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MYOPIC FISCAL OBJECTIVES AND LONG-RUN MONETARY EFFICIENCY

Gaetano Gaballo and Eric Mengus

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Centre for Economic Policy Research
33 Great Sutton Street, London EC1V 0DX, UK
Tel: +44 (0)20 7183 8801
www.cepr.org

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JEL Classification: E31, E52, E58, E62, E63

Keywords: Fiat money, Price level determination, Fiscal-monetary interactions, Seigniorage, Commitment, Ramsey plans

Gaetano Gaballo - gaballo@hec.fr
Hautes Etudes Commerciales de Paris and CEPR

Eric Mengus - mengus@hec.fr
Hautes Etudes Commerciales de Paris and CEPR

Myopic Fiscal Objectives and Long-Run Monetary Efficiency*

Gaetano Gaballo
HEC Paris and CEPR

Eric Mengus
HEC Paris and CEPR

March 7, 2023

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Keywords: fiat money, price level determination, fiscal-monetary interactions, inequality, redistribution, heterogeneous agents, incomplete markets.

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1 Introduction

Trading fiat money for goods can occur when people trust that the same trade will be possible at any future date. This confidence, which is the trademark of traditional currencies, has often been rationalized as the result of the commitment by some public authorities to preserve the value of money in the long run. In fact, such commitments are ubiquitous in monetary theories.[1] However, in reality, the presence of political cycles may undermine confidence in governments' long-run commitments. To avoid any such interference, in almost all economies, monetary policy has been delegated to an independent central bank mandated with an explicit objective of long-run price-stability (see Rogoff, 1985, and Walsh, 1995). But what are the conditions under which myopic fiscal goals may effectively threaten long-run monetary efficiency?

In this paper, we show that myopic redistribution concerns, stemming from one-period utility maximization, may actually sustain the socially efficient inflation rate as a by-product, even when public authorities have large fiscal spending needs and lack commitment ability or an explicit long-run goal.

To formally establish our point, we build on a textbook incomplete-market model: the monetary Overlapping Generations Model (OLG) *à la* Samuelson (1958),[2] and we enrich this model along two dimensions.

First, we introduce an alternative to money as a store-of-value: a storage technology with a socially inefficient fixed return, as in Sims (2013), which creates a meaningful private portfolio problem for private agents. By providing a lower bound to real returns, storage allows for closed-form expressions for off-equilibrium paths along which money progressively loses value, as in hyperinflations.[3] Importantly, these two saving vehicles

[1] A stream of literature has emphasized the commitment to back money with real resources as a way to prevent extreme events, such as hyperinflations. This can be a commitment to either a fractional currency backing, as in Obstfeld and Rogoff (1983), or future fiscal surpluses, as in the Fiscal Theory of the Price Level (Leeper, 1991; Woodford, 1994, 1995; Sims, 1994, 2013; Bassetto, 2002) (see Obstfeld and Rogoff, 2017 for a discussion of the two approaches.) Other studies have advocated the importance of commitment ability to prevent inefficient nominal fluctuations: this is, for example, the purpose of interest-rate rules satisfying the Taylor principle in New-Keynesian models, to which timeless policy makers should optimally commit.

[2] This model is well known to capture the self-fulfilling nature of the store-of-value role of money – a role of money studied by Wallace (1981b) and, more recently, by Asriyan et al. (2021). As Brunnermeier and Sannikov (2016) argue forcefully, it is also a natural benchmark to capture the redistributive impact of monetary policy (e.g., Doepke and Schneider, 2006; Sterk and Tenreyro, 2018; Auclert, 2019).

[3] As noticed by Sargent (1982), among others, during hyperinflation episodes, money may also stop playing its other roles, as a medium of exchange and a unit of account. For the same reason, notice that, to analyze hyperinflation situations, we then cannot simply assume that money enters into utility or is part of a feasibility constraint such as cash-in-advance. Also, we are not interested here in the properties of special assets that have intrinsic advantages in becoming dominant means of exchange (see Williamson

have different consequences for redistribution: whereas money can transfer consumption across generations, storage transfers consumption over time.

Second, we introduce a sequence of one-period authorities that have the power to tax and carry out money market operations – i.e., can buy and sell money. Each authority implements a policy in order to maximize its one-period objective, which includes the utility of present (but not future) households and its own consumption. Thus, authorities are completely myopic as they embody no consideration for the future.

In this model, absent any policy intervention, a monetary equilibrium exists where savings are fully monetary, but other equilibria are also possible.[4] There exists global indeterminacy because a complementarity in the decisions to save in money across generations feature equilibria in which the real value of money shrinks progressively in time as storage crowds it out. In addition, there exists local indeterminacy in that there are dynamically-stable multiple equilibria in which only money is used. Furthermore, in all these equilibria, monetary included, the rate of return is below the socially efficient level.

Our main result is that the sequence of myopic policy interventions, together with optimal private saving decisions, selects a unique equilibrium – one in which the socially efficient intertemporal rate of return on savings prevails – although public interventions do not pursue any intertemporal objective.

The intuition behind the result is the following. Private agents already evaluate intertemporal trade-offs efficiently without any externality; however, the absence of a market for consumption between old and young – the typical market incompleteness of OLG economies – allows for suboptimal equilibria where marginal utilities are not equalized, as the young consume more than the old. The optimal policy by the myopic authority provides for buying money backed by tax revenues to increase its real return until the old get the same level of consumption as the young. Thus, the policy nails down equality among marginal utilities realizing the same outcome as complete markets. In particular, we show that the optimality conditions stemming from private portfolio choices and myopic policy objectives reproduce the set of conditions that characterize the unique solution to the unconstrained long-run social planner problem.

Key to the result is also what the current authority cannot do. In fact, a myopic but unconstrained social planner would make agents consume any given stock of storage right

and Wright, 1980, for an overview).

[4]Since at least Wallace (1978), this multiplicity of equilibria has been interpreted as reflecting the self-fulfilling nature of money and the role of confidence in money exchanges.

away to maximize current consumption. Instead, the authority cannot control future price levels and, hence, current private saving decisions, which are then taken efficiently by private agents.

In the second part of the paper, we explore the robustness and limits of the uniqueness and efficiency results by working out tractable extensions of our baseline model. We show simple ways of introducing production through labor or capital that preserve both efficiency and uniqueness. We then show how different frictions in redistribution may preserve uniqueness, but not the efficiency of the equilibrium. Finally, we show that when taxes are constrained to not exceed an upper bound, both uniqueness and efficiency may get lost because of a conflict between public consumption and redistribution, which does not emerge otherwise.

Literature review. To the best of our knowledge, this is the first paper showing that short-run fiscal objectives jointly with optimal private saving decisions may lead to the efficient determination of the price level. In previous literature, the emphasis on long-run commitments prevented a full appreciation of the fact that, when policy implementation occurs through markets, the efficiency of intertemporal prices is assured through private agent choices. Thus, authorities do not necessarily need to care about intertemporal optimality for policy to select the efficient equilibrium.

Our paper relates to a famous literature on the interaction between monetary and fiscal policy, as pioneered by Sargent and Wallace (1981). In the same spirit, we study a framework in which the conduct of fiscal policy is crucial for monetary stability. In contrast to this literature, in our setting, the presence of a fiscal authority is not only a source of danger, but plays an active and essential role in preserving monetary stability. Consistent with Wallace (1981b), we show that interventions require fiscal backing. Yet this requirement does not imply fiscal interventions in equilibrium, but rather out of equilibrium.

On the one hand, our theory relates to Obstfeld and Rogoff (1983), in that an off-equilibrium intervention is essential to stabilize the money market, and, in principle, there could be no fiscal interventions along the equilibrium (this is the case with a discount factor equal to one, as discussed in the paper). On the other hand, Obstfeld and Rogoff (1983) (see also Wallace, 1981a) demonstrate that the mere ability to commit has such strong consequences that the presence of a fiscal authority may not even be necessary. Anyone endowed with commitment power can use an arbitrarily small redemption value to prevent fiat money from losing value. In our theory, instead, commitment has no role, whereas the ability to raise taxes is crucial.

Nicolini (1996) analyzes the mechanism of Obstfeld and Rogoff (1983, 2017) in a model in which a fiscal authority decides under discretion the implementation of a costly conversion facility for money. In his model, should hyperinflation occur, there is always a period in which the social costs of hyperinflation will exceed the fixed-cost of the conversion facility. As agents anticipate the intervention of the authority, hyperinflation does not occur, although the facility is not implemented along the equilibrium. In contrast to Nicolini (1996), we assume, as in Sims (2013), that agents face a portfolio choice that effectively constrains the authority's plan, in the spirit of Bassetto (2002). Absent such a feature, our model would always exhibit a unique equilibrium, even in the case of limits to fiscal capacity, consistent with Nicolini (1996).

More recent works about the determination of the price level include Benigno (2020) and Hall and Reis (2016), among others. Although all these works deviate from the typical framework of the fiscal theory of the price level, they are also concerned with the commitment to a particular rule for fiscal transfers without inquiring about its optimality and sub-game perfection.

In this respect, we are closer in spirit to Atkeson et al. (2010) and, more generally, to Bassetto (2005), who emphasize that policy implementation is not about committing to unconditional *actions*, but about committing to a strategy leading to feasible actions as a function of private agents' decisions. The optimal policy should then make privately suboptimal those actions that the authority finds undesirable and cannot directly control. In contrast to these papers, we do not assume any form of commitment on the side of the fiscal/monetary authority, consistent with Cochrane's (2011) discussion of credibility. We share this approach with Barthelemy and Mengus (2022), who investigate the social cost of the commitment required to implement a unique equilibrium in macroeconomic games.

Other papers investigate the effects of monetary policy rules in Overlapping Generation Models by postulating a demand for money, such as, recently, Asriyan et al. (2021). Another example is Tirole (1985), who considers a situation in which the government forces agents to invest some of their savings in an intrinsically worthless asset (that he labels "gold"). More generally, exogenous motives for money demand obtain by introducing a cash-in-advance or money-in-the-utility-function, as reviewed by Walsh (2010).

All these approaches rule out equilibria in which money loses value by assumption; these equilibria are, instead, the only source of multiplicity in our paper.

Another related stream of literature models money as one possible emerging medium of exchange in search and matching economies (Kiyotaki and Wright, 1989). Also, in those environments, one may formalize the idea that the government's commitment to implement a certain transaction can coordinate agents on the preferred medium of exchange as a unique equilibrium (Aiyagari and Wallace, 1997; Li and Wright, 1998). A natural interpretation of such a commitment is the fact that tax obligations can be carried out in money only as modeled by Starr (1974) among others. In any case, Malmberg and Öberg (2021) show theoretically that the constraint to pay tax in money is, in fact, neither a necessary nor a sufficient condition to ensure price level determination.

Our paper is also connected to the literature on multiplicity of equilibria and seignorage revenues initiated by Bruno and Fischer (1990). In contrast to them, we find that, because of private portfolio choices, higher equilibrium rates of inflation may be associated with lower seignorage income.

2 A Simple Model of Fiat Money

2.1 Physical environment

The economy is populated by equal-sized overlapping generations of atomistic agents and a sequence of short-sighted fiscal authorities. Time is discrete and indexed by $t \in \{0, 1, 2, ...\}$. The consumption good is homogeneous and perfectly divisible and it appears in each period as flows of endowments. Endowments are sufficiently larger in agents' first period of life that agents have an incentive to save. Saving can occur in two forms.

First, there exists a homogeneous and perfectly divisible asset called money, which is intrinsically worthless. Money exists in an initial physical stock M_0 in the economy. The fiscal authority can hold physical money and can also issue liabilities that are indistinguishable from physical money. Thus, at each time t, we have

$$M_t + M_{g,t} = M_0, \tag{1}$$

where M_t is the stock of money privately owned, whereas $M_{g,t}$ denotes the stock of money held by the authority. Only the latter can be negative, in which case the money held by the private sector must include both physical money and public liabilities.

The alternative to money is to store part of the endowment in a technology with a fixed real return as in Sims (2013). Every quantity of consumption goods stored at time t – namely, S_t – yields θS_t quantity of consumption goods available next period, where $\theta < 1$. Whereas trading money can transfer consumption across agents, storing transfers individual consumption across time, but at a cost. This storing cost makes the return on storage to be inefficiently low, allowing money trades to be *essential* for improving welfare in the sense of Wallace (1981b).[5]

Households. At each date, a new generation of homogeneous agents is born. Each agent lives two periods and then disappears. Agent $i \in (0,1)$ born at time t maximizes the following utility function:

$$U_{i,t} \equiv u(C_{i,y,t}) + \beta u(C_{i,o,t+1}), \tag{2}$$

where $C_{i,y,t} \geq 0$ and $C_{i,o,t+1} \geq 0$ are individual consumption in the first and second period, respectively; $\beta \in (0,1]$ is the discount factor and $u(\cdot) \in \mathcal{U}$ is the utility function. $\mathcal{U}$ denotes a set of continuous and differentiable functions $u(\cdot) : \mathbb{R}^+ \rightarrow \mathbb{R}$ with typical concavity properties – i.e., $u'(\cdot) > 0$, $u''(\cdot) < 0$ and $u'(0) \rightarrow \infty$, with $u'(\cdot)$ being a multiplicative function. Standard utility functions such as CRRA and CARA belong to this family.

The budget constraint of an agent i born in period t is:

$$C_{i,y,t} = W^y - T_t - S_{i,t} - \frac{M_{i,t}}{P_t}, \tag{3}$$

$$C_{i,o,t+1} = W^o + \theta S_{i,t} + \frac{M_{i,t}}{P_{t+1}}, \tag{4}$$

where W^y and W^o are endowments in consumption goods available to the agents when young and old, respectively; T_t is a (positive or negative) lump-sum real transfer paid by the young; $S_{i,t} \geq 0$ is the amount of goods stored in the first period; $P_t \geq 0$ is the equilibrium price of consumption in terms of money; and $M_{i,t} \geq 0$ is the quantity of money acquired by i when young at time t. The first generation is born at date 0, lives just one period, owns a stock of fiat money $M_0 > 0$, does not have storage, $S_{-1} = 0$, and has utility function $U_0 \equiv u(C_{o,1})$. Aggregate consumption, storage and monetary

[5]Our argument remains valid as far as money is used as a store of value in equilibrium, as it is, for example, when including capital with stochastic returns as in Brunnermeier and Sannikov (2016).

holdings are denoted by $C_{y,t} \equiv \int C_{i,y,t}di, C_{o,t} \equiv \int C_{i,o,t}, S_t \equiv \int S_{i,t}di$, and $M_t \equiv \int M_{i,t}di$, respectively.

The authorities. In analogy with households, we introduce a sequence of short-sighted authorities, each one solving a one-period problem. The authority in office at time t maximizes the following one-period objective function:

$$\mathbb{U}_t \equiv \int u(C_{i,y,t})di + \int u(C_{i,o,t})di + \tilde{\lambda} u(G_t), \tag{5}$$

which, it is worth noting, does not include any monetary target. The authority cares, instead, about the current flow of utilitarian welfare – i.e., the utility of the current young and old agents, but also the level of public spending – that is, its own consumption, G_t proportional to $\tilde{\lambda} \geq 0$.[6] Notice also that we assume for simplicity that the authority puts the same weights for the old and the young: what is important in our results is that the authority is sufficiently willing to transfer resources to the old generation/money holders from the young generation/taxpayers.

The budget of the authority is written as:

$$T_t + \frac{M_{g,t-1}}{P_t} = \frac{M_{g,t}}{P_t} + G_t. \tag{6}$$

That is, transfers plus the real value of money holdings from the previous period must equal public consumption and the real value of new money holdings. Because of (1), an increase in $M_{g,t}$ corresponds to a decrease in M_t. The budget constraint of the authority must hold in any state of the world – i.e., in and off equilibrium. In fact, the price level P_t is determined in the money market as described below, and taxes adjust to make sure that (6) always holds.

Money market. We describe in detail here how the market for money works. In the market, money demand expressed in terms of consumption goods has to match the real value of the supply of money. Formally, let us denote by $m_{i,t}$ the private real demand of money by agent i at time t, which cannot exceed available real resources

$$m_{i,t} \leq W^y - T_t - S_{i,t} - C_{i,y,t}.$$

[6]Note that G_t does not necessarily entail a "waste." The $\tilde{\lambda} u(G_t)$ component can be added to the utility of the agents without any impact on any private choice: in such a case, G_t denotes a public good whose provision is out of the control of the agents (for example, a public health good).

The aggregate quantity $m_t \equiv \int m_{i,t}\, di$ is, therefore, the quantity of goods owned by the young put up for exchange with money, at time t. By analogy, let us define the public real demand of money – i.e., the quantity of goods that the authority bids in exchange for money, at time t – as follows:

$$m_{g,t} \leq T_t - G_t$$

The private supply of money at time t is simply M_{t-1}— i.e., the money holdings of the old – as there is no alternative use. Let us further indicate by $M^S_{g,t} \geq 0$ the public supply of money. That is, the authority can be a buyer or a seller of money. We refer to $\Delta_t = \left(m_{g,t}, M^S_{g,t}\right)$ as the position of the authority on the money market.[7]

For a given available nominal supply of money $M_{t-1} + M^S_{g,t}$ and real money demand $m_t + m_{g,t}$ a *market-clearing price* P_t is such that

$$P_t \left(m_t + m_{g,t}\right) = M_{t-1} + M^S_{g,t}. \tag{7}$$

Finally, $M_{g,t} = M_{g,t-1} + m_{g,t}P_t - M^S_{g,t}$ and

$$M_t = m_t P_t \tag{8}$$

define the stocks of money held by the authority and the private sector, respectively, at the end of the trade.

2.2 Timing, market clearing and equilibrium

Let us now describe the economy as a game between households and the authority. We state timing assumptions and then formally define the strategic space of each actor in the economy and a notion of equilibrium.

Timing. In our economy, all actions taken at a given time t are set simultaneously. Each time is characterized by an aggregate state $\omega_t \equiv \{W^o, W^y, S_{t-1}, M_{t-1}\}$. We define a policy

[7] Note that one can rewrite the budget constraint of the authority uniquely in terms of the actions that the authority can perform:

$$T_t + \frac{M^S_{g,t}}{P_t} = m_{g,t} + G_t.$$

In words, the authority can tax or use seigniorage revenues to either consume or buy money from the private sector.

of the authority $\mathcal{P}_t \equiv (T_t, \Delta_t, G_t)$ as a collection of transfers imposed on the young,[8] and money market operations and public consumption that are implemented by the authority at time t.

Actions and continuation policies. A date-t strategy for the authority is a mapping $\sigma_{\mathcal{P},t} : \omega_t \mapsto \mathcal{P}_t$ from aggregate states to an action at date t decided by the time-t authority. We define $\sigma_{\mathcal{P}} \equiv \{\sigma_{\mathcal{P},\tau}\}_{\tau=1}^{\infty}$ as the policy plan of authorities. A date-t strategy for households is a mapping $\sigma_t : \omega_t \mapsto \{S_t, m_t\}$ from aggregate states to a portfolio choice of households at date-t.[9] We define $\sigma \equiv \{\sigma_\tau\}_{\tau=1}^{\infty}$ as the policy plan of households. We define $\sigma^t = \{\sigma_\tau\}_{\tau=t}^{\infty}$ and $\sigma_{\mathcal{P}}^t = \{\sigma_{\mathcal{P},\tau}\}_{\tau=t}^{\infty}$ as the continuation strategies of households and the authorities, respectively, from time t onward.

Equilibrium. Consistent with the literature on macroeconomic games (e.g., Ljungqvist and Sargent, 2018, among others), we use the concept of competitive equilibrium on the private agents' side and require the authority to implement the optimal policy. We restrict this to symmetric equilibria, without any loss of generality (see later remark).

Definition 1. For a given initial state ω_0, an *equilibrium* is a set of policy plans $(\sigma, \sigma_{\mathcal{P}})$ such that at any ω_t for $t \geq 1$, $\{S_t, m_t\} = \sigma_t(\omega_t)$ and $\mathcal{P}_t = \sigma_{\mathcal{P},t}(\omega_t)$ are such that:

(i) $\{S_t, m_t\}$ maximizes (2) subject to (3)–(4) for each i, taking prices (P_t, P_{t+1}) and taxes T_t as given;

(ii) $\mathcal{P}_t$ maximizes (5) subject to (6), taking $(\sigma^t, \sigma_{\mathcal{P}}^{t+1})$ as given;

(iii) P_t is determined by (7), M_t by (8);

(iv) market-clearing conditions for money (1) hold.

In an equilibrium, each individual choice at time t is a best response to the perfect foresight of the aggregate choice of other agents and the authority from time t onwards. As all of these agents are atomistic, this leads them to take price levels and taxes as given. In analogy, the authority at time t sets a best response to the perfect foresight of the aggregate choice of agents from time t onwards and to the policies of future authorities

[8] Here, we implicitly assume that the government cannot implement direct transfers to the old. As will become clear, the absence of such an instrument, by itself, will not prevent the authority from implementing its first-best allocation.

[9] Note that the money acquired by households, M_t, is an equilibrium object, as it depends on the price level. Defining the macroeconomic game with markets in this way is consistent with Bassetto (2002): only in this way we can make sure that this action is selected consistently with households' budget constraint off-equilibrium.

from time $t+1$ onwards. The short-sighted behavior of agents and authorities may lead to possible miscoordinations. On the one hand, agents are subject to coordination failures, as they take (current and future) aggregate actions as given.

Aggregate resources. The aggregate resource constraint

$$C_{y,t} + C_{o,t} + G_t = W^y + W^o + \theta S_{t-1} - S_t, \tag{9}$$

which holds in equilibrium because of Walras's law, shows that private storage decisions affect the availability of resources at a given time: the higher the storage, the lower real resources available. In this sense, portfolio choices of agents effectively put constraints on the feasibility of fiscal plans, in line with the point put forward by Bassetto (2002). In particular, as we will see later more formally, any positive level of S_t is sub-optimal from the point of view of the current authority as it implies lower resources for current consumption.

3 Optimal portfolios and optimal policy

In this section, we derive the optimal policies of the young and the authority and show how monetary policy gets implemented through the money market.

Optimal private portfolios. At each date t, the young generation decides how much to save and how to divide the resulting savings between storage and money holdings. We focus on symmetric equilibria. Let us denote by ρ_{t+1} the gross per-unit real return on real savings D_t defined as:

$$D_t \equiv S_t + m_t,$$

where m_t is the equilibrium real money holding as defined in (8). For a given $\sigma_{\mathcal{P}}$, the optimal level of real saving D_t is given implicitly by $u'\left(W^y - T_t - D_t\right) = \beta\rho_{t+1}u'\left(\rho_{t+1}D_t + W^o\right)$, whereas the split between money and storage is given by arbitrage between the equilibrium return on money Π_{t+1}^{-1} and storage θ. We can then state the following.

Lemma 1 (Optimal private-sector policy)**.** *For a given arbitrary policy plan $\sigma_{\mathcal{P}}$, the* *private-sector optimal policy $\sigma_t^* \in \sigma^*$ at any date $t \geq 1$ is given by*

$$S_t = 0, \quad m_t = D_t \qquad \textit{if} \quad \Pi_{t+1}^{-1} > \theta \qquad \textit{in which case} \quad \rho_{t+1} = \Pi_{t+1}^{-1}, \tag{10}$$

$$S_t + m_t = D_t \qquad \textit{if} \quad \Pi_{t+1}^{-1} = \theta \qquad \textit{in which case} \quad \rho_{t+1} = \theta, \tag{11}$$

$$S_t = D_t, \quad m_t = 0 \qquad \textit{if} \quad \Pi_{t+1}^{-1} < \theta \qquad \textit{in which case} \quad \rho_{t+1} = \theta, \tag{12}$$

where

$$D_t = \frac{W^y - T_t - R(\rho_{t+1})\rho_{t+1}^{-1} W^o}{1 + R(\rho_{t+1})}, \tag{13}$$

and $R(\rho_{t+1}) \equiv u'_{-1}(\beta\rho_{t+1})\,\rho_{t+1}$, with $\rho_{t+1} = \max\{\Pi_{t+1}^{-1}, \theta\}$ and u'_{-1} being the inverse of u'. Note $R(\rho) > \rho$ for any $\rho \in (0,1)$.

Proof. See Appendix A.1 $\square$

Savings choices are purely forward-looking: the young make savings decisions only by looking at future returns; current inflation is not relevant to their savings decision. Money and storage may coexist only insofar as they yield the same return.

To make the savings problem of the young non-trivial, we shall maintain that the endowment of the old is sufficiently small. This requirement is formally captured by:

$$W^y > R(\theta)\theta^{-1} W^o,$$

that is, at the minimal savings return θ, the young still have an incentive to save. This assumption captures the essence of OLG models, where the efficient transfer of resources through market transactions is prone to inefficient coordination failures.

Remark: The restriction to symmetric equilibrium is without loss of generality. Since returns on savings are determined by aggregate variables only (Π_{t+1}, θ), objectives are strictly concave, and budget sets are convex, there will be a unique solution to the individual saving problem – i.e., $D_{i,t} = D_t$ for each i. Nevertheless, the allocation of real returns between money and storage is a potential source of within-cohort heterogeneity when both yield the same return – that is, when $\Pi_t^{-1} = \theta$. We show in Appendices A.2 and A.3 that such heterogeneity is immaterial to the characterization of the set of equilibria.

Constrained-optimal myopic policy. We will now derive the optimal response of the myopic authority. The first step is to note that the budget constraint of the young individual can be rewritten independently of current real money demand $m_{i,t}$ and current taxes T_t, as the following lemma states.

Lemma 2. *The level of consumption by the young is given by:*

$$C_{y,t} = W^y - G_t - \Pi_t^{-1} m_{t-1} - S_t. \tag{14}$$

Proof. See Appendix A.2. □

This is a powerful implication because it shows that the consumption of the young is independent of any return ρ_{t+1}, discount factor β and utility function $u(\cdot)$, and depends only on storing choices, public consumption and real money holdings of the old.

We then show how the *current* authority implements monetary policy determining *current* (but not future!) inflation for given private sector choices. The simultaneous trades of agents and the authority on the money market determine the rate of inflation $\Pi_{t+1} \equiv P_{t+1}/P_t$ between period t and $t+1$. In particular, because of (7), we can state the following.

Lemma 3 (Implementation of Monetary Policy)**.** *For given* (P_{t-1}, m_{t-1}, m_t)*:*

$$\Pi_t = \frac{m_{t-1} + M_{g,t}/P_{t-1}}{m_t + m_{g,t}} \tag{15}$$

entails a surjective mapping from Δ_t *to* Π_t*.*

By offering more money on the market ($M_{g,t} > 0$), the authority pushes the price level up, producing inflation. In contrast, by demanding money against consumption ($m_{g,t} > 0$), the authority depresses the current price level, reducing inflation. Thus, for given private choices, choosing a position on the money market Δ_t amounts to choosing the *current* inflation Π_t. However, the current authority has no control on *future* inflation Π_{t+1}, which is what matters to current storing choices S_t as shown by Lemma 1.

By plugging (14) into the objective of the authority, we can easily derive the constrained-optimal policy of the authority as stated by the following proposition.

Proposition 4. *For a given portfolio policy* σ_t*, we can rewrite the constrained problem of* *the authority at time t as:*

$$\max_{\Pi_t, G_t} \left\{ u \underbrace{\left(W^y - G_t - \Pi_t^{-1} m_{t-1} - S_t\right)}_{=C_{y,t}} + u \underbrace{\left(\Pi_t^{-1} m_{t-1} + \theta S_{t-1} + W^o\right)}_{=C_{o,t}} + \tilde{\lambda} u(G_t) \right\}, \tag{16}$$

whose solution, once defined $\lambda = 1/(u')^{-1}(\tilde{\lambda})$, is given by:

- *$\Delta_t(\sigma_t)$ is such that $C_{y,t} = C_{o,t}$, that is, according to (15)*

$$\Pi_t(\sigma_t) = \frac{(2+\lambda) m_{t-1}}{W^y - (1+\lambda)(W^o + \theta S_{t-1}) - S_t} \qquad \text{if} \quad \lim_{m_{t-1} \to 0} C_{y,t} \geq \lim_{m_{t-1} \to 0} C_{o,t} \tag{17}$$

$$\Pi_t(\sigma_t) \to \infty \qquad \text{otherwise} \tag{18}$$

- *$G_t(\sigma_t)$ is such that $G_t = \lambda C_{o,t}$, that is,*

$$G_t(\sigma_t) = \frac{\lambda}{1+\lambda}(W^y - S_t - \Pi_t^{-1} m_{t-1}), \tag{19}$$

- *$T_t(\sigma_t)$ is such that (6) holds, that is,*

$$T_t(\sigma_t) = \frac{1}{1+\lambda} \Pi_t^{-1} m_{t-1} + \frac{\lambda}{1+\lambda}(W^y - S_t) - m_t. \tag{20}$$

Proof. See Appendix A.3. □

Expression (16) reveals the trade-offs at stake in the policy problem. According to (17), the optimal inflation level is the one that equalizes consumption of the young with that of the old. To increase the price level, the authority raises real resources by taxing the young generation and uses these resources to purchase money from the old, thus redistributing resources to them. A corner solution (18) emerges when the young consume less than the old at the autarky limit, $m_{t-1} \to 0$, in which case the authority would like to choose a negative money return to transfer resources from the latter to the former: given that this is unfeasible, $\Pi_t \to \infty$ obtains. The optimal amount of public consumption (19) is such that the marginal utility of consumption of the young is equal to the marginal utility of public consumption weighted by λ. In the case where $u = \log$, $\lambda = \tilde{\lambda}$. More generally, λ is an increasing function of $\tilde{\lambda}$ so that $\lambda = 0$ when $\tilde{\lambda} = 0$ and $\lambda \to \infty$ when $\tilde{\lambda} \to \infty$. Taxes (20) clear the budget constraint of the authority.

4 Equilibrium

In this section, we characterize the set of equilibria. First, we show that, in the absence of policy interventions, the economy exhibits multiple equilibria. We then demonstrate that the implementation of the constrained-optimal myopic policies leads to a single equilibrium in which money is the only savings asset and yields the efficient intertemporal rate of return.

4.1 Multiplicity in the absence of policy reaction

Let us first establish the benchmark in the absence of public policies – i.e., with $\mathcal{P}_t = (0,0,0)$ at each date t. In this case, by combining Lemma 3 and Lemma 1, we obtain that equilibrium inflation must satisfy:

$$\Pi_{t+1} = \frac{m_t}{m_{t+1}} = \frac{\frac{W^y - R(\rho_{t+1})\rho_{t+1}^{-1}W^o}{1+R(\rho_{t+1})} - S_t}{\frac{W^y - R(\rho_{t+2})\rho_{t+2}^{-1}W^o}{1+R(\rho_{t+2})} - S_{t+1}}, \tag{21}$$

given that $M_t = M_0$, and so $m_t = M_0/P_t$, for any $t \geq 1$. We can then easily check that, absent policy, a continuum of market equilibria exists, as the following proposition states.

Proposition 5. *For any $\{\lambda, \beta\}$ and initial conditions $M_0 > 0$ and $S_0 = 0$, without any policy – i.e., with $\sigma_{\mathcal{P}} = \{0,0,0\}$ for any $t \geq 1$ – a multiplicity of equilibria exist. In particular:*

i) ***Local indeterminacy of monetary equilibria*** *obtains when private-sector policies $\sigma_t^* \in \sigma^*$ given by $\{S_\tau = 0, m_\tau = D_t\}_{\tau=t}^{\infty}$ feature more than a sequence $\{\Pi_{\tau+1} < \theta^{-1}\}_{\tau=t}^{\infty}$ that satisfies (21) converging to $\Pi^* \equiv 1$.*

In the CRRA case $u(\cdot) = (\cdot)^{1-\sigma}/(1-\sigma)$ with $\sigma > 0$, local indeterminacy obtains when:

$$\left| \frac{(1+\sigma\beta^{1/\sigma}) + \frac{W^o}{W^y}(1-\sigma)}{(1-\sigma) + \frac{W^o}{W^y}(1+\sigma\beta^{-1/\sigma})} \right| < 1; \tag{22}$$

otherwise, a unique monetary equilibrium exists.

ii) ***Global indeterminacy of asymptotic autarky equilibria*** *obtains for each $s \geq 1$, such that the private-sector policy $\sigma_t^* \in \sigma^*$ is given by $\{S_\tau = 0, m_\tau = D_t\}_{\tau=1}^{s-1}$ with*

$\Pi_t \leq \theta^{-1}$ *for* $t \leq s$, *and by*

$$S_t = \frac{W^y - R(\theta)\theta^{-1}W^o}{1 + R(\theta)} - \theta m_t, \qquad \textit{and} \qquad m_{t+1} = \theta m_t$$

with $\Pi_{t+1} = \theta^{-1}$ *for* $t > s$, *with* $P_s \in (P^*, \theta^{-1}P^*)$ *and* $m_s = M_0/P_s$.

iii) ***An autarky equilibrium*** *exists where the private-sector policy* $\sigma_t^* \in \sigma^*$ *at any date* $t \geq 1$ *is given by* $m_t = 0$, $S_t = (W^y + R(\theta)\theta^{-1}W^o)/(1 + R(\theta))$ *and* $P_t \to \infty$.

Proof. See Appendix A.4 □

Without policy interventions, the model exhibits two different kinds of indeterminacy.

Local indeterminacy emerges when a continuum of monetary equilibria exists where storage is not used, but inflation remains bounded around its steady state. This occurs as there are local converging paths of inflation satisfying (21) for $S_t = S_{t+1} = 0$ at any t. Intuitively, local indeterminacy obtains when savings choices are sufficiently insensitive to inflation rates, which happens when income effects are sufficiently strong or β is sufficiently small.[10]

Global indeterminacy may arise instead due to the existence of storage equilibria in which money progressively loses value and consumption inequality between the young and the old emerges – asymptotically, these equilibria converge to autarky. The main force behind this kind of equilibria is the complementarity of storage decisions across generations: the more future agents invest in storage, the lower the future return on money, the larger the incentive of current agents to invest in storage.

In Figure 1, we illustrate a case in which the monetary equilibrium is unique and show its co-existence with storage equilibria. We assume that $u(\cdot) = \log(\cdot)$, $\beta = 1$, $\theta = 0.95$ and $W^y = W = 0.3$ and $W^o = 0$. A *monetary equilibrium* exists where agents never use storage. Agents then perfectly equalize consumption across periods. This equilibrium, which is denoted with a circle marker in Figure 1, is characterized by a constant real demand for money $m_t = W/(1 + R(1))$, constant prices $\Pi_t = 1$, and no storage.

In addition to this equilibrium, there also exist equilibria in which storage and money are both used, and storage progressively crowds out monetary savings. We call this kind of equilibria *asymptotic autarky equilibria*. As storage and money are used at the same

[10]For example, in the CRRA case with risk aversion σ, the condition of indeterminacy is maximal for $W^o = 0$ and reads as $\beta < ((\sigma - 2)/\sigma)^\sigma$; that is, it is characterized by a monotonically increasing upper bound on β as σ increases above 2, with the largest bound given by $\lim_{\sigma\to\infty}((\sigma-2)/\sigma)^\sigma \approx 1/e^2 = 0.13534$. This means that, at least in this example, local indeterminacy emerges only for a small discount factor.

time, in these equilibria, $\Pi_t = \theta^{-1}$ holds since arbitrage between the two savings assets must not be possible. Along these paths, real money demand follows the process:

$$m_{t+1} = \theta m_t, \tag{23}$$

that is, lower real money demand today depresses future real money demand, so that storage crowds out money as time goes on. In the end, storage converges to $\lim_{t\to\infty} S_t = W/2$. Given that $M_0/P_t = m_t$ in the absence of intervention and m_t converges to 0, money ultimately has no real value – i.e., $\lim_{t\to\infty} M_0/P_t = 0$. These equilibria are denoted with a cross marker in Figure 1. Importantly, notice that storage can jump in any period from zero to positive since there are positive levels of S_t compatible with $\Pi_t < \theta^{-1}$ for which $S_{t-1} = 0$ is optimal. In the figure we provide an example showing that storage jumps to a positive value at $S_{10} = 0.006$. However, for S_t to be positive, $\Pi_{t+1} = \theta^{-1}$, which implies that $S_{t+1} > S_t$. So, storage can jump from zero to positive at any period, but then it can never go back to zero.

An *autarky equilibrium* exists in the absence of policy interventions. It is represented by a single solid line in Figure 1. In this case, storage is maximal, and the real value of monetary savings is zero, with prices being infinitely large (so that inflation is not defined). Consumption profiles are the same as in an asymptotic autarky equilibrium with storage and money, as the return to savings is the same.

4.2 Equilibrium with Optimal Myopic Policies

We now turn to the case in which policy interventions are optimally chosen, as determined by Proposition 4. We first provide a set of equations characterizing the equilibrium outcome and then describe the equilibrium set. This set boils down to the only monetary equilibrium. We finally provide a discussion of why policy interventions lead to a single equilibrium.

Equilibrium Characterization. To start with, let us focus on the equilibrium conditions implied by the private sector. First, by combining the young generation's budget constraint (14) with the optimal level of taxes set by the authority in Proposition 4, we are able to compute the real demand for money at date t:

$$m_t = \frac{W^y + W^o - (2+\lambda)R(\rho_{t+1})\rho_{t+1}^{-1}W^o - (1+R(\rho_{t+1})(2+\lambda))S_t + \theta S_{t-1}}{(2+\lambda)R(\rho_{t+1})}. \tag{24}$$

Using (17) at date $t+1$, we can recover the actual law of motion for inflation as:

$$\Pi_{t+1} = \frac{1}{R(\rho_{t+1})} \frac{W^y + W^o - (2+\lambda)(R(\rho_{t+1})\rho_{t+1}^{-1})W^o + \theta S_{t-1} - (1+(2+\lambda)R(\rho_{t+1}))S_t}{W^y - (1+\lambda)(W^o + \theta S_t) - S_{t+1}}, \tag{25}$$

which must always hold in any equilibrium.

Now we investigate the equilibrium set once optimal policy is in play. Formally, this requires the equilibrium allocation to satisfy (25). As shown by the following proposition, this set of equilibria boils down to a unique equilibrium, one in which money is efficiently traded.

Proposition 6 (Global and local price level determination)**.** *For any $\{\lambda, \beta\}$, given endowments such that $W^y > (1+\lambda)W^o$, and initial state ω_0, there exists a unique equilibrium, $(\sigma^*, \sigma_{\mathcal{P}^*})$, in which money is efficiently traded. In such an equilibrium, at any $t \geq 1$:*

(i) $\sigma_t^* \in \sigma^*$ is such that: $S_t = 0$ and $m_t = \beta\frac{W^y-(1+\lambda)W^o}{2+\lambda}$;

(ii) $\mathcal{P}_t^* \in \sigma_{\mathcal{P}^*}$ is such that:

$$\begin{aligned}
\Pi_t &= \frac{1}{u'_{-1}\left(\beta\Pi_t^{-1}\right)\Pi_t^{-1}} = \beta \\
G_t &= \frac{\lambda}{2+\lambda}(W^y + W^o) \\
T_t &= \frac{1+\lambda-\beta}{2+\lambda}W^y - \frac{1-\beta(1+\lambda)}{2+\lambda}W^o; \text{ and}
\end{aligned}$$

(iii) the price level is given by $P_t = M_t/m_t$, where $M_t = \beta M_{t-1}$.

Furthermore, for any $S_t \in (0, (W^y - (1+\lambda)W^o)/(1+R(\theta))]$, a unique equilibrium exists in which consumption is equalized across living agents and storage shrinks over time at the socially efficient rate, reaching the steady state characterized by $(\sigma^*, \sigma_{\mathcal{P}^*})$.
Otherwise, when $W^y \leq (1+\lambda)W^o$, a unique equilibrium exists in which $m_t = S_t = 0$ for all $t \geq 1$ and $\Pi_t = \infty$ for all $t > 1$.

Proof. See Appendix A.5 □

The proposition states that, for the initial condition $S_0 = 0$, the optimal policy eliminates any possible inflation indeterminacy in monetary equilibria. Moreover it fixes the intertemporal rate of return in the monetary equilibrium – i.e., the inverse of inflation – equal to the discount factor β: this is indeed a socially efficient outcome in the spirit of the Friedman rule. To achieve this result, the authority taxes the young generation to buy money at a fixed rate. In particular, both private money holdings and prices shrink at a rate β consistent with a fixed real money demand.

The proposition also states that there exists a unique continuation equilibrium for any *given* positive S_t converging to the unique steady state. This equilibrium is characterized by a level of storage that shrinks towards zero at the socially efficient rate. We show in the proof of Appendix A.5 that any of the paths where $S_t > 0$ has to satisfy a second order differential equation:

$$R(\theta)S_{t+1} - (1 + R(\theta))\theta S_t + \theta^2 S_{t-1} = (R(\theta) - \theta)(W^y + W^o), \tag{26}$$

which does not depend on λ. The properties of (26) are key to understanding the implications of a jump to an off-equilibrium aggregate state $S_t > 0$ and why only one of such paths can be an equilibrium. We show that (26) effectively features a saddle-path for any given level of storage, so that only one stable path exists, which leads to the monetary steady state. Moreover, such a unique path provides for a deflation at the first period after a deviation to positive savings occurs, making the initial deviation from the monetary steady state $S_t > 0$ suboptimal. This result ensures that uniqueness obtains, not because a deviation from equilibrium would prevent the formation of any other equilibrium (as with non-Ricardian policies), but because such deviations are simply not optimal from an individual point of view.

Surprisingly, the existence and optimality of a unique monetary equilibrium is independent from the size of λ, provided the young have savings needs, i.e., $W^y > (1+\lambda)W^o$.[11] The key intuition for the irrelevance of λ is that the consumption of the government is a fraction of the consumption of the old (which, in this equilibrium, is equal to the consumption of the young), which can always be secured through taxes. As a result, whatever the level of λ, the authorities always induce the economy to stay in the monetary equilibrium,

[11] Notice that this condition is different from the one for which savings are positive in the absence of any policy intervention, that is $W^y > R(\theta)\theta^{-1}W^o$.

in which everyone, authorities included, are better off as overall consumption is larger.[12]

Finally, autarky is not an equilibrium with policy interventions. As we show in the proof of Proposition 6, in an autarky situation, the authority at time t has an incentive to exchange real resources for the money bought by the young at time $t-1$ (as a deviation from autarky), no matter how small the deviation is. This leads to an infinite return on money. To see this, suppose that a young individual at time $t-1$ buys an arbitrarily small but strictly positive amount of money, whereas no one else in either her cohort or the next cohort does – i.e., $m_{t-1} = \epsilon$ and $m_t = 0$, with $\epsilon > 0$ but arbitrarily small. According to (15), any combination $m_{g,t} > 0$ and $M^S_{g,t} = 0$, leads to $\Pi_t \to 0$ and then to an infinite return to money. Because of the profitability of any individual deviation from autarky, autarky cannot be an equilibrium.

Why does short-run redistribution ensure long-run efficiency? It is important to notice that, from the point of view of the current authority, any current positive level of storage is sub-optimal as it reduces the availability of resources for current consumption, as shown by (9). Thus, equilibrium allocations are not optimal from the point of view of the single authorities. Still, short-run redistribution entails the social first-best allocation (for given public consumption). What makes that possible?

The key to understand the mechanism is looking at the first-order conditions of the unconstrained social planner problem that maximizes the discounted sum of the authorities' utility flows. The solution, entailing the first-best allocation, is then the same as the one uncovered in Proposition 6. Formally, we have the following.

Proposition 7. *For given ω_{t-1}, the sequence $(\sigma^*, \sigma_{\mathcal{P}^*})$ solves the problem:*

$$\max_{\{C_{o,t}, C_{y,t}, G_t, S_t, M_t, P_t\}_{t\geq 1}} \sum_{t-1}^{\infty} \beta^{t-1} \mathbb{U}_t,$$

subject to the individuals' and authorities' budget constraints (3)–(4) and (6), and non-negativity constraints $M_t \geq 0$, $S_t \geq 0$, $P_t \geq 0$, at any $t \geq 1$.

Proof. See Appendix A.6. □

The first-order conditions of the social planner problem replicate the private agents' optimality conditions (10)–(13). This is expected as there is no externatility in the intertemporal trade-off evaluated by individuals. On top of that, and in contrast with

[12] This statement generalizes to the case in which the authority gives a sufficiently large relative weight to money holders – i.e., the old generation.

the set of private optimality conditions, the solution of the social planner provides for the equality of marginal utilities between the young and the old. This condition cannot be ensured in the OLG economy, since there is no market possible between young and old as the old cannot receive anything of use in exchange for current consumption. This is the typical market incompleteness in OLG economies.

The policy of the myopic authority, aimed at maximizing (5), makes sure that consumption – and so the marginal utilities – of old and young are equalized in any state of the world. It therefore replicates, jointly with private agents' optimality conditions (10)–(13), the set of conditions nailing down the first-best allocation, the same that a unconstrained social planner would choose. In this respect, it is instructive to note that (26) equally obtains from the first-order condition of a unconstrained planner problem where money is absent: $\max_{S_t} \{u(c_t) + \beta u(c_{t+1})\}$ subject to $c_t = \theta S_{t-1} - S_t$. Thus, the outcome of myopic and uncoordinated policy interventions is ensuring paths along which any given non-zero stock of storage is optimally consumed in time.

Contingent fiscal surplus: an illustration. In the fiscal theory of the price level, a commitment to a fixed fiscal surplus leads to price level determinacy (Sims, 2013, among others). In our model, we give microfoundations to a policy that generates a state-contingent fiscal surplus able to ensure not only determinacy, but also optimality, of the price level.

We illustrate this mechanism in Figure 2. In this figure, we plot the pure monetary equilibrium with circles, but also the continuation of an equilibrium for a *given* S_t with cross markers. The figure is produced with the same parametrization as in Figure 1, except that we now assume that $\lambda = 0.5$. Note that, along the pure monetary equilibrium, because the authority cares about its own consumption, private consumption is lower than in Figure 1, as taxes are raised. On the other hand, the case $\beta = 1$, plotted in the figure, corresponds to a monetary equilibrium in which inflation is equal to one and primary fiscal surplus is zero; that is, public spending is completely financed by taxes.

In analogy to Figure 1, we explore a potential equilibrium starting at $S_0 = 0$ with a jump to positive storage at $S_{10} = 0.006$. The dashed line with cross markers denotes the ideal path of storage satisfying (26) that would have sustained such a move. In analogy to the reasoning in absence of policy, positive storage at time $t = 10$ could be sustained only by a belief in higher storage at time $t = 11$, and so on. Along this path, the increase in storage by the young reduces the real value of private money demand and so generates downward pressure on money return. In this case, the authority reacts by taxing the young to buy money ($m_{g,t} > 0$ and $M_t^S = 0$) in order to sustain its value, and in doing that, it also ensures the consumption level of the old. By Lemma 3, for $\Pi_t = \theta^{-1}$, we get the analogue to (21) with optimal policy interventions:

$$T_t - G_t + m_t = \theta m_{t-1}, \tag{27}$$

where the additional term captures the intervention. In particular, the optimal real surplus decided by the authority in response to past storage choices evolves according to

$$\begin{aligned} T_t &= \left(1 + \lambda - R(\rho_{t+1})^{-1}\right) \frac{W}{2+\lambda} + \left(1 + R(\rho_{t+1})^{-1}\right) \left(\frac{S_t - \theta S_{t-1}}{2+\lambda}\right) & (28) \\ G_t &= \frac{\lambda}{2+\lambda} \left(W - S_t + \theta S_{t-1}\right). & (29) \end{aligned}$$

Thus, increasing storage goes along with increasing primary surplus and decreasing (but equally split) consumption. However, storage increases faster than without interventions, violating the constraint of positive consumption at some point, which is not possible. So these paths cannot be equilibria.

As the picture shows, for a *given* positive level of storage at time $t = 11$, there exists a unique continuation equilibrium, denoted by a solid dark line with cross markers, that satisfies (26) with storage decreasing at time $t = 12$ before converging to zero. Crucially, this implies, according to (25), that an inflation rate Π_{12} from period $t = 11$ to $t = 12$ drops much lower than $\theta^{-1} = 1/0.95$, producing a return on money strictly higher than the one on storage. For that rate of inflation, the young at $t = 11$ would have never optimally chosen to store any unit in storage! By anticipating that no individual would then rationally anticipate $S_{11} > 0$, no jump to positive storage at $S_{11} > 0$ can occur in equilibrium.

Along a path where storage decreases, the authority implements a negative surplus: it sells off money in its balance sheet and transfers seigniorage revenues to the young. To understand the optimality of this behavior, notice that a *given* amount of positive storage at time t increases the availability of resources available at time $t + 1$. As noted above, (26) entails an optimal decrease in storage that balances the utility of transferring resources to the next generation and the depreciation cost of waiting one more period before consuming.

Finally, notice that, in all paths, the authorities' objective of financing their own consumption is completely covered by taxes; thus, the primary surplus only results from the implementation of market operations to ensure consumption equality between living agents. In fact, as Figure 2 shows, in any path in- and out-of-equilibrium, consumption equality between young and old is ensured by the policy.

5 Robustness and Limits

From the discussion following Proposition 7, it is clear that any externality that prevents the intertemporal optimality guaranteed by (10)–(13), or the exact consumption equality embedded in the policy objective (5), breaks the efficiency of the monetary equilibrium. It is not obvious, however, how deviations from that benchmark may change the result of equilibrium uniqueness. In this section, we briefly review tractable extensions of the baseline setting, focusing on the simplest case $u(\cdot) = \log(\cdot)$, $W^y = W$, $W^o = 0$, $\beta = 1$ and $\lambda = 0$.

We will first present a simple way to incorporate production in our basic setting, through labor or capital, that preserve efficiency and uniqueness of equilibrium. Then, we will look at three cases of inefficient redistribution: distorted weights in the policy objective, distortionary taxes, and absence of individual-specific tax instruments. In all these cases we show that the efficiency result gets lost but uniqueness is preserved at least to some extent. Finally, we show that when taxes are capped below a given level, both the efficiency and the uniqueness of the equilibrium may get lost.

5.1 Production

Labor. To get an intuitive grasp of the robustness of our findings to the introduction of production, we assume that the income of the young is a function of the labor that the young provides, i.e. $W_t = L_t$ with L_t being the amount of labor. The utility of the young is given by $u_\ell(C_{y,t}, L_t) \equiv \log(C_{y,t}) - L_t^2/2$. The authority solves the following problem:

$$\max_{\Pi_t, T_t} \left\{ \log\left(L_t - T_t - S_t - m_t\right) - L_t^2/2 + \log\left(\Pi_t^{-1} m_{t-1} + \theta S_{t-1}\right) \right\}$$

where $T_t = (1-\tau)L_t$ with τ being a tax rate on labor income, with everything else staying unchanged. We should note first that given production choices and policies are determined simultaneously, tax rate and production are taken as given at equilibrium values by the young and the authority, respectively. This means that there is no out-of-equilibrium impact of the policy choice on the production choice. Thus, given production choices, the optimal policy still equalizes consumption between young and old exactly as in our baseline case. Thus, the inflation rate obtains as:

$$\Pi_{t+1} = \frac{L^* - 3S_t + \theta S_{t-1}}{L^* - \theta S_t - S_{t+1}},$$

where $L^* = \sqrt{2}$, which is consistent with (25). As a consequence, in this case, the results of Proposition 4 directly apply. We show in Appendix B that the invariant nature of equilibrium production is the outcome of the log consumption utility and quadratic labor disutility explored here.[13] In the general case, L^* is a function of the prevailing real return on savings, which in our setting can take only two steady state values: either β^{-1} in the efficient monetary equilibrium or θ when storage and money are used jointly.

Capital. Capital can be introduced similarly to Tirole (1985). To make this mapping explicit, suppose the young agent i can invest one unit of endowment to get one unit in capital, which yields consumption at decreasing returns to scale and fully depreciates one period after. Formally,

$$C_{y,t} = W - K_t - T_t - S_t - m_t \quad \text{and} \quad C_{o,t} = K_{t-1}^{\alpha} + \theta S_{t-1} - \Pi_t^{-1} m_{t-1}$$

where K_t denotes capital and $\alpha \in (0, 1)$ measures the degree in return to scale, with everything else staying unchanged. The young invest in capital up to the point its return $\alpha K_t^{\alpha-1}$ matches the return of the most viable option, i.e., $K(\rho) = (\alpha/\rho)^{\frac{1}{1-\alpha}}$ with $\rho = \max\{\Pi_{t+1}^{-1}, \theta\}$. For given ρ, inflation obtains as

$$\Pi_{t+1} = \frac{W - K(\rho) + K(\rho)^{\alpha} - 2\rho^{-1} K(\rho)^{\alpha} + \theta S_{t-1} - 3S_t}{W - K(\rho) - K(\rho)^{\alpha} - \theta S_t - S_{t+1}},$$

which intuitively obtains from (25) once relabeling $W^y = W - K(\rho)$ and $W^o = K(\rho)^{\alpha}$ in the baseline case, noting $R(\rho) = 1$ for the simple case. As in Tirole (1985), when storage is used, agents overinvest in capital to match the return of storage $K(\theta) = (\alpha/\theta)^{\frac{1}{1-\alpha}}$; when only money is used instead, agents invest in capital only up to $K(1) = \alpha^{\frac{1}{1-\alpha}} < K(\theta)$ to yield the same return of money. In the monetary equilibrium the return on money is efficient ($\rho = 1$) as the results of Proposition 6 apply.

[13] We work out the generic case with exogenous fluctuations in endowments in Appendix E.

5.2 Inefficient redistribution

Unequal weights. Suppose young and old are weighted differently in the authority's objective function. Formally, it solves the following problem:

$$\max_{\Pi_t, T_t} \left\{ \log\left(W - T_t - S_t - m_t\right) + \mu \log\left(\Pi_t^{-1} m_{t-1} + \theta S_{t-1}\right) \right\}$$

where $\mu \in \mathbb{R}^+$ denotes the weight given by the authority to old relative to young, with everything else staying unchanged. We show in Appendix C.1 that, in this case, the inflation rate obtains as

$$\Pi_{t+1} = \frac{W - (\mu + 2)S_t + \theta S_{t-1}}{\mu W - \mu S_{t+1} - \theta S_t},$$

from which we can easily see that a monetary equilibrium ($S_{i,t} = 0$ for any t) exists as long as $\mu \geq \theta$, i.e., the weight put on the old is sufficiently high. We prove in Appendix C.1 that this equilibrium is also unique. However, the equilibrium is no longer efficient, as the return on money $\Pi^{-1} = \mu$ is generally different from the discount rate $\beta = 1$. When instead $\mu < \theta$, the monetary equilibrium cannot exist as returns on storage exceed returns on money. This is the effect of policy engineering a too-high redistribution in favor of the young through a too-high inflation, which, however, discourages them from saving in money.

Distortionary taxation. Suppose now that taxation is distortionary in the sense that any unit in taxes paid by the young produces less than one unit of resources by the government. Formally, the budget constraint of the young is given by:

$$C_{y,t} = W - (1 + \eta)T_t - S_t - m_t$$

where $\eta \in \mathbb{R}^+$ measures a dead-weight loss in consumption for each unit of effective taxes collected by the government, with everything else staying unchanged. We show in Appendix C.2 that, in this case, the inflation rate obtains as

$$\Pi_{t+1} = (1 + \eta) \frac{W - 3S_t + (1 + \eta)\,\theta S_{t-1}}{W - \theta(1 - \eta)^2 S_t - (1 + \eta)S_{t+1}},$$

from which, similarly to the case above, we can easily see that a monetary equilibrium ($S_{i,t} = 0$ for any t) exists as long as $(1 + \eta)^{-1} \geq \theta$, i.e., the dead-weight loss is sufficiently small. In Appendix C.2 we prove that this is also the only equilibrium. But the equilibrium is inefficient as the return on money $\Pi^{-1} = (1+\eta)^{-1}$ is generally different from the discount rate $\beta = 1$ and does not exist in case such a return is sufficiently small, i.e., $(1+\eta)^{-1} < \theta$. This is because part of the income of the young gets lost through taxation, which yields lower real value for money.

Age-specific transfers. In our benchmark model, the authority is constrained to transfer resources to the old only through money purchases, still, it is able to equalize consumption across generations. This means that if direct fiscal transfers to the old were possible, the authority would be at most indifferent between achieving consumption equality through direct transfers or money purchases.

Here, we go one step further and showcase a situation in which money purchases dominate direct transfers. We assume that agents have utility function $\log C^i_{y,t} + \gamma_i \log C^i_{o,t}$ and differ in the discount factor γ_i in that a mass p of *savers* are such that $\gamma_s = 1$ and a mass $1-p$ of *consumers* are such that $\gamma_c = 0$. *Savers* save half of their disposable endowments, while *consumers* do not save at all. In this context, we consider the possibility of positive transfers (negative taxes) to the old, $T_{o,t} \leq 0$.[14] The key assumption is that the authority has no tax instrument to discriminate between the two types. This is a situation in which the budget set of the authority reads as: $T_{y,t} + T_{o,t} = \Pi_t^{-1} m_{t-1} - m_t$ and $C^s_{o,t} = \theta S_{t-1} + \Pi_t^{-1} m_{t-1} - T_{o,t}$, where we use the superscript s to denote consumption of *savers*, with everything else staying unchanged.

The resulting problem for the authority is:

$$\max_{\Pi_t, T_{y,t}, T_{o,t} \leq 0.} \left\{ \int \log\left(W - T_{y,t} - S_{i,t} - m_{i,t}\right) di + \int \log\left(m_{i,t-1}\Pi_t^{-1} + \theta S_{i,t-1} - T_{o,t}\right) di \right\},$$

yielding to the following first-order conditions for Π_t and $T_{o,t}$ (see Appendix C.3) :

$$\begin{array}{lcc} \Pi_t & : & \dfrac{1}{W - m_{t-1}\Pi_t^{-1} + T_{o,t} - S_t} = \dfrac{1}{1/p\left(m_{t-1}\Pi_t^{-1} + \theta S_{t-1}\right) - T_{o,t}} \\ T_{o,t} & : & \dfrac{1}{W - m_{t-1}\Pi_t^{-1} + T_{o,t} - S_t} = \Xi + \dfrac{p}{1/p\left(m_{t-1}\Pi_t^{-1} + \theta S_{t-1}\right) - T_{o,t}}, \end{array}$$

with Ξ being the Lagrange multiplier associated with $T_{o,t} \leq 0$. Subtracting both conditions yields $\Xi = (1-p)/(1/p\left(m_{t-1}\Pi_t^{-1} + \theta S_{t-1}\right) - T_{o,t})$. When $p < 1$, we obtain $\Xi > 0$

[14] A microfoundation of this constraint can be that agents of the "saver" type may pretend to be of the "consumer" type. See, for example, Mengus (2019) for a theory of asset purchases based on asymmetric information.

and $T_{o,t} = 0$. This means that redistribution through inflation can achieve higher welfare than redistribution through transfers to the old. On the one hand, as transfers are not individual-specific, redistribution through direct transfers channels funds to old *consumers*, whose consumption has no value. On the other hand, only *savers* self-select to hold money, so that redistribution through inflation selectively transfers resources to old *savers* only. Inflation in this case obtains as

$$\Pi_{t+1} = \frac{pW + \theta p S_{t-1} + (p^2 - 2(1+p))S_t}{pW - \theta S_t - pS_{t+1}} \frac{1}{2-p},$$

showing that the return on money Π_t^{-1} in the monetary equilibrium ($S_t = 0$ for any t) is equal to $2 - p > 1 > \theta^{-1}$, which is generally inefficient. As we show in Appendix C.3, the unique equilibrium is monetary as the same logic of Proposition 6 applies.

5.3 Limits to taxation

We finally want to illustrate a case where a multiplicity of equilibria where money is used may obtain. We assume that taxes $\tilde{T}_t$ cannot exceed an exogenous upper bound, i.e., $\tilde{T}_t \leq \bar{T}$. We also consider a positive weight on public consumption $\lambda \geq 0$. In this case inflation obtains as

$$\Pi_{t+1} = (1+\lambda)\frac{W - \tilde{T}_t - 2S_t}{W + \tilde{T}_{t+1} - 2(S_{t+1} + \theta\lambda S_t)} \qquad \text{with} \qquad \tilde{T}_t = \min\{T_t, \bar{T}\}$$

where T_t is defined as in (28), in the case of logarithmic preferences. One can check by substitution that the unbounded case $\bar{T} \to \infty$ is the same as (25). By losing the ability to change taxes in response to private saving choices, the authority loses the ability to influence the demand for savings and, thus, the consumption of the young. There is now a trade-off in the use of the price for money as an instrument. On the one hand, the authority may reduce consumption inequality by lowering the price for money. On the other hand, it can increase public expenditures by increasing the price for money. Which force prevails depends on the weight of public expenditures in the authority's objective relative to the bound $\bar{T}$. We study this case in full detail in Appendix D, whose results can be summarized as follows. We show that, provided

$$\hat{\pi} \equiv (1+\lambda)\frac{W - \bar{T}}{W + \bar{T}} \leq \theta^{-1},$$

an inefficient monetary equilibrium exists such that $\Pi_t = \hat{\pi}$ for any $t \geq 1$. Furthermore, when

$$\frac{\bar{T}}{W} < \frac{\lambda\theta}{2+\lambda\theta},$$

money-storage equilibria also exist where storage and the real value of private money holding steadily converges to

$$S = \frac{W + \bar{T} - \theta(1+\lambda)(W-\bar{T})}{2(1-\theta)} \leq \frac{W-\bar{T}}{2} \quad \text{and} \quad m = \frac{\theta\lambda(W-\bar{T})}{2(1-\theta)} \geq 0,$$

besides the autarky equilibrium. Figure 3 illustrates the different types of equilibria. We use the same parameter values as in Figure 3 but with taxes binding at $\bar{T} = 0.056$. This level of taxes is compatible with the existence of an inefficient monetary equilibrium and a money-storage equilibrium.

The inefficient monetary equilibrium is denoted by a solid line with circle markers in Figure 3. In this equilibrium, money is the only savings asset, but the level of inflation is generically inefficient, increasing in λ and decreasing in $\bar{T}$. Note that, in this equilibrium, the primary fiscal surplus is negative, indicating that the authority covers part of its spending by creating and selling money – i.e., generating seigniorage.

Money-storage equilibria are denoted by a solid line with cross markers in Figure 3. In these equilibria, money and storage are jointly used, but money never fully loses value. This is possible because, by selling money, the authority makes inflation equal to θ despite the fact that the young keep their real money demand constant. In the storage-money equilibrium, inflation is higher than in the inefficient monetary equilibrium; however, the primary fiscal surplus is *less* negative, showing that actual seigniorage revenues are lower. Effectively, in the storage-money equilibrium, the consumption by both the old and the authority is lower. This means that storage-money equilibria is the result of a coordination failure between private agents and the authority, entailing a Laffer curve of seigniorage.

6 Conclusion

In this paper, we show that the pursuit of short-term fiscal objectives can sustain long-term monetary efficiency. The main reason is that the policy ensures the static optimality condition ensuring equality in marginal utilities. OLG economies cannot meet this condition otherwise due to the incompleteness of the market for consumption. However, we also show that this result has limits and we provide examples of several dimensions along which it may not hold, e.g., due to unequal redistribution concerns or limits to taxation.

Finally, it is worth remarking that our modeling choice of short-sighted agents and authorities helps emphasize that our main result does not rely on any long-run optimality (transversality conditions), history-dependent strategies (trigger strategies) or time-inconsistent behavior (commitments). Nevertheless, their insights equally apply to infinite-horizon economies. In Appendix F, we show how our OLG economy in the absence of policy delivers the same allocation of a simple Bewley economy with infinitely-lived agents subject to income fluctuations, as it is well known since Townsend (1980).

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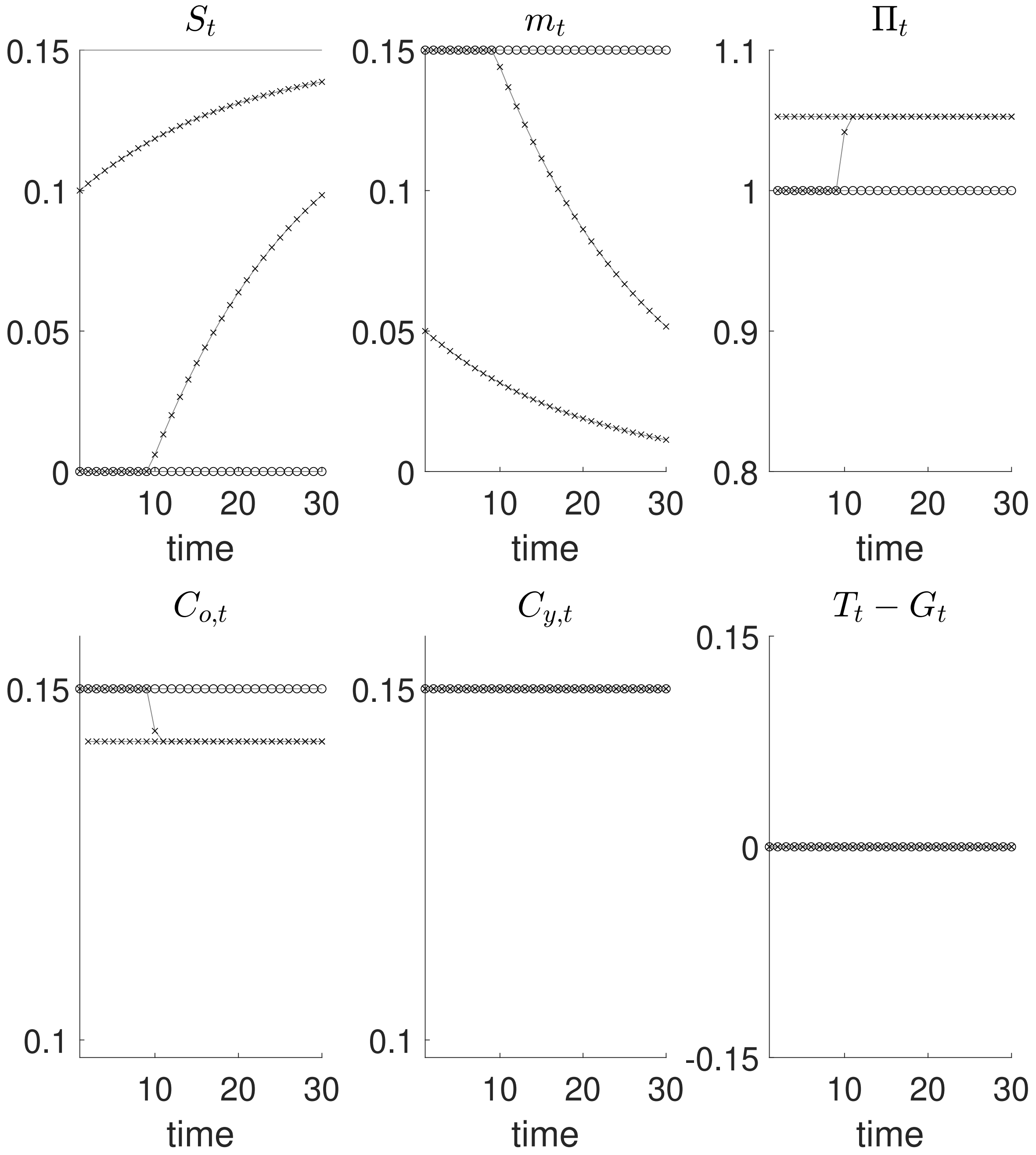

Figure 1 – **Global indeterminacy.** Equilibria with no policy interventions for $u(\cdot) = \log(\cdot)$, $\beta = 1$, $\theta = 0.95$, $W^y = 0.3, W^o = 0, \mathcal{P} = (0, 0, 0)$. Circles denote the monetary equilibrium; cross markers denote two asymptotic autarky equilibria: one that starts at $S_0 = 0.1$ and the other at $S_0 = 0$ with a jump at $S_{10} = 0.006$. Autarky, which is possible in this case, is denoted by a solid line.

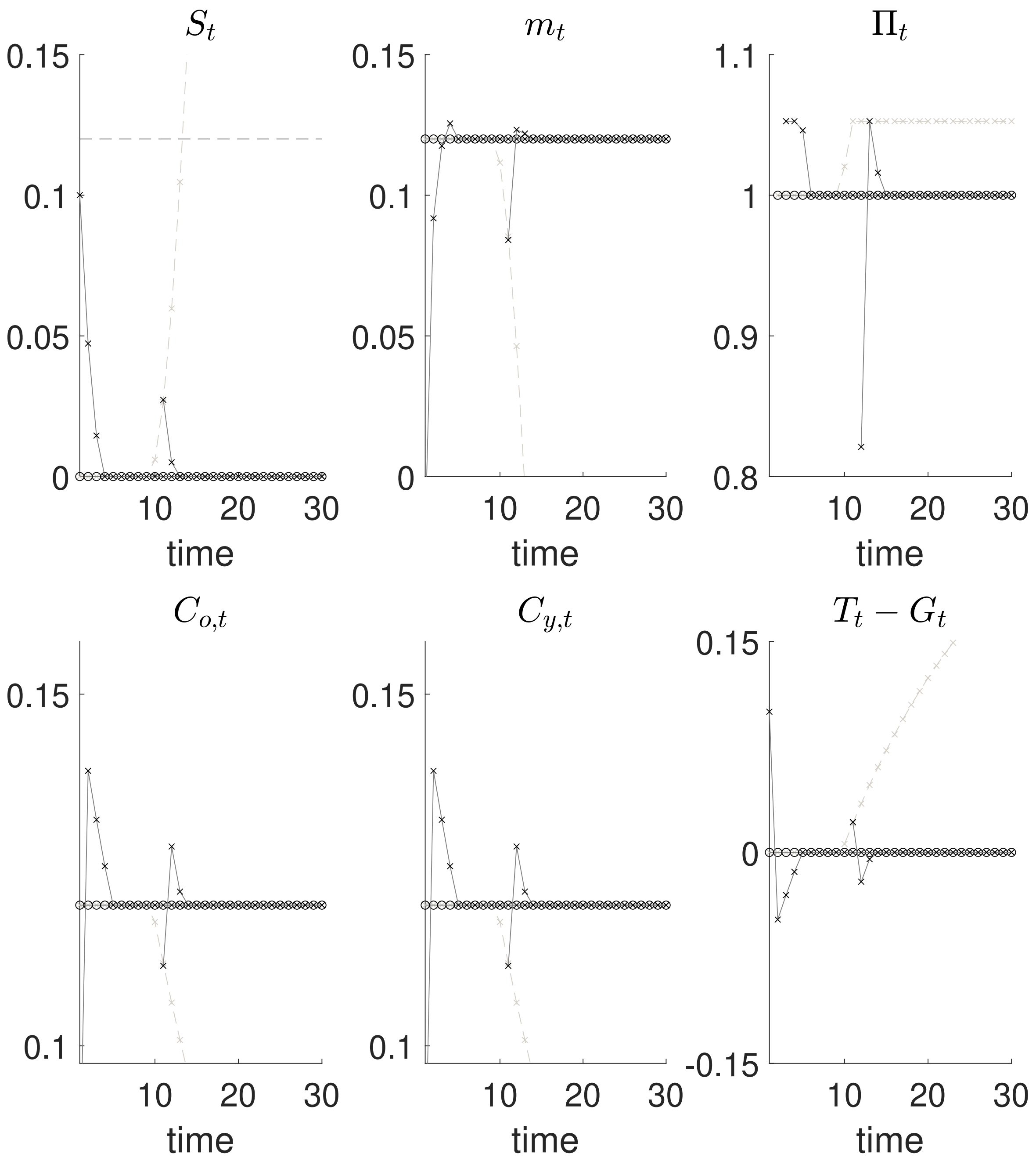

Figure 2 – Equilibria with optimal policy for $\beta = 1$, $\theta = 0.95$, $W = 0.3, \lambda = 0.5$. Circles denote the pure monetary equilibrium; cross markers denote two equilibria with storage: one that starts at $S_0 = 0.1$ and the other at $S_{11} = 0.027$. We also denote with a light grey dashed line the unfeasible path of an equilibrium that starts at $S_0 = 0$ with a jump at $S_{10} = 0.006$, which requires $S_{11} = 0.027$. Autarky, which is not possible in this case, is denoted by a simple dashed line.

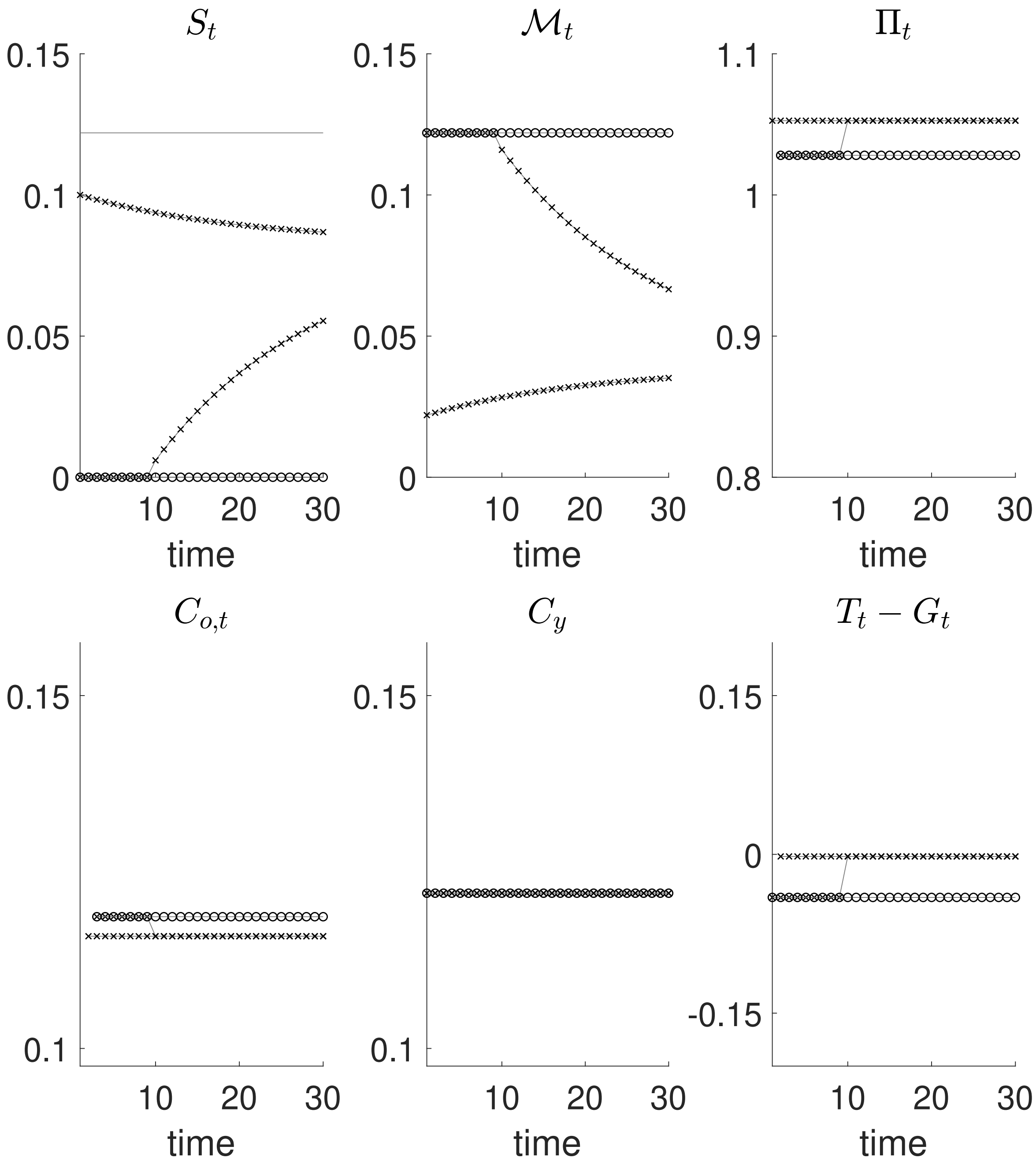

Figure 3 – Equilibria with optimal policy for $\beta = 1$, $\theta = 0.95$, $W = 0.3$, $\lambda = 0.5$ and $\bar{T} = 0.056$. Circles denote the monetary equilibrium; cross markers denote two money-storage equilibria: one that starts at $S_0 = 0.1$ and the other at $S_0 = 0$ with a jump at $S_{10} = 0.006$. Autarky, which is not possible in this case, is denoted by a simple dashed line.

Appendix

A Proofs

A.1 Proof of Lemma 1

Proof. Optimal private policies come directly from the first order condition $u'(W^y - T_t - D_t) = \beta\rho_{t+1}u'(\rho_{t+1}D_t + W^o)$.

To prove the statement about $R(\rho)$, note that $R(\rho) > \rho$ is equivalent to $u'_{-1}(\beta\rho) > 1$, which is true. Indeed, u' is decreasing since $u'' < 0$. As a result u'_{-1} is decreasing as well. Thus, given that $\beta\rho < 1$, we have $u'_{-1}(\beta\rho) > u'_{-1}(1)$. Given $u'(\cdot)$ are multiplicative, it has to be , $u'(1)u'(1) = u'(1)$ and as a result, $u'(1) = 1$ and $u'_{-1}(1) = 1$. This allows to conclude $u'_{-1}(\beta\rho) > 1$ whenever $\rho < 1$. □

A.2 Proof of Lemma 2

Proof. Use the market clearing condition for money (1) and the equilibrium value of real money holdings (8) into (6) and solve for T_t. Then substitute T_t into (3).

The potential heterogeneity noted in Remark **??** does not impact on this result. We can use $D_{i,t} = D_t$ – i.e. $m_{i,t} - m_t = S_t - S_{i,t}$ – to get (14) in case of a cross-sectional heterogeneity in $\{m_{i,t}, S_{i,t}\}$. □

A.3 Proof of Proposition 4

Proof. With symmetric private choices we have $C_{i,o,t} = C_{o,t} = \rho_{t+1}D_{t-1}$ for any i. Hence, given (14) in Lemma 2, it is immediate to show that the optimal $\Pi_t(\sigma_t)$ and $G_t(\sigma_t)$ are the ones that solve (16), i.e. the ones that equalize consumption between the young and the old, with T_t set to satisfy the budget constraint of the authority.

Let us here discuss how the potential heterogeneity noted in Remark **??** may impact on this result. This implies that the impact of marginal changes in inflation on the average utility of the old can be written as

$$\frac{\partial \int u(C_{i,o,t})di}{\partial \Pi_t} = -\int u'(C_{i,o,t})\frac{m_{i,t-1}}{\Pi_t^2}\,di$$

which introduces another potential concern for redistribution within the old. However, we note that, along an equilibrium we necessarily have $C_{i,o,t} = C_{o,t} = \rho_{t+1}D_{t-1}$ and so

$$-\int u'(C_{i,o,t})\frac{m_{i,t-1}}{\Pi_t^2}\,di = -u'(C_{o,t})\frac{\int m_{i,t-1}\,di}{\Pi_t^2}$$

which proves that the symmetry in private choices is without loss of generality for the characterization of the equilibrium set.

□

A.4 Proof of Proposition 5

Local indeterminacy. To show existence we work out the CRRA case. The law of motion of inflation in the absence of intervention is:

$$\Pi_t = \frac{m_{t-1}}{m_t} = \frac{1+\beta^{-1/\sigma}\Pi_{t+1}^{1/\sigma-1}}{1+\beta^{-1/\sigma}\Pi_t^{1/\sigma-1}} \frac{W^y-(\beta^{-1}\Pi_t)^{1/\sigma}W^o}{W^y-(\beta^{-1}\Pi_{t+1})^{1/\sigma}W^o}.$$

Note that a fixed point for this law of motion is $\Pi_t = \Pi_{t+1} = 1$. By writing this law of motion as a function $\Pi_t = f(\Pi_t, \Pi_{t+1})$, we can write the following partial derivatives:

$$\frac{\partial f(\Pi_t,\Pi_{t+1})}{\partial \Pi_{t+1}}|_{(\Pi_t,\Pi_{t+1})=(1,1)} = \frac{\beta^{-1/\sigma}\left[(1-\sigma)W^y + W^o\left(1+\sigma\beta^{-1/\sigma}\right)\right]}{\sigma(1+\beta^{-1/\sigma})(W^y-\beta^{-1/\sigma}W^o)}$$
$$\frac{\partial f(\Pi_t,\Pi_{t+1})}{\partial \Pi_t}|_{(\Pi_t,\Pi_{t+1})=(1,1)} = -\frac{\partial f(\Pi_t,\Pi_{t+1})}{\partial \Pi_{t+1}}|_{(\Pi_t,\Pi_{t+1})=(1,1)}$$

Around the fixed point, the dynamic of Π_t and Π_{t+1} is:

$$\Pi_{t+1} - 1 = \frac{1-\frac{\partial f(\Pi_t,\Pi_{t+1})}{\partial \Pi_t}|_{(\Pi_t,\Pi_{t+1})=(1,1)}}{\frac{\partial f(\Pi_t,\Pi_{t+1})}{\partial \Pi_{t+1}}|_{(\Pi_t,\Pi_{t+1})=(1,1)}}(\Pi_t - 1)$$

which implies:

$$\Pi_{t+1} - 1 = (\Pi_t - 1)\frac{W^y(1+\sigma\beta^{1/\sigma}) + W^o(1-\sigma)}{W^y(1-\sigma) + W^o(1+\sigma\beta^{-1/\sigma})}.$$

or with $X = W^o/W^y$:

$$\Pi_{t+1} - 1 = (\Pi_t - 1)\frac{(1+\sigma\beta^{1/\sigma}) + X(1-\sigma)}{(1-\sigma) + X(1+\sigma\beta^{-1/\sigma})}.$$

The condition for local convergence is then:

$$\left|\frac{(1+\sigma\beta^{1/\sigma}) + X(1-\sigma)}{(1-\sigma) + X(1+\sigma\beta^{-1/\sigma})}\right| < 1$$

Global indeterminacy. It is easy to note that $\Pi_{t+1} = 1 < \theta^{-1}$ and $S_t = 0$ for any t is an equilibrium; one in which money is always used and storage never. We refer to this equilibrium as the *pure monetary equilibrium.*

To check if there exist an equilibrium where storage is used jointly with money we should use the arbitrage condition in (1). For $S_t > 0$ at time t we must have $\Pi_{t+1} = \theta^{-1}$. In this case, (21) obtains as

$$S_{t+1} = \theta S_t + (1-\theta)\bar{S}, \tag{A.1}$$

with

$$\bar{S} \equiv \frac{W^y - R(\theta)\theta^{-1}W^o}{1+R(\theta)}$$

which implies $S_{t+1} \geq S_t$, given the limit $S_t \leq \bar{S}$ for each date t. Therefore we obtain that, if storage is used in one period, it must necessarily be used on a larger extent next period. In fact, an equilibrium for each initial level of storage $S_1 \in [0, \bar{S})$ (S_0 is not an optimal choice, i.e. (21) is not an equilibrium condition for S_0) exists such that storage is always used jointly with money. It is easy to show that in the long run, storage and the real money balance satisfy:

$$\lim_{t\to\infty} S_t = \frac{W}{2} \text{ and } \lim_{t\to\infty} \frac{M_0}{P_t} = lim_{t\to\infty} m_t = 0 \tag{A.2}$$

for any initial level of storage S_1, where the latter obtains as a consequence of the former because of the expression of D_t in 1. There are equilibria in which storage is always used, prices grows at a rate $1/\theta$ and money loses value in time until it eventually become worthless; let us call them the *asymptotic autarky equilibria*.

Importantly, all asymptotic autarky equilibria do not necessarily feature storage at date-0 and it is possible to construct asymptotic autarky equilibria where storage is not used until a certain date s after which it is always used. In fact, notice that $S_{s-1} = 0$ only requires that $\Pi_s < \theta^{-1}$, that is

$$0 \leq S_s < (1-\theta)\,\bar{S}.$$

Thus, at each date t, after having only used money in past periods, it is possible to start using storage. What is peculiar of the environment with constant endowment is that once storage is used it will used for ever; this is because, for a given S_t, (21) implies a certain S_{t+1} which has the property $S_{t+1} \geq S_t$, given the limit $S_t \leq \bar{S}$ for each t.

Finally, there also exists a *pure autarky equilibrium* defined as one in which $S_t = \bar{S}$ and $m_t = M_0/P_t = 0$ for each t in which money is never used and the price level is infinite and grows at a rate larger than $1/\theta$.

A.5 Proof of Proposition 6

1. Case $W^y > (1+\lambda)W^o$. We first start by the case $W^y \geq (1+\lambda)W^o$. As we will show, this ensures that savings is positive.

To establish our results, we use equations (24) and (25) to investigate how optimally chosen policies affect equilibrium outcomes. These two equations are derived as follows.

We substitute (17) into (20) to get

$$\begin{aligned}
T_t &= \frac{1}{1+\lambda}\left(\frac{W^y-(1+\lambda)(W^o+\theta S_{t-1})-S_t}{(2+\lambda)}\right)+\frac{\lambda}{1+\lambda}(W^y-S_t)-m_t,\\
&= \frac{1}{1+\lambda}\left(\frac{1}{2+\lambda}+\lambda\right)(W^y-S_t)-\frac{W^o+\theta S_{t-1}}{2+\lambda}-m_t,\\
&= \frac{1}{1+\lambda}\left(\frac{1+2\lambda+\lambda^2}{2+\lambda}\right)(W^y-S_t)-\frac{W^o+\theta S_{t-1}}{2+\lambda}-m_t,\\
&= \frac{1+\lambda}{2+\lambda}(W^y-S_t)-\frac{W^o+\theta S_{t-1}}{2+\lambda}-m_t,
\end{aligned}$$

We substitute for T by using the above into $m_t = D_t - S_t$ where D_t is given by (13) and get

$$
\begin{aligned}
m_t &= \frac{W^y - \left(\frac{1+\lambda}{2+\lambda}(W^y - S_t) - \frac{W^o+\theta S_{t-1}}{2+\lambda} - m_t\right) - R(\rho_{t+1})\rho_{t+1}^{-1}W^o}{1+R(\rho_{t+1})} - S_t \\
&= \frac{W^y - \frac{1+\lambda}{2+\lambda}W^y + \frac{1+\lambda}{2+\lambda}S_t + \frac{W^o+\theta S_{t-1}}{2+\lambda} + m_t - R(\rho_{t+1})\rho_{t+1}^{-1}W^o - (1+R(\rho_{t+1}))S_t}{1+R(\rho_{t+1})} \\
&= \frac{\frac{1}{2+\lambda}W^y + m_t + \frac{W^o-(2+\lambda)R(\rho_{t+1})\rho_{t+1}^{-1}W^o}{2+\lambda} + \frac{1}{2+\lambda}\theta S_{t-1} + \frac{1+\lambda-(2+\lambda)(1+R(\rho_{t+1}))}{2+\lambda}S_t}{1+R(\rho_{t+1})} \\
&= \frac{\frac{1}{2+\lambda}W^y + m_t + \frac{W^o-(2+\lambda)R(\rho_{t+1})\rho_{t+1}^{-1}W^o}{2+\lambda} + \frac{1}{2+\lambda}\theta S_{t-1} - \frac{1+(2+\lambda)R(\rho_{t+1})}{2+\lambda}S_t}{1+R(\rho_{t+1})}
\end{aligned}
$$

and, once solving for m_t we have

$$
m_t = \frac{M_t}{P_t} = \frac{W^y + W^o - (2+\lambda)R(\rho_{t+1})\rho_{t+1}^{-1}W^o + \theta S_{t-1} - (1+(2+\lambda)R(\rho_{t+1}))S_t}{(2+\lambda)R(\rho_{t+1})}.
$$

To get inflation, we use again (17) to get

$$
\Pi_t^{-1}m_{t-1} = \frac{M_{t-1}}{P_t} = \frac{W^y - (1+\lambda)(W^o + \theta S_{t-1}) - S_t}{(2+\lambda)}.
$$

Combining the last two we get

$$
\begin{aligned}
\Pi_{t+1} &= \frac{M_t}{P_t}\left(\frac{M_t}{P_{t+1}}\right)^{-1} \\
&= \frac{1}{R(\rho_{t+1})}\frac{W^y + (1-(2+\lambda)(R(\rho_{t+1})\rho_{t+1}^{-1}))W^o + \theta S_{t-1} - (1+(2+\lambda)R(\rho_{t+1}))S_t}{W^y - (1+\lambda)(W^o+\theta S_t) - S_{t+1}}.
\end{aligned}
$$

In the rest of this proof, we first show that there exists a pure monetary equilibrium where only money is traded. Then we show that neither an asymptotic autarky nor an autarky equilibrium exists. We use this latter result to investigate the continuation of an equilibrium after a deviation of the private sector (i.e. the private sector starts to use the storage technology) and we show that such a deviation is suboptimal.

a) The pure monetary equilibrium. Let us first show that there exists an equilibrium where only money is traded. More formally, the pure monetary equilibrium where $S_t = 0$ at each t is an equilibrium. This can be easily seen by checking that $S_t = 0$ at any t. In this case we have

$$
\Pi_{t+1} = \frac{1}{R(\Pi_{t+1}^{-1})}\frac{W^y + (1-(2+\lambda)(R(\Pi_{t+1}^{-1})\Pi_{t+1}))W^o}{W^y - (1+\lambda)W^o}
$$

that is

$$\begin{aligned}
R(\Pi_{t+1}^{-1})\Pi_{t+1}(W^y-(1+\lambda)W^o) &= W^y+(1-(2+\lambda)(R(\Pi_{t+1}^{-1})\Pi_{t+1}))W^o \\
R(\Pi_{t+1}^{-1})\Pi_{t+1}(W^y-(1+\lambda-2-\lambda)W^o) &= W^y+W^o \\
R(\Pi_{t+1}^{-1})\Pi_{t+1}(W^y+W^o) &= W^y+W^o \\
u'_{-1}(\beta\Pi_{t+1}^{-1})\Pi_{t+1}^{-1}\Pi_{t+1} &= 1 \\
u'_{-1}(\beta\Pi_{t+1}^{-1}) &= 1 \\
\beta\Pi_{t+1}^{-1} &= 1 \\
\Pi_{t+1} &= \beta,
\end{aligned}$$

where $\Pi_{t+1}=\beta<\theta^{-1}$ at any t. The property $u'_{-1}(\beta\Pi_{t+1}^{-1})=1$ implies $\beta\Pi_{t+1}^{-1}=1$ is a property of multiplicative functions. Note that money holdings in this equilibrium are such that $m_t=D_t=\beta\frac{W^y-(1+\lambda)W^o}{2+\lambda}$.

b) Non existence of any equilibria where storage is used. Let us now show that there cannot be an equilibrium where storage is positive at some date $T\geq 0$. Let us proceed by contradiction. Storage is initially at $S_0=0$. Suppose, that storage becomes positive at period $t>1$ so that $S_{t-1}=0$. If this is part of an equilibrium we should have that necessarily, $\theta=\Pi_{t+1}^{-1}$. In such a case, we can use (24) and (25) to show

$$\begin{aligned}
\Pi_{t+1} &= \frac{1}{R(\rho_{t+1})}\frac{W^y+(1-(2+\lambda)(R(\rho_{t+1})\rho_{t+1}^{-1}))W^o+\theta S_{t-1}-(1+(2+\lambda)R(\rho_{t+1}))S_t}{W^y-(1+\lambda)(W^o+\theta S_t)-S_{t+1}} \\
R(\theta)\theta^{-1} &= \frac{W^y+(1-(2+\lambda)(R(\theta)\theta^{-1})W^o+\theta S_{t-1}+(1+(2+\lambda)R(\theta))S_t}{W^y-(1+\lambda)(W^o+\theta S_t)-S_{t+1}}
\end{aligned}$$

$$\begin{aligned}
(R(\theta)\theta^{-1}-1)(W^y+W^o)-R(\theta)(1+\lambda)S_t-R(\theta)\theta^{-1}S_{t+1} &= \theta S_{t-1}-(1+(2+\lambda)R(\theta))S_t \\
(R(\theta)\theta^{-1}-1)(W^y+W^o)-R(\theta)\theta^{-1}S_{t+1} &= \theta S_{t-1}-(1+R(\theta))S_t
\end{aligned}$$

i.e. that storage S_{t+1} has to satisfy the following second order differential equation:

$$R(\theta)S_{t+1}-(1+R(\theta))\theta S_t+\theta^2S_{t-1}=(R(\theta)-\theta)(W^y+W^o).$$

Then necessarily $S_{t+1}=(1+R(\theta))\theta S_t+(R(\theta)-\theta)(W^y+W^o)>S_t$, so that $\theta=\Pi_{t+2}^{-1}$ also holds. Standard results on second order difference equations point out that, as $\theta<1$ and $\theta/R(\theta)<1$ are the two roots of the associated characteristic equation[15], and so on, finally showing that $\{S_{t+\tau}\}_\tau$ is on a monotonically increasing path converging to $\bar{S}=(W^y+W^o)/(1-\theta)$. Given $\bar{S}$ value higher than available endowments, but consumption cannot be negative, so a contradiction obtains.

c) Non existence of a pure autarky equilibrium. Here we prove that an equilibrium in which real money balance are valueless starting at some date t – i.e. $m_\tau=M_\tau/P_\tau=0$ starting a $t\geq\tau$ – does not exist.

Indeed, suppose that there exists a date t such that $m_t=0$. Optimal policy at date-$t+1$ is, according to Proposition 4, to set date $t+1$ inflation at 0. This then implies that the return on money is infinite and exceeds the return on storage.

[15]We have shown that indeed $R(\rho)>\rho$ in A.1.

Such 0 inflation rate is implemented when $m_t = 0$ simply by setting $m_{g,t+1} > 0$ (and thus $M^S_{t+1} = 0$) at date $t+1$, whatever the value of m_{t+1}.

Moreover, this results extends by continuity to any arbitrarily small deviation: suppose that one agent deviates so that $m_t = \epsilon$ with $\epsilon > 0$ arbitrarily small. Date-$t+1$ inflation rate $\Pi_{t+1} = \epsilon/(W^y - (1+\lambda)(\theta S_{t-1} + W^o) - S_t)$ is also arbitrarily close to 0 when ϵ is close to 0 – the denominator in fact is strictly positive as we show below in Lemma A.1 for the relevant case $W^y > (1+\lambda)W^o$. As a result, the return on money also exceeds the return on storage θ, thus making the deviation profitable.

Lemma A.1. *$W^y - (1+\lambda)(\theta S_{t-1} + W^o) - S_t \geq 0$ with equality if and only if $W^y = (1+\lambda)W^o$.*

Proof. At date 1, we have:

$$S_1 \leq D_1 = \frac{W^y - T_1 - R(\rho_2)\rho_2^{-1}W^o}{1 + R(\rho_2)}.$$

with $T_1 = (1+\lambda)/(2+\lambda)W^y - (W^o - S_1)/(2+\lambda) - D_1$. We then obtain that:

$$S_1 \leq \frac{W^y + W^o - (2+\lambda)R(\rho_2)\rho_2^{-1}W^o}{(2+\lambda)R(\rho_2)} \leq \frac{W^y - (1+\lambda)W^o}{1 + (2+\lambda)R(\rho_2)}$$

which yields the result at date 1.

Suppose that $S_{t-1} \leq \Theta_{t-1}(W^y - (1+\lambda)W^o)$. As before, we obtain that:

$$S_t \leq \frac{W^y - (1+\lambda)W^o + \theta S_{t-1}}{1 + (2+\lambda)R(\rho_{t+1})}$$

and thus that

$$S_t \leq \frac{1 + \Theta_{t-1}\theta}{1 + (2+\lambda)R(\rho_{t+1})}.$$

As a result:

$$\Theta_t \leq \frac{1 + \Theta_{t-1}\theta}{1 + (2+\lambda)R(\rho_{t+1})}.$$

Let us consider then the sequence defined by $\eta_1 = \Theta_1$ and

$$\eta_t = \frac{1 + \eta_{t-1}\theta}{1 + (2+\lambda)R(\theta)}.$$

This sequence converges from below to $1/(1 + (2+\lambda)R(\theta) - \theta)$.

Let us then show that an upper bound to $(1+\lambda)\theta S_{t-1} + S_t$ is $W^y - (1+\lambda)W^o$, that is:

$$(1+\lambda)\theta\Theta_{t-1} + \Theta_t \leq 1$$

Writing this inequality with η_t, we find that it is implied by:

$$\eta_{t-1} \leq \frac{\lambda + 2}{1 + (1+\lambda)(1 + (2+\lambda)R(\theta))}$$

As $\eta_{t-1} \leq 1/(1+(2+\lambda)R(\theta)-\theta)$, this boils down to comparing

$$\frac{1}{1+(2+\lambda)R(\theta)-\theta} \leq \frac{2+\lambda}{1+(1+\lambda)(1+(2+\lambda)R(\theta))}.$$

which is satisfied as: $1+(2+\lambda)R(\theta) \geq 1+(2+\lambda)\theta$. □

d) Uniqueness of the equilibrium continuation starting at a given $S_t > 0$. Suppose that $S_t > 0$ is part of an equilibrium. Let us show that this contradicts date-t agent's optimality condition.

To do this, we shall prove that the equilibrium continuation after a deviation S_t leads to a path for storage $\{S_t, ..., S_{t+n}\}$ so that S_τ with $\tau \in \{1, .., n-1\}$ is decreasing and $S_{t+n} = 0$, taking S_t as given. We show the existence of this path in d.1). We then show in d.2) that such a decreasing path leads money to have a strictly better return than storage at date t. That is $S_t > 0$ cannot be part of an equilibrium with constant endowment: the young at date $t-1$ are therefore better off not investing in storage.

Finally, we show that the path is unique in d.3).

d.1.) The equilibrium continuation features decreasing storage. Let us first show that the continuation equilibrium after storage S_t features decreasing storage.

First, notice that, if such an equilibrium path for storage exists, then storage goes to 0 at some point. More formally, there exists $n > 0$ such that $S_{t+n} = 0$. Suppose that it is not the case and storage is always used, then S_τ converges to $S = W/(1-\theta)$, which is not feasible as we showed in b).

Second, for any n we can construct a path for storage $\{S_t, ..., S_{t+n}\}$ such that $S_{t+n} = 0$ and, at any date $\tau \in \{t+1, ...t+n-1\}$,

$$R(\theta)S_{\tau+1} - (1+R(\theta))\theta S_\tau + \theta^2 S_{\tau-1} = (R(\theta)-\theta)(W^y + W^o). \tag{A.3}$$

Indeed, this latter equation defines a linear difference equation of order 2 for S_τ and $S_{t+n} = 0$ as well as S_t define two boundary conditions. As a result, there exists a unique path $\{S_{t+1}, ..., S_{t+n-1}\}$ solving the linear difference equation combined with the boundary conditions.

Our objective here is showing that any potential continuation of equilibrium leads to a decreasing path for storage. More precisely, let us show that the sequence of S_τ is decreasing:

Lemma A.2. *Suppose that there exists $t+n$ such that $S_{t+n} = 0$ and for all τ such that $t < \tau < t+n$*

$$R(\theta)S_{\tau+1} + \theta^2 S_{\tau-1} - (R(\theta)-\theta)(W^y + W^o) = \theta(1+R(\theta))S_\tau$$

then $S_t > S_{t-1} > ... > S_{t+n} = 0$.

Proof. We proceed by iteration. Let us first show that $S_{t+n-2} > S_{t+n-1}$. At date $t+n-1$, we have

$$R(\theta)S_{t+n} + \theta^2 S_{t+n-2} - (R(\theta)-\theta)(W^y + W^o) = \theta(1+R(\theta))S_{t+n-1}$$

Using the fact that $S_{t+n}=0$, we can then write:

$$\frac{\theta}{1+R(\theta)}S_{t+n-2}-\frac{R(\theta)-\theta}{\theta(1+R(\theta))}(W^y+W^o)=S_{t+n-1}$$

Given

$$\frac{\theta}{1+R(\theta)}<1 \quad \text{and} \quad \frac{R(\theta)-\theta}{\theta(1+R(\theta))}(W^y+W^o)>0,$$

then $S_{t+n-1}<S_{t+n-2}$.

Suppose that that $S_{t+n-1}<S_{t+n-2}<...<S_\tau$. Let us show that $S_{\tau-1}>S_\tau$. We can write at date τ:

$$\frac{R(\theta)}{\theta}S_{\tau+1}+\theta S_{\tau-1}-\frac{R(\theta)-\theta}{\theta}(W^y+W^o)=(1+R(\theta))S_\tau$$

□

d.2.) Optimal portfolio decision at date t. We show here that having a decreasing path for storage after a deviation S_t leads to a return on money at date t so that $S_t>0$ is suboptimal.

Suppose indeed that $S_t>0$. For this to happen, we need that the return $\rho_{t+1}=\theta$. Let us show that this is not consistent with households' optimal portfolio decision.

At time t, the return on money after a deviation S_t is (as $S_{t-1}=0$):

$$\frac{P_t}{P_{t+1}}=R(\rho_{t+1})\frac{W^y-(1+\lambda)(W^o+\theta S_t)-S_{t+1}}{W^y+(1-(2+\lambda)(R(\rho_{t+1})\rho_{t+1}^{-1}))W^o-(1+(2+\lambda)R(\rho_{t+1}))S_t}$$

Using Lemma A.2, $S_{t+1}<S_t$ and the return satisfies:

$$\frac{P_t}{P_{t+1}}>R(\rho_{t+1})\frac{W^y+W^o-(2+\lambda)(W^o+\theta S_t)-S_t(1-\theta)}{W^y+W^o-(2+\lambda)(R(\rho_{t+1})\rho_{t+1}^{-1})W^o-(1+(2+\lambda)R(\rho_{t+1}))S_t}$$

Suppose that $\rho_{t+1}=\theta$ that would be consistent with $S_t>0$. We have:

$$\frac{P_t}{P_{t+1}}>R(\theta)\frac{W^y+W^o-(2+\lambda)(W^o+\theta S_t)-S_t(1-\theta)}{W^y+W^o-R(\theta)\theta^{-1}(2+\lambda)(W^o+\theta S_t)-S_t}$$

As $R(\theta)\theta^{-1}>1$, $S_t>0$ and $\theta<1$, the right-hand term is strictly larger than θ: there is there an arbitrage possibility with money, a contradiction with household's optimal portfolio decision.

d.3.) Uniqueness of the equilibrium continuation after S_t. Let us show that there exists a unique continuation of an equilibrium after S_t.

In what we describe above, for every integer n', we can build a unique sequence solving (A.3) that we denote $\{S_t^{n'},...,S_{t+n'}^{n'}\}$ so that $S_{t+n'}^{n'}=0$, with n' being the number of periods that storage needs to get back to 0. However, in an equilibrium, the sequence should also be such that $S_{t+n'}=0$ is optimal, which requires $S_{t+n'-1}$ to satisfy $\theta^2S_{t+n'-1}<(R(1)-\theta)(W^y+W^o)$, i.e. that money return is strictly higher than return on storage.

Let us show that this implies that there exists a unique n, such that storage goes back to 0, that is $S_{t+n}=0$ and $\theta^2S_{t+n-1}^n<(R(1)-\theta)(W^y+W^o)$. To this purpose, let us show the following lemma:

Lemma A.3. *S^n_{t+n-1} is decreasing with n.*

Proof. First, let us show that $S^1_t = S_t > S^2_{t+1}$. Indeed, $\theta^2 S_t - (R(\theta) - \theta)(W^y + W^o) = \theta(1 + R(\theta))S^2_{t+1} > S_t$.

Let us extend this proof to n. To this purpose, let us note that:

$$\theta^2 S^n_{t+n-2} - (R(\theta) - \theta)(W^y + W^o) = \theta(1 + R(\theta))S^n_{t+n-1}$$
$$\theta^2 S^{n+1}_{t+n-1} - (R(\theta) - \theta)(W^y + W^o) = \theta(1 + R(\theta))S^{n+1}_{t+n}$$

As a result, $S^{n+1}_{t+n} < S^n_{t+n-1}$ if and only $S^{n+1}_{t+n-1} > S^n_{t+n-2}$. Let us investigate whether $S^{n+1}_{t+n-1} > S^n_{t+n-2}$. To this purpose, let us note that:

$$R(\theta)(S^{n+1}_{t+n} - S^n_{t+n-1}) + \theta^2(S^{n+1}_{t+n-2} - S^n_{t+n-3}) = \theta(1 + R(\theta))(S^{n+1}_{t+n-1} - S^n_{t+n-2})$$
$$(1 + R(\theta))(S^{n+1}_{t+n} - S^n_{t+n-1}) = \theta(S^{n+1}_{t+n-1} - S^n_{t+n-2})$$

We can infer two results from these equations. On the one hand, there exists $A_{t+n-1}(\theta) > 1$ such that $A(\theta)(S^{n+1}_{t+n-1} - S^n_{t+n-2}) = (S^{n+1}_{t+n-2} - S^n_{t+n-3})$. On the other hand, $S^{n+1}_{t+n-1} > S^n_{t+n-2}$ if and only if $S^n_{t+n-3} > S^{n+1}_{t+n-2}$.

Let us proceed by iteration: suppose that there exists $A_\tau(\theta) > 1/\theta$ such that for some τ:

$$R(\theta)(S^{n+1}_{\tau} - S^n_{\tau-1}) + \theta^2(S^{n+1}_{\tau-2} - S^n_{\tau-3}) = \theta(1 + R(\theta))(S^{n+1}_{\tau-1} - S^n_{\tau-2})$$
$$A(\theta)(S^{n+1}_{\tau} - S^n_{\tau-1}) = \theta(S^{n+1}_{\tau-1} - S^n_{t+n-2})$$

We then obtain:

$$\theta^2(S^{n+1}_{\tau-2} - S^n_{\tau-3}) = \theta\left((1 + R(\theta)) - \frac{R(\theta)}{A(\theta)}\right)(S^{n+1}_{\tau-1} - S^n_{\tau-2})$$

As a result there exists $A_{\tau-1}(\theta) > 1$ such that:

$$(S^{n+1}_{\tau-2} - S^n_{\tau-3}) = A_{\tau-1}(\theta)(S^{n+1}_{\tau-1} - S^n_{\tau-2})$$

In the end, we obtain by iteration that

$$S^{n+1}_{t+n-1} - S^n_{t+n-2} = A_{t+n-1}(\theta) \times .... \times A_{t+1}(\theta)(S^{n+1}_{t+1} - S_t)$$

Given that all the As are positive and $S_t > S^{n+1}_{t+1}$, we then obtain that $S^{n+1}_{t+n} < S^n_{t+n-1}$. As a result, S^n_{t+n-1} is a decreasing function of n. □

Given that for all τ, we have

$$\frac{\theta}{R(\theta)} S_{\tau-1} - \frac{R(\theta) - \theta}{R(\theta)\theta}(W^y + W^o) \geq S_\tau,$$

we can find a sufficiently large n' such that $\theta^2 S^{n'}_{t+n'-1} < (R(1) - \theta)(W^y + W^o)$. Using Lemma A.3, there exists a unique n such that the sequence of storage decisions $\{S_t, ..., S_{t+n}\}$ is such that $\theta^2 S^n_{t+n-1} < (R(1) - \theta)(W^y + W^o)$ and $\theta^2 S^n_t > (R(1) - \theta)(W^y + W^o)$ at any previous date.

e) Intertemporal efficiency of storage. Let us finally look at the intertemporal efficiency of storage decisions. As we show, the path for storage resulting from optimal policy and private decisions is the unique solution to: for any $t \geq 0$:

$$R(\theta)S_{t+1} - (1 + R(\theta))\theta S_t + \theta^2 S_{t-1} = (R(\theta) - \theta)(W^y + W^o),$$

for some initial S_0 and $\lim_{t\to\infty} S_t = 0$. Let us compare this allocation to what a central planner would do when confronted to an intertemporal allocation problem having access to storage,– in this approach, it is as if the central planner can perfectly redistribute resources within periods – that is:

$$\max_{\{S_t\}_{t\geq 0}} \sum_{t\geq 0} \beta^t u(c_t)$$

$$c_t = \theta S_{t-1} + W^y + W^o - S_t,$$
$$S_t \geq 0,$$
$$c_t \geq 0.$$

The first order condition to this problem is:

$$\beta^t u'(c_t) = \eta_t + \mu_t$$
$$\eta_t = \eta_{t+1}\theta + \Gamma_t$$

with Γ_t the Lagrange multiplier associated with $S_t \geq 0$, η_t the one associated with the budget constraint and μ_t the one associated with $c_t \geq 0$.

When $S_t > 0$ and $c_t > 0$, $\Gamma_t = \mu_t = 0$ and rewriting the first order condition yields:

$$\theta S_{t-1} + W^y + W^o - S_t = u'_{-1}(\beta\theta)\left(\theta S_t + W^y + W^o - S_{t+1}\right).$$

When arranging terms and multiplying both sides by θ, we find:

$$R(\theta)S_{t+1} - ((1 + R(\theta))\theta S_t + \theta^2 S_{t-1} = (R(\theta) - \theta)(W^y + W^o). \tag{A.4}$$

If this holds in any future period, S_t converges to $(W^y + W^o)/((1-\theta)$, this leads to $c_t = 0$. In addition, the transversality condition for the central planner problem writes:

$$\lim_{t\to\infty} \beta^t u'(c_t)S_t = 0$$

which implies, as $u' > 0$, that S_t converges to 0. This transversality condition then implies that (A.4) does not hold in any future period and S_t goes back to 0 in a finite number of periods.

The path of storage resulting from the equilibrium between private decisions and optimal policy is the unique path that solves (A.4) and also converges back to the monetary equilibrium in which $S_t = 0$. As a result, this path also solves the central planner's problem.

2. Case where $W^y \leq (1 + \lambda)W^0$. Let us first show that $m_t = 0$ for all t.

Suppose instead that $D_t > 0$ for some period t. From Proposition 4, if both $m_t > 0$ then $\Pi_{t+1} < 0$, which contradicts optimality. As a result, $\Pi_{t+1} = \infty$ and $m_t = 0$.

As a result, $T_t = \frac{\lambda}{1+\lambda}(W^y - S_t)$ and

$$D_t = S_t = \frac{W^y + \lambda S_t - (1 + \lambda)R(\theta)\theta^{-1}W^o}{(1 + R(\theta))(1 + \lambda}$$

and then:

$$S_t = \frac{W^y - (1+\lambda)R(\theta)\theta^{-1}W^o}{1 + (1+\lambda)R(\theta)}$$

As $W^y - (1+\lambda)R(\theta)\theta^{-1}W^o \leq 0$, this cannot be and we have $S_t = 0$.

A.6 Proof of Proposition 7

Let us now show that $(\sigma_{\mathcal{P}^*}, \sigma^*)$ leads to the authority's first best allocation. To start with, let us first write the date-t problem of the social planner:

for a given S_0 and M_0,

$$\max_{\{C_{o,t}, C_{y,t}, G_t, S_t, M_t, P_t\}_{t\geq 1}} \sum_{t=1}^{\infty} \beta^{t-1} \Big(u(C_{o,t}) + u(C_{y,t}) + \tilde{\lambda} u(G_t) \Big)$$

$$\begin{aligned}
\text{s.t. } & C_{o,t} - W^o - \theta S_{t-1} - \frac{M_{t-1}}{P_t} \leq 0 && (\zeta_t) \\
& C_{y,t} - W^y + S_t + \frac{M_t}{P_t} + T_t \leq 0 && (\mu_t) \\
& \frac{M_{t-1}}{P_t} - \frac{M_t}{P_t} + G_t - T_t \leq 0 && (\gamma_t) \\
& M_t \geq 0 && (\eta_t) \\
& S_t \geq 0 && (\omega_t) \\
& P_t \geq 0 && (\epsilon_t)
\end{aligned}$$

In this problem, we have already take into account the market clearing condition for money. The first order conditions of this problem are:

$$\begin{aligned}
C_{o,t}: &\quad \beta^{t-1}u'(C_{o,t}) = \zeta_t \\
C_{y,t}: &\quad \beta^{t-1}u'(C_{y,t}) = \mu_t \\
G_t: &\quad \beta^{t-1}\tilde{\lambda}u'(G_t) = \gamma_t \\
S_t: &\quad \theta\zeta_{t+1} = \mu_t + \omega_{t+1} \\
M_t: &\quad \frac{-\zeta_{t+1} + \gamma_{t+1}}{P_{t+1}} = \frac{-\mu_t + \gamma_t}{P_t} + \eta_t \\
T_t: &\quad \mu_t - \gamma_t = 0 \\
P_t: &\quad (-\zeta_t + \gamma_t)M_{t-1} + (\mu_t - \gamma_t)M_t + P_t^2\epsilon_t = 0
\end{aligned}$$

where ζ_t, μ_t, γ_t, ω_t, η_t, ϵ_t are Lagrangian associated to the constraints as showed above.

It is inefficient to use only the storage technology, so that $M_t > 0$. In this case, $\eta_t = 0$ and, combining the focs for T_t and M_t, we obtain that at each t:

$$\begin{aligned}
C_{y,t} &= C_{o,t} \\
G_t &= \lambda C_{o,t}
\end{aligned}$$

with $\lambda = 1/(u')^{-1}(\tilde{\lambda})$, are optimal as provided by $\mathcal{P}_t^*$. T_t is then the one satisfying the budget constraint. A social planner would equalize consumption of the young and the old generations and choose public consumption as a fraction of them. Note that a solution also exists in which the consumption of the old and the young are not equalized, money is used and $P_t \to \infty$. Those are the same conditions characterizing the authority's optimal time-consistent policy.

Finally, note that portfolio decisions that are solutions to Lemma 1 are also solutions of the first order conditions of the private sector, thus satisfying the constraints of the Ramsey problem. To see this, let us note that

$$\frac{u'(C_{y,t})}{\beta u'(C_{o,t+1})} = \frac{\mu_t}{\zeta_{t+1}} = \rho_{t+1}$$

where ρ_{t+1} is the equilibrium return on savings as defined in the text. This is the same optimality conditions for private saving choices (i.e. on D_t). Let us now turn to portfolio composition. According to the first order condition for S_t, we note that $S_t > 0$, i.e. $\omega_{t+1} = 0$, if and only if $\rho_{t+1} = \theta$. As private agents do, the social planner would use storage only when the return on savings is θ. This demonstrates that $(\sigma^t_{\mathcal{P}^*}, \sigma^*)$ entails the authority's first best allocation.

B Production

For this case, the simultaneous timing of actions in our model is key. This implies that when a young evaluates her labor supply, her expectation about the tax rate is the equilibrium one. Symmetrically, in setting its optimal policy, an authority expects equilibrium labor choices.

Labor choice for given τ_t We assume that young households can produce goods using labor. More precisely, by supplying L_t units of labor, households obtain $W(L_t)$ goods. Yet, supplying labor comes at a welfare cost $v(L_t)$. Such a disutility of labor is assumed to be increasing and strictly convex in labor L_t. The problem solved by households is:

$$\max_{C_{y,t},C_{o,t+1},L_t,S_t,M_t} u\left(C_{y,t}\right) - v\left(L_t\right) + \beta u\left(C_{o,t+1}\right)$$
$$\text{s.t. } C_{y,t} + S_t + \frac{M_t}{P_t} = W\left(L_t\right) - T_t$$
$$C_{o,t+1} = \theta S_t + \frac{M_t}{P_{t+1}}$$

where, we will assume: $W\left(L_t\right) = L_t$, $u\left(\cdot\right) = \frac{(\cdot)^{1-\sigma}}{1-\sigma}$, $v\left(\cdot\right) = \frac{1}{1+\gamma}\left(\cdot\right)^{1+\gamma}$, $T_t = (1-\tau_t)L_t$ with $\sigma > 0, \gamma > 0$ and $\tau_t \in (0,1)$.

For a giver real return r on savings, the program of the household can be written:

$$\max \frac{C_{y,t}^{1-\sigma}}{1-\sigma} - \frac{L_t^{1+\gamma}}{1+\gamma} + \beta\frac{C_{o,t}^{1-\sigma}}{1-\sigma} \quad \text{s.t. } C_{y,t} + \frac{C_{o,t}}{r} = \tau_t L_t.$$

The first order conditions to this problem are:

$$\begin{aligned} C_{y,t}^{-\sigma} &= r\beta C_{o,t}^{-\sigma} \\ L_t^{\gamma} &= \tau_t C_{y,t}^{-\sigma} \end{aligned}$$

Plugging the first condition in the budget constraint, we can solve for $C_{o,t}$ and $C_{y,t}$:

$$
\begin{aligned}
C_{o,t} &= \frac{1}{(r\beta)^{-\frac{1}{\sigma}} + \frac{1}{r}} \tau_t L_t = r \frac{(r\beta)^{\frac{1}{\sigma}}}{r + (r\beta)^{\frac{1}{\sigma}}} \tau_t L_t \\
C_{y,t} &= r \frac{1}{r + r^{\frac{1}{\sigma}} \beta^{\frac{1}{\sigma}}} \tau_t L_t
\end{aligned}
$$

Note with $r \leq 1/\beta$ we have $C_{y,t} \geq C_{o,t}$. Plugging in the second condition, we get

$$
C_{y,t} = \frac{r}{r + r^{\frac{1}{\sigma}} \beta^{\frac{1}{\sigma}}} \tau_t L_t = \frac{r}{r + r^{\frac{1}{\sigma}} \beta^{\frac{1}{\sigma}}} \tau_t^{\frac{1+\gamma}{\gamma}} C_{y,t}^{-\frac{\sigma}{\gamma}} \quad \Rightarrow \; C_{y,t} = \left(\frac{r}{r + r^{\frac{1}{\sigma}} \beta^{\frac{1}{\sigma}}} \right)^{\frac{\gamma}{\gamma+\sigma}} \tau_t^{\frac{1+\gamma}{\gamma+\sigma}}
$$

or, solving for $\tau_t L_t$,

$$
\tau_t L_t = \tau_t^{\frac{1+\gamma}{\gamma}} C_{y,t}^{-\frac{\sigma}{\gamma}} = \tau_t^{\frac{1+\gamma}{\gamma}} \left(\frac{r}{r + r^{\frac{1}{\sigma}} \beta^{\frac{1}{\sigma}}} \right)^{-\frac{\sigma}{\gamma+\sigma}} \tau_t^{-\frac{\sigma(1+\gamma)}{\gamma(\gamma+\sigma)}} = \left(\frac{r}{r + r^{\frac{1}{\sigma}} \beta^{\frac{1}{\sigma}}} \right)^{-\frac{\sigma}{\gamma+\sigma}} \tau_t^{\frac{1+\gamma}{\gamma+\sigma}}.
$$

Thus, the optimal L_t, given an expected real rate r and expected τ_t, is

$$
L_t = \left(\frac{r}{r + r^{\frac{1}{\sigma}} \beta^{\frac{1}{\sigma}}} \right)^{-\frac{\sigma}{\gamma+\sigma}} \tau_t^{\frac{1-\sigma}{\gamma+\sigma}}.
$$

Notice that, with $\sigma = 1$, L_t does not depend on τ_t and r as we obtain that $L_t = (1+\beta)^{\frac{1}{1+\gamma}}$. In the case where $\beta = 1$ and $\gamma = 1$, we find the results in the core of the text. Except with these parameters, the optimal labor supply is a function of expected τ_t and r.

Fiscal Policy for given L_t Let us suppose that $1 - \tau_t$ is the fraction of production taxed by the government, so that $T_t = (1 - \tau_t) L_t$. The budget constraint of the government can be rewritten:

$$
(1 - \tau_t) L_t - \Pi_t^{-1} m_{t-1} + m_t = 0
$$

that we plug in the budget constraint of the young to get

$$
C_{y,t} = L_t - (1 - \tau_t) L_t - S_t - m_t = L_t - S_t - \Pi_t^{-1} m_{t-1}.
$$

We then obtain that the objective of the authority is:

$$
\left\{ u \left(L_t - S_t - \Pi_t^{-1} m_{t-1} \right) - v(L_t) + u \left(\Pi_t^{-1} m_{t-1} + \theta S_{t-1} \right) \right\}
$$

The resulting optimal inflation rate is:

$$
\Pi_t = \frac{2 m_{t-1}}{L_t - S_t - \theta S_{t-1}}
$$

and the tax rate satisfies: $1 - \tau_t = (S_t - \theta S_{t-1})/L_t$. In particular, this policy is such that consumption is equalized across generations $C_{o,t} = C_{y,t}$. The same kind of arguments that we used to obtain equilibrium uniqueness then apply to this case.

C Inefficient Redistribution

C.1 Unequal weights

The authority solves the following problem:

$$\max_{\Pi_t, T_t} \left\{ \log \left(W - T_t - S_t - m_t \right) + \mu \log \left(\Pi_t^{-1} m_{t-1} + \theta S_{t-1} \right) \right\} \quad \text{s.t. } T_t = \Pi_t^{-1} m_{t-1} - m_t$$

with $\mu \geq 0$ being the weight of old relative to young in the authority's objective. Substituting the budget constraint $T_t = \Pi_t^{-1} m_{t-1} - m_t$ in the consumption of the young, the first order condition with respect to Π_t^{-1} writes:

$$\frac{m_{t-1}}{W - \Pi_t^{-1} m_{t-1} - S_t} = \mu \frac{m_{t-1}}{\Pi_t^{-1} m_{t-1} + \theta S_{t-1}}$$

which implies:

$$\frac{(1+\mu) m_{t-1}}{\mu W - \mu S_t - \theta S_{t-1}} = \Pi_t$$

Using optimal consumption equation $C_{y,t} = (W - T_t)/2 = m_t + S_t$, we can solve for the level of taxes:

$$T_t = \mu S_t + (\mu - 1) m_t - \theta S_{t-1},$$

where money demand is given by:

$$m_t = \frac{W - (\mu + 2) S_t + \theta S_{t-1}}{\mu + 1}.$$

Let us check whether an equilibrium can form in which storage is used. In this case, the inflation rate has to satisfy:

$$\frac{W - (\mu + 2) S_t + \theta S_{t-1}}{\mu W - \mu S_{t+1} - \theta S_t} = \Pi_{t+1} = \theta^{-1}$$

This leads to the following equation for S_t:

$$\mu S_{t+1} - \theta(\mu + 1) S_t + \theta^2 S_{t-1} = (\mu - \theta) W \tag{C.1}$$

The steady state solution to this equation is:

$$\bar{S} = \frac{W}{1 - \theta}$$

Note that such level of storage is not compatible with the non-negativity constraint on money holdings as

$$m = \frac{W - (\mu + 2 - \theta) \bar{S}}{\mu + 1} \geq 0$$

requires that:

$$W \geq \frac{\mu + 2 - \theta}{1 - \theta} W$$

which cannot be as $\mu + 2 > 1$.

Let us now find the solutions to (C.1). The discriminant associated to this equation is positive and writes:

$$\theta^2(\mu+1)^2 - 4\mu\theta^2 = \theta^2(\mu-1)^2 \geq 0$$

As a result, the two roots are:

$$r_1 = \frac{\theta(\mu+1) + \theta(\mu-1)}{2\mu} = \frac{\theta}{\mu}$$
$$r_2 = \frac{\theta(\mu+1) - \theta(\mu-1)}{2\mu} = \theta$$

As a result, the solutions to (C.1) are of the form:

$$S_t = A\theta^t + B\frac{\theta^t}{\mu^t} + \bar{S}$$

First case: $\theta < \mu$. In this case, S_t converges to $\bar{S}$, which cannot be, as we showed.

Second case: $\theta > \mu$. In this case, either S_t converges to $\bar{S}$ or diverges to ∞, which cannot be either.

Let us indeed show that S_t converges to $\bar{S}$ or diverges to ∞. This happens when $B \geq 0$ (if $B > 0$, S_t diverges to ∞ and if $B = 0$, S_t converges to $\bar{S}$). Suppose indeed that $B < 0$. This implies that S_t ultimately becomes negative, which cannot be. Instead, there exists a date t such that $S_t = 0$. However, at date t, the expected return is, when $S_t = 0$:

$$\frac{W + \theta S_{t-1}}{\mu W} = \Pi_{t+1}$$

That $S_t = 0$ and $m_t > 0$ requires that this return is larger than θ^{-1}. However, this implies that $\theta \frac{W+\theta S_{t-1}}{W} < \mu$ and so $\mu > \theta$ which is inconsistent with $\mu < \theta$.

Finally, note that a pure monetary equilibrium is not possible when $\mu < \theta$ as the inflation rate would be such that:

$$\frac{1}{\mu} = \Pi_{t+2} < \theta^{-1}$$

which would require $\mu > \theta$.

C.2 Distortionary Taxation

When utility is logarithmic, the authority must solve

$$\max_{\Pi_t, T_t} \left\{ \log\left(W_t - (1+\eta)T_t - S_t - m_t\right) + \log\left(\Pi_t^{-1} m_{t-1} + \theta S_{t-1}\right) \right\}$$
$$T_t = \Pi_t^{-1} m_{t-1} - m_t$$

Substituting the budget constraint in the objective function, we obtain:

$$\max_{\Pi_t} \left\{ \log \left(W_t - (1+\eta)\Pi_t^{-1} m_{t-1} - S_t + \eta m_t \right) + \log \left(\Pi_t^{-1} m_{t-1} + \theta S_{t-1} \right) \right\}$$

The first order condition with respect to the inflation rate yields the following optimal rate:

$$\Pi_t = \frac{2(1+\eta)m_{t-1}}{W_t - S_t + \eta m_t - (1+\eta)\theta S_{t-1}}$$

and the optimal level of taxes is so that:

$$T_t = \frac{W_t - S_t - (2+\eta)\, m_t - (1+\eta)\theta S_{t-1}}{2(1+\eta)}$$

As a result, the money demand is equal to

$$\begin{aligned} m_t &= \frac{\beta}{1+\beta} \left(W - (1+\eta) T_t \right) - S_t \\ &= \frac{\beta W - (2+\beta)\, S_t + (1+\eta)\theta\beta S_{t-1}}{2 - \beta\eta} \end{aligned}$$

Using this expression, we obtain that:

$$\begin{aligned} \Pi_{t+1} &= \frac{M_t}{P_t} \frac{P_{t+1}}{P_t} \frac{P_t}{M_t} \\ &= -(1+\eta) \frac{\beta W - 2S_t - \beta S_t + \theta\beta S_{t-1} + \theta\beta\eta S_{t-1}}{S_{t+1} - W + \theta S_t + \eta S_{t+1} + \theta\eta S_t - \theta\beta\eta^2 S_t - \theta\beta\eta S_t} \end{aligned}$$

Let us investigate whether an equilibrium can form in which storage is always used. This implies:

$$\theta^{-1} = -(1+\eta) \frac{\beta W - 2S_t - \beta S_t + \theta\beta S_{t-1} + \theta\beta\eta S_{t-1}}{S_{t+1} - W + \theta S_t + \eta S_{t+1} + \theta\eta S_t - \theta\beta\eta^2 S_t - \theta\beta\eta S_t}$$

We then obtain the following second order equation for storage: The steady state solution to this equation is:

$$\bar{S} = \frac{W}{(1-\theta)(1+\eta)}$$

The characteristic polynomial for the difference equation is:

$$x^2 - \theta\left(\beta + \beta\eta + 1\right) x + \theta^2 \beta \left(\eta + 1\right) = 0$$

whose roots are

$$\theta\beta(1+\eta), \theta$$

The solutions for S_t are of the form:

$$S_t = \bar{S} + A\theta^t + B(\theta\beta(1+\eta))^t$$

This sequence then converges to $\bar{S}$ when $B = 0$, when $\theta\beta(1+\eta) < 1$. Otherwise, S_t either diverges to ∞ or $-\infty$. The first case is clearly not possible. In the second case, this would mean that storage ultimately goes to 0. Suppose that this happens at date t.

The expected return in this case is:

$$\frac{W-(1+\eta)S_{t+1}}{(1+\eta)\left(\beta W+(1+\eta)\theta\beta S_{t-1}\right)}$$

This return is below θ as:

$$W\left(1-(1+\eta)\,\beta\theta\right)<\theta\left((1+\eta)^2\left(\theta\beta S_{t-1}\right)\right)+(1+\eta)S_{t+1}$$

when $(1+\eta)\,\beta\theta>1$. As a result, it is not optimal at date t not to use storage and B cannot be negative.

As a result S_t converges to $\bar{S}$. Let us investigate whether this is feasible. In particular, money holdings have to be positive:

$$\begin{aligned} m_t &= \frac{\beta}{1+\beta}\left(W-(1+\eta)T_t\right)-S_t \\ &= \frac{\beta W-(2+\beta)\,S+(1+\eta)\theta\beta S}{2-\beta\eta}\geq 0. \end{aligned}$$

which happens whenever:

First case $\beta<\frac{2}{\eta}$. In this case, the denominator is positive so that it has to be

$$\beta W-(2+\beta)\,S+(1+\eta)\theta\beta S>0$$

which we get, when $2+\beta>\theta\beta(1+\eta)$, when:

$$\bar{S}<\hat{S}\equiv\frac{\beta}{2+\beta-\theta\beta\,(1+\eta)}W,$$

which can never be the case since $\bar{S}<\hat{S}$ implies that

$$\frac{\beta}{2+\beta-\theta\beta\,(1+\eta)}>\frac{1}{(1-\theta)\,(1+\eta)}$$

which happens only when $\beta>2/\eta$.

Alternatively, $2+\beta<\theta\beta(1+\eta)$ is not possible when $\beta\eta<2$. Indeed, this would imply that:

$$2+\beta<\theta(2+\beta)$$

which is not possible as $\theta<1$.

Second case $\beta>\frac{2}{\eta}$. In this case, the denominator is negative so that it has to be

$$\beta W-(2+\beta)\,S+(1+\eta)\theta\beta S<0.$$

Now, with $\eta\beta>2$, we have $2+\beta<\theta\beta(1+\eta)$. In this case, $m>0$ when

$$\bar{S}<\hat{S}\equiv\frac{\beta}{2+\beta-\theta\beta\,(1+\eta)}W,$$

which is never true as $\bar{S}$ is positive and $\hat{S}$ negative.

C.3 Age-specific transfers

Preference heterogeneity. To begin with, agents can differ in their preferences. This can translate into heterogeneous savings. Let us elaborate an example of such heterogeneity.

Let us assume that agents' preferences are as follows: $u(C^i_{y,t}, C^i_{o,t}) = \log C^i_{y,t} + \gamma_i \log C^i_{o,t}$ with heterogeneous γ_i. We also assume that a group of mass p of agents are such that $\gamma_i = 1$ – *savers*, in which case $i = s$ – and the rest are such that $\gamma_i = 0$ – *consumers*, in which case $i = c$. The former agents save half of their endowment net of taxes to be consumed in the second period of their life – as in the benchmark model – , while the latter do not save at all. We assume that the transfer to the old has to be positive.

As a result, consumption of savers while being young are:

$$C^s_{y,t} = m^s_t + S^s_t = \frac{W - T_{y,t}}{2} \text{ and } C^c_{y,t} = W - T_{y,t},$$

where $C^s_{y,t}$ is the consumption of savers and $C^s_{y,t}$ the consumption of consumers, with $S^s_t = S_t$ and $m^s_t = m_t$ as only savers save. The government's budget constraint is:

$$T_{y,t} + T_{o,t} + m_t = m_{t-1}\Pi_t^{-1}$$

and, thus:

$$C^s_{y,t} = m^s_t + S^s_t = \frac{W + m_t - m_{t-1}\Pi_t^{-1} + T_{o,t}}{2} \text{ and } C^c_{y,t} = W + m_t - m_{t-1}\Pi_t^{-1} + T_{o,t}.$$

Integrating the first equality across all savers yields:

$$m_t + S_t = p\frac{W - m_{t-1}\Pi_t^{-1} + m_t + T_{o,t}}{2}.$$

We then obtain that:

$$m_t = \frac{p}{2-p}\left(W - m_{t-1}\Pi_t^{-1} + T_{o,t}\right) - \frac{2}{2-p}S_t$$

Let us compute the consumption of the old, starting with savers:

$$C^s_{o,t} = \Pi_t^{-1}m^s_{t-1} + \theta S^s_{t-1} - T_{o,t}.$$

As $pm^s_{t-1} = \int m^i_{t-1}di = m_{t-1}$ and $pS^s_{t-1} = \int S^i_{t-1}di = S_{t-1}$

$$C^s_{o,t} = \frac{\Pi_t^{-1}m_{t-1} + \theta S_{t-1}}{p} - T_{o,t}$$

In contrast, as they do not save, consumers only consume $C^c_{o,t} = -T_{o,t}$.

We can plug this value into the expressions for agents' consumption levels so that the current demand of money m_t disappears:

$$C^s_{y,t} = \frac{1}{2-p}\left(W - m_{t-1}\Pi_t^{-1} + T_{o,t} - S_t\right) \tag{C.2}$$

$$C^c_{y,t} = \frac{2}{2-p}\left(W - m_{t-1}\Pi_t^{-1} + T_{o,t} - S_t\right) = 2C^s_{y,t} \tag{C.3}$$

The resulting problem for the authority is:

$$\max_{\Pi_t, T_{o,t}\leq 0} \left\{ p\log\left(C^s_{y,t}\right) + (1-p)\log\left(2C^s_{y,t}\right) + p\log\left(\frac{m_{t-1}\Pi_t^{-1} + \theta S_{t-1}}{p} - T_{o,t}\right)\right\}.$$

The first order conditions with respect to Π_t and $T_{o,t}$ are as follows:

$$\frac{m_{t-1}}{W - m_{t-1}\Pi_t^{-1} + T_{o,t} - S_t} = \frac{m_{t-1}}{\frac{m_{t-1}\Pi_t^{-1}+\theta S_{t-1}}{p} - T_{o,t}}$$

$$\frac{1}{W - m_{t-1}\Pi_t^{-1} + T_{o,t} - S_t} = \Xi + p\frac{1}{\frac{m_{t-1}\Pi_t^{-1}+\theta S_{t-1}}{p} - T_{o,t}}$$

with $C^s_{y,t}$ and $C^c_{y,t}$ defined by equations (C.2) and (C.3) and Ξ the Lagrange multiplier associated with $-T_{o,t} \geq 0$.

Taking together the two conditions, we obtain:

$$\Xi = \frac{1-p}{\frac{m_{t-1}\Pi_t^{-1}+\theta S_{t-1}}{p} - T_{o,t}}$$

so that, when $p < 1$, $\Xi > 0$ and $T_{o,t} = 0$. As a result of these conditions, we obtain the following expression for $m_{t-1}\Pi_t^{-1}$:

$$m_{t-1}\Pi_t^{-1} = \frac{pW - \theta S_{t-1} - pS_t}{1+p},$$

which allows to also rewrite m_t as follows:

$$m_t = \frac{1}{(2-p)(1+p)}\left(pW + \theta p S_{t-1} + (p^2 - 2(1+p))S_t\right).$$

The inflation rate at $t+1$ can be expressed as function of storage. Using the no-arbitrage condition between money and storage, we find:

$$\frac{pW + \theta p S_{t-1} + (p^2 - 2(1+p))S_t}{pW - \theta S_t - pS_{t+1}}\frac{1}{2-p} = \theta^{-1}$$

which leads to:

$$(2-p-\theta)W = (2-p)S_{t+1} + \theta(p-3)S_t + \theta^2 S_{t-1}.$$

As in the benchmark case, the sequences S_t satisfying this equation are of the following form, for $p < 1$:

$$S_t = \lambda_1\theta^t + \lambda_2\left(\frac{\theta}{2-p}\right)^t + \frac{2-p-\theta}{2-p-\theta+\theta(p-2)+\theta^2}W.$$

As θ and $\theta/(2-p)$ are both below 1, S_t converges to $\frac{2-p-\theta}{2-p-\theta+\theta(p-2)+\theta^2}W$. Given that $\theta(p-2)+\theta^2=\theta(\theta+p-2)<0$, we then obtain that S_t is ultimately above $W/2$. We can then use the same logic as for the proof of Proposition 6.

D Fixed or bounded taxes

D.1 Fixed taxes

We first explore the equilibrium outcome when the authority cannot change taxes in reaction to saving choices.

Optimal policy. Fixing taxes amounts to restricting the policy's space to $\hat{\mathcal{P}}_t=(\Delta_t,G_t,\bar{T})$, where taxes on the young $T_t=\bar{T}$ are taken fixed.

As in the benchmark case, the first step is to rewrite the consumption of the young independently of date-t variables:

$$C_{y,t}=W-\bar{T}-S_t-m_t=\frac{R(\rho_{t+1})(W-\bar{T})}{1+R(\rho_{t+1})}, \tag{D.1}$$

and real saving then writes as $S_t+m_t=C_{y,t}/R(\rho_{t+1})$. By combining the authority's budget constraint with the saving equation $S_t+m_t=C_{y,t}/R(\rho_{t+1})$, we obtain spending G_t as follows:

$$G_t=\frac{W+R(\rho_{t+1})\bar{T}}{1+R(\rho_{t+1})}-S_t-\Pi_t^{-1}m_{t-1}. \tag{D.2}$$

Consumption and portfolio choices are still as described in Section 3. Instead, the policy is different. In analogy with the first part of Proposition 4, the optimal policy at date t is given by $\hat{\mathcal{P}}_t^*=\operatorname{argmax}\{\mathbb{U}_t\}$.The current flow of utility in the authority's objective is now given by:

$$\mathbb{U}_t=u\left(C_{y,t}\right)+u\underbrace{\left(\Pi_t^{-1}m_{t-1}+\theta S_{t-1}\right)}_{=C_{o,t}}+\tilde{\lambda}u\underbrace{\left(\frac{W+R(\rho_{t+1})\bar{T}}{1+R(\rho_{t+1})}-S_t-m_{t-1}\Pi_t^{-1}\right)}_{=G_t}, \tag{D.3}$$

where $C_{y,t}$, according to (D.1), is independent from policy. The solution to this problem, $\hat{\mathcal{P}}_t^*=\{\Delta_t,G_t,\bar{T}\}$, is Markovian and given by

- $\Delta_t(\sigma_t)$ is such that:

$$\Pi_t = \frac{(1+\lambda)m_{t-1}}{\frac{W+R(\rho_{t+1})\bar{T}}{1+R(\rho_{t+1})}-\lambda\theta S_{t-1}-S_t} \qquad if \quad \lim_{m_{t-1}\to\infty}\lambda C_{o,t}\leq\lim_{m_{t-1}\to\infty}G_t, \tag{D.4}$$

$$\Pi_t \to \infty \qquad \text{otherwise}, \tag{D.5}$$

according to (15);

- $G_t(\sigma_t)$ given by (D.2)

at any t, where, again, $\lambda=1/u'_{-1}(\tilde{\lambda})$. There is now a trade-off in the use of the price for money as an instrument. On the one hand, the authority may reduce consumption inequality by lowering the price for money. On the other hand, it can increase public expenditures by increasing the price for money. Which force prevails depends on the initial level and the importance of public expenditures. We have the following.

Proposition D.1. *For any β and initial conditions $M_0 > 0$ and $S_0 = 0$, multiple equilibria exist depending on $\{\bar{T}, \lambda\}$.*

(a) *Provided that*

$$\hat{\pi} \equiv (1+\lambda)\frac{W-\bar{T}}{W+\bar{T}} \leq \theta^{-1},$$

an inefficient monetary equilibrium exists such that, for any $t \geq 1$:

(i) $\sigma_t^* \in \sigma^*$ is such that:

$$S_t = 0 \quad \text{and} \quad m_t = \frac{W-\bar{T}}{1+R(\Pi^{-1})}, \tag{D.6}$$

(ii) $\mathcal{P}_t^* \in \sigma_{\hat{\mathcal{P}}^*}$ is such that:

$$\Pi_t = \hat{\pi}, \tag{D.7}$$

$$G_t = \frac{(1+\lambda)(W+R(\hat{\pi}^{-1})\bar{T}) - (W+\bar{T})}{(1+R(\hat{\pi}^{-1}))(1+\lambda)}; \tag{D.8}$$

(iii) the price level is given by $P_t = M_t/m_t$ where $M_t = \hat{\pi}M_{t-1}$.

(b) *Furthermore, when*

$$\frac{\bar{T}}{W} < \frac{\lambda\theta}{1+R(\theta)+\lambda\theta},$$

- *a money-storage equilibrium also exists for each $s \geq 1$ such that, (D.6)-(D.8) holds for $t < s$, and for any $t \geq s$:*

  (i) $\sigma_t^* \in \sigma^*$ is such that:

$$S_t = \theta S_{t-1} + \frac{W+R(\theta)\bar{T} - \theta(1+\lambda)(W-\bar{T})}{1+R(\theta)},$$

$$m_t = \frac{W-\bar{T}}{1+R(\theta)} - S_t \quad \text{with} \quad \lim_{t\to\infty} m_t = \frac{\theta\lambda(W-\bar{T}) + (R(\theta)-1)\bar{T}}{(1-\theta)(1+R(\theta))} \geq 0;$$

  (ii) $\mathcal{P}_t^* \in \sigma_{\hat{\mathcal{P}}^*}$ is such that:

$$\Pi_t = \theta^{-1} = \frac{1}{R(\theta)},$$

$$G_t = \lambda\theta\frac{W-\bar{T}}{1+R(\theta)}; \quad \text{and}$$

  (iii) the price level is given by $P_t = M_t/m_t$ where $M_t = \theta^{-1}M_{t-1}$.

- *and an autarky equilibrium exists where $\sigma_t^* \in \sigma^*$ is such that $S_t = (W-\bar{T})/(1+R(\theta))$, $m_t = 0$, and $\mathcal{P}_t^* \in \sigma_{\hat{\mathcal{P}}^*}$ such that $\Pi_t > \theta^{-1}$, $G_t = \bar{T}$ and $P_t \to \infty$, for any $t \geq 1$.*

Proof. See Appendix D.3. □

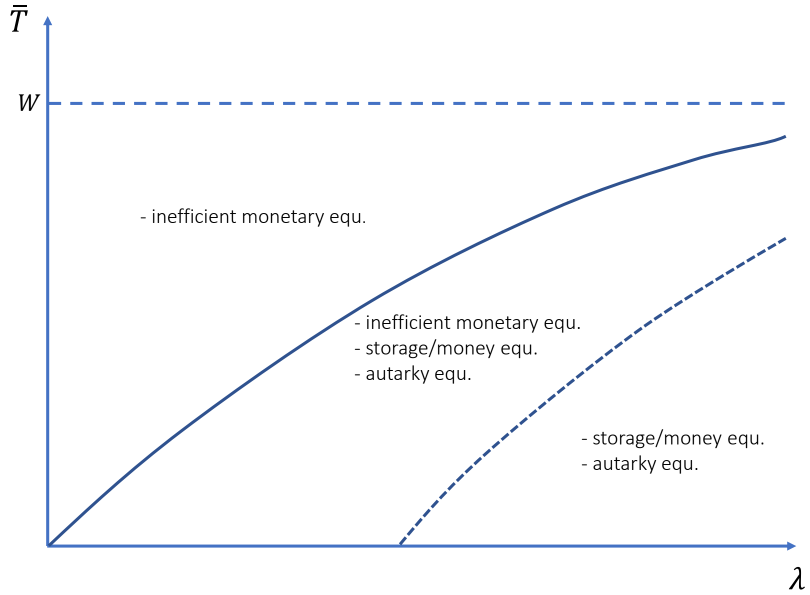

Figure D.1 – Equilibria sets in the space $(\lambda, \bar{T})$.

The proposition shows the role of fiscal capacity in nailing down the set of equilibria. The conditions for the existence of equilibria are illustrated in Figure D.1. The set of equilibria crucially depends on the level of $\bar{T}$ – the fiscal capacity of the authority – and λ – the importance of public spending. With fixed taxes, the authority systematically uses its seigniorage power to balance the consumption of the old vis-à-vis public expenditure. When taxes are fixed too low relative to the importance of public spending, the authority can only adjust expenditures to purchase money, and, thus, it trades off the welfare gains of money trading against its cost of cutting expenditures. Whenever this equilibrium exists, the possibility of autarky also exists. The trade-off between public spending and agents' consumption does not arise when taxes can be freely set, as, then, the authority has sufficient tools to adjust its expenditures. In such a case, the authority sustains the value of money to improve the total amount of consumption goods available at that time and sets taxes to ensure the fraction that it needs.

It is instructive to remark that, in the extreme case where the authority cannot tax and puts no weight on its spending ($\bar{T} = \lambda = 0$), the set of equilibria in Proposition D.1 coincides with the set of equilibria in the absence of policy intervention. This means that in the absence of a fiscal counterpart, the authority cannot do better than the market, in line with Wallace (1981b).

A Laffer curve of seigniorage. In the storage-money equilibrium, inflation is higher than in the inefficient monetary equilibrium; however, the primary fiscal surplus is *less* negative, showing that actual seigniorage revenues are lower. Effectively, in the storage-money equilibrium, the consumption by both the old and the authority is lower. This means that storage-money equilibrium is the result of a coordination failure between private agents and the authority, entailing a sort of Laffer curve of seigniorage.

Let us expand on the reasons behind the coordination failure. Suppose that the young decide to save in storage. This action increases the resources available in the next period and, in particular, the consumption of the old. Since the authority wants to equalize the marginal utility of its own consumption and that of the old, it sells money to drain resources from the old. This increases inflation until it matches the return on storage, making the young indifferent between saving in money or storage, as in the absence of policy. However, whereas in the absence of policy, the equilibrium inflation rate is achieved by subsequent decreases in private money demand, in this equilibrium, it is achieved by subsequent increases in money supply. This is what allows real money demand to stay constant at a level lower than in the inefficient monetary equilibrium. In analogy with the Laffer curve of taxation, we can interpret the lower level of real money holdings as a lower "tax base" of seignorage, which pushes the authority to tax more money holdings to extract resources. This higher "tax rate" corresponds to a higher inflation rate. The expectation of such a higher inflation rate makes the more intense use of storage self-fulfilling and the willingness to tax more through more seignorage self-defeating. The agents would all benefit from being in the monetary equilibrium, but, individually, the optimality of their portfolio may lead them to use storage. The authority would also benefit from being in the monetary equilibrium to expand its tax base and obtain a higher revenue from seignorage, but it cannot because it does not control future authorities' decisions.

This inability to resist taxation through seigniorage also underlies the existence of the autarky equilibrium. When taxes are too low, and, thus, government expenditures are low as well, the government may even have the incentive to drive the price level to negative values so as to tax money holdings. Since negative price levels are not feasible, prices are sent to infinity: such an incentive prevents any credible deflation, which is what the optimal unconstrained policy was able to generate in the absence of private demand for money. As a consequence, autarky can be an equilibrium outcome.

D.2 Upper bound on taxation

In this subsection, we generalize the above findings by considering a bound on taxation; that is, we explore the case in which taxes are fully flexible conditional on being lower than a certain cap. We show that, when this constraint is sufficiently tight, multiple equilibria can emerge, as characterized in Proposition D.1; otherwise, equilibria as described in Proposition 6 hold.

We now assume that taxes on the young generation have to satisfy:

$$T_t \leq \hat{T}, \tag{D.9}$$

at any t, with $\hat{T} \geq 0$.[16] Leveraging on the results that we have already derived, we can show the following proposition:

Proposition D.2. *When*

$$\hat{T} \geq \frac{\lambda + 1 - \beta}{\lambda + 2} W,$$

(i) *the constraint* (D.9) *does not bind in equilibrium;*

(ii) *a unique equilibrium exists where only money is used as described by Proposition 6.*

Otherwise, when

$$\hat{T} \leq \frac{\lambda + 1 - \beta}{\lambda + 2} W,$$

(i) *the constraint* (D.9) *always binds in equilibrium; and*

[16] Such an upper on taxation may result from an extreme form of distortionary cost of taxation, where it is costless to set taxes up to $\hat{T}$ but arbitrarily large for any higher value.

(ii) *the set of equilibria is described by Proposition D.1 with taxes T_t fixed at $\hat{T}$; in particular, only money is used in equilibrium when also*

$$\hat{T} \geq \frac{\lambda\theta}{1 + R(\theta) + \lambda\theta} W.$$

Proof. See Appendix D.4 □

When its fiscal capacity is constrained, the authority faces a trade-off between monetary stability and its expenditures, as in the case of fixed taxes (Proposition D.1). When the constraint is tight enough, this trade-off results in multiple equilibria, and monetary stability cannot necessarily be ensured.

When this bound is sufficiently large, $\hat{T}/W \geq (\lambda + 1 - \beta)/(\lambda + 2)$, the monetary equilibrium without money creation is the single equilibrium, as in Proposition 6. In this case, the constraint $T_t \leq \hat{T}$ does not bind in equilibrium. Interestingly, the constraint can bind off-equilibrium, when $(1 + \lambda - R(\theta)^{-1})/(2 + \lambda) \geq \hat{T}/W \geq (\lambda + 1 - \beta)/(\lambda + 2)$, but this does not prevent uniqueness of the equilibrium.

Whenever, $\hat{T}/W \leq (\lambda + 1 - \beta)/(\lambda + 2)$, the constraint $T_t \leq \hat{T}$ always binds, in- and off-equilibrium. As a result, we are back to a situation as with fixed taxes, described in Proposition D.1. In such a situation, when $\hat{T}/W \geq \lambda\theta/(1 + R(\theta) + \lambda\theta)$, the monetary equilibrium is still the unique equilibrium, but one in which the authority is creating money to finance its expenditures given that, with taxes only, the level of spending would be suboptimal. It is only when $\hat{T}/W < \lambda\theta/(1 + R(\theta) + \lambda\theta)$ that multiple equilibria may emerge.

D.3 Proof of Proposition D.1

The proof works as follows. First, as in the benchmark case, we find equations that equilibrium variables have to solve. Then we use these equations to provide conditions under which different equilibria may arise.

Equilibrium characterization. Using (D.4), we get the actual law of motion of inflation of the real value of savings and inflation as:

$$m_t = \frac{W - \bar{T}}{1 + R(\rho_{t+1})} - S_t \tag{D.10}$$

$$\Pi_{t+1} = \frac{P_{t+1}}{P_t} = \frac{(1+\lambda)\left(W - \bar{T}\right) - (1 + R(\rho_{t+1}))(1+\lambda)S_t}{W + \bar{T} - (1 + R(\rho_{t+1}))\left(\lambda\theta S_t + S_{t+1}\right)} \tag{D.11}$$

provided $W + \bar{T} \geq (1 + R(\rho_{t+1}))(\lambda\theta S_t + S_{t+1})$, otherwise we have $m_{t+1} \to 0$ and $\Pi_{t+1} \to \infty$. We are ready now to investigate investigate how optimally chosen policies affect equilibrium outcomes.

The pure monetary equilibrium. The pure monetary equilibrium where $S_t = 0$ at each t is still an equilibrium provided $(1+\lambda)(W-\bar{T})/(W+\bar{T}) < \theta^{-1}$. This can be easily seen by checking that $S_t = 0$ at any t implies $\Pi_{t+1} = (1+\lambda)(W-\bar{T})/(W+\bar{T})$ at any t from (D.11). In turn, $S_t = 0$ requires that $\Pi_{t+1} \leq \theta^{-1}$, thus implying that $(1+\lambda)(W-\bar{T})/(W+\bar{T})$ does not exceed θ^{-1}. We then obtain m_t from (D.1) with $S_t = 0$ and G_t from (D.2).

In case $\Pi_{t+1} > \theta^{-1}$ implies $S_t > 0$, so that a pure monetary equilibrium does not exist in that case.

Existence of asymptotic storage equilibria. We investigate now whether there are equilibria where both money and storage are used. $S_t > 0$ implies $\Pi_t = \theta^{-1}$ at t that, is:

$$S_t = \theta S_{t-1} + \frac{W + R(\theta)\bar{T} - \theta(1+\lambda)(W - \bar{T})}{1 + R(\theta)}$$

Let us first consider the case $\theta(1+\lambda)(W - \bar{T}) < W + R(\theta)\bar{T}$. In such a case, $S_t > 0$ implies $S_{t+\tau} > 0$ for $\tau \geq 1$. However, an equilibrium where $S_t > 0$ for each $t \geq \tau$ requires a sequence $\{S_t\}_{t=1}^{\infty}$ converging monotonically to

$$\bar{S} = \frac{W + R(\theta)\bar{T} - \theta(1+\lambda)(W - \bar{T})}{(1 + R(\theta))(1-\theta)}.$$

As previously noted, to be feasible, $\bar{S}$ should satisfy $\bar{S} \leq \left(W - \bar{T}\right)/(1 + R(\theta))$. As a result, a necessary condition to be an equilibrium is:

$$\bar{T} \leq \frac{\lambda\theta}{1 + R(\theta) + \lambda\theta} W.$$

Otherwise, an equilibrium where money and storage are jointly used does not exist.

Similarly to the case without any policy, all asymptotic storage equilibria do not necessarily feature storage at date-0 and it is possible to construct asymptotic storage equilibria where storage is not used until a certain date s after which it is always used. In fact, notice that $S_{s-1} = 0$ only requires that $\Pi_s \leq \theta^{-1}$, that is

$$0 \leq S_s < \frac{W + R(\theta)\bar{T} - \theta(1+\lambda)(W - \bar{T})}{1 + R(\theta)}.$$

Thus, at each date t, after having only used money in past periods, it is possible to start using storage. Also here once storage is used it will used for ever.

In the case when $\theta(1+\lambda)(W - \bar{T}) > W + R(\theta)\bar{T}$, the sequence of storage S_t converges to a negative value; however this violates the constraint $S_t \geq 0$. Thus, in this case, an equilibrium where storage is used with money does not exist.

Finally, let us note that when condition

$$\bar{T} < \frac{\theta\lambda}{1 + R(\theta) + \theta\lambda} W$$

is satisfied, the condition $(1+\lambda)(W - \bar{T})/(W + \bar{T}) \leq \theta^{-1}$ is also satisfied. Indeed, this latter condition is a decreasing function of $\bar{T}$ and the condition is satisfied for $\bar{T} = \frac{\theta\lambda}{2+\theta\lambda} W > \frac{\theta\lambda}{1+R(\theta)+\theta\lambda} W$.

Existence of pure autarky equilibria. We study here the conditions for the existence of a pure autarky equilibrium – i.e. one in which $m_t = 0$ for any t. Without loss of generality, we consider period 1. Suppose that $m_1 = 0$. The optimal rate of inflation at date 2 is:

$$\Pi_2 = \frac{(1+\lambda)m_1}{\frac{W + R(\theta)\bar{T}}{1 + R(\theta)} - \lambda\theta S_1 - S_2}$$

$$\Pi_2 = \frac{(1+\lambda)(1 + R(\theta))m_1}{W + R(\theta)\bar{T} - (\lambda\theta + 1)(W - \bar{T})}$$

using the fact that, In autarky, $S_1 = S_2 = (W - \bar{T})/(1 + R(\theta))$. The denominator is strictly positive when:

$$\bar{T} > \frac{\lambda\theta}{1 + R(\theta) + \lambda\theta} W,$$

in which case $\Pi_2 = 0$ when $m_1 = 0$. As a result, agents are strictly better off not to store. Otherwise, when

$$\bar{T} \leq \frac{\lambda\theta}{1 + R(\theta) + \lambda\theta} W,$$

D.4 Proof of Proposition D.2

To prove this Proposition, we first determine conditions under which the constraint on T_t may bind. We split this investigation depending on whether storage is used in equilibrium.

(Potential) equilibria without storage. Suppose that storage is never used in equilibrium. From Proposition 6, the unconstrained level of taxes is $(1 - \beta + \lambda)/(\lambda + 2)W$. Depending on the value of $\hat{T}$, this level of taxes is then constrained by $\hat{T}$. When $\hat{T} \geq (1-\beta+\lambda)/(\lambda+2)W$, there exists a monetary equilibrium as described by Proposition 6. Otherwise, a monetary equilibrium exists under the condition of Proposition D.1.

(Potential) Equilibria with storage. Suppose that storage is used at some date. Let us first show the following lemma.

Lemma D.3. *At date t, suppose that storage is used and that the constraint binds:*

$$\frac{\lambda + 1 - R(\theta)^{-1}}{2 + \lambda} W + \frac{1 + R(\theta)^{-1}}{2 + \lambda}(S_t - \theta S_{t-1}) \geq \hat{T} \tag{D.12}$$

Then the constraint binds for $t + 1$.

Proof. Suppose that storage is used at date t. As a result $S_t > 0$ and the expected return is θ.

Given that the constraint binds at date t, we can use $R(\theta)(S_{t+1} - \theta S_t) - \theta(S_t - \theta S_{t-1}) = (R(\theta) - \theta)W$ to replace $(S_t - \theta S_{t-1})$ in the constraint to obtain:

$$\frac{(\lambda + 1 - R(\theta)^{-1})}{2 + \lambda} W + \frac{(1 + R(\theta)^{-1})}{2 + \lambda} \left(\frac{R(\theta)}{\theta}(S_{t+1} - \theta S_t) - \left(\frac{R(\theta)}{\theta} - 1 \right) W \right) \geq \hat{T} \tag{D.13}$$

As a result, the constraint also binds when:

$$\left(\frac{R(\theta)}{\theta} - 1 \right) (S_{t+1} - \theta S_t - W) \tag{D.14}$$

This is always satisfied as $R(\theta) > \theta$ and $S_t < W/(1 - \theta)$. □

As a result of Lemma D.3, either the constraint never binds during or after storage is used or it always binds. If it never binds along these paths, the proof of Proposition 6 implies that these paths cannot be an equilibrium outcome. If the constraint always binds, Proposition D.1 implies that these paths can be equilibrium outcomes only when storage is always used and that:

$$\hat{T} \leq \frac{\theta(1+\lambda)-1}{R(\theta)+\theta(1+\lambda)}W. \tag{D.15}$$

Finally, note that when $\hat{T} \geq \frac{1-\beta+\lambda}{\lambda+2}$, no such paths can be equilibrium outcomes as:

$$\hat{T} \geq \frac{1-\beta+\lambda}{\lambda+2}W \geq \frac{\lambda}{2+\lambda}W \geq \frac{\theta(1+\lambda)-1}{R(\theta)+\theta(1+\lambda)}W. \tag{D.16}$$

E Fluctuations in endowments

In this section we will look at the case of time-varying endowment. We will initially look at the dynamics in the absence of policy and then study the optimal policy reaction. For simplicity we will restrict to the case of log-preferences.

E.1 Absence of policy

Before exploring how time-varying endowment affect the optimal policy, let us review briefly what changes absent policy, i.e. with $\mathcal{P}_t = (0,0,0)$ at each t. In the case of fluctuations in endowment, we have

$$\Pi_{t+1} = \frac{W_t - 2S_t}{W_{t+1} - 2S_{t+1}}.$$

This modification has important consequences on the set of equilibria. As a first result note that the pure monetary equilibrium may not exists any longer. In fact, $S_t = 0$ for any $t \geq 1$ is not an equilibrium when $\Pi_{\tau+1} = W_\tau / W_{\tau+1} \geq \theta^{-1}$. In this case, there not exist an equilibrium where $S_\tau = 0$ because the return on storage *necessarily* exceeds the one on money; on the contrary, S_τ must be strictly positive.

On the other hand, it is possible now an equilibrium where $S_\tau > 0$ and $S_{\tau+1} = 0$. To see this notice that, after substituting $m_\tau = W_\tau/2 - S_\tau$,(23) now becomes

$$S_{\tau+1} = \theta S_\tau + \frac{W_{\tau+1} - \theta W_\tau}{2},$$

so that, when $\theta W_\tau > W_{\tau+1}$, there exists a value of S_τ, namely

$$0 < \hat{S}_\tau \equiv \frac{\theta W_\tau - W_{\tau+1}}{2\theta} < \frac{W_\tau}{2},$$

such that $S_{\tau+1} = 0$, provided $\hat{S}_\tau < W_\tau/2$. The reason is that, when all savings are in money and the endowment decreases sufficiently fast, the return on money may fall below the return on storage, making storing more attractive instead. In general however, while a jump to positive storage is made necessary in situations of a strong decrease in endowment, the return to pure monetary savings is not. In analogy to the case with constant endowment, a multiplicity of equilibria exists that satisfy (E.1) where storage is positive also at date $t+1$.

E.2 Optimal policy reaction with fluctuations in endowment

We build the optimal policy in this case based on two elements.

First, whenever $S_t > 0$ we have that

$$\Pi_{t+1} = \frac{W_t + \theta S_{t-1} - (3+\lambda)S_t}{W_{t+1} - \theta(1+\lambda)S_t - S_{t+1}} = \theta^{-1},$$

or

$$S_t = \frac{S_{t+1} + \theta^2 S_{t-1} - W_{t+1} + \theta W_t}{2\theta}, \tag{E.1}$$

that is, the return on money must be the same than the return on storage.

Second, whenever $S_t = 0$ instead optimal saving choices require

$$\Pi_{t+1} = \frac{W_t + \theta S_{t-1}}{W_{t+1} - S_{t+1}} < \theta^{-1},$$

or

$$\theta^2 S_{t-1} < W_{t+1} - \theta W_t - S_{t+1}, \tag{E.2}$$

that is, the return on money is higher than the return on storage.

These two elements, which do not depend on λ, nail down the unique equilibrium consistent with an optimal policy response. The following proposition characterise the equilibrium path of storage, for arbitrary sequence of endowments, that, for an arbitrary initial S_t at some t, converges to zero storage in a finite time.

Proposition E.1. *For any $\{\lambda, \beta\}$, and a given $S_t \in (0, W_t)$ and sequence of endowments $\{W_\tau\}_{\tau=t+1}^{\infty}$, the sequence of $\{S_\tau\}_{\tau=t+1}^{\infty}$ characterising an equilibrium with optimal policy is such that:*

i) given the unique $n^* \in \mathbb{N}$ that satisfies the following inequality:

$$\frac{n^* W_{t+1+n^*} - \sum_{i=0}^{n^*-1} \theta^{n^*-i} W_{t+1+i}}{\theta^{n^*}} < \theta S_t < \frac{(n^*+1) W_{t+2+n^*} - \sum_{i=0}^{n^*} \theta^{n^*+1-i} W_{t+1+i}}{\theta^{n^*+1}},$$

ii) then there are at least n^* successive storage values $\{S_{t+1+n^*-n}\}_{n=1}^{n^*}$ given by:

$$S_{t+1+n^*-n} = \frac{n\theta^n \theta S_{t+n^*-n} + n\theta^n W_{t+1+n^*-n} - \sum_{i=1}^{n} \theta^{n-i} W_{t+1+n^*-n+i}}{(1+n)\,\theta^n},$$

before the pure monetary state $(S_{t+n^*+1}, S_{t+n^*+2}) = (0, 0)$.

Proof. To build our solution let us suppose that there is a t such that $S_t > 0$, $S_{t+1} = 0$ and $S_{t+2} = 0$ and work backwards. According to (E.1) we get

$$S_t = \frac{\theta^2 S_{t-1} + \theta W_t - W_{t+1}}{2\theta} \tag{E.3}$$

from which it is obvious that, to get $S_t > 0$ either W_t is sufficiently big or it must be $S_{t-1} > 0$. Moreover, according to (E.2) it should be

$$S_t = \frac{\theta^2 S_{t-1} + \theta W_t - W_{t+1}}{2\theta} < \frac{W_{t+2} - \theta W_{t+1}}{\theta^2}$$

i.e.

$$\frac{W_{t+1}-\theta W_t}{\theta} < \theta S_{t-1} < \frac{2W_{t+2}-\theta W_{t+1}-\theta^2 W_t}{\theta^2}. \tag{E.4}$$

If the inequality is satisfied with $S_{t-1}=0$ then only at t storage is positive in an equilibrium with $S_{t+1}=S_{t+2}=0$. Otherwise, it must be that also $S_{t-1}>0$.

Consider then, $S_{t-1}>0$. Applying iteratively (E.3), this requires that

$$S_{t-1}=\frac{S_t+\theta^2 S_{t-2}-W_t+\theta W_{t-1}}{2\theta}=\frac{2\theta^3 S_{t-2}+2\theta^2 W_{t-1}-\theta W_t-W_{t+1}}{3\theta^2} \tag{E.5}$$

from which it is obvious that either W_{t-1} is sufficiently big or it must be $S_{t-2}>0$. Because of (E.4) and (E.5) it must be that

$$\frac{2W_{t+1}-\theta W_t-\theta^2 W_{t-1}}{\theta^2} < \theta S_{t-2} < \frac{3W_{t+2}-\theta W_{t+1}-\theta^2 W_t-\theta^3 W_{t-1}}{\theta^3}$$

If the inequality is satisfied with $S_{t-2}=0$ then from $t-1$ to t storage is positive in an equilibrium with $S_{t+1}=S_{t+2}=0$. Otherwise, it must be that also $S_{t-2}>0$.

By iterating we have

$$S_{t+1-n}=\frac{n\theta^n\theta S_{t-n}+n\theta^n W_{t+1-n}-\sum_{i=1}^n\theta^{n-i}W_{t+1-n+i}}{(1+n)\,\theta^n}\geq 0,$$

which requires that either W_{t+1-n} is sufficiently large or S_{t-n} must be positive. In particular, it should be

$$\frac{nW_{t+1}-\sum_{i=0}^{n-1}\theta^{n-i}W_{t+1-n+i}}{\theta^n} < \theta S_{t-n} < \frac{(n+1)W_{t+2}-\sum_{i=0}^{n}\theta^{n+1-i}W_{t+1-n+i}}{\theta^{n+1}} \tag{E.6}$$

If the inequality is satisfied with $S_{t-n}=0$ then from $t-n+1$ to t storage is positive in an equilibrium with $S_{t+1}=S_{t+2}=0$. Otherwise, it must be that also $S_{t-n}>0$.

The last step of the proof is to verify that

$$\theta S_{t+1-n}=\theta\frac{n\theta^n\theta S_{t-n}+n\theta^n W_{t+1-n}-\sum_{i=1}^n\theta^{n-i}W_{t+1-n+i}}{(1+n)\,\theta^n} < \frac{nW_{t+2}-\sum_{i=0}^{n-1}\theta^{n-i}W_{t+2-n+i}}{\theta^{n+1}},$$

leads to (E.6). This follows from

$$\begin{aligned}
\theta\frac{n\theta^n\theta S_{t-n}}{(1+n)\,\theta^n} &< \frac{nW_{t+2}-\sum_{i=0}^{n-1}\theta^{n-i}W_{t+2-n+i}}{\theta^n}-\theta\frac{n\theta^n W_{t+1-n}-\sum_{i=1}^n\theta^{n-i}W_{t+1-n+i}}{(1+n)\,\theta^n}\\
\frac{n\theta^{n+1}\theta S_{t-n}}{(1+n)\,\theta^n} &< \frac{(1+n)nW_{t+2}-(1+n)\sum_{i=1}^n\theta^{n+1-i}W_{t+1-n+i}-n\theta^{n+1}W_{t+1-n}+\sum_{i=1}^n\theta^{n+1-i}W_{t+1-n+i}}{(1+n)\,\theta^n}\\
\theta S_{t-n} &< \frac{(1+n)nW_{t+2}-n\sum_{i=1}^n\theta^{n+1-i}W_{t+1-n+i}-n\theta^{n+1}W_{t+1-n}}{n\theta^{n+1}}\\
\theta S_{t-n} &< \frac{(1+n)W_{t+2}-\sum_{i=0}^n\theta^{n+1-i}W_{t+1-n+i}}{\theta^{n+1}}
\end{aligned}$$

which is the same as (E.6). So we conclude that our recursive formulation indeed holds at any t for given a n.

Now to recover the expressions in the proof we need to operate an appropriate change of variables to express as a given the initial positive level of storage. In practice, whereas the proof defines a sequence of storage expressed as $\{S_{t-n},S_{t-n+1},....S_{t-1},S_t,0,0\}$ the proposition defines the same sequence relabelling time indexes to be $\{S_t, S_{t+1}, ...S_{t+n^*-1}, S_{t+n^*}, 0, 0\}$. □

We finally state here the solution for a sequence of constant endowments, which are the ones considered in the main text. The uniqueness of such paths is established in A.5 (point d.3).

Corollary E.2. *For any $\{\lambda, \beta\}$, and a given $S_t \in (0, W_t)$ and sequence of endowments $W_t = W$ at any t, the sequence of $\{S_\tau\}_{\tau=t+1}^{\infty}$ characterising an equilibrium with optimal policy is such that:*

i) given the unique $n^* \in \mathbb{N}$ that satisfies the following inequality:

$$\frac{n^* - \sum_{i=0}^{n^*-1} \theta^{n^*-i}}{\theta^{n^*}} W < \theta S_t < \frac{(n^*+1) - \sum_{i=0}^{n^*} \theta^{n^*+1-i}}{\theta^{n^*+1}} W,$$

ii) then there are at least n^* successive storage values $\{S_{t+1+n^*-n}\}_{n=1}^{n^*}$ given by:

$$S_{t+1+n^*-n} = \frac{n\theta^n \theta S_{t+n^*-n} + \left(n\theta^n - \sum_{i=1}^{n} \theta^{n-i}\right) W}{(1+n)\,\theta^n},$$

before the pure monetary state $(S_{t+n^*+1}, S_{t+n^*+2}) = (0, 0)$.

F Infinite-horizon economy with no policy

In this Appendix, we show a simple model with infinite-horizon agents reproducing the same form of market incompleteness of OLG economies. This mapping is well known since Townsend (1980) – a model that, according to Ljungqvist and Sargent (2018) (chap. 28), "can be viewed as a simplified version" of a stochastic Bewley economy.

Consider a simple endowment economy with two types of infinitely-living agents j initially endowed with two different quantities of a non storable endowment, $q \in \{H, L\}$, with $\bar{q} = \{H, L\}/q$ and $H > L$. Each period the distribution of endowments flips. No credit contracts are possible, however agents can buy an intrinsically worthless asset called money available in a stock M so that $M = \sum_j M_{j,t}$ at any t. The problem of the agent j is :

$$V(q_t) = u(C(q_t)) + \beta V(\bar{q}_{t+1})$$

subject to

$$C(q_t) = q_t - \frac{M_{j,t} - M_{j,t-1}}{P_t}$$

with $M_{j,t} \geq 0$, where $V(q)$ denotes the present value of having a quantity of endowment q, $\beta \in (0, 1)$ is a discount factor and $C(q)$ is consumption contingent on having endowment quantity q. In a stationary equilibrium, we have

$$V(q_t) = \frac{u(C(q_t)) + \beta u(C(\bar{q}_{t+1}))}{1 - \beta^2}.$$

The first order conditions with respect to $M_{j,t}$ are:

$$-\frac{1}{1-\beta^2} u'(C(q_t)) \frac{1}{P_t} + \frac{1}{1-\beta^2} \beta u'(C(\bar{q}_{t+1})) \frac{1}{P_{t+1}} + \nu_{q,t} = 0 \quad \text{with} \quad \nu_{q,t} M_{q,t} = 0$$

where ν_q is the Lagrangian associated with the no short selling constraint of the type having endowment q.

An equilibrium with stationary consumption levels – $C(q) = C(q_t)$ for any t – exists featuring money trade. In this equilibrium the high endowment type $j = H$ optimally buys all money so that:

$$\frac{P_{t+1}}{P_t} = \beta \frac{u'\left(L + \frac{M}{P_{t+1}}\right)}{u'\left(H - \frac{M}{P_t}\right)}, \quad M_{H,t} = M, \quad \nu_H = 0$$

where $u'(C(H)) = \beta u'(C(L))$ and $P_t = P_{t+1}$, and the low endowment type $j = L$ has a binding short-selling constraint in that

$$\nu_L = \frac{1}{1-\beta^2} \frac{1}{P_{t+1}} \left(u'\left(L + \frac{M}{P_t}\right) \frac{P_{t+1}}{P_t} - \beta u'\left(H - \frac{M}{P_{t+1}}\right)\right) > 0,$$

unless the knife-edge case $\beta = 1$. The departure from optimality obtains as the high-endowment type internalizes the utility of the low-endowment type only intertemporally, whereas a social planner would care about it without any time discount. As the low-endowment type is constrained, the setting reproduces the logic of the OLG model presented in the text.

At least one other solution exists where money is worthless, which is the following:

$$P_t \to \infty, \quad \nu_H = 0, \nu_L = 0,$$

characterizing the allocation in autarky. In this case, there is a larger consumption inequality between households as there are no transfers in autarky. Thus, as in the OLG model, this infinite-horizon economy features inequality as a function of trade in money.