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JEL Classification: D83, D85, L82

Keywords: Social Networks, threshold models, homophily, Diffusion, information transmission, influencers

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Competing Diffusions in a Social Network*

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1 Introduction

The process by which new ideas and behaviors spread through a population has long been a fundamental topic of inquiry in the social sciences. One key question concerns how network structure favors the spread and survival of some products, ideas, behaviors, and technologies over others. In this paper, we address this question by studying the diffusion of two competing horizontally differentiated "goods" (e.g., products, ideas, technologies, languages, political beliefs) that propagate through a network of individuals.

We develop a dynamic network model of diffusion in which individuals choose which good to adopt in each period depending on how many of their connections in the network have adopted the same product in the previous period. Individuals are either a mass-market type (type M) or of a niche-market type (type N), whereby each type exhibits a preference for adopting their own type of good; hence, goods are horizontally differentiated. The distinguishing feature of the mass-market good is that there is a greater share of the population which is of type M. If we think of technology adoption, then an established technology (with a set of generally appealing attributes) may correspond to the mass-market, while a technology with a specialised set of attributes (that appeal to a minority of the population) may correspond to the niche-market. In terms of language, the majority and minority correspond to a mass- and niche-market, respectively.

In our model, agents are embedded in a network described by a degree distribution (i.e., the distribution of the number of friends in the population)[1] and a level of homophily α_M and α_N for each type (tendency for each type to be connected to/friends with a person who is the same type). We take a mean-field approximation to study the diffusion on the network where, in each period, an individual draws her friends at random from the population. The type of each friend is determined by the level of homophily of the individual and the relative fraction of each type in the population. Each individual observes the choices of her friends from the previous period and then chooses which good to adopt in the current period. As in many contagion models, agents use a threshold rule (denoted by $d(k)$), which, in our case, determines for each type of agent $i = M, N$ how many of their k friends need to have adopted their own-type good in the previous period for i to adopt that same good.

We start with a general threshold rule whereby the only restrictions we impose are that $d(k)$ is less than 50% of the total number of friends and is weakly increasing in k.[2] For example, in the case $k = 1$, an individual adopts the good that they observe from their single friend. We find that at least one of three types of equilibria exist: either an "extreme" niche-market or mass-market equilibrium, whereby only one good survives in equilibrium, or a mixed equilibrium, by which both competing goods prevail in steady state. A key quantity for determining the stability of extreme or mixed steady states is the number of friends of individuals who adopt their preferred product when they observe a single friend adopting it. Networks that are sparse with respect to this quantity (either because everyone has few friends or there are very few people who use a threshold of a single friend) exhibit these extreme steady states. On the other hand, denser networks (with respect to this quantity) exhibit stable mixed steady states.

We investigate some of the properties of our general threshold model through simu-

[1] For an overview of the network literature, see Vega-Redondo (2007), Newman (2010), Jackson (2008), and Jackson et al. (2017).

[2] This implies that agents are more inclined to adopt goods of their own type because they require less than 50% of their friends to be using the product.

lations of a regular network with no homophily across different values of the threshold $d(k) = d$ for all k. We identify three main properties. First, the competing diffusions tend to confer an advantage to the mass-market good whereby the steady state (when it is unique ($d = 1$) or is the unique mixed steady state ($d = 2, 3, 4$)) amplifies the mass-market good. Furthermore, when both extreme equilibria exist, the basin of attraction for the mass-market extreme is larger than the niche-market extreme steady state. Second, denser networks support mixed steady-state equilibria. Moreover, the basin of attraction for the mixed steady state when it emerges is large, suggesting a distinct phase transition in the behavior of our model. Third, higher thresholds (in which case goods are closer substitutes) favor the extreme steady states—that is, mixed steady-states are less likely to exist and, when they do, have smaller basins of attraction.

Our simulations suggest that our model has a uniquely stable steady state when $d = 1$. Thus, in the second part of the paper, we focus on the case whereby $d(k) = 1$ for all k and when the homophily parameter is the same for both types, such that $\alpha_M = \alpha_N = \alpha$. First, we establish that there is indeed a unique stable steady state, which is either an extreme type-M steady state or a mixed steady state. The sufficient conditions that determine which steady state prevails form a comparison of the expected number of friendships in the population with a threshold that is a function of the model parameters. When the network is relatively disconnected, then the mass-market good is the only good that is adopted/observed in steady state. Above this threshold, the niche good survives, but the mass-market good is always amplified by the social network—that is, the fraction of mass-market (niche) adoption is greater (less) than the fraction of mass-market (niche) individuals. In this case, we find that increasing the connectivity or homophily of the social network reduces the prevalence/amplification of the mass-market good.

In applications where the "good" is information or a norm of behavior, we are also interested in the differences in the observational patterns between the two types, not just their respective adoption decisions, since these observations may influence their views/opinions about alternatives that they themselves do not adopt. Our second set of results focuses on the systematic differences in the observational patterns for the two types of individual. We find that systematic differences arise through a combination of two forces: (i) systematic differences in what each type of individual *observes* and (ii) differences in the type of content that each type of individual *adopts* in steady state. Intuitively, increasing homophily acts through both channels and increases the differences in what people observe. However, less obviously, a more dense network (through an FOSD shift in the degree distribution) or a larger niche market also increases the differences in what people observe through the second channel—that is, the change to the steady state adoption behavior.

We extend our framework to study two strategic decision-making settings. First, we consider how two influencing agents (the "influencers") can promote the prevalence of each type of good. In our model, an influencer of one type directly spreads observations of their product among consumers to influence their adoption decision. The niche influencer is shown to be relatively more effective at changing the behavior of niche individuals than the mass-market influencer is at changing the behavior of mass-market individuals. Since the returns are higher for the niche influencer (and we assume that neither agent has a cost advantage), in equilibrium, the niche influencer invests more.[3] Second, we allow

[3]This result that the niche agents exert the largest effort to "replicate" themselves is reminiscent of some results in the literature on cultural transmission of values, beliefs, and norms (e.g., Bisin and Verdier, 2001; Bénabou and Tirole, 2006; Tabellini, 2008). For example, in Bisin and Verdier (2001), with

consumers to make a costly investment to increase the degree of homophily amongst their own connections because they value observing their own type of product. We demonstrate that, when the society is not sufficiently well connected, a steady-state equilibrium with zero homophily exists. However, if the cost of homophily effort is low enough, there also exists another equilibrium with strictly positive homophily. Moreover, if the society is sufficiently connected, then, in an equilibrium with positive homophily, the niche-type consumers exhibit greater homophily than mass-market-type consumers.

In the final part of the paper, we extend our model to three goods located on a line, "left," "middle," and "right," motivated by classical models of differentiated competition on a line in political economy (goods are political ideologies/policies) and industrial organization (goods are products). We assume that each agent always prefers her own type, but the middle good has a pairwise advantage over the other two more "extreme" goods (left and right). We find that this pairwise advantage is a potential source of amplification for the middle type; in particular, when there is an equal fraction of consumers of each type (or a greater fraction of the middle type than either extreme), the stable steady state amplifies the middle. On the other hand, when the extreme types are more prevalent than the middle, then, in sufficiently dense networks, the extremes become amplified relative to the middle. As above, steady states, whereby a type is not adopted, occur in sparse networks. When the fraction of people of the middle type is less than either of the extreme types, then the two sources of amplification (greater prevalence and pairwise advantage) counteract each other. When this is true in sparse networks, we find that the stable steady state is highly sensitive to changes in the underlying model parameters and can dramatically shift between steady states where either the middle type of goods floods the market or the extreme types flood the market for small changes in the underlying parameters.

2 Related literature

Our model contributes to different strands of the literature.

2.1 Threshold models of contagion

There is a large body of literature across economics, sociology, applied mathematics, computer science and epidemiology that has developed network models of contagion with thresholds. In this literature, two types of models have emerged: those in which the network is fixed and perfectly observed and those in which the network is random and unknown.[4]

2.1.1 Fixed networks

Granovetter (1978) and Schelling (1978) were the first to formulate basic mathematical models for the mechanisms by which ideas and behaviors diffuse through a population.

two types, parents can exert effort to influence the "replication" of their type in the society. However, in these models, there is usually no homophily and no explicit network, so that everyone meets everyone with the same probability, and the density of the network has not impact on the reproduction of a given trait.

[4] For an overview of both literatures, see Kleinberg (2007), Jackson (2008), Easley and Kleinberg (2010), Jackson and Yariv (2011), Jackson and Zenou (2015), Pastor-Satorras et al. (2015), and Wang et al. (2019).

Granovetter (1978) proposed the *linear threshold model*, in which an individual becomes active if and only if the current absolute number of active neighbors is equal to or exceeds the corresponding threshold. Watts (2002) developed a novel threshold model (later named the Watts threshold model) that showed that it is the fraction instead of the absolute number of active neighbors that matters.

In economics, the seminal paper is that of Morris (2000), which built on earlier work by Blume (1993), Ellison (1993), and Young (1998). The basic idea is as follows. Consider a model with two actions: 1 and 0. Each agent has a certain number of friends, and the benefits of taking action 1 (e.g., adopting a new technology) increase as an increasing number of these other individuals adopt it. In such a case, this agent takes action 1 once a sufficient fraction of her neighbors (threshold) has taken action 1. Morris (2000) found that "cohesive" groups act as a barrier to diffusion because everyone in a highly cohesive group has most of their friends within the group; thus, no one in the group will adopt until others in the group also adopt, making it hard for the technology to penetrate the group. In this context, Morris (2000) showed how the possibility of contagion depends on the network structure.[5]

There are more recent papers that have shown the importance of network structure in contagion. Reich (2020) showed how the size of complementarities moderates the role of group cohesion, while Jackson and Storms (2018) studied how coordination incentives divide the population into "behavioral communities." Similarly, Leister et al. (2022) found that networks can be partitioned into communities, in which agents with high "social connectedness" are more likely to adopt—that is, agents who have a high degree and who are connected to agents from other communities who also adopt.

2.1.2 Random networks

There is also a body of literature on diffusion and contagion in which the network is not known and thus random.[6,7] This approach uses a mean-field approximation; that is, it assumes that the role of interactions at any instant depends only on the fraction of agents who have adopted. These are in essence models of diffusion through random meetings, rather than diffusion through a fixed network (as in Section 2.1.1).

Let us explain how these network models of diffusion work using Jackson and Yariv (2005, 2007, 2011). In these models, agents play a coordination game with their neighbors, and the authors analyzed the dynamics using tools from game theory. As stated above, instead of a known network of interactions, in the random network literature, it is assumed that players are unsure about the network that will be in place in the future but have some idea of the number of interactions that they will have. In particular, the set of players is fixed, but the network is *unknown* when players choose their actions. A player knows her own degree k, when choosing an action, but does not yet know the realized network.

[5]See also Acemoglu et. al. (2011).

[6]See, for example, the work by Pastor-Satorras and Vespignani (2000), Newman (2002), and Alon et al. (2004).

[7]These models are related to other literature that is more common in epidemiology based on models in the form Susceptible-Infected, which includes Susceptible-Infected-Susceptible (SIS), Susceptible-Infected-Removed (SIR), and their many additional variants. These models help to quantify important aspects of disease transmission. They typically do not have an explicit network structure and are based on random matching derived from exogenously specified probabilities of being in particular states (for an overview, see Bailey, 1975; Anderson and May, 1991; Jackson, 2008).

Jackson and Yariv (2006, 2007) studied diffusion properties using the more tractable *mean-field* analysis. Mean-field models are examples of the more general class of models based on population-level arguments, which have proven to be tractable and yielded valuable insights. These models require the same treatment of players with the same degree and cost structure, regardless of their position in the network. They show that the strategies of players for adopting an action over the other can be represented by *threshold functions*, which depend on the degree k and on the cost of adopting. In particular, if the game is of strategic complements, then they show that agents take action 1 over action 0 if the fraction of neighbors among their k friends taking action 1 is above this threshold. Jackson and Yariv then showed how changes in *network structure*, captured by the degree distribution, affect this equilibrium (see also López-Pintado, 2006, 2008, 2012; Jackson and Rogers, 2007; Galeotti et al., 2010; Campbell, 2013; Sadler, 2020).[8]

In all these models (fixed and random networks), contrary to our approach, the authors have not examined how the network affects the competition between two actions and under which condition both actions survive in a stable steady-state equilibrium.[9] There is a recent strand of literature in computer science and epidemiology that studies these issues, which we review next.

2.1.3 Contagion and diffusion with competing actions

There is a body of non-economic literature that has studied the "survival" of competing actions or goods. The crucial concept is cross-immunity, namely the possibility that being infected by one pathogen confers partial or total immunity against others. Depending on the network topology, for some values of the parameters, it is possible to find a steady state in which the two processes coexist, each having a finite prevalence.[10]

Two important papers in this strand of literature are those of Wei et al. (2012, 2013) who examine the intertwined propagation of two competing "memes" (or viruses, rumors, products, etc.) across interconnected agents by extending the susceptible-infected-susceptible (SIS) model to construct a novel propagation scheme. One of the contributions of this paper is to introduce the notion of *composite network*, which is defined as a single set of nodes with two distinct types of edge interconnecting them (i.e., multiplex networks).[11] This network is random, and it is assumed that all nodes are passive and follow the same propagation model. They show under which condition on the topology of both networks either one meme or both memes survive in the stable steady-state equilibrium. This research has extended previous models that either studied a single epidemic on a single topology (e.g., Pastor-Satorras and Vespignani, 2002; Wang et al., 2003; Ganesh et al., 2005; Beutel et al., 2012; Prakash et al., 2012a) or studied two pathogens, but on the same topology and under the assumption that the two viruses appear one after the

[8]These kind of models have been applied to explain contagious failures that spread among financial institutions during a financial crisis (Allen and Gale, 2000; Elliott et al., 2014), breakdowns that spread through the nodes of a power grid or communication network during a widespread outage (Asavathiratham et al., 2001), and the course of an epidemic disease as it spreads through a human population (Anderson and May, 1991).

[9]In Jackson and Yariv (2005, 2007), there are some parameter values for which the two actions survive in steady state, but this was not the focus of their paper.

[10]For an overview, see Pastor-Satorras et al. (2015) and Wang et al. (2019).

[11]See Sahneh and Scoglio (2014) and Yang et al. (2017), who also extended the SIS epidemic model to a model with a competing pair of viruses over a two-layer network, where network layers represent the distinct transmission routes of the viruses.

other (Newman, 2005).[12]

We contribute to these different strands of the literature by introducing horizontal differentiated heterogeneity in the types of agents and goods, which impacts on the way information is transmitted and on the equilibrium characterization. In particular, horizontal differentiation results in agents' threshold rules being defined with respect to own-type, whereby differing systematically between individuals of different types. It also naturally leads us to consider a wide variety of questions that do not necessarily arise in this strand of literature. In particular, we address the following new issues: (*i*) the tendency for equilibria to systematically favor one or other product with respect to the relative proportion of types in the population, (*ii*) the differences in the conditions that lead to one or other extreme equilibrium emerging, (*iii*) the endogenous emergence of homophily through strategic decisions by individuals and the consequences of homophily in the dimension of differentiation, (*iv*) the impact of competing self-interested influencers that engage in influence activity to increase the prevalence of each type, (*v*) the notion of amplification (non-amplification) of a type of good in steady state relative to the share of those types in the population, and (*vi*) an extension to three-types to analyze the factors that drive political polarization.

2.2 Learning on networks

Our paper is also related to the literature on learning in networks (for an overview of this literature, see Jackson, 2008; Goyal, 2012; Möbius and Rosenblat, 2014; Golub and Sadler, 2016) and, more generally, to the literature on sequential learning, in particular the papers by Banerjee (1993), Ellison and Fudenberg (1995), Banerjee and Fudenberg (2004), and, more recently, Wolitzky (2018) and Tabasso (2019). In this literature, successive generations of agents use information from either the experiences or observations of the choices of earlier generations to guide their own decisions and form their own beliefs. By looking at the convergence of this dynamic process, different authors have shown the conditions under which there is a reinforcement of opinions over time, which may lead to polarization in steady state. For example, Golub and Jackson (2010), using the DeGroot model in which agents correct their heterogeneous initial opinions by averaging the opinions of their neighbors, showed that if the network is strongly connected, then all agents in the network will converge to the same norm such that there will be no polarization of beliefs and opinions in the long run.

Compared to this strand of literature, our model is quite different. First, each individual does not form an opinion by averaging the opinion of her neighbors, as is usually the case in the learning-in-networks literature. Here, the process of making a recommendation is based on homophily and on a (strong) preference for one's own type of content. Second, we study market rather than individual effects and thus can examine under which conditions niche-market goods emerge. Finally, because of the tractability of our model, we can also study the impact of influencers on steady-state equilibrium and how individuals choose their degree of homophily and extend the model to three types of goods or political ideologies.

[12] See also Buldyrev et al. (2010) and Prakash et al. (2012b) who studied multiple virus propagation on a simple fair-play single network and the effects of cascades in inter-dependent networks, respectively.

2.3 Strategic communication on networks

There is a strand of literature on strategic communication on networks that models not only the spread of conflicting news or beliefs on a network but also the underlying incentives of agents to acquire and communicate them (Hagenbah and Koessler, 2010; Ambrus et al., 2013; Galeotti et al., 2013; Calvó-Armengol et al., 2015; Bloch et al., 2018; Bénabou et al., 2020; Egorov and Sonin, 2020).

Compared to this literature, in our model, not only is communication not strategic but the modeling of the network is richer (for example, the network is a tree in Bloch et al. (2018) and Bénabou et al. (2020)). More importantly, because we consider a general degree disribution, we are able to examine the impact of the density of the network on the diffusion and survival of the different goods in the steady-state equilibrium. Furthermore, in our model, agents do not learn mechanically; they make decisions, albeit not strategic ones. Indeed, by prefering one action over the other, we can determine the threshold in terms of the number of friends taking an action under which agents choose the same action. Also, because the model is relatively simple, we can extend it to introduce strategic agents that promote their types, endogeneize homophily, consider three goods or ideologies, and study the condition under which "extreme" ideology ideas "contaminate" the population and survive in the long run. These issues have not been addressed by the strategic-communication-on-network literature. Thus, we view our model as complementary to this strand of literature.

3 General model

We follow the random-network literature (e.g., Jackson and Yariv, 2005, 2007, 2011) by assuming that players are unsure about the network they are facing but have some idea of the number of interactions that they will have. To fix ideas, think of choosing between a new and an old software program, either of which is only useful in interactions with other people who have also adopted the same software, but without being sure of whom one will interact with in the future.

3.1 Model

There is a unit mass of individuals who are either of a mass-market type (type M) or niche-market type (type N). Mass-market type individuals have a preference for mainstream goods[13] (e.g., an incumbent software program), while niche-market type individuals have a preference for more specialized goods (e.g., a new software program). A fraction $\rho > 1/2$ of consumers is of the mass-market type M, while $1 - \rho$ is of the niche-market type N. Goods are *horizontally differentiated* in the sense that neither good is clearly superior to the other (e.g., a new and an old software program, where the new program is not "better" than the old one, it is just different). The distinguishing characteristic of the mass-market good is that there is a greater share of the population that is of type M.

[13]We use the term "good," but, as in the non-economic literature, we could have used the term "meme," which the Merriam–Webster dictionary defines as "an idea, behavior, style, or usage that spreads from person to person within a culture." The term was coined by the evolutionary biologist Richard Dawkins in his 1976 book, *The Selfish Gene*. Dawkins argued that the meme is to cultural transmission what the gene is to biological transmission.

Time is discrete $t = 1, 2, \ldots$. In each period, an individual draws at random a number of friends k from a distribution $\{p_k\}$, and she observes their choice of good during the previous period.[14] An individual draws each friend uniformly at random from the measure of similar types with probability α_j, for $j = M, N$, and draws a friend uniformly at random from the population with probability $1 - \alpha_j$. In other words, we take a *mean field approach*,[15] such that each agent re-draws her neighborhood in every time period randomly from the population (subject to some propensity to be matched to neighbors of their own type). As stated above, this corresponds to an approximation of a situation in which individuals have a large group of friends K but only observe a subset $k << K$ of them at each period of time.[16] The parameter α_j measures the extent of *homophily* in friendships, since higher α_j means that individual $j = M, N$ is more likely to have a friend of the same type.[17] When $\alpha_j = 0$ for $j = M, N$, all friendships are drawn uniformly at random from the population (no homophily), and when $\alpha_j = 1$ for $j = M, N$, all friendships are between individuals of the same type.

Agents use a threshold adoption rule to decide which good to use in each period. In period t, we assume that an individual adopts her own type of good provided that at least $d(k)$ of her friends were using it during the previous period $t-1$; otherwise she adopts the other good. The threshold is given by a discrete function $d(k)$ in the number of friends k of an individual for which we assume $d(k)$ is non-decreasing, $d(k) \leq k/2$ for each $k > 1$ and $d(1) = 1$.[18] Importantly, the assumption $d(k) \leq k/2$ for each $k > 1$ embodies the notion of horizontal differentiation in our setting (i.e., agents are more inclined to adopt goods of their own type because they require less than 50% of their friends have been using the product in the previous period in order to choose it in the current period).

This process is similar to the threshold contagion models with binary actions developed both in economics and in other fields (see Section 2.1) whereby, in the case of strategic complements, there is a threshold $d(k)$ such that if more than $d(k)$ neighbors choose action 1, then the player prefers to choose action 1; otherwise, the player chooses action 0. Observe that, in our model, the goods are substitutes; that is, an agent chooses to adopt one or other in each period. For example, agents choose between a new and an old software program but do not use both concurrently. However, agents may switch between using one and the other over time. This is consistent with many models of cascades and diffusion in networks in that agents may switch multiple times (Blume, 1993; Ellison, 1993; Young, 2006; Montanari and Saberi, 2010; Adam et al., 2012), which means that choices are *not irreversible*.[19]

[14] We assume that everyone interacts with at least one person, such that $p_0 = 0$, and a positive fraction interacts with two or more people, such that $p_1 < 1$.

[15] A mean-field version of a model is a deterministic approximation of the statistical system where interactions take place at their expected rates. See Vega-Redondo (2007) or Jackson (2008) for some discussion of these techniques in network analysis.

[16] It should be clear that the results of the model would not be affected if, instead of having agents living forever and making adoption decisions at each period of time, we had assumed that each agent was active in terms of consumption only during one period (when young) and then inactive in the second period (when old) in which each agent recommends to her k friends (who are young) the good she has consumed in the previous period. See Section 3.4.4 for more details.

[17] One of the most observed behavior is the tendency for individuals to predominantly interact with people who are similar to themselves. This particular tendency –known as *homophily* or *assortativity*– has a rich intellectual history in sociology (see McPherson et al., 2001, for a review of the literature documenting this tendency).

[18] Since $k = 1, 2, \cdots$ takes discrete values, $d(k)$ is a discrete function.

[19] Examples of diffusion models where choices are irreversible include those by Watts (2002), Goyal et

Denote by $x_{M,t-1}$, the ex ante probability that a type-M individual *observes* a randomly chosen friend adopting the type-M good in the previous period $t-1$ and by m_t, the ex ante probability that a type-M individual *adopts* a good of type M in period t. Similarly, define the ex ante probabilities $x_{N,t}$ and n_t that a type-N individual observes a friend adopting a type-N in period $t-1$ and adopts goods of type N in period t, respectively. The probabilities $x_{M,t}, x_{N,t}$ may be written in terms of the probabilities of each type adopting their own-type of good m_t, n_t, as follows:

$$x_{M,t} = \alpha_M m_t + (1-\alpha_M)(\rho m_t + (1-\rho)(1-n_t)) \tag{1}$$
$$x_{N,t} = \alpha_N n_t + (1-\alpha_N)\left((1-\rho)n_t + \rho(1-m_t)\right), \tag{2}$$

where the probabilities m_t and n_t evolve according to[20]

$$m_t = 1 - \sum_k p_k \sum_{j=0}^{d(k)-1} \binom{k}{j} (1-x_{M,t-1})^{k-j} (x_{M,t-1})^j \tag{3}$$
$$n_t = 1 - \sum_k p_k \sum_{j=0}^{d(k)-1} \binom{k}{j} (1-x_{N,t-1})^{k-j} (x_{N,t-1})^j \tag{4}$$

To understand these equations, consider first $x_{M,t}$ (expression (1)). In period $t+1$, a type-M individual observes a friend of the same type with probability α_M who has adopted a type-M good in the period t with probability m_t. Otherwise, she observes a friend uniformly at random from the population (with probability $1-\alpha_M$), and then with probability ρ, the friend is of type M, while with probability $1-\rho$, she is of type N. Each of these types has adopted a type-M good in period t with probability m_t and n_t, respectively. One may similarly understand the relationship for a type N in (2).

Now, consider the evolution of the likelihood that a type-M consumer adopts good M in period t in equation (3). In period t, a type-M consumer with k friends adopts the type-N content only if she observes that *at most* $d(k)-1$ of her friends had adopted the type-N good in the previous period. With probability p_k, a consumer finds k friends, and the probabilities that each friend adopted type-M and type$-N$ good in the previous period are $x_{M,t-1}$ and $1-x_{M,t-1}$ respectively. The expectation of this quantity is given by:

$$\sum_k p_k \sum_{j=0}^{d(k)-1} \binom{k}{j} (1-x_{M,t-1})^{k-j} (x_{M,t-1})^j .$$

Thus, the expression for adopting a type-M good by a type-M individual is 1 minus this expectation. A similar explanation is true for n_t (expression (4)).

al. (2019) and Campbell et al. (2020).

[20]Both of these equations can be equivalently represented as sums from $d(k)$ to k, such that

$$m_t = \sum_k p_k \sum_{j=d(k)}^{k} \binom{k}{j} (1-x_{M,t-1})^{k-j} (x_{M,t-1})^j$$

and

$$n_t = \sum_k p_k \sum_{j=d(k)}^{k} \binom{k}{j} (1-x_{N,t-1})^{k-j} (x_{N,t-1})^j .$$

3.2 General characterization result

A steady-state equilibrium (n^*, m^*) satisfies $n_{t-1} = n_t = n^*$ and $m_{t-1} = m_t = m^*$. We say that there is a *mass-market equilibrium* when $x_M^* = m^* = 1$ and $x_N^* = n^* = 0$ and a *niche-market equilibrium* when $x_M^* = m^* = 0$ and $x_N^* = n^* = 1$. We say that there is *mixed-market equilibrium* when $\rho < x_M^* < m^* < 1$ and $0 < x_N^* < n^* < 1$.

Define the following positive values:

$$B_N = \frac{1 - [\alpha_N + (1-\alpha_N)(1-\rho)]\, p_1}{1 - (1-\alpha_M)(1-\rho) - [\alpha_M(1-\rho) + \alpha_N \rho]\, p_1} \tag{5}$$

and

$$B_M = \frac{1 - [\alpha_M + (1-\alpha_M)\rho]\, p_1}{1 - (1-\alpha_N)\rho - [\alpha_M(1-\rho) + \alpha_N \rho]\, p_1}. \tag{6}$$

Denote by $\hat{k} \geq 1$ the largest k, such that $d(k) = 1$, and by $\sum_{k=1}^{\widehat{k}} kp_k$ the truncated mean for all individuals for which $d(k) = 1$.[21] We have the following result:

Theorem 1.

- *If* $\sum_{k=1}^{\widehat{k}} kp_k < B_N$*, then the niche-market steady-state equilibrium is asymptotically stable. Otherwise, when* $\sum_{k=1}^{\widehat{k}} kp_k > B_N$*, it is unstable.*
- *If* $\sum_{k=1}^{\widehat{k}} kp_k < B_M$*, then the mass-market steady-state equilibrium is asymptotically stable. Otherwise, when* $\sum_{k=1}^{\widehat{k}} kp_k > B_M$*, it is unstable.*
- *If* $\sum_{k=1}^{\widehat{k}} kp_k > \max\{B_M, B_N\}$*, then there exists at least one mixed-market steady-state equilibrium that is asymptotically stable.*

The "extreme"good (either niche or mass) markets are always steady-state equilibria, since if every agent chooses an extreme action, say niche, then everybody will choose the same action. The key question we ask in Theorem 1 is if these steady-state equilibria are (asymptotically) stable; that is, if a small fraction of agents adopts the other good, do the dynamics of the system take it to the extreme good steady state? Consider the extreme niche-market equilibrium and let us show under which condition it is stable. In a niche-market equilibrium, independent of their type, all agents only observe others adopting the niche good (i.e., $x_M^* = 0$ and $x_N^* = 1$) and so adopt a niche good themselves (i.e., $m^* = 0$ and $n^* = 1$). Suppose a small fraction of individuals in the neighborhood of this steady state adopts the type-M good. These individuals are then potentially observed by others in the following period. The conditions for the stability of this steady state can be understood by whether, in expectation, the fraction of people adopting the alternative good is increasing or decreasing. In the neighborhood of the niche-market steady state, this is determined by the fraction of type-M individuals with a threshold of $d(k) = 1$ (and any niche individuals with a degree of 1), since all other individuals must observe two or more people adopting for them to adopt themselves.

In other words, we look at all agents with threshold $d(k) = 1$ and thus consider the mean number of friends among them, such that $\sum_{k=1}^{\widehat{k}} kp_k$. If this truncated mean is small

[21] When $\hat{k} = \infty$, $d(k) = 1$, for all k, and $\sum_{k=1}^{\infty} kp_k = \mathbb{E}[k]$, the expected or mean number of friends for each individual.

enough, the niche-market equilibrium is stable. The same reasoning applies for the mass-market equilibrium. When $\sum_{k=1}^{\widehat{k}} kp_k$ is large enough, only a mixed-market equilibrium in which both goods survive in equilibrium is possible. Thus, the presence (lack thereof) of individuals with threshold $d(k) = 1$ is important for the stability of the extreme steady states. In particular, smaller $\widehat{k}$ (more people who require at least two observations of their own type) makes it easier to satisfy the condition for the extreme equilibria to be stable.

More generally, Theorem 1 shows that if the truncated mean $\sum_{k=1}^{\widehat{k}} kp_k$ is smaller than B_M and B_N,[22] then each individual extreme steady-state equilibrium, namely the mass- or niche-market, is asymptotically stable. This provides a threshold on connectedness beneath which niche- and mass-market goods cannot survive separately in equilibrium. Interestingly, when $\sum_{k=1}^{\widehat{k}} kp_k > \max\{B_M, B_N\}$, then neither the niche-market nor the mass-market extreme steady-state equilibrium is stable. In this case, only the mixed-market equilibrium is stable. In other words, higher connectedness makes possible the existence of both communities of niche- and mass-market goods because consumers of both markets find a sufficient number of like-minded friends to whom they pass on their type's information and preserve their market share.

Let us provide more intuition on the results in Theorem 1. Assume that $p_1 = 0$, such that all agents have at least two friends, and no homophily ($\alpha_M = \alpha_N = 0$). Then, since $\rho > 1/2$, $B_N = 1/\rho < 2$ and $B_M = 1/(1-\rho) > 2$, where ρ is the fraction of mainstream individuals. Thus, when the network is such that $B_N < \sum_{k=1}^{\widehat{k}} kp_k < B_M$, only the mass-market goods will survive in steady state. Consequently, this mechanism will tend to *amplify* the majority action. When there is homophily, this is not necessary true, since $B_N = 1/[(1-\alpha_M)(1-\rho)]$ and $B_M = 1/[(1-\alpha_N)\rho]$, where the denominators are respectively the probability that a friend of a type-M and type-N individual is of the same type. Indeed, we see that α_M and α_N are negatively related to B_N and B_M, respectively, and that B_N increases with ρ while B_M decreases with ρ. When individuals are not too homophilous, or when the fraction of mainstream individuals is not too large (it has to be greater than 1/2), then we are more likely to have a niche-market or a mass-market equilibrium. In this case, the threshold condition may be written as

$$\sum_{k=1}^{\widehat{k}} kp_k(Pr[\text{friend same type}]) \lesseqgtr 1.$$

If $\epsilon \approx 0$ individuals of a given type are choosing their own type in period t, then, in period $t+1$, approximately $\epsilon \sum_{k=1}^{\widehat{k}} kp_k(Pr[\text{friend same type}])$ choose it in the following period. When this is bigger than ϵ, then, the dynamics move away from the extreme steady state, which cannot be stable.

3.3 Simulations for a regular network

In Theorem 1, we provide sufficient conditions for the existence and asymptotic stability of the extreme and mixed steady states. There are cases in which multiple steady states are asymptotically stable, and so comparative static exercises are not always possible

[22] We cannot rank B_M and B_N, since they depend on α_N and α_M, and we make no assumption on whether α_N is greater or smaller than α_M. However, when $\alpha_N = \alpha_M = \alpha$, then $B_N > B_M$, since $\rho > 1/2$.

in the general framework that we consider.[23] Indeed, as we show below, both extremes may co-exist and so even comparative static exercises on the "highest" and "lowest" steady states become problematic. Rather, in this section, we present what we feel are informative simulations of the behavior of our system for a regular network ($p_k = 1$ for some k). In particular, we are interested in understanding how features such as a uniform threshold rule $d(k) = d$ for all k and degree k of the regular network affect the types/number/characteristics of asymptotically stable steady states that emerge and, in the cases in which there are more than one, the relative sizes of the basins of attraction for each.

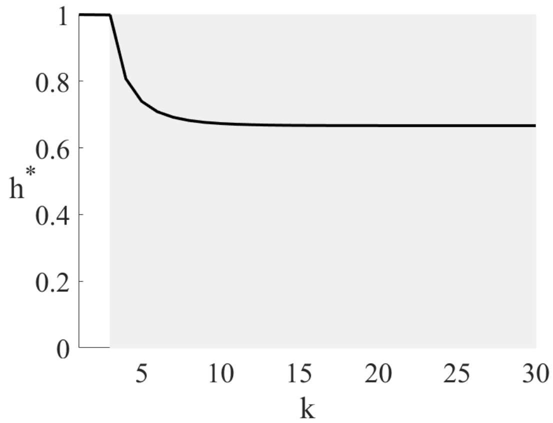
Our first set of simulations considers a regular network with zero homophily $\alpha_M = \alpha_N = 0$. When there is zero homophily, our system of two equations may reduce to a single equation in which the relevant state is the fraction h_t of the population adopting the mass-market good, and a steady state is denoted by h^*. In each case, we consider a grid of starting points for h_0 and find the steady state to which each starting point converges. The results are displayed in Figure 1. Each panel shows the set of steady states reached and, for each set of starting values h_0 that approach each state, the basin of attraction. Each panel also shows these outcomes for different degrees k holding fixed a given threshold rule $d(k) = d$ for all k. We consider different values for the threshold from $d = 1$ (panel (a)), $d = 2$ (panel (b)), $d = 3$ (panel (c)), to $d = 1$ (panel (d)). The case $d = 1$ has a unique stable steady state indicated by the solid line. In the remaining cases $d = 2, 3, 4$, both extreme steady states exist for all values of k, and the basins of attraction for each are separated by the dashed line shown in the figures. We also see that, beyond a threshold value of k, a mixed steady state also exists (as well as the two extremes), in which case the basin of attraction for the mixed steady state separates the basins of attraction for the two extremes.

It is possible for both extreme equilibria and a single mixed equilibrium to arise in this setting. When $d = 1$, a unique stable steady state arises for all values $h_0 > 0$ (panel (a)). In this case, the uniquely stable steady state is the mass-market steady state in low-degree networks; however, above a threshold degree, the unique stable steady state becomes mixed and decreases as the degree increases. The uniqueness and tractability of this case lends itself to further analysis, which we pursue in Section 4. The three other cases $d = 2, 3, 4$ are all qualitatively similar to one another. Both extreme steady states are always present, and then beyond a threshold degree a third mixed steady state emerges.

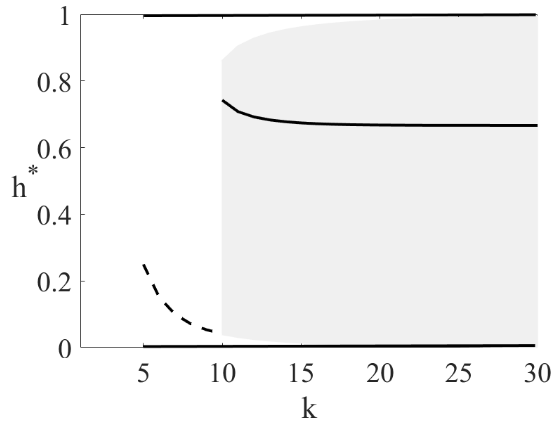
We emphasize three properties of our system from these simulations. First, in all cases, when the mixed steady state emerges, it has a large basin of attraction; that is, a non-trivial set of starting points $h_0 \in (0, 1)$ results in the mixed steady state. Moreover, in the cases where $d > 1$, the mixed steady state that emerges is strictly away from the extremes (that is, unlike the case where $d = 1$, where the mixed steady state emerges close to the mass-market extreme and gradually moves away as k increases). This suggests that for $d > 1$, the system exhibits a phase transition whereby around a critical density k^{crit}, the system distinctly changes behavior from always exhibiting extreme steady states to exhibiting a mixed steady state with a non-negligible basin of attraction.

Second, the process of diffusion confers an advantage to the mass-market good. This is evident in two features of the simulations. Indeed, across all simulations, the basin of attraction for the mass-market steady state is larger than the niche-market steady state,

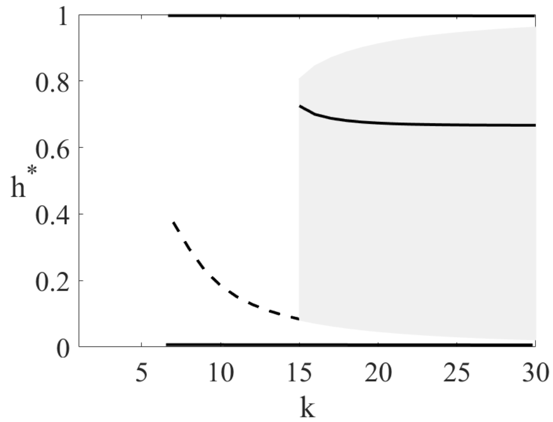
[23] In Section 4, we consider the case $d(k) = 1$ and show that there is essentially a unique steady state; we are also able to obtain comparative static results.



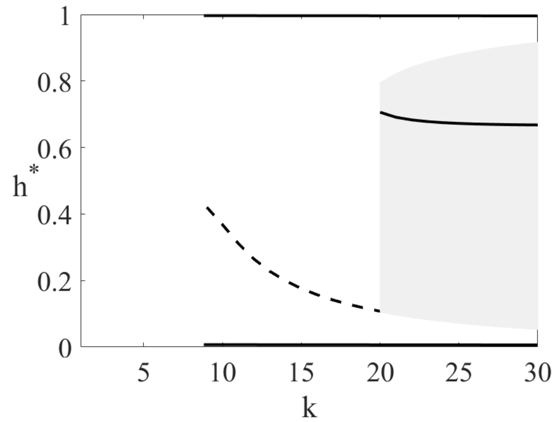
(a) $d(k) = 1$ for all $k > 2$



(b) $d(k) = 2$ for all $k > 4$



(c) $d(k) = 3$ for all $k > 6$



(d) $d(k) = 4$ for all $k > 8$

Figure 1: Steady states h^* (solid black lines) as a function of k ($p_k = 1$) for cases $d(k) = 1, 2, 3, 4$; dashed line separates basins of attraction to extreme steady states, and the gray area gives the basin of attraction to interior steady states

and, when the mixed steady state exists, it amplifies the mass-market good relative to the share of type-M people in the population, such that $h^* > \rho$. This characteristic is particularly pronounced in the case that $d = 1$, since all steady states exhibit the property that $h^* > \rho$, which we investigate further in Section 4.

Third, higher thresholds for adopting one's own type of good (larger d) confers advantages to the extreme steady states. Specifically, it makes the threshold density for the mixed steady state to exist greater and, when it does exist, the basin of attraction for the mixed steady state is smaller. One way to interpret a higher threshold is that it corresponds to a reduction in the amount of horizontal differentiation between the two goods.[24] This suggests that as the goods become closer substitutes, it becomes less likely that both can concurrently exist. The reason for the lack of co-existence is that this similarity tends to re-enforce the dynamics that lead to one or the other of the extremes.

3.4 Applications

We believe our model can fit a number of applications. Here, we provide some examples.

3.4.1 Adoption

There is a large body of literature on adoption and peer effects (for an overview, see Munshi, 2008; Chuang and Schechter, 2015; Breza, 2016). Usually, the decision is binary and is composed of whether or not to adopt a new technology or a new software (see, e.g., Leister et al., 2022). Our model could also be interpreted in the same way. Consider an old mainstream technology or software program (mass-market good) that most people are using and a new technology or software program (niche-market good), and agents have to decide which one to adopt. Because we assume that goods are differentiated, the new technology and the new software program are not "better" than the old ones; they are just different. Also, we assume that the new and old technologies are substitutes, so that they cannot be used together. For example, in many countries, farmers are encouraged to adopt a new environment-friendly technology, which, in terms of production, is not necessarily more efficient than the old one.

3.4.2 Language

Another application for our model is to think of "type" as a preference for a language spoken. Consider, for example, a minority and a majority language and assume that $d(k) = 1$, for all k, such that $\hat{k} = \infty$. Each agent is either born with a minority or a majority language and strongly prefers to speak her own language. Thus, each agent will choose her language in each period and stick to her preferred choice as long as at least one person in her neighborhood speaks it. If nobody speaks her preferred language, then, quite naturally, she switches to the other language in order to be able to communicate with her neighbors.

[24] For example, the threshold for a type-N individual to adopt a type-M good is $k - d$, and so the difference in the thresholds for each type to adopt the type-M good is given by $k-2d$, which is decreasing in d.

3.4.3 Political news content

Our model can be applied to news content by differentiating between mass-market and niche-market news. Indeed, with the rise of social media, different views are flourishing. Some people are promoting *niche* views that impact small groups within the population. Others are discussing more *mainstream* ideas that influence a vast majority of the population. A natural extension of our model to political ideologies is to consider a discrete version of competition on a line using three types of content/views L ("left"), M ("middle"), and R ("right"). In Section 5, we extend our model to include three political ideologies whereby the central "middle" type has an advantage relative to the two extreme types in pairwise comparisons. Our model can then be interpreted as a political news content model. We show the conditions under which the social mechanism tends to amplify the middle and those under which it will tend to amplify the extremes. In the latter case, we describe this as *polarization*, that is, when the left and the right ideology dominate the news market at the expense of mainstream ideology.

3.4.4 Irreversible choice

In some instances, switching may be quite costly after the initial adoption decision. In such cases, individuals tend to stay with the same product for an extended period of time. One may reinterpret our model for these types of goods. Instead of having agents living forever and making adoption decisions at each period of time, one may assume that each agent is active in terms of consumption only during one period (when young) and then is inactive in the second period (when old). In the first period (when young), each agent decides which good to consume based on the observations she makes about the decisions made in the previous period by her connections. In the second period (when old), following the information transmission described above, an agent may be observed by a new set of connections (who are young) in the social network. In each period of time, a new generation replaces the old generation with the same fraction ρ of mass-market individuals and niche individuals.[25] Thus, each period represents a different set of people/consumers who observe a set of people who recently made a choice. Mathematically, our model is equivalent, but the interpretation is different. Indeed, now agents only make adoption/consumption decision once (when young). Also, a consumer born in period t (young) has $k \geq 0$ friends amongst the (old) consumers who consumed a good in period $t-1$.[26] For example, in this interpretation of our model, agents will decide to choose an occupation/labor supply/education (e.g., becoming a doctor) if they have enough social connections with individuals from the previous generation who have previously chosen this occupation/labor supply/education (e.g., enough connections to doctors) and thus act as a role model for them.[27]

[25] Thus, we consider only intergenerational communication as information flows from one generation to the other. Following Campbell et al. (2020), we could relax this assumption and consider both inter- (old to young) and intragenerational (young to young) communication.

[26] We assume that friendships are formed uniformly at random between individuals in subsequent periods. This means that there is no correlation between the number of friends an individual observes (indegrees) and the number of people who observe that individual's choice (outdegrees). Following Campbell et al. (2020), we could relax this assumption by allowing for some correlation between indegrees and outdegrees.

[27] There is empirical evidence for this. For example, Olivetti et al. (2020) showed that women's labor supply in the U.S. is strongly influenced by the labor supply of their friends' mothers when they were teenagers.

4 Single threshold

To obtain a unique stable steady state, perform comparative statics exercises, and have a tractable setting for investigating further extensions, we set $d(k) = 1$ for all k (this implies $\hat{k} = \infty$). This means that each individual of type $j = M, N$ will adopt goods that are of the same type as her own provided that she observes at least *one* connection adopting this type of good in the previous period. Otherwise, in the event that she only observes the other type of good, she adopts the other type of good. From an economic perspective, this means that agents of a certain type have a *strong preference* for consuming and recommending a good of the same type. Thus, for them to adopt another good, they need all their friends to have adopted it in the previous periods. In the general model of Section 3, this was not necessarily true. We also assume that homophily is the same for all agents, such that $\alpha_N = \alpha_M = \alpha$.

4.1 Main results

4.1.1 Model

The dynamics of our system form a special case of the general model of Section 3 whereby $d(k) = 1$ for all k. The quantities $x_{M,t}, x_{N,t}, m_t, n_t$ are defined as before and can be written as follows:

$$\begin{aligned} x_{M,t} &= \alpha m_t + (1-\alpha)(\rho m_t + (1-\rho)(1-n_t)) & (7) \\ x_{N,t} &= \alpha n_t + (1-\alpha)\left((1-\rho)n_t + \rho(1-m_t)\right), & (8) \end{aligned}$$

where the probabilities m_t and n_t evolve according to:

$$\begin{aligned} m_t &= 1 - \sum_k p_k \left[1 - x_{M,t-1}\right]^k & (9) \\ n_t &= 1 - \sum_k p_k \left[1 - x_{N,t-1}\right]^k . & (10) \end{aligned}$$

Equations (7) and (8) are exactly the same as (1) and (2) when $\alpha_N = \alpha_M = \alpha$. Similarly, equations (9) and (10) are the same as (3) and (4) when $d(k) = 1$ for all k. Indeed, consider the evolution of the likelihood that a type-M consumer adopts good M in period t in (9). In period t, a type-M consumer adopts the type-N good only if *every* friend in the previous period adopts the type-N good. Thus, the expression for adopting the type-M good is 1 minus the probability that she only observes her friends adopting the type-N good in period $t-1$ (the summation term). A similar explanation is true for n_t (expression (10)).

4.1.2 Equilibrium

A steady-state equilibrium (n^*, m^*) satisfies $n_{t-1} = n_t = n^*$ and $m_{t-1} = m_t = m^*$.[28] Writing n^* and m^*, the implicit functions defined by the steady state conditions (by substituting the expressions for $x_{M,t}, x_{N,t}$ and dropping time indexes in the above equations)

[28] Observe that when n^* and m^* are determined, we can calculate the steady-state values $x_{N,t} = x_{N,t-1} = x_N^*$ and $x_{M,t} = x_{M,t-1} = x_M^*$.

are given by

$$n^* = 1 - \sum_k p_k \left[(1-\alpha)\left((1-\rho)(1-n^*) + \rho m^*\right) + \alpha(1-n^*)\right]^k \tag{11}$$

$$m^* = 1 - \sum_k p_k \left[(1-\alpha)(\rho(1-m^*) + (1-\rho)n^*) + \alpha(1-m^*)\right]^k \tag{12}$$

We assume that the starting point of the system is somewhere on the interior $0 < m_0, n_0 < 1$. Our analysis focuses on the steady state that occurs from any such point. As we will show, this point will be unique and identical, given $\{p_k\}, \alpha, \rho$, for any starting point on the interior. Associated with any steady state (n^*, m^*) are corresponding values for x_m^* and x_n^*.

Define the prevalence of the mass-market good by

$$h^* = (1-\rho)(1-n^*) + \rho m^*, \tag{13}$$

giving the steady-state probability that a randomly drawn individual in the population adopts the mass-market good. Then, $1 - h^* = \rho(1-m^*) + (1-\rho)n^*$ gives the probability that a randomly drawn individual adopts the niche-market good. The quantity h^* measures the relative prevalence of each type of good. In this way, higher (lower) h^* corresponds to the mass- (niche-) market, constituting a higher fraction of adoption.

Define the positive value

$$B := \left[\alpha + (1-\alpha)(1-\rho)\frac{1 - p_1\alpha}{1 - p_1(\alpha + (1-\alpha)\rho)}\right]^{-1}. \tag{14}$$

Let $\mathbb{E}[\cdot]$ denote the expectation operator over p_k. We can now characterize our model's stable steady states.

Proposition 1. *There is a unique stable steady-state equilibrium (n^*, m^*) for any starting point $0 < m_0, n_0 < 1$, which is characterized as follows:*

1. *If $\mathbb{E}[k] \leq B$, then $x_M^* = m^* = 1$, $x_N^* = n^* = 0$, and $h^* = 1$.*
2. *If $\mathbb{E}[k] > B$, then $\rho < x_M^* < m^* < 1$, $0 < x_N^* < n^* < 1$ and $\rho < h^* < 1$.*

In the general model (Section 3), we found that it was possible to have multiple stable steady-state equilibria (Theorem 1). In contrast, we find here that in the case $d(k) = 1$, there is a unique stable steady-state mass-market equilibrium (Proposition 1, part 1). The combination of these two results demonstrates that a necessary condition for multiple stable steady-state equilibria is $d(k) > 1$ for some k.[29]

The main insights from Proposition 1 are that the social transmission of adoption amplifies some goods at the expense of others, and this amplification confers a disproportionate advantage to mass-market goods. We see that in all cases, the steady-state pattern of adoption in the population overrepresents the mass-market goods relative to the fraction of the population of this type. For low $\mathbb{E}[k]$ below the threshold B, the mass-market good swamps that of the niche-market, and we end up with a steady-state equilibrium for which $x_M^* = m^* = 1$ and $x_N^* = n^* = 0$, so that only mass-market goods prevail in the market. If, instead, individuals observe a sufficiently large number of friends

[29]Observe that, in Proposition 1, we use a different concept of stability than in Theorem 1. That is, starting at any interior values of m_0 and n_0, we examine which equilibria are globally stable.

in expectation, so that $\mathbb{E}[k]$ is above B, then the niche-market good survives. However, the steady-state fraction adopting the mass-market good exceeds the fraction of these types in the population, such that $h^* > \rho$. For networks above a minimum level of density given by B, the niche good can survive in a "sub-community," but its share in the population will still be below the population share ρ of the majority type.

To gain some intuition for this amplification mechanism, assume that every agent takes her preferred action and has two neighbors at $t = 0$ and that there is no homophily ($\alpha = 0$). Then, the minority agents (population share $1 - \rho$) will have two majority neighbors with probability ρ^2, and the majority agent will have two minority neighbors with probability $(1 - \rho)^2$. Only these two populations will change their action in the next period. Hence, the share of the majority agents will change by $(1 - \rho)\,\rho^2 - \rho\,(1 - \rho)^2 > 0$. Consequently, this mechanism tends to amplify the majority action.[30]

It is interesting that p_1 plays such a prominent role in the formula for B. Indeed, consider the case where $p_1 = 1$ and hence the network degenerates to pairs of agents, since all agents have only one link. In this case, without homophily ($\alpha = 0$), we get $B = 1$ and hence a knife-edge case (since $\mathbb{E}[k] = 1$). This makes a lot of sense if we take as a starting point at $t = 0$ again the case where each agent takes her preferred action. In this case, minority agents with a majority partner will switch and vice versa, and the net change in majority share is: $(1 - \rho)\,\rho - \rho\,(1 - \rho) = 0$. Thus, the amplification mechanism fails exactly for pairs (whose share is governed by p_1) but works otherwise.

Finally we examine how the threshold B is related to the primitives of the model. First, B increases with the prominence of the mass market, as captured by ρ. Indeed, when ρ increases, the fraction of type-M consumers increases, and an equilibrium with $m^* = 1$ and $n^* = 0$ is more likely to emerge. An increase in homophily α reduces B; hence, homophily decreases the burden placed on the niche consumers to sufficiently socialize and perpetuate the niche good. Combining these insights, for a more niche good (corresponding to larger ρ) to survive, it requires that homophily be greater in the population.

4.1.3 Comparative statics

We can also study the comparative statics properties of the unique interior stable steady-state equilibrium. In what follows, we denote $\underline{k} := \min_k\{k : p_k > 0\}$ the maximum lower bound in terms of degree to the support of $\{p_k\}$.

Proposition 2. *Assume* $\mathbb{E}[k] > B$. *Then*

1. *an increase in the degree of homophily α increases both the probability a niche individual receives and recommends niche-market content, such that n^* and x_N^*, respectively, and decreases the prevalence of mass-market content h^*, with* $\lim_{\alpha\to 1} m^* = \lim_{\alpha\to 1} n^* = \lim_{\alpha\to 1} x_M^* = \lim_{\alpha\to 1} x_N^* = 1$, *and* $\lim_{\alpha\to 1} h^* = \rho$;

2. *a first-order stochastic dominance (FOSD)*[31] *change to the distribution of friendships p_k increases both the probability a niche individual receives and recommends*

[30] This is a special case in which $p_1 = 0$ and $\alpha = 0$, such that $B = 1/(1-\rho) > 2$, and hence Proposition 1, part 1, applies, since $\mathbb{E}[k] = 2 < B$.

[31] Consider the distributions $\{p_{k,1}\}$ and $\{p_{k,2}\}$. The concept of *first-order stochastic dominance* captures the idea that $\{p_{k,1}\}$ is obtained by shifting the mass from $\{p_{k,2}\}$ to place it on higher values. Thus, $\{p_{k,1}\}$ first-order stochastically dominates $\{p_{k,2}\}$ if $\sum_k f(k)p_{k,1} \geq \sum_k f(k)p_{k,2}$ for all nondecreasing functions $f(k)$. Thus, it requires a higher expectation of all nondecreasing functions.

niche-market content n^* and x_N^* and decreases the prevalence of mass-market content h^*, with $\lim_{\underline{k}\to\infty} x_M^* = \alpha + (1-\alpha)\rho$ and $\lim_{\underline{k}\to\infty} x_N^* = \alpha + (1-\alpha)(1-\rho)$;

3. *an increase in ρ increases m^*, h^*, and x_M^*, but decreases n^* and x_N^*.*

First, we find increasing homophily α and increasing the density of the network increases (reduces) the prevalence of a niche- (mass-) market good.[32] Interacting with similar individuals or more individuals helps propagate the niche-market good and in part offsets the disadvantage that the social transmission mechanism confers on it vis-à-vis the mass-market good. As homophily or density becomes extreme, α converges to 1 or $\underline{k} \to \infty$, each type of individual adopts their own type of good, and the relative share of goods is reflective of the share of types in the population.

In the case of type-M individuals, an increase in homophily α or the density of the network has an ambiguous effect on the probability m^* that type-M individuals adopt type-M goods. Indeed, on the one hand, type-M individuals observe the decisions of more individuals or more like-minded individuals, but, on the other hand, when ρ is not too large, the increase in the prevalence of the niche good (from the aforementioned effect above) reduces h^*. When α goes to 1 or $\underline{k} \to \infty$, the first effect dominates the other, and thus α has a positive impact on m^*, while when α is small or the network is relatively sparse, the effect is the opposite.

Finally, when ρ, the fraction of consumers who are of the mass-market type M, increases, then quite naturally m^* increases and n^* decreases. The net effect is an increase in the prevalence of the mass-market good h^*.

4.1.4 Differences in exposure between types

So far, we have focused on the steady-state adoption behavior of each type of individual within the population. In some applications, such as where the good composed of information/media content and where an individual's decision to adopt corresponds to a choice about what that person shares with their friends, then a dimension of interest is to understand the steady-state patterns of observation for individuals (i.e., what information/media content they get to see) and the systematic differences in the population. In this section, we consider a metric for the difference between what each type observes about others' choices; that is, the difference in the composition of goods observed by each type of individual. This metric is defined as the difference in the probability that each type observes the type-M good from a randomly chosen friend in steady state. It is given by[33]

$$P^*(m^*, n^*) = |x_M^* - (1 - x_N^*)\,|,$$

where

$$x_M^* = \alpha m^* + (1-\alpha)h^*, \tag{15}$$

$$x_N^* = \alpha n^* + (1-\alpha)(1-h^*). \tag{16}$$

[32] Using a very different model (the susceptible-infected-susceptible (SIS) framework), Tabasso (2019), who modeled the simultaneous diffusion of multiple pieces of information on a network, found a similar result: Higher connectivity benefits predominantly the "meme" that is preferred by a minority of the population.

[33] We could have a similar definition in terms of type N with $P^*(m^*, n^*) = |x_N^* - (1 - x_M^*)\,|$. This will lead to the same definition of sytematic differences given in (17).

In particular, x_M^* is the steady-state probability for an individual of type M of observing goods of type M, while $1 - x_N^*$ is the steady-state probability that a type-N individual observes goods of type M (equations (16) and (15)). By direct calculation, we find that[34]

$$P^*(m^*, n^*) = \alpha\,(m^* + n^* - 1)\,. \tag{17}$$

Observe that our measure of systematic differences $P^*(m^*, n^*)$ measures the *divergence* between the two types of agent. In this section, we study the comparative statics of $P^*(m^*, n^*)$ and so restrict the analysis to the cases where $P^*(m^*, n^*)$ is non-zero (i.e., $\alpha > 0$). The closed-form expression in Proposition 1 allows us to directly see how homophily and market composition determine whether or not the systematic consumption differences $P^*(m^*, n^*)$ can be sustained in equilibrium. As a result, we will also assume that $\mathbb{E}[k] > B$ so that $P^*(m^*, n^*)$ is non-zero.

Proposition 3. *Assume $\alpha > 0$ and $\mathbb{E}[k] > B$. Then, an increase in homophily (increase in α), an improvement in connectivity (through a FOSD change to the distribution of friendships p_k), or a larger niche market (smaller ρ) results in an increase in $P^*(m^*, n^*)$.*

The first comparative statics result of this proposition shows that as each individual observes more similar individuals (individuals are more homophilous, i.e., higher α), our measure of systematic differences in the population increases. This captures both the direct effect of homophily on $P^*(m^*, n^*)$ and the indirect effect through the changes to the steady-state adoption probabilities m^* and n^*. One can immediately observe from the relationship in equation (17) that the direct effect is positive. The indirect effect is somewhat less obvious, as homophily has a positive effect on n^* but an ambiguous effect on m^* (Proposition 2). Nonetheless, the net effect is positive because the positive effect on n^* offsets any negative effects of homophily on $P^*(m^*, n^*)$ through m^*. This result shows that greater homophily in the social network creates larger differences in the patterns in what the types observe.

The second part of the proposition considers the effect of improved connectivity of the network through an FOSD change to p_k. Our measure of consumption differences between types is a combination of the difference in who each type observes (as captured by homophily α) and differences in the adoption behavior of each type (as captured by $m^* + n^* - 1$). The effect of connectivity on $P^*(m^*, n^*)$ acts entirely through its effect on the adoption behavior (through m^* and n^*). In Proposition 1, we have seen that when the network is not very connected, such that $\mathbb{E}[k] \le B$, then every individual adopts the mass-market good; this drives $P^*(m^*, n^*)$ to zero. When the network becomes sufficiently connected, such that $\mathbb{E}[k] > B$, then each type exhibits different adoption behavior ($m^* \neq 1 - n^*$), and as the network becomes better connected (by having an FOSD change to p_k), these differences become greater, thereby increasing the differences between types. To summarize, conditional on the network being sufficiently connected, denser networks increase the systematic differences $P^*(m^*, n^*)$.

Finally, in Proposition 3, we show that a decrease in ρ, the fraction of type-M individuals in the society, leads to an increase in $P^*(m^*, n^*)$. In this case, the more balanced the population is (i.e., it is closer to a 50–50 mass-market–niche-market), the greater are the differences between types.

[34] In the proof of Proposition 1, we show that $m^* + n^* > 1$; thus, we do not require the absolute value notation in $P^*(m^*, n^*)$.

4.2 Influencers

In this section, we analyze how strategic actors ("influencers")[35] may influence the prevalence of each type of good. In a number of contexts, there may be parties that wish to promote some kinds of goods over others. For example, if we think of the application in terms of technology adoption (Section 3.4.1), the strategic actors will be the ones that push the adoption of their technology, such as a new environment-friendly technology. If we think of language (Section 3.4.2) and, say, we are in an English-speaking country (so the majority language is English), then some strategic actors, such as the French Institute, will try to promote the French language (minority language). If we consider the application in terms of political parties (Section 3.4.3), then the strategic actors will be likely to be "partisan media" or "party propaganda."

We address three questions: (*i*) Does the social network mechanism lead one or other influencer (niche-market versus mass-market) to invest more in influencing activities? (*ii*) Will banning influencing activities lead to more niche-market or mass-market goods being adopted? (*iii*) Does influencing activity affect $P^*(m^*, n^*)$, the degree of difference in what each type observes?

There are two strategic players (influencers), namely a mass-market player and a niche-market player, whereby each makes up-front investment, e_M and e_N, respectively, at cost $C(\cdot)$, where $C'(0) = 0$, $C'(1-\rho) > 1$, and $C''(\cdot) > \overline{C}$, where $\overline{C}$ is a positive constant that guarantees a certain level of convexity of the cost function. The aim of these strategic actors is to increase the prevalence of their own respective good. This means that the investment e_i, $i = M, N$, from a strategic player of type i can change the adoption of an individual of the same type only in the situation which she has exclusively observed adoption of the other type. In other words, if a niche-market influencer invests effort e_N, then when a niche individual has only observed her friends adopting the mass-market good in the previous period, she will be influenced by the investment effort and adopt a niche good with probability e_N (and the mass-market good with probability $1 - e_N$). The same reasoning applies to a mass-market influencer who exerts effort e_M. Clearly, when $e_N = e_M = 0$, we are back to the benchmark model.

Each influencer seeks to maximize the fraction of the population adopting her good in the steady state subject to the costs of investment. This implicitly assumes that these two strategic players are perfectly patient in the sense that they place zero weight on the behavior on the path to the steady state. This assumption preserves a good deal of tractability for the analysis while, we believe, still capturing the elements of the trade-offs facing sufficiently patient players.

Under these assumptions, the benefits for each player may be captured by h^* and $1 - h^*$ for the mass-market and niche-market player, respectively, where $h^* = \rho m^* + (1-\rho)(1-n^*)$. We assume that the costly investments are to spread additional observations of mass- or niche-market goods to their respective populations during each period. The strategic players choose e_M and e_N, respectively, where e_i represents the fraction of the population that the player reaches. The dynamic equations that describe the evolution of n_t and m_t are now given by

[35] We use the word "influencer" to denote a strategic actor of a given type that tries to influence consumers of the same type to adopt the same type of good. Even though the term may be wrongly interpreted because of its use in social media, it has to be clear that here it means "persuader," namely someone of a given type who is making a costly effort to "influence" the consumption decision of individuals of the same type.

$$n_t = 1 - (1 - e_N) f(z_{N,t-1}) \tag{18}$$
$$m_t = 1 - (1 - e_M) f(z_{M,t-1}), \tag{19}$$

where $f(x) = \sum_k p_k x^k$, $z_{N,t-1} := 1 - x_{N,t-1} = \alpha(1 - n_{t-1}) + (1-\alpha)h_{t-1}$, and $z_{M,t-1} := 1 - x_{M,t-1} = \alpha(1 - m_{t-1}) + (1-\alpha)(1 - h_{t-1})$. When $e_N = e_M = 0$, we are back to equations (10) and (9). In steady state, these equations become

$$n^* = 1 - (1 - e_N) f(z_N^*) \tag{20}$$
$$m^* = 1 - (1 - e_M) f(z_M^*). \tag{21}$$

We see that the investments e_M and e_N only appear in their own respective equations for m^* and n^*. Indeed, the direct effect of an investment e_i only changes a consumer's behavior by reaching an individual of the same type who has exclusively observed adoption of the other type.[36] Clearly, if an individual of type $i = M, N$ has observed at least one friend adopting a good of the same type, then the strategic actor of the other type has no impact on this individual. The cost function is in terms of the total mass of players that each investment affects. We assume that the cost function is the same for the two influencers; that is, it is equal to $C(\cdot)$ for both players. This implies that neither player has a cost advantage in reaching a mass of players of their own type. The mass-market influencer solves the following program:

$$\max_{e_M} \{h^*(e_M, e_N) - C(\rho e_M)\}, \tag{22}$$

while, for the niche influencer, we have

$$\max_{e_N} \{1 - h^*(e_M, e_N) - C((1-\rho)e_n)\}. \tag{23}$$

The first-order condition for each player is[37]

$$C'(\rho e_M^*) = \frac{f(x_M^*)}{\Delta}[1 - \alpha(1 - e_N)f'(x_N^*)] \tag{24}$$
$$C'((1-\rho)e_N^*) = \frac{f(x_N^*)}{\Delta}[1 - \alpha(1 - e_M)f'(x_M^*)], \tag{25}$$

where

$$\Delta = \begin{pmatrix} \alpha(1 - (1 - e_N^*)f'(x_N^*))(1 - (1 - e_M^*)f'(x_M^*)) \\ -(1-\alpha)(1 - ((1-\rho)(1 - e_N^*)f'(x_N^*) + \rho(1 - e_M^*)f'(x_M^*))) \end{pmatrix} > 0.$$

Our first result establishes that there is a unique Nash equilibrium in effort choices by the two players and that when there is no homophily, the niche influencer invests more in influence activities, but the mass-market good is nonetheless more prevalent.

Proposition 4. *There is a unique interior Nash equlibrium in investments $(e_M^*, e_N^*) \in (0,1)^2$. Moreover, when $\alpha = 0$, the niche influencer puts in more effort than the mass-market influencer, such that $(1-\rho)e_N^* > \rho e_M^*$, and the fraction of the population forwarding mass-market goods is greater than a half, namely $h^* > \frac{1}{2}$.*

[36] Note that an investment affects both steady state values m^* and n^* through a combination of the direct and indirect effects of the investments.

[37] We provide a derivation of these conditions in Appendix B.

The social network mechanism confers a benefit to mass-market content, and so it is more likely that a niche-market individual only observes adoption of the mass-market good than a mass-market individual only observing adoption of the niche good. Hence, niche investments are relatively more likely to change the behavior of niche individuals than mass-market investments are to change the behavior of mass-market individuals. The returns are higher for the niche influencer. Since, by assumption, there are no cost advantages to either influencer, in equilibrium, the niche influencer invests more.

Observe that since $\rho > 1 - \rho$, the result in Proposition 4 not only implies that the niche influencer invests more than the mass-market influencer (i.e., $e_N^* > e_M^*$) but also that the niche influencer affects a larger percentage of the total population, such that $(1-\rho)\, e_N^* > \rho e_M^*$, which requires the niche influencer to spend *much* more effort than the mass-market influencer.

In the following proposition, we consider the impact of banning influence activities.

Proposition 5. *When $\alpha = 0$, the fraction of individuals adopting the niche good is higher when there are influencers, namely $h\,(0,0) > h\,(e_M^*, e_N^*)$.*

Proposition 5 shows that in the absence of homophily, banning influencing activities increases (decreases) the steady state prevalence of a mass (niche) market good. Indeed, the niche-market influencer invests more in equilibrium, and so, when she cannot persuade her consumers, the shift in the steady state is towards more mass-market goods. This unambiguously benefits the mass-market player because it results in a better steady state and reduces its investment costs. On the other hand, the impact on the niche-market player is ambiguous, as it may also benefit from the change: Although the steady state worsens under the ban, it does not incur the costs of influencing activities.

Under our measure of systematic consumption differences $P^*(m^*, n^*)$ (see (17)), a necessary condition for non-zero differences in what each type observes is a positive level of homophily. In Proposition C1 in Appendix C.1, we consider this as an extension to the impact of the influencers on $P^*(m^*, n^*)$. We show that $P^*(m^*, n^*)$ approaches its upper bound α as influencing investments increase to their maximum values. Hence, it also establishes that $P^*(m^*, n^*)$ increases as the costs of investment vary from prohibitively expensive where it is banned to inexpensive where the investments approach their upper bound $e_M^* = e_N^* = 1$. The second part of the proposition establishes that as the social network becomes highly connected, it concurrently crowds out influencing activities and increases $P^*(m^*, n^*)$.

We investigate these issues further in an example illustrated in Figure 2. The figure shows the equilibrium investments by each strategic player and the level of $P^*(m^*, n^*)$ at the equilibrium and in the case without influencers where $e_M = e_N = 0$. The network is regular where $p_k = 1$ for $k \geq 2$; costs are parametrized by a cost function $C(x) = 2x^2$, homophily is positive $\alpha = 0.1$, and the mass/niche market parameter is $\rho = 2/3$. First, we observe that the niche-market influencer invests more in equilibrium (as per our result without homophily in Proposition 4) and that the equilibrium investments by both players are decreasing in the connectivity of the social network. Second, our measure of systematic differences $P^*(m^*, n^*)$ is greater under influencers than without them $(P(e_M^*, e_N^*) > P(0,0))$ and is increasing in the connectivity of the social network. Finally, the impact of the influencers on $P^*(m^*, n^*)$ $(P(e_M^*, e_N^*) - P(0,0))$ is decreasing in the connectivity of the social network. Moreover, it is greatest in the case where the mass-market steady state exists in the absence of influencers (where $k = 2$). As k decreases,

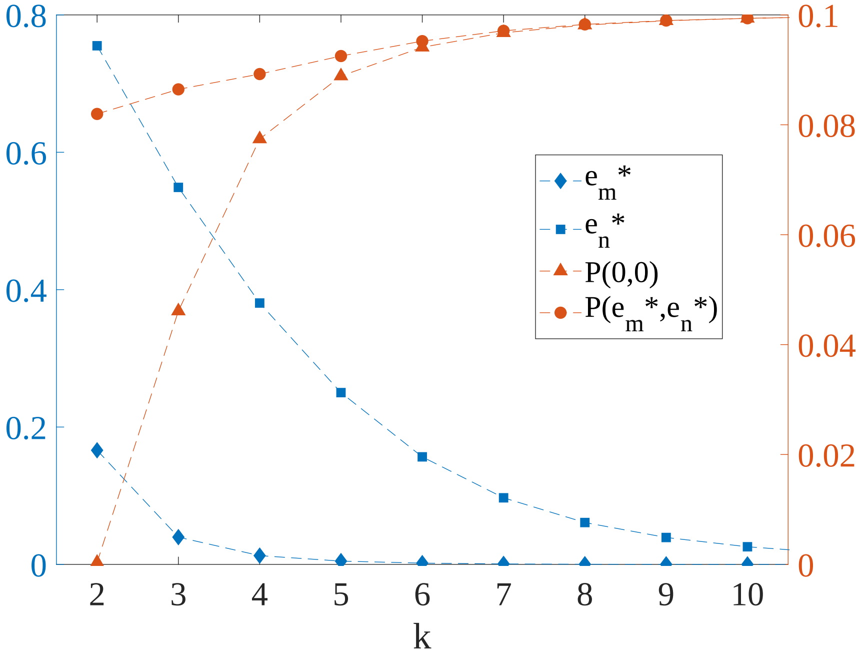


Figure 2: Influencers and systematic consumption differences

the niche market player vigorously invests in influencing activities, as the steady state is otherwise tending towards the mass-market steady state.

We see that influencing investments provides an influential channel for determining systematic differences in what different types observe, $P^*(m^*, n^*)$, in addition to the connectivity and homophily channels established in the baseline model. Moreover, it is precisely in the settings where connectivity is low (and so $P^*(m^*, n^*)$ would otherwise be small) that influencing activities are the most vigorously pursued and the impact is the greatest. In our example, the endogenous response of the influencers largely offsets the change in $P^*(m^*, n^*)$ that would otherwise have occurred at lower levels of connectivity of the social network.

4.3 Homophily

In many environments, individuals have a great deal of discretion over who with whom they interact/observe and share information. One factor that may affect the choice of who to be connected with is the likely adoption decision of the other individual. In particular, a common observation in many environments is that people tend to interact with like-minded others (McPherson et al., 2001; Currarini et al., 2009). In this section, we consider how the endogenous choice of homophily by individuals and the social network mechanism affect the goods that are adopted.

We allow agents to make a costly investment to increase the degree of homophily amongst their connections. We will assume it is costly to increase homophily. For an individual $i = M, N$, the costs are given by $kD(\alpha_i)$, where k is the number of friends. We assume that $D'(\alpha_i) > 0$, $D'(0) = 0$ and $D''(\alpha_i) > F$, where F is a positive constant that guarantees enough convexity for the cost function. For tractability, we simplify the network by only considering *regular* networks in which everyone has the same number of friends $k \geq 2$.

We assume that agents maximize the expected number of friends that adopt the good of their own type subject to the costs of homophily. For a mass-market individual, we have

$$\max_{\alpha_M} \left\{k\left[\alpha_M m^* + (1-\alpha_M)h^* - D(\alpha_M)\right]\right\}, \tag{26}$$

while for a niche individual,

$$\max_{\alpha_N} \left\{k\left[\alpha_N n^* + (1-\alpha_N)(1-h^*) - D(\alpha_N)\right]\right\}. \tag{27}$$

We note that the steady state quantities m^*, n^*, h^* are all a function of the equilibrium levels of homophily chosen, but each individual takes these quantities, as given as she has no influence over them. Our steady-state equilibrium is determined by the solution to

$$\begin{aligned} n^* &= 1-\left[\alpha_N(1-n^*)+(1-\alpha_N)h^*\right]^k & (28)\\ m^* &= 1-\left[\alpha_M(1-m^*)+(1-\alpha_M)(1-h^*)\right]^k, & (29)\end{aligned}$$

where $h^* = \rho m^* + (1-\rho)(1-n^*)$. In Proposition C2 in Appendix C.2, we prove that, for a given pair of homophily levels α_M and α_N, there is a unique stable steady-state equilibrium (m^*, n^*).

We denote the steady-state equilibrium as a function of the actions of niche and mass-market individuals by $m^*(\alpha_M, \alpha_N), n^*(\alpha_M, \alpha_N), h^*(\alpha_M, \alpha_N)$. An equilibrium is a pair (α_M^*, α_N^*) that satisfies

$$\alpha_M^* = \arg\max_{\alpha_M} \left\{k\left[\alpha_M m^*(\alpha_M^*, \alpha_n^*) + (1-\alpha_M)h^*(\alpha_M^*, \alpha_N^*) - D(\alpha_M)\right]\right\} \tag{30}$$

and

$$\alpha_N^* = \arg\max_{\alpha_N} \left\{k\left[\alpha_N n^*(\alpha_M^*, \alpha_N^*) + (1-\alpha_N)(1-h^*(\alpha_M^*, \alpha_N^*)) - D(\alpha_N)\right]\right\}. \tag{31}$$

The first order conditions are given by

$$\begin{aligned} D'(\alpha_M^*) &= (1-\rho)\left[m^*(\alpha_M^*, \alpha_N^*) + n^*(\alpha_M^*, \alpha_N^*) - 1)\right] & (32)\\ D'(\alpha_N^*) &= \rho\left[m^*(\alpha_M^*, \alpha_n^*) + n^*(\alpha_M^*, \alpha_n^*) - 1\right]. & (33)\end{aligned}$$

Our first result (see Proposition C3 in Appendix C.2) shows that when the underlying network is not well connected or the niche is particularly small, then there may be no homophily in the choices of individuals.

The zero homophily equilibrium corresponds to a situation in which the mass-market good is the only good adopted in the market. When this is the case, there are no returns for an individual of either type from increasing their level of homophily because everyone is sharing the same type of good. In this case, our measure of the difference in the composition of goods seen by each type of individual, $P^*(m^*, n^*)$, is minimal and equal to zero. The condition $k \leq \frac{1}{1-\rho}$ demonstrates that it is a lack of connectivity in the underlying network that allows this equilibrium to exist. The empirical prediction of the model is that a lack of a homophily is associated with a lack of connectivity (or number of friends) and small niche (high ρ). It is important to note that these results are driven by the equilibrium prevalence of each type of good and not the difficulty of finding a particular type of individual.

The following proposition characterizes the equilibrium level of homophily when the underlying network is sufficiently well connected.

Proposition 6. *Suppose $k > \frac{1}{1-\rho}$. Then, there exists an equilibrium with positive levels of homophily for both types of individual. Moreover, the niche-type players exhibit greater homophily than the mass-type agents, such that $\alpha_N^* > \alpha_M^*$.*

In sufficiently well-connected networks, the mass-market good will not flood the market, and so there is always some amount of benefit to connect with individuals who are similar to oneself. Moreover, the niche individuals benefit the most from this because it is relatively more difficult for these individuals to find people who adopt the type-N good. Hence, in equilibrium, the niche individuals exhibit a greater degree of homophily than mass-market individuals.

5 Application to political economy

In Section 3.4.3, we proposed an interpretation of the goods in our model in terms of political news content as a decision over what type of content to forward on to one's friends. Here, we adopt this interpretation and extend our model to a three-type spectrum of political ideologies by considering types L ("left"), M ("middle"), and R ("right"). The key characteristic in this environment is that the middle type will have an advantage in pairwise comparisons relative to the extreme types left and right. In this section, we are particularly interested in understanding the amplification of the extremes relative to the middle, or vice versa.

5.1 A model with three political ideologies

Time is discrete $t = 1, 2, ...$ The population of mass 1 consists of a measure ρ of type M and $(1-\rho)/2$ of types L and R each for $0 < \rho < 1$. Each individual has k friends amongst individuals in the previous period, where k is drawn from a distribution p_k (where $p_k > 0$ for some $k \geq 2$), and to maintain tractability, we assume zero homophily (i.e., $\alpha = 0$). In each period, an individual receives a recommendation of content from each of her friends from the previous period. She views all of these items and then recommends/forwards the content that is the closest match to her type on to her friends in the next period. An individual ranks products in the following manner depending on her own type:

Type	Ranking
L	$L \succ M \succ R$
M	$M \succ L \sim R$
R	$R \succ M \succ L$

Each agent prefers her own type of news to any other type. Further, a type-L or a type-R individual prefers the middle news content to news from the other extreme. On the other hand, a type-M individual is indifferent to the news of types L and R. As in the model in the previous section, any agent will recommend her own type of news if she receives at least one recommendation of her type of news from her k friends. Otherwise, amongst the content that she receives, she will follow her ranking (described above) in terms of preferences of type of news content. Finally, in the event that a middle type only receives type-L or a type-R content, and so is indifferent, then she will choose each with 50–50 probability. Once again, we study the stable steady state consumption bundles of each type of individual and the resulting pattern of recommendations as $t \to \infty$.

5.2 Steady-state equilibrium

As above, we assume a mean-field approximation; that is, each type in period t draws k friends uniformly at random from amongst the consumers in the previous period $t-1$. Denote by $y_{M,t}$ the probability that a friend drawn uniformly at random from period t recommends product M. Observe that $y_{M,t}$ is different to $x_{M,t}$, as defined in the previous sections (see (1) or (7)), because there is no homophily, and thus, all types are equally likely to *receive* each type of news. Their type only matters when deciding which product to *recommend* to their friends. Then, utilizing the symmetry of the problem, we can write the probability of a friend recommending each of the other products by $\frac{1-y_{M,t}}{2}$. The probability $y_{M,t}$ evolves according to

$$\begin{aligned} y_{M,t} &= \sum_k p_k \left(\rho \left[1-(1-y_{M,t-1})^k \right] + (1-\rho) \left[1 - \left(\frac{1-y_{M,t-1}}{2} \right)^k - \left[1 - \left(\frac{1+y_{M,t-1}}{2} \right)^k \right] \right] \right) \\ &= \sum_k p_k \left(\rho \left[1-(1-y_{M,t-1})^k \right] + (1-\rho) \left[\left(\frac{1+y_{M,t-1}}{2} \right)^k - \left(\frac{1-y_{M,t-1}}{2} \right)^k \right] \right), \end{aligned}$$

where $\rho\left(1-(1-y_{M,t-1})^k\right)$ is the probability that an individual is of type M and receives at least one recommendation about her most preferred content (i.e., content of type M) from k individuals drawn uniformly at random from the population at $t-1$. The term $(1-\rho)\left[\left(\frac{1+y_{M,t-1}}{2}\right)^k - \left(\frac{1-y_{M,t-1}}{2}\right)^k\right]$ is the probability that an individual is of type L or type R and recommends the type-M content. In this last case, this can be written as the probability that the individual (type L or type R) neither hears (*i*) about their preferred content nor (*ii*) exclusively about their least preferred product (product R for a type-L individual and product L for a type-R individual) from k individuals drawn uniformly at random from the population at $t-1$.

Our object of interest is the steady state of our dynamics determined by $y_M^* = \lim_{t\to\infty} y_{M,t}$. Define

$$C_{LR} = \rho\mathbb{E}[k] + (1-\rho)\sum_{k=1}^{\infty} p_k k \left(\frac{1}{2}\right)^{k-1} \tag{34}$$

$$C_M = \frac{1-\rho}{2}\mathbb{E}[k] + \frac{1+\rho}{2}p_1. \tag{35}$$

In what follows, we denote $\underline{k} := \min_k\{k : p_k > 0\}$ the maximum lower bound in terms of degree to the support of $\{p_k\}$.

Proposition 7. *There is a unique stable steady-state equilibrium y_M^*, which is characterized as follows:*

1. *If $C_{LR} \leq 1$, then $y_M^* = 0$; that is, there is a unique stable steady state for which only the extreme news content L and R exist.*

2. *If $C_M \leq 1$, then $y_M^* = 1$; that is, there is a unique stable steady state for which only the middle news content M exists.*

3. *Otherwise, when $C_{LR} > 1$ and $C_M > 1$, then $0 < y_M^* < 1$; that is, all three news contents co-exist at the unique stable steady-state equilibrium. Moreover,*

(a) *if* $\rho \geq 1/3$*, then* $y_M^* > \rho$*;*

(b) *if* $p_k > 0$ *for some* $k > 2$*, then there exist positive numbers* $\bar{\rho}$ *and* $\hat{k}$ *such that if the extremes are sufficiently prevalent* $\rho < \bar{\rho}$ *or the network is sufficiently dense* $\underline{k} > \hat{k}$*, then the steady state amplifies the extremes* $y_M^* < \rho$*.*

This proposition shows that either the middle content or the extreme content may be amplified in the unique steady-state equilibrium. In particular, it is possible for either the extremes or the middle to dominate the market or for a mixed steady state to emerge. A sufficient condition for a mixed steady state is a well-connected network $\mathbb{E}[k] > \max\{\frac{2}{1-\rho}, \frac{1}{\rho}\}$; hence, the steady states in which the middle or the extremes dominate may only emerge in less well connected networks.

In the previous section, the prevalence of the mass-market type was an important force for amplifying that type of good/content. In our three-type model, there is a new force at work: The middle product M holds an advantage over the extreme products L, R in pairwise comparisons. We can see the effect of this new force by considering the case in which there are equal fractions of each type in the population ($\rho = \frac{1}{3}$), thereby shutting down the prevalence force. In this case, the advantage in pairwise comparisons results in the middle content being amplified. In a sufficiently sparse network where $\mathbb{E}[k] < 3 - 2p_1$, then this advantage allows the middle to dominate the extremes, such that $y_M^* = 1$. In fact, for $\rho \geq \frac{1}{3}$, the middle content is amplified because the middle is both more prevalent in the population and has the advantage in pairwise comparisons.

In populations in which the middle type is less frequent than the extreme $\rho < \frac{1}{3}$, the two forces are countervailing. Indeed, extreme types are more prevalent in the population, but the middle content has an advantage in pairwise comparisons. In this case, either the extremes or the middle content may be amplified depending on the strength of each. We find that in sufficiently well-connected networks, or in networks with a sufficiently large (resp. small) fraction of extreme (resp. middle) types, the extremes are amplified relative to the middle.

Finally, the two threshold values C_{LR} and C_M reveal that the stable steady state is highly sensitive to the connectivity of the network when ρ is small and when the network is not well-connected, $\mathbb{E}[k] \approx 2$. To illustrate this sensitivity, define two threshold values $\rho_{CL} = \max\{\rho | C_{LR} \leq 1\}$ and $\rho_M = \min\{\rho | C_M \leq 1\}$ and consider networks in which almost everyone has two friends, such that $p_2 = 1 - \epsilon$ and $p_3 = \epsilon$, where $\epsilon \approx 0$. In this case, the threshold values are given by $\rho_{LR} = \frac{\epsilon}{4+5\epsilon} < \rho_M = \frac{\epsilon}{2}$, and the difference (size of the range where the stable steady state is between the extremes) is $\epsilon\left(\frac{1}{2} - \frac{1}{4+\epsilon}\right)$. We can readily observe that as the network approaches one, where $p_2 = 1$ ($\epsilon \to 0$), then both thresholds approach 0. This suggests that for networks in which ρ is small and the density of the network is such that $\mathbb{E}[k] \approx 2$, the steady state can change dramatically in response to relatively small changes in network density.

Our model with three types of news content enables us to explore the phenomenon of *polarization* of news content through the process of forwarding/recommendating content to one's friends. Our results show that this may occur in populations in which the middle type is less prevalent than the extreme types and the network is sufficiently connected; as ρ drops below 1/3, both the left and right groups, having measures $(1-\rho)/2$, continue to embody a minority of the overall population. However, from Proposition 7, part 2(b), we see that the portion of news that is propagated in steady state is above the left and right's population shares, with y_M^* dropping below ρ. In other words, in a very connected world, as society is predominantly comprised of partisan groups and the middle is shrinking, news of the moderate center quickly diminishes, while news of the two partisan groups flood the economy. Thus, by introducing three types of news content, we are able to explain under which conditions social media tends to *polarize* towards more extreme content at the expense of more central content. Moreover, in networks that are not well connected, then the shift from a steady state in which the middle content predominates to one in which the extremes predominate may be dramatic. Small shifts in the fraction of the middle type can lead to very different steady states.

To summarize, the pairwise advantage of the middle leads it to be amplified when types are balanced (ρ large enough). If extremes are sufficiently more prevalent (ρ low enough), then the extremes are amplified relative to the middle. This suggests that steady state content goes through a transitory range in ρ where $\frac{\partial y_M^*}{\partial \rho} > 1$; that is, steady state y_M^* grows faster than ρ.

5.3 Numerical simulations

We further illustrate Proposition 7 by conducting numerical simulations to evaluate y_M^* (the steady-state fraction of the middle ideology) for different numbers of friends k in a regular network and different values of ρ, the fraction of individuals of type M.[38] Figure 3 shows the steady state transition between $y_M^* = 0$ and $y_M^* = 1$ as ρ increases. For each value of k, there is a neighborhood close to $\rho = 0$ and $\rho = 1$ where the steady states $y_M^* = 0$ and $y_M^* = 1$ occur. Then, consistent with our observations from the previous section, there is a region in between where the steady state transitions relatively "quickly" between these two extremes, where $\frac{\partial y_M^*}{\partial \rho} > 1$. Further, we see that the speed of this transition becomes more rapid for lower values of k.

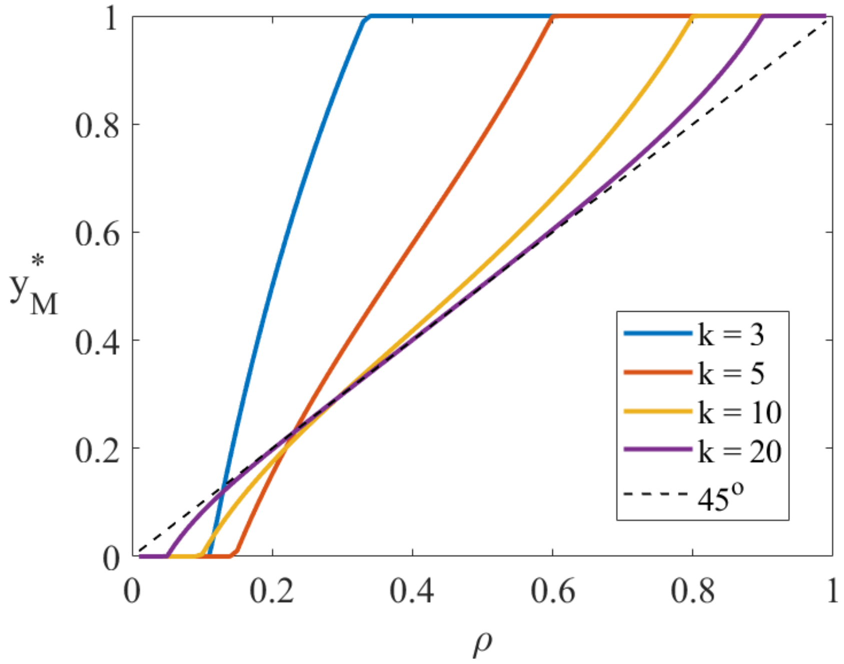


Figure 3: Steady-state values of y_M^* for different numbers of friends k

[38]For these numerical simulations, we consider a regular network with $p_1 = 0$, which implies that $\mathbb{E}[k] = k \geq 2$.

Figure 4 shows the transition of the steady state as k increases for different compositions of the middle type in the population ρ. When the population is equally balanced across types or contains more of the middle type than the extremes, such that $\rho \geq 1/3$, we are in the cases of Proposition 7, parts 2 and 3(a); that is, the middle mainstream ideology dominates the market $y_M^* > \rho$ for any value of k and decreases towards a balanced steady state $y_M^* \to \rho$ in dense networks with large k.

On the other hand, for low $\rho < 1/3$, a non-monotonic relationship exists between k and the steady state y_M^*. At $k = 2$, the middle floods the market $y_M^* = 1$; however, there is a dramatic shift in the steady state for higher values of k. In these cases, for sufficiently well-connected networks ($k \geq 4$ for $\rho = 1/6$ and $k \geq 3$ for $\rho = 1/10$), the extreme ideologies dominate the market, such that $y_M^* < \rho$. Indeed, in the case of $\rho = 1/10$, the extremes may completely flood the market (i.e., $y_M^* = 0$) such that the middle type of content does not survive in networks where $3 \leq k \leq 9$. This implies that the steady-state value of y_M^* is highly sensitive to changes in density when the network is relatively sparse (low k).

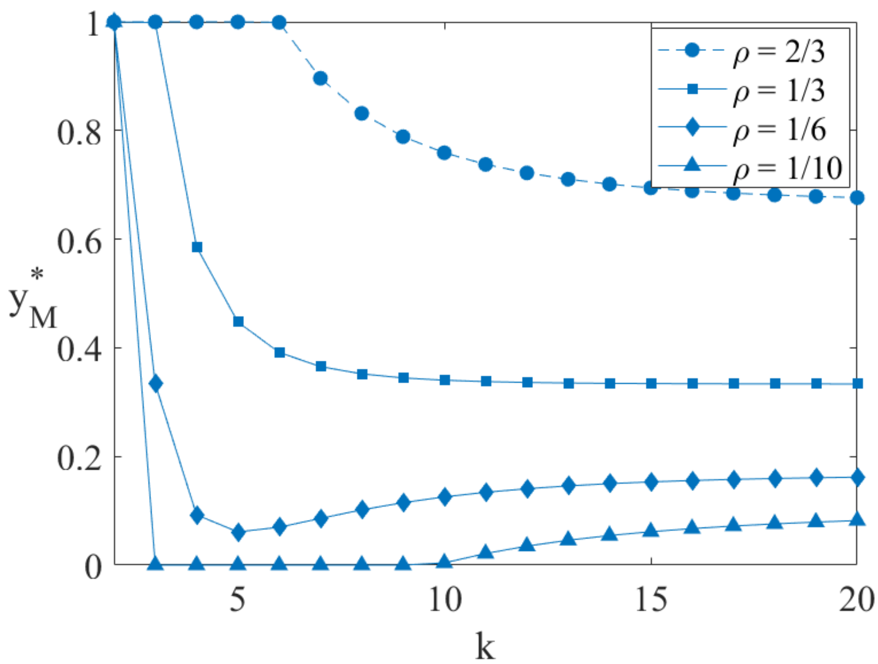


Figure 4: Steady-state values of y_M^* for different values of ρ

6 Conclusion

Using a mean-field approximation, we developed a social network diffusion model to understand its role in promoting or suppressing particular types of goods. In a setting in which everyone uses a single friend threshold, we found that there is a unique stable steady state. Contrasting a niche and a mass-market good, we found that social networks promote mass-market goods at the expense of niche ones. We showed that either greater connectivity or greater homophily increase the prevalence of niche-market goods. We also investigated how social networks affect the difference between types in the goods that each individual observes. The same forces that promote the prevalence of niche-market goods tend to also increase these differences.

When we introduced strategic actors that influence consumers of the same type, we found that this may be an additional channel for increasing the difference between types in the goods that each individual observes. We also allowed consumers to choose the degree of homophily amongst their connections and demonstrated that niche-market individuals exhibit greater homophily than mass-market consumers.

Finally, we extend our model to a three-type political ideologies by considering types left, middle, and right, with the middle having an advantage in pairwise comparisons. We showed that when the prevalence of the middle content is quite small, then the more extreme views of the left and the right will flourish and dominate the market.

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Appendix

A Proofs of the results in the main text

Proof of Theorem 1. The dynamics of the system are described by:

$$
\begin{aligned}
x_t &= \sum_k p_k \sum_{j=0}^{d(k)-1} \binom{k}{j} \left(1 - w_{n,t-1}\right)^{k-j} \left(w_{n,t-1}\right)^j \\
y_t &= \sum_k p_k \sum_{j=0}^{d(k)-1} \binom{k}{j} \left(1 - w_{m,t-1}\right)^{k-j} \left(w_{m,t-1}\right)^j
\end{aligned}
$$

where

$$
\begin{aligned}
w_{n,t-1} &= \alpha_n \left(1 - x_{t-1}\right) + \left(1 - \alpha_n\right) \left[\left(1 - \rho\right) \left(1 - x_{t-1}\right) + \rho y_{t-1}\right] \\
w_{m,t-1} &= \alpha_m \left(1 - y_{t-1}\right) + \left(1 - \alpha_m\right) \left[\left(1 - \rho\right) x_{t-1} + \rho \left(1 - y_{t-1}\right)\right]
\end{aligned}
$$

substituting in $w_{n,t-1}, w_{m,t-1}$ into the right-hand side and defining

$$
\begin{aligned}
f_x\left(x_{t-1}, y_{t-1}\right) &= \sum_k p_k \sum_{j=0}^{d(k)-1} \binom{k}{j} \left(1 - \left(\alpha_n \left(1 - x_{t-1}\right) + \left(1 - \alpha_n\right) \left[\left(1 - \rho\right) \left(1 - x_{t-1}\right) + \rho y_{t-1}\right]\right)\right)^{k-j} \\
&\quad \times \left(\alpha_n \left(1 - x_{t-1}\right) + \left(1 - \alpha_n\right) \left[\left(1 - \rho\right) \left(1 - x_{t-1}\right) + \rho y_{t-1}\right]\right)^j \\
f_y\left(x_{t-1}, y_{t-1}\right) &= \sum_k p_k \sum_{j=0}^{d(k)-1} \binom{k}{j} \left(1 - \left(\alpha_m \left(1 - y_{t-1}\right) + \left(1 - \alpha_m\right) \left[\left(1 - \rho\right) x_{t-1} + \rho \left(1 - y_{t-1}\right)\right]\right)\right)^{k-j} \\
&\quad \times \left(\alpha_m \left(1 - y_{t-1}\right) + \left(1 - \alpha_m\right) \left[\left(1 - \rho\right) x_{t-1} + \rho \left(1 - y_{t-1}\right)\right]\right)^j
\end{aligned}
$$

we can write this system as

$$
\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} f_x\left(x_{t-1}, y_{t-1}\right) \\ f_y\left(x_{t-1}, y_{t-1}\right) \end{bmatrix} = f\left(x_{t-1}, y_{t-1}\right)
$$

where we note that $f_j\left(x_{t-1}, y_{t-1}\right)$ for $j = m, n$ defines a $C1$ map $R^2 \to R^2$ and define the Jacobian matrix of the system by $A:$

$$
A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} \frac{\partial f_x}{\partial x_{t-1}} & \frac{\partial f_x}{\partial y_{t-1}} \\ \frac{\partial f_y}{\partial x_{t-1}} & \frac{\partial f_y}{\partial y_{t-1}} \end{bmatrix}
$$

Each element (see derivation in lemma C9) is given by

$$\begin{aligned}
\frac{\partial f_x}{\partial x_{t-1}} &= \left(\alpha_n + (1-\alpha_n)(1-\rho)\right) \sum_k p_k \left(-\frac{k!}{(k-d(k)-1)!d(k)!} (1-w_n)^{k-d(k)} (w_n)^{d(k)-1} \right) \geq 0 \\
\frac{\partial f_x}{\partial y_{t-1}} &= -\left((1-\alpha_n)\rho\right) \sum_k p_k \left(-\frac{k!}{(k-d(k)-1)!d!} (1-w_n)^{k-d(k)} (w_n)^{d(k)-1} \right) \leq 0 \\
\frac{\partial f_y}{\partial x_{t-1}} &= -(1-\alpha_m)(1-\rho) \sum_k p_k \left(-\frac{k!}{(k-d(k)-1)!d(k)!} (1-w_m)^{k-d(k)} (w_m)^{d(k)-1} \right) \leq 0 \\
\frac{\partial f_y}{\partial x_{t-1}} &= \left(\alpha_m + (1-\alpha_m)\rho\right) \sum_k p_k \left(-\frac{k!}{(k-d(k)-1)!d(k)!} (1-w_m)^{k-d(k)} (w_m)^{d(k)-1} \right) \geq 0
\end{aligned}$$

where the inequalities are strict for $0 < w_j < 1$ $j = m, n$.

A useful result is Theorem 4.11 on pg 221 from Discrete Chaos 2nd Ed. by Saber N. Elyadi:

Theorem 4.11 *Let $f : G \subset R^2 \to R^2$ be a $C1$ map, where G is an open subset of R^2, X^* is a fixed point of f, and $A = Df(X^*)$. Then the following statements hold true:*

1. *If $\rho(A) < 1$, then X^* is asymptotically stable.*
2. *If $\rho(A) > 1$, then X^* is unstable.*
3. *If $\rho(A) = 1$, then X^* may or may not be stable*

Where $\rho(A) = \max\{|\lambda_1|, |\lambda_2|\}$, and f and A are defined as they are above. Hence, if $\rho(A) < 1$ (> 1) at a steady state $X^* = (x^*, y^*)$ of our system then the steady state is asymptotically stable (unstable).

Lemma A1. *If* $\max\{a_{11}, a_{22}\} > 1$ *then* $\rho(A) > 1$. *Otherwise, suppose* $\max\{a_{11}, a_{22}\} \leq 1$ *and* $(1-a_{22})(1-a_{11}) > a_{12}a_{21}$ *(*$< a_{12}a_{21}$*) then* $\rho(A) < 1$ (> 1).

Proof. The characteristic equation for the 2×2 matrix A is

$$\lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21})$$

with roots λ_1, λ_2

$$\begin{aligned}
\lambda_1 &= \frac{(a_{11} + a_{22}) + \sqrt{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})}}{2} \\
\lambda_2 &= \frac{(a_{11} + a_{22}) - \sqrt{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})}}{2}
\end{aligned}$$

We know from lemma C9 that $a_{11}, a_{22} > 0$ and $a_{12}, a_{21} < 0$, hence, both roots are real and $\rho(A) = \lambda_1$. Now, note that if $a_{11}, a_{22} > 1$ then this immediately implies $\lambda_1 > 1$. Now, suppose $\max\{a_{11}, a_{22}\} \leq 1$ and that

$$\begin{aligned}
(1-a_{22})(1-a_{11}) &> a_{12}a_{21} \\
1 - (a_{11} + a_{22}) + (a_{11}a_{22} - a_{12}a_{21}) &> 0 \\
(a_{11} + a_{22})^2 + 4(1 - (a_{11} + a_{22})) &> (a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21}) \\
(2 - (a_{11} + a_{22}))^2 &> (a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})
\end{aligned}$$

which implies that

$$\begin{aligned}
2-(a_{11}+a_{22}) &> \sqrt{(a_{11}+a_{22})^2-4(a_{11}a_{22}-a_{12}a_{21})} \\
1 &> \frac{(a_{11}+a_{22})+\sqrt{(a_{11}+a_{22})^2-4(a_{11}a_{22}-a_{12}a_{21})}}{2} \\
1 &> \lambda_1
\end{aligned}$$

the same steps apply for the case $(1-a_{22})(1-a_{11})<a_{12}a_{21}$ with the opposite inequality. □

Returning to the main proof. In the niche market steady state $(x^*,y^*)=(0,1)$, $w_n^*=1$, $w_m^*=0$ and the Jacobian is given by:

$$A(0,1)=\begin{bmatrix} (\alpha_n+(1-\alpha_n)(1-\rho))\,p_1 & -(1-\alpha_n)\,\rho p_1 \\ -(1-\alpha_m)(1-\rho)\sum_{k=1}^{\widehat{k}} kp_k & (\alpha_m+(1-\alpha_m)\,\rho)\sum_{k=1}^{\widehat{k}} kp_k \end{bmatrix}$$

If $\sum_{k=1}^{\widehat{k}} kp_k<B_n$:

$$\begin{aligned}
\sum_{k=1}^{\widehat{k}} kp_k &< \frac{1-(\alpha_n+(1-\alpha_n)(1-\rho))\,p_1}{1-(1-\alpha_m)(1-\rho)-p_1\left[\alpha_m(1-\rho)+\alpha_n\rho\right]} \\
\sum_{k=1}^{\widehat{k}} kp_k &< \frac{1-(\alpha_n+(1-\alpha_n)(1-\rho))\,p_1}{\rho+\alpha_m-\rho\alpha_m-\alpha_m p_1+\rho\alpha_m p_1-\rho\alpha_n p_1} \\
\sum_{k=1}^{\widehat{k}} kp_k\left[(1-\alpha_m)(1-\rho)(1-\alpha_n)\,\rho p_1\right] &< \begin{array}{c}\left[1-(\alpha_n+(1-\alpha_n)(1-\rho))\,p_1\right] \\ \times\left[1-(\alpha_m+(1-\alpha_m)\,\rho)\sum_{k=1}^{\widehat{k}} kp_k\right]\end{array} \\
a_{12}a_{21} &< (1-a_{11})(1-a_{22})
\end{aligned}$$

and we conclude by lemma A1 above that $\rho(A)<1$ and so (using Theorem 4.11 above) that the steady state is aymptotically stable. Similarly the same steps apply for the case where $\sum_{k=1}^{\widehat{k}} kp_k>B_n$ to find that $\rho(A)>1$ and the steady state is unstable.

In the mass market steady state $(x^*,y^*)=(1,0)$, $w_n^*=0$, $w_m^*=1$ and the Jacobian is given by:

$$A(1,0)=\begin{bmatrix} (\alpha_n+(1-\alpha_n)(1-\rho))\sum_{k=1}^{\widehat{k}} kp_k & -(1-\alpha_n)\,\rho\sum_{k=1}^{\widehat{k}} kp_k \\ -(\alpha_m+(1-\alpha_m)\,\rho)\,p_1 & (1-\alpha_m)(1-\rho)\,p_1 \end{bmatrix}$$

We note that the niche market condition is the same as the mass market condition with the subscripts m,n switched and parameter ρ replaced by $1-\rho$. Then, applying identical steps to the above we establish the result.

Define the steady state relations $x^*(y)$ and $y^*(x)$ by:

$$\begin{aligned}
x^* &= f_x(x^*,y) \\
y^* &= f_y(x,y^*)
\end{aligned}$$

In the above $f_x(x^*,y)$ is a continuous function that is increasing in x^* and decreasing in y and $f_x(0,1)=0$; $f_x(1,0)=1$. Hence, for each $y\in[0,1]$ there exists a non-empty set $x^*(y)=\{x:0\leq x\leq 1;x=f_x(x,y)\}$.

Define $\underline{x}^*(0) = \min\{x : 0 \leq x \leq 1; x = f_x(x, 0)\}$ and $\overline{x}^*(1) = \max\{x : 0 \leq x \leq 1; x = f_x(x, 1)\}$. By virtue of $\underline{x}^*(0)$ being the minimum value of x that satisfies $x = f_x(x, 0)$ and $f_x(0, 0) > 0$ we conclude that $f_x(x, 0) > x$ for all $x < \underline{x}^*(0)$, and by virtue of $\overline{x}^*(1)$ being the maximum value of x that satisfies $x = f_x(x, 1)$ and $f_x(0, 1) < 0$ we conclude that $f_x(x, 0) < x$ for all $x > \overline{x}^*(1)$. Now, $\frac{\partial f_x(x^*, y)}{\partial y} < 0$ for $0 < y < 1$ so $f_x(\underline{x}^*(0), y) > \underline{x}^*(0)$ and $f_x(\overline{x}^*(1), y) < \overline{x}^*(1)$ and by the continuity of $f_x(x, y)$ in x and y and $\frac{\partial f_x(x^*, y)}{\partial y} < 0$ then for every $x \in [\overline{x}^*(1), \underline{x}^*(0)]$ there is a unique $0 \leq y \leq 1$ such that $x = f_x(x, y)$.

Define a function $g(x) : [\overline{x}^*(1), \underline{x}^*(0)] \to [0, 1]$ as the solution y to $x = f_x(x, y)$ for each $x \in [\overline{x}^*(1), \underline{x}^*(0)]$. The function $g(x)$ is continuous in x on $[\overline{x}^*(1), \underline{x}^*(0)]$ and by definition $g(\overline{x}^*(1)) = 1, g(\underline{x}^*(0)) = 0$. We can (similarly to $g(x)$) define a function $h(y)$ using $f_y(x, y^*)$ that is continuous on $\left[\overline{y}^*(1), \underline{y}^*(0)\right]$ and $h(\overline{y}^*(1)) = 1, h\left(\underline{y}^*(0)\right) = 0$ where the quantities are defined as they are above with x, y interchanged.

By definition any interior point (x^*, y^*) such that $(x^*, g(x^*)) = (h(y^*), y^*)$ is a steady state of our system. We know that the points $(\overline{x}^*(1), 1)$ and $(\underline{x}^*(0), 0)$ lie on $g(x)$ and are connected by a continuous function in x and similarly the points $\left(0, \underline{y}^*(0)\right)$ and $(1, \overline{y}^*(1))$ lie on $h(y)$ and are connected by a continuous function in y. Hence, the graphs $(x^*, g(x^*))$ and $(h(y^*), y^*)$ are guaranteed to intersect on the interior of $[0, 1] \times [0, 1]$ provided that they do not intersect on the boundary at $(0, 1)$ or $(1, 0)$. In the event that one or both do intersect on the boundary then a sufficient condition to guarantee that they will intersect on the interior is that the orientation of the slope at the exterior points is such that

$$\frac{dg}{dx}\frac{dh}{dy} > 1$$

where $\frac{dg}{dy} = \frac{a_{12}}{1-a_{11}}, \frac{dh}{dx} = \frac{a_{21}}{1-a_{22}}$. Hence, a sufficient condition for the existence of an interior steady state is that each extreme steady state is unstable $\sum_{k=1}^{\widehat{k}} kp_k > B_m, B_n$, such that in the neighborhood of $(0, 1)$ $g(x) > h^{-1}(x)$ and in the neighborhood of $(1, 0)$ $g(x) < h^{-1}(x)$.

Finally, we establish our result by showing that, under this condition, there is at least one interior steady state where $\max\{a_{11}, a_{22}\} < 1$ and

$$\frac{dg}{dx}\frac{dh}{dy} < 1$$

implying that

$$a_{12}a_{21} < (1 - a_{11})(1 - a_{22})$$

In particular, the graphs $(x, g(x))$ and $(h(y), y)$ cross at least once in $[\overline{x}^*(1), \underline{x}^*(0)] \times \left[\overline{y}^*(1), \underline{y}^*(0)\right]$. By virtue of $g(x), h(y)$ being functions (i.e. single valued) then if there is a single point of crossing $\frac{dg}{dx}\frac{dh}{dy} < 1$ and $\frac{dg}{dx}, \frac{dh}{dy} < 0$ or in the case where there are multiple points of crossing then at least one has the property $\frac{dg}{dx}\frac{dh}{dy} < 1$. Hence, by lemma A1 that steady state is asymptotically stable.

Proof of Proposition 1. We prove the result under heterogeneous homophily (α_n, α_m) and lobbying efforts (e_n, e_m). Denote $a_n \equiv 1 - e_n$ and $a_m \equiv 1 - e_m$. We first establish the necessary and sufficient condition, $\mathbb{E}[k] > B$, for there to exists a unique interior steady state; no interior steady states exists when the condition is violated. We then show that all steady states are globally stable.

Define variables $x_t := 1 - n_t$ and $y_t := 1 - m_t$, which yields the equivalent system:

$$x_t = a_n \sum_k p_k \left[(1-\alpha_n)((1-\rho)x_{t-1} + \rho(1-y_{t-1})) + \alpha_n x_{t-1}\right]^k, \tag{A.1}$$

$$y_t = a_m \sum_k p_k \left[(1-\alpha_m)(\rho y_{t-1} + (1-\rho)(1-x_{t-1})) + \alpha_m y_{t-1}\right]^k. \tag{A.2}$$

Dropping time subscripts to focus on the steady state, the system can be written:

$$x = a_n \sum_k p_k \left[(1-\alpha_n)h + \alpha_n x\right]^k, \tag{A.3}$$

$$y = a_m \sum_k p_k \left[(1-\alpha_m)(1-h) + \alpha_m y\right]^k. \tag{A.4}$$

where, at the steady-state, $h^* = H(x^*, y^*) := (1-\rho)x^* + \rho(1-y^*)$. (A.3) and (A.4) define implicit functions $x(h)$ and $y(h)$. The functions are continuous and monotone increasing and decreasing over $h \in [0,1]$, respectively, with $x(0) = 0$, $x(1) = 1$, $y(0) = 1$ and $y(1) = 0$. We verify these properties below.

For uniqueness of an interior steady state, it suffices to show existence of at most one h^* solving (A.3), (A.4) and $h^* = H(x(h^*), y(h^*))$. For this, we show that (i) $H(x(h), y(h))$ is continuous, (ii) $\lim_{h\to 0} H(x(h), y(h)) \geq 0$, (iii) $\lim_{h\to 1} H(x(h), y(h)) \leq 1$, and (iv) $\frac{d^3}{dh^3} H(x(h), y(h)) \geq 0$, which imply $H(x(h), y(h))$ crosses the 45-degree line at at-most one unique point $h^* \in (0,1)$. (i) follows immediately from $x(h)$ and $y(h)$ continuous. (ii) and (iii) clearly hold for $x, 1-y, \rho \in [0,1]$.

To show (iv), we first show $\frac{\partial^3 x}{\partial h^3} > 0$. Define $f(z) := \sum_k p_k z^k$, and note that $\frac{\partial^n}{\partial z^n} f^z \geq 0$ for all $n \geq 0$. Then define $z_x := (1-\alpha_n)h + \alpha_n x$, giving $\frac{\partial z_x}{\partial h} = (1-\alpha_n) + \alpha_n \frac{\partial x}{\partial h}$, and $\frac{\partial z_x}{\partial h} = \alpha_n \frac{\partial^2 x}{\partial h^2}$. Repeated implicit differentiation of (A.3) gives:

$$\frac{\partial x}{\partial h} - \frac{\partial z_x}{\partial h} a_n f'(z_x) = 0, \tag{A.5}$$

$$\frac{\partial^2 x}{\partial h^2} - \alpha_n a_n \frac{\partial^2 x}{\partial h^2} f'(z_x) - a_n \left(\frac{\partial z_x}{\partial h}\right)^2 f''(z_x) = 0, \tag{A.6}$$

$$\frac{\partial^3 x}{\partial h^3} - \alpha_n a_n \frac{\partial^3 x}{\partial h^3} f'(z_x) - \alpha_n a_n \frac{\partial^2 x}{\partial h^2}\frac{\partial z_x}{\partial h} f''(z_x) - 2\frac{\partial z_x}{\partial h}\alpha_n a_n \frac{\partial^2 x}{\partial h^2} f''(z_x) - a_n \left(\frac{\partial z_x}{\partial h}\right)^3 f'''(z_x) = 0. \tag{A.7}$$

Now, (A.5) gives:

$$\frac{\partial x}{\partial h} = \frac{(1-\alpha_n)a_n f'(z_x)}{1 - \alpha_n a_n f'(z_x)} > 0,$$

the inequality holding by $(1-\alpha_n)a_n f'(z_x) > 0$, and by $1 - \alpha_n a_n f'(z_x) > 0$ holding where equality (A.3) holds, for each $h \in (0,1)$, because $a_n f((1-\alpha_n)h) > 0$, $a_n f((1-\alpha_n)h + \alpha_n) < 1$ and f is convex. This verifies that $x(h)$ is increasing and continuous, and establishes that $\frac{\partial z_x}{\partial h} > 0$. (A.6) gives:

$$\frac{\partial^2 x}{\partial h^2} = \frac{\left(\frac{\partial z_x}{\partial h}\right)^2 a_n f''(z_x)}{1 - \alpha_n a_n f'(z_x)} > 0,$$

the inequality holding by $a_n f''(z_x) > 0$, and the above. (A.7) gives:

$$\frac{\partial^3 x}{\partial h^3} = \frac{3\alpha_n \frac{\partial z_x}{\partial h}\frac{\partial^2 x}{\partial h^2} a_n f''(z_x) + \left(\frac{\partial z_x}{\partial h}\right)^3 a_n f'''(z_x)}{1 - \alpha_n a_n f'(z_x)} > 0,$$

the inequality holding by $\frac{\partial z_x}{\partial h} > 0$, $f''(z_x), f'''(z_x) > 0$, and by the above.

Still for (iv), we next show $\frac{\partial^3 y}{\partial h^3} < 0$. Now define $z_y := (1-\alpha_m)(1-h) + \alpha_m y$, giving $\frac{\partial z_y}{\partial h} = -(1-\alpha_m) + \alpha_m \frac{\partial y}{\partial h}$, and $\frac{\partial z_y}{\partial h} = \alpha_m \frac{\partial^2 y}{\partial h^2}$. Repeated implicit differentiation of (A.4) gives analogous expressions to (A.5), (A.6) and (A.7) but with y's in place of "x" and z_y's in place of "z_x". However, (A.5) now gives:

$$\frac{\partial y}{\partial h} = \frac{-(1-\alpha_m)a_m f'(z_y)}{1-\alpha_m a_m f'(z_y)} < 0,$$

the inequality holding by $-(1-\alpha_m)a_m f'(z_y) < 0$, and by $1 - \alpha_m a_m f'(z_y) > 0$ holding where equality (A.4) holds, for each $h \in (0,1)$, because $a_m f((1-\alpha_m)(1-h)) > 0$, $a_m f((1-\alpha_m)(1-h)+\alpha_m) < 1$ and f is convex. This verifies that $y(h)$ is decreasing and continuous, and establishes that $\frac{\partial z_y}{\partial h} < 0$. (A.6) gives:

$$\frac{\partial^2 y}{\partial h^2} = \frac{\left(\frac{\partial z_y}{\partial h}\right)^2 a_m f''(z_y)}{1-\alpha_m a_m f'(z_y)} > 0,$$

the inequality holding by $f''(z_y) > 0$, and the above. Finally, (A.7) can be written:

$$\frac{\partial^3 y}{\partial h^3} = \frac{\partial z_y}{\partial h} \frac{3\alpha_m \frac{\partial^2 y}{\partial h^2} a_m f''(z_y) + \left(\frac{\partial z_y}{\partial h}\right)^2 a_m f'''(z_y)}{1-\alpha_m a_m f'(z_y)} < 0,$$

the inequality holding by the above.

It follows that:

$$\frac{d^3}{dh^3} H(x(h), y(h)) = (1-\rho)\frac{\partial^3 x}{\partial h^3} - \rho \frac{\partial^3 y}{\partial h^3} > 0.$$

We have shown that $H(x(h), y(h))$ crosses the 45-degree line at at-most one unique point $h^* \in (0,1)$.

Before establishing the condition characterizing an interior steady state (where $h^* \in (0,1)$), first note that, setting $y = 1-\rho$ on the right-hand-side of (A.4), and by $x^* \leq 1 - y^*$ in the unique steady state, we have $H(x^*, 1-\rho) < H(\rho, 1-\rho) = \rho$, while setting $y = 1-\rho$ on the left-hand-side of (A.4) gives:

$$\begin{aligned} y &= 1-\rho = a_m \sum_k p_k \left[(1-\alpha_m)(1-H(x, 1-\rho)) + \alpha_m(1-\rho)\right]^k \\ &> a_m \sum_k p_k \left[(1-\alpha_m)(1-H(\rho, 1-\rho)) + \alpha_m(1-\rho)\right]^k = 1-\rho. \end{aligned}$$

This implies that $y^* < 1-\rho$ for condition (A.4) to hold with equality, or equivalently $m^* > \rho$. $n^* > 0$ when $h^* \in (0,1)$ then follows directly from condition (A.3), or equivalently $m^* > \rho$.

We now derive the condition characterizing when the steady state is interior, that is, $h^* \in (0,1)$, which implies $x^*(h^*) < 1$ and equivalently $n^* > 0$ (i.e. news of the niche market persists in steady state). Remember:

$$\begin{aligned} \frac{\partial}{\partial x} h &= \frac{(1-\alpha_n)a_n \sum_k k p_k \left((1-\alpha_n)h + \alpha_n x\right)^{k-1}}{1 - \alpha_n a_n \sum_k k p_k \left((1-\alpha_n)h + \alpha_n x\right)^{k-1}}; \\ \frac{\partial}{\partial y} h &= -\frac{(1-\alpha_m)a_m \sum_k k p_k \left((1-\alpha_m)(1-h) + \alpha_m y\right)^{k-1}}{1 - \alpha_m a_m \sum_k k p_k \left((1-\alpha_m)(1-h) + \alpha_m y\right)^{k-1}}. \end{aligned}$$

Note that:

$$\lim_{h\to 1}\frac{\partial}{\partial x}h = \frac{(1-\alpha_n)a_n\mathbb{E}[k]}{1-\alpha_n a_n\mathbb{E}[k]};\lim_{h\to 1}\frac{\partial}{\partial y}h = -\frac{(1-\alpha_m)a_m p_1}{1-\alpha_m a_m p_1},$$

These give:

$$\begin{aligned}\lim_{h\to 1}\frac{d}{dh}H(x(h),y(h)) &= \lim_{h\to 1}\left[(1-\rho)\frac{\partial}{\partial h}x - \rho\frac{\partial}{\partial h}y\right]\\ &= \left((1-\rho)\frac{(1-\alpha_n)a_n\mathbb{E}[k]}{1-\alpha_n a_n\mathbb{E}[k]} + \rho\frac{(1-\alpha_m)a_m p_1}{1-\alpha_m a_m p_1}\right)\end{aligned}$$

When $\lim_{h\to 1}\frac{d}{dh}H(x(h),y(h)) < 1$ then the unique steady state satisfies $h^* < 1$ and is therefore interior. Rearranging gives:

$$\mathbb{E}[k] > B(\alpha_n,\alpha_m) := \frac{1}{a_n}\frac{1-a_m p_1(\alpha_m+(1-\alpha_m)\rho)}{1-a_m p_1(\alpha_m(1-\rho)+\alpha_n\rho)-(1-\alpha_n)\rho}.$$

When $\alpha_n = \alpha_m$ and $a_n = a_m = 1$ we obtain

$$\mathbb{E}[k] > B := B(\alpha,\alpha) = \left[\alpha + (1-\alpha)(1-\rho)\frac{1-p_1\alpha}{1-p_1(\alpha+(1-\alpha)\rho)}\right]^{-1}.$$

We next establish the steady state condition $h^* \geq \rho$ when $\mathbb{E}[k] > B$ as a lemma.

Lemma A2. $h^* \geq \rho$

Proof of Lemma A2. We prove the lemma by showing that $H(x(\rho),y(\rho)) \geq \rho$. Then, given $H(x(h),y(h))$ crosses the 45-degree line at most once from above, the lemma follows. The quantities $x(\rho),y(\rho)$ are defined as:

$$\begin{aligned}f^{-1}(x)-\alpha x &= (1-\alpha)\rho\\ f^{-1}(y)-\alpha y &= (1-\alpha)(1-\rho)\end{aligned}$$

where the left-hand side is an increasing concave function of its argument and is equal to 0 at 0. Hence, using the line defined by the points $(y,(1-\alpha)(1-\rho))$ and $(x,(1-\alpha)\rho)$, the following relationship holds

$$(1-\alpha)(1-\rho) \geq \frac{(1-\alpha)[\rho-(1-\rho)]}{x(\rho)-y(\rho)}y(\rho)$$

rearranging

$$\frac{x(\rho)}{y(\rho)}\frac{1-\rho}{\rho} \geq 1$$

which implies that $H(x(\rho),y(\rho)) \geq \rho$ thereby establishing the result. □

With $h^* \geq \rho$ by Lemma A2, $\rho < x_m^*$ follows as a corollary by writing $x_m^* = \alpha m^* + (1-\alpha)h^*$ and applying $m^* \geq \rho$, while $x_m^* < m^*$ by the definition of x_m^* and applying $m^* > 1-n^*$. Similarly, $0 < x_n^* < n^*$ by the definition of x_n^* and applying $m^* > 1-n^*$.

Toward establishing global stability, we next show the unique steady state is locally stable, a necessary condition for global stability. Again allow $\alpha_n \neq \alpha_m$. For this, writing

$x^*(y)$ and $y^*(x)$ the implicit steady-state solutions to (A.1) and (A.2), respectively, and $x^{*-1}(x)$ the inverse of the former. Then:

$$
\begin{aligned}
& \frac{d}{dm}n^*(m) \cdot \frac{d}{dn}m^*(n)\Big|_{(n^*,m^*)} < 1 \\
\Leftrightarrow \quad & \frac{d}{dy}x^*(y) \cdot \frac{d}{dx}y^*(x)\Big|_{(x^*,y^*)} < 1 \\
\Leftrightarrow \quad & \frac{d}{dx}y^*(x)\Big|_{x^*} > \frac{d}{dx}x^{*-1}(x))\Big|_{x^*}.
\end{aligned}
$$

Implicit differentiation gives:

$$
\begin{aligned}
\lim_{x \to 1} \frac{dy^*}{dx} &= \frac{-(1-\alpha_m)(1-\rho)a_m p_1}{1-((1-\alpha_m)\rho + \alpha_m)a_m p_1}, \\
\lim_{x \to 1} \frac{dy}{dx^*} &= \left(\lim_{y \to 0} \frac{dx^*}{dy}\right)^{-1} = \frac{1-((1-\alpha_n)(1-\rho)+\alpha_n)a_n\mathbb{E}[k]}{-(1-\alpha_n)\rho a_n \mathbb{E}[k]},
\end{aligned}
$$

the second equality following from the inverse function theorem. Then, it suffice for the steady state (x^*, y^*) to be stable for:

$$
\lim_{x\to 1} \tfrac{dy^*}{dx} > \lim_{x\to 1} \tfrac{dy}{dx^*} \quad if\ \mathbb{E}[k] > B,
$$

$$
\lim_{x\to 1} \tfrac{dy^*}{dx} \leq \lim_{x\to 1} \tfrac{dy}{dx^*} \quad if\ \mathbb{E}[k] \leq B,
$$

as then $x^*(y)$ and $y^*(x)$ must intersect at (x^*, y^*) such that $\frac{d}{dx}y^*(x^*) > \frac{d}{dx}x^{*-1}(x^*)$, both when (x^*, y^*) and when $(x^*, y^*) = (1, 0)$. With algebra, the second strict inequality at limits $x \to 1$ and $y \to 0$ becomes:

$$
\mathbb{E}[k] > \frac{1 - a_m p_1(\alpha_m + (1-\alpha_m)\rho)}{1 - a_m p_1(\alpha_m(1-\rho) + \alpha_n \rho) - (1-\alpha_n)\rho} = B(\alpha_n, \alpha_m).
$$

We now show global stability, provided the starting point (x_0, y_0) is interior. Define the dynamic system $(x_t, y_t) = g(x_{t-1}, y_{t-1})$ by equations (A.1) and (A.2) where it is straightforward to verify that g is continuous in $(x, y) \in [0,1] \times [0,1]$. The quantities $x^*(y_{t-1})$ and $y^*(x_{t-1})$ give the unique interior fixed point of $g_z : [0,1] \to [0,1]$ for $z = x, y$ where $x = g_x(x, y_{t-1})$ and $y = g_y(x_{t-1}, y)$; when there is more than one fixed point (this may only occur when the argument is 0) then it is defined as the minimum. Define $\underline{x}(x_{t-1}, y_{t-1}) = \min\{x_{t-1}, x^*(y_{t-1})\}$ and $\bar{x}(x_{t-1}, y_{t-1}) = \max\{x_{t-1}, x^*(y_{t-1})\}$ and similarly define $\underline{y}(x_{t-1}, y_{t-1}), \bar{y}(x_{t-1}, y_{t-1})$. We now show the following property of the dynamic system.

Lemma A3. *Suppose $x_{t-1} \neq x^*(y_{t-1})$ then $g_x(x_{t-1}, y_{t-1}) \in (\underline{x}(x_{t-1}, y_{t-1}), \bar{x}(x_{t-1}, y_{t-1}))$ and, similarly, suppose $y_{t-1} \neq y^*(x_{t-1})$ then $g_y(x_{t-1}, y_{t-1}) \in (\underline{y}(x_{t-1}, y_{t-1}), \bar{y}(x_{t-1}, y_{t-1}))$.*

Proof. We observe that $g : [0,1]^2 \to [0,1]^2$, $g_x(x_{t-1}, y_{t-1})$ and $g_y(x_{t-1}, y_{t-1})$ are increasing and convex in x_{t-1} and y_{t-1} respectively. Hence, if $x_{t-1} < x^*(y_{t-1})$ then $x_{t-1} < g_x(x_{t-1}, y_{t-1}) < x^*(y_{t-1})$ and if $x_{t-1} > x^*(y_{t-1})$ then $x^*(y_{t-1}) < g_x(x_{t-1}, y_{t-1}) < x_{t-1}$. The same argument applies for the y coordinate completing the proof. □

Take the case where the steady state is interior: $0 < m^*, n^*, x^*, y^* < 1$. Define a line segment $\tilde{y}_1(\tilde{x}_1)$ between the steady state (x^*, y^*) and point $(1,1)$ by $\tilde{y}_1 = \frac{1-y^*}{1-x^*}\tilde{x}_1 + \frac{y^*-x^*}{1-x^*}$ for $\tilde{x}_1 \in [x^*, 1]$. Now, define a second line $\tilde{y}_2(\tilde{x}_2)$ for $\tilde{x} \in [0, x^*]$ in the following way. For each $x \in [x^*, 1]$ find the point $(x^*(\tilde{y}_1(x)), \max\{x^{*-1}(x), 0\})$; this point corresponds with the south-west corner of the dashed box in Figure A1. First, $x^*(\tilde{y}_1(x))$ is continuous and strictly decreasing in x for all $x^* \le x \le 1$, $x^*(\tilde{y}_1(1)) = 0$ and $x^*(\tilde{y}_1(x^*)) = x^*$. Second, note that we have defined the y-coordinate equal to 0 when the inverse of equation (A.3) $x^{*-1}(x)$ gives a negative solution. Finally, there exists $\bar{x} \le 1$ such that $x^{*-1}(x) > 0$ for $x < \bar{x}$ and $x^{*-1}(x)$ is strictly decreasing in x. The line segments $\tilde{y}_1, \tilde{y}_2, x^*(y)$ segment the interior of (x, y) into four regions about the steady state as in the diagram below where the regions are labeled A, B, C, D:

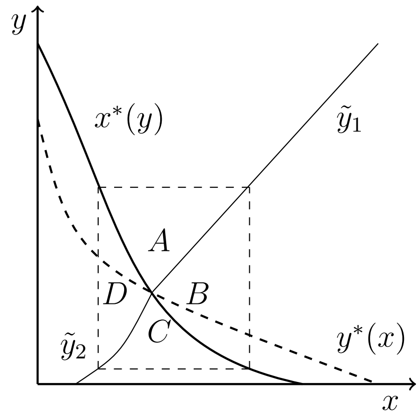

Figure A1: Four Segments

Now define a function $L(x,y) : [0,1]^2 \to \Re$ as follows:

$$L(x,y) = \begin{cases} y^* - y & \text{if } (x,y) \in A \\ y^* - \tilde{y}_1(x) & \text{if } (x,y) \in B \\ y^* - \tilde{y}_1(x^*(y)) & \text{if } (x,y) \in C \\ y^* - x^{*-1}(x) & \text{if } (x,y) \in D \end{cases} \tag{A.8}$$

The dashed box in Figure A1 defines one isoquant of L.

For any interior starting point (x_0, y_0) we can define $X = \{(x,y) : L(x,y) \ge L(x_0, y_0)\}$ where X is a compact subset $X \subseteq \Re^2$. Moreover, the function $L(x,y) : X \to \Re$ is continuous in x, y and $L(x,y) \le 0$ with equality if and only if $(x,y) = (x^*, y^*)$. We now show the following two lemmas

Lemma A4. *Suppose* $(x_{t-1}, y_{t-1}) \in (0,1) \times [0,1]$ *and* $x_{t-1} \neq x^*(y_{t-1})$ *then* $L(g(x_{t-1}, y_{t-1})) > L(x_{t-1}, y_{t-1})$.

Proof. We construct the following upper x'', y'' and lower x', y' bounds from the value $L(x_{t-1}, y_{t-1})$:

$$\begin{aligned} y'' &= y^* - L(x_{t-1}, y_{t-1}) \\ x'' &= \tilde{y}_1^{-1}(y'') \\ y' &= \max\{x^{*-1}(x''), 0\} \\ x' &= x^*(y'') \end{aligned}$$

We note that from the definition of L that (x_{t-1}, y_{t-1}) lies on the boundary of $[x', x''] \times [y', y'']$ and $L(x,y) > L(x_{t-1}, y_{t-1})$ for any point on the interior $(x,y) \in (x', x'') \times (y', y'')$. We also note that $x^*(y)$ and $y^*(x)$ cross at most once in $(0,1) \times (0,1)$ where $\left|\frac{\partial x^*}{\partial y}\right| < (\left|\frac{\partial y^*}{\partial x}\right|)^{-1}$ so $y' < y^*(x) < y''$ for all $x \in [x', x'']$. Now, by lemma A3 we have that $g_y(x_{t-1}, y_{t-1}) \in (\underline{y}, \bar{y}) \subset (y', y'')$ and for $x_{t-1} \neq x^*(y_{t-1})$ then $g_x(x_{t-1}, y_{t-1}) \in (\underline{x}, \bar{x}) \subset (x', x'')$ so $g(x_{t-1}, y_{t-1}) \in (x', x'') \times (y', y'')$ and hence $L(g(x_{t-1}, y_{t-1})) > L(x_{t-1}, y_{t-1})$. □

Lemma A5. *Suppose* $(x_{t-1}, y_{t-1}) \in (0,1) \times [0,1]$, $x_{t-1} = x^*(y_{t-1})$ *and* $y_{t-1} \neq y^*(x_{t-1})$ *then* $L(g(g(x_{t-1}, y_{t-1}))) > L(x_{t-1}, y_{t-1})$.

Proof. In this case $g_x(x_{t-1}, y_{t-1}) = x_{t-1}$ and using lemma A3 $g_y(x_{t-1}, y_{t-1}) \in (\underline{y}, \bar{y})$, hence, $L(g(x_{t-1}, y_{t-1})) = L(x_{t-1}, y_{t-1})$ and $g_x(x_{t-1}, y_{t-1}) \neq x^*(g_y(x_{t-1}, y_{t-1}))$. We now apply lemma A4 to conclude that $L(g(g(x_{t-1}, y_{t-1}))) > L(g(x_{t-1}, y_{t-1})) = L(x_{t-1}, y_{t-1})$ □

To establish our result for an interior steady state and interior starting point (x_0, y_0) we define $X = \{(x,y) : L(x,y) \geq L(x_0, y_0)\}$ and $h(x,y) = g(g(x,y))$. X is a compact subset $X \subseteq \Re^2$. Moreover, the function $L(x,y) : X \to \Re$ is continuous in x, y, $L(x,y) \leq 0$ with equality if and only if $(x,y) = (x^*, y^*)$, and $h(x,y)$ is a continuous mapping $X \to X$ by virtue of g having the same properties. By lemmas A4 and A5 $L(h(x,y)) > L(x,y)$ for all $(x,y) \in X/(x^*, y^*)$ and $L(h(x^*, y^*)) = L(x^*, y^*)$. The conditions of Lemma 6.2 of Stokey and Lucas (1989) are satisfied for X and L as defined here, $g = h$ and $\bar{x} = (x^*, y^*)$. Therefore, (x^*, y^*) is the globally stable steady state of h and hence g in X.

For the case where the steady state is $m^* = 1 = 1 - x^*, n^* = 0 = 1 - y^*$ we define:

$$\tilde{L}(x,y) = -\max\{y, x^{*-1}(x)\} \tag{A.9}$$

Lemma A6. *Suppose* $(x_{t-1}, y_{t-1}) \in (0,1) \times [0,1]$ *and* $x_{t-1} \neq x^*(y_{t-1})$ *then* $\tilde{L}(g(x_{t-1}, y_{t-1})) > \tilde{L}(x_{t-1}, y_{t-1})$.

Proof. We note that from the definition of $\tilde{L}$ that (x_{t-1}, y_{t-1}) lies on the boundary of the set $\{(x,y) : \tilde{L}(x,y) \geq \tilde{L}(x_{t-1}, y_{t-1})\}$. Moreover, $\tilde{L}(x,y) > \tilde{L}(x_{t-1}, y_{t-1})$ for any (x,y) such that $x > x^*(\tilde{L}(x,y))$ and $y < \tilde{L}(x,y)$. In the case that $m^* = 1 = 1 - x^*, n^* = 0 = 1 - y^*$ then $y^*(x) < x^{*-1}(x)$ $\forall x \in (0,1)$, hence, $y^*(x) < \tilde{L}(x_{t-1}, y_{t-1})$ $\forall x \in (0,1)$. Now, by lemma A3 $g_y(x_{t-1}, y_{t-1}) \in (0, \tilde{L}(x_{t-1}, y_{t-1})$ and $g_x(x_{t-1}, y_{t-1}) \in (x^*(\tilde{L}(x_{t-1}, y_{t-1})), 1)$. □

Lemma A7. *Suppose* $(x_{t-1}, y_{t-1}) \in (0,1) \times [0,1]$, $x_{t-1} = x^*(y_{t-1})$ *then* $\tilde{L}(g(g(x_{t-1}, y_{t-1}))) > \tilde{L}(x_{t-1}, y_{t-1})$.

Proof. In this case, $g_x(x_{t-1}, y_{t-1}) = x_{t-1}$ and observing that $y_{t-1} \neq y^*(x_{t-1}$ we can use lemma A3 to conclude that $g_y(x_{t-1}, y_{t-1}) \in (0, y_{t-1})$. Hence, $\tilde{L}(g(x_{t-1}, y_{t-1})) = \tilde{L}(x_{t-1}, y_{t-1})$ and $g_x(x_{t-1}, y_{t-1}) \neq x^*(g_y(x_{t-1}, y_{t-1}))$. We can now apply lemma A6 to conclude that $\tilde{L}(g(g(x_{t-1}, y_{t-1}))) > \tilde{L}(g(x_{t-1}, y_{t-1})) = \tilde{L}(x_{t-1}, y_{t-1})$ completing the proof. □

To establish our result for the mass market steady state and interior starting point (x_0, y_0) we define $X = \{(x,y) : \tilde{L}(x,y) \geq \tilde{L}(x_0, y_0)\}$ and $h(x,y) = g(g(x,y))$. X is a compact subset $X \subseteq \Re^2$. Moreover, the function $\tilde{L}(x,y) : X \to \Re$ is continuous in x, y, $\tilde{L}(x,y) \leq 0$ with equality if and only if $(x,y) = (1,0)$, and $h(x,y)$ is a continuous mapping $X \to X$ by virtue of g having the same properties. By lemmas A6 and A7 $\tilde{L}(h(x,y)) > \tilde{L}(x,y)$ for all $(x,y) \in X/(x^*, y^*)$ and $\tilde{L}(h(1,0)) = \tilde{L}(1,0)$. The conditions of Lemma 6.2 of Stokey and Lucas (1989) are satisfied for X and $\tilde{L}$ as defined here, $g = h$

and $\bar{x} = (1, 0)$. Therefore, $(1, 0)$ is the globally stable steady state of h and hence g in X. $\square$

Proof of Proposition 2. We use the multivariate implicit function theorem. Define z_x and z_y as above; $h = (1-\rho)x + \rho(1-y)$. Write the system (A.3) and (A.4):

$$\begin{bmatrix} g_x(x,y) \\ g_y(x,y) \end{bmatrix} = \begin{bmatrix} x - \sum_k p_k \left[(1-\alpha)h + \alpha x\right]^k \\ y - \sum_k p_k \left[(1-\alpha)(1-h) + \alpha y\right]^k \end{bmatrix}.$$

$g_x(x^*, y^*) = 0$ and $g_y(x^*, y^*) = 0$ then defines the steady state. The Jacobian of the system is:

$$J = \begin{bmatrix} 1 - ((1-\alpha)(1-\rho)+\alpha)f'(z_x) & (1-\alpha)\rho f'(z_x) \\ (1-\alpha)(1-\rho)f'(z_y) & 1 - ((1-\alpha)\rho+\alpha)f'(z_y) \end{bmatrix}$$

which has inverse:

$$J^{-1} = \frac{1}{|J|}\begin{bmatrix} 1 - ((1-\alpha)\rho+\alpha)f'(z_y) & -(1-\alpha)\rho f'(z_x) \\ -(1-\alpha)(1-\rho)f'(z_y) & 1 - ((1-\alpha)(1-\rho)+\alpha)f'(z_x) \end{bmatrix}.$$

We know that $|J| > 0$ by stability of the steady state.

The comparative statics with respect to α is then given by:

$$\begin{aligned} \begin{bmatrix} \frac{\partial x^*}{\partial \alpha} \\ \frac{\partial y^*}{\partial \alpha} \end{bmatrix} &= -J^{-1}\begin{bmatrix} \frac{\partial g_x(x^*,y^*)}{\partial \alpha} \\ \frac{\partial g_y(x^*,y^*)}{\partial \alpha} \end{bmatrix} = -J^{-1}\begin{bmatrix} -\rho(1-(x^*+y^*))f'(z_x^*) \\ (1-\rho)(1-(x^*+y^*))f'(z_y^*) \end{bmatrix} \\ &= \frac{-1}{|J|}(1-(x^*+y^*))\begin{bmatrix} \rho f'(z_x^*)(1-f'(z_y^*)) \\ (1-\rho)f'(z_y^*)(1-f'(z_x^*)) \end{bmatrix}. \end{aligned}$$

Now, $\frac{\partial}{\partial h}H(x(h^*), y(h^*)) < 1$ from the proof of Theorem 1. Also from the proof of Theorem 1, we can write:

$$\frac{\partial}{\partial h}H(x(h^*), y(h^*)) = (1-\rho)\frac{(1-\alpha)f'(z_x)}{1-\alpha f'(z_x)} + \rho\frac{(1-\alpha)f'(z_y)}{1-\alpha f'(z_y)}.$$

Therefore, either $f'(z_x) < 1$ or $f'(z_y) < 1$ must hold for $\frac{\partial}{\partial h}H(x(h^*), y(h^*)) < 1$ to obtain. By $y^* \leq x^*$ and $y^* < 1/2$, $h^* - (1-h^*) = (1-2y^*) + 2(1-\rho)(x^*-y^*) > 0$, and therefore $z_x^* > z_y^*$. It must therefore be that $f'(z_y^*) < 1$. With $|J| > 0$, $1 > x^* + y^*$ and $f'(z_x^*) > 0$, this gives $\frac{\partial x^*}{\partial \alpha} < 0$, or equivalently $\frac{\partial n^*}{\partial \alpha} > 0$. And, to show $\frac{\partial h^*}{\partial \alpha} < 0$:

$$\begin{aligned} \frac{\partial h^*}{\partial \alpha} &= (1-\rho)\frac{\partial x^*}{\partial \alpha} - \rho\frac{\partial y^*}{\partial \alpha} \\ &= \frac{-(1-\rho)\rho}{|J|}(1-(x^*+y^*))\left(f'(z_x^*)(1-f'(z_y^*)) - f'(z_y^*)(1-f'(z_x^*))\right) \\ &= \frac{-(1-\rho)\rho}{|J|}(1-(x^*+y^*))\left(f'(z_x^*) - f'(z_y^*)\right) \\ &< 0, \end{aligned}$$

the inequality following from $f'(z_x^*) - f'(z_y^*) > 0 \Leftrightarrow z_x^* > z_y^* \Leftrightarrow$:

$$\begin{aligned} (1-\alpha)h^* + \alpha x^* &> (1-\alpha)(1-h^*) + \alpha y^* \\ \alpha(x^*-y^*) &> (1-\alpha)(1-2h^*), \end{aligned}$$

which holds by $x^* > y^* \Leftrightarrow n^* < m^*$ and by $h^* > \rho > 1/2$.

To show $\lim_{\alpha\to 1} m^* = \lim_{\alpha\to 1} n^* = 1$, note that $\lim_{\alpha\to 1} f(z_x) = f(x)$ and $\lim_{\alpha\to 1} f(z_y) = f(y)$. $x - f(x) = 0$ and $y - f(y) = 0$ both having solutions 1 and 0. By $\frac{\partial m^*}{\partial \alpha} > 0$, $\lim_{\alpha\to 1} m^* = 0$ is excluded. For $\lim_{\alpha\to 1} n^*$, $\lim_{\alpha\to 1} y^* = 0$ implies that $f'(z_y) = p_1 < 1$, and therefore, $\lim_{\alpha\to 1} \frac{\partial x^*}{\partial \alpha} < 0$, equivalently $\lim_{\alpha\to 1} \frac{\partial n^*}{\partial \alpha} > 0$, which excludes $\lim_{\alpha\to 1} n^* = 0$.

We now show the claim on FOSD shifts to $\{p_k\}$. Any FOSD shift to $\{p_k\}$ can be decomposed into multiple shifts in probability densities $\epsilon > 0$ from $p_{k'}$ to $p_{k''}$ for some $k'' > k'$. Then, given this ϵ, we can write the system:

$$\begin{aligned} x &= \left(\begin{array}{c} ((p_{k'} - \epsilon)\left[(1-\alpha)h + \alpha x\right]^{k'}) + ((p_{k''} + \epsilon)\left[(1-\alpha)h + \alpha x\right]^{k''}) \\ + \sum_{k\neq k',k''} p_k \left[(1-\alpha)h + \alpha x\right]^{k} \end{array} \right), \\ y &= \left(\begin{array}{c} ((p_{k'} - \epsilon)\left[(1-\alpha)(1-h) + \alpha y\right]^{k'}) + ((p_{k''} + \epsilon)\left[(1-\alpha)(1-h) + \alpha y\right]^{k''}) \\ + \sum_{k\neq k',k''} p_k \left[(1-\alpha)h + \alpha y\right]^{k} \end{array} \right). \end{aligned}$$

The comparative statics with respect to ϵ is then given by:

$$\begin{aligned} \begin{bmatrix} \frac{\partial x^*}{\partial \epsilon} \\ \frac{\partial y^*}{\partial \epsilon} \end{bmatrix} &= -J^{-1} \begin{bmatrix} \frac{\partial g_x(z_x^*, z_y^*)}{\partial \epsilon} \\ \frac{\partial g_y(z_x^*, z_y^*)}{\partial \epsilon} \end{bmatrix} = -J^{-1} \begin{bmatrix} z_x^{*k'} - z_x^{*k''} \\ z_y^{*k'} - z_y^{*k''} \end{bmatrix} \\ &= \frac{-1}{|J|} \begin{bmatrix} (1 - ((1-\alpha)\rho + \alpha) f'(z_y^*))\delta_x - (1-\alpha)\rho f'(z_x^*)\delta_y^* \\ -(1-\alpha)(1-\rho) f'(z_y)\delta_x^* + (1 - ((1-\alpha)(1-\rho) + \alpha) f'(z_x^*))\delta_y^* \end{bmatrix}. \end{aligned}$$

where $\delta_x := z_x^{k'} - z_x^{k''} > 0$ and $\delta_y := z_y^{k'} - z_y^{k''} > 0$.

$$\begin{aligned} \frac{\partial h}{\partial \epsilon} &= (1-\rho)\frac{\partial x}{\partial \epsilon} - \rho \frac{\partial y}{\partial \epsilon} \\ &= \frac{-1}{|J|}\left(\delta_x (1-\rho)(1 - \alpha f'(z_y)) - \delta_y \rho (1 - \alpha f'(z_x))\right) > 0 \\ &\Leftrightarrow \frac{\delta_x}{\delta_y}\frac{1-\rho}{\rho} > \frac{1 - \alpha f'(z_x)}{1 - \alpha f'(z_y)}. \end{aligned} \tag{A.10}$$

The right-hand-side of (A.10) is less than one at (z_x^*, z_y^*), which is shown in the proof of Proposition 1. Now:

$$\begin{aligned} \frac{\delta_x^*}{\delta_y^*} \geq \frac{z_x^*}{z_y^*} &= \frac{(1-\alpha)h^* + \alpha x^*}{(1-\alpha)(1-h^*) + \alpha y^*} \\ &\geq \frac{(1-\alpha)\rho + \alpha x^*}{(1-\alpha)(1-\rho) + \alpha y^*} = \frac{\rho}{1-\rho}\left[\frac{(1-\alpha)\frac{1-\rho}{y^*} + \alpha \frac{x^*}{y^*}\frac{1-\rho}{\rho}}{(1-\alpha)\frac{1-\rho}{y^*} + \alpha}\right] \\ &\geq \frac{\rho}{1-\rho} \end{aligned}$$

where the first inequality follows from $\frac{\delta_x}{\delta_y}$ decreasing in k'' and increasing in k' upon $k'' \to \infty$, the second inequality follows from Lemma A2, and the third inequality follows from $\frac{x^*}{y^*}\frac{1-\rho}{\rho} > 1$ which is shown in the proof of Lemma A2. Therefor the left-hand-side of (A.10) is above one.

Finally, to show n^* increases with FOSD shifts to $\{p_k\}$, a decrease in h^* decreases each term in the right-hand-side of (A.3) (i.e. for each k). Moreover, an FOSD shift in $\{p_k\}$ shifts probabilities to larger k, also decreasing the right-hand-side of (A.3). Therefore, x^* unambiguously decreases, or equivalently n^* increases.

For the last claim, the comparative statics with respect to ρ is then given by:

$$\begin{aligned}
\begin{bmatrix} \frac{\partial x^*}{\partial \rho} \\ \frac{\partial y^*}{\partial \rho} \end{bmatrix} &= -J^{-1} \begin{bmatrix} \frac{\partial g_x(x^*,y^*)}{\partial \rho} \\ \frac{\partial g_y(x^*,y^*)}{\partial \rho} \end{bmatrix} = -J^{-1} \begin{bmatrix} -(1-\alpha)(1-(x^*+y^*))f'(z_x^*) \\ (1-\alpha)(1-(x^*+y^*))f'(z_y^*) \end{bmatrix} \\
&= \frac{-1}{|J|}(1-\alpha)(1-(x^*+y^*)) \begin{bmatrix} -f'(z_x^*)(1-\alpha f'(z_y^*)) \\ f'(z_y^*)(1-\alpha f'(z_x^*)) \end{bmatrix}.
\end{aligned}$$

$x^* \geq y^*$ implies $x^* \leq 1 - y^*$, equivalently $x^* + y^* \leq 1$. $1 - \alpha f'(z_x^*)$ where equality (A.3) holds, and $1 - \alpha f'(z_y^*)$ where equality (A.4) holds; see proof of Proposition 1. Therefore, $\frac{\partial x^*}{\partial \rho} > 0$ and $\frac{\partial y^*}{\partial \rho} < 0$, equivalently $\frac{\partial n^*}{\partial \rho} < 0$ and $\frac{\partial m^*}{\partial \rho} > 0$. $\frac{\partial h^*}{\partial \rho} > 0$ follows immediately from the definition of h^*. Moreover:

$$\frac{\partial x_n^*}{\partial \alpha} = \alpha \frac{\partial n^*}{\partial \alpha} - (1-\alpha)\frac{\partial h^*}{\partial \alpha} + \rho(m^* + n^* - 1)$$

Each term of the right-hand-side is positive, giving $\frac{\partial x_n^*}{\partial \alpha} > 0$. Likewise:

$$\frac{\partial x_n^*}{\partial \epsilon} = \alpha \frac{\partial n^*}{\partial \epsilon} - (1-\alpha)\frac{\partial h^*}{\partial \epsilon}$$

Each term of the right-hand-side is positive, giving $\frac{\partial x_n^*}{\partial \epsilon} > 0$, and thus a First-Order Stochastic Dominance change to the distribution of friend- ships p_k increases x_n^*. Next, with $m^*, n^* \to 1$ as $\underline{k} \to \infty$, we have:

$$\begin{aligned}
\lim_{\underline{k} \to \infty} x_m^* &= \alpha + (1-\alpha)\rho \\
\lim_{\underline{k} \to \infty} x_n^* &= \alpha + (1-\alpha)(1-\rho)
\end{aligned}$$

Lastly:

$$\begin{aligned}
\frac{\partial x_m^*}{\partial \rho} &= \alpha \frac{\partial m^*}{\partial \rho} + (1-\alpha)\frac{\partial}{\partial \rho}\left(\rho m^* + (1-\rho)(1-n^*)\right) \\
&= -\alpha \frac{\partial y^*}{\partial \rho} + (1-\alpha)\left(-\rho \frac{\partial y^*}{\partial \rho} + (1-\rho)\frac{\partial x^*}{\partial \rho}\right) + (1-\alpha)(m^* + n^* - 1) \\
&= -\frac{\partial y^*}{\partial \rho} + (1-\alpha)(1-\rho)\left(\frac{\partial x^*}{\partial \rho} + \frac{\partial y^*}{\partial \rho}\right) + (1-\alpha)(m^* + n^* - 1) \\
&= -\frac{\partial y^*}{\partial \rho} + (1-\alpha)(1-\rho)\left(\frac{1}{|J|}(1-\alpha)(1-(x^*+y^*))(f'(z_x^*) - f'(z_y^*))\right) \\
&\quad +(1-\alpha)(m^* + n^* - 1)
\end{aligned}$$

With $\frac{\partial y^*}{\partial \rho} < 0$, $f'(z_x^*) > f'(z_y^*)$ and $m^* + n^* > 1$, each term on the right-hand-side is positive, giving $\frac{\partial x_m^*}{\partial \rho} > 0$. Likewise:

$$\begin{aligned}
\frac{\partial x_n^*}{\partial \rho} &= \alpha\frac{\partial n^*}{\partial \rho} + (1-\alpha)\frac{\partial}{\partial \rho}\left((1-\rho)n^* + \rho(1-m^*)\right) \\
&= -\alpha\frac{\partial x^*}{\partial \rho} + (1-\alpha)\left(-(1-\rho)\frac{\partial x^*}{\partial \rho} + \rho\frac{\partial y^*}{\partial \rho}\right) - (1-\alpha)(m^*+n^*-1) \\
&= (1-\alpha)\frac{\partial y^*}{\partial \rho} - \alpha\frac{\partial x^*}{\partial \rho} - (1-\alpha)(1-\rho)\left(\frac{\partial x^*}{\partial \rho} + \frac{\partial y^*}{\partial \rho}\right) - (1-\alpha)(m^*+n^*-1) \\
&= (1-\alpha)\frac{\partial y^*}{\partial \rho} - \alpha\frac{\partial x^*}{\partial \rho} - (1-\alpha)(m^*+n^*-1) \\
&\quad -(1-\alpha)(1-\rho)\left(\frac{1}{|J|}(1-\alpha)(1-(x^*+y^*))(f'(z_x^*) - f'(z_y^*))\right)
\end{aligned}$$

With $\frac{\partial y^*}{\partial \rho} < 0$, $\frac{\partial x^*}{\partial \rho} > 0$, $m^*+n^* > 1$ and $f'(z_x^*) > f'(z_y^*)$, each term on the right-hand-side is negative, giving $\frac{\partial x_n^*}{\partial \rho} < 0$. □

Proof of Proposition 3. We construct the comparative statics of P^* with respect to α, ϵ and ρ using the proof of Proposition 2, and establish their signs to be negative, positive, and negative, respectively.

First:

$$\begin{aligned}
\frac{dP^*}{d\alpha} &= (m^*+n^*-1) + \alpha\left(\frac{\partial x^*}{\partial \alpha} + \frac{\partial y^*}{\partial \alpha}\right) \\
&= (m^*+n^*-1)\left(1 + \frac{\alpha}{|J|}(\rho f'(z_x^*) + (1-\rho)f'(z_y^*) - f'(z_x^*)f'(z_y^*))\right),
\end{aligned}$$

so, $\frac{dP^*}{d\rho} > 0$ if and only if:

$$|J| > -\alpha(\rho f'(z_x^*) + (1-\rho)f'(z_y^*) - f'(z_x^*)f'(z_z^*)). \tag{A.11}$$

From the proof of Proposition 2 we may calculate:

$$|J| = 1 - ((1-\alpha)(1-\rho)+\alpha)\, f'(z_x^*) - ((1-\alpha)\rho + \alpha)\, f'(z_y^*) + \alpha f'(z_x^*)f'(z_z^*),$$

and thus (A.11) is equivalent to:

$$1 > (1-\rho)f'(z_x^*) + \rho f'(z_y^*).$$

To show this inequality, from the proof of Proposition 2 we have:

$$\frac{\partial}{\partial h}H(x(h^*), y(h^*)) = (1-\rho)\frac{(1-\alpha)f'(z_x^*)}{1-\alpha f'(z_x^*)} + \rho\frac{(1-\alpha)f'(z_y^*)}{1-\alpha f'(z_y^*)} < 1.$$

If we define $g(z) \equiv \frac{(1-\alpha)z}{1-\alpha z}$, a convex function which is increasing for $z \geq 0$ and satisfies $g(0) = 0$ and $g(1) = 1$, this can be written:

$$(1-\rho)g(f'(z_x^*)) + \rho g(f'(z_y^*)) < 1.$$

By Jensen's inequality, we have:

$$1 > (1-\rho)g(f'(z_x^*)) + \rho g(f'(z_y^*)) > g((1-\rho)f'(z_x^*) + \rho f'(z_y^*)),$$

which implies $1 > (1-\rho)f'(z_x^*) + \rho f'(z_y^*)$.

Second:

$$\begin{aligned}
\frac{dP^*}{d\epsilon} &= -\alpha\left(\frac{\partial x^*}{\partial \epsilon} + \frac{\partial y^*}{\partial \epsilon}\right) \\
&= \frac{1}{|J|}\left((1-f'(z_y^*))\delta_x + (1-f'(z_x^*))\delta_y\right) \\
&= \frac{(\delta_x+\delta_y)}{|J|}\left(1-\left(\frac{\delta_y}{\delta_x+\delta_y}f'(z_x^*) + \frac{\delta_x}{\delta_x+\delta_y}f'(z_y^*)\right)\right).
\end{aligned}$$

From the Proposition 2, $\delta_x/\delta_y \geq \rho/(1-\rho)$, or equivalently $\rho \leq \frac{\delta_x}{\delta_x+\delta_y}$. With $1 > ((1-\rho)f'(z_x^*)) + \rho f'(z_y^*)$ from above and $f'(z_y^*) < f'(z_x^*)$, these give $1 > \frac{\delta_y}{\delta_x+\delta_y}f'(z_x^*) + \frac{\delta_x}{\delta_x+\delta_y}f'(z_y^*)$, and thus $\frac{dP^*}{d\epsilon} \geq 0$.

Third:

$$\begin{aligned}
\frac{dP^*}{d\rho} &= -\alpha\left(\frac{\partial x^*}{\partial \rho} + \frac{\partial y^*}{\partial \rho}\right) \\
&= \frac{(1-\alpha)(1-(x^*+y^*))}{|J|}\left(-(f'(z_x^*) - f'(z_y^*)))\right) < 0,
\end{aligned}$$

as $f'(z_y^*) < f'(z_x^*)$ shown in the proof of Proposition 2.

□

Proof of Proposition 4. The proof of Proposition 1 shows that there is a unique locally stable steady state $m^*(e_N, e_M), n^*(e_N, e_M)$ for any pair $(e_N, e_M) \in [0,1) \times [0,1)$. We now proceed to establish that $\exists \overline{C}$ such that $C'' > \overline{C}$ each player's objective is concave, the first order condition is sufficient for the optimal effort choice and the best responses functions are continuous. The second derivative of each player's objective are:

$$\rho\frac{\partial^2 m^*}{\partial e_M^2} - (1-\rho)\frac{\partial^2 n^*}{\partial e_M^2} - \frac{\partial^2 C(\rho e_M)}{\partial e_M^2} \tag{A.12}$$

$$(1-\rho)\frac{\partial^2 n^*}{\partial e_N^2} - \rho\frac{\partial^2 m^*}{\partial e_N^2} - \frac{\partial^2 C((1-\rho)e_N)}{\partial e_N^2} \tag{A.13}$$

Hence, a sufficient condition for concavity under the condition $C'' > \overline{C}$ is that the magnitude of $\frac{\partial^2 m^*}{\partial e_M^2}, \frac{\partial^2 n^*}{\partial e_M^2}, \frac{\partial^2 m^*}{\partial e_N^2}, \frac{\partial^2 n^*}{\partial e_N^2}$ are bounded. In each case, the second derivative is a bounded term (given by $\frac{1}{|J|^2}$ where $|J|$ is the determinant of the Jacobian of the system of equations given in equation B) multiplied by a second term that is a sequence of product, addition or subtraction operations involving $\alpha, \rho, e_m, e_n, f(z_n), f(z_m), f'(z_n), f'(z_m), f''(z_n), f''(z_m)$ and the first derivatives $\frac{\partial m^*}{\partial e_m}, \frac{\partial n^*}{\partial e_m}, \frac{\partial m^*}{\partial e_n}, \frac{\partial n^*}{\partial e_n}$. All these terms are themselves bounded and so the entire term comprises a finite sequence of product, addition and subtraction operations will also be bounded. The best response of each player may be written as

$$\rho\frac{\partial m^*}{\partial e_m} - (1-\rho)\frac{\partial n^*}{\partial e_m} - \rho C'(\rho e_m) = 0 \tag{A.14}$$

$$(1-\rho)\frac{\partial n^*}{\partial e_n} - \rho\frac{\partial m^*}{\partial e_n} - (1-\rho)C'((1-\rho)e_n) = 0 \tag{A.15}$$

and via the implicit function theorem we find that:

$$\begin{aligned}
\frac{\partial BR_m(e_n)}{\partial e_n} &= -\frac{\rho\frac{\partial^2 m^*}{\partial e_m \partial e_n} - (1-\rho)\frac{\partial^2 n^*}{\partial e_m \partial e_n}}{\rho\frac{\partial^2 m^*}{\partial e_m^2} - (1-\rho)\frac{\partial^2 n^*}{\partial e_m^2} - \rho C''(\rho e_m^*)} \\
\frac{\partial BR_n(e_m)}{\partial e_m} &= -\frac{(1-\rho)\frac{\partial^2 n^*}{\partial e_m \partial e_n} - \rho\frac{\partial^2 m^*}{\partial e_m \partial e_n}}{(1-\rho)\frac{\partial^2 n^*}{\partial e_n^2} - \rho\frac{\partial^2 m^*}{\partial e_n^2} - (1-\rho)C''((1-\rho)e_n^*)}
\end{aligned}$$

The same argument that bounded the value of the second derivatives also bounds $\frac{\partial^2 m^*}{\partial e_m \partial e_n}$ and $\frac{\partial^2 n^*}{\partial e_m \partial e_n}$. It is then straightforward to observe that for $\delta < 1$ $\exists \overline{C}$ such that

$$\left|\frac{\partial BR_m(e_n)}{\partial e_n}\right|, \left|\frac{\partial BR_n(e_m)}{\partial e_m}\right| < \delta$$

and hence the best response functions are continuous and there exists a unique Nash equilibrium (e_m^*, e_n^*) where they coincide. We can verify that the equilibrium will be interior by first observing that for any $e_m \in [0,1]$ the right-hand side of equation (25) is positive and the marginal costs of investment go to zero for $e = 0$ so $BR_n(0) > 0$. Second, we observe that when $e_n > 0$, the right-hand side of equation (24) is strictly positive and so $BR_m(e_n) > 0$ for $e_n > 0$. These two observations rule out any equilibria where either player invests 0. Finally, our assumption on the convexity of the cost function guarantees that $e_n = 1$ or $e_m = 1$ is not part of an equilibrium.

First, observe in equation (A.18) that a steady state is increasing in the mass-market investment e_m and decreasing in the niche investment e_n.

We now proceed to establish that the niche-market player invests more than the mass-market player when $\alpha = 0$. In this case, the steady state relationships for m, n and h may be written as:

$$\begin{aligned}
n^* &= 1 - (1-e_n) f(h^*) && \text{(A.16)} \\
m^* &= 1 - (1-e_m) f(1-h^*) && \text{(A.17)} \\
h^* &= \rho - \rho(1-e_m) f(1-h^*) + (1-\rho)(1-e_n) f(h^*) && \text{(A.18)}
\end{aligned}$$

First, for $\rho e_m = (1-\rho)e_n = \mu$ and $h^* = \frac{1}{2}$ the right-hand side of equation (A.18) is greater than $\frac{1}{2}$. Then, using the properties of the right-hand side of equation (A.18) (mapping $[0,1] \to [0,1]$, continuous, positive first and third derivatives) shown in the proof of Proposition 1 we may conclude that $h^*(\frac{\mu}{\rho}, \frac{\mu}{1-\rho}) > \rho$.

Second, observe in equations (A.16), (A.17) and (A.18) that a steady state is increasing in the mass-market investment e_m and decreasing in the niche investment e_n. Now, to establish the results by way of contradiction, suppose $(1-\rho)\, e_n^* < \rho e_m^*$. The two points above then imply that the steady state $h^*(e_m^*, e_n^*) > \frac{1}{2}$. However, this is a contradiction of the first order conditions for each player (equations (24) and (25)), where, $h^*(\rho e_m^*, (1-\rho)e_n^*) > \frac{1}{2} \Rightarrow (1-\rho)\, e_n^* > \rho e_m^*$. Therefore, we conclude that $(1-\rho)\, e_n^* > \rho e_m^*$.

To show that $h^* > \frac{1}{2}$, by way of contradiction suppose this was not the case $h^* < \frac{1}{2}$. The first order conditions of the two players equations for the case $\alpha = 0$ simply to:

$$\begin{aligned}
C'(\rho e_m^*) &= \frac{f(1-h^*)}{\Delta} && \text{(A.19)} \\
C'((1-\rho)\, e_n^*) &= \frac{f(h^*)}{\Delta} && \text{(A.20)}
\end{aligned}$$

These immediately implies that $(1-\rho)\, e_n^* < \rho e_m^*$ if $h^* < \frac{1}{2}$ which contradicts our earlier result $(1-\rho)\, e_n^* > \rho e_m^*$. Thus establishing that $h^* > \frac{1}{2}$. $\square$

Proof of Proposition 5. Choose any A such that $\rho e_m^* < A < (1-\rho)\, e_n$. Now write $\widetilde{n}\,(h, A) = 1 - \left(1 - \frac{A}{1-\rho}\right) f(h)$ and $\widetilde{m}\,(h, A) = 1 - \left(1 - \frac{A}{\rho}\right) f\,(1-h)$. We can find $\frac{\partial \widetilde{n}}{\partial A}$ and $\frac{\partial \widetilde{m}}{\partial A}$ as $\frac{1}{1-\rho} f(h)$ and $\frac{1}{\rho} f\,(1-h)$ respectively and can conclude that the function $H = \rho \widetilde{m}\,(h, A) + (1-\rho)\, \widetilde{n}\,(h, A)$ is decreasing in A because $\frac{\partial H}{\partial A} = f\,(1-h) - f(h) < 0$ for $h > \frac{1}{2}$. Now, we can conclude (using the properties of H shown in Proposition 1) that the value $h^*\,(A, A)$ that satisfies $\rho \widetilde{m}\,(h^*, A) + (1-\rho)\,(1 - \widetilde{n}\,(h^*, A)) = h^*$ is also decreasing in A. Finally, to establish the result, we note that $\rho e_m^* < A < (1-\rho)\, e_n^*$ and so increasing e_n from $\frac{A}{1-\rho}$ to e_n^* and decreasing e_m from $\frac{A}{\rho}$ to e_m^* will reduce h so $h^*\,(e_m^*, e_n^*) < h^*\,(A, A) < h^*\,(0, 0)$. $\square$

Proof of Proposition 6. Define $\hat{\alpha}_m(\alpha_n)$ as the implicit solution for α_m in equation (32) and $\hat{\alpha}_n(\alpha_m)$ as the implicit solution for α_n in equation (33). For sufficient convexity of D both are continuous functions $[0,1] \to [0,1]$. Hence there exists a point on $[0,1] \times [0,1]$ where equations 32 and 33 are satisfied. Moreover, when $E[k] > \frac{1}{1-\rho}$ we have $\hat{\alpha}_m(0), \hat{\alpha}_n(0) > 0$ and $\hat{\alpha}_m(1), \hat{\alpha}_n(1) < 1$ such that the equilibrium is interior. Finally, the returns to homophily for a niche individual shown on the right-hand side of equation 33 is greater than the returns to homophily for mass-market individuals shown in equation 32 since $\rho > \frac{1}{2}$. This immediately implies that the niche individuals will exhibit greater homophily $\alpha_n^* > \alpha_m^*$ in equilibrium. $\square$

Proof of Proposition 7. We work with the properties of the function $y_{M,t}\,(y_{M,t-1})$. First, observe that there are fixed points at 0 and 1 where $y_{M,t}\,(y_{M,t-1}) = y_{M,t-1}$. Second, the derivative is given by:

$$\begin{aligned} \frac{\partial y_{M,t}}{\partial y_{M,t-1}} &= \sum_k p_k \left(\rho \left[k\,(1 - y_{M,t-1})^{k-1} \right] + (1-\rho) \left[\frac{k}{2} \left(\frac{1 + y_{M,t-1}}{2} \right)^{k-1} + \frac{k}{2} \left(\frac{1 - y_{M,t-1}}{2} \right)^{k-1} \right] \right) \\ &> 0 \end{aligned}$$

We find that the function $y_{M,t}\,(y_{M,t-1})$ is increasing for all $y_{M,t-1} \in [0,1]$. Taking limits, we also find:

$$\lim_{y_{M,t-1} \to 0^+} \frac{\partial y_{M,t}}{\partial y_{M,t-1}}\,(y_{M,t-1}) = \rho \mathbb{E}[k] + (1-\rho) \sum_{k=1}^{\infty} p_k k \left(\frac{1}{2} \right)^{k-1} = C_{LR}$$

$$\lim_{y_{M,t-1} \to 1^-} \frac{\partial y_{M,t}}{\partial y_{M,t-1}}\,(y_{M,t-1}) = p_1 + \frac{1-\rho}{2}\,(\mathbb{E}[k] - p_1) = C_M.$$

Third, consider the curvature

$$\begin{aligned} \frac{\partial^2 y_{M,t}}{\partial y_{M,t-1}^2} &= \sum_k p_k \left(-\rho \frac{1}{2} \left[k\,(k-1)\,(1 - y_{M,t-1})^{k-2} \right] \right. \\ &+ (1-\rho) \left[\frac{k\,(k-1)}{4} \left(\frac{1 + y_{M,t-1}}{2} \right)^{k-2} - \frac{k\,(k-1)}{4} \left(\frac{1 - y_{M,t-1}}{2} \right)^{k-2} \right] \end{aligned}$$

(A.22)

This expression is increasing in $y_{M,t-1}$ for all $y_{M,t-1} \in (0,1)$, is positive as $y_{M,t-1} \to 1^-$ by $p_k > 0$ for some $k > 2$ and negative as $y_{M,t-1} \to 0^+$. Hence there exists a threshold value $\widehat{y}_{M,t-1}$ such that the function $y_{M,t}\left(y_{M,t-1}\right)$ is concave for $y_{M,t-1} < \widehat{y}_{M,t-1}$ and convex for $y_{M,t-1} > \widehat{y}_{M,t-1}$ for $y_{M,t-1} \in (0,1)$. Given these properties and $y_{M,t}(y) = y$ for $y = 0, 1$ we conclude that a stable steady state equilibrium exists and is unique for $y_M^* \in [0,1]$. Moreover, we have the following mutually exclusive cases:

1. $y_M^* = 0$ if $C_{LR} \leq 1$,
2. $y_M^* = 1$ if $C_M \leq 1$,
3. $0 < y_M^* < 1$ otherwise.

where $C \equiv p_1 + \frac{1-p_1}{\gamma} > 1$, where $\gamma = 1 - \rho$. We now proceed to the final two components of the Proposition. We first develop the following lemma.

Lemma A8. *If $p_k > 0$ for some $k > 2$, then $y_{M,t}\left(\rho\right) > \rho$ for all $\rho \geq 1/3$.*

Proof. Starting with the condition:

$$y_{M,t}\left(\rho\right) = \sum_k p_k \left[\rho\left[1-\left(1-\rho\right)^k\right] + \left(1-\rho\right)\left[\left(\frac{1+\rho}{2}\right)^k - \left(\frac{1-\rho}{2}\right)^k\right]\right] > \rho, \quad \text{(A.23)}$$

Now consider the term:

$$\rho\left[1-\left(1-\rho\right)^k\right] + \left(1-\rho\right)\left[\left(\frac{1+\rho}{2}\right)^k - \left(\frac{1-\rho}{2}\right)^k\right] \quad \text{(A.24)}$$

We show this term is weakly positive for all $k \geq 2$ and $\rho \geq \rho$ and holding strictly for some k where $p_k > 0$. With some algebra, this term can be rearranged to:

$$\rho + \left(1-\rho\right)\left[\left(\frac{1+\rho}{2}\right)^k - \left(\frac{1-\rho}{2}\right)^k - \rho\left(1-\rho\right)^{k-1}\right] \quad \text{(A.25)}$$

Which is strictly positive whenever:

$$\left(\frac{1}{2}\right)^k \left(\left(\frac{1+\rho}{1-\rho}\right)^k - 1\right) > \frac{\rho}{1-\rho}.$$

and equal to zero whenever the condition holds with equality. For $k = 1$ the relationship holds with equality. Now taking the derivative of the left-hand side with respect to k we find:

$$\left[\ln\left(\frac{1+\rho}{1-\rho}\right) + \ln\left(\frac{1}{2}\right)\right]\left(\frac{1}{2}\right)^k \left(\frac{1+\rho}{1-\rho}\right)^k - \ln\left(\frac{1}{2}\right) \quad \text{(A.26)}$$

Which is strictly positive for all $\rho \geq \frac{1}{3}$. $\square$

By the above lemma, $y_{M,t}(\rho) > \rho$ for $\rho \geq \frac{1}{3}$, and thus, combined with the other characteristics of $y_{M,t}(y_{M,t-1})$ introduced above, $y_{M,t}(y_{M,t-1})$ will pass the 45-degree line to the right of ρ.

We now show the final statement of the proposition. This follows from noting that:

$$\lim_{\rho \to 0^+} C_{LR} = \sum_{k=1} k \left(\frac{1}{2}\right)^k < 1 \tag{A.27}$$

provided that $p_k > 0$ for some $k > 2$, hence there will be steady state below ρ in this neighborhood.

□

B Derivation of the first-order conditions for "influencers" (Section 4.2)

Taking the derivative of equations (22) and (23) with respect to e_M and e_N respectively produces the following first order conditions for each player:

$$\rho\frac{\partial m^*}{\partial e_M} - (1-\rho)\frac{\partial n^*}{\partial e_M} - \rho C'(\rho e_M) = 0 \tag{B.1}$$

$$(1-\rho)\frac{\partial n^*}{\partial e_N} - \rho\frac{\partial m^*}{\partial e_N} - (1-\rho)C'((1-\rho)e_N) = 0 \tag{B.2}$$

We use the multivariate implicit function theorem to find the partial derivatives of the steady-state quantities with respect to the investments. Define z_x and z_y as in the proof of Proposition 1, that is, $z_x := (1-\alpha)\,h + \alpha x$ and $z_y \equiv (1-\alpha)\,(1-h) + \alpha y$, where $x \equiv 1-n$ and $y := 1-m$, which are defined in equations (A.1) and (A.2). Also, $h = (1-\rho)x + \rho(1-y)$ and $a_N \equiv 1-e_N$ and $a_M \equiv 1-e_M$. Then, the system of equations (20) and (21) can be written as:

$$\begin{bmatrix} g_x(x,y) \\ g_y(x,y) \end{bmatrix} = \begin{bmatrix} x - a_N \sum_k p_k \left[(1-\alpha)h + \alpha x\right]^k \\ y - a_M \sum_k p_k \left[(1-\alpha)(1-h) + \alpha y\right]^k \end{bmatrix}.$$

where $g_x(x^*, y^*) = 0$ and $g_y(x^*, y^*) = 0$ defines the steady state. The Jacobian of the system is:

$$J = \begin{bmatrix} 1 - ((1-\alpha)(1-\rho) + \alpha)a_N f'(z_x) & (1-\alpha)\rho a_N f'(z_x) \\ (1-\alpha)(1-\rho)a_M f'(z_y) & 1 - ((1-\alpha)\rho + \alpha)a_M f'(z_y) \end{bmatrix}$$

which has inverse:

$$J^{-1} = \frac{1}{|J|}\begin{bmatrix} 1 - ((1-\alpha)\rho + \alpha)a_M f'(z_y) & -(1-\alpha)\rho a_N f'(z_x) \\ -(1-\alpha)(1-\rho)a_M f'(z_y) & 1 - ((1-\alpha)(1-\rho) + \alpha)a_N f'(z_x) \end{bmatrix}.$$

The determinant $|J| > 0$ by stability of the steady state. The comparative statics with respect to a_M and a_N are then given by:

$$\begin{bmatrix} \frac{\partial x^*}{\partial a_M} \\ \frac{\partial y^*}{\partial a_M} \end{bmatrix} = -J^{-1}\begin{bmatrix} \frac{\partial g_x(x^*,y^*)}{\partial a_M} \\ \frac{\partial g_y(x^*,y^*)}{\partial a_M} \end{bmatrix} = -J^{-1}\begin{bmatrix} 0 \\ -f(z_y) \end{bmatrix} \tag{B.3}$$

$$= \frac{f(z_y)}{|J|}\begin{bmatrix} -(1-\alpha)\rho a_N f'(z_x) \\ 1 - ((1-\alpha)(1-\rho) + \alpha)a_N f'(z_x) \end{bmatrix}; \tag{B.4}$$

$$\begin{bmatrix} \frac{\partial x^*}{\partial a_N} \\ \frac{\partial y^*}{\partial a_N} \end{bmatrix} = -J^{-1}\begin{bmatrix} \frac{\partial g_x(x^*,y^*)}{\partial a_N} \\ \frac{\partial g_y(x^*,y^*)}{\partial a_N} \end{bmatrix} = -J^{-1}\begin{bmatrix} -f(z_x) \\ 0 \end{bmatrix} \tag{B.5}$$

$$= \frac{f(z_x)}{|J|}\begin{bmatrix} 1 - ((1-\alpha)\rho + \alpha)a_M f'(z_y) \\ -(1-\alpha)(1-\rho)a_M f'(z_y) \end{bmatrix}. \tag{B.6}$$

It is straightforward to observe that $\frac{\partial x^*}{\partial a_N} = \frac{\partial n^*}{\partial e_N}, \frac{\partial x^*}{\partial a_M} = \frac{\partial n^*}{\partial e_M}, \frac{\partial y^*}{\partial a_N} = \frac{\partial m^*}{\partial e_N}$ and $\frac{\partial y^*}{\partial a_M} = \frac{\partial m^*}{\partial e_M}$ and so we may substitute the relationships in equations (B.6) and (B.4) into the first order conditions in equations (B.1) and (B.2) for the respective partial derivatives. Rearranging and defining $\Delta \equiv |J|$ establishes equations (24) and (25) in the main text.

C Additional results

C.1 Influencers (Section 4.2)

Recall that $\underline{k} := \min_k\{k : p_k > 0\}$. We have the following result:

Proposition C1. *When $\alpha > 0$, then, Polarization $P(e_m^*, e_n^*) \to \alpha$ as $e_m^* \to 1$ and $e_n^* \to 1$. $e_m^* \to 0$, $e_n^* \to 0$ and $P(e_m^*, e_n^*) \to \alpha$ as $\underline{k} \to \infty$.*

Proof. To establish the first result, it is straightforward to observe that as $e_m, e_n \to 1$ then the solution to the system of equations 20 and 21 goes to $m^* = n^* = 1$ and polarization $P(e_m^*, e_n^*) = \alpha(m^* + n^* - 1) \to \alpha$. To establish the second result, we observe that when $f(x) = x^k$ then $lim_{k\to\infty} f(x) \to 0$ for all $x < 1$ so again the solution to the system of equations 20 and 21 goes to $m^* = n^* = 1$ and polarization $P(e_m^*, e_n^*) = \alpha(m^*+n^*-1) \to \alpha$ for $\underline{k} \to \infty$. Finally, we note that this also implies that $lim_{\underline{k}\to\infty} h^* = \rho$ and so the right hand side of equations A.19 and A.20 goes to zero. This establishes that the equilibrium investments e_m^*, e_n^* also go to zero in this limit. $\square$

C.2 Homophily (Section 4.3)

We have the following results:

Proposition C2. *For a given pair of homophily levels α_m and α_n, there is a unique stable steady-state equilibrium (m^*, n^*).*

Proof. The proof of Proposition 1 includes the case of different homophily parameters α_m, α_n. $\square$

Proposition C3. *There exists an equilibrium with zero homophily if and only if $k \leq \frac{1}{1-\rho}$.*

Proof. When $E[k] \leq \frac{1}{1-\rho}$ then $m^*(0,0) = h^*(0,0) = 1$ and $n^*(0,0) = 0$. Moreover, this satisifies the first order conditions given in equations (32) and (33). To show the converse, suppose there is an equilibrium with zero effort. This implies that the right-hand side of equations (32) and (33) are 0. The only steady state where this is possible is $m^* = 1, n^* = 0$ which from Proposition 1 implies our result. $\square$

C.3 Useful Lemma

Lemma C9. *The sign of the partial derivatives of the mapping f are*

$$\begin{aligned}
\frac{\partial f_x}{\partial x_{t-1}} &\geq 0 \\
\frac{\partial f_x}{\partial y_{t-1}} &\leq 0 \\
\frac{\partial f_y}{\partial x_{t-1}} &\leq 0 \\
\frac{\partial f_y}{\partial y_{t-1}} &\geq 0
\end{aligned}$$

where a sufficient condition for the inequality to be strict is $w_{n,t-1}, w_{m,t-1} \neq 0, 1$.

Proof. First, we present a useful result for the first derivative of a function of the form: □

$$\begin{aligned} f\left(d\left(k\right),w\right) &= \sum_{k} p_k \sum_{j=0}^{d(k)-1} \binom{k}{j} \left(1-w\right)^{k-j} \left(w\right)^{j} \\ &= \sum_{k} p_k \sum_{j=0}^{d(k)-1} \frac{k!}{(k-j)!j!} \left(1-w\right)^{k-j} \left(w\right)^{j} \end{aligned}$$

Taking the first derivative we find:

$$\frac{\partial f}{\partial w} = \sum_{k} p_k \sum_{j=0}^{d(k)-1} \frac{k!}{(k-j)!\,(j-1)!} \left(1-w\right)^{k-j} \left(w\right)^{j-1} - \frac{k!}{(k-j-1)!j!} \left(1-w\right)^{k-j-1} \left(w\right)^{j}$$

Consider two consecutive terms in the sum $j = z, z+1$. For $j = z$

$$\frac{k!}{(k--1)!\,(z-1)!} \left(1-w\right)^{k-z} \left(w\right)^{z-1} - \frac{k!}{(k-z-1)!z!} \left(1-w\right)^{k-z-1} \left(w\right)^{z}$$

and for $j = z+1$

$$\frac{k!}{(k-z-1)!\,(z)!} \left(1-w\right)^{k-z-1} \left(w\right)^{z} - \frac{k!}{(k-z)!\,(z+1)!} \left(1-w\right)^{k-z} \left(w\right)^{z+1}$$

where the second term for $j = z$ cancels out the first term for $j = z+1$. Hence, the evaluation of the summation

$$\sum_{j=0}^{d(k)-1} \frac{k!}{(k-j)!\,(j-1)!} \left(1-w\right)^{k-j} \left(w\right)^{j-1} - \frac{k!}{(k-j-1)!j!} \left(1-w\right)^{k-j-1} \left(w\right)^{j}$$

results in only second term for the upper limit $j = d\left(k\right) - 1$, note that the first term for the lower limit is 0. The derivative is given by:

$$\frac{\partial f}{\partial w}\left(d\left(k\right),w\right) = \sum_{k} p_k \left(-\frac{k!}{(k-d\left(k\right))!\,(d\left(k\right)-1)!} \left(1-w\right)^{k-d(k)} \left(w\right)^{d(k)-1}\right)$$

Using the chain rule we evaluate $\frac{\partial f_x}{\partial x_{t-1}} = \frac{\partial f_x}{\partial w_{n,t-1}} \frac{\partial w_{n,t-1}}{\partial x_{t-1}}$ and $\frac{\partial f_x}{\partial y_{t-1}} = \frac{\partial f_x}{\partial w_{n,t-1}} \frac{\partial w_{n,t-1}}{\partial y_{t-1}}$ where the above result can be used to evaluate

$$\frac{\partial f_x}{\partial w_{n,t-1}} = \sum_{k} p_k \left(-\frac{k!}{(k-d\left(k\right))!\,(d\left(k\right)-1)!} \left(1-w_{n,t-1}\right)^{k-d(k)} \left(w_{n,t-1}\right)^{d(k)-1}\right)$$

and we can readily observe that:

$$\begin{aligned} \frac{\partial w_{n,t-1}}{\partial x_{t-1}} &= -\alpha_n - \left(1-\alpha_n\right)\left(1-\rho\right) < 0 \\ \frac{\partial w_{n,t-1}}{\partial y_{t-1}} &= \left(1-\alpha_n\right)\rho > 0 \end{aligned}$$

Thus, for $0 < w_{n,t-1} < 1$ $\frac{\partial f_x}{\partial w_{n,t-1}} < 0$ and hence

$$\begin{aligned} \frac{\partial f_x}{\partial x_{t-1}} &> 0 \\ \frac{\partial f_x}{\partial y_{t-1}} &< 0 \end{aligned}$$

Similarly,

$$\frac{\partial f_y}{\partial w_{m,t-1}} = \sum_k p_k \left(- \frac{k!}{(k - d(k))!\,(d(k) - 1)!} (1 - w_{m,t-1})^{k-d(k)} (w_{m,t-1})^{d(k)-1} \right)$$

and

$$\begin{aligned} \frac{\partial w_{m,t-1}}{\partial x_{t-1}} &> (1 - \alpha_m)(1 - \rho) \\ \frac{\partial w_{m,t-1}}{\partial y_{t-1}} &< -\alpha_m - (1 - \alpha_m)\rho \end{aligned}$$

so, for $0 < w_{n,t-1} < 1$ $\frac{\partial f_y}{\partial w_{m,t-1}} < 0$ and hence

$$\begin{aligned} \frac{\partial f_y}{\partial x_{t-1}} &< 0 \\ \frac{\partial f_y}{\partial y_{t-1}} &> 0 \end{aligned}$$

$\square$