th and 95th percentile bounds, these are the dashed lines, for any given path of relative human wealth. The solid lines on the same figure reproduce the data and the single simulation of one draw from Figure 5. The figure illustrates that, given our estimates of the magnitude of sunspot uncertainty from the data, a path for relative consumption inequality in which some generations have lifetime median income that is 12% higher than other is not particularly unusual and birth cohorts separated by twenty-five years could easily find themselves better or worse off than earlier generations by as much as 30%.

10.2. *Discussion of the Mechanism*

What are the key features of our approach that allow us to reconcile our two sets of facts?

First, the interaction of a low intertemporal elasticity of substitution with a hump-shaped income profile creates the possibility for multiple autarkic steady-state equilibria in addi-

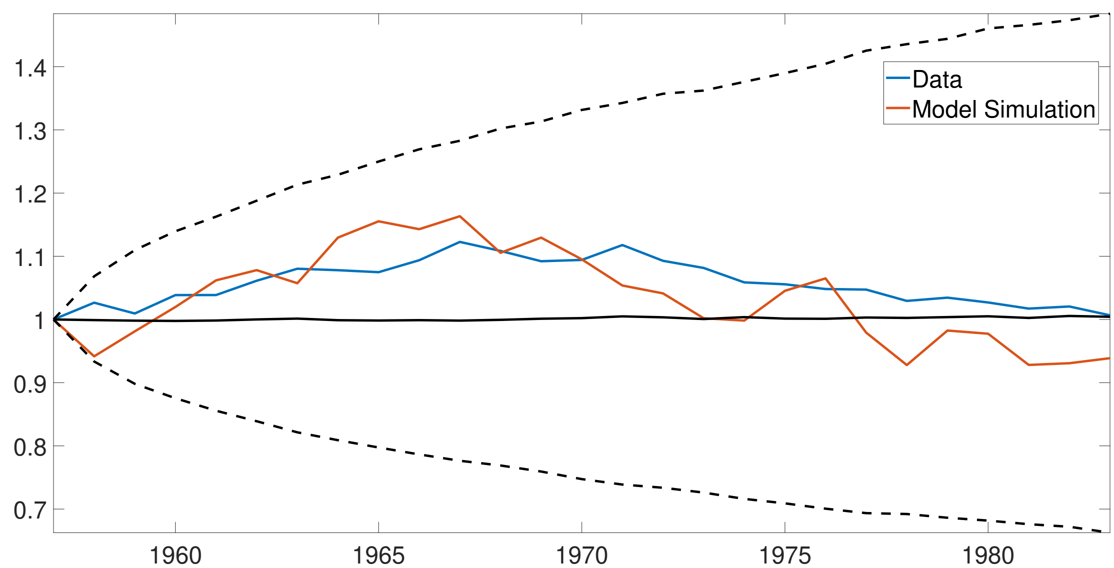

Figure 6.: Distribution of Generational Inequality

Note: This figure plots the distribution of relative human wealth paths from our estimated model. We simulate 100,000 paths of aggregate human wealth from our model of the same sample size as the U.S. data. For each path, we normalize the first period's value to be one. The red line plots one particular sample path realization which mimics the data on U.S. median lifetime income. The dashed black lines are the 5th and 9th percentiles of the simulated distribution of relative human wealth paths.

tion to the golden rule. This is in contrast to most existing calibrated OLG models in which there is a single autarkic steady state equilibrium.

Second, our generalization of preferences to the Epstein-Zin class is important to our exercise because it allows us to fit the coefficient of relative risk aversion to the equity premium while freeing the value of the intertemporal elasticity of substitution to be low enough for the existence of three steady-states. We find that to understand asset market data, we need very large sunspot shocks which are uninsurable for newborns. The magnitude of this inequality displays significant highly persistent time series variation even though the primary source of aggregate shocks is an i.i.d. sunspot.

We find, in our model, that the relative wealth of different cohorts is significantly impacted by whether they are born into a state of the world in which the price earnings ratio is high or low. This feature of our model captures the idea that if a particular cohort enters the job market during a severe recession, it has a strong negative impact on their lifetime earnings and consumption.

11. CONCLUSION

We have constructed and estimated a perpetual youth model of an endowment economy with dynamically complete markets and aggregate shocks. Our work makes three principal contributions to the literature.

First, our theoretical work presents the first discrete-time solution of the problem of a long-lived agent with Epstein-Zin preferences as a function of the moments of the endowment profile and of current and future prices. Previous macro models that use Epstein-Zin preferences have exploited the representative agent assumption to simplify the solution. Our contribution to this literature will permit researchers to construct more general models with multiple types of agents and can potentially be generalized to allow for multiple commodities.

Second, we have proved the existence of multiple autarkic steady-state equilibria in the perpetual youth model when agents have a hump-shaped endowment profile and when the intertemporal elasticity of substitution is less than a critical value that depends on the income profile and the preference and aggregate endowment parameters. We established that one of the autarkic steady-states is dynamically inefficient and we demonstrated that this fact permits the construction of equilibria that are driven principally by non-fundamental shocks to beliefs. By exploiting sunspots and indeterminacy, we are able to explain three asset market puzzles: the low safe rate of interest, excess volatility of asset prices and a large equity premium.

Third, we compared estimated versions of our model with and without sunspot equilibria and we showed that the indeterminate equilibrium provides a significantly better fit to U.S. data from 1990Q1-2019Q4. Although our model can explain how fiscal and monetary policy influence generational inequality, it cannot explain feedback effects from fiscal and monetary policy to real GDP since we assume that all GDP movements are generated by an exogenous stochastic process.[14]

[14] In ongoing research, we are generalizing these results to a production economy with the goal of comparing alternative mechanisms of policy transmission from nominal to real variables.

APPENDIX A: PROOF OF PROPOSITION 1

PROBLEM 2:

$$v^j = \max_{W^{j\prime}} \left[(C^j)^\rho + \beta\pi(m^j)^\rho\right]^{\frac{1}{\rho}}, \tag{14}$$

$$m^j = \left[\mathbb{E}(v^{j\prime})^{\rho\theta}\right]^{\frac{1}{\rho\theta}}, \tag{15}$$

$$\pi\mathbb{E}\left[Q' W^{j\prime}\right] = W^j - C^j, \tag{16}$$

$$W^j = H^j. \tag{17}$$

We seek to prove that the value function v^j and the policy function C^j that solve Problem 1 are given by the expressions

$$C^j = \psi^{-1} W^j \quad \text{and} \quad v^j = \psi^{\frac{1-\rho}{\rho}} W^j, \tag{18}$$

where ψ is the inverse propensity to consume out of wealth and where ψ satisfies the recursion, $\psi = 1 + \pi\beta^{\frac{1}{1-\rho}} \left(\mathbb{E}\left[\psi'^{\frac{(1-\rho)\theta}{(1-\rho\theta)}} Q'^{\frac{\rho\theta}{\rho\theta-1}}\right]\right)^{\frac{1-\rho\theta}{(1-\rho)\theta}}$. The proof proceeds in five steps.

STEP 1 We show that the wealth of a person with the income share defined in Eq. (1) evolves according to Eq. (16).

STEP 2 We show that our conjectured solution obeys the envelope condition.

STEP 3 We show that the Euler equation implies the following two lemmata

LEMMA 1: *In the optimal program*

$$m^j = \beta^{\frac{1}{1-\rho}} \left(\mathbb{E}\left[\psi'^{\frac{(1-\rho)\theta}{(1-\rho\theta)}} Q'^{\frac{\rho\theta}{\rho\theta-1}}\right]\right)^{\frac{1-\rho\theta}{(1-\rho)\rho\theta}} C^j. \tag{19}$$

Lemma 1 is proved in Appendix B.

LEMMA 2: *In the optimal program*

$$C^{j\prime} = C^j \beta^{\frac{1}{1-\rho}} \left(Q'^{\frac{1}{\rho\theta-1}} \psi'^{\frac{\theta-1}{1-\rho\theta}}\right) \left(\mathbb{E}\left[\psi'^{\frac{(1-\rho)\theta}{(1-\rho\theta)}} Q'^{\frac{\rho\theta}{\rho\theta-1}}\right]\right)^{\frac{1-\theta}{(1-\rho)\theta}}. \tag{20}$$

Lemma 2 is proved in Appendix C.

STEP 4 Using Lemma 1 we show that if $C^j = \psi^{-1} W^j$, and if ψ satisfies the recursion $\psi = 1 + \pi\beta^{\frac{1}{1-\rho}} \left(\mathbb{E}\left[\psi'^{\frac{(1-\rho)\theta}{(1-\rho\theta)}} Q'^{\frac{\rho\theta}{\rho\theta-1}} \right] \right)^{\frac{1-\rho\theta}{(1-\rho)\theta}}$, that the budget constraint, Eq. (16), holds at consecutive dates.

STEP 5 Using Lemma 2 we show that the value function has the functional form given by the equality $v^j = \psi^{\frac{1-\rho}{\rho}} W^j$.

PROOF OF PROPOSITION 1: We prove each step in turn.

STEP 1 Define, I^j; the endowment at date t of a member of cohort j conditional on surviving to date t, as I^j, $I^j = \frac{1}{1-\pi}\left(\kappa_1 \lambda_1^{t-j} + \kappa_2 \lambda_2^{t-j}\right)(1-\tau)Y$, and define the human wealth of cohort j by the recursion

$$H^j = I^j + \pi \mathbb{E}\left[Q' H^{j\prime} \right], \tag{21}$$

where H^j, I^j and Q are functions of the state S. Define $W^j = A^j + H^j$, where A^j is the value of Arrow securities owned by a member of cohort j that have positive value in the state S. It follows from the budget constraint of a member of cohort j that

$$\pi \mathbb{E}\left[Q' A^{j\prime} \right] = A^j + I^j - C^j. \tag{22}$$

Combining equations (21) – (22) gives the wealth evolution equation,

$$\pi \mathbb{E}\left[Q' W^{j\prime} \right] = W^j - C^j. \tag{23}$$

This establishes STEP 1.

STEP 2 The envelope condition is that $\frac{\partial v^j}{\partial W^j} = \frac{\partial v^j}{\partial C^j}\frac{\partial C^j}{\partial W^j}$. Using Equations (14) and (18) $\frac{\partial v^j}{\partial W^j} = \psi^{\frac{1-\rho}{\rho}} = \left(\frac{v^j}{C^j}\right)^{\frac{1-\rho}{\rho}} \psi^{-1} = \frac{\partial v^j}{\partial C^j}\frac{\partial C^j}{\partial W^j}$. This establishes STEP 2. Appendices B and C establish STEP 3.

STEP 4 Use Eq. (18) to replace W^j and $W^{j\prime}$ with ψC^j and $\psi' W_j{}'$ in Eq. (23),

$$\pi \mathbb{E}\left[Q' \psi' C^{j\prime} \right] = \psi C^j - C^j. \tag{24}$$

Use Lemma 2 to replace $C^{j\prime}$

$$\pi\mathbb{E}\left[Q'\psi'\left\{C^j\beta^{\frac{1}{1-\rho}}\left(Q'^{\frac{1}{\rho\theta-1}}\psi'^{\frac{\theta-1}{1-\rho\theta}}\right)\left(\mathbb{E}\left[\psi'^{\frac{(1-\rho)\theta}{(1-\rho\theta)}}Q'^{\frac{\rho\theta}{\rho\theta-1}}\right]\right)^{\frac{1-\theta}{(1-\rho)\theta}}\right\}\right]=\psi C^j-C^j. \tag{25}$$

Cancel terms in C^j and rearrange and consolidate terms to give,

$$\psi=1+\pi\beta^{\frac{1}{1-\rho}}\left(\mathbb{E}\left[\psi'^{\frac{(1-\rho)\theta}{(1-\rho\theta)}}Q'^{\frac{\rho\theta}{\rho\theta-1}}\right]\right)^{\frac{1-\rho\theta}{(1-\rho)\theta}}. \tag{26}$$

This establishes STEP 4.

STEP 5 Define the function u^j by the recursion,

$$u^j=\left[(C^j)^\rho+\beta\pi(m^j)^\rho\right]^{\frac{1}{\rho}}, \tag{27}$$

$$m^j=\left[\mathbb{E}(u^{j\prime})^{\rho\theta}\right]^{\frac{1}{\rho\theta}}. \tag{28}$$

u^j is the utility attached to an arbitrary stochastic sequence $\{C_t^j\}_{t\geq j}$. Use Lemma 1 to replace m^j in Eq. (27), and rearrange terms to give

$$u^j=C^j\left[1+\pi\beta^{\frac{1}{1-\rho}}\left(\mathbb{E}\left[\psi'^{\frac{(1-\rho)\theta}{(1-\rho\theta)}}Q'^{\frac{\rho\theta}{\rho\theta-1}}\right]\right)^{\frac{1-\rho\theta}{(1-\rho)\theta}}\right]^{\frac{1}{\rho}}, \tag{29}$$

Using Eq. (26) and the conjecture $C^j=\psi^{-1}W^j$, it follows that the expression for the optimal value, $v^j(W^j)$, is given by Eq. (30)

$$v^j(W^j)=C^j\psi^{\frac{1}{\rho}}=\psi^{\frac{1-\rho}{\rho}}W^j. \tag{30}$$

This establishes STEP 5.

Q.E.D.

APPENDIX B: PROOF OF LEMMA 1

Differentiating the value function, Eq. (14) w.r.t. ${W^j}'$ leads to the expression

$$\frac{\partial v^j}{\partial {W^j}'} = \frac{\partial v^j}{\partial C^j}\frac{\partial C^j}{\partial {W^j}'} + \frac{\partial v^j}{\partial m^j}\frac{\partial m^j}{\partial {v^j}'}\frac{\partial {v^j}'}{\partial {W^j}'} = 0, \tag{31}$$

where the partial derivatives of v^j and m^j w.r.t. ${W^j}'$ are taken using the functions defined by equations (15), (16) and the conjecture, Eq. (18). These expressions are, $\frac{\partial v^j}{\partial C^j} = \left(\frac{v^j}{C^j}\right)^{1-\rho}$, $\frac{\partial C^j}{\partial {W^j}'} = -\pi\chi' Q'$ $\frac{\partial v^j}{\partial {m^j}'} = \beta\pi\left(\frac{v^j}{m^j}\right)^{1-\rho}$, $\frac{\partial m^j}{\partial {v^j}'} = \chi'\left(\frac{m^j}{{v^j}'}\right)^{1-\rho\theta}$, $\frac{\partial {v^j}'}{\partial {W^j}'} = {\psi'}^{\frac{1-\rho}{\rho}}$, where χ' is the conditional probability that state S' occurs. Substituting these expressions into Eq. (31) canceling terms and rearranging terms gives,

$$Q' = \beta {C^j}^{1-\rho}\left(m^j\right)^{\rho(1-\theta)}\left({v^j}'\right)^{\rho\theta-1}{\psi'}^{\frac{1-\rho}{\rho}}. \tag{32}$$

Take the term in ψ' to the left-hand-side, raise the equation to the power $\frac{\rho\theta}{\rho\theta-1}$ and take date t conditional expectations of both sides,

$$\mathbb{E}\left[{\psi'}^{\frac{\theta(1-\rho)}{1-\rho\theta}}{Q'}^{\frac{\rho\theta}{\rho\theta-1}}\right] = \beta^{\frac{\rho\theta}{\rho\theta-1}}{C^j}^{\frac{(1-\rho)\rho\theta}{(\rho\theta-1)}}\left(m^j\right)^{\frac{\rho(1-\theta)\rho\theta}{\rho\theta-1}}\mathbb{E}\left[\left({v^j}'\right)^{\rho\theta}\right]. \tag{33}$$

Simplify this expression using the fact that $\mathbb{E}\left[\left({v^j}'\right)^{\rho\theta}\right] = \left({m^j}'\right)^{\rho\theta}$, to give

$$\mathbb{E}\left[{\psi'}^{\frac{\theta(1-\rho)}{1-\rho\theta}}{Q'}^{\frac{\rho\theta}{\rho\theta-1}}\right] = \beta^{\frac{\rho\theta}{\rho\theta-1}}{C^j}^{\frac{(1-\rho)\rho\theta}{(\rho\theta-1)}}\left(m^j\right)^{\frac{(\rho-1)\rho\theta}{\rho\theta-1}}. \tag{34}$$

Rearranging and raising both sides to the power $\frac{1-\rho\theta}{(1-\rho)\rho\theta}$

$$m^j = \beta^{\frac{1}{1-\rho}}\left(\mathbb{E}\left[{\psi'}^{\frac{(1-\rho)\theta}{1-\rho\theta}}{Q'}^{\frac{\rho\theta}{\rho\theta-1}}\right]\right)^{\frac{1-\rho\theta}{(1-\rho)\rho\theta}}C^j. \tag{35}$$

This establishes Lemma 1.

APPENDIX C: PROOF OF LEMMA 2

Using Eq.(35) we have the following expression for $(m^j)^{\rho(1-\theta)}$

$$\left(m^j\right)^{\rho(1-\theta)} = \beta^{\frac{\rho(1-\theta)}{1-\rho}} \left(\mathbb{E}\left[\psi'^{\frac{(1-\rho)\theta}{1-\rho\theta}} Q'^{\frac{\rho\theta}{\rho\theta-1}}\right]\right)^{\frac{(1-\theta)(1-\rho\theta)}{(1-\rho)\theta}} (C^j)^{\rho(1-\theta)}. \quad (36)$$

Use this expression to replace m^j in Eq. (32), consolidate terms in β and C^j, and use Eq. (18) to replace $v^{j\prime}$ by $C^{j\prime}\psi'^{\frac{1}{\rho}}$ to give

$$Q' = \beta^{\frac{1-\rho\theta}{1-\rho}} C^{j\,1-\rho\theta} \left\{\left(\mathbb{E}\left[\psi'^{\frac{(1-\rho)\theta}{1-\rho\theta}} Q'^{\frac{\rho\theta}{\rho\theta-1}}\right]\right)^{\frac{(1-\theta)(1-\rho\theta)}{(1-\rho)\theta}}\right\} \left(C^{j\prime}\psi'^{\frac{1}{\rho}}\right)^{\rho\theta-1} \psi'^{\frac{1-\rho}{\rho}}. \quad (37)$$

Simplifying further and rearranging gives

$$C^{j\prime} = C^j \beta^{\frac{1}{1-\rho}} \left(Q'^{\frac{1}{\rho\theta-1}} \psi'^{\frac{\theta-1}{(1-\rho\theta)}}\right) \left(\mathbb{E}\left[\psi'^{\frac{(1-\rho)\theta}{1-\rho\theta}} Q'^{\frac{\rho\theta}{\rho\theta-1}}\right]\right)^{\frac{1-\theta}{(1-\rho)\theta}}. \quad (38)$$

This establishes Lemma 2.

APPENDIX D: PROOF OF PROPOSITION 2

We begin by establishing that aggregate human wealth obeys a simple recursive relationship. We assume that, conditional on survival, the cohort of newborns is endowed with the after-tax income streams, for $i = \{1, 2\}$,

$$\kappa_1\{\lambda_1^{s-t}\}_{s=t}^{\infty}(1-\tau)Y_s \quad \text{and} \quad \kappa_2\{\lambda_2^{s-t}\}_{s=t}^{\infty}(1-\tau)Y_s, \quad (39)$$

where $\kappa_1 = \alpha_1(1-\lambda_1\pi)$, $\kappa_2 = \alpha_2(1-\lambda_2\pi)$, and $\alpha_1 + \alpha_2 = 1$. Note that the $\frac{1}{1-\pi}$ from (1) drops out since we are integrating over the measure $1-\pi$ of newborn agents. Define the type i after-tax human wealth, H_i^t, owned by cohort t at date t for $i \in \{1, 2\}$, $H_i^t = \alpha_i(1-\lambda_i\pi)(1-\tau)\mathbb{E}\left[\sum_{k=t}^{\infty}(\lambda_i\pi)^{k-t} Q_t(S_k) Y_k\right]$, where $Q_t(S_k)$ is the date t price of a claim to one commodity in state S_k, for $k > t$.

At date t there are π^{t-j} surviving members of cohort $j \leq t$ each of whom owns a claim to a fraction λ_i^{t-j} of the type i income stream of a new-born. It follows that the type i human wealth at date t of cohort j is given by the expression $H_i^j = (\lambda_i \pi)^{t-j} H_i^t, \quad \text{for all} \quad j \leq t.$ Adding up this expression over all cohorts $j = -\infty, \ldots, t$ gives the following expressions for type i aggregate human wealth

$$H_i = \frac{1}{1-\lambda_i \pi} H_i^t, \tag{40}$$

and notice that H_i has a recursive representation, using prime notations, as $H_i = \alpha_i(1-\tau)Y + \lambda_i \pi \mathbb{E}\left[Q' H_i'\right]$. Define the human wealth ratio, h_i for $i = 1,2$ $h_i \equiv \frac{H_i}{Y}$, where Y is aggregate GDP and the h_i follow the recursion $h_i = \alpha_i(1-\tau) + \lambda_i \pi_i \mathbb{E}\left[\gamma' Q' h_i'\right]$.

Next, we establish that Eq. (41),

$$\phi(X, X') \equiv \left(\frac{\pi \beta^{\frac{1}{1-\rho}} c_L}{\gamma \Big(c - \psi^{-1}\left[(1-\lambda_1 \pi) h_1 + (1-\lambda_2 \pi) h_2\right] \Big) z_L^{\frac{\theta-1}{(1-\rho)\theta}} \psi^{\frac{1-\theta}{1-\rho\theta}}} \right)^{1-\rho\theta}, \tag{41}$$

is a valid representation for the pricing kernel.

We begin with Eq. (38), which holds for all individuals at alive in two consecutive date-state pairs, Let $C_t = \sum_j C_t^j$ be the aggregate consumption of all people alive at date t. Let $\mathcal{A}(t, t+1)$ denote the index set of all individuals alive at dates t and $t+1$ and note that

$$\sum_{j \in \mathcal{A}(t,t+1)} C_t^j = \pi C_t. \tag{42}$$

This expression recognizes that a measure π of people alive at date t survive into period $t+1$. Next, note that

$$\sum_{j \in \mathcal{A}(t,t+1)} C_{t+1}^j = C_{t+1} - C_{t+1}^{t+1}, \tag{43}$$

where C_{t+1}^{t+1} denotes the consumption of generation $t+1$ at date $t+1$. These individuals own no financial assets but, from Eq. (40) they own a fraction $1-\lambda_i \pi$ of type i human wealth. Using the expression for the policy function from Eq. (18) it follows that

$$C_{t+1}^{t+1} = \psi_{t+1}^{-1}\left[(1-\lambda_1\pi)H_{1,t+1} + (1-\lambda_2\pi)H_{2,t+1}\right] \tag{44}$$

Summing equation (38) over all $j \in \mathcal{A}(t,t+1)$, using equations (42), (43) and (44) gives

$$\begin{aligned} C' - \psi'^{-1}\left[(1-\lambda_1\pi)H_1'\right. &\left. + (1-\lambda_2\pi)H_2'\right] \\ &= C\pi\beta^{\frac{1}{1-\rho}}\left(Q'^{\frac{1}{\rho\theta-1}}\psi'^{\frac{\theta-1}{1-\rho\theta}}z^{\frac{1-\theta}{(1-\rho)\theta}}\right), \end{aligned} \tag{45}$$

where $z = \mathbb{E}\left[\psi'^{\frac{(1-\rho)\theta}{1-\rho\theta}}Q'^{\frac{\rho\theta}{\rho\theta-1}}\right]$. Rearranging,

$$Q'^{\frac{1}{1-\rho\theta}}\psi'^{\frac{1-\theta}{1-\rho\theta}}z^{\frac{\theta-1}{(1-\rho)\theta}} = \frac{\pi\beta^{\frac{1}{1-\rho}}C}{C' - \psi'^{-1}\left[(1-\lambda_1\pi)H_1 + (1-\lambda_2\pi)H_2\right]}. \tag{46}$$

Divide the top and bottom of the right hand side by $(\alpha_1+\alpha_2)Y$, rearrange terms and lag the equation by one period to give

$$\begin{aligned} Q \equiv & \\ &\left(\frac{\pi\beta^{\frac{1}{1-\rho}}c_L}{\gamma\left(c - \psi^{-1}\left[(1-\lambda_1\pi)h_1 + (1-\lambda_2\pi)h_2\right]\right)z_L^{\frac{\theta-1}{(1-\rho)\theta}}\psi^{\frac{1-\theta}{1-\rho\theta}}}\right)^{1-\rho\theta}, \end{aligned} \tag{47}$$

where c_L and c are the ratios of consumption to GDP at dates $t-1$ and t. This completes the proof of the functional form of the function $Q = \phi(X, X')$.

APPENDIX E: PROOF OF PROPOSITION 4

A steady-state goods market equilibrium is characterized by the equality,

$$\bar{\psi}^{-1}\left(\bar{h}_1 + \bar{h}_2 + \frac{\bar{b}^\delta(1+\delta\bar{p}^\delta)}{\bar{\Pi}\gamma}\right) = 1 - \bar{g}. \tag{48}$$

The left-hand-side of this expression is the demand for consumption goods and the right-hand-side is the supply of consumption goods. Both variables are written as ratios to GDP. In an autarkic steady-state, $b^{\delta}(\bar{Q}) = 0$ and $\bar{g} = \bar{\tau}$: these conditions imply that, $\frac{h(Q)}{1-\bar{g}} = \frac{\psi(Q)}{1-\bar{\tau}}$, where the functions $h(Q)$ and $\psi(Q)$ are written out explicitly in Eqn. (49)

$$\frac{h(Q)}{1-\bar{g}} \equiv \left[\frac{\alpha}{1-\pi\lambda_1\bar{\gamma}Q} + \frac{(1-\alpha)}{1-\pi\lambda_2\bar{\gamma}Q}\right] = \left[\frac{1}{1-\pi\beta^{\frac{1}{1-\rho}}Q^{\frac{\rho}{\rho-1}}}\right] \equiv \frac{\psi(Q)}{1-\bar{\tau}}, \tag{49}$$

and where $h(q) \equiv h_1(Q) + h_2(Q)$.

Define the compound parameters, $\delta_1 \equiv \frac{1}{1-\pi\lambda_1\bar{\gamma}}$, $\delta_2 \equiv \frac{1}{1-\pi\lambda_2\bar{\gamma}}$, $\delta_b \equiv \frac{1}{1-\pi\beta^{\frac{1}{1-\rho}}}$, $\Delta \equiv \delta_1 - \delta_2$, $\rho_c \equiv -\frac{\log(\pi\beta)}{\log(\lambda_1\bar{\gamma})}$, and the following inequalities,

$$\alpha > 1, \qquad \delta_b > \delta_1 > \delta_2, \qquad \delta_1 - \Delta(1-\alpha) > \delta_b, \qquad \rho < \rho_c < 0. \tag{50}$$

The proof that there are three autarkic steady-states proceeds in steps.

1. Note that the functions $h(Q) : (0, Q_1) \to \mathbb{R}_+$ and $\psi(Q) : (0, Q_1] \to \mathbb{R}_+$ are continuous.
2. Next we establish that h and ψ are increasing. The derivative of h is given by the expression

$$h_Q = \frac{\alpha\pi\lambda_1\bar{\gamma}}{(1-\pi\lambda_1\bar{\gamma}Q)^2} - \frac{(1-\alpha)\pi\lambda_2\bar{\gamma}}{(1-\pi\lambda_2\bar{\gamma}Q)^2} > 0, \tag{51}$$

where the inequality follows since $\alpha > 1$. The derivative of ψ is given by the expression

$$\psi_Q = \frac{\rho}{\rho-1}\frac{\pi\beta^{\frac{1}{1-\rho}}Q^{\frac{1}{\rho-1}}}{\left(1-\pi\beta^{\frac{1}{1-\rho}}Q^{\frac{\rho}{\rho-1}}\right)^2} > 0, \tag{52}$$

where the inequality follows since $\rho < 0$. This establishes that both functions are increasing.
3. Now consider the derivatives of h and ψ evaluated at $Q = 0$. These are given by the expressions, $0 < \{h_Q\}_{Q\to 0} \to \pi\bar{\gamma}(\alpha\lambda_1 + (1-\alpha)\lambda_2) < \infty$, and $0 < \{\psi_Q\}_{Q\to 0} \to \infty$, which establishes that for small ε, $\psi(\varepsilon) > h(\varepsilon)$.

4. Now consider the inequalities (50) which imply that $Q_b < Q_1$ where Q_b is the asymptote of the function ψ and Q_1 is the asymptote of the function h. Since both functions are increasing, it follows that, as $Q \to Q_b$, $\psi(Q) \to \infty > h(Q)$.
5. We have established that there is a trivial equilibrium at $Q = 0$ and that $\psi(Q) > h(Q)$ close to $Q = 0$ and close to $Q = Q_b$. Now note that

$$h(1) - \psi(1) = \delta_1 - \Delta(1-\alpha) - \delta_b > 0, \tag{53}$$

where the inequality follows from assumption (50).

We have established that h and ψ are continuous functions and that h starts below ψ, is above ψ for $Q = 1$ and drops below ψ at $Q = Q_b$. It follows that the functions must cross at least twice and for large enough negative values of ρ there are two non-trivial autarkic equilibria. When inequalities (50) hold, $\bar{Q}_{au1} < \bar{\gamma}^{-1}$ and $\bar{Q}_{au_2} > \bar{\gamma}^{-1}$. These inequalities establish that $\bar{Q}_{au_1}$ is dynamically efficient and $\bar{Q}_{au_2}$ is dynamically inefficient as claimed in Proposition 4. □.

APPENDIX: REFERENCES

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