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Abstract

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Network Games Made Simple*

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Abstract

Most network games assume that the best response of a player is a *linear* function of the actions of her neighbors; clearly, this is a restrictive assumption. We developed a theory called sign-equivalent transformation (SET) underlying the mathematical structure behind a system of equations defining the Nash equilibrium. By applying our theory, we reveal that many network models in the existing literature, including those with *nonlinear* best responses, can be transformed into games with best-response potentials after appropriate restructuring of equilibrium conditions using SET. Thus, through our theory, we produce a unified framework that provides conditions for existence and uniqueness of equilibrium for most network games with both linear and nonlinear best-response functions. We also provide novel economic insights for both the existing network models and the new ones we develop in this study.

JEL classification codes: C62, C72, D85, H41, Z13.
Keywords: network games, nonlinear best responses, sign-equivalent transformation, variational inequalities, best-response potential.

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1 Introduction

Peer effects matter in education (Epple and Romano, 2011; Sacerdote, 2014), crime (Warr, 2002; Lindquist and Zenou, 2019), performance at the workplace (Herbst and Mas, 2015), obesity (Christakis and Fowler, 2007), environmentally friendly behavior (Kyriakopoulou and Xepapadeas, 2021), and depression (Giulietti et al., 2022), among other outcomes. Most of the aforementioned peer-effect studies used the *linear-in-means model* in which each agent was *linearly* affected by the *mean* action of her reference group. The game theory foundation of the linear-in-means model is a network model that shows the best-response function of each agent to be linear and proportional to the mean action of her peers (see, e.g., Patacchini and Zenou, 2012; Boucher, 2016; Kline and Tamer, 2020; Ushchev and Zenou, 2020; Boucher et al., 2022).

In this paper, we provide a new methodology that relaxes the linearity assumption of this model. A general methodology exists that solves theoretical models with *linear best-response functions* (Jackson and Zenou, 2015); however, despite few existing network models with nonlinear best-response functions (e.g., Baetz, 2015; Allouch, 2015; Melo, 2019; Parise and Ozdaglar, 2019),[1] a general methodology that could solve these models in a unified manner does not exist yet.[2] In this paper, we propose such a methodology based on variational inequalities (VI) and sign-equivalent transformation (SET). Through many illustrations, we show that this new methodology is simple to apply, and it can provide a unified approach to analyze many of the existing nonlinear network models in the literature. In addition, we propose new network models that can be accommodated using the VI and SET techniques.

Many equilibrium models in economics and operations research can be formulated as a VI problem.[3] In particular, if a game is well-behaved, there is a well-known equivalence between determining the Nash equilibrium of continuous actions in pure strategy and solving a related VI problem where the operator in the VI, called the game Jacobian operator, is the (minus of the) gradients of players' payoffs in the underlying game (Lemma 1).

Next, we introduce an ordinal equivalent relation on VI that we called SET, which has the property of preserving the set of solutions on any rectangular domain. That is, if we consider two VI problems defined on the same (rectangular) domain, these two VIs are considered *sign equivalent* if the sign of the operators defining these VIs is the same at every point (Theorem 1). The combination of Lemma 1 and Theorem 1 is particularly useful for applications in well-behaved games. Indeed, starting from a game with an initial game Jacobian operator, we construct a new

[1] Melo (2019) and Parise and Ozdaglar (2019) also provided a general methodology to solve network models with nonlinear best-response functions. We explain the differences with our methodology in Section 5.1.

[2] Allen et al. (2022) study the equilibrium properties of network models with heterogeneous agents that include both linear and nonlinear best-response functions. Their main result is to characterize the equilibrium properties of these models based on a single statistic of the matrix of the strength of economic interactions: its spectral radius. Their techniques are different than ours and rely on (an extension of) the contraction mapping theorem.

[3] For overviews, see Nagurney (1999) and Konnov (2007).

VI whose operator is sign equivalent to the initial operator. Then, by finding the solution to the new VI, we obtain the solutions of the Nash equilibrium of the original game. Furthermore, many ways for conducting sign equivalent transformations exist. In Proposition 1, we identify different types of operations that preserve sign equivalence.

The theory of SET is applicable to well-behaved games. In this paper, we are particularly interested in games played on networks.[4] Consider a VI problem on a *network* (games on networks) with relatively general best-response functions (which include nonlinear best responses). In Theorem 2, we show that we can reorganize these best-response functions to obtain sign-equivalent ones that have the property of being integrable. Integrability of an operator is an appealing property since the solution to the corresponding VI is also the solution to the mathematical programming problem of minimizing a function, whose gradient is equal to the operator on the domain.[5] In other words, we have identified a *best-response potential* for the original game. We give the exact potential of these best-response functions and show that the Nash equilibrium of this class of games is one that maximizes this best-response potential function. This theorem shows that, in many classical network models, finding Nash equilibria can be equivalent to a much simpler optimization problem, in which the existence, uniqueness, and stability problems can be easily identified.[6]

Finally, we revisit many classical network games with nonlinear best responses, which include both games with strategic complements and substitutes. For each of these games, we can transform an original VI problem into a much simpler one by identifying certain SETs. This allowed us to construct a best-response potential function of the original network game, from which various properties of Nash equilibrium can be easily derived.

To illustrate our methodology, we start with general preferences and the simple possible network: the dyad; that is, a complete network with two players. We explain each step of our SET methodology and derive conditions that must be met for a unique Nash equilibrium to exist; that is, we explicitly construct the potential best-response function with its Hessian matrix being positive definite (under certain conditions). This simple example shows that we can solve many *aggregative network games*; that is, games in which the underlying network is complete.

Then, we consider network games with an arbitrary network structure with nonlinear best responses and *strategic complements*. We start with the model of Baetz (2015) and provide a condition under which there exists a unique interior Nash equilibrium for any possible network. Interestingly, by generalizing the model of Baetz (2015), we can solve all network games with linear best-response functions and strategic complements, since they are a particular case of the utility function considered in Baetz (2015).

[4] We briefly mention applications beyond network games in Section 5.4.

[5] This function can be computed using a line integral by Green's Theorem.

[6] Technically, an SET finds a suitable reoganization of the original equilibrium conditions into a new form, such that the Hessian matrix becomes symmetric, hence integrable.

Next, we consider network games with nonlinear best responses and *strategic substitutes*. We start with a generalized version of the linear public-good model of Bramoullé and Kranton (2007) (BK hereafter) by having convex costs, instead of linear costs, and more general spillover effects.[7] Because of the convexity of the cost, the best-response functions are, generally, nonlinear and it becomes difficult to solve the game using the techniques used in BK, where the best responses are linear. Following our SET framework, we construct a best-response potential, which facilitates the equilibrium analysis. In particular, we show that only interior solutions exist if the marginal cost is zero at zero (which is true for many specifications of convex costs). Thus, contrary to BK, there are no corner solutions, which implies that the equivalence between the maximal independent set and specialized equilibrium in BK does not hold anymore.

Furthermore, in terms of economic implications, when we consider the complete network and the star network, we show that the equilibrium properties are drastically different when costs are convex instead of being linear. For the complete network, there is a unique Nash equilibrium in our framework, whereas there is a continuum of equilibria in BK. In the star network, we show that the center node always exerts strictly less effort than the periphery nodes; however, this is not always true in BK. We also identify a reverse relationship between a player's equilibrium action and its neighboring nodes in the sense of set inclusion under any equilibrium. Focusing on nested-split graphs, we show that, at any Nash equilibrium, players with lower degree always exert higher effort, which is not necessary true in BK.

Finally, we consider network games with nonlinear best responses and *strategic substitutes* for the private provision of public goods, which, among others, have been considered by the public economics literature (Bergstrom, Blume, and Varian, 1986; complete network and $\delta = 1$)[8] and the network literature (Allouch, 2015; and network and $\delta = 1$). We generalize both models by considering an arbitrary network and a δ that takes any positive value between 0 and 1; that is, $\delta \in (0, 1]$. Using our SET methodology, we provide conditions under which there is a unique Nash equilibrium of our extended game. This model is richer and highlights the role of δ in the provision of public goods.

Related literature

Network games. As stated previously, we contribute to the literature on network games (for overviews, see Jackson and Zenou, 2015; Bramoullé and Kranton, 2016). Most papers in this literature consider linear-quadratic utility function. An exception is the paper by Bramoullé et al. (2014), who considered non-quadratic payoffs with linear best responses. They also introduced the notion of best-response potential; however, it is a quadratic function in their setting. Another exception is the study by Bourlès et al. (2017) who introduced a novel transformation to the equi-

[7]Indeed, in BK, the intensity of spillover effects δ is assumed to be equal to 1, whereas we assume that $\delta \in (0, 1]$.

[8]In this model, δ captures the intensity of the impact of the sum of the public-good provisions of a consumer's neighbors on own utility.

librium system to obtain a simple, integrable system that leads to the best-response potential of the altruism network game. According to our terminology, the transformation they adopt is sign equivalent.

We believe we are the first to provide a unified and systematic methodology that solves network games with nonlinear best-response functions. Other researchers have used VI to solve network games with nonlinear best-response functions (Melo, 2019; Parise and Ozdaglar, 2019) without using SET, and thus best-response potentials, which limit their applicability.[9]

Our methodology has the advantage of simplicity and easy applicability. All we need to do is define an ordinal equivalent relation on VIs, which can preserve the solution set by our Theorem 1. Since SET is a large set, it gives us sufficient flexibility to change the original VI into a SET VI to simplify the Nash equilibrium (NE) problem. For many well-known games on networks, we show that the new VI is integrable; hence, there is a potential function associated with it. Then, showing that this constructed potential function is a strictly convex function suffices. Furthermore, we can prove that the only domains that preserve the solution sets for arbitrary pair of ordinal equivalent VIs are the rectangular domains. Since many economic models naturally have either a lower bound and/or an upper bound on each effort x_i, the strategy space is usually a rectangular domain.[10]

Potential games. Beyond network games, potential function approaches have been widely adopted in several classes of games in the literature (Monderer and Shapley, 1996), such as the congestion game (Rosenthal, 1973; Nisan et al., 2007), Cournot oligopoly (Slade, 1994), aggregative games (Dubey et al., 2006; Jensen, 2010), and beauty contest (Huo and Pedroni, 2020), among others.[11]

Potential games have also been used in network games. However, as stated previously, this literature mostly deals with network games with linear best-response functions and shows that the game has an *exact* potential. Having an exact potential imposes strong restrictions on the underlying network models (see Monderer and Shapley, 1996). In this study, we show that even if the original game does not have an exact potential (which is usually the case if the best responses are not linear), we can still construct a best-response potential after using appropriate SET operations. Thus, we can establish equilibrium properties (such as existence, uniqueness, and stability) by just analyzing this constructed best-response potential.

The rest of the paper is as follows. In Section 2, we introduce the concept of VI and SET. In Section 3, we provide our main theoretical result, that is, how to solve network games with nonlinear best-response functions using best-response potentials. In Section 4, we show how we

[9]In Section 5.1, we discuss in details the differences between our model and that of Melo (2019) and Parise and Ozdaglar (2019).

[10]Indeed, our results apply to any rectangular domain $[a_1, b_1] \times \cdots \times [a_n, b_n]$ with the possibility of $a_i = -\infty$ and/or $b_i = +\infty$.

[11]See Voorneveld (2000) for discussion of games with best-response potentials and Ewerhart (2020) for necessary (and some sufficient) conditions for the existence of ordinal potentials in smooth games.

can use our new methodology to solve network games with nonlinear best-response functions with strategic complements (Section 4.1) and substitutes (Sections 4.2 and 4.3). In Section 5, we discuss the strengths and limitations of our approach. Finally, Section 6 concludes our study. All proofs can be found in the Appendix.

2 Variational inequalities and sign equivalence

We first introduce variational inequalities (VI hereafter) in Section 2.1, followed by defining sign equivalence and mentioning its implications in Section 2.2.

2.1 VI and Nash equilibrium

Given a nonempty closed convex set $K \subset R^m$ and a continuous mapping $\mathbf{F} = (F_1, \cdots, F_m)'$ from K to R^m,[12] the VI problem, $VI(K, \mathbf{F})$, is to determine a vector $\mathbf{x}^* \in K \subset R^m$, such that

$$\langle (\mathbf{x} - \mathbf{x}^*), \mathbf{F}(\mathbf{x}^*) \rangle \geq 0, \quad \forall \mathbf{x} \in K, \tag{1}$$

where $\langle ., . \rangle$ denotes the inner product. Let $Sol(K, \mathbf{F})$ denote the solution set, and $\#|Sol(K, \mathbf{F})|$ denote the cardinality of the solutions.[13]

VI provides a convenient tool for our analysis, since many economic problems using equilibrium notation can succinctly be reformulated using VI.[14] In particular, a well-known reformulation of Nash equilibrium exists while using VI in well-behaved games.

Definition 1. *A normal form game $\Gamma = (u_i, K_i)_{i \in N}$ among $N = \{1, \cdots, n\}$ players is well-behaved if (i) the strategy space of each player K_i is a closed and convex subset of R^{m_i}; (ii) the payoff functions are twice continuously differentiable; and (iii) for each player i, for any $x_{-i} \in \prod_{j \neq i} K_j$, $u_i(x_i, x_{-i})$ is concave in $x_i \in K_i$.*

Lemma 1. *Suppose Γ is a well-behaved game (Definition 1). Then, $\mathbf{x}^* = (x_1^*, \cdots, x_n^*) \in \prod_i K_i$ is a pure-strategy Nash equilibrium of Γ if and only if $\mathbf{x}^*$ solves the $VI(K, \mathbf{F})$ with*[15]

$$\mathbf{F}(\mathbf{x}) \equiv - \begin{pmatrix} \nabla_{x_1} u_1(x) \\ \cdots, \\ \nabla_{x_n} u_n(x) \end{pmatrix} \quad \textit{and} \quad K = \prod_{i=1}^{n} K_i, \tag{2}$$

[12] Let R^n_+ and R^n_{++} denote the set of nonnegative and positive vectors, respectively. Transpose of a matrix $\mathbf{A}$ is denoted by $\mathbf{A}'$. All bold symbols refer to vectors and matrices.

[13] Geometrically, $\mathbf{x}^*$ is a solution to the $VI(K, \mathbf{F})$ if and only if $-\mathbf{F}(\mathbf{x}^*) \in N_C(x^*)$, where $N_C(\mathbf{F})$ denotes the normal cone of K at $\mathbf{x}$ defined by $N_C(\mathbf{x}) = \{\mathbf{y} \in R^m : \langle \mathbf{y}, \mathbf{z} - \mathbf{x} \rangle \leq 0, \ \forall \mathbf{z} \in K\}$. See Nagurney (1999).

[14] VI was initially introduced in the literature to succinctly characterize critical points for minimization programs with constraints. By now, VI is a well-studied subject (Facchinei and Pang, 2007). Many economic problems can be reformulated using VI. For overview, see Nagurney (1999) and Konnov (2007).

[15] For a multivariable function $f(z, y)$ with $\mathbf{z} = (z_1, \cdots, z_p) \in \mathbf{R}^p$ and $\mathbf{y} = (y_1, \cdots, y_q) \in \mathbf{R}^q$, $\nabla_z f = (\partial f / \partial z_1, \cdots, \partial f / \partial z_p)'$ denotes the gradient (column) vector of f with respect to z.

where $\mathbf{F}$ *is called the game Jacobian of* Γ.[16]

Lemma 1 is applicable to many games studied in applied and theoretical papers because the conditions stated in Definition 1 are very mild. For instance, in games with continuous actions, the differentiability assumption in Definition 1 items (i) and (ii) are usually satisfied. In addition, the concavity requirement in Definition 1 (iii) facilitates the equilibrium analysis by focusing on local deviations instead of global ones.[17]

As a bridge connecting VI and Nash equilibrium (NE), Lemma 1 is useful because we will employ the results of VI to analyze Nash equilibrium in (network) games. For instance, the uniqueness of the solution to the $VI(K, \mathbf{F})$ implies the uniqueness of NE of Γ, and the comparative statics analysis of this VI informs us how the NE changes with parameters.[18] From now on, we can work directly with the game Jacobian $\mathbf{F}$ of the game. In addition, Lemma 1 lays out the basis of our subsequent VI-based theory called sign equivalence.[19]

2.2 Sign equivalences: Definition, implications, and constructions

Consider two VI problems, $VI(K, \mathbf{F})$ and $VI(K, \tilde{\mathbf{F}})$, defined on the same domain $K \subset R^m$,[20] where $\mathbf{F} = (F_1(x), \cdots, F_m(x))'$ and $\tilde{\mathbf{F}} = (\tilde{F}_1(x), \cdots, \tilde{F}_m(x))'$.

Definition 2. *$VI(K, \mathbf{F})$ is said to be* sign equivalent *to $VI(K, \tilde{\mathbf{F}})$ on K if, for every i,*

$$F_i(\mathbf{x}) \lesseqqgtr 0 \text{ if and only if } \tilde{F}_i(\mathbf{x}) \lesseqqgtr 0, \quad \forall \mathbf{x} \in K. \tag{3}$$

The definition is easy to check as follows: for any $\mathbf{x}$ in K, $F_i(\mathbf{x})$ and $\tilde{F}_i(\mathbf{x})$ must have the same sign. The requirement for *sign equivalence* is fairly weak as the exact values of $F_i(\mathbf{x})$ and $\tilde{F}_i(\mathbf{x})$ at $\mathbf{x}$ could differ dramatically as long as the signs of $F_i(\mathbf{x})$ and $\tilde{F}_i(\mathbf{x})$ are the same.

Definition 3. *K is called rectangular if $K = \prod_{i=1}^m [a_i, b_i]$ with $-\infty \le a_i < b_i \le +\infty$.*

The following theorem demonstrates the implications of sign equivalence on the solution sets of VIs.

[16] The dimension of $\nabla_{x_i} u_i(x)$ is m_i, so the dimension of $\mathbf{F}$ is $\sum_{i=1}^n m_i$, which is the same as that of K.

[17] Although we think of the requirements of Definition 1 as very mild, they rule out games with discrete actions. Furthermore, one can relax the concavity requirement in Definition 1 (iii) by imposing weaker conditions such as quasi-concavity.

[18] This Lemma (or its variation) appears in several recent economic papers (see, e.g., Ui, 2016; Melo, 2019; Parise and Ozdaglar, 2019).

[19] In Lemma 1, both interior solution and corner solution are analyzed symmetrically.

[20] Throughout the paper, we assume that the domain K of a VI is nonempty, convex, and closed. These assumptions are usually satisfied in most of the economic applications. Note that K is not necessarily compact.

Theorem 1. *Suppose* $\mathbf{F}$ *and* $\tilde{\mathbf{F}}$ *are sign equivalent on K.*

(i) Both VIs have the same set of solutions in $int(K)$*, the interior of K, that is,*[21]

$$Sol(K, \mathbf{F}) \cap int(K) = Sol(K, \tilde{\mathbf{F}}) \cap int(K).$$

This means that if $\mathbf{x}^* \in int(K)$*, then* $\mathbf{x}^*$ *solves* $VI(K, \mathbf{F})$ *if and only if* $\mathbf{x}^*$ *solves* $VI(K, \tilde{\mathbf{F}})$*.*

(ii) In addition, if K is rectangular*, then, two VIs have the same solution set, that is,*

$$Sol(K, \mathbf{F}) = Sol(K, \tilde{\mathbf{F}}).$$

This means that any $\mathbf{x}^* \in K$ *that solves* $VI(K, \mathbf{F})$ *must solve* $VI(K, \tilde{\mathbf{F}})$*, and vice versa.*

Theorem 1 illustrates that the solutions of two sign-equivalent VIs must be the same within the interior of K. Furthermore, if K is rectangular (Definition 2), their solutions (regardless of interior or corner ones) must be the same. Note that a solution to $VI(K, \mathbf{F})$ might be in the interior of K or on its boundary. Theorem $1(i)$ has broad applicability as the domain K is arbitrary. However, it only compares the solutions in the interior of K and remains silent about the solution on the boundary (see Remark 1). The implication of Theorem $1(ii)$ is strong as it compares all the solutions across two problems but it imposes additional mild rectangularity assumption on K.

Our theory of sign equivalence is particularly powerful when the domain is rectangular, that is, a product of intervals. In Definition 3, we allow for the possibility that $a_i = -\infty$ and/or $b_i = +\infty$. For instance, the Euclidean space R^m, its first quadrant $R^m_+ = [0, \infty)^m$, and the unit box $[0, 1]^m$ are all rectangular. Furthermore, if K_j is rectangular for each $j = 1, \cdots, n$, then the Cartesian product $\prod_{j=1}^n K_j$ is also rectangular (the product domain naturally arises in VI formulation of equilibrium in games; see Lemma 1).

Remark 1. *Suppose interior solutions are the main interest, which is true for many applied works, then, for any domain K, Theorem 1(i) is applicable. Furthermore, we could obtain the same result in Theorem 1 (i) by weakening sign equivalence to zero equivalence.*[22]

Remark 2. *Theorem 1(i) is silent about the solution possibly on the boundary. In fact, sign equivalence does not always preserve the solutions on the boundary of K when K is not rectangular (see Example A1(i) in Section A of the Online Appendix for an illustration).*

Furthermore, in Theorem A1, we show that the rectangular domain is the only domain that preserves the solution set to VI under any sign equivalent transformations. Thus, it shows the necessity of considering rectangular domains in Theorem 1(ii).

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[21] A point x is in the interior of K if there exists $\epsilon > 0$ such that the ball centered at x with radius ϵ is in K. We let $int(K)$ denote the set of interior points of K. Note that it is possible that $Sol(K, \mathbf{F}) \cap int(K)$ is empty.

[22] F_i is zero equivalent to $\tilde{F}_i$ if $F_i(\mathbf{x}) = 0 \iff \tilde{F}_i(\mathbf{x}) = 0, \forall \mathbf{x} \in K$. Note that this notation is weaker than sign equivalence (Definition 2), since it is possible that $F_i(\mathbf{y}) > 0$ but $\tilde{F}_i(\mathbf{y}) < 0$ for some $\mathbf{y} \in K$.

The combination of Lemma 1 and Theorem 1 is particularly useful for applications in well-behaved games. To see this, let $\mathbf{F}$ be the game Jacobian of Γ, as defined in Lemma 1. Suppose we can construct $\tilde{\mathbf{F}}$, which is sign equivalent to $\mathbf{F}$ (see Proposition 1 below for sign preserving operations), then, we can obtain information about the Nash equilibrium of Γ from the constructed $VI(K, \tilde{\mathbf{F}})$. In particular, any interior equilibrium of Γ must solve $VI(K, \tilde{\mathbf{F}})$. Furthermore, if each player's strategy space K_i is rectangular, then $K = \prod_i K_i$ is also rectangular; thus, *any* equilibrium of Γ (no matter whether it is in the interior or on the boundary of K) must solve $VI(K, \tilde{\mathbf{F}})$ and vice versa.

The scope of the applicability of Theorem 1 hinges on the following two critical prerequisites:

(A) How can we construct "many" sign equivalent transformations of $\mathbf{F}$?

(B) Among these sign-equivalent problems, is it possible to identify a particular one that is simpler and easier (in some appropriate sense) to analyze than the original one?

In general, issue (B) is complicated, since it depends on the specific structure of the original problem $VI(K, \mathbf{F})$. However, within a large class of network games, we are able to exploit the network structure to systematically construct a "nice" $\tilde{\mathbf{F}}$, as given in the next section, which helps us shed light on the equilibrium of the original network game.

To address issue (A), we identify several types of operations of $\mathbf{F}$ that preserve the sign equivalence. Since sign equivalent requires component-wise comparison of $\mathbf{F}$ and $\tilde{\mathbf{F}}$, in the operations defined below, we define it on each component F_i.[23]

SET_1: Scalar multiplication by a positive function:

$$F_i(\mathbf{x}) \to \tilde{F}_i(\mathbf{x}) = \alpha_i(\mathbf{x}) F_i, \tag{4}$$

where $\alpha_i(\mathbf{x}) > 0, \forall \mathbf{x} \in K$.

SET_2: Composition with a sign-preserving function:

$$F_i(\mathbf{x}) \to \tilde{F}_i(\mathbf{x}) = \kappa(F_i(\mathbf{x})) \tag{5}$$

where κ is a scalar function such that $\kappa(t) \lesseqqgtr 0$ if and only if $t \lesseqqgtr 0$. Note that κ is not necessarily monotone.

SET_3: Suppose $F_i(\mathbf{x}) = A(\mathbf{x}) - B(\mathbf{x})$ for some functions $A(\cdot)$ and $B(\cdot)$. Define

$$\tilde{F}_i(\mathbf{x}) = \begin{cases} h^+(A(\mathbf{x})) - h^+(B(\mathbf{x})), & \text{where } h^+(\cdot) \text{ is strictly increasing;} \\ h^-(B(\mathbf{x})) - h^-(A(\mathbf{x})), & \text{where } h^-(\cdot) \text{ is strictly decreasing.} \end{cases} \tag{6}$$

[23]Note that the transformation from F_i to $\tilde{F}_i$ can differ from that of F_j to $\tilde{F}_j$ for $j \neq i$.

It is clear that each of these three operations preserves the sign. Since sign equivalence, as a binary relation on all VIs defined on K, is transitive,[24] any composition of these operations must preserve the sign. Proposition 1 summarizes these observations.

Proposition 1. *$\mathbf{F}$ and $\tilde{\mathbf{F}}$ are sign equivalent if, for each i, we can obtain $\tilde{F}_i(\mathbf{x})$ from $F_i(\mathbf{x})$ through a sequence of operations using SET_1, SET_2, or SET_3 defined previously.*

In view of Proposition 1, these operations are called sign-equivalent transformations (SETs). Of course, the list of the aforementioned transformations is by no mean exhaustive. Hence, other well-suited SETs for specific applications might exist.

Remark 3. *Fix a game $\Gamma = (u_i, K_i)_{i\in N}$. Then, suppose we modify the payoff $u_i(x)$ to $\tilde{u}_i(x) = \phi_i(u_i(x))$ to define a new game $\tilde{\Gamma} = (\tilde{u}_i, K_i)_{i\in N}$, where the scalar function $\phi_i(\cdot)$ is strictly increasing, that is, $\phi_i' > 0$, then, the Jacobian $\tilde{\mathbf{F}}$ of the new game $\tilde{\Gamma}$ is sign equivalent to the Jacobian $\mathbf{F}$ of Γ (see Lemma 1). In fact, $\tilde{F}_i(\mathbf{x})$ can be obtained from $F_i(\mathbf{x})$ using SET_1 by setting $\alpha_i(\mathbf{x}) = \phi_i'(u_i(\mathbf{x})) > 0$ as $\nabla_{x_i}\tilde{u}_i(\mathbf{x}) = \phi_i'(u_i(\mathbf{x}))\nabla_{x_i}u_i(\mathbf{x})$ by the chain rule. Obviously, Γ and $\tilde{\Gamma}$ have the same set of pure-strategy equilibrium, a result consistent with Theorem 1.*[25]

Remark 4. *Though for different purposes, special cases of SET_1 are explicitly used in the literature (see Rosen, 1965; Ui, 2008).*[26] *In defining diagonal concavity, Rosen (1965) considered multiplying each $F_i(\mathbf{x})$ by a positive constant γ_i, which clearly was a special case of SET_1 with $\alpha_i(\mathbf{x}) \equiv \gamma_i$. Ui (2008), in defining his γ-monotonicity of the VI, considered multiplying $F_i(\mathbf{x})$ by $\gamma_i(x_i)$, a positive function depending only on x_i, but not on x_{-i}. Incidentally, this operation used by Ui (2008) is also a special case of SET_1. Both Rosen (1965) and Ui (2008) neither considered other types of SETs nor formally discussed the impact of these SETs on the solution sets of VIs as in our Theorem 1.*

Remark 5. *As the pivotal step in analyzing a network altruism game, Bourlès et al. (2017) employed a novel "logarithmic" transformation of the equilibrium conditions in their setting to a simple, integrable system, which eventually led to an explicitly constructed best-response potential of the underlying game. Incidentally, the transformation they adopted is a special case of SET_3 with $h^+(z) = \ln(z)$.*[27]

[24] That is, if $VI(K, \mathbf{F})$ is "sign equivalent" to $VI(K, \tilde{\mathbf{F}})$, then $VI(K, \tilde{\mathbf{F}})$ is "sign equivalent" to $VI(K, \mathbf{F})$. Moreover, if $VI(K, \mathbf{F})$ is "sign equivalent" to $VI(K, \tilde{\mathbf{F}})$ and $VI(K, \tilde{\mathbf{F}})$ is "sign equivalent" to $VI(K, \hat{\mathbf{F}})$, then $VI(K, \tilde{\mathbf{F}})$ is "sign equivalent" to $VI(K, \hat{\mathbf{F}})$. Note that the binary relation is not complete, since not any pair of VIs are comparable.

[25] Such a monotonic transformation of payoff naturally arises in some economic settings. For instance, when ϕ_i is affine, then u_i and $\tilde{u}_i$ are two equivalent utility representations of i's preference over (mixed) strategy profile. When ϕ_i is increasing and concave, it reflects risk aversion of player i. See, for instance, Weinstein (2016) on the impact of risk attitude (risk aversion and risk loving) on strategic behavior in games, and Weinstein (2017) for discussions of substitutes and complements in classical consumer theory.

[26] These two papers focus on concave games (which satisfy our Definition 1), but have different research questions. In particular, they are interested in identifying conditions of the modified VI to obtain results such as uniqueness and integrability. Instead, we consider a broader class of operations of SET and study the implications on the solution to VI.

[27] Roughly speaking, they rewrite the equilibrium conditions (see their paper for the notation):

$$0 \leq t_{ij} \perp (u_i'(y_i) - a_{ij}u_j'(y_j)) \geq 0, \forall i, j \tag{7}$$

3 Network games and best-response potentials

Consider a network g and a normal-form network game $\Gamma^g = (u_i, \mathbf{G}, K_i)_{i\in N}$. Here, K_i is the strategy space of agent i, u_i is her payoff, and $\mathbf{G} = (g_{ij})_{n\times n}$ is the adjacency matrix of the network g in which $g_{ij} = 1$ if and only if i and j are directly connected, and $g_{ij} = 0$, otherwise. We also assume that $g_{ii} = 0$ (no self-loops) and $g_{ij} = g_{ji} \in \{0, 1\}$ (undirected and unweighted network). Denote the set of neighbors of i by N_i, that is, $N_i = \{\text{all } j \mid g_{ij} = 1\}$, and the degree of i by d_i, that is, $d_i = |N_i|$.

3.1 A general result

We consider a class of network games in which a player's payoff depends on the actions of her direct neighbors. We present a general theorem demonstrating how to use SET to obtain simple ways of equilibrium characterization and establish existence and uniqueness results. In particular, we are able to provide solutions to network games when the best-response functions are *not* linear, which is a well-known difficult problem in the literature.

Theorem 2. *Consider a network game $\Gamma^g = (u_i, \mathbf{G}, K_i)_{i\in N}$ with a single-dimension strategy space $K_i = [a_i, b_i] \subseteq R$ and $K = \prod_i K_i$. Assume that, for each i, there exists a scalar δ, and some continuous functions $h_i(\cdot)$, $s_i(\cdot)$ and $R_i(\cdot)$, such that player i's optimal decision x_i, given each x_{-i}, implicitly satisfies*[28]

$$h_i(x_i) = R_i\left(s_i(x_i) + \delta \sum_{j=1}^{n} g_{ij} x_j\right). \tag{9}$$

Assume that $R_i(\cdot)$ is either strictly increasing or strictly decreasing. Then, $\mathbf{x}^* = (x_1^*, \cdots, x_n^*)$ in int(K) solves (9) if and only if $\mathbf{x}^*$ is a stationary point of $\phi(\mathbf{x})$, where

$$\phi(\mathbf{x}) := \sum_{i=1}^{n} \int_{x_i^0}^{x_i} \left\{ R_i^{-1}\left(h_i(z_i)\right) - s_i(z_i) \right\} dz_i - \frac{1}{2}\delta \sum_{i=1}^{n}\sum_{j=1}^{n} g_{ij} x_i x_j. \tag{10}$$

The intuition behind Theorem 2 is simple. Consider a network game $\Gamma^g = (u_i, \mathbf{G}, K_i)_{i\in N}$ for which the best-response function of each player i is given by (9). By inverting $R_i(\cdot)$ in (9), we obtain:

$$\underbrace{R_i^{-1}(h_i(x_i)) - s_i(x_i) - \delta \sum_{j=1}^{n} g_{ij} x_j}_{=\partial\phi/\partial x_i, \text{ for } \phi(\mathbf{x}) \text{ defined in (10)}} = 0, \forall i.$$

---

as

$$0 \le t_{ij} \perp (\ln u_i'(y_i) - \ln a_{ij} - \ln u_j'(y_j)) \ge 0, \forall i, j. \tag{8}$$

Note that the first function in (7), $u_i'(y_i) - a_{ij} u_j'(y_j)$, is sign equivalent to the second function in (8), $(\ln u_i'(y_i) - \ln a_{ij} - \ln u_j'(y_j))$ using SET_3 with $h^+(z) = \ln(z)$.

[28] Assuming $h_i' - R_i' s_i' \neq 0$, we have a unique solution of x_i to (9) for any x_{-i}.

Thus, the gradient of $\phi(\mathbf{x})$ defined in (10) vanishes if and only if (9) holds for any i. To determine the Nash equilibrium of Γ^g, one needs to find critical points $\mathbf{x}$ of $\phi(\mathbf{x})$ in (10). Note that Theorem 2 is a direct consequence of Theorem 1 as the operation we adopted here is a special case of SET (more precisely SET_3). This theorem is quite general but, at the same time, quite simple.

Observe that, in Theorem 2, the definition of the rectangular domain K is important. For economic applications, this rectangular domain may have some limitations; however, for the unidimensional domain, it is a very natural one. For some non-rectangular domains, we can still use our SET approach to show the existence and uniqueness of equilibrium.

We can summarize our results using Figure 1. Consider a normal-form, well-behaved game $\Gamma = (u_i, K_i)_{i \in N}$ (Definition 1). Finding the pure-strategy Nash equilibrium $\mathbf{x}^*$ of this game is equivalent for $\mathbf{x}^*$ to solve the VI problem $VI(K, \mathbf{F})$ defined in (1), where $\mathbf{F}$ is the Jacobian matrix of this game Γ (Lemma 1). We can then have a sign-equivalent transformation of $VI(K, \mathbf{F})$ by using $VI(K, \tilde{\mathbf{F}})$ instead of $VI(K, \mathbf{F})$ (Definition 2) because the Nash equilibrium solution $\mathbf{x}^*$ is the same for $VI(K, \mathbf{F})$ and $VI(K, \tilde{\mathbf{F}})$ for interior solutions and, if K is rectangular (Definition 3), it is the same for all solutions (Theorem 1). In particular, many transformations of $\mathbf{F}$ lead to a sign-equivalent $\tilde{\mathbf{F}}$ (Proposition 1). Now, consider a network game $\Gamma^g = (u_i, \mathbf{G}, K_i)_{i \in N}$, which has the same properties as those of the aforementioned game above. Then, if the best-response function of this game using $\tilde{\mathbf{F}}$ can be written as given in (9), so that it can be integrated and equal to $\phi(\mathbf{x})$, then, the potential best response $\mathbf{x}^*$ in (10), which is a maximizer of $\phi(\mathbf{x})$, is a Nash equilibrium of this game (Theorem 2).

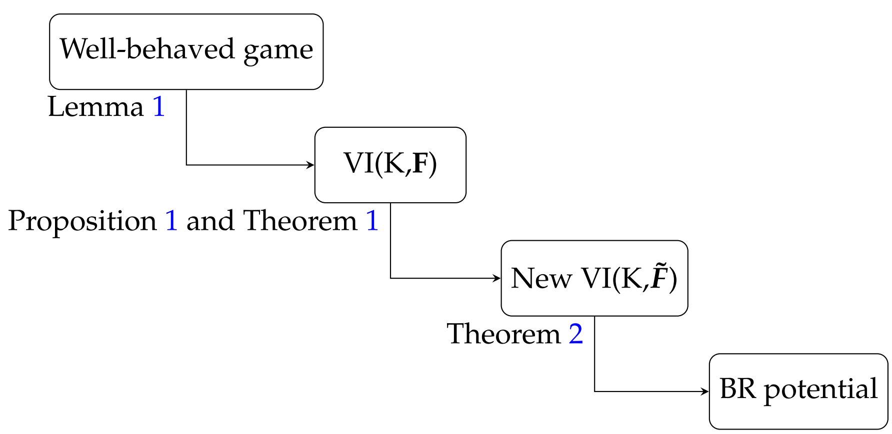

Figure 1: A summary of our methodology

3.2 Example: A dyad network

Now, let us illustrate Theorem 2 and Figure 1 using a simple example.

Consider a dyad network g^d with $N = 2$ and $g_{12} = g_{21} = 1$. Define the network game $\Gamma^{g^d} = (u_i, \mathbf{G}^d, K_i)_{i \in N}$, where $\mathbf{G}^d$ is the adjacency matrix of the dyad network g^d, and assume that

Γ^{g^d} is well behaved (Definition 1). Applying the first-order conditions yield the following:

$$x_1 = R_1(x_2), \tag{11}$$
$$x_2 = R_2(x_1), \tag{12}$$

where $R_i(.)$ is the best-response function of player $i = 1, 2$. Clearly, these best-response functions are a particular case of (9) when $h_i(x_i) = x_i$, $s_i(x_i) = 0$, $\delta = 1$, and the network is a dyad. Assume that $R_i(\cdot)$ is invertible for each $i = 1, 2$, then, we can rewrite these equations as follows:

$$x_2 = R_1^{-1}(x_1) := f_1(x_1), \tag{13}$$
$$x_1 = R_2^{-1}(x_2) := f_2(x_2), \tag{14}$$

where $f_i(.)$ denotes the inverse function of $R_i(\cdot)$ for each i.

We can define the VI problem $VI(K, \mathbf{F})$, where $K = R^2$ and $\mathbf{F}$ is given by:

$$\mathbf{F}(\mathbf{x}) = -\begin{pmatrix} R_1(x_2) - x_1 \\ R_2(x_1) - x_2 \end{pmatrix}. \tag{15}$$

Observe that $\mathbf{F}(\mathbf{x})$ is not integrable because its Jacobian matrix is not symmetric: $\frac{\partial F_1(\mathbf{x})}{\partial x_2} = -R_1'(x_2) \neq -R_2'(x_1) = \frac{\partial F_2(\mathbf{x})}{\partial x_1}$. Thus, none of the functions has gradient equal to $\mathbf{F}(\mathbf{x})$. Now, the "trick" is to transform this first-order condition into a new form that is integrable and has a best-response potential. Indeed, SET of $VI(K, \mathbf{F})$ is $VI(K, \tilde{\mathbf{F}})$, where $\tilde{\mathbf{F}}$ is given by:

$$\tilde{\mathbf{F}}(\mathbf{x}) = -\begin{pmatrix} f_1(x_1) - x_2 \\ f_2(x_2) - x_1 \end{pmatrix}. \tag{16}$$

Indeed, it should be clear that, for every i,

$$F_i(\mathbf{x}) \lesseqqgtr 0 \Leftrightarrow \tilde{F}_i(\mathbf{x}) \lesseqqgtr 0, \quad \forall \mathbf{x} \in K. \tag{17}$$

It is easily verified that the Jacobian matrix of $\tilde{\mathbf{F}}$ is now symmetric, since $\frac{\partial \tilde{F}_1(\mathbf{x})}{\partial x_2} = 1 = \frac{\partial \tilde{F}_2(\mathbf{x})}{\partial x_1}$. An SET changes the best-response function in a way that the Jacobian matrix becomes symmetric. By doing so, the operator becomes integrable and is associated with a potential function. Thus, to find the Nash equilibrium $\mathbf{x}^* = (x_1^*, x_2^*)$ of the original network game $\Gamma^{g^d} = (u_i, \mathbf{G}^d, K_i)_{i \in N}$, we can solve the VI problem $VI(K, \tilde{\mathbf{F}})$.

Following Theorem 2, we can use a line integral to $\tilde{\mathbf{F}}$ to obtain the best-response potential function $\phi(\mathbf{x})$ given by:

$$\phi(\mathbf{x}) = \int_{x_1^0}^{x_1} f_1(z_1) dz_1 + \int_{x_2^0}^{x_2} f_2(z_2) dz_2 - x_1 x_2, \tag{18}$$

with the property that $\nabla \phi = \tilde{\mathbf{F}}$. Clearly, the Nash equilibrium $\mathbf{x}^* = (x_1^*, x_2^*)$ solves the system of equations (11) and (12) if and only if $\mathbf{x}^*$ is a critical point of $\phi(\mathbf{x})$. The Hessian matrix of $\phi(\mathbf{x})$, or, equivalently, the Jacobian of $\tilde{\mathbf{F}}$, is given by:

$$\mathbf{H}\left[\phi(\mathbf{x})\right] = \begin{bmatrix} f_1'(x_1) & -1 \\ -1 & f_2'(x_2) \end{bmatrix}. \tag{19}$$

Note that the slope of $f_i(.)$ is the inverse of the slope of $R_i(.)$. To show that the Nash equilibrium is unique, we can impose a condition to make the Hessian matrix positive (negative) definite, so that $\phi(\mathbf{x})$ is strictly convex (concave). For strategic complements in which both best-response functions R_1 and R_2 are strictly increasing (or equivalently $f_i'(.) > 0, i = 1,2$), $\mathbf{H}\left[\phi(\mathbf{x})\right]$ is positive definite when $f_1'(x_1)f_2'(x_1) > 1$, or equivalently $R_1'(x_1)R_2'(x_2) < 1$. Under this condition, $\phi(\mathbf{x})$ is strictly convex, and a critical point must be unique. For strategic substitutes in which both best-response functions R_1 and R_2 are strictly decreasing (or equivalently $f_i'(.) < 0, i = 1,2$), $\mathbf{H}\left[\phi(\mathbf{x})\right]$ is negative definite when $|f_1'(x_1)f_2'(x_1)| > 1$, or, equivalently, $|R_1'(x_1)R_2'(x_2)| < 1$. Under this condition, $\phi(\mathbf{x})$ is strictly concave and a critical point must be unique.

For more general network settings, we can construct the potential function in a similar way and impose similar restrictions to guarantee the uniqueness and existence of the equilibrium of the original network game. We elaborate more on this in Section 4 for several network games with nonlinear best response functions.

3.3 Discussion: The scope of Theorem 2

A few remarks on Theorem 2 are given below.

First, note that this theorem is applicable to very broad settings, since it includes network games with both strategic complements and substitutes and also network games with nonlinear best-response functions.

Second, Theorem 2 does not always require an explicit expression of the best response of player i, x_i, as a function of x_{-i}. What is important is that i's best response explicitly solves (9) for some functions $h_i(\cdot)$, $R_i(\cdot)$, and $s_i(\cdot)$. Moreover, we only require that $R_i(.)$ is invertible, which covers either strictly increasing or decreasing $R_i(.)$. The invertibility of $R_i(.)$ can be a consequence of standard monotone comparative statics results, such as those in the studies by Milgrom and Shannon (1994) and Edlin and Shannon (1998).

Third, the network structure g is arbitrary. Thus, this Theorem takes a specific form when the network structure g takes a particular shape. In particular, Theorem 2 also contributes to the literature on *aggregative games*. To see that, when g is a complete network, Theorem 2 is applicable when the best response of player i is either (i) $x_i = \tau_i(X - x_i)$ for some $\tau_i(\cdot)$, or (ii) $x_i = \rho_i(X)$ for some $\rho_i(\cdot)$, where $X = \sum_{k=1}^{n} x_k$ is the aggregate action. Therefore, aggregative games with either invertible τ_i or ρ_i satisfy our condition and, thus, Theorem 2 presents a closed-form best-response potential function for these aggregative games.[29]

[29] ρ_i is often called the fitting-in curve of player i in aggregative games (see, e.g., Dubey et al., 2006; Jensen, 2010).

4 Solving network games with nonlinear best-response functions

Given Theorem 2, we would like to show how general our results are and how we can apply them to standard network games with nonlinear best response functions, which are usually difficult to solve. For all these applications, we assumed that the network is undirected, so that the adjacency matrix **G** is symmetric. We highlight two features of our approach: (i) tractability even for general models and (ii) new predictions and novel insights.

4.1 Network games with strategic complementarities

Consider the following network game Γ^B of Baetz (2015). There are n players connected in a network g with the utility function:[30]

$$u_i(\mathbf{x}, g) = v\left(\sum_j g_{ij}x_j\right)x_i - \frac{1}{2}x_i^2, \tag{20}$$

where $v(\cdot)$ is strictly increasing. The best-response function of i takes the following form

$$x_i = v\left(\sum_j g_{ij}x_j\right). \tag{21}$$

Baetz (2015) assumes that $v(0) = 0$, $v'(0) > 1$, $0 \leq \lim_{x\to\infty} v'(x) < 1/(n-1)$ and $v''(\cdot) < 0$ and shows that, under these assumptions, there are exactly two equilibria for any network g: (*i*) a trivial equilibrium in which all agents choose 0 and (*ii*) a nontrivial equilibrium that is interior. The uniqueness of the non-trivial equilibrium in the study of Baetz (2015) relies on a fixed-point theorem in Kennan (2001), which puts some restrictions on $v(\cdot)$, in particular, on the concavity of $v(\cdot)$.

Now, let us use our technique to show the uniqueness of equilibrium of Γ^B under a different set of conditions. First, we can define the VI problem $VI(K, \mathbf{F})$, where $K = R^n_+ = [0, \infty)^n$ and $\mathbf{F}$ is the Jacobian matrix of this game with

$$F_i(\mathbf{x}) = -\frac{\partial u_i(\mathbf{x}, g)}{\partial x_i} = x_i - v\left(\sum_j g_{ij}x_j\right). \tag{22}$$

Observe that the original game is not a potential because the Jacobian of $F(\mathbf{x})$ is not symmetric, since $\frac{\partial F_i(\mathbf{x})}{\partial x_j} = -v'\left(\sum_j g_{ij}x_j\right) \neq -v'\left(\sum_i g_{ji}x_i\right) = \frac{\partial F_j(\mathbf{x})}{\partial x_i}$. This is true even if the network is a dyad,

While we primarily focus on network games, several classes of non-network games, such as Cournot quantity competition and differentiated product Bertrand competition with linear demand, are aggregate games; hence, our results are directly applicable. See Section 5.4.

[30] Baetz (2015) provides other forms of utility functions leading to the same best responses in (21).

since $x_i \neq x_j$. Now, the "trick" is to transform this first-order condition into an integrable one with a best-response potential. Indeed, a SET of $VI(K, \mathbf{F})$, using SET_3, is $VI(K, \tilde{\mathbf{F}})$ with

$$\tilde{F}_i(\mathbf{x}) = v^{-1}(x_i) - \sum_j g_{ij} x_j, \tag{23}$$

where, as in Baetz (2015), we assume that $v(\cdot)$ is strictly increasing. It is easily verified that the Jacobian matrix of $\tilde{\mathbf{F}}$ is now symmetric, since $\frac{\partial \tilde{F}_i(\mathbf{x})}{\partial x_j} = -g_{ij} = -g_{ji} = \frac{\partial \tilde{F}_j(\mathbf{x})}{\partial x_i}$. Thus, following Theorem 2, we can construct a best-response potential $\phi^B(\mathbf{x})$ as follows:

$$\phi^B(\mathbf{x}) = \sum_{i=1}^{n} \int_0^{x_i} v^{-1}(z_i) dz_i - \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} g_{ij} x_i x_j, \tag{24}$$

where $\nabla \phi^B(\mathbf{x}) = \tilde{\mathbf{F}}(\mathbf{x})$ in equation (23). To show that there exists a unique Nash equilibrium, we need to demonstrate under which condition the Hessian matrix of $\phi^B(\mathbf{x})$ is positive definite (so that $\phi^B(\mathbf{x})$ is strictly convex). The Hessian matrix of ϕ^B is given by $\mathbf{H}\left[\phi^B(\mathbf{x})\right] = \mathbf{D} - \mathbf{G}$, where $\mathbf{D}$ is a diagonal matrix with $1/v'(v^{-1}(x_i))$ on the diagonal of row i. $\mathbf{H}\left[\phi(\mathbf{x})\right]$ is positive definite if the following condition holds: $\min_z \left\{\frac{1}{v'(z)}\right\} - \lambda_{\max}(\mathbf{G}) > 0$, where $\lambda_{\max}(\mathbf{G})$ is the largest eigenvalue of $\mathbf{G}$. This is equivalent to:

$$\max_z \{v'(z)\} < \frac{1}{\lambda_{\max}(\mathbf{G})}. \tag{25}$$

We summarize our finding in the following proposition:

Proposition 2. *Suppose $v(0) \geq 0$, $v'(z) > 0$ for $z \geq 0$, and condition (25) holds. Then, there exists a unique equilibrium of the game Γ^B.*

Compared to Baetz (2015), we have a unique equilibrium, and we only assume (25) and the fact that $v(\cdot)$ is strictly increasing. On the contrary, Baetz (2015) had to assume concavity ($v''(\cdot) < 0$), $v(0) = 0$, and $v'(0) > 1$ and end up with two equilibria. In our framework, to obtain a unique equilibrium, we do not need to impose these conditions, which are stronger than ours.

Remark 6. *Proposition 2 holds with heterogeneous marginal benefit function $v_i(\cdot)$ as long as the same set of assumptions stated in Proposition 2 hold for each i.*

Remark 7. *Consider an extension of (20) where the utility function is now given by: $u_i(\mathbf{x}, g) = v_i\left(\sum_j g_{ij} x_j\right) x_i - \frac{1}{2} x_i^2$, where $\delta > 0$. Then, all network games with linear best-response functions are a particular case of this utility function and the results in Proposition 2 and Remark 6 hold for $v_i(z) \equiv \theta_i + \delta z$. In particular, the corresponding condition (25) reduces to $0 < \delta < \frac{1}{\lambda_{\max}(\mathbf{G})}$, which is the standard regularity condition imposed in these network games (see, e.g., Ballester et al., 2006).*

4.2 Network games with strategic substitutes: Public goods on networks with convex costs

Consider an undirected network g of n agents, where each player i exerts effort $x_i \geq 0$ ($K = R_+^n = [0,\infty)^n$) by maximizing the following utility function:

$$u_i^{BKC}(\mathbf{x}, g) = b(x_i + \delta \sum_j g_{ij} x_j) - c(x_i), \tag{26}$$

where $b(\cdot)$ captures the individual benefit of exerting effort as well as the impact of peers' efforts on own utility; $\delta \in (0,1]$ is the intensity of peers' effort on own benefit; and $c(x_i)$ is the private cost of effort. When $\delta = 1$ and $c(x_i) = x_i$, (26) reduces to the utility function in Bramoullé and Kranton (2007), which we denote by $u_i^{BK}(\mathbf{x}, \mathbf{g})$.[31] Moreover, we denote the game of Bramoullé and Kranton (2007) by $\Gamma^{BK} = (u_i^{BK}, \mathbf{G}, R_+)_{i\in N}$ and our game by $\Gamma^{BKC} = (u_i^{BKC}, \mathbf{G}, R_+)_{i\in N}$.[32]

The game Γ^{BKC} extends Γ^{BK} by considering a convex instead of a linear cost and an imperfect substitution, $\delta \in (0,1]$ instead of perfect substitutable $\delta = 1$. Next we use our SET techniques to solve this game with strategic substitutes and nonlinear best response functions, and show the key differences with the original model of Bramoullé and Kranton (2007).

Assumption 1. *$b(\cdot)$ is strictly increasing and strictly concave, and $c(\cdot)$ is strictly increasing and weakly convex. That is, $b'(\cdot) > 0, b''(\cdot) < 0, c'(\cdot) > 0$, and $c''(\cdot) \geq 0$. Furthermore, we impose the standard Inada conditions, that is, $b'(0) > c'(0)$ and $\lim_{x_i\to+\infty} b'(x_i) < \lim_{x_i\to+\infty} c'(x_i)$.*

Denote the *autarky* solution by k^*, which solves $\max_{k\geq 0} b(k) - c(k)$. By Assumption 1, $k^* > 0$ is uniquely defined by $b'(k^*) = c'(k^*)$. By Assumption 1, $u_i^{BKC}(\mathbf{x}, g)$ is strictly concave in x_i, and thus, the unique maximizer (including corner solutions) satisfies the following complementary slackness condition:

$$c'(x_i) - b'(x_i + \delta \sum_j g_{ij} x_j) = 0, \text{ if } x_i > 0,$$

$$c'(x_i) - b'(x_i + \delta \sum_j g_{ij} x_j) \geq 0, \text{ if } x_i = 0,$$

or, equivalently in the following compact form:

$$0 \leq x_i \perp \left\{ c'(x_i) - b'(x_i + \delta \sum_j g_{ij} x_j) \right\} \geq 0.$$[33]

[31] In Section C of the Online Appendix, we solve the original model of Bramoullé and Kranton (2007) using our methodology.

[32] The superscript BKC refers to the model of Bramoullé and Kranton (2007) but with convex costs and $\delta \in (0,1]$.

[33] Here, we adopt this notation to succinctly state the following three conditions:

$$x_i \geq 0, \quad \left\{ c'(x_i) - b'(x_i + \delta \sum_j g_{ij} x_j) \right\} \geq 0, \textit{and} \quad x_i \times \left\{ c'(x_i) - b'(x_i + \delta \sum_j g_{ij} x_j) \right\} = 0.$$

Thus, we can define the VI problem $VI(R^n_+, \mathbf{F})$, where $\mathbf{F}$ is the Jacobian matrix of this game with

$$F_i(\mathbf{x}) = -\frac{\partial u_i^{BKC}(\mathbf{x}, g)}{\partial x_i} = c'(x_i) - b'(x_i + \delta \sum_j g_{ij} x_j). \tag{27}$$

This game is well-behaved (Definition 3) and, by Lemma 1, $\mathbf{x}^*$ is a Nash equilibrium if it solves $VI(R^n_+, \mathbf{F})$. However, $\mathbf{F}$ is not integrable because the Jacobian of $\mathbf{F}$ is not symmetric.[34] Therefore, we cannot solve the $VI(R^n_+, \mathbf{F})$ using a potential approach, as there is no function $\phi(\mathbf{x})$ such that $\nabla\phi(\mathbf{x}) = \mathbf{F}$. However, using a sign-equivalent transformation (SET), we can transform the $VI(R^n_+, \mathbf{F})$ into a new one $VI(R^n_+, \tilde{\mathbf{F}})$ with

$$\tilde{F}_i(\mathbf{x}) = -\left\{ x_i + \delta \sum_j g_{ij} x_j - b'^{-1}(c'(x_i)) \right\}.$$

Indeed, since $b'(\cdot)$ is strictly decreasing, we can do this transformation using SET_3 (see (6)). By Proposition 1, $F_i(\mathbf{x})$ and $\tilde{F}_i(\mathbf{x})$ are signed equivalent and, since $K = R^n_+$ is rectangular, by Theorem 1, all solutions (including corner solutions) of $VI(R^n_+, \mathbf{F})$ and $VI(R^n_+, \tilde{\mathbf{F}})$ are the same. In other words, we need to solve:

$$0 \leq x_i \perp \left\{ b'^{-1}(c'(x_i)) - x_i - \delta \sum_{j \in N_i} g_{ij} x_j \right\} \geq 0.$$

Observe that now the Jacobian of $\tilde{\mathbf{F}}$ is symmetric and thus $\tilde{\mathbf{F}}$ is integrable.[35] As a result, we can use Theorem 2 by noticing that $R_i^{-1}(t) = b^{-1}(t)$, $h_i(z_i) = c'(z_i)$, and $s_i(z_i) = z_i$ to obtain the following best-response potential function, for $\mathbf{x} \geq 0$:[36]

$$\phi^{BKC}(\mathbf{x}) = \sum_{i=1}^n \int_0^{x_i} \left\{ b'^{-1}(c'(z_i)) \right\} dz_i - \frac{1}{2} \sum_{i=1}^n x_i^2 - \frac{1}{2} \delta \sum_{i=1}^n \sum_{j=1}^n g_{ij} x_i x_j. \tag{28}$$

Now, i is easy to impose conditions to establish the existence and uniqueness of a Nash equilibrium. It is easily verified that the Hessian of $\phi^{BKC}(\mathbf{x})$ (or equivalently the Jacobian of $\tilde{\mathbf{F}}$) is equal to: $\mathbf{H}\left[\phi^{BKC}(\mathbf{x})\right] = \mathbf{D} - (\mathbf{I} + \delta\mathbf{G})$, where $\mathbf{I}$ is the identity matrix and $\mathbf{D}$ is a diagonal matrix with $c''(x_i) / \left[b''\left(b'^{-1}(c'(x_i))\right)\right] < 0$ on the diagonal of row i. Define $\lambda_{\min}(\mathbf{G})$ as the lowest eigenvalue of the adjacency matrix $\mathbf{G}$. Hence, we have the following:

Proposition 3. *Consider the network public-good game Γ^{BKC} and suppose that Assumption 1 holds. If, in addition, $\delta\lambda_{\min}(\mathbf{G}) + 1 > 0$, then (i) $\phi^{BKC}(\mathbf{x})$ is strictly concave and (ii) there exists a unique Nash equilibrium in Γ^{BKC}.*

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[34] For two linked nodes i and j, that is, $g_{ij} = g_{ji} = 1$, $\partial F_i(\mathbf{x})/\partial x_j = -\delta b''(x_i + \delta \sum_j g_{ij} x_j) \neq \partial F_j(\mathbf{x})/\partial x_i = -\delta b''(x_j + \delta \sum_k g_{jk} x_k)$.

[35] For two linked nodes i and j, that is, $g_{ij} = g_{ji} = 1$, $\partial \tilde{F}_i(\mathbf{x})/\partial x_j = \partial \tilde{F}_j(\mathbf{x})/\partial x_i = -\delta$.

[36] Note that, by construction, $\frac{\partial \phi_i^{BKC}(\mathbf{x})}{\partial x_i} = -\left\{ x_i + \delta \sum_{j \in N_i} g_{ij} x_j - b'^{-1}(c'(x_i)) \right\} = \tilde{F}_i(\mathbf{x})$.

Remark 8. *All the results in Proposition 3 hold with heterogeneous $c_i(\cdot)$ and $b_i(\cdot)$ as long as Assumption 1 holds for $b_i(\cdot)$ and $c_i(\cdot)$.*

Remark 9. *Our result about the existence of the best-response potential is more general than that of Bramoullé and Kranton (2007), which only considered linear cost. Indeed, assume $c(x) = x$. If, the autarky solution is $k^* = b'^{-1}(1)$, then, $\phi^{BKC}(\mathbf{x})$ reduces to*

$$\phi^{BK}(\mathbf{x}) = \sum_i (k^* x_i - \frac{1}{2}x_i^2) + \delta \sum_{ij} g_{ij} x_i x_j,$$

which is a quadratic function. In this case, $\phi^{BK}(\mathbf{x})$ is strictly concave if and only if the Hessian matrix $-(\mathbf{I} + \delta \mathbf{G})$ is positive definite or, equivalently, $\delta \lambda_{\min}(\mathbf{G}) + 1 > 0$.[37]

When the cost $c(.)$ is strictly convex, we assume the following.

Assumption 2. *$b'(0) > c'(0) = 0$.*

Remark 10. *Consider the network public-good game Γ^{BKC} and suppose that Assumptions 1 and 2 hold. Then, any NE must be interior since, at zero effort, the marginal benefit of public good is still strictly positive, while the marginal cost is zero. Hence, contrary to Bramoullé and Kranton (2007), there cannot be a specialized equilibrium where some agents provide zero effort and others exert k^* (the autarky contribution). This is true regardless of whether $\delta = 1$ or $\delta \in (0, 1)$.*

This remark shows that the discussion about specialized equilibrium in Bramoullé and Kranton (2007) is not valid for the case with nonlinear cost under Assumption 2. In particular, the equivalence between maximal independent set (of certain degree) and Nash equilibrium, which is at the center of Bramoullé and Kranton (2007), does not hold anymore.

Remark 11. *When $\mathbf{G}$ is directed, the Jacobian of $\tilde{\mathbf{F}}$ may not be symmetric anymore, and thus, $\tilde{\mathbf{F}}$ is not integrable; therefore, there is no best-response-potential function $\phi^{BKC}(\mathbf{x})$ associated with it. However, uniqueness is still guaranteed when $\mathbf{I} + \delta \mathbf{G}$ is a $\mathbf{P}-$matrix, no matter the values of $b(.)$ and $c(.)$.*[38] *Indeed, when $\mathbf{G}$ is symmetric, $\mathbf{I} + \delta \mathbf{G}$ is a $\mathbf{P}-$matrix, if and only if $\mathbf{I} + \delta \mathbf{G}$ is positive definite, which is true if and only if $\delta \lambda_{\min}(\mathbf{G}) + 1 > 0$. When $\mathbf{G}$ is asymmetric, a sufficient (not often necessary) condition for $\mathbf{I} + \delta \mathbf{G}$ to be a $\mathbf{P}-$matrix is that $\delta \lambda_{\min}(\mathbf{G}^S) + 1 > 0$, where $\mathbf{G}^S = (\mathbf{G} + \mathbf{G}')/2$ is the symmetric part of $\mathbf{G}$.*

Proposition 4. *Consider the network public-good game Γ^{BKC} and suppose that Assumptions 1 and 2 hold. In addition, assume that $\delta = 1$. Then the following statements hold:*

(i) Any Nash equilibrium $\mathbf{x}^*$ is interior, that is, $x_i^* > 0$, for all i.

(ii) At any Nash equilibrium $\mathbf{x}^*$,

[37] See Section C of the Online Appendix.

[38] See Section B of the Online Appendix for a definition of a $\mathbf{P}-$matrix.

(a) $N_i \subseteq (\subsetneq) N_j, \implies x_i^* \geq (>) x_j^*$.

(b) $\bar{N}_i \subseteq (\subsetneq) \bar{N}_j, \implies x_i^* \geq (>) x_j^*$, *where* $\bar{N}_i = N_i \cup \{i\}$.

(iii) *For the complete network, there exists a unique Nash equilibrium in which every agent exerts the same effort* $\bar{\mathbf{x}}^*$, *where* $\bar{\mathbf{x}}^*$ *uniquely solves* $b'(n\bar{\mathbf{x}}^*) = c'(\bar{\mathbf{x}}^*)$.

(iv) *For the star network ($n \geq 3$), in any Nash equilibrium, the center node exerts strictly less effort than any of the periphery nodes, while all periphery nodes exert the same effort.*

These results are in sharp contrast with those of Bramoullé and Kranton (2007) where the cost function is linear and $\delta = 1$. For the complete network, Bramoullé and Kranton (2007) predict a specialized equilibrium in which a single node exerts k^*, while all the remaining nodes free ride by choosing zero. In addition, there is a continuum of equilibria. Indeed, $(x_1^*, \cdots, x_n^*)$ is a Nash equilibrium as long as the sum of x_i^* adds up to k^*. In our Γ^{BKC} game, when the network is complete, there is a unique equilibrium.[39] In the *star network*, Bramoullé and Kranton (2007) show that only specialized profiles are equilibria and there are just two Nash equilibria: either the center or the three agents at the periphery are specialists. This implies, in particular, that the center node can exert either a higher or a lower effort than the periphery nodes. In Proposition 4(iv), we show that this is impossible: the center node always exerts strictly less effort than the periphery nodes.

Definition 4. *A nested-split graph (NSG) is a network g^{NSG} such that, for any three different indices i, j, k, if $g_{ij} = 1$ and $d_k \geq d_j$, then $g_{ik} = 1$.*

As shown by Mahadev and Peled (1995) and König et al. (2014), a nonempty network is a nested-split graph (NSG)[40] if the set of nodes can be partitioned into several classes, where players in the same class have the same degree and those in the "upper" class are linked to every agent in the "lower" classes.

Corollary 1. *Consider the network public-good game Γ^{BKC} with the same assumptions as imposed in Proposition 4. If the network g is a Nested Split Graph (NSG), then, at any Nash equilibrium $\mathbf{x}^*$, players with lower degree exert higher effort, that is, $d_i < d_j \Longrightarrow x_i^* > x_j^*$.*

Note that the complete network and the star are both NSG. Corollary 1 is quite general and the prediction is very sharp, since, in any Nash equilibrium, the effort of a player must be a decreasing function of her degree. Note that this corollary does not rely on the uniqueness of the NE.

Remark 12.

- *In Proposition 4 and Corollary 1, we assume $\delta = 1$. In fact, these results hold for any $\delta \in (0, 1]$.*

[39] Here, we do not impose the condition $\delta\lambda_{\min}(\mathbf{G}) + 1 > 0$ as in Proposition 3.

[40] There are several equivalent definitions of NSG; see, for instance, König et al. (2014), Billand et al. (2015), Nordvall Lagerås and Seim (2016), and Belhaj et al. (2016).

- *In Assumption 2, we impose $c'(0) = 0$, which implies that any Nash equilibrium is interior. If $c'(0) > 0$, then the Nash equilibrium might be in the corner. In this case, the strict inequalities in Corollary 1 do not hold anymore, but we still have the following property: $d_i < d_j$ implies $x_i^* \geq x_j^*$.*

4.3 Network games with strategic substitutes: Private provision of public goods

In this section, we simultaneously extend the pure-public goods provision models of Bergstrom, Blume, and Varian (1986) and Allouch (2015) on the private provision of public goods in networks by considering an arbitrary network and $\delta \in (0, 1]$. Indeed, Bergstrom et al. (1986) (referred to as BBV) consider a *complete network* and $\delta = 1$, while Allouch (2015) (referred to as AL) allows for any network but imposes the condition $\delta = 1$. We use our SET techniques to solve this general game with strategic substitutes and nonlinear best-response functions and show several key differences with the findings of Bergstrom et al. (1986) and Allouch (2015).

Consider an undirected network g of n agents. Each individual i consumes two goods: a *private good* x_i and a *local public good* q_i. The utility function of i is equal to:[41]

$$u_i^{ALD}(\mathbf{x}, \mathbf{q}, g) = u_i(x_i, q_i + \delta Q_{-i}), \tag{29}$$

where $Q_{-i} = \sum_j g_{ij} q_j$ is the sum of public good provisions of consumer i's neighbors and $\delta \in (0, 1]$ captures the intensity of the impact of Q_{-i} on i's marginal utility. The budget constraint of each consumer i is given by:

$$x_i + q_i = w_i, \quad x_i \geq 0, \tag{30}$$

where $w_i > 0$ is the income of player i (note that, for simplicity, we have normalized the prices of both goods to 1).

Each individual i chooses $x_i \geq 0$ and $q_i \geq 0$ that maximize the utility function (29) under the budget constraint (30). When $\delta = 1$ and the network is complete (i.e., $g = g^C$), we are back to the utility function used in Bergstrom et al. (1986), which we denote by $u_i^{BBV}(\mathbf{x}, \mathbf{q}, g^C)$. When $\delta = 1$ and the network is arbitrary, the utility function is the one given in Allouch (2015), which we denote by $u_i^{AL}(\mathbf{x}, \mathbf{q}, g)$. Moreover, we denote the game of Bergstrom et al. (1986) by $\Gamma^{BBV} = (u_i^{BBV}, \mathbf{G}^C, R_+)_{i \in N}$, that of Allouch (2015) by $\Gamma^{AL} = (u_i^{AL}, \mathbf{G}, R_+)_{i \in N}$, and ours by $\Gamma^{ALD} = (u_i^{ALD}, \mathbf{G}, R_+)_{i \in N}$.

Assumption 3. *$u_i(\cdot)$ is continuous, strictly increasing in x_i and q_i, and is strictly quasi-concave.*

Define $Q_i = q_i + \delta Q_{-i}$. Then, each individual i solves the following program:

$$\begin{aligned} &\max_{x_i, Q_i} u_i(x_i, Q_i) \\ &s.t.\ x_i + Q_i = w_i + \delta Q_{-i}, \quad x_i \geq 0,\ Q_i \geq \delta Q_{-i}. \end{aligned} \tag{31}$$

[41] The superscript ALD refers to the model of Allouch (2015) but with $\delta \in (0, 1]$.

If we ignore the constraint $Q_i \geq \delta Q_{-i}$, then solving this maximization problem allows us to interpret Q_i as the Marshallian demand for the public good; hence, it can be written as $Q_i = \gamma_i(w_i + \delta Q_{-i})$, where $w_i + \delta Q_{-i}$ may be interpreted as consumer i's *social income* and $\gamma_i(\cdot)$ as consumer i's *Engel curve*. Introducing back the constraint $Q_i \geq \delta Q_{-i}$ in this maximization program leads to $Q_i = \max\{\gamma_i(w_i + \delta Q_{-i}), \delta Q_{-i}\}$. The solution of this program is equal to:

$$q_i^* = \begin{cases} 0 & \text{if } \gamma_i(w_i + \delta Q_{-i}) - \delta Q_{-i} < 0 \\ \gamma_i(w_i + \delta Q_{-i}) - \delta Q_{-i} & \text{if } \gamma_i(w_i + \delta Q_{-i}) - \delta Q_{-i} \geq 0 \end{cases} \tag{32}$$

In other words,

$$q_i^* = \max\{0, \gamma_i(w_i + \delta Q_{-i}) - \delta Q_{-i}\} := f_i(\mathbf{q}). \tag{33}$$

Then, the Nash equilibrium of this game Γ^{ALD} is a vector $\mathbf{q}^* = (q_1^*, \cdots, q_n^*)$ such that $q_i^* = f_i(\mathbf{q}^*)$ for every i.

Now, let us apply our SET technique using the VI formulation to show the uniqueness of Nash equilibrium of this game. First, we can define the VI problem $VI(R_+^n, \mathbf{F})$ with

$$F_i(\mathbf{q}) = q_i + \delta Q_{-i} - \gamma_i(w_i + \delta Q_{-i}). \tag{34}$$

It is easy to see that the best responses of player i (equations (32) and (33)) can be written as follows:

$$0 \leq \mathbf{q} \perp F(\mathbf{q}) \geq 0, \tag{35}$$

or, equivalently,

$$q_i^* \geq 0,\ F_i(q_i^*, q_{-i}) \geq 0,\ q_i^* \times F_i(q_i^*, q_{-i}) = 0, \text{ for each } i.$$

Define $\eta_i(z_i) = z_i - \gamma_i(z_i)$ as the Engel curve for the private good. Clearly, $\eta_i'(z_i) = 1 - \gamma_i'(z_i) \in (0, 1)$. In particular, $\eta_i(z_i)$ is invertible. Let $\zeta_i(.)$ denote the inverse of $\eta_i(z_i)$. We construct a series of SETs of F_i:

$$F_i(\mathbf{q}) = q_i + \delta Q_{-i} - \gamma_i(w_i + \delta Q_{-i}) \geq 0 \Leftrightarrow q_i - w_i + w_i + \delta Q_{-i} - \gamma_i(w_i + \delta Q_{-i}) \geq 0$$

$$\Leftrightarrow q_i - w_i + \eta_i(w_i + \delta Q_{-i}) \geq 0 \Leftrightarrow w_i + \delta Q_{-i} - \zeta_i(w_i - q_i) \geq 0.$$

Define the function $\tilde{F}_i(\mathbf{q})$ as follows:

$$\tilde{F}_i(\mathbf{q}) = w_i + \delta Q_{-i} - \zeta_i(w_i - q_i).$$

By construction, $F_i(\mathbf{q})$ and $\tilde{F}_i(\mathbf{q})$ are signed equivalent, that is, $F_i(\mathbf{q}) \lesseqgtr 0$ if and only if $\tilde{F}_i(\mathbf{q}) \lesseqgtr 0$ for any $\mathbf{q} \in K = \prod_i [0, w_i]$; see Definition 2. Thus, the solutions of $VI(K, \mathbf{F})$ and $VI(K, \tilde{\mathbf{F}})$ are the same by Theorem 1. Therefore, the Nash equilibrium is equivalent to the following $VI(R_+^n, \tilde{\mathbf{F}})$:

$$0 \leq \mathbf{q} \perp \tilde{\mathbf{F}}(\mathbf{q}) \geq 0, \tag{36}$$

It is easily verified that $\tilde{\mathbf{F}}$ is integrable, since

$$\frac{\partial \tilde{F}_i(\mathbf{q})}{\partial q_j} = \frac{\partial \tilde{F}_j(\mathbf{q})}{\partial q_i} = \delta g_{ij} = \delta g_{ji}.$$

This holds for any inverse function $\zeta_i(.)$. Since the Jacobian of $\tilde{\mathbf{F}}$ is symmetric, this game has a best-response potential function $-\phi^{ALD}(\cdot)$, where:[42]

$$\phi^{ALD}(\mathbf{q}) = -\sum_i \int_{q_i^0}^{q_i} \{\zeta_i(w_i - z_i) - w_i\}\, dz_i + \frac{1}{2}\delta \sum_{i=1}^{n}\sum_{j=1}^{n} g_{ij} q_i q_j, \tag{37}$$

with $\nabla \phi^{ALD}(\mathbf{q}) = \tilde{\mathbf{F}}(\mathbf{q})$.

Proposition 5. *Suppose that Assumption 3 holds and for each $i = 1, ..., n$,*

$$\frac{1}{1 - \gamma_i'(w_i + \delta Q_{-i})} + \delta \lambda_{\min}(\mathbf{G}) > 0. \tag{38}$$

Then, there exists a unique Nash equilibrium of the game Γ^{ALD}.

In the proof of Proposition 5, we show that condition (38) implies the strict convexity of $\phi^{ALD}(\mathbf{q})$, which immediately gives the uniqueness result stated in the proposition. More generally, Proposition 5 provides a unified and simple proof of uniqueness. For instance, in the model of BBV,[43] the condition (38) requires that for each individual, both goods are strictly normal, that is, $0 < \gamma_i' < 1$, which is exactly the condition imposed in BBV. In Allouch (2015), where $\delta = 1$ and the network is general, condition (38) reduces to the network normality assumption stated in his paper for $\delta = 1$, that is, $1 + \frac{1}{\lambda_{\min}(\mathbf{G})} < \gamma_i' < 1$. Our model is more general since δ can take any value between 0 and 1.

Note that Proposition 5 is also useful to understand the results developed in this literature on the private provision of public goods regarding the effects of income redistribution. To see this, we define the potential function ϕ^{ALD} in the space of private good consumption $\mathbf{x}$ instead of public good consumption $\mathbf{q}$, that is,

$$\widetilde{\phi}^{ALD}(\mathbf{x}) := \phi^{ALD}(\mathbf{w} - \mathbf{q}).$$

Clearly, the minimizer of $\widetilde{\phi}^{ALD}(\mathbf{x})$ over $\prod_i [0, w_i]$ gives the equilibrium private consumption of agents. Under condition (38), $\widetilde{\phi}^{ALD}(\mathbf{x})$ is strictly convex and thus, has a unique minimizer.

In the model of BBV, one can further simplify $\widetilde{\phi}^{ALD}(\mathbf{x})$ as follows:

$$\widetilde{\phi}^{BBV}(\mathbf{x}) := \left(\sum_i \int_{x_i^0}^{x_i} \zeta_i(z_i) dz_i \right) - \bar{w}(\sum_i x_i) + \sum_{i \neq j} x_i x_j, \tag{39}$$

[42] Alternatively, we can use Theorem 2 to obtain ϕ^{ALD}.

[43] where $\delta = 1$ and $\mathbf{G} = K_n$ (complete network) with a minimal eigenvalue that is equal to -1.

where $\varpi = \sum w_i$ is the aggregate income (recall that ζ_i is the inverse of i's Engel curve for the private good). Since $\widetilde{\phi}^{BBV}(\mathbf{x})$ depends on the income vectors only through the aggregate income ϖ, any local income redistribution should have no impact on the consumption of private goods for all agents; it also has no effect on the aggregate public good $Q = \sum_i q_i$ as $Q = \varpi - \sum_i x_i$.[44]

Corollary 2. *In the BBV's model, any local income redistribution is neutral in the sense that it has no impact on the equilibrium consumption bundles of all agents.*

Such a neutrality result in the setting of BBV is a standard result (see, among others, Warr, 1983 and Bergstrom, Blume, and Varian, 1986). However, we believe that our approach provides a new perspective. In particular, we use the special structure of the best-response potential that we have explicitly constructed. Thus, one can adopt our approach to explore the implications of income redistribution in general settings (such as in Allouch, 2015).[45]

5 Discussions

5.1 Comparisons with Melo (2019) and Parise and Ozdaglar (2019)

The papers by Melo (2019)[46] and Parise and Ozdaglar (2019) are related to our model, since both use VI to solve network games with nonlinear best-response functions. Basically, they prove the uniqueness of the solution to VI (i.e., Nash equilibrium) by finding conditions in which the mapping $\mathbf{F}$ is monotone (or equivalent the Jacobian of $\mathbf{F}$ is a $\mathbf{P}-$matrix). These conditions are normally verified by showing that the Jacobian of $\mathbf{F}$ belongs to a certain class of matrices, such as $\mathbf{PD}-$ or $\mathbf{P}-$matrices.[47] We have a different but complementary approach by determining SET, that is, transforming best responses into best-response potentials.[48] Indeed, we define an ordinal equivalent relation on VIs, which preserves the solution set. This gives us plenty of room to reorganize the original VI to simplify the Nash equilibrium problem. Furthermore, for many network games, we show that the new VI is integrable, hence there is a potential function. More importantly, SET substantially simplifies the general problem and is very simple to apply. Indeed, in Section 4, in many games, we show that the new VI (SET) is integrable and hence, there is a potential function. This implies that our Theorem 2 has many economic applications. On the contrary, the conditions

[44]Under a local income redistribution $\mathbf{t} = (t_1, \cdots, t_n)$, agent i's income becomes $t_i + w_i$, where $\sum_j t_j = 0$ and $|t_j|$ is small (so that the set of active contributors to the public goods does not change (see Bergstrom et al., 1986; Andreoni, 1989, 1990; Allouch, 2015).

[45]One can show that in our general model ALD, where $\delta \in (0, 1]$, local income redistribution neutrality does not hold under a generic δ.

[46]See also Melo (2022).

[47]See Section B of the Online Appendix for a definition of these matrices.

[48]Observe that the property of the $\mathbf{F}$ matrix is not necessarily preserved under SET. Indeed, even if $\mathbf{F}$ and $\tilde{\mathbf{F}}$ are sign equivalent, the properties of the Jacobian matrix of $\mathbf{F}(\mathbf{x})$ and $\tilde{\mathbf{F}}(\mathbf{x})$ can change dramatically. For example, if $\mathbf{A}$ is a $\mathbf{P}-$matrix, then $\tilde{\mathbf{A}}$ may fail to be a $\mathbf{P}-$matrix.

imposed by Melo (2019) and Parise and Ozdaglar (2019) are more difficult to verify in many applications. In Section G of the Online Appendix, we provide a simple example where we show that the conditions for equilibrium uniqueness in Melo (2019) and Parise and Ozdaglar (2019) are much more restrictive than ours. However, we believe that our approach is complementary to theirs as we could first use our SET transformation and then check the uniqueness of the Nash equilibrium using the techniques developed by Melo (2019) and Parise and Ozdaglar (2019).

Observe that, even if the potential function is not concave, such a potential function is still useful for economic selection mechanism and the dynamics of best-response potentials along the lines of standard benefits of potential games (see, e.g., Morris and Ui, 2004, 2005).

5.2 The applicability of our approach and its limitations

As shown in Section 4, Theorem 2 is relatively easy to apply to many network games with nonlinear best responses. Following are the strengths of our approach and its limitations.

5.2.1 Directed versus undirected networks

We assumed that the network adjacency matrix $\mathbf{G}$ is *undirected*. Although we believe this is natural in many real-world applications (e.g., social links), there are some scenarios when the network matrix is not symmetric, that is, the network is *directed*. Our SET transformation and Theorem 1 do not rely on the symmetry of the adjacency matrix $\mathbf{G}$; thus, we can combine SET and standard VI results to simplify the game. However, for Theorem 2 and, in particular, for the integrability of the potential, we need the symmetry of $\mathbf{G}$. Note that, even if Theorem 2 cannot be directly applied, we can still solve games with directed networks using SET.

5.2.2 Row-normalized and weighted networks

In Theorem 2, we normalized $\delta_i = \delta$ for every i in (9). This was without any loss of generality as we could redefine $s_i(.)$ and $R_i(.)$ appropriately to set $\delta_i = \delta$. Consequently, Theorem 2 could incorporate network games with *local average* (see, e.g., Ushchev and Zenou, 2020) instead of *local aggregate*, that is, we could have $\sum_j \frac{g_{ij}}{d_i} x_j$ in (9) (where d_i is the degree of i) instead of $\sum_j g_{ij} x_j$.[49] For instance, for the local-average model, the utility function in the study of Baetz (2015) would be given by $u_i(\mathbf{x}, \mathbf{g}) = v\left(\sum_j \frac{g_{ij}}{d_i} x_j\right) x_i - \frac{1}{2} x_i^2$ instead of (20).

In addition, instead of a $(0,1)$ adjacency matrix $\mathbf{G}$, Theorem 2 could easily incorporate *weighted networks* with $g_{ij} \in [0,1]$.

[49] We focus on the mathematical applicability of Theorem 2 in both models. See Ushchev and Zenou (2020) for an illustration of the differences in terms of economics predictions between the two network models.

5.3 Multiple activities and multiple networks

Even though, we cannot derive general results when agents make efforts in different activities or are embedded in multiple networks, we can still solve some specific games having these features.

First, we can easily use our SET approach to solve network games with multiple activities. In Section D of the online Appendix, we use our SET methodology to solve for the game of Chen et al. (2018), who consider multiple activities in a network game with strategic complementarities but linear best-response functions.

Second, to illustrate the fact that our SET approach can incorporate a network game with multiple networks and nonlinear best-response functions, we would like to present a new network game (which has not been studied by the literature) with *two public goods* and *two networks*.

Consider the private provision of two local public goods q_i and y_i. For each public good, there is an undirected network that connects n agents for the consumption of the public good. For the public good q_i, the corresponding undirected n-players network is denoted by g with adjacency matrix $\mathbf{G}$ while, for the public good y_i, the corresponding undirected n-players network is denoted by h with adjacency matrix $\mathbf{H}$. Each player i has an exogenous endowment of income $w_i > 0$, which can be allocated between two public goods.[50] The payoff of player i has a strictly convex preference given by

$$u_i^{NEW}(\mathbf{q}, \mathbf{y}, \mathbf{g}, \mathbf{h}) = u_i(q_i + Q_{-i}, y_i + Y_{-i}), \tag{40}$$

where $Q_{-i} = \delta \sum_j g_{ij} q_j$ and $Y_{-i} = \mu \sum_j h_{ij} y_j$ are the sums of public good provisions of consumer i's neighbors, and $\delta > 0$ and $\mu > 0$ capture the intensity of the network effects. The budget constraint of each consumer i is given by:[51]

$$q_i + y_i = w_i, \quad q_i \geq 0,\, y_i \geq 0. \tag{41}$$

Each player i chooses q_i and y_i that maximize $u_i(q_i + Q_{-i}, y_i + Y_{-i})$ under the budget constraint (41). To isolate the impact of network structure on the provision of the two public goods, we assume that both public goods are symmetric ex ante, that is, $u_i(q_i + Q_{-i}, y_i + Y_{-i}) = u_i(y_i + Y_{-i}, q_i + Q_{-i})$.[52]

Under this symmetry assumption, the solution to this maximization program must satisfy:

$$(q_i^*, y_i^*) = \begin{cases} (0, w_i) & \text{if } \frac{w_i - Q_{-i} + Y_{-i}}{2} \leq 0, \\ \left(\frac{w_i - Q_{-i} + Y_{-i}}{2}, \frac{w_i + Q_{-i} - Y_{-i}}{2}\right) & \text{if } \frac{w_i - Q_{-i} + Y_{-i}}{2} \in (0, w_i) \\ (w_i, 0) & \text{if } \frac{w_i - Q_{-i} + Y_{-i}}{2} \geq w_i. \end{cases} \tag{42}$$

[50] For simplicity, we ignore the consumption of private good.

[51] For simplicity, we have normalized the prices of both public goods to be 1.

[52] Many utility functions satisfy this condition. For example, for any increasing and concave function $v_i(.)$, this is true for $u_i(q_i + Q_{-i}, y_i + Y_{-i}) = v_i(q_i + Q_{-i}) + v_i(y_i + Y_{-i})$. This is also true for Cobb-Douglas preference $u_i(q_i + Q_{-i}, y_i + Y_{-i}) = (q_i + Q_{-i})^{b_i}(y_i + Y_{-i})^{b_i}$, for any constant $b_i > 0$. Note that we do not require identical preferences across players.

This result is intuitive. In the region with positive supplies of both goods, which happens when $Q_{-i} \leq w_i + Y_{-i}$ and $Y_{-i} \leq w_i + Q_{-i}$, the supply of q_i is decreasing in Q_{-i} (strategic substitution) and increasing in Y_{-i} (strategic complementarity), since the consumer i has a taste for variety in consumption. However, if $Q_{-i} \geq w_i + Y_{-i}$, there is excess supply of public good q_i than y_i by i's neighbors; thus, player i prefers to contribute to zero for the public good q_i. The same reasoning applies to the consumption of the public good y_i.

In Section E of the Online Appendix, we solve this game using our SET methodology by first determining the potential function $\phi(\mathbf{q})$ of this game (see (E8)) and then, by showing that under a network regularity condition, $2 + \lambda_{\min}(\delta\mathbf{G} + \mu\mathbf{H}) > 0$, there is a unique Nash equilibrium. We also characterize this NE when the solutions $\mathbf{q}^*$ and $\mathbf{y}^*$ are interior.

5.4 Applications beyond network games

Our SET theory is applicable for not only network games but also any smooth games. Here, we discuss its application to industrial organization by studying a standard pricing game of an oligopoly market with discrete choices. We show the benefit of using our SET methodology in demonstrating the uniqueness of the Bertrand-Nash equilibrium.

Consider a market with $n \geq 2$ firms, where each sells a differentiated product. Fore each firm $i = 1, 2, \cdots, n$, let p_i denote the price charged by firm i, p_{-i}, the price charged by other firms, and p_0 the price charged by the outside option, which is exogenously fixed (depending on the utility obtained from the outside option). Using a discrete choice model, we can derive the demand for a firm i, which is given by $Q^i(p_i, p_{-i}, p_0)$. Consider the Bertrand pricing competition game among n firms with payoff given by

$$\pi_i(p_i, p_{-i}) = (p_i - c_i)Q^i(p_i, p_{-i}), \; i = 1, 2, \cdots, n.$$

In Section F of the Online Appendix, using standard assumptions, we show that there exists a unique Nash equilibrium of this game using our SET methodology by finding a suitable SET of the game Jacobian of this Bertrand-pricing game (see Lemma F5 and Theorem F4). We believe that our SET methodology can be applied to many other economics problems such as contest models, rent-seeking games, and Cournot game with nonlinear demand.[53]

6 Conclusion

In this paper, we developed the concept of sign-equivalent transformation (SET) to analyze games played on networks with nonlinear best responses. First, we introduced SET, which is an ordinal

[53] For the latter, we could assume an (inverse) demand function given by $p_i(Q) = (A - q_i - \sum g_{ij}q_j)^a$, $a > 0$, a linear marginal cost $C_i(q_i) = c_i q_i$, and, fixing q_{-i}, a payoff that needs to be strictly quasi-concave in q_i. Notice that when $a = 1$, this game has an exact potential (Monderer and Shapley, 1996).

equivalent relation on variational inequalities (VI) and has the property of preserving the set of solutions on any rectangular domain. Second, we considered a VI problem on a network game and showed that we can rewrite the best-response functions to obtain sign-equivalent ones, which has the property of being best-response potentials. We showed that finding Nash equilibria can be equivalent to a much simpler problem, where the existence, uniqueness, stability problems can be easily identified. Finally, we explained how to apply our methodology to many well-known network games with general payoffs and nonlinear best responses.

Compared to the existing approaches in the literature, our SET method is much simpler to apply and delivers sharper results by expanding its applicability. Indeed, Theorem 2 is very general and simple to use for anyone who wants to apply SET to network models. Moreover, by providing a unified approach, SET gives a new perspective on issues such as existence, uniqueness, and structure of equilibria.[54] We believe that the methodological contribution of this approach is valuable for researchers who are interested in theory and technical aspects of equilibrium[55] as well as researchers interested in application and use of our methodology to solve applied network issues. By using and extending different network models from the literature, we showed how to apply our SET methodology to a large class of network games.

A possible direction for future research is to study interventions/targeting policy in network games (such as those in Ballester et al., 2006 and Galeotti et al., 2020). One advantage of having a best-response potential is that we can transform the intervention/targeting problem on networks, which is normally an optimization program with (Nash) equilibrium constraints, into an optimization program with optimality constraints, that is, potential maximization. The latter is simpler to handle than the former. Other possible directions of research would be to incorporate incomplete information in our model or to have a general theory of multi-dimensional strategy space or multiplex networks.

We believe that having a general methodology solving for *nonlinear network models* is important for empirical applications, since the linear-in-means model, the workhorse model in empirical work on peer/network effects, imposes a particular *linear* relationship between the outcome of an individual and the (mean) outcomes of the other individuals in the group.[56] Nonlinear models relax this assumption by assuming a nonlinear relationship instead, which is often more realistic and can also allow to test for linearity as a special case.

[54] Observe that best-response potential is not only useful for the theoretical analysis of existence, uniqueness, and stability of equilibrium but also for its application. With best-response potential, the computation of Nash equilibrium, usually in fixed-point equations, is reduced to a maximization program, which is well analyzed in numerical computations. There are fast algorithm, such as Newton gradient descent, that we can compute if the objective function is concave, which happens to be true in many applications mentioned in this paper.

[55] When starting with multiple equilibria, our best-response potential function approach can be used as a selection criterion to identify a stable equilibrium.

[56] See Boucher et al. (2022) who empirically showed that the correct empirical model is not linear in many activities.

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Appendix: Proofs

Proof of Lemma 1: The proof is standard. See, for instance, chapter one of Facchinei and Pang (2007). □

Before proving Theorem 1, we need the following well-known result about VI on rectangular domains.

Lemma 2.

(i) When $K = \prod[a_i, b_i]$ *with* $-\infty \le a_i < b_i \le +\infty$, x^* *solves* $VI(K, \mathbf{F})$ *if and only if, for every* i,

$$\begin{aligned} x_i^* = a_i &\implies F_i(x^*) \ge 0, \\ a_i < x_i^* < b_i &\implies F_i(x^*) = 0, \\ x_i^* = b_i &\implies F_i(x^*) \le 0. \end{aligned} \tag{43}$$

(ii) When $K = R_+^n = [0, +\infty)^n$, x^* *solves* $VI(K, F)$ *if and only if fore every* i,

$$x_i^* \ge 0, \quad F_i(x^*) \ge 0, \quad x_i \times F_i(x^*) = 0$$

which is called **Nonlinear Complementarity Problem** $NCP(F)$*, and can be rewritten using the following compact form*

$$0 \le x^* \perp F(x^*) \ge 0.$$

(iii) When $K = R^n$*, then* x^* *solves* $VI(K, F)$ *if and only if* $F(x^*) = 0$.

Proof of Theorem 1:

Theorem 1 (i) follows from a simple observation that $\mathbf{x}^*$ is a solution to the $VI(K, \mathbf{F})$ and $\mathbf{x}^*$ is interior point of K, then it must be the case that $\mathbf{F}(\mathbf{x}^*) = 0$. Then, $\tilde{\mathbf{F}}(\mathbf{x}^*) = 0$ as well by sign equivalence, thus $\mathbf{x}^*$ is also the solution of $VI(K, \tilde{\mathbf{F}})$. Theorem 1 (ii) further shows that under rectangularity assumption, the solution on the boundary must be preserved under sign equivalence.

To prove item (i), we first prove that

$$Sol(K, \mathbf{F}) \cap int(K) = \{\mathbf{x}^* \in int(K), \mathbf{F}(\mathbf{x}) = 0\}.$$

Note that any $\mathbf{x}^*$ which satisfies $\mathbf{F}(\mathbf{x}^*) = 0$ must satisfy (1) for any $\mathbf{x} \in K$; thus it is a solution of $VI(K, \mathbf{F})$. Hence, $\{\mathbf{x}^* \in int(K), \mathbf{F}(\mathbf{x}^*) = 0\} \subseteq Sol(K, \mathbf{F}) \cap int(K)$. To show the opposite, suppose that $\mathbf{x}^* \in int(K)$ is a solution of $VI(K, \mathbf{F})$, then $\langle \mathbf{F}(\mathbf{x}^*), \mathbf{x} - \mathbf{x}^* \rangle \ge 0, \forall \mathbf{x} \in K$ by (1). Since $\mathbf{x}^*$ is an interior point of K, there exists $\epsilon > 0$, such that $\mathbf{x} = \mathbf{x}^* + \epsilon \mathbf{z} \in K$ for any $\mathbf{z}$ with $||\mathbf{z}|| = 1$. Therefore, $\langle \mathbf{F}(\mathbf{x}^*), (\mathbf{x}^* + \epsilon \mathbf{z}) - \mathbf{x}^* \rangle = \epsilon \langle \mathbf{F}(\mathbf{x}^*), \mathbf{z} \rangle \ge 0$ for any $\mathbf{z}$ with $||\mathbf{z}|| = 1$. Then, it must be the case that $\mathbf{F}(\mathbf{x}^*) = 0$ (otherwise we can choose $\mathbf{z} = -\mathbf{F}(\mathbf{x}^*)/||\mathbf{F}(\mathbf{x}^*)||$). By sign equivalence of $\mathbf{F}$ and $\tilde{\mathbf{F}}$ on K, $\{\mathbf{x}^* \in int(K), \mathbf{F}(\mathbf{x}) = 0\} = \{\mathbf{x}^* \in int(K), \tilde{\mathbf{F}}(\mathbf{x}) = 0\}$, the result just follows.

For part (ii), we assume that K is rectangular, i.e., $K = \prod_i [a_i, b_i]$. By Lemma 2(i), we have that $\mathbf{x}^* = (x_1^*, \cdots, x_n^*) \in Sol(K, \mathbf{F})$ if and only if, for every i,

$$\begin{aligned} x_i^* = a_i &\implies F_i(x^*) \geq 0, \\ a_i < x_i^* < b_i &\implies F_i(x^*) = 0, \\ x_i^* = b_i &\implies F_i(x^*) \leq 0, \end{aligned} \tag{44}$$

which, from the definition of sign equivalence, implies the following:

$$\begin{aligned} x_i^* = a_i &\implies \tilde{F}_i(x^*) \geq 0, \\ a_i < x_i^* < b_i &\implies \tilde{F}_i(x^*) = 0, \\ x_i^* = b_i &\implies \tilde{F}_i(x^*) \leq 0. \end{aligned} \tag{45}$$

Again, the above condition, by Lemma 2(i), implies that $\mathbf{x}^* \in Sol(K, \tilde{\mathbf{F}})$. Thus, we have shown that any solution of $VI(K, \mathbf{F})$ is also a solution of $VI(K, \tilde{\mathbf{F}})$. By the same logic, any solution of $VI(K, \tilde{\mathbf{F}})$ is also a solution of $VI(K, \mathbf{F})$, implying the solution set must be the same, that is, $Sol(K, \mathbf{F}) = Sol(K, \tilde{\mathbf{F}})$. □

Proof of Proposition 1: It follows immediately from the discussion in the main text. □

Proof of Proposition 2: It follows from the discussion in the main text. □

Proof of Proposition 3: First, the existence of equilibrium can be shown by a standard fixed point theorem on a compact rectangular $[0, k^*]^N$, where k^* is the autarky solution. To show uniqueness, it suffices to prove that the Hessian $\mathbf{H}\left[\phi^{BKC}(\mathbf{x})\right]$ is negative definite. Since $\mathbf{H}\left[\phi^{BKC}(\mathbf{x})\right] = \mathbf{D} - (\mathbf{I} + \delta\mathbf{G})$, where $\mathbf{I}$ is the identity matrix and $\mathbf{D}$ is a diagonal matrix with $c''(x_i) / \left[b''\left(b'^{-1}(c'(x_i))\right)\right] < 0$ on the diagonal of row i. $\mathbf{H}\left[\phi^{BKC}(\mathbf{x})\right]$ is negative definite if all its eigenvalues are negative. Since all terms in the diagonal matrix $\mathbf{D}$ are negative, a sufficient condition for $\mathbf{H}\left[\phi^{BKC}(\mathbf{x})\right]$ to be negative definite is that $-(\mathbf{I} + \delta\mathbf{G})$ is negative definite, or equivalently, $\delta\lambda_{\min}(G) + 1 > 0$, which is assumed in Proposition 3. □

Proof of Proposition 4: Define $f(x, t) := b(x + t) - c(x)$, $x \geq 0, t \geq 0$. By Assumptions 1 and 2, $f(\cdot)$ is strictly concave in x and $f_x(0, t) = b'(t) - c'(0) > 0$. Let $\theta(t)$ denote the unique maximizer of the following optimization problem

$$\max_{x \geq 0} f(x, t) = b(x + t) - c(x).$$

Clearly, $\theta(t) > 0$ and satisfies the FOC with equality, $f_x(\theta(t), t) = 0$, i.e., $b'(\theta(t) + t) = c'(\theta(t))$. Moreover $f_{xt} = b''(x + t) < 0$, so $\theta(t)$ is strictly decreasing in t.

Suppose $\mathbf{x}^*$ is a Nash equilibrium. We have

$$x_i^* = \theta\left(\sum_{k\in N_i} x_k^*\right),$$

which is clearly positive. This shows item (i).

For item $2(i)$, if $N_i \subseteq (\subset) N_j$, then $\sum_{k\in N_i} x_k^* \leq (<) \sum_{k\in N_j} x_k^*$ since $x_k^* > 0$ for each k by item (i). Therefore, $x_i^* = \theta(\sum_{k\in N_i} x_k^*) \geq (>) \theta(\sum_{j\in N_j} x_j^*) = x_j^*$, as $\theta(\cdot)$ is strictly decreasing.

For item $2(ii)$, note that the FOC for player i implies that

$$b'(x_i^* + \sum_{k\in N_i} x_k^*) = c'(x_i^*).$$

Similarly, for j, we have

$$b'(x_j^* + \sum_{k\in N_j} x_k^*) = c'(x_j^*).$$

Now suppose $\bar{N}_i \subseteq (\subset) \bar{N}_j$, then $x_i^* + \sum_{k\in N_i} x_k^* \leq (<) x_j^* + \sum_{k\in N_j} x_k^*$. Since $b(\cdot)$ is concave, so $b'(\cdot)$ is strictly decreasing, $b'(x_i^* + \sum_{k\in N_i} x_k^*) \geq (>) b'(x_j^* + \sum_{k\in N_j} x_k^*)$. Therefore, $c'(x_i^*) \geq (>) c'(x_j^*)$, or equivalently $x_i^* \leq (<) x_j^*$.

Consider, now, the complete network in item (iii). We have $\bar{N}_i = \bar{N}_j$ equals the whole network, so in any NE, the effort must be symmetric by item (ii). The common effort x^* must satisfy $b'(nx) = c'(x)$, which has a unique and finite solution by Assumption 1.

For the star network, let agent 1 denote the star, and agents $2, 3, \cdots, n$ the periphery nodes. In any NE, since each of the periphery node is only connected to the center, by item $2(i)$, all the periphery nodes choose the same efforts. Moreover, $\bar{N}_1 = \{1, 2, \cdots, n\} \supsetneq \bar{N}_2 = \{1, 2\}$ (recall $n \geq 3$), so by item $2(ii)$, $x_1^* < x_2^* = \cdots x_n^*$. □

Proof of Corollary 1: First, let us state some properties of NSG (König et al., 2014). We can always partition nodes into different classes D_k.

Suppose nodes i and j belong to the same class D_k (which implies that $d_i = d_j$), we would like to show that they must have the same effort. Suppose this class D_k is a dominant set, then $\bar{N}_i = \bar{N}_j$; hence by item (ii) of Proposition 4, $x_i^* = x_j^*$. If the class D_k is in the independent class, then $N_i = N_j$, so by item (i) of Proposition 4, , $x_i^* = x_j^*$.

Now consider a pair of nodes i, j with $d_i < d_j$, so that they cannot belong to the same class. Then either $N_i \subsetneq N_j$ or $\bar{N}_i \subsetneq \bar{N}_j$ by the definition of NSG (König et al., 2014). Moreover, any of i' neighbor is necessarily j's neighbor from the definition of NSG, and j has a neighbor m, which is not directly linked to i. In other words, $N_i \subset N_j$, implying $x_i^* < x_j^*$ by item (ii) of Proposition 4. □

Proof of Proposition 5: Program (36) is equivalent to:

$$\min_{\mathbf{q}\geq 0} \phi^{ALD}(\mathbf{q}) \tag{46}$$

where $\phi^{ALD}(\mathbf{q})$ is defined in (37), that is,

$$\phi^{ALD}(\mathbf{q}) = -\sum_i \int_{q_i^0}^{q_i} \{\zeta_i(w_i - z_i) - w_i\}\, dz_i + \frac{1}{2}\delta \sum_{i=1}^{n}\sum_{j=1}^{n} g_{ij} q_i q_j.$$

The Hessian matrix of $\phi^{ALD}(\mathbf{q})$, by direct computation, is given by:

$$\mathbf{H}\left[\phi^{ALD}(\mathbf{q})\right] = diag\{\zeta_1'(w_1 - q_1), \cdots, \zeta_n'(w_n - q_n)\} + \delta\mathbf{G} \tag{47}$$

$$= diag\{\zeta_1'(w_1 - q_1), \cdots, \zeta_n'(w_n - q_n)\} + \delta\lambda_{\min}(\mathbf{G})\mathbf{I} + \delta\mathbf{G} - \delta\lambda_{\min}(\mathbf{G})\mathbf{I}.$$

The smallest eigenvalue of $(\delta\mathbf{G} - \delta\lambda_{\min}(\mathbf{G})\mathbf{I})$ is 0. Hence, $(\delta\mathbf{G} - \delta\lambda_{\min}(\mathbf{G})\mathbf{I})$ is positive semidefinite. We need to show that $\{diag\{\zeta_1'(w_1 - q_1), \cdots, \zeta_n'(w_n - q_n)\} + \delta\lambda_{\min}(G)\mathbf{I}\}$ is positive definite. Observe that $\zeta_i'(w_i - q_i) = \frac{1}{1-\gamma_i'}$. Then, if for each $i = 1, ..., n$,

$$\zeta_i'(w_i - q_i) + \delta\lambda_{\min}(\mathbf{G}) = \frac{1}{1-\gamma_i'} + \delta\lambda_{\min}(\mathbf{G}) > 0,$$

$\mathbf{H}\left[\phi^{ALD}(\mathbf{q})\right]$ is positive definite. So $\phi^{ALD}(\mathbf{q})$ is strictly convex. Thus, the solution to (46) is unique. □

(Not-for-Publication) Online Appendix for Network Games Made Simple

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A More on rectangular domains

Given a nonempty closed and convex set K, we say that K is a Sign Equivalent Domain (SED) or SET invariant if $VI(K, \mathbf{F},)$ and $VI(K, \tilde{\mathbf{F}})$ has the same solution set whenever $\mathbf{F}$ and $\tilde{\mathbf{F}}$ are sign-equivalent.

Theorem A1. *A nonempty closed and convex set K is SED if and only if K is a rectangular region.*

Proof of Theorem A1: The if direction is shown in Theorem 1 (ii).

Consider the only if direction. Suppose K is a SED. If K equals the whole space R^n, which is rectangular, we are done. For now, let us assume K is not equal to R^n. We need the following Claim.

Claim #. *Suppose K is a SED. For any $\mathbf{x} \notin K$, there exists a rectangular region $Rec(\mathbf{x})$ such that $\mathbf{x} \notin Rec(\mathbf{x})$ but $K \subseteq Rec(\mathbf{x})$.*

Fix any $\mathbf{x} \notin K$, let $\mathbf{y}$ be the orthogonal projection of $\mathbf{x}$ on K, i.e., $\mathbf{y} = Proj_K[\mathbf{x}] = \arg\min_{\mathbf{z}\in K} ||\mathbf{z} - \mathbf{x}||$. Then, geometrically, we obtain

$$\langle (\mathbf{y} - \mathbf{x}), (\mathbf{z} - \mathbf{y}) \rangle \geq 0, \forall \mathbf{z} \in K.$$

In other words, $\mathbf{y}$ is a solution to $VI(K, \mathbf{F})$,where the mapping $\mathbf{F}(z) = \underbrace{\mathbf{y} - \mathbf{x}}_{:=\mathbf{w}=(w_1,\cdots,w_n)}$ is a constant mapping. Pick $t_i > 0$ for all i. Consider another constant mapping $\mathbf{F}' = \mathbf{w}' = (t_1 w_1, \cdots, t_n w_n)$, which is sign equivalent to $\mathbf{F}$. Since K is SED, $\mathbf{y}$ must be a solution to $VI(K, \mathbf{F}')$, i.e.,

$$\langle \mathbf{w}', (\mathbf{z} - \mathbf{y}) \rangle = \sum_j t_j (y_j - x_j)(z_j - y_j) \geq 0, \forall \mathbf{z} \in K.$$

Since the inequality holds for any positive $\{t_i\}$, we must have

$$\begin{cases} z_i - y_i \geq 0 & \text{if } y_i - x_i > 0, \\ z_i - y_i \leq 0 & \text{if } y_i - x_i < 0. \end{cases}$$

Define a closed rectangular region $Rec(x) = \prod_i \mathcal{I}_i(x)$ where

$$\mathcal{I}_i(x) = \begin{cases} [y_i, +\infty) & \text{if } y_i - x_i > 0; \\ (-\infty, y_i] & \text{if } y_i - x_i < 0; \\ (-\infty, +\infty) & \text{if } y_i - x_i = 0. \end{cases} \tag{A1}$$

Therefore, $z_i \in \mathcal{I}_i(x)$ by (A1) for all i, i.e., $\mathbf{z} \in Rec(x)$. Since it holds for any $\mathbf{z} \in K$, we have $K \subseteq Rec(\mathbf{x})$. Also, since $\mathbf{x} \neq \mathbf{y}$ (otherwise $\mathbf{x}$ would be in K), it must be the case that, for some i, either $y_i < x_i$ or $y_i > x_i$. In either case, x_i is not in $\mathcal{I}_i(x)$ by the definition of $\mathcal{I}_i(x)$ in (A1). Therefore, Claim # is proved.

Claim # is true for any x not in K. We now have the following relationships

$$K \subseteq \cap_{x \notin K} Rec(\mathbf{x}) \subseteq K,$$

while the last inequality follows because for any $\mathbf{x} \notin K$, $\mathbf{x} \notin Rec(\mathbf{x})$, so $K^c \subseteq \cup_{\mathbf{x} \in K^c} Rec(\mathbf{x})^c$, implying $(\cup_{\mathbf{x} \in K^c} Rec(\mathbf{x})^c)^c = \cap_{\mathbf{x} \notin K} Rec(\mathbf{x}) \subseteq K$. Here a set S's complement is denoted as S^c. As a result, $K = \cap_{\mathbf{x} \notin K} Rec(\mathbf{x})$. Since the intersection of closed rectangular regions is a closed rectangular region, we have now shown that K is rectangular. □

Example A1. *Consider* $\mathbf{F}^a(\mathbf{x}) = \begin{pmatrix} a(x_1 - 2) \\ (x_2 - 2) \end{pmatrix}$*, parametrized by* $a > 0$*. It is easy to see that, for any* $a', a'' > 0$*,* $\mathbf{F}^{a'}(\mathbf{x})$ *is sign equivalent to* $\mathbf{F}^{a''}(\mathbf{x})$ *at any* $\mathbf{x} = (x_1, x_2) \in R^2$*.*

(i) Let $K_1 := \{\mathbf{x} = (x_1, x_2) \in R^2 : x_1^2 + x_2^2 \leq 1\}$*. Then,* $Sol(K_1, \mathbf{F}^a)$ *varies with* a*.*[1]

(ii) Let $K_2 := [-1, 1] \times [-1, 1]$*. Then,* $Sol(K_2, \mathbf{F}^a)$ *does not vary with* a*.*[2]

Proof of results in Example A1: The Jacobian matrix $\mathbf{F}^a$ is $\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix}$, which is symmetric and positive definite. Thus $\mathbf{F}^a$ is integrable. By Theorem 2, solving $VI(K, \mathbf{F})$ is equivalent to solving

$$\min_{x \in K} \phi(x_1, x_2) := \frac{a}{2}(x_1 - 2)^2 + \frac{1}{2}(x_2 - 2)^2. \tag{A2}$$

By strict convexity of $\phi(x_1, x_2)$, the minimizer must exist and it is unique whenever the domain K is compact and convex.

Consider the domain K_1. For any $a > 0$, the unique solution to $VI(K_1, \mathbf{F}^a)$ is $(\sin(t^*), \cos(t^*))$, where $t^*(a)$ is the the unique solution to

$$a(\sin(t) - 2)\cos(t) - (\cos(t) - 2)\sin(t) = 0,$$

on the interval $[0, \pi/2]$. As a increases from 0^+ to $+\infty$, t^* increases from 0 to $\pi/2$. For example, when $a = 1$, $t^* = \pi/4$, so $\mathbf{x}^*$ is equal to $(\sqrt{1/2}, \sqrt{1/2})$. The result in case (1) is consistent with Theorem 1(i) as the solution is always on the boundary of K_1.

[1] Indeed, for each $a > 0$, the solution of $VI(K_1, \mathbf{F}^a)$ is unique. Furthermore,

- when $a = 1$, the unique solution of $VI(K_1, \mathbf{F}^a)$ is $(\sqrt{1/2}, \sqrt{1/2})$;
- when $a \to 0^+$, the unique solution approaches $(0, 1)$;
- when $a \to +\infty$, the solution approaches $(1, 0)$.

[2] Indeed, $(1, 1)$ is the unique solution to $VI(K_2, \mathbf{F}^a)$ for any a.

Now consider the domain K_2. We directly verify that $x^* = (1,1)$ is the unique solution for any $a > 0$:

$$\langle \mathbf{x} - \mathbf{x}^*, \mathbf{F}^a(\mathbf{x}^*) \rangle = \underbrace{(x_1 - 1)}_{\leq 0} a(1-2) + \underbrace{(x_2 - 1)}_{\leq 0}(1-2) \geq 0, \quad \forall\, x \in K_2 = [-1,1] \times [-1,1],$$

as $x_1 \in [-1,1], x_2 \in [-1,1]$. In particular, the solution $\mathbf{x}^*$ does not vary with $a > 0$. The result in case (2) is consistent with Theorem 1(ii) as K_2 is a rectangular domain. □

Example A1 does not contradict Theorem 1, since for any $a > 0$, the solution to $VI(K, \mathbf{F}^a)$ lies on the boundary of K. Example A1 also demonstrates that the solution of VI on the boundary of K may not be invariant under sign equivalence, if we drop the rectangularity of K.

B Different classes of matrices

Let $\mathcal{M}^{n\times n}$ denote the set of $n-$dimensional square matrices. We introduce some families of matrices used in the paper.

Definition B5.

1. *A symmetric matrix $\mathbf{S}$ is such that $\mathbf{S}' = \mathbf{S}$. We denote by $\mathcal{S}$ the set of symmetric matrices: $\mathcal{S} = \{\mathbf{S} \in \mathcal{M}^{n\times n} | \mathbf{S}' = \mathbf{S}\}$.*

2. *A diagonal matrix $\mathbf{D}$ is a matrix in which all off-diagonal entries are zero. We denote by $\mathcal{D}$ the set of diagonal matrices, by $\mathcal{D}^+$, the set of diagonal matrices with positive diagonal entries, and by $\mathcal{D}^0$, the set of diagonal matrices with nonnegative diagonal entries.*

3. *A symmetric matrix $\mathbf{Q}$ is a $\mathbf{P}-$matrix if it is positive definite.*[3] *Equivalently, $\mathbf{Q}$ is a $\mathbf{P}-$matrix if and only if every principal minor of $\mathbf{Q}$ is positive. We denote by $\mathcal{P}$ the set of $\mathbf{P}-$matrices, that is, $\mathcal{P} = \{\mathbf{Q} \in \mathcal{M}^{n\times n} | \forall \mathbf{x} \in R^n \backslash \{0\}, \exists k, \text{such that } \mathbf{x}_k(\mathbf{Q}\mathbf{x})_k > 0\}$.*

4. *A symmetric matrix $\mathbf{Q}$ is a $\mathbf{PD}-$matrix if it is positive definite. We denote by $\mathcal{PD}$ the set of positive definite matrices, that is, $\mathcal{PD} = \{\mathbf{Q} \in \mathcal{M}^{n\times n} | \mathbf{x}'\mathbf{Q}\mathbf{x} > 0, \forall \mathbf{x} \in R^n \backslash \{0\}\}$ and by $\mathcal{PD}^0$ the set of positive semidefinite matrices, that is, $\mathcal{PD}^0 = \{\mathbf{Q} \in \mathcal{M}^{n\times n} | \mathbf{x}'\mathbf{Q}\mathbf{x} \geq 0, \forall \mathbf{x} \in R^n\}$. Moreover, we denote by $\mathcal{SPD} = \mathcal{PD} \cap \mathcal{S}$ the set of symmetric and positive definite matrices,*[4] *and by $\mathcal{SPD}^0 = \mathcal{PD}^0 \cap \mathcal{S}$ the set of symmetric and positive semidefinite matrices.*

We collect some well-known relationships between these classes of matrices (see, for instance, Facchinei and Pang, 2007). The proofs are standard, hence omitted.

[3] A positive definite matrix is a symmetric matrix with all positive eigenvalues.

[4] Equivalently, $\mathbf{Q} \in \mathcal{SPD}$ if and only if $\mathbf{Q}$ is symmetric and all its eigenvalues are positive. Moreover, $\mathbf{Q} \in \mathcal{PD}$ (resp. $\mathcal{PD}^0$) if and only if the symmetric part of $\mathbf{Q}$, i.e., $(\mathbf{Q} + \mathbf{Q}')/2$, is in $\mathcal{SPD}$ (resp. $\mathcal{SPD}^0$).

Lemma B3.

1. $\mathcal{SPD} \subset \mathcal{PD} \subset \mathcal{P}$;

2. $\mathcal{D}^+ \subset \mathcal{D}^0 \subset \mathcal{SPD} \subset \mathcal{PD}$;

3. $\mathcal{P} \cap \mathcal{S} = \mathcal{SPD}$;

4. *If* $\mathbf{D} \in \mathcal{D}^0$, $\mathbf{Q} \in \mathcal{P}$, *then* $\mathbf{D} + \mathbf{Q} \in \mathcal{P}$;

5. *If* $\mathbf{D} \in \mathcal{D}^+$, $\mathbf{Q} \in \mathcal{P}$, *then* $\mathbf{DQ} \in \mathcal{P}$ *and* $\mathbf{QD} \in \mathcal{P}$.[5]

Lemma B4. *The following conditions are equivalent for a* $\mathbf{P}-$*matrix.*

1. $\mathbf{A}$ *is a* $\mathbf{P}-$*matrix.*

2. *Every principal minor of* $\mathbf{A}$ *has a strictly positive determinant.*

This lemma implies that, for a network g, if $\mathbf{G}$ is symmetric and $\delta > 0$, then $\mathbf{I} - \delta\mathbf{G}$ is a $\mathbf{P}-$matrix if and only if $1 - \delta\lambda_{\max}(\mathbf{G}) > 0$ and $\mathbf{I} + \delta\mathbf{G}$ is a $\mathbf{P}-$matrix if and only if $1 + \delta\lambda_{\min}(\mathbf{G}) > 0$.

C Public goods on networks: A linear public-good network game with strategic substitutes

Consider the model by Bramoullé et al. (2014). For $x_i \geq 0$, the payoff is given by:

$$u_i^{BKD}(\mathbf{x}, g) = a_i x_i - \frac{1}{2}x_i^2 - \delta \sum_j g_{ij} x_j,$$

for $\delta > 0$ and network g.[6] This is the similar to Ballester et al. (2006) except the sign of δ reversed.

Bramoullé et al. (2014) show that this game has the following exact potential function:

$$\phi(\mathbf{x}) = \sum_i (a_i x_i - \frac{1}{2}x_i^2) - \delta \sum_{ij} g_{ij} x_i x_j.$$

Moreover this $\phi(\mathbf{x})$ is concave if and only if $(\mathbf{I} + \delta\mathbf{G})$ is positive definite, which is equivalent to $1 + \delta\lambda_{\min}(\mathbf{G}) > 0$, where $\lambda_{\min}(\mathbf{G})$ is the lowest eigenvalue of $\mathbf{G}$. Under this condition, there is a unique Nash equilibrium.

[5]Note that if $\mathbf{D} \in \mathcal{D}^+$ and $\mathbf{Q} \in \mathcal{PD}$, then it is not always the case that $\mathbf{DQ} \in \mathcal{PD}$ or $\mathbf{QD} \in \mathcal{PD}$. For instance, consider $\mathbf{D} = \begin{bmatrix} 10 & 0 \\ 0 & 1/10 \end{bmatrix} \in \mathcal{D}^+$, $\mathbf{Q} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \in \mathcal{SPD}$, then $\mathbf{DQ} = \begin{bmatrix} 20 & 10 \\ 0.1 & .2 \end{bmatrix} \in \mathcal{P}$ but $DQ \notin \mathcal{PD}$.

[6]There exists game with different payoffs but the same best responses; see Bramoullé and Kranton (2007).

Remark C1. *For this type of game with strategic subsitutes, the constraint $x_i \geq 0$ may be bidding in equilibrium. But potential function/VI approach can deal with this constrain very easily. In fact, there is a unique equilibrium even if we impose $x_i \in [\underline{k}_i, \bar{k}^i]$ with $0 \leq \underline{k}_i < \bar{k}_i \leq +\infty$, as long as the potential $\phi(\mathbf{x})$ remains strictly concave.*

D Network games with strategic complementarities and multiple activities

Consider the model of Chen et al. (2018), which is a multiple-activity version of network games with the following linear quadratic payoff:[7]

$$\begin{aligned} u_i(\mathbf{x}_i, \mathbf{x}_{-i}) &= a_i^A x_i^A + a_i^B x_i^B - \left\{ \frac{1}{2}(x_i^A)^2 + \frac{1}{2}(x_i^B)^2 + \beta x_i^A x_i^B \right\} \\ &\quad + \delta \sum_{j=1}^{n} g_{ij} x_i^A x_j^A + \delta \sum_{j=1}^{n} g_{ij} x_i^B x_j^B. \end{aligned} \tag{D3}$$

where $\mathbf{x}_i = (x_i^A, x_i^B) \in R^2$, and $\mathbf{x}_{-i} := (x_1^A, x_1^B, ..., x_{i-1}^A, x_{i-1}^B, x_{i+1}^A, x_{i+1}^B, ..., x_n^A, x_n^B)$ as the decisions selected by players other than i, parameter $\delta > 0, \beta \in (-1, 1)$.

Define the mapping $F : R^{2n} \to R^{2n}$ as follows:

$$-F(\mathbf{x}^A, \mathbf{x}^B) = \left(\frac{\partial u_1}{\partial x_1^A}, \cdots, \frac{\partial u_n}{\partial x_n^B}, \frac{\partial u_1}{\partial x_1^B}, \cdots, \frac{\partial u_n}{\partial x_n^B}\right)^T.$$

Note that,

$$-F(\mathbf{x}^A, \mathbf{x}^B) = \begin{bmatrix} \alpha^A \\ \alpha^B \end{bmatrix} - \underbrace{\begin{bmatrix} \mathbf{I} - \delta \mathbf{G} & \beta \mathbf{I} \\ \beta \mathbf{I} & \mathbf{I} - \delta \mathbf{G} \end{bmatrix}}_{:=\mathbf{\Omega}} \begin{bmatrix} \mathbf{x}^A \\ \mathbf{x}^B \end{bmatrix}, \tag{D4}$$

Since the network is undirected, the matrix $\mathbf{\Omega}$ is symmetric. In fact, this game has a potential function as follows:

$$\phi(\mathbf{x}^A, \mathbf{x}^B) := \begin{bmatrix} \mathbf{x}^A & \mathbf{x}^B \end{bmatrix} \begin{bmatrix} \alpha^A \\ \alpha^B \end{bmatrix} - \frac{1}{2} \begin{bmatrix} \mathbf{x}^A & \mathbf{x}^B \end{bmatrix} \begin{bmatrix} \mathbf{I} - \delta \mathbf{G} & \beta \mathbf{I} \\ \beta \mathbf{I} & \mathbf{I} - \delta \mathbf{G} \end{bmatrix} \begin{bmatrix} \mathbf{x}^A \\ \mathbf{x}^B \end{bmatrix}. \tag{D5}$$

Chen et al. (2018) show that $\mathbf{\Omega}$ is positive definite if and only if

$$1 - |\beta| - \delta \lambda_{\max}(\mathbf{G}) > 0. \tag{D6}$$

Note that under condition (D6), ϕ is a strictly concave function.

[7]It is straightforward to extend the model in Chen et al. (2018) with two activities to any number of activities.

Corollary D3. $\mathbf{x}^*$ *is a NE in game defined by (D3) if and only if F defined in (D4) satisfies* $F(\mathbf{x}^*) = 0$. *Under condition (D6), the unique maximizer* $\mathbf{x}^* = \arg\max \phi(x)$ *defined in (D5) is the unique Nash equilibrium.*

Proof of Corollary D3: Since $\beta \in (-1, 1)$, each payoff u_i is concave in $\mathbf{x}_i$. By Lemma 1, $\mathbf{x}^*$ is a NE if and only if x^* solves in $VI(R^{2n}, F)$. Since the domain R^{2n} has no boundary point, any solution to $VI(R^{2n}, F)$ must be interior, hence $F(\mathbf{x}^*) = 0$. The second implication follows from Theorem 2, since the Jacobian of $\mathbf{F}$ is exactly $\boldsymbol{\Omega}$, which is positive definite under condition (D6). □

Remark D2. *If we impose nonnegative constrains on efforts, then the NE is equivalent to solve* $0 \leq \mathbf{x}^* \perp \mathbf{F}(\mathbf{x}^*) \geq 0$, *or* $VI(R^{2n}_+, \mathbf{F})$. *The uniqueness of equilibrium is still obtained under the same condition (D6). Chen et al. (2018) also consider the case with* $\delta < 0$, *or settings with more than two activities, cross-activity externalities, etc. It is straightforward to extend our analysis to those settings.*

E A public-good game with two different networks

Consider the network game of Section 5.3 with two networks $\mathbf{g}$ and $\mathbf{h}$ and utility function (40).

The condition (42) is equivalent to the following for q_i:

$$(q_i - q_i^*)F_i(\mathbf{x}) \geq 0, \forall x_i \in [0, w_i].$$

Define $K_i = [0, w_i]$, $F_i(\mathbf{x}) = 2q_i - (w_i - Q_{-i} + Y_{-i}) = 2q_i - w_i + \delta \sum_j g_{ij}q_j - \mu \sum_j h_{ij}(w_j - q_j)$, and $\mathbf{F} = (F_1, \cdots, F_n)^T$.

Therefore, finding a Nash equilibrium to the game in Section 5.3 is equivalent to solving the following $VI(K, \mathbf{F})$:

$$\langle (\mathbf{q} - \mathbf{q}^*), \mathbf{F}(\mathbf{q}^*) \rangle \geq 0, \forall \mathbf{q} \in [0, w_1] \times \cdots \times [0, w_n] = \prod K_i = K.$$

Note that

$$\frac{\partial \mathbf{F}}{\partial \mathbf{x}} = 2\mathbf{I} + \delta \mathbf{G} + \mu \mathbf{H}, \tag{E7}$$

which is symmetric. Therefore, $\mathbf{F}$ is integrable. Consequently, we can reformulate the above VI as the following minimization program:

$$\min \phi(\mathbf{q}), \ s.t \ \mathbf{q} \in K$$

with

$$\phi(\mathbf{q}) = \mathbf{q}^T\mathbf{q} - (\mathbf{w} + \mu \mathbf{H}\mathbf{w})^T \mathbf{q} + \frac{1}{2}\mathbf{q}^T(\delta \mathbf{G} + \mu \mathbf{H})\mathbf{q}. \tag{E8}$$

Here $\mathbf{w} = (w_1, \cdots, w_n)$ represents the vector of incomes. It's easy to check that $\nabla \phi(\mathbf{q}) = \mathbf{F}(\mathbf{q})$.

Theorem E2. *The network game with two networks* $\mathbf{g}$ *and* $\mathbf{h}$ *and utility function* (40) *has a best response potential function* $-\phi(\mathbf{q})$*, where* $\phi(\mathbf{q})$ *is given by* (E8)*.*

Note that this theorem holds without any assumption on δ, μ, $\mathbf{G}$, and $\mathbf{H}$. Since $-\phi(\mathbf{q})$ is continuous, we can always obtain a global maximum on the compact set K, which is a NE of the original game. Thus, we obtain the following result:

Proposition E1. *There exists at least one Nash Equilibrium for this game.*

Observe that $\phi(\mathbf{q})$ defined in (E8) is a quadratic function of $\mathbf{q}$. Further, the Hessian matrix of $\phi(\mathbf{q})$ is equal to $2\mathbf{I} + \delta\mathbf{G} + \mu\mathbf{H}$ (see (E7)), which is positive definite if and only if $2 + \lambda_{\min}(\delta\mathbf{G} + \mu\mathbf{H}) > 0$.

Assumption 4 (Network Regularity)**.** $2 + \lambda_{\min}(\delta\mathbf{G} + \mu\mathbf{H}) > 0$.

Theorem E3. *When Assumption 4 holds,* $\phi(\mathbf{q})$*, given by* (E8)*, is strictly convex on the rectangle region* K*. Thus, there is a unique Nash equilibrium* $\mathbf{q}^*$*, which is the global minimum of* $\phi(\mathbf{q})$ *on the region* K*.*

Whenever the solution is interior, we can have the closed-form solution of the Nash equilibrium:

$$\mathbf{q}^* = [2\mathbf{I} + (\delta\mathbf{G} + \mu\mathbf{H})]^{-1}(\mathbf{I} + \mu\mathbf{H}\mathbf{G})\mathbf{w}, \tag{E9}$$

and

$$\mathbf{y}^* = \mathbf{w}^* - \mathbf{q}^* = [2\mathbf{I} + (\delta\mathbf{G} + \mu\mathbf{H})]^{-1}(\mathbf{I} + \delta\mathbf{G})\mathbf{w}. \tag{E10}$$

Corollary E4. *Assume that both* $\mathbf{G}$ *and* $\mathbf{H}$ *are regular networks with degree* d_g*and* d_h*, respectively. Assume also that each player* i *has the same endowment* $w_i = w, \forall i$*. There exists a Nash equilibrium in which each player* i *chooses*

$$q_i^* = \frac{1 + \mu d_h}{2 + \delta d_g + \mu d_h} w \text{ and } y_i^* = \frac{1 + \delta d_g}{2 + \delta d_g + \mu d_h} w.$$

Remark E3. *When Assumption 4 is violated (that is, the network effects* δ *and* μ *are too large), there still exists at least one NE, since* $\phi(\mathbf{q})$ *is continuous and thus, we always obtain a global maximum on the compact set* K*. But a critical point of* $-\phi(\mathbf{q})$ *may be a saddle point, instead of a maximizer. Furthermore, there might be multiple equilibria.*

F Pricing equilibrium in differentiated products

The purpose of this section is not to show new results but to demonstrate the usefulness of our SET methodology for solving some known results.

Consider a market with $n \geq 2$ firms; each sells a differentiated product. Fore each firm $i = 1, 2, \cdots, n$, let p_i denote the price charged by firm i, p_{-i}, the price charged by other firms, and p_0

the price charged by the outside option, which exogenously fixed (depends on the utility obtained from the outside option). Using a discrete choice model, we can derive the demand for a firm i, which is given by $Q^i(p_i, p_{-i}, p_0)$. We have the following (standard) assumptions:

1. **Assumption E1**: $Q^i(.)$ is **log-concave** in p_i, i.e., $\frac{\partial^2 \ln Q^i}{\partial p_i^2} \leq 0$;
2. **Assumption E2**: $Q^i(.)$ is **log-supermodular** in (p_i, p_j) for $j \neq i$, i.e, $\frac{\partial^2 \ln Q^i}{\partial p_i \partial p_j} \leq 0, \forall j \neq i$;
3. **Assumption E3** (**homogeneity**) For every i and every ϵ,

$$Q^i(p_1 + \epsilon, p_2 + \epsilon, \cdots, p_n + \epsilon, p_0 + \epsilon) = Q^i(p_1, p_2, \cdots, p_n, p_0).$$

By definition of discrete choice models, Assumption E3 holds automatically (ϵ_i is a random variable that represents all else that affects consumers i's choosing product i). Assumptions E1 and E2 are satisfied for many demand functions derived from discrete choice models (Anderson et al., 1992).[8]

Consider the pricing competition game among n firms with payoff given by

$$\pi_i(p_i, p_{-i}) = (p_i - c_i)Q^i(p_i, p_{-i}), \; i = 1, 2, \cdots, n.$$

Observe that Assumption E1 implies that π_i is log-concave in p_i, hence it is quasi-concave in p_i. The existence of pricing equilibrium is obtained under standard methods by either using a fixed point argument or, under certain mild conditions, by applying classical results from supermodular games (see, e.g., Milgrom and Roberts (1990); Vives (1990); Caplin and Nalebuff (1991)).

Let us now focus on the uniqueness of a Nash equilibrium (NE) using our SET methodology. Clearly, if $\mathbf{p}^*$ is a NE, then the following first-order conditions must hold

$$\frac{\partial \pi_i}{\partial p_i} = 0 \implies Q^i + (p_i - c_i)Q^i_i = 0, \quad i = 1, 2, \cdots, n. \tag{F11}$$

where $Q^i_i := \partial Q^i / \partial p_i$. Define

$$F_i(\mathbf{p}) := p_i - c_i + \frac{Q^i}{Q^i_i} = p_i - c_i + \frac{1}{(\ln Q^i)_i} \tag{F12}$$

Clearly, the solutions to equations (F11) is the same as the solutions to

$$F_i(\mathbf{p}) = 0, \quad i = 1, 2, \cdots, n. \tag{F13}$$

To demonstrate the uniqueness of a NE, let us show that the system of equations (F13) has at most one solution.

[8] In particular, they hold for multinomial logit demands, that is,

$$Q^i(p_i, p_{-i}, p_0) = \frac{e^{\epsilon_i - p_i}}{e^{\epsilon_0 - p_0} + \sum_{k=1}^n e^{\epsilon_k - p_k}}, \; i = 1, 2, \cdots, n.$$

See Nocke and Schutz (2018, 2019) for applications of exploit best-response/transformed potential in multiproduct oligopoly competition with generalized logit and CES demands.

Lemma F5. *The mapping* $\mathbf{F} = (F_1(\mathbf{p}), F_2(\mathbf{p}), \cdots, F_n(\mathbf{p}))$ *from* $\mathbf{R}^n$ *to* $\mathbf{R}^n$ *is univalent, that is,* $\mathbf{F}(\mathbf{x}) = \mathbf{F}(\mathbf{y})$, $\forall \mathbf{x}, \mathbf{y} \in \mathbf{R}^n$ *implies* $\mathbf{x} = \mathbf{y}$.

Proof: Let $\mathbf{J}(\mathbf{p}) = (J(\mathbf{p})_{ij})$ denote the $n \times n$ Jacobian matrix of the mapping $\mathbf{F}$. We have:

$$J_{ij}(\mathbf{p}) = \frac{\partial F_i}{\partial p_j} = 1_{\{i=j\}} - \frac{(\ln Q^i)_{ij}}{[(\ln Q^i)_i]^2}.$$

Suppose, $i \neq j$, $1_{\{i=j\}} = 0$, and $(\ln Q^i)_{ij} \geq 0$ by Assumption E2. Thus, $J_{ij} \leq 0$. Moreover $(\ln Q^i)_{ii} \leq 0$ by Assumption E1, so $J_{ii} \geq 1 > 0$ for each i.

Moreover, if we take the sum of each row of $\mathbf{J}$, we obtain:

$$\sum_{j=1}^{n} J_{ij} = 1 - \frac{\sum_{j=1}^{n} (\ln Q^i)_{ij}}{[(\ln Q^i)_i]^2}.$$

By Assumption E3, we have:

$$Q^i(p_1 + \epsilon, p_2 + \epsilon, \cdots, p_n + \epsilon, p_0 + \epsilon) = Q^i(p_1, p_2, \cdots, p_n, p_0),$$

or, equivalently,

$$\ln Q^i(p_1 + \epsilon, p_2 + \epsilon, \cdots, p_n + \epsilon, p_0 + \epsilon) = \ln Q^i(p_1, p_2, \cdots, p_n, p_0).$$

Taking the derivative with respect to ϵ and let $\epsilon = 0$ yields

$$\sum_{j=1}^{n} (\ln Q^i)_j = -(\ln Q^i)_0$$

Taking derivative with respect to p_i on both sides yields

$$\sum_{j=1}^{n} (\ln Q^i)_{ji} = -(\ln Q^i)_{0i} \leq 0 \tag{F14}$$

where the last equality follows from Assumption E2. As a result,

$$\sum_{j=1}^{n} J_{ij} = 1 - \frac{\sum_{j=1}^{n} (\ln Q^i)_{ij}}{[(\ln Q^i)_i]^2} \geq 1 > 0.$$

We have shown that the off diagonal entries are non-positive, the sum of each row is strictly positive, $\mathbf{J}$ is a strictly (row) diagonally dominant matrix, therefore it is $\mathbf{P}$ matrix.[9] Since this is true for any $\mathbf{p}$, thus, by the global univalence theorem of Gale and Nikaido (1965), the mapping $\mathbf{F}$ is injective. □

We have the following straightforward uniqueness result.

[9] See Section B of the Online Appendix for the definition and various properties of the $\mathbf{P}$ matrices.

Theorem F4. *There is at most one pure strategy Nash equilibrium in the pricing game.*

Observe that, the key step to show the uniqueness of a Nash equilibrium is to rewrite the FOC in an equivalent form (SET) to apply the univalence theorem. If we, instead, define $\tilde{F}_i = Q^i + (p_i - c_i)Q^i_i$, the univalence of $\tilde{\mathbf{F}}$ will also imply the uniqueness of equilibrium but it is not clear how to show the **P** matrix property of this new mapping $\tilde{\mathbf{F}}$ using Assumptions E1-E3.

Assume that $a_i \geq c_i$, as none of the firm would charge price below the marginal cost. Assume also a price cap for each firm i equal to b_i, which may result from either vertical restriction such as RPM, or government regulation. We have the following result:

Theorem F5. *Suppose that, for each firm i, p_i is restricted to the interval $[a_i, b_i]$ with $-\infty \leq a_i < b_i \leq +\infty$. Then, there is at most one pure strategy Nash equilibrium in the pricing game with restricted prices.*

Proof: For restricted pricing game, the FOCs are not always satisfied with equality, but there exists appropriate sign restrictions on the derivative of π_i at the left and right boundary points a_i, b_i. That is,

$$\frac{\partial \pi^i}{\partial p_i} = Q^i + (p_i - c_i)Q^i_i \begin{cases} = 0 \text{ if } p^*_i \in (a_i, b_i); \\ \leq 0 \text{ if } p^*_i = a_i; \\ \geq 0 \text{ if } p^*_i = b_i. \end{cases} \tag{F15}$$

Equivalently,

$$-F_i(\mathbf{p}) := -(p_i - c_i + \frac{Q^i}{Q^i_i}) = -(p_i - c_i + \frac{1}{(\ln Q^i)_i}) = \begin{cases} = 0 \text{ if } p^*_i \in (a_i, b_i) \\ \leq 0 \text{ if } p^*_i = a_i \\ \geq 0 \text{ if } p^*_i = b_i \end{cases} \tag{F16}$$

as $Q^i_i < 0$.

The system of inequalities in (F16) is called a **nonlinear complementary problem** and denoted $CP(-\mathbf{F}(\cdot), K)$, with $K = \prod_i [a_i, b_i]$, which is a special case of **variational inequalities** (VI).

Since the Jacobian matrix **J** of mapping **F** is **P** matrix at every point, standard results of Variational Inequalities (VI) show that the solution to (F16) is either a singleton or empty. Thus, there exists at most one price equilibrium. □

G An example with different conditions for uniqueness: Comparing our approach with that of Melo (2019) and Parise and Ozdaglar (2019)

In this section, we give an example explaining the differences between our SET approach and that of Melo (2019) and Parise and Ozdaglar (2019). Consider a dyad example of peer effects with linear quadratic payoffs ($\delta > 0$); that is, the utility function is given by:

$$\begin{aligned} u_1(x_1, x_2) &= x_1 - \frac{1}{2}x_1^2 + \delta x_1 x_2, \\ u_2(x_1, x_2) &= x_2 - \frac{1}{2}x_2^2 + \delta x_2 x_1. \end{aligned}$$

The game Jacobian is

$$\mathbf{F} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} -\partial u_1/\partial x_1 \\ -\partial u_2/\partial x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 1 - \delta x_2 \\ x_2 - 1 - \delta x_1 \end{bmatrix}.$$

From Ballester et al. (2006), there is a unique interior equilibrium if and only if $0 < \delta < 1$, in which case, the Jacobian matrix of $\mathbf{F}$,

$$\mathbf{J} = \frac{\partial \mathbf{F}}{\partial \mathbf{x}} = \begin{bmatrix} 1 & -\delta \\ -\delta & 1 \end{bmatrix} = \mathbf{I} - \delta \mathbf{G}$$

is positive definite.

(1) To draw some comparison with Melo (2019), consider a modified problem where we scale player 2's payoff by $\beta > 0$, that is, $\tilde{u}_2(x_1, x_2) = \beta u_2(x_1, x_2)$. The game Jacobian now becomes

$$\tilde{\mathbf{F}} = \begin{bmatrix} \tilde{F}_1 \\ \tilde{F}_2 \end{bmatrix} = \begin{bmatrix} x_1 - 1 - \delta x_2 \\ \beta(x_2 - 1 - \delta x_1) \end{bmatrix}.$$

Clearly, $\beta > 0$ should not affect the set of equilibrium. In fact, $\mathbf{F}$ is sign equivalent to $\tilde{\mathbf{F}}$ as $\tilde{F}_2 = \beta F_2$. Thus, using our SET technique, $\delta < 1$ is still the condition for uniqueness for this modified model.

However, to obtain uniqueness of the solution to VI, Melo (2019) requires the strict monotonicity of the operator $\tilde{\mathbf{F}}$, i.e.,

$$\langle \tilde{\mathbf{F}}(\mathbf{x}) - \tilde{\mathbf{F}}(\mathbf{x}), \mathbf{x} - \mathbf{y} \rangle > 0, \forall \mathbf{x} \neq \mathbf{y}.$$

Note that the Jacobian of $\tilde{\mathbf{F}}$ is

$$\tilde{\mathbf{J}} = \frac{\partial \tilde{\mathbf{F}}}{\partial \mathbf{x}} = \begin{bmatrix} 1 & \\ & \beta \end{bmatrix} \mathbf{J} = \begin{bmatrix} 1 & -\delta \\ -\beta\delta & \beta \end{bmatrix}$$

is non symmetric. The strict monotonicity imposed by Melo (2019) requires that $\tilde{\mathbf{J}}$ (or its symmetric part $(\tilde{\mathbf{J}} + \tilde{\mathbf{J}}')/2$) is positive definite, which boils down to the following condition on δ:

$$\delta^2 < \frac{4\beta}{(1+\beta)^2}. \tag{G17}$$

For instance, when $\beta = 4$, the condition becomes $\delta < 0.8$, which is more restrictive than the original condition $\delta < 1$. Even worse, $\frac{4\beta}{(1+\beta)^2} \to 0$ as $\beta \to 0^+$ or $+\infty$. Thus, we can rule out any $\delta \in (0, 1)$ by choosing a suitable β.

(2) To draw comparison with Parise and Ozdaglar (2019), consider a modified game where player 2 is now risk-averse with payoff $\hat{u}_2(x_1, x_2) = -e^{-u_2(x_1,x_2)}$. The game Jacobian now becomes

$$\hat{\mathbf{F}} = \begin{bmatrix} \hat{F}_1 \\ \tilde{F}_2 \end{bmatrix} = \begin{bmatrix} x_1 - 1 - \delta x_2 \\ e^{-u_2(x_1,x_2)}(x_2 - 1 - \delta x_1) \end{bmatrix}.$$

In fact, $\mathbf{F}$ is sign equivalent to $\hat{\mathbf{F}}$ as $\hat{F}_2 = e^{-u_2(x_1,x_2)} F_2$. Note that the Jacobian of $\hat{\mathbf{F}}$ is

$$\hat{\mathbf{J}} = \frac{\partial \hat{\mathbf{F}}}{\partial \mathbf{x}} = \begin{bmatrix} 1 & -\delta \\ -e^{-u_2}\delta & e^{-u_2} \end{bmatrix} + \begin{bmatrix} 1 & \\ & e^{-u_2}(x_2 - 1 - \delta x_1) \end{bmatrix} \begin{bmatrix} 1 & -\delta \\ -\delta x_2 & x_2 - 1 - \delta x_1 \end{bmatrix}.$$

Although $\hat{F}_2$ is sign equivalent to F_2 as $\hat{F}_2 = e^{-u_2(x_1,x_2)} F_2$, its gradient is not sign equivalent, since $\nabla \hat{F}_2 \neq e^{-u_2(x_1,x_2)} \nabla F_2$ by the chain rule.

To obtain uniqueness of the solution to VI, Parise and Ozdaglar (2019) require that at any point (x_1, x_2) the operator $\hat{\mathbf{F}}$ is a **P-mapping** (or its Jacobian $\hat{\mathbf{J}}$ is a P-matrix[10]). We can directly check that the required condition is violated for $\delta = 0.9$ at $(x_1, x_2) = (0, 2)$. However, in the original game (and also with our SET), $\delta = 0.9$ is allowed, since we only require that $\delta < 1$.

Note that in both modified games, the corresponding pure strategy equilibrium is equal to the one in the original game, which is in consistent with our SET approach as $\mathbf{F}$, $\tilde{\mathbf{F}}$, and $\hat{\mathbf{F}}$ are all sign equivalent. Therefore, we think of our approach as complementary to that of Melo (2019) and Parise and Ozdaglar (2019) as we can first use our SET transformation and then check the uniqueness of the VI using the techniques in Melo (2019) and Parise and Ozdaglar (2019).

[10]That is, every principal minor is positive. See Definition B5 and Lemma B4 in online appendix B