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JEL Classification: E44, E62

Keywords: Incomplete Financial Markets, Debt, Interest, Growth, Ponzi Games, Heterogeneous Agents

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Public Debt and the Balance Sheet of the Private Sector*

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December 2022

Abstract

Is the interest rate of an economy naturally higher than the rate of growth? Are permanent government budget deficits sustainable? This paper argues that the answer to these questions depends on the interplay between the balance sheets of the private and public sector and the weight of the corporate sector in political decision making. We propose a simple growth model with incomplete markets and heterogeneous agents, featuring households and firms that face non-insurable idiosyncratic productivity shocks. More government debt reduces corporate leverage, increases the risk free rate r and decreases the growth rate g. An appropriate combination of public debt and taxes can implement the constrained social welfare optimum. The weight of firms in the government's welfare function determines whether $r < g$ or $r > g$ at the optimum, with quite different dynamics in these two regimes.

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1 Introduction

The rate of growth (g) and the rate of interest (r) are central for the analysis of efficiency and welfare of an economy. In this paper, we show how these two variables depend on the interplay between private and public debt and argue that fiscal policy can affect their relative magnitudes by influencing the balance sheet of the private sector through the volume of public debt and the level of taxes. In particular, we focus on two variables that influence the relationship between r and g and investigate how they impact the distributive consequences of public debt: corporate leverage and the weight of the corporate sector in governmental decision-making.

For this, we develop an analytically tractable dynamic macroeconomic model along classical lines of Merton (1971), Dumas (1989), and more recently He and Krishnamurty (2012) and Brunnermeier and Sannikov (2014), featuring incomplete financial markets and two types of risk averse agents: households and owners of firms facing idiosyncratic productivity shocks. Firms finance their investments by issuing securities that are bought by households. But financial markets are incomplete because firms' profits cannot be contracted upon. Hence, the only tradeable security that can be issued by firms (and the government) is straight debt. For the same reason, firms cannot be taxed on their profits, only on their net wealth.

In our closed economy[1], issuing public debt and distributing the proceeds to firms and households has three effects: a balance sheet effect, an interest rate effect, and a growth effect. The balance sheet effect reduces the leverage of firms and increases the incentives of firms to undertake risky investments at a given risk free interest rate. To clear the market for capital, the risk free interest rate increases. This buffers the risk that owners of firms are bearing. Finally, issuing public debt increases the aggregate wealth of the private sector, which stimulates aggregate consumption but has a negative impact on output growth. The optimal level of government debt balances these different effects. Our welfare criterion is a weighted sum of the preferences of firms' owners and households. We show that optimal fiscal policy maintains a constant debt to GDP ratio and taxes firms at a higher rate than households. At the welfare optimum, the interest rate can be greater or smaller than the growth rate, depending on the political weight of firms' owners.

This model integrates different strands of the macroeconomic literature on fiscal policy with agent heterogeneity which we review in the next section. To this literature, we make several contributions.

First, our analytically tractable dynamic model generates simple mechanics for

[1]The case of a small open economy is briefly discussed in the concluding section.

fiscal policy, interest rates, and growth[2]. Second, we add to previous work showing how issuing government bonds may improve social welfare when agents face uninsurable idiosyncratic risks. We argue that issuing public debt and distributing it to firms and households in appropriate proportions allows the government to reduce the leverage of firms and buffer their losses. Up to some level, this outweighs the growth reducing effect of increased aggregate consumption due to higher individual wealth. Since corporate bonds wash out in the aggregate balance sheet of the private sector, public bonds are needed to achieve these welfare gains.

Third, since idiosyncratic risks are only borne by firms, the preferences of firms' owners over interest rates and growth paths of the economy are different from those of households. Thus the welfare optimum depends on the weight of firms in the objective of the government.

Fourth, the implementation of the welfare optimum necessitates different tax rates for firms and households. We show that firms should be the primary recipient of the investment stimulus from public debt, but that tax rates should be lower for households than for firms.

Fifth, optimal fiscal policy targets both the growth rate and the interest rate of the economy. If the welfare weight of firms is sufficiently large, the real interest rate is higher than the real growth rate of GDP, and the opposite occurs when the welfare weight of firms is small. In the first case ($g < r$), the intertemporal budget constraint of the government binds. The value of outstanding debt at each date is equal to the net present value of all future primary surpluses. This is consistent with the arguments put forward by Cochrane (2019),(2022). We call this case the "Cochrane Case". In the second case, which we call the "Blanchard Case"[3], $r < g$ and there is a permanent and growing primary deficit, the government's intertemporal budget constraint does not bind and, perhaps surprisingly, the public debt-to-GDP ratio is small. The government optimally runs a Ponzi scheme: it repays old debt by issuing new one.

The rest of this paper is organized as follows. In the next section, we provide a more detailed discussion of the literature. The model is set out in Section 3. In Section 4, we characterize the individually optimal decisions. The equilibrium analysis is presented in Section 5, while Section 6 develops the welfare analysis. In Section 7, we discuss the welfare improvement that can be generated by public debt issuance and its implications for redistribution through taxes. The implications of fiscal policy for optimal growth, interest rates, and the government budget are explored in Section 8. Section 9 presents a brief outlook on further research. Appendix A shows how to implement aggregate consumption profiles as the general equilibrium of our model through the appropriate choice of fiscal policy. For completeness, we

[2]As we explain in the concluding section, it is tractable enough to allow several simple extensions.
[3]See Blanchard (2019).

provide detailed calculations behind the results of Section 4 in Appendix B.

2 Relation to the literature

As noted above, our paper is related to several strands of the academic literature.

First, the overlapping generation literature dating back to Diamond (1965) has examined how fiscal policy influences the relations between three crucial macroeconomic variables: the real interest rate (r), the growth rate of output (g) and the marginal product of capital (μ). This literature has recently been subject to intense debates about the sustainability of fiscal policy in the United States and other countries, focusing on when and why governments can run prolonged deficits without being forced to rely on taxation when $r < g$ – so-called "Ponzi schemes" – in the presence of uncertain production returns (Blanchard and Weil (2001), Blanchard (2019), and Jiang et al. (2019)). Abel et al. (1989) and Hellwig (2021) examine whether the conditions for dynamic inefficiency have to be based on the returns of all assets or only on the return of the safe asset.

Second, and more related to our work, a recent strand of literature re-examines this question in settings with infinitely-lived agents using continuous-time methods from asset pricing. It also provides ways of endogenizing r, g and μ. Building on the seminal contributions by He and Krishnamurty (2012), Brunnermeier and Sannikov (2014), and Di Tella (2017), this literature considers economies with aggregate risk and studies the emergence and amplifications of financial crises, as well as the role of intermediaries in this dynamic. While technically similar, our work considers neither aggregate risk nor intermediaries. The long-run focus of our work is, in fact, closer to the OLG literature discussed above, and the absence of aggregate risk in our model makes it possible to derive stationary states and to undertake an explicit welfare analysis. In this sense our work is similar to Brunnermeier et al. (2021) who focus on how to integrate a bubble term representing government expenditures – without ever raising taxes for them – into the fiscal theory of the price level in the presence of idiosyncratic risks and incomplete markets. They determine what they call the optimal"bubble mining rate", which implies an optimal rate of issuing government debt. Brunnermeier et al. (2022) extend this approach and resolve the "public debt valuation puzzle", by noting that the price of debt is procyclical, since the bubble term rises in bad times. Their focus is neither on redistribution through taxation, nor on the role of the political weight of different interest groups.

Reis (2021) considers a continuum of households hit by idiosyncratic depreciation shocks to their capital. Since households face borrowing constraints, relatively unproductive and risky firms can operate. This creates a misallocation of resources. Non-insurable idiosyncratic risks and the misallocation of resources create demand for public debt as a safe asset and as an alternative form of savings. Reis (2021) identifies the determinants of the upper limit of spending that can be financed by debt. Similar to Reis (2021), we provide a model with uninsurable idiosyncratic production risks, but different from his work, there are no borrowing constraints. To the contrary, debt markets are frictionless, but because of idiosyncratic risks firms are willing to operate the capital stock of the economy only if the interest rate is sufficiently low. Public debt is not a new asset, it is a perfect substitute for safe corporate debt. Hence, unlike in Reis (2021), issuing public debt boosts the amount of corporate equity and raises r.

Third, a sizable literature examines which role government debt can play beyond allowing firms to buffer their losses. Woodford (1990) shows how issuing highly liquid public debt can increase the flexibility of liquidity-constrained agents to respond to variations in income and spending opportunities, thereby increasing economic efficiency. Aiyagari and McGrattan (1998) develop a model in which households face a borrowing constraint, which generates a precautionary savings motive. Government debt loosens borrowing constraints and enhances the liquidity of households, which improves consumption smoothing. The authors also stress the cost of higher government debt via adverse wealth distribution, incentive effects and crowding-out effects on investment. The benefits and costs of public debt determine the optimal quantity of debt. In the presence of moral hazard for firms and of optimal contracts between firms and outside investors, Holmström and Tirole (1998) show that no public supply of liquidity is necessary as long as intermediaries coordinate the use of scarce private liquidity and no aggregate uncertainty is present. Angeletos et al. (2016) explore how debt can be used as collateral or a liquidity buffer in order to ease financial frictions. Since public debt lowers the liquidity premium but increases the cost of borrowing for the government, there exists a long-run optimal level of public debt. In our paper, there are no borrowing constraints nor liquidity constraints. Public debt allows firms to buffer their losses but they have to pay a higher interest rate on their own debt.

Fourth, a strand of macroeconomic models discusses uninsurable idiosyncratic income shocks and incomplete financial markets. Prominent references are Bewley (1983), Imrohoroğlu (1989), Huggett (1993) and in particular Aiyagari (1994). The seminal paper of Aiyagari (1994), in which households self-insure against idiosyncratic income fluctuations by buying shares of aggregated capital, is widely used to examine the impact of household heterogeneity when markets are incomplete. This literature is large and was surveyed by Heathcote et al. (2009) and Krueger et al. (2016). Recently, Krueger and Uhlig (2022) have characterized analytically the stationary equilibrium in a continuous-time version with one-sided commitment to insurance contracts. In our model, only firms are subject to uninsurable idiosyncratic productivity risks. If there is no public debt, the leverage of firms and the risk-free interest rate at equilibrium are entirely determined by the relative wealth of firms and households. By issuing public debt, the government can modify the aggregate balance sheet of the private sector and transform the portfolio problem of firms, such that the losses of owners are buffered better.

Fifth, our work is complementary to the literature initiated by the seminal contributions by Bernanke and Gertler (1989), Bernanke et al. (1996), and Kiyotaki and Moore (1997) on macroeconomic models with leveraged agents. In this literature, firms or banks need sufficient net worth to credibly commit to the repayment obligations stipulated in their credit contracts. In our paper, firms are completely free to borrow and lend in a frictionless market, but only undertake uninsurable risky investments at a given risk-free interest rate if their leverage is sufficiently low.

3 The Model

3.1 The Macroeconomic Environment

The economy features a mass 1 continuum of competitive firms, owned and controlled by their shareholders, a mass 1 continuum of households who do not own shares, and a government. Time is continuous: $t \in [0, \infty)$. There is only one physical good that can be consumed or invested and is taken as a numéraire. There is one financial asset, namely risk-free debt, that can be issued by firms and the government. This debt is real and its unit price is normalized to one: one unit of it can always be exchanged for one unit of the good, i.e. debt can be issued and retired without frictions or costs. The equilibrium between supply and demand of debt at each date t determines the interest rate at date t, denoted r_t.

Firms are run by managers who act in the interest of their shareholders. Hence, we ignore all agency conflicts between managers and shareholders, as in much of the macroeconomic literature. Firms are individually risky in the sense that they produce random output at each point in time. We assume that these random outputs are not publicly observable, which implies that firms cannot insure their risk away and that their equity cannot be traded.[4] Firms can only issue debt, which turns out to be risk free. In equilibrium, default never occurs in our model. Similarly, the government finances public expenditures by issuing risk-free debt and taxing (or subsidizing) the wealth of households and firms (i.e. their equity).

The physical good is initially held by households and firms, but only firms can invest in productive technologies. Households cannot, and so they sell their initial endowments to firms in exchange for debt. They receive interest payments

[4]So strictly speaking these are entrepreneurial firms with no outside equity. However, since there is no manager-shareholder conflict, it is straightforward to generalize the model to the case where also households can hold fully diversified portfolios of shares. This is why we sometimes call the set of all firms the "corporate sector". The important restriction is that owners of firms cannot diversify away their idiosyncratic risks by holding each others' equity.

on their savings and decide continuously how much to consume. Households are identical, and not subject to individual shocks. Without loss of generality, we can aggregate them into a single representative household (the "household sector"), denoted by the superscript H. Firms have more complex decision problems: they can continuously adjust their investments and debt levels, and decide how much dividends to pay their owners for consumption.

The government has to finance an exogenous level of public expenditures and can redistribute wealth between the two sectors, households and firms, by means of taxes and subsidies[5]. These fiscal instruments are choice variables of the government. The dynamics of government debt is determined by the difference between interest payments on outstanding debt and primary surpluses (total tax revenues minus government expenditures).[6]

3.2 The Formal Set-up

There is a continuum of firms $i \in [0, 1]$, with initial endowments (equity) $\tilde{e}^i$. Aggregate equity is denoted by

$$\tilde{E} = \int_0^1 \tilde{e}^i di.$$

The representative household has initial endowment $\tilde{H}$. At time $t = 0$, the government can redistribute initial endowments and issue debt B_0, which is distributed to households and firms. After redistribution, net wealth is H_0 for the representative household, e_0^i for firm i, and aggregate equity is

$$E_0 = \int_0^1 e_0^i di.$$

The aggregate wealth of the private sector is

$$E_0 + H_0 = K_0 + B_0,$$

where $K_0 = \tilde{E} + \tilde{H}$ is the initial stock of capital. Thus the government can modify the balance sheet of the private sector, and increase its net wealth by issuing debt. However the government cannot produce output, so the aggregate capital stock of the economy is still K_0. Just as in the famous article of Barro (1974), we will examine whether government debt as a financial asset can increase overall welfare.

At each date t, firm i chooses its volume of productive assets k_t^i, financed by equity e_t^i and debt d_t^i. Instantaneous output is

$$k_t^i[\mu dt + \sigma dz_t^i], \tag{1}$$

[5]For simplicity of exposition, we will usually use the word "taxes", with the interpretation that negative taxes are subsidies.

[6]The present paper assumes that taxes are linear wealth taxes. In a companion paper, we show that this actually implements the optimal direct mechanism if firms' output is privately observable and firms' owners can divert profits.

where $\mu > 0$ is the average instantaneous return of the corporate sector net of depreciation, $\sigma \geq 0$ is the volatility of the instantaneous return, and the z_t^i are firm-specific i.i.d. Brownian motions. Since production shocks are independent, they wash out in the aggregate, and aggregate production at time t is

$$Y_t = \mu K_t, \tag{2}$$

where K_t is aggregate capital at date t. We assume that government expenditures are an exogenous fraction γK_t of aggregate capital stock. Their share of GDP is thus an exogenous constant γ/μ.[7] Government debt evolves according to

$$\dot{B}_t = \gamma K_t + r_t B_t - T_t, \tag{3}$$

where the dot represents the time derivative, r_t is the instantaneous risk-free interest rate, and T_t is net aggregate tax revenue (tax revenue minus subsidies) at time $t > 0$.

The balance sheet equation of firm i at time t is

$$k_t^i = d_t^i + e_t^i. \tag{4}$$

We allow debt d_t^i to be negative, in which case the firm has no debt but invests in bonds issued by other firms or the government.

Given e_t^i, at each date t, firm i chooses its investment k_t^i and its dividend payout c_t^i to be consumed by its shareholders. At each date, the firm pays a linear tax $\tau_t^E e_t^i$ on its equity. The representative household chooses its consumption flow c_t^H and pays a linear tax $\tau_t^H H_t$ on its wealth. Recall that tax rates can be negative, in which case they represent subsidies. Households and shareholders maximize the expected discounted utility of consumption

$$\int_t^\infty e^{-\rho s} \log c_s^k ds, \quad k = i, H,$$

where $\rho > 0$ is the discount rate, the same for households and firms.[8]

4 Individual Decisions

We next characterize the solutions of the household's and the firms' problems. These are standard and yield well-known solutions going back to Merton (1971). For completeness, we add the detailed calculations in Appendix B.

[7]We do not model the social utility generated by these expenditures explicitly and, therefore, say nothing about their optimal level.

[8]The model can be fully solved for CRRA utilities. The analysis becomes significantly more complex, but the main result regarding the welfare improvement by issuing public debt continues to hold.

4.1 Households

Net of initial lump sum taxes, the representative household has initial net worth $H_0 > 0$ at time $t = 0$, no further income later, and saves via corporate and government bonds, which are perfect substitutes. There is no other form of savings, since the good cannot be stored.[9] Hence the household chooses a consumption path $c^H = \left(c_t^H\right)_{t\geq 0}$ that solves

$$\max_{c^H} \int_0^\infty e^{-\rho t} \log c_t^H dt,$$

subject to the equation of motion of wealth

$$\dot{H}_t = (r_t - \tau_t^H)H_t - c_t^H. \tag{5}$$

This is a standard problem. At the optimum,

$$c_t^H = \rho H_t \tag{6}$$

for all $t \in [0, \infty)$, and the value function of the household's problem is

$$\rho V^H(t, H_t) = e^{-\rho t} \log(\rho H_t) + \int_t^\infty e^{-\rho s} \left(r_s - \tau_s^H - \rho\right) ds. \tag{7}$$

Note that equations (5) and (6) imply that H_t is always positive.

4.2 Firms

To simplify the exposition, we first assume $r_t < \mu$ and then verify that this is always the case in equilibrium. With initial equity $e_0^i > 0$ at $t = 0$, the firm's flow of funds is given by

$$k_t^i[\mu dt + \sigma dz_t^i] = [r_t d_t^i + \tau_t^E e_t^i + c_t^i]dt + de_t^i, \tag{8}$$

where the left-hand side represents earnings before interest and taxes and the right-hand side is the sum of interest payments, taxes, consumption of equity holders (dividends), and the change in equity as a residual. (8) reflects the simple corporate accounting identity:

$$\text{EBIT} = \text{interest} + \text{taxes} + \text{dividends} + \text{retained earnings}.$$

The firm then chooses a path k_t^i, d_t^i, c_t^i, $t \geq 0$ that solves

$$\max_{k^i, d^i, c^i} \mathbb{E} \int_0^\infty e^{-\rho s} \log c_s^i ds$$

[9] Our results continue to hold when the good can be stored and the real interest rate is larger than the return from storage.

subject to the balance sheet constraint (4) and the law of motion (8) for each $t \geq 0$. The Bellman Equation yields the standard solution

$$\begin{aligned} c_t^i &= \rho e_t^i & (9) \\ k_t^i &= \frac{\mu - r_t}{\sigma^2} e_t^i & (10) \end{aligned}$$

and the stochastic law of motion for firm equity

$$de_t^i = \left[\left(\frac{\mu - r_t}{\sigma} \right)^2 + r_t - \tau_t^E - \rho \right] e_t^i dt + \frac{\mu - r_t}{\sigma} e_t^i dz_t^i. \tag{11}$$

Condition (10) implies that the capital-to-equity ratio k_t^i / e_t^i is identical across firms. Firms continuously adjust their debt levels, but they all keep the capital-to-equity ratio at the same value

$$x_t \equiv \frac{k_t^i}{e_t^i} = \frac{\mu - r_t}{\sigma^2} \tag{12}$$

Condition (10) also implies that if we had $k_t^i \leq 0$ for one i, this would hold for all i and therefore yield $K_t \leq 0$ in the aggregate, which justifies our initial assumption $r_t < \mu$. Further, note that default never occurs. Indeed, an application of Itô's Lemma gives

$$d \log(e_t^i) = \frac{de_t^i}{e_t^i} - \frac{\sigma^2 x_t^2}{2} dt = \left(r_t - \tau_t^E - \rho + \frac{\sigma^2 x_t^2}{2} \right) dt + \sigma x_t dz_t^i, \tag{13}$$

which implies

$$e_t^i = e_0^i exp \left(\int_0^t (r_s - \tau_s^E - \rho + \frac{\sigma^2 x_s^2}{2}) ds + \sigma \int_0^t x_s dz_s^i \right) > 0. \tag{14}$$

The value function is the same for all firms (because they all face the same tax rate), and by (13) equal to

$$\rho V^E(t, e) = e^{-\rho t} \log(\rho e) + \int_t^{\infty} e^{-\rho s} \left(r_s - \tau_s^E - \rho + \frac{\sigma^2}{2} x_s^2 \right) ds. \tag{15}$$

Note the similarity with the value function for households, the difference being the last term in the integral, which comes from the optimization of investment in the risky technology.

5 The Macroeconomic Equilibrium

5.1 Aggregates

By (5) and (6), households' aggregate wealth H_t follows the law of motion

$$\dot{H}_t = \left(r_t - \tau_t^H - \rho \right) H_t. \tag{16}$$

Hence, aggregate household wealth is always strictly positive. This wealth is entirely invested in risk free debt, and the household is indifferent between public debt and corporate debt. Let D_t^H and B_t^H denote the households' holdings of private and public debt, respectively. This yields the households' balance sheet

$$H_t = D_t^H + B_t^H \tag{17}$$

Individual balance sheets of firms follow random trajectories but, thanks to the Law of Large Numbers, the aggregate balance sheet of the corporate sector is deterministic. Denoting by B_t^E the firms' aggregate holdings of public debt (which may be negative), it is simply given by:

Assets	Liabilities
K_t	D_t^H
B_t^E	E_t

(18)

Aggregating individual investment rules (10) yields aggregate capital as

$$K_t = \frac{\mu - r_t}{\sigma^2} E_t, \tag{19}$$

which produces gross domestic product Y_t, as defined in (2). Note that $K_t > 0$ at all times, which implies that $r_t < \mu$, as we have assumed.

By the individual laws of motion of firm equity (11) and the Law of Large Numbers, the equation of motion of aggregate corporate equity is

$$\begin{aligned} \dot{E}_t &= \int_0^1 \left(\left(\frac{\mu - r_t}{\sigma} \right)^2 + r_t - \tau_t^E - \rho \right) e_t^i di \\ &= \left(\left(\frac{\mu - r_t}{\sigma} \right)^2 + r_t - \tau_t^E - \rho \right) E_t & (20) \\ &= \left(r_t - \tau_t^E - \rho \right) E_t + (\mu - r_t) K_t, & (21) \end{aligned}$$

where the last equality follows from (19). As noted in Section 4, all firms have the same leverage target and they adjust their debt continuously: after a positive productivity shock, they invest more and issue more debt; after a negative shock, they do the opposite. This leads to the aggregate law of motion of equity (20).

Government debt is $B_t = B_t^H + B_t^E$ and evolves according to (3), $\dot{B}_t = \gamma K_t + r_t B_t - T_t$, where aggregate tax receipts (or subsidy expenditures if negative) at date $t > 0$ are given by

$$T_t = \tau_t^H H_t + \tau_t^E E_t. \tag{22}$$

Note that we allow B_t to be negative, but this will never be optimal.

5.2 Fiscal Policy and Equilibrium

Equilibrium requires markets to clear at all times, given the fiscal policy in place. Here, fiscal policy consists of two parts:

- at date 0, the government issues debt B_0 and distributes lump-sum subsidies L^H to households and L^E to firms,
- at all further dates $t > 0$ the government collects instantaneous wealth taxes at rates τ_t^H for households and τ_t^E for firms.

The government cannot create real goods, but it can boost the private sector's balance sheet by creating paper assets. Correspondingly, the government's balance sheet identity at $t = 0$ is $B_0 = L^H + L^E$.

In equilibrium, the interest rate path r_t makes the aggregate balance sheet constraint of the economy hold at each point in time t. Consolidating the aggregate firm balance sheet (18) with the households' balance sheet equation (17) yields the private sector's balance sheet

Assets	Liabilities
K_t	H_t
B_t	E_t

(23)

Note that the aggregate balance sheet is deterministic – there is no aggregate risk in our economy. In equilibrium, all changes must be consistent:

$$\dot{K}_t + \dot{B}_t = \dot{H}_t + \dot{E}_t \tag{24}$$

for all t. Using (16), (19), (21), (3), and (22), condition (24) can be written as

$$\begin{aligned} \dot{K}_t &= \dot{H}_t + \dot{E}_t - \dot{B}_t \\ &= \left(r_t - \tau_t^H - \rho\right) H_t + \left(r_t - \tau_t^E - \rho\right) E_t + (\mu - r_t)K_t - \gamma K_t - r_t B_t + T_t \\ &= (\mu - \gamma)K_t - \rho(H_t + E_t), \end{aligned} \tag{25}$$

which is the economy's IS equation (equality of investment and net savings).

At each date t, the four aggregate variables K_t, B_t, E_t, H_t are linked by the balance sheet identity (23). In fact, by the homogeneity of the firms' investment problem, only two state variables are sufficient: the capital-equity ratio x_t as defined in (12), and $h_t = \frac{H_t}{E_t}$, the ratio of household wealth over firm equity. Note that $x_t > 1$ if and only if $D_t^H - B_t^E > 0$, i.e. if firms are net borrowers. In this case, $x_t - 1$ is the firms' debt to equity ratio. For simplicity of exposition, we will often refer to x_t as corporate leverage. If $x_t < 1$, firms have zero leverage and are net lenders.[10]

The trajectories of the two state variables (x_t, h_t) completely determine all aggregate variables (output, consumption, and investment) in equilibrium.[11] In fact,

[10] This can only happen if $B_t > H_t$, which means that public debt exceeds the total wealth of households.

[11] Equation (14) shows the dynamics of individual equity positions, which depends on idiosyncratic shocks dz_t^i.

by (25), the equilibrium growth rate g_t of capital (and thus GDP) is

$$\begin{aligned} g_t = \frac{\dot{K}_t}{K_t} &= \mu - \gamma - \rho \frac{H_t + E_t}{K_t} \\ &= \mu - \gamma - \rho \frac{h_t + 1}{x_t}. \end{aligned} \tag{26}$$

By (16), aggregate household wealth grows according to

$$\frac{\dot{H}_t}{H_t} = \mu - \rho - \tau_t^H - \sigma^2 x_t \tag{27}$$

and aggregate equity, by (20), according to

$$\frac{\dot{E}_t}{E_t} = \mu - \rho - \tau_t^E - \sigma^2 x_t (1 - x_t).$$

Finally, by (3) and (22), the evolution of government debt B_t is given by

$$\frac{\dot{B}_t}{B_t} = \mu - \sigma^2 x_t + \frac{\gamma x_t - \tau_t^H h_t - \tau_t^E}{1 + h_t - x_t}$$

as long as $B_t \neq 0$, i.e. as long as $x_t \neq h_t + 1$, and by (3) and (22) for all points (x_t, h_t) with $x_t = h_t + 1$. Given this direct relation between equilibria and the $x_t - h_t$ - trajectories, it is useful to characterize the dynamic system (x_t, h_t) in more detail.

The initial values of the state variables are given by the government lump sum transfers at date 0:

$$h_0 = \frac{H_0}{E_0} = \frac{\tilde{H} + L^H}{\tilde{E} + L^E}, \tag{28}$$

$$x_0 = \frac{K_0}{E_0} = \frac{\tilde{H} + \tilde{E}}{\tilde{E} + L^E}. \tag{29}$$

The dynamics of the state variables for $t > 0$ is then determined by the instantaneous tax rates. Indeed, the definitions of x_t and h_t imply:

$$\dot{h}_t = \left(\frac{\dot{H}_t}{H_t} - \frac{\dot{E}_t}{E_t} \right) h_t, \tag{30}$$

$$\dot{x}_t = \left(\frac{\dot{K}_t}{K_t} - \frac{\dot{E}_t}{E_t} \right) x_t. \tag{31}$$

By (16), (20), and (38),we have

$$\dot{h}_t = (\tau_t^E - \tau_t^H - \sigma^2 x_t^2) h_t. \tag{32}$$

Similarly, using (25) in (31) yields

$$\dot{x}_t = (\sigma^2 x_t^2 - \rho)(1 - x_t) + (\tau_t^E - \gamma) x_t - \rho h_t. \tag{33}$$

If the system (32) and (33) has a solution that stays in the interior of the positive (x, h) quadrant, then this solution yields an equilibrium of our economy, as shown above. Conversely, any equilibrium of our economy yields a solution of (32) and

(33) in the interior of the positive quadrant.[12] Going one step further, an inspection of (28)-(29) and (32)-(33) shows that any differentiable trajectory of (32) and (33) in the interior of the positive (x, h) quadrant can be obtained by an appropriate fiscal policy:

- The lump sum transfers L^E and L^H are determined by initial values (x_0, h_0):

$$L^E = \frac{1}{x_0}(\tilde{H} + \tilde{E}) - \tilde{E}, L^H = \frac{h_0}{x_0}(\tilde{H} + \tilde{E}) - \tilde{H}. \tag{34}$$

- Instantaneous tax rates are given by

$$\tau_t^E = \gamma + \sigma^2 x_t(x_t - 1) + \rho(\frac{1 + h_t}{x_t} - 1) + \frac{\dot{x}_t}{x_t}, \tag{35}$$

$$\tau_t^H = \tau_t^E - \sigma^2 x_t^2 - \frac{\dot{h}_t}{h_t}. \tag{36}$$

Thus we have established:

Proposition 1. *For any differentiable trajectory (x_t, h_t) in $\mathbb{R}^2_{++}$ there is a choice of fiscal policy (L^E, L^H) and (τ_t^E, τ_t^H) such that the general equilibrium under this policy exists, is unique, and generates (x_t, h_t).*

Note that the converse of Proposition 1 is not true: not every choice of fiscal policy (L^E, L^H) and (τ_t^E, τ_t^H) is sustainable, in the sense that it yields a dynamic system whose solution stays in $\mathbb{R}^2_{++}$ forever.

The simple characterization of equilibrium through trajectories $(x_t, h_t) \in \mathbb{R}^2_{++}$ makes it possible to describe some key policy variables and relations succinctly. In fact, by (23), public debt at time t is positive iff

$$1 + h_t - x_t > 0.$$

We will refer to the locus of points $(x, h) \in \mathbb{R}^2_+$ with $1+h-x = 0$ as the "Zero-Debt Line" (ZDL).

The government debt-to-GDP ratio at date t can be expressed as

$$\delta_t \equiv \frac{B_t}{Y_t} = \frac{1 + h_t - x_t}{\mu x_t}. \tag{37}$$

Simple calculations then yield explicit relations between the main aggregate variables of our economy, which we collect in the following proposition.

Proposition 2. *In equilibrium, at any date t:*

1. *The interest rate is a linearly decreasing function of firm leverage:*

$$r_t = \mu - \sigma^2 x_t. \tag{38}$$

[12] Note that by (32) the trajectory never hits the x_t-axis. Constellations in which it hits the h_t-axis in finite time are uninteresting.

2. *Output growth is a linearly decreasing function of the debt-to-GDP ratio:*

$$g_t = \mu - \gamma - \rho - \rho\mu\delta_t. \tag{39}$$

3. *The interest rate is smaller than the growth rate, $r_t < g_t$, if and only if*

$$\gamma x_t + \rho(h_t + 1) < \sigma^2 x_t^2. \tag{40}$$

As an illustration, Figure 1 shows a particular trajectory of the state variables (32)-(33) (in blue) in a diagram showing the interest-growth boundary (40) (in magenta). In this equilibrium, the economy starts out with zero private debt ($x_0 = 1$), positive public debt, and $r_t > g_t$. It then increases private debt and reduces public debt until it crosses the Zero-Debt Line when public debt becomes negative, and finally reaches the region where $r_t < g_t$. The equilibrium corresponds to stationary tax rates $\tau^E = 0.3$ and $\tau^H = 0.2$. Note that in this equilibrium, the share of household wealth in total private wealth, $H_t/(H_t + E_t)$, is initially increasing and then converges monotonically to 0. This does not necessarily mean, however, that household wealth decreases in absolute terms, i.e. that households are becoming worse off over time. In fact, an inspection of (27) shows that this depends on the productivity of capital μ, and occurs if and only if μ is sufficiently small.

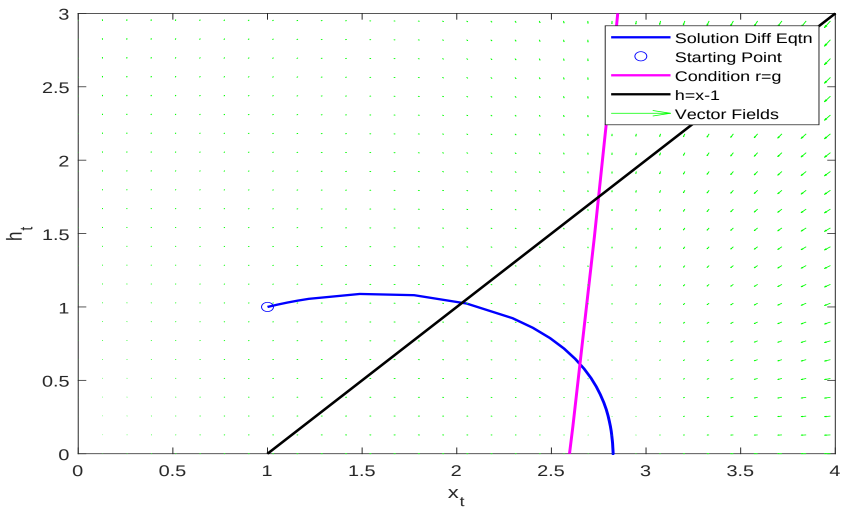

Figure 1: A (x_t, h_t) trajectory for $\rho = 0.01, \sigma = 0.2, \gamma = 0.1$.

6 Welfare

We have seen that any general equilibrium with linear taxes defines a differentiable trajectory $(x_t, h_t)_{t=0}^{\infty}$ in the strictly positive quadrant, and that any such trajectory can be implemented as a general equilibrium of the economy with an appropriate fiscal policy. In order to assess the intertemporal preferences of households over fiscal policies, we therefore express the value functions (7) and (15) as a function of these trajectories.[13]

6.1 Indirect Utilities

Given equation (6), the optimal consumption rule of households, their value function as of time 0 is

$$\begin{aligned}\rho V^H(0, H_0) &= \rho \int_0^\infty e^{-\rho t} \log(\rho H_t) dt \\ &= \rho \int_0^\infty e^{-\rho t} (\log(\rho E_t) + \log h_t) dt \end{aligned} \tag{41}$$

by the definition of h_t.

Equityholders' utilities depend on the way initial capital is shared between them. When it is shared equally, they all have the same expected utility at date 0, namely, using Equation (20),

$$\begin{aligned}\rho V^E(0, E_0) &= \log(\rho E_0) + \int_0^\infty e^{-\rho t} \left[\frac{\dot{E}_t}{E_t} - \frac{\sigma^2}{2} x_t^2\right] dt \\ &= \rho \int_0^\infty e^{-\rho t} \left[\log(\rho E_t) - \frac{\sigma^2}{2\rho} x_t^2\right] dt \end{aligned} \tag{42}$$

By the definition of K_t and its law of motion (26), we have

$$\begin{aligned}\rho \int_0^\infty e^{-\rho t} \log E_t dt &= \rho \int_0^\infty e^{-\rho t} [\log K_t - \log x_t] dt \qquad (43) \\ &= \log K_0 + \frac{1}{\rho}(\mu - \gamma) - \rho \int_0^\infty e^{-\rho t} \left[\frac{h_t + 1}{x_t} + \log x_t\right] dt\end{aligned}$$

Inserting (43) into (41) and (42),

$$\begin{aligned}\rho V^H &= \log(\rho K_0) + \frac{\mu - \gamma}{\rho} \\ &\quad -\rho \int_0^\infty e^{-\rho t} \left[\frac{h_t + 1}{x_t} + \log x_t - \log h_t\right] dt \\ \rho V^E &= \log(\rho K_0) + \frac{\mu - \gamma}{\rho} \\ &\quad -\rho \int_0^\infty e^{-\rho t} \left[\frac{h_t + 1}{x_t} + \log x_t + \frac{\sigma^2}{2\rho} x_t^2\right] dt.\end{aligned}$$

This shows that households and firms have very different preferences over the trajectories of (x_t, h_t). Households' utility is maximum when x_t, h_t go to infinity such that $h_t < x_t < h_t + 1$. Equityholders, on the other hand, achieve maximum utility when $h_t = 0$ and x_t equals the unique positive solution x_{min} of the equation

$$\sigma^2 x^3 + \rho x - \rho = 0. \tag{44}$$

[13] The general welfare optimization problem consists of two parts. First, optimal consumption-production plans must be identified for all agents regardless of policy and prices, and second, one must show that this optimum can be implemented by linear wealth taxes. In Appendix B, we show how to implement arbitrary **aggregate** consumption profiles for both types of agents by general equilibrium with linear wealth taxes, which makes it possible to use the approach of the previous two sections. The general problem of finding optimal **individual** consumption profiles is more difficult, and is developed in a companion paper.

6.2 Welfare Optima and the Pareto Frontier

The social optimum must take these diverging preferences into account. In this vein, we assume that social welfare is a weighted average of firms' and households' utilities:

$$W = \alpha V^E + (1-\alpha)V^H$$

where $0 < \alpha < 1$ is the weight put by the government on the corporate sector. Using the above expressions of V^H and V^E, we have

$$W = \log(\rho K_0) + \frac{\mu - \gamma}{\rho} - \int_0^\infty e^{-\rho t}\left[\frac{1+h_t}{x_t} + \log x_t - (1-\alpha)\log h_t + \alpha\frac{\sigma^2 x_t^2}{2\rho}\right] dt.$$

The expression under the integral is bounded and can be maximized pointwise.[14] Hence, W is maximum for constant values of x_t and h_t, namely x^* and h^*, which are uniquely determined by the first-order conditions[15]

$$(1-\alpha)x^* = h^* \tag{45}$$

$$\frac{\sigma^2}{\rho}x^{*3} + x^* - \frac{1}{\alpha} = 0. \tag{46}$$

For any $0 < \alpha \leq 1$, equation (46) has a unique positive solution x^*. Equation (45) then determines h^*. Hence, there is a unique welfare maximum that corresponds to a stationary point of the (x_t, h_t) - dynamics.[16] Furthermore,

Proposition 3. *When $\sigma > 0$, optimal government debt is strictly positive:*

$$1 + h^* > x^*.$$

Proof. $1 + h^* - x^* = 1 - \alpha x^*$ by (45). The polynomial in (46) is increasing, and strictly positive for all $x \geq 1/\alpha$. Hence, $x^* < 1/\alpha$ and thus $1 + h^* - x^* > 0$. □

Hence, the welfare maximum is not compatible with balanced budgets, and a government wishing to implement this maximum through fiscal policy must issue a positive amount of safe debt. Specifically, Proposition 1 implies that the welfare optimum (45)-(46) can be implemented by some combination of initial lump sum transfers (L^{E*}, L^{H*}) and instantaneous tax rates (τ^{E*}, τ^{H*}), which follow from (35) and (36):

$$\tau^{E*} = \gamma + \sigma^2 x^*(x^* - 1) + \rho(\frac{1+h^*}{x^*} - 1) \tag{47}$$

$$\tau^{H*} = \tau^{E*} - \sigma^2(x^*)^2 = \gamma - \sigma^2 x^* + \rho(\frac{1+h^*}{x^*} - 1). \tag{48}$$

[14] This is because of the stationary nature of all decision problems.

[15] It is straightforward to verify that the first-order conditions determine the unique global maximum.

[16] Note that W is well defined for any bounded and piecewise continuous trajectory (x_t, h_t) which would be implemented by more general fiscal policies involving multiple lump sum transfers. Since W is maximum for a constant (x_t, h_t), our restriction to a single episode of lump sum transfers is without loss of generality.

Note that the welfare optimum is independent of the government expenditure coefficient γ, while the taxes needed to implement it are not.

By eliminating α between (45) and (46) we obtain a representation of the (constrained) Pareto frontier in the (h, x) plane:

$$h(x) = x - \frac{\rho}{\rho + \sigma^2 x^2} \tag{49}$$

for $x \geq x_{min}$, where x_{min} is the lower bound given by (44). By (49), each $x \geq x_{min}$ defines a constrained Pareto allocation, which is a steady state with constant interest rate $r = \mu - \sigma^2 x$.

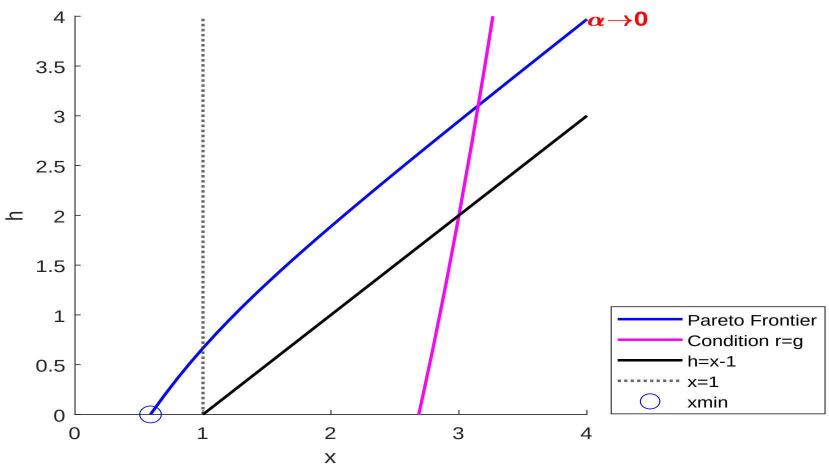

Figure 2: The Pareto frontier in (x, h) space for $\rho = 0.01, \sigma = 0.2, \gamma = 0.1$.

Figure 2 shows the Pareto frontier in the (x, h) plane, for the same values of ρ, σ, and γ as Figure 1.

When $\sigma > 0$, the Pareto frontier lies entirely above the Zero Debt Line $h + 1 - x = 0$, and it converges to the diagonal $h = x$ for $x \to \infty$. The Zero Debt Line corresponds to the unconstrained Pareto frontier: when there are no frictions, idiosyncratic risks can be eliminated, which is equivalent to taking $\sigma = 0$. In this case, optimal public debt is zero. In the general case, by (35), (36), (45) and (46), the instantaneous tax/subsidy rates that implement the second best allocations are

$$\tau^E = \gamma + \sigma^2 x^2 - \frac{\sigma^4 x^3}{\rho + \sigma^2 x^2} \tag{50}$$

$$\tau^H = \gamma - \frac{\sigma^4 x^3}{\rho + \sigma^2 x^2}. \tag{51}$$

A simple inspection shows that the lower bound for the Pareto Frontier satisfies $x_{min} < 1$. Hence, there are Pareto Optima with $K^* < E^*$, i.e. in which firms are net lenders. This means that the situation mentioned in footnote 10 can not only arise, but even be optimal. This is the case if α is large, i.e. if fiscal policy caters strongly to firms' interests.

6.3 Laisser-Faire

A passive government does not engage in fiscal policy or redistribution. We therefore define a Laisser-Faire (LF) policy by three features: (i) $B_t = 0$ for all t (balanced budget), (ii) $L^H = L^E = 0$ (no lump-sum redistribution), and (iii) $\tau_t^H = \tau_t^E = \tau_t$ (equal taxation).

Laisser-Faire therefore implies $T_t = \tau_t(H_t + E_t) = \tau_t K_t$. Together with the balanced-budget constraint $T_t = \gamma K_t$ this implies that taxes are constant,

$$\tau_t = \gamma.$$

The trajectory (x_t, h_t) under LF is entirely contained in the Zero-Debt-Line $x = h + 1$ and starts at $x_0 > 1$. Therefore we can focus on the variable x_t alone. Its equation of motion (46) simplifies to

$$\dot{x}_t = -\sigma^2 x_t^2 (x_t - 1).$$

Thus x_t converges monotonically to 1 and h_t to 0.

Interestingly, Pareto Optima are not necessarily Pareto improvements over the Laisser-Faire. This is depicted in the generic Figure 3, which displays the Pareto frontier in utility space, i.e. the $V^H - V^E$ - plane. By construction, the LF is not Pareto optimal: it is represented by the two values LF^E and LF^H for V^E and V^H, respectively.

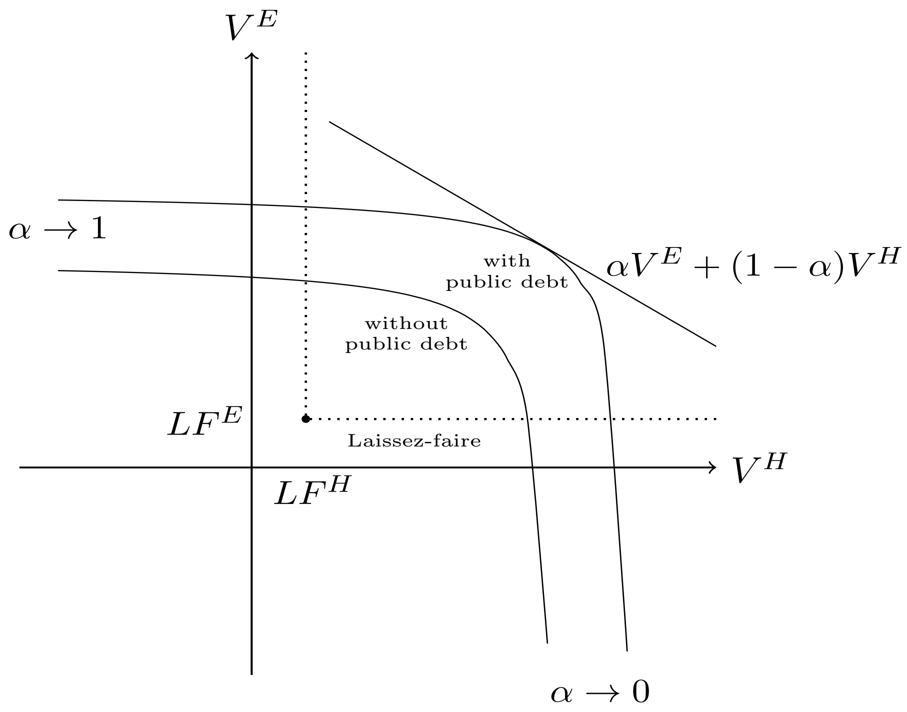

Figure 3: The Pareto frontiers and Laisser-Faire in utility space.

Figure 3 also shows the following two properties, which follow formally from an inspection of (41) and (42) together with (45) and (46):

$$\lim_{\alpha \to 0} V^E(x^*, h^*) = \lim_{\alpha \to 1} V^H(x^*, h^*) = -\infty.$$

Hence, the allocation that maximizes W is a Pareto improvement over Laisser-Faire when α is intermediary. As the figure illustrates, when α is large, households strictly prefer Laisser-Faire to the welfare optimum, while firms strictly prefer Laisser-Faire when α is small.

Note that there exist Pareto improvements over the Laisser-Faire even if fiscal policy is constrained by a balanced budget at each point in time, i.e. no public debt is issued. This constrained Pareto frontier is also displayed in Figure 3 and will be discussed next.

7 Public Debt, Taxes, and Redistribution

7.1 Pareto Improving Debt

By Proposition 3, Pareto optimal public debt must be strictly positive at all times. The following thought experiment illustrates this basic result, by explicitly constructing the Pareto improvement that is possible in a situation of balanced government budgets.

As discussed in the previous section, under balanced budgets, $h_t = x_t - 1$ and the welfare function becomes, up to a constant,

$$W_{BB} = \int_0^\infty e^{-\rho t} \left[(1-\alpha)\log(x_t - 1) - 1 - \log x_t - \alpha \frac{\sigma^2 x_t^2}{2\rho} \right] dt.$$

Maximizing W_{BB} yields the maximal welfare that can be achieved without issuing public debt. Again, the expression is maximized at a steady state allocation, characterized by the unique solution $x_{BB} > 1$ of the first order condition

$$\frac{\sigma^2}{\rho}(x^3 - x^2) + x = \frac{1}{\alpha}.$$

Note the similarity with the equation defining x^*, (46), and that private leverage is higher here: $x_{BB} > x^*$. x_{BB} corresponds to the following allocation of initial wealth: $E_0 = \frac{K_0}{x_{BB}}$ and $H_0 = K_0 - E_0$. The growth rate of output is $g = \mu - \gamma - \rho$, and up to additive constants, the expected continuation utilities can be written as follows:

$$\begin{aligned} V_{BB}^H &= \frac{1}{\rho}[\log H_0 - H_0 - E_0] \\ V_{BB}^E &= \frac{1}{\rho}[\log E_0 - H_0 - E_0 - \frac{\sigma^2}{2\rho E_0^2}]. \end{aligned}$$

To understand how this allocation can be Pareto improved, suppose that the government issues a small amount of debt and distributes it to the two categories of agents, so that both ΔE_0 and ΔH_0 are positive. The government also adjusts the tax rates, so that the economy remains in the new steady state. The first order change in households' utility is such that

$$\rho \Delta V_{BB}^H = \frac{\Delta H_0}{H_0} - \Delta(H_0 + E_0),$$

corresponding to the difference between the relative wealth increase and the total wealth increase (equal to new government debt), which reduces growth. The equivalent term for firm equity is

$$\rho \Delta V_{BB}^E = \frac{\Delta E_0}{E_0} - \Delta(H_0 + E_0) + \frac{\sigma^2}{\rho E_0^3}\Delta E_0,$$

where a new term appears, corresponding to the reduction in the risk premium that follows the decrease in private leverage. Hence, it is possible to distribute the additional wealth created by the government in such a way that both types of agents benefit, as long as the following two conditions hold:

$$h_0 < \frac{\Delta H_0}{\Delta E_0} < h_0 + \frac{\sigma^2 x_0^3}{\rho}.$$

This is only possible in an economy with frictions, where idiosyncratic risk cannot be eliminated and $\sigma > 0$. In a frictionless economy, the welfare enhancing role of government debt disappears. Moreover, in the frictionless economy, the welfare optimum is achieved by initial lump sum redistribution, and no further redistribution takes place. Indeed, by (50) and (51), $\tau^E = \tau^H = \gamma$ when $\sigma = 0$. Since firms and households earn the same return on their investments ($\mu = r$) in the frictionless case, they face the same tax rate to finance public expenditures.

Hence, issuing public debt and distributing it to the private sector in adequate proportions has three effects: a balance sheet effect, an interest rate effect, and a growth effect. The balance sheet effect reduces the leverage of firms and increases the incentives of firms to undertake risky investments at a given risk-free interest rate. To clear the market for capital, the risk free interest rate thus increases. This buffers the portfolio risk that owners of firms are bearing.[17] Finally, the higher wealth created by the new asset increases consumption of all agents and thus has a negative impact on output growth. All these effects occur jointly and feed back into each other.[18]

However, since these wealth increases must accrue to firms in order to trigger the balance sheet effect, it is necessary to balance them by continuously redistributing

[17] Note that these comparative statics refer to a change from a constrained optimal stationary allocation (x_{BB}, h_{BB}) on the Zero Debt Line (ZDL) to the welfare optimum (x^*, h^*). An arbitrary change from any initial allocation $(\tilde{x}, \tilde{h})$ on the ZDL to the welfare optimum, of course, cannot be signed, as there must be redistribution according to the weight α.

[18] Although public debt increases private consumption and thus decreases private investment, public debt does not "crowd out" private investment in the traditional sense (see Blanchard (2008)). "Crowding out" usually refers to the substitution of private spending by public spending, which is impossible in our model, where government expenditure is exogenous.

wealth from firms to households to maintain steady state growth. In fact, as we show in the next subsection, in the optimum, the initial asset injection must entirely accrue to firms, which in turn has a strong redistributionary consequence in terms of ongoing taxation.

7.2 Optimal Fiscal Policy

To clarify the role of government debt and taxes in our economy further, consider first the case without financial frictions, in which idiosyncratic risks can be diversified away, so that we can effectively take $\sigma = 0$. As we already saw, the optimal allocation is then implemented by redistributing initial wealth, so that $E_0 = \alpha K_0$ and $H_0 = (1-\alpha)K_0$, and having zero government debt at all times. To keep the economy in steady state, taxes should be equal across firms and households: $\tau^E = \tau^H = \gamma$.

Suppose now that as of date 0, frictions appear such that it is not possible anymore to eliminate idiosyncratic risks and $\sigma > 0$.[19] The optimal response of the government to this shock is to issue debt for an amount $B_0(\sigma) = (\frac{1}{x^*} - \alpha)K_0$ and to distribute it exclusively to the firms. Indeed, the optimality conditions (45)-(46) imply

$$H_0^*(\sigma) = (1-\alpha)K_0.$$

Together with the aggregate balance sheet identity (23), this implies

$$E_0^*(\sigma) = \alpha K_0^*(\sigma) + B_0^*(\sigma).$$

Thus the firms are initially the only beneficiaries of government intervention. The following result shows that, in any optimal allocation, households are subsidized ever after in order to compensate for the initial imbalance.

Proposition 4. *In any optimal allocation, households are subsidized, in the sense that they contribute less to public expenditures than their share in the social welfare function:* $\tau^{H*}H_t < (1-\alpha)\gamma K_t$ *for all* t.

Proof. By (45), the claimed inequality is equivalent to $\tau^{H*} < \gamma$. Equation (48) implies that

$$\tau^{H*} - \gamma = -\sigma^2 x^* + \rho(\frac{1}{x^*} - \alpha),$$

where we have again used (45). Replacing α by $\frac{\rho}{\rho x^* + \sigma^2 (x^*)^3}$, the above equation can be rewritten as

$$\tau^{H*} - \gamma = -\frac{\sigma^4 (x^*)^3}{\rho + \sigma^2 (x^*)^2} < 0.$$

[19]This thought experiment corresponds to the traditional one in macroeconomic classics such as Bernanke et al. (1996) and Kiyotaki and Moore (1997), where a stationary equilibrium is shocked unexpectedly. The specific shock analyzed here is the same as in Di Tella (2017). In fact, citing from his paper, introducing "an aggregate uncertainty shock that increases idiosyncratic risk in the economy" ... "can create balance sheet recessions." Different from Di Tella (2017), we are interested in long-run market imperfections rather than recessions.

$\square$

Proposition 4 states that households contribute less than their "fair" share of public expenditures.[20]

7.3 Debt

In order to illustrate the different regimes of debt in the welfare optimum in more conventional terms, it is useful to consider the steady state debt-to-GDP ratio. Evaluating (37) at the welfare optimum, by (45), the optimal debt-to-GDP ratio is

$$\delta^* = \frac{1-\alpha x^*}{\mu x^*} = \frac{\sigma^2 x^*}{\mu(\rho + \sigma^2 x^{*2})}, \tag{52}$$

which is strictly positive by Proposition 3. Our analysis identifies the determinants of the debt-to-GDP ratio and shows how it depends on the political influence of firm interests (captured by parameter α). Differentiating (46) shows that x^* is decreasing in α. from the second equation of (52), it is straightforward to see that δ^* is a quasi concave function of x^*. Hence, the optimal debt-to-GDP ratio is also single-peaked in α, which is summarized in the following proposition.

Proposition 5. *The optimal debt-to-GDP ratio is a strictly quasi concave function of the political weight of firm interests, with maximum at* $\widehat{\alpha} = \min(1, \frac{\sigma}{2\sqrt{\rho}})$*. It converges to* 0 *for* $\alpha \to 0$*.*

Proof. Differentiating (52) shows that δ^* as a function of x is strictly quasiconcave, with maximum at $x = \sqrt{\rho}/\sigma$. An inspection of (44) shows that $x_{\min} \geq \sqrt{\rho}/\sigma$ if and only if $\sqrt{\rho}/\sigma \leq \frac{1}{2}$. Since $x^* \in [x_{\min}, \infty)$, this shows that δ^* as given by (52) is strictly decreasing in x^* if $\sqrt{\rho} \leq \frac{\sigma}{2}$ and strictly quasiconcave with maximum at $\sqrt{\rho}/\sigma$ otherwise. The rest of the proposition follows because $\frac{dx^*}{d\alpha} < 0$ and by inserting $x^* = \sqrt{\rho}/\sigma$ into equation (46). $\square$

Figure 4 illustrates Proposition 5 by plotting the optimal debt-to-GDP ratio as a function of the political weight of firm equity. The figure uses values for ρ and μ that are in the standard range of the literature, and shows how sensitive the optimal debt-to-GDP ratio is to different values of the volatility of idiosyncratic productivity risk. Of course, it is not easy to calibrate σ in the present model. Nevertheless, calibrations for idiosyncratic productivity shocks have been the subject of various studies, and recent work, for instance, by Bloom et al. (2018) or Arellano et al. (2019), has provided estimates for such shocks. Bloom et al. (2018) report that the yearly variance of plant-establishment-level TFP shocks in the US in a non-recession time was 0.198. In order to use these estimates for a numerical illustration, one

[20] It does not say that $\tau^{H*} < 0$. However, the proposition implies that this is the case if γ is sufficiently small.

needs additional information about how much of the volatility is not insurable, which is hard to assess. But the value can serve as an upper bound.

When $\sigma < 2\sqrt{\rho}$, we have $\widehat{\alpha} < 1$ in Proposition 5. Given the preceding discussion, this seems to be the empirically relevant range in our framework.[21] Figure 4 therefore displays the inverse U-shape to be expected according to Proposition 5. Debt-to-GDP is largest if the interests in the economy are relatively balanced, and decreases if one group becomes more and more dominant.

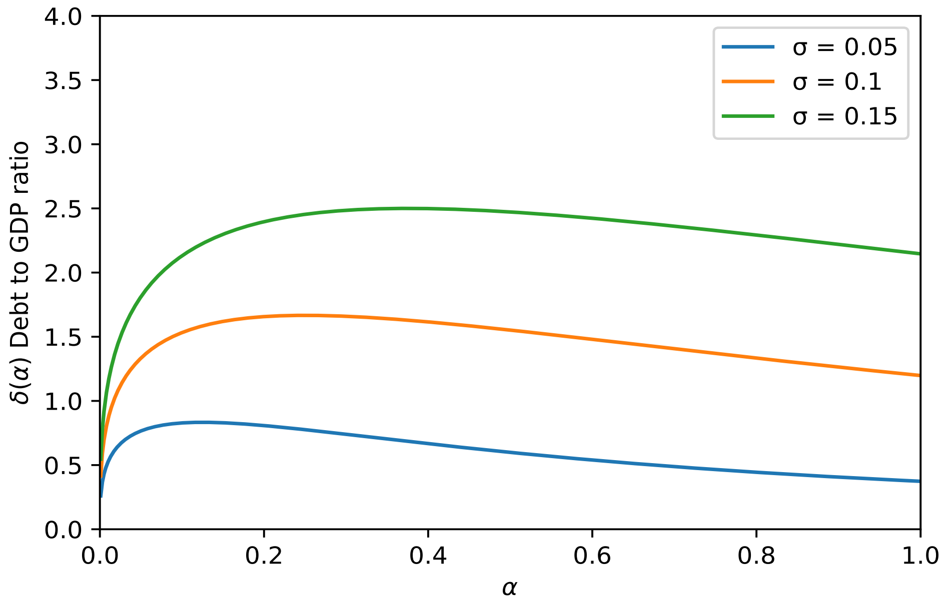

Figure 4: Debt-to-GDP ratio for $\rho = 0.04, \mu = 0.15$ and different values for σ.

In our model, corporate debt is safe because steady state equity follows a geometric Brownian motion and therefore never reaches zero: firms do not default. Hence, when the government issues public debt, it does *not* create a new type of (safe) asset: government debt is exactly as good as existing corporate debt. However, public debt is valuable because it allows firms to reduce their risk exposure. One interpretation is that firms can buffer some of their losses by holding public debt on the asset side of their balance sheet. Another equivalent interpretation is that firms reduce their leverage by buying back some of their equity. A necessary requirement for our analysis, of course, is the credibility of the government's promise to never default. But since the government is assumed to maximize social welfare, which is achieved in the steady state with sustainable debt issuance, there is neither a reason for the government to default nor for the private sector to refuse buying new government debt. Not defaulting is time-consistent for our benevolent government.

[21]For example, it comprises all combinations $\rho \geq 0.02$ and $\sigma \leq 0.28$.

Extending our model, though, in the spirit of the seminal papers of Calvo (1988) and Cole and Kehoe (2000), one can ask whether default can be a problem. Suppose for example that for whatever reason – for instance, coordination failures in debt issuance auctions – there is a possibility at a particular point in time t that the private sector refuses to roll over public debt, since it anticipates default of the government in the future. But since the government relies on taxation of wealth, even this would not cause default. By the basic balance sheet identity, $B_t = H_t + E_t - K_t$, which is strictly smaller than $H_t + E_t$. Hence, off the equilibrium the government can confiscate sufficient private wealth in emergency taxation to stop such a debt run in the first place.[22]

8 Interest, Growth, and the Dynamics of the Government Budget

8.1 Interest

It is straightforward to apply the steady state conditions (38) and (46) to the determinants of interest rates in Proposition 2.

Proposition 6. *The optimal interest rate $r^* = \mu - \sigma^2 x^*$ is an increasing function of μ and α and a decreasing function of ρ. It is negative if μ or α are sufficiently small.*

Proposition 6 sheds some interesting light on the recent debate about the observation that real interest rates have indeed fallen over the last decades and have reached negative territory in a variety of industrialized countries. At the center of most explanations is the observation that the amount of savings, relative to investment demand, has changed. While some explanations put emphasis on the origin of changes in savings, others put more emphasis on changes in productivity or put emphasis on both. One prominent voice is Rachel and Summers (2019), who stress that these secular movements are for a larger part a reflection of changes in saving and investment propensities. They argue that the industrialized world will probably face a longer period of secular stagnation with sluggish growth and low real interest rates.[23]

Our results point to structural factors that might contribute to low real interest rates. For instance, and consistent with Proposition 6, permanent shifts in the objectives of policy-making with respect to risk-bearing versus non-risk-bearing agents

[22] Things would be slightly more complicated if government debt constituted a fully liquid real promise. But even then one can show that there is sufficient tax backing out of equilibrium if σ is not too large.

[23] For discussions and evidence how to differentiate whether rising income inequality or an aging of the population have contributed to an increase of savings see e.g. Mian et al. (2021), and see also von Weizsäcker and Krämer (2019) on how technological progress and demography may have jointly contributed to a secular decline in real interest rates.

can induce a secular decline and even negative values of real interest rates. Proposition 6 is also consistent with the suggested link between aggregate productivity and interest rates. Moreover, our results qualify the standard logic that higher savings rates lead to lower real interest rates. If ρ declines and thus the saving rate increases, the real interest rate increases. This occurs since the risk-bearing corporate sector operates with a larger share of wealth in the form of equity and is thus willing to absorb savings by households at a higher interest rate. Simply focusing on household savings may therefore not suffice to address the secular stagnation problem.

8.2 Growth

We now turn to the determinants of the optimal growth rate g^*, obtained by evaluating (26) at the optimal stationary levels (x^*, h^*).

Proposition 7. *(i) At the optimum, the growth rate g^* and private leverage x^* are related by*

$$g^* = \mu - \gamma - \rho - \frac{\rho\sigma^2 x^*}{\rho + \sigma^2 x^{*2}}. \tag{53}$$

(ii) As a function of α, g^* is strictly quasiconvex with minimum at $\widehat{\alpha} = \min(1, \frac{\sigma}{2\sqrt{\rho}})$.
(iii) When $\alpha \to 0$, $x^* \to \infty$ and the optimal growth rate converges to the Modified Golden Rule rate $\mu - \gamma - \rho$.

Proof. (53) follows from substituting h^* from (49) into the expression for growth, (26). The rest follows as in the proof of Proposition 5. □

Proposition 7 is the mirror image of Proposition 5. It shows that the political weights in the welfare function may have a non-monotonic impact on growth and this impact is moderated by impatience and risk, ρ and σ. As argued after Proposition 5, plausible parameter values imply that $\widehat{\alpha} < 1$, i.e. that growth is minimized at interim values of α. But by (53) growth is unambiguously maximized for $\alpha \to 0$, i.e. if corporate interests become irrelevant.

While taxation and redistribution ensure that the wealth of firms and of households increase at the same rate on average, there is growing inequality among firms. Indeed, by the standard theory of Brownian motion, if all firms start out with equity $e_0^i = e_0$ at time 0, then, in any optimum (x, h), equity e_t^i at time t as given by (14) is log-normally distributed with mean and variance

$$\begin{aligned} E[e_t] &= e_0 \exp g^* t \\ var(e_t) &= e_0^2 \left[\exp 2g^* t\right] \left[\exp(\sigma^2 x^2 t) - 1\right]. \end{aligned}$$

Thus the coefficient of dispersion of firms wealth grows over time:

$$\frac{\sqrt{var(e_t)}}{E[e_t]} = \sqrt{\exp(\sigma^2 x^2 t) - 1}.$$

Firms' heterogeneity is endogenous in our economy: even if the initial redistribution of capital equalizes initial wealth among firms, the impossibility to tax individual profits implies that the coefficient of dispersion of the distribution of firms' wealth necessarily grows over time.

The preceding results and the description of welfare optima in Section 6.2 now make it possible to fully characterize the optimal relation between the growth and the interest rate in our model.

Proposition 8. *(i) At the welfare optimum, $g^* > r^*$ if and only if*

$$2\alpha\left(\rho+\gamma\right) + \left(\rho+\gamma+\alpha\right)\sqrt{\alpha\left(1+\frac{\gamma}{\rho}\right)} < \sigma^2. \tag{54}$$

(ii) The left hand side of formula (54) being increasing in α, the growth rate is more likely to be higher than the interest rate when the political weight of the corporate sector α is small.

Proof. From (38), (26), and (45) we have

$$r^* - g^* = \frac{\rho}{x^*} - \sigma^2 x^* + \rho(1-\alpha) + \gamma.$$

Using (46), this implies

$$\frac{x^{*2}}{\rho}(r^* - g^*) = \left(1 - \alpha + \frac{\gamma}{\rho}\right) x^{*2} + 2x^* - \frac{1}{\alpha}.$$

Hence, we have $r^* < g^*$ iff $x^* < \widetilde{x}$, where $\widetilde{x}$ is the unique positive solution to

$$x^2 + \frac{2}{y}x - \frac{1}{\alpha y} = 0, \tag{55}$$

i.e.

$$\widetilde{x} = \frac{1}{y}\left[\sqrt{1+\frac{y}{\alpha}} - 1\right],$$

where $y \equiv 1-\alpha+\frac{\gamma}{\rho}$. Again, using the definition of x^* in (46), which can be written as

$$f(x) \equiv x^3 + \frac{\rho}{\sigma^2}x - \frac{\rho}{\alpha\sigma^2} = 0,$$

the condition $x^* < \widetilde{x}$ is equivalent to $f(\widetilde{x}) > 0$. Substituting and using (55) twice shows that this is the case if and only if

$$\left(4\alpha + y + \frac{\rho\alpha}{\sigma^2}y^2\right)\left[\sqrt{1+\frac{y}{\alpha}} - 1\right] > 2y + \frac{\rho}{\sigma^2}y^3.$$

In a number of straightforward steps, this inequality can be re-written as (54). □

Proposition 8 provides precise information about the determinants of the difference between the interest and the growth rate at the welfare optimum. As discussed in the introduction, historically, the case $g > r$ seems to be more relevant than the opposite one. This has important consequences for the sustainability of government deficits, as we discuss below. In particular, the prediction of Proposition 8 is that the growth rate will optimally exceed the interest rate when the private propensity to consume ρ, the size of the public sector γ, and the political weight of corporate interests α are low, and when idiosyncratic production risk σ is large. These predictions are independent of the productivity of capital, μ.

As noted above, x^* decreases monotonically in α and becomes large when $\alpha \to 0$. Hence, the comparative statics variation of α allows us to plot the optimal debt-to-GDP ratio δ against x^*, optimal corporate leverage. Figure 5, which mirrors Figure 4, plots this relation, which is independent of γ, under the assumption $\sigma < 2\sqrt{\rho}$, which implies $\widehat{\alpha} < 1$ in Propositions 5 and 7 and thus the inverse U-shape of the curve. The figure shows that, at the welfare optimum, corporate leverage and the public debt-to-GDP ratio are not comonotonic. In fact, there is an interior maximum of δ, corresponding to an interior value of α. Public debt-to-GDP is first relatively low, for low levels of corporate leverage, then it increases, and later declines. It is monotonically decreasing for sufficiently high levels of corporate leverage.

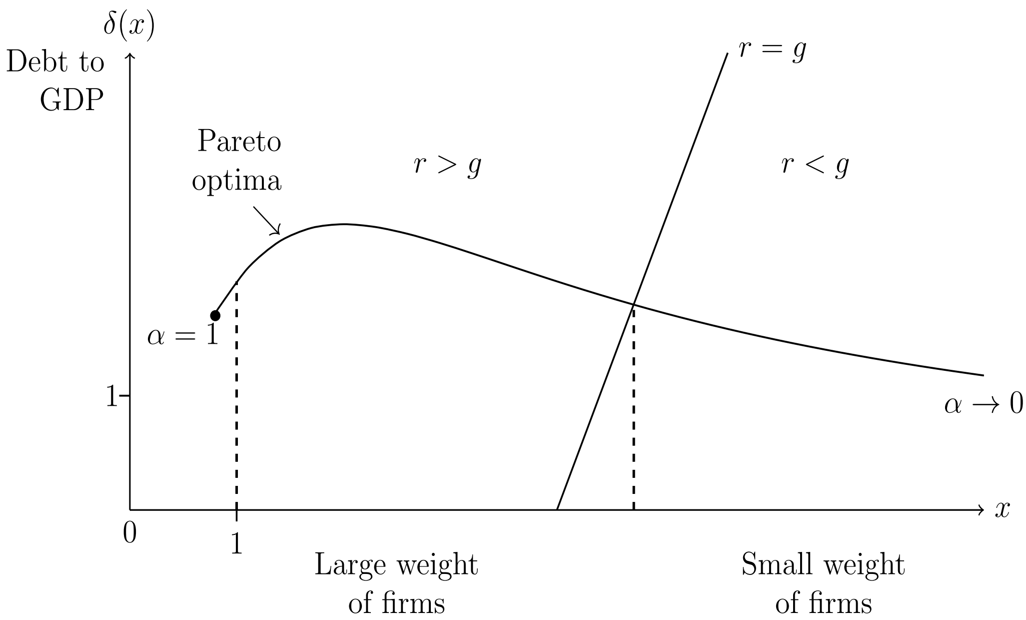

Figure 5: Regimes for parameter values $\rho = 0.04, \sigma = 0.1, \mu = 0.1$.

Figure 5 also illustrates the insight of Proposition 8 that depending on the welfare weight of firms, the economy can be in different regimes $r - g > 0$ or

$r - g < 0$, a question to which we turn now.

8.3 The Sustainability of Fiscal Policy

In many OECD countries, real rates of return on safe assets have been below growth rates for some time now. Yet mean rates of return on risky assets have been above growth rates. Whether this represents an instance of dynamic inefficiency in overlapping generation frameworks has been addressed in a series of important contributions and we refer to Hellwig (2021) and Dumas et al. (2022) for recent discussions and assessments.

Whether $r < g$ or not is a central question in current debates about the sustainability of the US' and other countries' fiscal policy. From an asset pricing perspective, Cochrane (2019) describes the limits of public deficits by noting that in models with infinitely-lived agents, "[t]he market value of government debt equals the present discounted value of primary surpluses." In conformity with our results, Cochrane (2022) argues that under complete financial markets ($\sigma = 0$ in our model), a permanent relationship $r < g$ is theoretically implausible, and empirically unlikely when r and g are measured correctly.[24] On the other hand, Blanchard (2019) adopts a more positive view on the theoretical possibility of $r < g$ and investigates the potential and limitations of a large fiscal expansion at little or no fiscal cost.

In our model of an economy with idiosyncratic production risk and imperfect macroeconomic risk-sharing, the return on safe debt r can fall below g. If buffering losses of owners of firms has less weight in the welfare function, public debt issuance and reduction of corporate leverage are less important. As a consequence, firms are only willing to invest in risky production if the real interest rate is sufficiently low. Hence, there is a role for government policy by actively reducing r in such cases. Figure 5 summarizes one our main insight that the relationship between r and g is a consequence of the weight of firm owners in the welfare function and the associated optimal debt issuance and taxation.

Our analysis is consistent with both views about the dynamics of the government budget and shows how to reconcile them. The government's flow budget constraint at date t, (3), can be written as

$$\dot{B}_t = \gamma K_t + r_t B_t - T_t = r B_t - S_t, \tag{56}$$

where S_t is the primary surplus. Consider an arbitrary steady state (not necessarily optimal) and let r and g be the associated interest and growth rates, respectively. Since in steady state, all endogenous quantities evolve at the same rate, we have

$$B_t = e^{gt} B_0 \quad \text{and} \quad S_t = e^{gt} S_0. \tag{57}$$

[24] Cochrane (2022) provides a comprehensive account how the $r < g$ debate is connected to the fiscal theory of the price level.

Discounting and integrating (56) between dates 0 and some later date T yields:[25]

$$B_0 = \int_0^T S_t e^{-rt} dt + B_T e^{-rT}. \tag{58}$$

This relation can be viewed as the balance sheet identity for the public sector, with liabilities B_0 and two types of assets as follows:

Assets	Liabilities
$X_0 = \int_0^T S_t e^{-rt} dt$	B_0
$Y_0 = B_T e^{-rT}$	

where we let $T \to \infty$.

As in our previous discussion, we can distinguish two cases. The first case is $r > g$. Then Y_0 tends to zero when T tends to ∞, and we thus obtain the standard relationship that the value of debt equals the net present value of future primary surpluses, as argued by Cochrane (2019). The second case is $r < g$. Then Y_0 tends to $+\infty$ and X_0 tends to $-\infty$. Hence, in the limit the balance sheet identity $X_0 + Y_0 = B_0$ is not well defined. However, we can interpret $B_T e^{-rT}$ as a form of intangible asset for the government, which can be attributed to its capacity to borrow again in the future and may be called government "goodwill". In fact, our analysis shows that it is rather the government's "eternal power to issue safe debt" – rather than to tax – that creates this intangible asset. As long as the government can convince investors of its capacity to sustain a high enough level of growth, this intangible asset has a positive value.

To illustrate this point, we can consider the following simple example where the government does not raise any taxes but can still sustain a positive debt level and positive public expenditures. Take for simplicity $H_t \equiv 0$ (no households), and assume $0 < \gamma - \rho < \frac{1}{3}\sigma^2$. In this economy, in the absence of public debt, the private sector balance sheet is $K_t = E_t$. Consider the dynamics of $x_t = 1 - \frac{B_t}{E_t}$. By equation (33), this quantity evolves as $\dot{x}_t = \sigma^2 \phi(x_t)$, where

$$\phi(x) = -x^3 + x^2 - \frac{\gamma - \rho}{\sigma^2} x - \frac{\rho}{\sigma^2}.$$

It is easy to show that the equation $\phi(x) = 0$ has two solutions $x_- < x_+$ on (0,1). Moreover ϕ changes sign twice on this interval. Therefore, x_+ is a locally stable equilibrium of x_t: if the economy starts close to it, it converges to it. In this economy, public debt and public expenditures are thus sustainable even if the government never raises taxes.

In light of the results of Sections 7.3 - 8.2 and under the assumption that σ is

[25]Which discount rate should be used for the government budget constraint has been the subject of recent work. Brunnermeier et al. (2021), and Reis (2021) offer particular rationales for using discount rates different from r.

not too large,[26] we can therefore distinguish two polar cases for the influence of firm interests on the sustainability of government deficits. First, if α is small, $g > r$ in equilibrium, and the government runs increasing budget deficits that it covers by taxes and rolling over ever increasing public debt. Nevertheless, by Proposition 5 the public debt-to-GDP ratio is small. Second, if α is large, we have $g < r$ in equilibrium, "[t]he market value of government debt equals the present discounted value of primary surpluses" (Cochrane (2019)), and the public debt-to-GDP ratio is intermediary. For medium values of α, the public debt-to-GDP ratio is large, the growth rate is low, and the sign of $r - g$ depends on γ and σ as given by (54). Hence, government deficits have a "Cochranian" interpretation or a "Blanchardian" one, depending on α.

9 Conclusion

We have presented a simple model in which government debt issuance affects corporate leverage, and thus the investment and growth dynamics of the economy, through changes in the mix of private and public debt. It highlights how the weights of firm owners and households in the government welfare function impacts the relationship between r and g. In this sense, interest, growth, and public debt are a matter of redistributionary political tradeoffs.

Our model also allows many extensions. First, our paper is part of normative macroeconomic theories in which government debt serves a socially desirable purpose. Yet, the view that the amount of government debt, and in particular the rise of government debt over the past decades, was the optimal response is strongly debated. For instance Yared (2019) provides a comprehensive account of political economy theories of government debt and how these theories may explain a substantial part of the long-term trend in government debt accumulation. Adding political factors, e.g. political turnover between households and equity holders when embedding the current model in a simple election framework would reveal how much of the welfare increasing role of government debt is preserved in a democracy.

Second, one can embed our analysis in a monetary version of the model, (Gersbach et al. (2022)), in which central bank reserves play the same role of safe asset as government debt in the current model. Since the financial crisis of 2007/2009, the reserves of commercial banks in the US, the UK, Japan, and in the Euro Area have strongly increased, albeit to different degrees. Our preliminary results support the argument that banks' holding a significant amount of central bank reserves is

[26] We need the inequality in (54) to be reversed for $\alpha = 1$, which implies an upper bound on σ. This is consistent with our discussion of parameter ranges in Section 7.3, and in particular with the assumption $\sigma < 2\sqrt{\rho}$ that ensures $\widehat{\alpha} < 1$ in Proposition 5. If σ is large (which is implausible empirically), we have $r < g$ for all α.

desirable from a welfare perspective when banks face significant uninsurable idiosyncratic risks.

Third, the model can be embedded in a straightforward way in a small open-economy context. Then, the interest rate is exogenous to our economy, but the amount of physical capital available for risky investments can be increased by borrowing in international capital markets. Then public debt issuance reduces corporate leverage, and may spur higher investments and growth as long as repayment of international borrowing is ensured.

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Appendix A: Implementation of Aggregate Consumption Profiles

Let $\mathbf{C}^k = (C_t^k)_{t=0}^{\infty}$, $k = H, E$, denote strictly positive, differentiable aggregate consumption profiles of households and equityholders, respectively.[27] Let $\tilde{H}$ be the initial endowment of the representative household and $\tilde{E}$ the (identical) endowments of equity holders, respectively. Here we show under what conditions and how one can construct a fiscal policy with linear wealth taxes such that these aggregate consumption profiles arise in the corresponding general equilibrium.

Let $\mathbf{C} = \mathbf{C}^E + \mathbf{C}^H$ denote total aggregate consumption. Clearly, for these consumption profiles to be feasible it must be possible to produce them. Aggregate capital evolves according to

$$\dot{K}_t = (\mu - \gamma)K_t - C_t \tag{A1}$$

Since any efficient consumption plan must use all initial endowments, $K_0 = \tilde{H} + \tilde{E}$. Hence, integrating (A1) by standard methods,

$$K_t = (\tilde{H} + \tilde{E})e^{(\mu-\gamma)t} - \int_0^t e^{(\mu-\gamma)(t-s)} C_s ds \tag{A2}$$

For $(\mathbf{C}^E, \mathbf{C}^H)$ to be feasible, it is necessary that $K_t > 0$ for all t, which is equivalent to

$$\int_0^t e^{(\mu-\gamma)(t-s)} C_s ds < (\tilde{H} + \tilde{E})e^{(\mu-\gamma)t} \text{ for all } t \geq 0$$

which in turn is equivalent to

$$\int_0^\infty e^{-(\mu-\gamma)s} C_s ds \leq \tilde{H} + \tilde{E} \tag{A3}$$

Definition. *An aggregate consumption profile* $(\mathbf{C}^E, \mathbf{C}^H)$ *is called admissible if both components are strictly positive, differentiable, and aggregate consumption satisfies (A3).*

Condition (A3) is a modified transversality condition; consumption profiles that do not satisfy it cannot be sustained by the economy's productive capacity given in (1). The conditions of positivity and differentiability are needed to define law of motions in line with the preceding analysis.[28]

As the following proposition shows, admissibility is not only necessary, but also sufficient to implement an aggregate consumption profile as an equilibrium outcome.

[27] Of course, equityholders' individual consumption streams are risky. We will identify individual consumption streams that aggregate to $\mathbf{C}^E$. Note that this distinction is not necessary for households, whose consumption stream is certain.

[28] It is possible to work with piecewise differentiable profiles.

Proposition. *Suppose* $(\mathbf{C}^E, \mathbf{C}^H)$ *is admissible. Then there is a unique set of policy parameters,*

$$L^H = \tilde{H} - \frac{1}{\rho} C_0^H \tag{A4}$$

$$L^E = \tilde{E} - \frac{1}{\rho} C_0^E \tag{A5}$$

$$\tau_t^H = \mu - \rho - \sigma^2 x_t - \frac{\dot{C}_t^H}{C_t^H} \tag{A6}$$

$$\tau_t^E = \mu - \rho - \sigma^2 x_t + \sigma^2 x_t^2 - \frac{\dot{C}_t^E}{C_t^E} \tag{A7}$$

where

$$x_t = \rho \frac{K_t}{C_t^E} \tag{A8}$$

and K_t *is given by (A2), such that* $(\mathbf{C}^E, \mathbf{C}^H)$ *are the aggregate consumption profiles arising in the unique general equilibrium with these policy parameters.*

Proof. If $(\mathbf{C}^E, \mathbf{C}^H)$ can be decentralized, individually optimal consumption (6) and (9) implies that aggregate household net worth and firm equity are

$$H_t = \frac{1}{\rho} C_t^H, \; E_t = \frac{1}{\rho} C_t^E \tag{A9}$$

and (19) implies $r_t = \mu - \sigma^2 x_t$.

By (A9), x_t as defined in (A8) then is the aggregate capital-equity ratio. By (A2), x_t is fully determined by $(\mathbf{C}^E, \mathbf{C}^H)$.

Again by (A9), $\dot{H}_t / H_t = \dot{C}_t^H / C_t^H$, and (16) implies that if $(\mathbf{C}^E, \mathbf{C}^H)$ can be decentralized, τ_t^H must be given by (A6). (A7) follows similarly. Because of (A9), (A4) follows from

$$\frac{1}{\rho} C_0^H = H_0 = \tilde{H} - L^H$$

and (A5) by a similar argument. Equilibrium public debt then is

$$\begin{aligned} B_t &= H_t + E_t - K_t \\ &= \frac{1}{\rho} C_t - K_t \end{aligned} \tag{A10}$$

Under the fiscal policy defined by (A4) - (A8), the aggregate quantities thus defined are consistent with the individual decision rules (6), (9), and (10) derived in Section 4, evaluated at the interest rate $r_t = \mu - \sigma^2 x_t$. Market clearing at all times is implied by (A10). We thus have identified the unique general equilibrium that implements the aggregate consumption profiles $(\mathbf{C}^E, \mathbf{C}^H)$. □

Appendix B: The Individual Decision Problems

For completeness, this appendix provides a detailed solution to the individual optimization problems in Section 4 that were only sketched in the main text.

B.1 Households

Suppose that the representative household has initial net worth n_0^H at time $t = 0$, no further income later, and can only save via safe debt. Consider the variation of the household's decision problem in which the household starts out at time $t \geq 0$ with net worth $n > 0$. It chooses a consumption path c_s^H, $s \geq t$, to solve the standard consumption problem

$$\begin{aligned}
&\max_{c^H} \int_t^\infty e^{-\rho s} \log c_s^H ds \\
&dn_s^H = \left((r_s - \tau_s^H)n_s^H - c_s^H\right) ds \qquad \text{(B1)}\\
&n_t^H = n \\
&n_s^H \geq 0
\end{aligned}$$

Denote the optimal consumption path for this problem by $c_s^H(t, n)$.

Remark 1. *The problem is homogeneous and invariant to scaling. Hence, if $c_s^H = c_s^H(t, n)$, $s \geq t$, is an optimal path for the problem with initial condition $n_t^H = n$, then αc_s^H, $s \geq t$, is an optimal path for the problem with initial condition $n_t^H = \alpha n$, for $\alpha > 0$.*

Hence, any optimal path satisfies

$$c_s^H(t, n) = c_s^H(t, 1)n$$

Let $V^H(t, n)$ be the value function of the problem. Homogeneity implies

$$\begin{aligned}
V^H(t, n) &= \int_t^\infty e^{-\rho s} \log c_s^H(t, n) ds \\
&= \frac{e^{-\rho t}}{\rho} \log n + v^H(t) \qquad \text{(B2)}
\end{aligned}$$

where

$$v^H(t) = \int_t^\infty e^{-\rho s} \log c_s^H(t, 1) ds \qquad \text{(B3)}$$

is independent of n.

Ignoring the non-negativity conditions (which will be satisfied at the optimum), the Bellman Equation of the household's problem is

$$\frac{\partial V^H}{\partial t} + \max_c \left[e^{-\rho t} \log c + \frac{\partial V^H}{\partial n} \left((r_t - \tau_t^H)n - c\right)\right] = 0$$

From (B2), we have

$$\frac{\partial V^H}{\partial n} = \frac{e^{-\rho t}}{\rho n}$$

such that the Bellman Equation becomes

$$\frac{\partial V^H}{\partial t} + \max_c \left[e^{-\rho t} \log c + \frac{e^{-\rho t}}{\rho n} \left((r_t - \tau_t^H) n - c \right) \right] = 0 \tag{B4}$$

It is easy to see that the first-order condition

$$c = \rho n \tag{B5}$$

is necessary and sufficient for the maximization problem in (B4). In particular, (B5) implies that $c > 0$. The Bellman Equation thus is equivalent to

$$-e^{-\rho t} \log n + \dot{v}^H(t) + e^{-\rho t} \left[\log \rho n - 1 + \frac{r_t - \tau_t^H}{\rho} \right] = 0$$

which is equivalent to

$$\dot{v}^H(t) = \frac{e^{-\rho t}}{\rho} \left[\rho - \rho \log \rho - r_t + \tau_t^H \right]$$

This can be integrated explicitly to yield

$$\rho v^H(t) = (1 - \log \rho) \left(1 - e^{-\rho t} \right) - \int_0^t e^{-\rho s} (r_s - \tau_s^H) ds + \rho v^H(0) \tag{B6}$$

By (B5), if $n_s^H(t, n)$ is on the trajectory generated by $c_s^H(t, n)$, $s \geq t$, the optimal policy is

$$c_s^H(t, n) = \rho n_s^H(t, n) \tag{B7}$$

Hence, inserting (B7) into (B1) yields the law of motion for household savings with initial value 1 at time $t = 0$, $n_s^H(0, 1)$, as

$$\frac{d n_s^H(0, 1)}{ds} = (r_s - \tau_s^H - \rho) n_s^H(0, 1)$$

Integrating yields

$$\log n_s^H(0, 1) = \int_0^s (r_\tau - \tau_\tau^H - \rho) d\tau \tag{B8}$$

where the constant of integration in (B8) is $\log n_0^H(0, 1) = \log 1 = 0$, by the construction of v.

Inserting (B7) and (B8) into (B3) yields, for $t = 0$,

$$\begin{aligned} v^H(0) &= \int_0^\infty e^{-\rho s} (\log \rho + \log n_s^H(0, 1)) ds \\ &= \frac{\log \rho}{\rho} + \int_0^\infty e^{-\rho s} \int_0^s (r_\tau - \tau_\tau^H - \rho) d\tau ds \\ &= \frac{\log \rho}{\rho} - \frac{1}{\rho} + \frac{1}{\rho} \int_0^\infty e^{-\rho \tau} (r_\tau - \tau_\tau^H) d\tau \end{aligned}$$

Combining this with (B6) yields

$$\rho v^H(t) = -(1 - \log \rho) e^{-\rho t} + \int_t^\infty e^{-\rho s} (r_s - \tau_s^H) ds$$

which together with (B2) yields the households' value function as

$$\rho V^H(t, n) = e^{-\rho t} \left(\log(\rho n) - 1 \right) + \int_t^\infty e^{-\rho s} \left(r_s - \tau_s^H \right) ds$$

which is (7) in the main text.

B.2 Firms

Net of initial lump sum taxes, at time $t = 0$ firm i has an initial equity position $e_0^i > 0$. Consider the variation where a firm starts at time t with equity $e^i > 0$. It chooses a path $k_s^i, e_s^i, c_s^i,\ s \geq t$ such as to

$$\max_{k^i, e^i, c^i} \mathbb{E} \int_t^\infty e^{-\rho s} \log c_s^i ds$$

$$de_s^i = \left[(\mu - r_s)k_s^i + (r_s - \tau_s^E)e_s^i - c_s^i\right] ds + \sigma k_s^i dz_s^i \tag{B9}$$

$$e_t^i = e^i$$

$$e_s^i \geq 0$$

where equation (B9) is the flow of funds equation (8) in the main text, after substituting out $d_s^i = k_s^i - e_s^i$ from the balance sheet equation (4). Denote the value function of the problem by $V^E(t, e^i)$.

Since as in the household problem the feasible set is homogeneous, any solution is invariant to scaling, and we must have, at the optimum,

$$(k_s^i(t, e^i), c_s^i(t, e^i)) = (k_s^i(t, 1)e^i, c_s^i(t, 1)e^i)$$

Therefore,

$$V^E(t, e^i) = \frac{e^{-\rho t}}{\rho} \log e^i + v^E(t) \tag{B10}$$

where

$$v^E(t) = \mathbb{E} \int_t^\infty e^{-\rho s} \log c_s^i(t, 1) ds \tag{B11}$$

is independent of e^i.

We first solve the unconstrained problem, in which we ignore the non-negativity constraint on e_s^i. In this case, the Bellman Equation is

$$\frac{\partial V^E}{\partial t} + \max_{k,c} \left[e^{-\rho t} \log c + \frac{\partial V^E}{\partial e} \left((\mu - r_t)k + (r_t - \tau_t^E)e^i - c \right) + \frac{\partial^2 V^E}{\partial e^2} \frac{\sigma^2}{2} k^2 \right] = 0$$

From (B10), we have

$$\frac{\partial V^E}{\partial e} = \frac{e^{-\rho t}}{\rho e^i}$$

$$\frac{\partial^2 V^E}{\partial e^2} = -\frac{e^{-\rho t}}{\rho (e^i)^2}$$

The Bellman Equation therefore becomes

$$\frac{\partial V^E}{\partial t} + max_{k,c} e^{-\rho t} \left[\log c + \frac{1}{\rho e^i} \left((\mu - r_t)k + (r_t - \tau_t^E)e^i - c \right) - \frac{1}{2\rho (e^i)^2} \sigma^2 k^2 \right] = 0 \tag{B12}$$

and the first-order conditions

$$c = \rho e^i \tag{B13}$$

$$k = \frac{\mu - r_t}{\sigma^2} e^i \tag{B14}$$

are necessary and sufficient for the maximum in (B12). In particular, (B13) implies that $c > 0$.[29] The Bellman Equation therefore is equivalent to

$$\begin{aligned} -e^{-\rho t}\log e^i + \dot{v}^E(t) + e^{-\rho t}\left[\log \rho e^i - 1 + \frac{r_t - \tau_t^E}{\rho} + \frac{(\mu - r_t)^2}{2\rho\sigma^2}\right] &= 0 \\ \Leftrightarrow \dot{v}^E(t) = e^{-\rho t}\left[1 - \log\rho - \frac{r_t - \tau_t^E}{\rho} - \frac{(\mu - r_t)^2}{2\rho\sigma^2}\right] \end{aligned}$$

This is a deterministic ODE that can be integrated explicitly to yield

$$\rho v^E(t) = (1 - \log\rho)\left(1 - e^{-\rho t}\right) - \int_0^t e^{-\rho s}\left(r_s - \tau_s^E + \frac{(\mu - r_s)^2}{2\sigma^2}\right) ds + \rho v^E(0) \tag{B15}$$

From (B13)-(B14), if $e_s^i = e_s^i(t, e^i)$ is on a trajectory generated by $c_s^i(t, e^i)$ and $k_s^i(t, e^i)$, $s \geq t$, the optimal policy is

$$c_s^i(t, e^i) = \rho e_s^i \tag{B16}$$

$$k_s^i(t, e^i) = \frac{\mu - r_s}{\sigma^2} e_s^i \tag{B17}$$

Hence, inserting (B16) and (B17) into the equation of motion (B9) yields the (random) law of motion for firm equity, with $s \geq t$ and $e_t^i = e_t^i(t, e^i) = e^i$, as

$$de_s^i = \left[\left(\frac{\mu - r_s}{\sigma}\right)^2 + r_s - \tau_s^E - \rho\right] e_s^i ds + \frac{\mu - r_s}{\sigma} e_s^i dz_s^i \tag{B18}$$

$$\equiv (\beta_s - \rho) e_s^i ds + \gamma_s e_s^i dz_s^i \tag{B19}$$

where we have set, for simplicity,

$$\beta_s = \left(\frac{\mu - r_s}{\sigma}\right)^2 + r_s - \tau_s^E \tag{B20}$$

$$\gamma_s = \frac{\mu - r_s}{\sigma} \tag{B21}$$

We must determine $v^E(0)$. From (B11), using (B17), we have

$$\begin{aligned} v^E(0) &= \mathbb{E}\int_0^\infty e^{-\rho s}\log \rho e_s^i(0,1) ds \\ &= \frac{\log\rho}{\rho} + \mathbb{E}\int_0^\infty e^{-\rho s}\log e_s^i(0,1) ds \end{aligned} \tag{B22}$$

[29]Note that for the argument to work, there is no need to impose the condition $r_t < \mu$ at this stage.

Applying Itō's Lemma to (B19),

$$\begin{aligned} d\log e_s^i &= \frac{1}{e_s^i}de_s^i - \frac{1}{2\left(e_s^i\right)^2}\gamma_s^2\left(e_s^i\right)^2 ds \\ &= \left(\beta_s - \rho - \frac{1}{2}\gamma_s^2\right)ds + \gamma_s dz_s^i \end{aligned}$$

For $e_s^i = e_s^i(0,1)$, where by definition $e_0^i = 1$, this means that with probability 1,

$$\log e_s^i(0,1) = \int_0^s \left(\beta_\tau - \rho - \frac{1}{2}\gamma_\tau^2\right)d\tau + \int_0^s \gamma_\tau dz_\tau^i$$

By the definition of the stochastic integral, under standard integrability assumptions for r_s,

$$\mathbb{E}\int_0^s \gamma_\tau dz_\tau^i = 0$$

for every s. The expectation in (B22) therefore is

$$\begin{aligned} \mathbb{E}\int_0^\infty e^{-\rho s}\log e_s^i(0,1)ds &= \int_0^\infty e^{-\rho s}\int_0^s \left(\beta_\tau - \rho - \frac{1}{2}\gamma_\tau^2\right)d\tau ds \\ &= -\frac{1}{\rho} + \int_0^\infty e^{-\rho s}\int_0^s \left(\beta_\tau - \frac{1}{2}\gamma_\tau^2\right)d\tau ds \\ &= -\frac{1}{\rho} + \frac{1}{\rho}\int_0^\infty e^{-\rho\tau}\left(\beta_\tau - \frac{1}{2}\gamma_\tau^2\right)d\tau \end{aligned}$$

Inserting this into (B22) and using (B20)-(B21),

$$\rho v^E(0) = \log\rho - 1 + \int_0^\infty e^{-\rho s}\left(r_s - \tau_s^E + \frac{(\mu - r_s)^2}{2\sigma^2}\right)ds$$

Combining this with (B15) yields

$$\rho v^E(t) = -e^{-\rho t}(1 - \log\rho) + \int_t^\infty e^{-\rho s}\left(r_s - \tau_s^E + \frac{(\mu - r_s)^2}{2\sigma^2}\right)ds \tag{B23}$$

Finally, inserting (B23) into the value function (B10), yields

$$\rho V^E(t, e^i) = e^{-\rho t}(\log\rho e^i - 1) + \int_t^\infty e^{-\rho s}\left(r_s - \tau_s^E + \frac{(\mu - r_s)^2}{2\sigma^2}\right)ds$$

which is (15) in the main text.