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Abstract

We model Artificial Intelligence (AI) as self-learning capital: Its productivity rises by its use and by training with data. In a three-sector model, an AI sector and an applied research (AR) sector produce intermediates for a final good firm and compete for high-skilled workers. AR development benefits from inter-temporal spillovers and knowledge spillovers of agents working in AI, and AI benefits from application gains through its use in AR. The economy converges to a steady state and displays a sequence of four tipping points in the transition: First, entrepreneurs and second, high-skilled workers drive the accumulation of self-learning AI, which will later be re-balanced by reverse movements to the AR sector (third and fourth). In the steady state, AI accumulates autonomously due to application gains from AR. We show that suitable tax policies induce socially optimal movements of workers between sectors. In particular, we provide a macroeconomic rationale for an AI-tax on AI-producing firms, once the accumulation of AI has sufficiently progressed.

JEL Classification: E24, E13, O33, O41

Keywords: applied research, artificial intelligence, growth, Labor Market Transitions, Learning Capital, Tech Giants

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Artificial Intelligence as Self-Learning Capital*

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1 Introduction

Motivation

Artificial Intelligence (AI) is on the rise: The last two decades have seen a rapid increase of transistors in electronic devices and households using the internet and social networks. The resulting computational power and availability of data have brought up what some call the "AI revolution" (Makridakis, 2017). After several periods called "AI winter" (Floridi, 2020; Hendler, 2008), when AI received little attention and funding, IT specialists now face an abundance of data and can use algorithms to perform increasingly complicated tasks. These tasks range from facial recognition to composing pieces of music and painting, as well as developing blueprints for products. New, powerful computers perform such operations in seconds.

AI is the set of learning algorithms and their subsequent deployment in software tools and digital platforms. Many consumer products rely heavily on AI, such as booking portals, streaming websites and smartphones, to name only a few. Also, the world of production and commerce is being reshaped by AI at the hands of so called "Tech Giants", such as Facebook, Google or Apple. Figure 1 displays the evolution of revenues for five major Tech Giants in the Western world from 2005-2020:[1] Apple, Alphabet, Amazon, Facebook, and Microsoft, showing that their revenues increased rapidly. After 2015, we find an exponential increase in revenues for Amazon, Alphabet and Facebook, while Microsoft is showing a constant linear trend over the entire time period. Apple's revenues had a drastic increase around 2010, with linear progress before and after. Furthermore, the AI Index Report (Zhang et al., 2021) shows that the AI hiring rate[2] increased persistently in the years 2016-2020.

As early as 1955, John McCarthy states the following in a proposal on the development on AI: *"Probably a truly intelligent machine will carry out activities which may best be described as self-improvement"* (McCarthy et al., 1955, p.1). Nowadays, AI is an increasingly diverse field, ranging from picture recognition of functional magnetic resonance imaging in the medical sector, to smart robots in industrial production or personalized advertising in business marketing. Nevertheless, the human-machine relationship still plays a significant role, as many human and manual factors are key to develop machines. However, the extent to which autonomous and self-learning systems can be built is increasingly investigated, e.g. in the field of automatic speech recognition and natural language processing by AI programs such as Alexa, Siri or Google Assistant (Ponnusamy et al., 2020).

[1]Revenue and Inflation data is taken from https://www.macrotrends.net [Accessed on 05.11.21]. We use the year 2005 as the base year for deflating USD revenues.

[2]The (country-specific) AI hiring rate is defined as the number of LinkedIn members, who include AI knowledge in their skill set and obtain a new position at a new employer in a month, divided by the total number of LinkedIn members (Zhang et al., 2021).

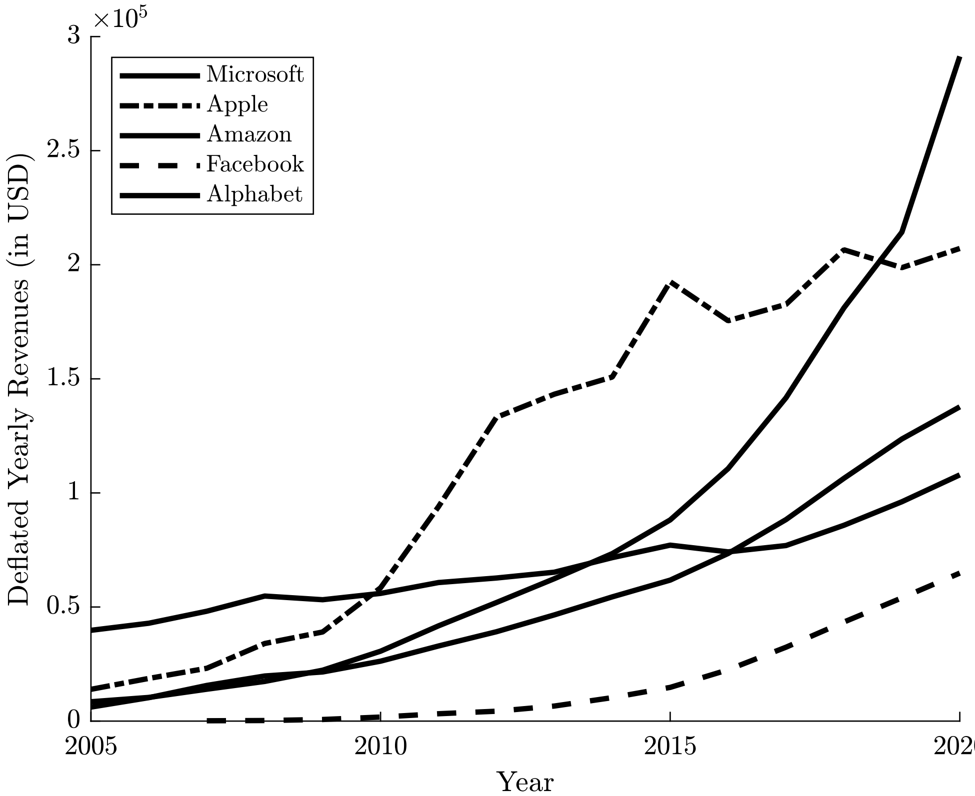

Figure 1: Yearly Revenues of Tech Giants (in $).

AI optimists expect that machine learning will complement human skills in the workplace, with collaborations between AI and humans reaching new heights of productivity. While AI performs tasks that require consistency and the processing of large amounts of data, humans can concentrate on the tasks for which they (still) have a comparative advantage, such as creative thinking, intuition and social interaction. A less optimistic view is that AI will reinforce inequality. With AI being able to perform more and more cognitive tasks, some experts think that AI puts jobs and professions at risk which have not been automated yet and were considered unfit for automation until now.[3]
The preceding considerations entail three main questions: Does AI drive technological progress and growth and if yes, how? Is there too much or too little AI? How could policy foster socially desirable transitions to AI-based economies and correct ensuing mis-allocations of high-skilled workers across AR and AI?

Model

To address these questions, we construct a growth model with self-learning AI. The model comprises three sectors, a final good sector, an AI sector and an AR sector. The final good sector uses AI and AR as intermediates. While AR is conducted in a perfectly competitive environment, there is monopolistic competition in the AI sector. The AR

[3]The dangers of AI for fairness, privacy, democratic institutions as well as various adverse social consequences of AI are discussed by e.g. Acemoglu and Restrepo (2021) and Gersbach (2020).

sector produces blueprints for the development and commercialization of products. Firms in the AI sector, which we call Tech Giants, produce AI algorithms whose quality increases through their application in AR, entailing that AI can be interpreted as a type of self-learning capital. A final good is produced, using labor, AI, AR and physical capital. The three stocks—AI, AR and physical capital—increase through application, research and savings, respectively. We focus on the development of an economy, starting from a state without AI, to examine workers' transitions between sectors, the dynamics of the development of AI and AR, and to specify a Balanced Growth Path (BGP).

Our model focuses primarily on labor market decisions of agents with heterogeneous skills. Such decisions yield transitions between sectors, resulting from increasing levels of AI. We assume three different types of agents. First, we have entrepreneurial-skilled individuals who are able to work in all sectors. They are called "entrepreneurs", as they are qualified to start running AI firms. Second, high-skilled individuals can work in all sectors, but are lacking entrepreneurial skills. Third, low-skilled workers can only work in the final good firm. Agents can decide in which sector to work, provided that they have a sufficient skill-endowment to work in the AI sector or the AR sector. Hence, the AI sector and AR sector compete for entrepreneurial-skilled and high-skilled workers. The model involves two-sided spillovers between the AI sector and AR sector. On the one hand, AI benefits from its application in AR, where AI algorithms can exploit their self-learning capabilities. On the other hand, there are knowledge spillovers of agents working in the AI sector on the development of AR.

Findings and Implications

Our analysis reveals three insights. First, until the economy converges to a steady state, the economy runs through five regimes characterized by four tipping points. Starting from a point where all workers are employed in final good production, entrepreneurial-skilled individuals first have an incentive to move to the AI sector, as they receive a profit share by running AI firms and thus can receive an overall income that is higher than elsewhere. Once AI has reached a certain level, which leads to increasing wages in this sector, an employment in AI becomes attractive for high-skilled individuals. Subsequently, when the self-learning potential of AI is sufficiently exhausted, employment in AR becomes more attractive and high-skilled individuals transition from AI to AR. Finally, also entrepreneurs move from AI to AR and the economy converges to a steady state in which all high-skilled workers and entrepreneurs are employed in AR.

Second, the social planner's solution yields transitions between sectors at tipping points that differ from the ones in a decentralized economy. This is the case due to five phenomena: (i), A slow-fast *Wage Effect*, which is the markdown on wages resulting from monopolistic distortions in the AI sector in a decentralized economy. It delays the entry of high-skilled workers in the AI sector, but also leads to their premature transition from

AI to AR. (ii), Entrepreneurs benefit from AI profits, which is called the *Profit Effect* and which motivates entrepreneurs to start investing in AI. For entrepreneurs, the *Profit Effect* neutralizes the *Wage Effect*. Hence, entrepreneurs' transitions are not distorted despite the monopolistic competition in the AI sector. (iii), Agents in a decentralized economy do not take into account *Knowledge Spillovers* from AI on AR development. This causes a delay in the transition of entrepreneurial-skilled and high-skilled workers to AI and a too fast transition back to AR. (iv), The social planner takes into account that AI can increasingly use its self-learning capabilities with a growing stock of AR, which we call *Application Gains*. (v), The social planner takes into account the influence of the AR stock today on the AR stock tomorrow, which we define as *Inter-temporal Spillovers*. The two effects (iv) and (v) result in earlier allocations of high-skilled workers and entrepreneurs from AI to AR in the social planner solution than in a decentralized economy. Taken all five together, the social planner's solution converges to the steady state sooner than in a decentralized economy.
Third, socially optimal transitions between the sectors can be implemented by three policy instruments: (i) an early development of AI can be promoted through a subsidy that corrects for the *Wage Effect* and the *Knowledge Spillovers*, (ii) the *Profit Effect* can be corrected by a profit tax, (iii) once a sufficient level of AI has been reached, the AI subsidization turns into an AI-tax. This tax ensures that agents do not remain in AI software development for too long, but move to the AR sector at the point when AI can grow to a sufficient degree due to self-learning, without the help of human labor. An AI-tax optimally balances the *Application Gains* of AI through a rising AR stock against the *Inter-temporal Spillovers* that fosters future AR from rising AR today. To sum up, we provide a macroeconomic rationale for imposing a tax on the price for AI (an AI-tax) once the accumulation of AI has sufficiently progressed.

Organization
The paper is organized as follows: In Section 2, we relate our research to the literature. In Section 3, we introduce the model. After defining equilibrium conditions in the economy in Section 4 in a decentralized economy, we focus on potential tipping points to set a light on transitional dynamics between the sectors in Section 5. Subsequently, we define long run steady state conditions in Section 6 and consider the social planner's solution of our optimization problem in Section 7. We investigate possible policy interventions in Section 8, and present a numerical example to illustrate our model in Section 9. Subsequently, we provide potential extensions in Section 10, discuss the results and interpretations of our model in Section 11 and conclude in Section 12.

2 Relation to the Literature

Definition of AI and AR

Alan Turing, one of the pioneers in the field of computer science and machine intelligence, was asking himself as early as 1950 to what extent a “thinking machine” could be developed (Turing, 1950). Above all, the question of the extent to which computers can imitate human activities is still one of the most challenging questions in this field of research. Understanding the capability of machines to learn and to form habits of ”intelligent” behavior has been in the spotlight of many fields such as information science, psychology or philosophy (Michalski et al., 2013).

We define AI as the development of algorithms and their deployment in software tools and digital platforms that can simplify tasks such as logical or search operations, pattern recognition, inference or planning. In contrast to pre-programmed machines whose actions are based on logic and ”brute-force” solution formation (Makridakis, 2017), we focus on the attribute that programs can improve their output (i.e. learn) from experience: So called ”machine learning systems” are designed to improve themselves over time (Brynjolfsson et al., 2017). Especially machine learning algorithms like ”neural networks” can be trained and improved through repeated application using real data. Typical applications of AI are for instance speech recognition, computer vision, natural language processing or heuristic classification. The level of AI can increase itself by e.g. deep machine learning or reinforcement learning (Lu, 2020). Thus, AI can be interpreted as a kind of self-learning and intangible capital.

AR is defined by the OECD (2002) [p. 30] as the ”original investigation undertaken in order to acquire new knowledge. It is, however, directed primarily towards a specific practical aim or objective” and is a major element of Research and Development (*R&D*). Moreover, ”the results of AR are intended primarily to be valid for a single or limited number of products, operations, methods or systems. In short, AR commercializes ideas. The knowledge or information derived from it is often patented but may be kept secret” (OECD, 2002, p. 78). The economic implications of ”learning by doing” by human beings were already assessed as early as 1962 by Arrow (1962). He points out that ”learning is a product of experience” [p.155] and highlights its importance for productivity increases. Hatch and Dyer (2004) come to the conclusion that human capital has a significant impact on learning, but is most valuable when it is firm-specific and inimitable. Indeed, the learning feature of human capital guarantees a competitive advantage for a firm as long as the application of what was learned remains in the firm. In contrast to learning of human capital, we focus on the self-learning features of intangible AI assets i.e. algorithms that can be further improved only by their widespread use in other firms. In our model, we presume that the higher the stock of AR, the larger the application area for AI and the more the development of AI will benefit.

AI in the Literature

Many predictions have been made about the development of AI and its effect on technological, social-economic and general factors (Makridakis, 2017). Referring to economic literature, we especially contribute to two strands of literature related to AI.

On the one side, there is automation literature in which AI is understood as another form of automation, i.e. as the replacement of human labor by physical capital (Autor and Salomons, 2018). Yet, the question arises which impact an increase in learning algorithms, which can take over many human activities, will have on the labor market (Furman and Seamans, 2019). The industrial and digital revolutions have already had a long-lasting impact on labor markets and on the significance of different industries. The emergence of AI will especially affect the tasks performed by middle-skilled or high-skilled workers (Lu, 2020). Graetz and Michaels (2018) come to a similar conclusion regarding the effect of "autonomous, flexible and versatile" robots on the labor market. In their empirical study covering the years 1993-2007, no negative effect of robots on total employment can be observed, but the share of low-skilled workers decreases. Bessen (2019) suggests that the impact of automation will cause disruptive re-allocations of jobs between industries in spite of the absence of large-scale unemployment and job elimination patterns. For this reason, Furman and Seamans (2019) generally recommend the establishment of a "technology office", an agency that can assist policymakers in decision-making in order to ideally integrate the benefits and limitations of AI into economic policy decisions. Acemoglu and Restrepo (2018a), Acemoglu and Restrepo (2018b), Hémous and Olsen (2014) and Irmen (2021) provide task-based models where capital is able to handle an increasing number of tasks, leading to an increasing wage gap between high-skilled and low-skilled workers. This literature builds on Zeira (1998), who pioneered in linking economic growth with the effect of automation on the working force. In Aghion et al. (2017), the authors incorporate AI in a model taking into account Baumol's cost disease, i.e. the fact that the aggregate productivity of an economy is determined by the least productive factor and not by the most productive one.[4]

On the other side, another strand of literature assesses the unique features of AI, which distinguish it from automation. Particularly, it has the characteristic that it can build on its own and produce new ideas by the creative act of recombining existing knowledge (Weitzman, 1998). Agrawal et al. (2018) construct a model where AI helps researchers to carry the "burden of knowledge" (Jones, 2009) by finding the most promising combination of existing know-how. Jones and Tonetti (2020) refer to the so called "economics of data" and interpret data as a factor that improves the quality of an idea and can be used by

[4] Baumol (1967) presumes that the labor productivity will increase less in the service sector than in the primary sector. Since the role of this tertiary sector has become more important over the last decades, this can be one reason that aggregate productivity growth has been falling.

several firms in a non-rival way to produce blueprints.
We contribute to this literature as follows: We model AI as self-learning capital that is developed by workers with a sufficient skill level. We advance the hypothesis that self-learning of AI via its application in AR, as well as knowledge spillovers from workers in AI on AR development, fuel economic growth and induce sufficiently-skilled workers to move to AI production and later to AR. We study the emergence and subsequent evolution of the AI sector with Tech Giants. The main focus of the paper is to analyze the tipping points in the labor movements of heterogeneously-skilled agents and to assess how economic policy could help to induce socially optimal transitions. We will provide a macroeconomic rationale for an AI-tax.

3 Model

We build a three-sector model in which the outputs of the AR sector and the AI sector are used as intermediates in the final good firm. Growth is driven by technological progress achieved by the development of AI algorithms and AR blueprints. An individual lives forever and is indexed by the discrete parameter $\eta \in \{U, H, E\}$. Depending on their index, individuals have the qualification to work in different sectors. In particular, we assume three disjoint labor forces, characterized by the index η. First, we have the group of entrepreneurial-skilled individuals with index $\eta = E$, who are able to work in all sectors and can run new AI businesses and make up the amount l^E of the total labor force. Second, there is the group of high-skilled agents with index $\eta = H$ and mass l^H, who can work in all sectors, but are not eligible to start running AI firms, as they lack entrepreneurial characteristics. Third, low-skilled workers with index $\eta = L$ and who make up the amount l^U of the total labor force, can only work in the final good firm. The total labor force is defined as $L = l^U + l^H + l^E$. We consider a continuum of individuals of mass one represented by the interval $[0, 1]$, so that $L = 1$.
The model incorporates a simple task-complexity skill relationship, where workers with a higher skill level can execute more tasks and tasks with higher complexity.[5] Conditional on being able to work in a particular sector, workers are equally productive. It means in effect that when workers with different skill levels work in the final good firm, they have the same productivity. Correspondingly, high-skilled agents, working in AI or AR, have the same productivity as entrepreneurial-skilled individuals, if both groups decide to work in the same sector.

[5]For a treatment of more refined task-complexity-skill relationships see Gersbach and Schmassmann (2019).

3.1 AI Sector

We assume that there are $N \geq 1$ distinct firms (Tech Giants) in the AI sector. Firms can employ workers, with skills $\eta \in \{E, H\}$, to produce AI algorithms which are used as intermediates in the final good production. There is monopolistic competition in the AI sector and each Tech Giant produces a variant of AI. This may reflect the current market structure, with several large firms dominating the market. AI is a type of learning capital and improves itself through applications in the real world, in particular, through its use in AR. Hence, the effective AI stock of firm j evolves according to

$$A_{t,j}^{S} = R_{t-1}^{q}(1 + \theta_A l_{t,j}^{A,D}), \tag{1}$$

where $l_{t,j}^{A,D}$ is the amount of labor demanded and $\theta_A \in (0, 1)$ the corresponding worker productivity parameter in the AI sector. Throughout the paper, we use the notation S for the index representing the supply, whereas D refers to the demand. Accordingly, $A_{t,j}^{S}$ is the supply of intermediates produced by AI firm j in period t. Furthermore, R_{t-1} is the stock of blueprints produced in the AR sector until period $t - 1$. Through the application of AI in AR, AI learns how to perform better. The higher the stock of AR blueprints, the more applications are provided for AI to exploit its self-learning feature and thus the higher the level of AI. Still, new AI algorithms need to be produced by software developers l_t^A, working in the AI sector. In line with the arguments of Agrawal et al. (2018), the function depicting the accumulation of AI is not linear but concave in AR, entailing that marginal benefits from spillovers from AR to AI, defined by q, are declining, which is reflected by assuming $q \in (0, 1)$.[6] In other words, self-learning of AI through practical application in AR has diminishing returns.

Profit maximization of AI-producing firms boils down to profit maximization for each period, since the current production of AI does not depend on the previous AI production, but only on the AR stock R_{t-1} of the previous period which cannot be influenced by AI firms. Applying Equation (1) yields the profits of an AI firm j

$$\Pi_{t,j}^{A} = p_{t,j} A_{t,j}^{S} - w_t^A l_{t,j}^{A,D} = p_{t,j} R_{t-1}^{q}(1 + \theta_A l_{t,j}^{A,D}) - w_t^A l_{t,j}^{A,D}, \tag{2}$$

where $p_{t,j}$ is the price firm j sets for its AI intermediate and w_t^A is the wage workers in the AI sector receive. Since workers in the AI sector are equally productive at all AI firms, there is a single wage w_t^A in the AI sector. Individual outputs of firms are aggregated to a composite AI supply, defined by the following aggregate CES function (Acemoglu, 2009)

[6] Already Arrow (1962) assumed that learning through repeated application has diminishing returns. We discuss the special case $q = 1$ in more detail in Appendix C, as self-learning of AI may display constant returns to scale in an extreme scenario.

with constant elasticity of substitution σ between AI variants from different AI firms:

$$A_t^S = \left(\sum_{j=1}^{N} \pi_j^{\frac{1}{\sigma}} (A_{t,j}^S)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}},$$

with the weights π_j such that $\sum_{j=1}^{N} \pi_j^{\frac{1}{\sigma}} = 1$. The "distribution parameter π_j" (Klump et al., 2012) can be interpreted as the share of an AI variant j in the provision of the composite AI supply. We assume that the set of entrepreneurs l^E is partitioned into N disjunct subsets of measure $\frac{l^E}{N}$, with each of the N groups owning a single AI firm. Hence each AI firm is owned by infinitely many entrepreneurs of mass $\frac{l^E}{N}$, with each entrepreneur receiving the share $\frac{N}{l^E}$ of firm j's profit $\Pi_{t,j}^A$. We assume a certain degree of substitutability between the AI variants and suppose that $\sigma \in \mathbb{R}_{>1}$. Thus, the final good firm may replace an AI intermediate from a specific firm with variants from different AI-producing firms.

3.2 AR Sector

We consider a finite number of symmetric and price-taking firms in the AR sector, that operate in a competitive market to produce AR blueprints. Thus, we can restrict ourselves to the behavior of a single representative firm producing the AR intermediate. The representative AR firm produces blueprints that accumulate over time. Only skilled workers with $\eta \in \{H, E\}$ can be hired for the development of AR blueprints which are used as intermediates in the final good production. AR blueprints are sold to the final good firm in each period. In each period, the entire accumulated knowledge, stored in the AR blueprints from the previous period R_{t-1}, can be used in an open-source manner. This is equivalent to saying that the protection of blueprints generated by the representative AR firm lasts one period. There are several interpretations for this duration of protection. For instance, the complexity of the intermediate takes one period for competitors to replicate. Another interpretation is that intermediates may be protected by patents which last one period or become valueless through new intermediates after one period. The representative AR firm has the capability to transform this open-source knowledge in period t into a new intermediate that can be sold to the final good production. Consequently, the stock of blueprints produced by the representative AR firm in period t is given by

$$R_t^S = R_{t-1}(1 + \theta_R l_t^{R,D} + \psi_A l_t^{A,D}), \tag{3}$$

where $l_t^{R,D}$ and $l_t^{A,D}$ are the demand for workers in AR and in AI, respectively. The parameter $\theta_R \in (0, 1)$ depicts the workers' productivity in AR, whereas $\psi_A \geq 0$ measures spillovers from AI to AR. We assume that workers are more productive in performing

AR than in developing new AI and thus assume that $\theta_R > \theta_A$. We allow for knowledge spillovers from AI on AR via the working force $l_t^{A,D}$ in AI. In this respect, we follow Balconi and Laboranti (2006) that there are knowledge exchanges of researchers who produce new technologies and consequently take into account that AR development benefits from cooperating with software developers from AI. We assume that $\theta_R > \psi_A > 0$ to model that the direct effect of the labor demand of AR has a larger effect on R_t than AI spillovers via $l_t^{A,D}$. The profit of the representative AR firm will be denoted by

$$\Pi_t^R = \gamma_t R_t^S - w_t^R l_t^{R,D}, \tag{4}$$

where the stock of blueprints, produced by $l_t^{R,D}$ workers, is given by (3) and γ_t is the price for AR intermediates at period t and w_t^R is the wage for workers in AR. Since the AR sector is competitive, the representative AR firm takes γ_t and w_t^R as given—in any equilibrium with an active AR sector such that the wage for a worker in AR equals its marginal benefit, which yields the following:

$$w_t^R = \gamma_t R_{t-1} \theta_R. \tag{5}$$

Otherwise, due to the linearity of producing the next AR level, the representative firm would either demand an infinite amount of workers or none. Both constellations cannot occur in equilibrium. With this condition, we will see in Section 4 that the representative AR firm makes positive profits. We assume that ownership of the representative AR firm is uniformly-distributed in the society so that each agent obtains a profit share. This assumption is irrelevant for the evolution of the economy, but of course matters for the income distribution.

3.3 Final Good Production

Finally, we introduce a final good firm where labor, capital, AI and AR are used to produce a consumption good Y_t in the following way:

$$Y_t = B \left(l_t^D + \phi_R R_t^D + \phi_A \left(\sum_{j=1}^{N} \pi_j^{\frac{1}{\sigma}} (A_{t,j}^D)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \right)^{1-\alpha} (K_t^D)^{\alpha}, \tag{6}$$

where l_t^D is the demand for low-skilled workers, K_t^D stands for the demand for physical capital, $A_{t,j}^D$ for the demand for AI algorithms developed by firm j and R_t^D for the demand for AR blueprints. The parameter $\alpha \in [0, 1)$ represents the share of capital in production and ϕ_A and ϕ_R[7] capture the relative advantage of AI and AR over low-skilled labor, respectively. Total factor productivity is defined as $B \in \mathbb{R}^+$.

[7] We set $\phi_A > \phi_R$, such that AI has a comparative advantage over AR in production.

The final good sector is populated by a continuum of price-taking firms. Since the production function displays constant returns to scale, we can restrict ourselves to the behavior of a single representative firm producing the final good. We assume that AR and AI intermediates fully depreciate after utilization in the final good production and thus have to be re-acquired in every period in their latest version. This is a stark assumption. It suffices that the depreciation is sufficiently strong such that the final good firm is better off by using the current intermediates than older ones. The profits Π_t of a final good firm depend on the production Y_t minus the costs for the input factors, namely low-skilled workers, capital, AI and AR

$$\Pi_t = Y_t - w_t l_t^D - r_t K_t^D - \sum_{j=1}^{N} p_{t,j} A_{t,j}^D - \gamma_t R_t^D. \tag{7}$$

The marginal costs of low-skilled labor and capital are given by the wage w_t in the final good firm and the interest rate r_t, respectively. Throughout the paper, we normalize the price of the final good to one. In addition, the final good firm pays the prices $p_{t,j}$ and γ_t for the AI and AR intermediates, respectively.

3.4 Household Optimization

We assume infinitely many agents of mass 1. Individuals of any skill level do not only work, but also save and consume, maximizing the following life-time utility:

$$U_\eta = \sum_{t=0}^{\infty} \beta^t u(c_{t,\eta}), \tag{8}$$

where $u(c_{t,\eta})$ is some instantaneous concave utility function, depending on agent η's consumption $c_{t,\eta}$ in period t. The parameter β is the discount factor of individual consumption. The individual endowment consists of an inelastic labor supply fixed at $l^U + l^H + l^E = L = 1$ and the capital supply $K_{t,\eta}^S$ which an agent with skill level η rents out to firms in period t. Some part of the capital depreciates at rate δ, so that an individuals' capital stock evolves according to

$$K_{t+1,\eta}^S = (1-\delta) K_{t,\eta}^S + s_{t,\eta}, \tag{9}$$

where $s_{t,\eta}$ are the savings made by an agent with skill level η in period t. Since entrepreneurs and high-skilled agents can work in different sectors, the budget constraints depend on the labor allocation. Figure 2 shows the structure of the economy for a constellation where high-skilled workers are employed in AR, entrepreneurial-skilled workers are employed in AI and low-skilled agents work in the final good firm. Figure 2 illustrates the spillovers effects and the flow of intermediates to the final good firm. All agents receive an equal share of the AR profits and the final good profits, whereas entrepreneurs share AI profits among themselves. Entrepreneurs (E) obtain the AI wage, high-skilled workers (H) obtain the wage from the AR sector and low-skilled agents (U) obtain the wage from the final good firm. In this labor market constellation, the budget constraint of a single agent with index η in period t is as follows:[8]

$$c_{t,\eta} + s_{t,\eta} = w_t + r_t K_{t,\eta} + \Pi_t + \Pi_t^R \quad \text{for} \quad \eta \in \{U\} \quad \text{in final good production,} \tag{10}$$

$$c_{t,\eta} + s_{t,\eta} = w_t^R + r_t K_{t,\eta} + \Pi_t + \Pi_t^R \quad \text{for} \quad \eta \in \{H\} \quad \text{in AR,} \tag{11}$$

$$c_{t,\eta} + s_{t,\eta} = w_t^A + r_t K_{t,\eta} + \Pi_t + \Pi_t^R + \frac{N}{l^E}\Pi_{t,j}^A \quad \text{for} \quad \eta \in \{E\} \quad \text{in AI,} \tag{12}$$

$$\text{with} \quad K_{0,\eta} \quad \text{given,}$$

where w_t, w_t^A and w_t^R are the wages in the final good sector, AI sector and AR sector, respectively. In line with this notation, Π_t and Π_t^R are the sector-specific profits of the final good firm and the AR sector, while $\Pi_{t,j}^A$ are the profits of AI firm j, of which infinitely many entrepreneurs of mass $\frac{l^E}{N}$ share ownership.

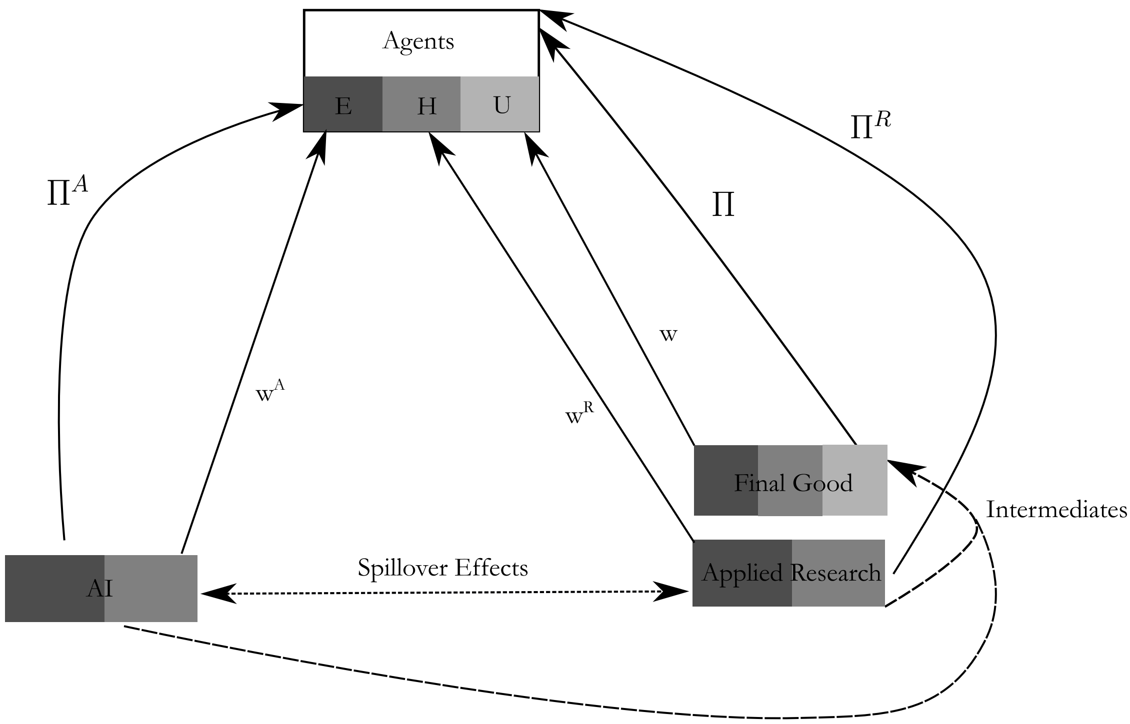

Figure 2: Diagram of the Structure of the Economy.

Note that profits of AI are allocated symmetrically, so that each entrepreneur receives the same share of the AI profits Π_t^A. The profits of the final good firm, Π_t, and from the AR sector, Π_t^R, are evenly-distributed to all agents in the economy. Since the measure of individuals is one, aggregate profits in the final good firm and AR sector are equal to

[8]Since individuals will optimally rent out all capital $K_{t,\eta} = K_{t,\eta}^S$, where $K_{t,\eta}$ is the capital an individual with skill level η has rented out to firms, we will only use $K_{t,\eta}$.

per-capita profits. We maximize individual lifetime utility (8) subject to the evolution of the capital stock (9) and the individual budget constraint—(10), (11) or (12)—which depends on the sector of employment. This yields the usual Euler equation which is indeed independent of the individual skill level[9]

$$\frac{u'(c_{t,\eta})}{u'(c_{t+1,\eta})} = \beta(1 - \delta + r_{t+1}). \tag{13}$$

4 Equilibrium Definition

Depending on the employment of the heterogeneously-skilled agents in the economy, there are different equilibria in our model. Therefore, we have to distinguish between equilibrium conditions that are valid for all employment constellations and conditions for specific constellations. General equilibrium conditions are as follows:

Definition 1

An equilibrium is a sequence of prices $(\gamma_t, \{p_{t,j}\}_{j=1}^N, r_t, w_t, w_t^A, w_t^R)_{t=0}^\infty$ *and a set of allocations* $(\{c_{t,\eta}, s_{t,\eta}, K_{t,\eta}\}_{\eta\in\{U,H,E\}}, R_t, \{A_{t,j}\}_{j=1}^N, l_t, l_t^R, \{l_{t,j}^A\}_{j=1}^N, Y_t, \Pi_t, \{\Pi_{t,j}^A\}_{j=1}^N,$ $\Pi_t^R)_{t=0}^\infty$ *that maximize the individuals' utilities, the profits of the AI firms and the profits of the representative final good firm and AR firm. The conditions that clear the goods market, the labor market and the market for AI and AR are denoted by*

$$\sum_{\eta\in\{U,H,E\}} (c_{t,\eta} + K_{t+1,\eta}) =$$

$$(1-\delta) \sum_{\eta\in\{U,H,E\}} K_{t,\eta} + w_t l_t + w_t^A l_t^A + w_t^R l_t^R + \sum_{j=1}^N \Pi_{t,j}^A + \Pi_t + \Pi_t^R, \tag{14}$$

$$l_t + l_t^R + \sum_{j=1}^N l_{t,j}^A = l_t + l_t^R + l_t^A = L, \tag{15}$$

$$K_t^D = K_t^S = K_t \quad and \quad A_{t,j}^D = A_{t,j}^S = A_{t,j} \quad , \quad R_t^D = R_t^S = R_t \ \forall j, \forall k, \forall t; \tag{16}$$

where $A_{t,j}^S$ and R_t^S are the intermediate supplies in period t by firm j operating in the monopolistic AI sector or by the representative firm in the competitive AR sector, respectively. Equation (14) can be interpreted as the aggregate budget constraint. We notice that Condition (15) and Condition (16) are market clearing conditions for labor, capital, AI and AR. Finally, we recall that the price $p_{t,j}$ is set by monopolistic AI firms and γ_t is the competitive price in the AR sector. In addition to these general conditions, we define equilibrium conditions which depend on the employment distribution of the heterogeneously-skilled agents across the sectors.

[9]A detailed derivation of the Euler equation in a decentralized economy is provided in Appendix A.

Definition 2

We recall that $l^U + l^H + l^E = L = 1$ *and distinguish between the following mutually exclusive labor market equilibrium constellations:*

$$l_t = L, \qquad l_t^A = 0, \qquad l_t^R = 0; \tag{17}$$

$$l_t = l^U + l^H, \qquad l_t^A = l^E \qquad l_t^R = 0; \tag{18}$$

$$l_t = l^U, \qquad l_t^A = l^H + l^E, \qquad l_t^R = 0; \tag{19}$$

$$l_t = l^U, \qquad l_t^A = l^E, \qquad l_t^R = l^H; \tag{20}$$

$$l_t = l^U, \qquad l_t^A = 0, \qquad l_t^R = l^H + l^E. \tag{21}$$

The five constellations capture all combinations how the different skill groups can be possibly allocated across sectors in an equilibrium. Which constellation will arise depends on the state of the economy and in particular on the development of AR, which determines the relationship between wages in AI, AR and the final good sector. Different relationships between the wages are associated with one of the employment patterns (17) to (21). This will be characterized in Section 5.

4.1 Intra-period Equilibrium

We will next look at intra-period conditions that hold in all constellations. In each sector, firms maximize their profits in each period.

Final Good Firm

By maximizing the profit function of the final good firm (7) with respect to the inputs K_t^D and l_t^D, we obtain the following conditions for the interest rate and the wage in the final good firm:

$$r_t = \alpha \frac{Y_t}{K_t^D}, \tag{22}$$

$$w_t = (1-\alpha) \frac{Y_t}{l_t^D + \phi_A A_t^D + \phi_R R_t^D}. \tag{23}$$

We next turn to the representative AR firm. Differentiating (7) with respect to R_t^D yields the following inverse demand function for an AR intermediate of the representative firm:

$$\gamma_t = (1-\alpha)\phi_R \frac{Y_t}{l_t^D + \phi_A A_t^D + \phi_R R_t^D}. \tag{24}$$

Combining (23) with (24) yields

$$\gamma_t = \phi_R w_t. \tag{25}$$

Maximizing (7) with respect to $A^D_{t,j}$ yields the following price for an AI intermediate of firm j:

$$p_{t,j} = (1-\alpha)\phi_A \frac{Y_t}{l^D_t + \phi_A A^D_t + \phi_R R^D_t} \left(\frac{A^D_{t,j}}{\pi_j A^D_t} \right)^{\frac{-1}{\sigma}}. \tag{26}$$

Combining this finding with Equation (23), we deduce that

$$p_{t,j} = \left(\frac{A^D_{t,j}}{\pi_j A^D_t} \right)^{\frac{-1}{\sigma}} \phi_A w_t. \tag{27}$$

Rewriting the price for an AI intermediate, defined by (27), we see that the inverse demand of a final good firm for an AI intermediate j is given by

$$A^D_{t,j} = \left(\frac{p_{t,j}}{\phi_A w_t} \right)^{-\sigma} \pi_j A^D_t. \tag{28}$$

AI Sector

An intermediate-producing AI firm, operating in a monopolistically competitive market, takes the inverse demand of the final good firm (28) as given and maximizes its profit

$$\begin{aligned} \Pi^A_{t,j} &= p_{t,j} A^D_{t,j} - w^A_t l^{A,D}_{t,j}, \\ &= p_{t,j} \left(\frac{p_{t,j}}{\phi_A w_t} \right)^{-\sigma} \pi_j A^D_t - \left(\frac{A^D_{t,j}}{R^q_{t-1}} - 1 \right) \frac{w^A_t}{\theta_A}, \\ &= p^{1-\sigma}_{t,j} (\phi_A w_t)^\sigma \pi_j A^D_t - \left(\left(\frac{p_{t,j}}{\phi_A w_t} \right)^{-\sigma} \frac{\pi_j A^D_t}{R^q_{t-1}} - 1 \right) \frac{w^A_t}{\theta_A}, \end{aligned} \tag{29}$$

where we substituted $l^{A,D}_{t,j}$ using the AI production function (1). Moreover, we used the price for an AI intermediate, from (27). We maximize (29) with respect to $p_{t,j}$ and obtain the price AI firm j sets

$$p_{t,j} = \frac{\sigma w^A_t}{\theta_A (\sigma - 1) R^q_{t-1}}, \tag{30}$$

where we find that the price is set as a mark-up $\sigma/(\sigma-1)$ over the marginal cost of producing a new unit of AI, which requires $1/\theta_A R^q_{t-1}$ units of labor compensated with the wage w^A_t. By equating $p_{t,j}$ from (30) with (27), we deduce that wages in the AI sector are given by

$$w^A_t = \frac{(\sigma-1)\theta_A \phi_A w_t R^q_{t-1}}{\sigma} \left(\frac{A^D_{t,j}}{\pi_j A^D_t} \right)^{\frac{-1}{\sigma}}. \tag{31}$$

The higher the substitutability between AI variants, the higher the markdown on wages. By inserting (30) into (2), we obtain

$$
\begin{aligned}
\Pi^A_{t,j} &= \frac{w^A_t}{\sigma - 1}\left[\frac{\sigma}{\theta_A} + l^{A,D}_{t,j}\right], \\
&= p_{t,j} R^q_{t-1}\left[1 + \frac{\theta_A l^A_{t,j}}{\sigma}\right]. \qquad (32)
\end{aligned}
$$

The profits of AI firms are the sum of two components: On the one hand, there are the revenues from selling the existing stock of AI that the entrepreneurs do not have to produce and thus do not have to pay wages for, i.e. $p_{t,j}R^q_{t-1}$. On the other hand, there are the revenues from the newly created amount of AI, of which the share $1/\sigma$ is turned into profit due to the mark-down on wages. The respective term is $p_{t,j}R^q_{t-1}\theta_A l^A_{t,j}/\sigma$. Implicitly, we assume that different AI entrepreneurs in different periods maximize profits in the AI sector. We summarize important comparative static properties in the following proposition:

Proposition 1
Profits of an AI firm increase in the productivity parameter θ_A and decrease with a higher elasticity of substitution σ. The higher the existing stock of AR, given by R_{t-1}, the price $p_{t,j}$ for each intermediate sold, and the more workers $l^A_{t,j}$ develop new algorithms, the higher the profits of an AI firm.

AR Sector
The representative intermediate-producing AR firm takes the price of its output (25) as given and maximizes its profit. Substituting γ_t from (25) into (5), we deduce that wages in the AR sector are given by

$$
w^R_t = w_t \phi_R R_{t-1} \theta_R. \qquad (33)
$$

When we insert (5) into (4), we obtain

$$
\begin{aligned}
\Pi^R_t &= \gamma_t R_t - w^R_t l^{R,D}_t, \\
&= \gamma_t \left(R_t - R_{t-1}\theta_R l^{R,D}_t\right), \\
&= \gamma_t R_{t-1}\left(1 + \theta_R l^{R,D}_t + \psi_A l^A_t - \theta_R l^{R,D}_t\right), \\
&= \gamma_t R_{t-1}(1 + \psi_A l^A_t). \qquad (34)
\end{aligned}
$$

We observe that the representative AR firm makes positive profits due to the freely accessible AR blueprints R_{t-1} from the previous period and the spillovers from the AI sector. We summarize relevant comparative statics properties in the following proposition:

Proposition 2

Profits of the representative AR firm increase with larger spillovers from the AI sector, given by $\psi_A l_t^A$. The higher the existing stock of AR, R_{t-1}, and the price γ_t for each blueprint sold, the higher the profits.

Clearly, each intra-period equilibrium in the AI sector is symmetric, since the demand from the final good firm is the same for each variant of AI, $A_{t,j} = A_t \, \forall j$. It follows that $l_{t,j}^A = \frac{l^E}{N} \forall j$, i.e, the labor supply of entrepreneurs, is equally divided among the N AI firms. It is irrelevant whether a single entrepreneur is hired by the firm he co-owns or by another AI firm. Yet, it is convenient to simplify the presentation by assuming that entrepreneurs are hired by the firm they own.[10] Combining our findings on the intra-period equilibrium conditions from the final good firm and the AI sector and the AR sector, we obtain:

Proposition 3

Given R_{t-1} and K_{t-1}, there exists a unique and symmetric intra-period equilibrium. In such an equilibrium, the general conditions (14)-(16) and of the mutually exclusive and constellation-specific conditions (17)-(21) hold. The unique intra-period equilibrium is defined by the following conditions on the wages, prices and profits that hold in all labor market constellations:[11]

	Final Good	**AI**	**AR**
Wage	$w_t = \frac{(1-\alpha)Y_t}{l_t^D + \phi_A A_t^D + \phi_R R_t^D}$	$w_t^A = \frac{(\sigma-1)}{N\sigma}\theta_A \phi_A w_t R_{t-1}^q$	$w_t^R = w_t \theta_R \phi_R R_{t-1}$
Price	1	$p_t = \frac{\phi_A w_t}{N}$	$\gamma_t = \phi_R w_t$
Profit	$\Pi_t = 0$	$\Pi_{t,j}^A = \frac{\phi_A w_t R_{t-1}^q}{N}\left[1 + \frac{\theta_A l_{t,j}^A}{\sigma}\right]$	$\Pi_t^R = \phi_R w_t R_{t-1}[1 + \psi_A l_t^A]$

Table 1: Intra-period Conditions on the Wages, Prices and Profits.

As N firms are operating in the AI sector the overall profit in the AI sector is $N\Pi_t^A$. Since the profit of the final good firm exhibits constant returns to scale and the market for the final good is competitive, the profit of the final good firm is zero and $\Pi_t = 0$.[12]

[10] The symmetric outcome implies that $\pi_j^{\frac{1}{\sigma}} = \frac{1}{N}$. Combining this finding with the symmetry of AI variants and Equation (26), it has to hold that $p_{t,j} = p_t$. Thus, the more firms operate under monopolistic competition in the AI sector, the higher the markdown on wages.

[11] In each equilibrium, the interest rate is given by $r_t = \alpha \frac{Y_t}{K_t^D}$.

[12] This can be verified by inserting (22), (23), (25) and (30) into (7).

5 Endogenous Rise of AI

Using the unique equilibrium for each labor market constellation denoted above, we next consider transitions of agents between the sectors to determine the sequential order of the constellations. Our objective is to analyze the transition dynamics in the economy before reaching a steady state. We study the development of the employment in the specific sectors, depending on the skill level of certain workers. We derive strict conditions for tipping points that characterize the transitions of agents between the sectors.

5.1 First Tipping Point

At $t = 0$, all agents—irrespective of their abilities—work in the final good firm. This state is defined as the initial constellation, given by Condition (17). To be precise, we assume the following environment: R_0 is sufficiently small, so that w_t^R and w_t^A are smaller than w_t and no high-skilled worker has an incentive to change the sector.[13] AI firms already exist and their profits go to the entrepreneurs. However, entrepreneurs do not work in the AI sector, denoted by $l_t^A = 0$, and thus the stock of AI is stagnant. Yet, entrepreneurs could decide to change their sector of employment. Hence, they initially have three options. First, they can remain in the production of the final good, earn w_t and obtain the constant and aggregate profits of AI firms, given by $N\Pi_t^A$. In all constellations, where nobody is employed in AI, the aggregate profits in the AI sector are as follows:

$$\sum_{j=1}^{N} \Pi_{t,j}{}^A = \phi_A w_t R_{t-1}^q. \tag{35}$$

In this way, we obtain what will be referred to as the overall income of entrepreneurs if they work in the final good firm

$$\sum_{j=1}^{N} \Pi_{t,j}^A + w_t l^E.$$

Recall that l^E denotes the amount of entrepreneurs in the labor force. Second, entrepreneurs can move to the AR sector, earn the wage payments w_t^R from the representative AR firm and obtain the aggregate AI profits Π_t^A. Third, entrepreneurs can commit their labor to AI firms. A single AI firm generates profits and wage payments which go to the entrepreneurs that own this firm and work at the firm:

$$\Pi_{t,j}^A + w_t^A l_{t,j}^A = \frac{1}{N} w_t \phi_t R_{t-1}^q \left[1 + \theta_A l_{t,j}^A\right].$$

[13] This can be guaranteed by setting R_0 such that $\frac{(\sigma-1)}{N\sigma}\theta_A\phi_A R_0^q < 1$ and $\theta_R\phi_R R_0 < 1$.

This holds for all $l^A_{t,j}$, i.e. independently of how many entrepreneurs work in the single AI firm. Thus, all AI firms generate the total income of entrepreneurs which is

$$\sum_{j=1}^{N} \Pi^A_{t,j} + \sum_{j=1}^{N} w^A_t l^A_{t,j} = \phi_A w_t R^q_{t-1} \left[1 + \theta_A l^A_{t,j}\right] = \sum_{j=1}^{N} \Pi^A_{t,j} + l^E w^A_t,$$

as $\sum_{j=1}^{N} l^A_{t,j} = l^E$. If a group of $\frac{l^E}{N}$ entrepreneurs works in an AI firm, it increases the level of AI in that firm by $A_{t,j} = R^q_{t-1}(1 + \theta_A \frac{l^E}{N})$. If entrepreneurs move to the AI sector, they receive a higher profit, as they work in the development of new AI algorithms. In this case, the aggregate profits in the AI sector are given by

$$\sum_{j=1}^{N} \hat{\Pi}^A_{t,j} = \phi_A w_t R^q_{t-1} \left[1 + \frac{\theta_A l^e}{N\sigma}\right]. \tag{36}$$

Thus, entrepreneurs move to the AI sector if[14]

$$\sum_{j=1}^{N} \Pi^A_{t,j} + w_t l^E < \sum_{j=1}^{N} \hat{\Pi}^A_{t,j} + w^A_t l^E, \tag{37}$$

where the left-hand-side represents their potential income if they stay in the final good firm and the right-hand-side the one if they move to the AI sector.[15] Substituting (36) and (35) into (37) and using the equilibrium wages specified in Table 1, we obtain

$$\left(1 - \frac{\sigma-1}{\sigma N}\phi_A \theta_A R^q_{t-1}\right) < R^q_{t-1}\frac{\phi_A \theta_A}{N\sigma}, \qquad \text{or simply}$$

$$R^*_{t_1} > \left(\frac{N}{\phi_A \theta_A}\left(\frac{1}{\sigma} + \frac{\sigma-1}{\sigma}\right)^{-1}\right)^{\frac{1}{q}}. \tag{38}$$

The lower bound on the level of AR blueprints needed for the development of AI to take place—the first tipping point—will be denoted by $R^*_{t_1}$. If R_t is below $R^*_{t_1}$, entrepreneurs remain in the final good production and do not move to the AI sector to actively increase the level of AI. The product of the parameters ϕ_A and θ_A and the AR stock of the preceding period, R^q_{t-1}, can be understood in the following way: When an agent with one unit of labor moves from the final good firm to the AI sector, he forfeits the wage w_t. In the AI sector, he creates algorithms which, in turn, produce the final good and

[14] We formulate the decision of the entrepreneurs as a group decision, since all entrepreneurs face the same decision. The individual problem is $\Pi^A_{t,j}\frac{N}{l^E} + w_t < \hat{\Pi}^A_{t,j}\frac{N}{l^E} + w^A_t$—which is equivalent to (37) and also yields (38).

[15] In general, all individuals are price-takers and wage-takers, and thus do not consider their potential impact on those variables if they change the sector. For instance, entrepreneurs consider w_t as independent of their choice and thus as being the same in both cases, i.e. whether they remain in the final good production or change to the AI sector.

effectively replace him. He increases the level of AI by one unit, multiplied with the productivity θ_A. In final good production, AI is ϕ_A-times more productive than a unit of labor. Finally, the increase of AI by using one unit of additional labor is scaled by R^q_{t-1} due to the spillovers from AR. Thus, in total, AI benefits by $R^q_{t-1}\theta_A\phi_A$.
In addition, in (38), a markdown $\frac{\sigma-1}{\sigma}$ on wages, originating from monopolistic competition in the AI sector which we call the *Wage Effect*, plays an important role. As we assume that $\sigma > 1$, it holds that $\frac{\sigma-1}{\sigma} < 1$ and that the *Wage Effect*—in an isolated assessment—delays transitions of entrepreneurs from final good production to AI due to monopolistic markdowns on wages compared to a competitive market. We observe that the higher the *Wage Effect*, the lower the markdown on wages in AI and the earlier the transition of entrepreneurial-skilled agents from the final good firm to AI.
Due to the profit share of entrepreneurs in the monopolistic AI sector, entrepreneurs additionally include the term $\frac{1}{\sigma}$ in their decision as they reap the profits from AI, which we interpret as the *Profit Effect*. As we assume that $\sigma > 1$, the *Profit Effect*—in an isolated assessment—leads to an earlier transitions of entrepreneurs than of high-skilled workers from final good production to AI. The higher σ, the better AI variants can be substituted for each other, and entrepreneurs can therefore only skim off lower profits, as they can only charge smaller markups on AI prices. This can also be verified by noting that $\frac{\partial \Pi^A_t}{\partial \sigma} \leq 0$. Therefore, the higher the *Profit Effect*, the earlier the transition of entrepreneurs from the final good firm to AI. We note that $\frac{1}{\sigma} + \frac{\sigma-1}{\sigma} = 1$. Hence, the combined influence of the *Wage Effect* and *Profit Effect* shows that the monopolistic distortion in the AI sector has no effect on the timing of the transition of entrepreneurs from final good production to AI. The two forces offset each other since entrepreneurs are the recipients of all revenues of AI, in the form of profits or wages that they pay themselves. Moreover, the variable N in (38) captures the scarcity of entrepreneurs, as they have to be allocated between AI firms. The decision of entrepreneurs to move from the final good sector to the AI sector is therefore determined by the technology to produce AI and the scarcity of entrepreneurs. They start working in the AI sector once it is productive enough. After passing this first tipping point $R^*_{t_1}$, the economy reaches the second labor market constellation, given by Condition (18). Entrepreneurs work in the AI sector, whereas all other workers are still employed in the final good firm. In order to ensure that entrepreneurs move from the final good firm to the AI sector and not to the AR sector, it has to hold that

$$w^R_{t_1} l^E + \sum_{j=1}^{N} \Pi^A_{t,j} < w_{t_1} l^E + \sum_{j=1}^{N} \Pi^A_{t,j} < w^A_{t_1} l^E + \sum_{j=1}^{N} \hat{\Pi}^A_{t,j}, \quad \text{which simplifies to}$$

$$\frac{1}{\theta_R \phi_R} > R^*_{t_1} > \left(\frac{N}{\phi_A \theta_A} \left(\frac{1}{\sigma} + \frac{\sigma - 1}{\sigma} \right)^{-1} \right)^{\frac{1}{q}} = \left(\frac{N}{\phi_A \theta_A} \right)^{\frac{1}{q}}, \tag{39}$$

where the second inequality stems from (38). Note that $w_{t_1}^R l^E + \sum_{j=1}^N \Pi_{t,j}^A < w_{t_1} l^E + \sum_{j=1}^N \Pi_{t,j}^A$ is equivalent to $w_{t_1} > w_{t_1}^R$, which is fulfilled if $\frac{1}{\theta_R \phi_R} > R_{t_1}^*$. Thus, Inequality (39) determines restrictions on the vector of parameters values for $\{q, \theta_R, \phi_R, \theta_A, \phi_A, N\}$ that we assume to be fulfilled in the following. Inequality (39) is fulfilled for instance if $\theta_R \phi_R$ is not too large compared to $\theta_A \phi_A$.

5.2 Second Tipping Point

Note that once entrepreneurs have dedicated their labor to AI development at period t_1, the spillovers from AI to AR, characterized by (3), kick in and the stock of AR blueprints increases. Hence, as long as entrepreneurs work in the AI sector, the level of AI and AR will increase in every period, leading to a larger amount of blueprints. The next question is *when* workers with a skill index $\eta \in \{H\}$ decide to relocate from the final good firm to the AI sector. Their choice is simpler than that of entrepreneurs, as they only compare w_t and w_t^A, taking w_t as given under both choices. Workers have an incentive to work in the AI sector if it holds that $w_t^A > w_t$, implied by

$$\frac{(\sigma - 1)}{N\sigma} \theta_A \phi_A R_{t-1}^q > 1, \tag{40}$$

where we used the equilibrium wages in the final good firm and the AI sector, specified in Table 1. This condition is quite intuitive and denotes the relationship between the marginal product of labor in the final good production and the wages paid to high-skilled workers in the AI sector. Hence, $w_t^A > w_t$ holds if

$$R_{t_2}^* > \left(\frac{N}{\phi_A \theta_A} \left(\frac{\sigma - 1}{\sigma} \right)^{-1} \right)^{\frac{1}{q}}. \tag{41}$$

The level $R_{t_2}^*$ of AR blueprints is reached in period t_2 and defines the second tipping point in our model. We observe that $R_{t_2}^*$ is very similar to $R_{t_1}^*$, apart from the term $\frac{1}{\sigma}$. The reason for this is that, unlike entrepreneurs, high-skilled workers do not receive a share of the AI profits. Yet, we again note that the previously explained *Wage Effect*, given by $\frac{(\sigma-1)}{\sigma} < 1$, delays the transition of high-skilled agents from final good production to AI in a decentralized economy. From t_2 on, a labor market equilibrium constellation, defined by Condition (19), occurs. Entrepreneurial-skilled and high-skilled agents are employed in the AI firm, whereas low-skilled agents work in the final good production.
Comparing (38) with (41), we see that $t_1 < t_2$. As in the case for entrepreneurs, we have to ensure that high-skilled agents move from the final good firm to the AI sector and not

to the AR sector. Therefore, it has to hold that

$$w_{t_2}^A > w_{t_2} > w_{t_2}^R \quad \text{and thus}$$

$$\frac{1}{\phi_R \theta_R} > R_{t_2}^* > \left[\frac{N}{\phi_A \theta_A} \left(\frac{\sigma - 1}{\sigma} \right)^{-1} \right]^{\frac{1}{q}}. \tag{42}$$

Note that since the right-hand side of (42) is greater than that of (39), the latter is always fulfilled if the former holds. This means that if high-skilled workers prefer the AI sector to the AR sector and Condition (42) is fulfilled, it implies that entrepreneurs also prefer the AI sector to the AR sector and (39) is fulfilled.

5.3 Third Tipping Point

Now, we compare the wages in AI and AR to assess which sector pays the higher wage. Unlike wages in the AR sector, wages in the AI sector benefit only to a reduced extent from a rising stock of AR blueprints R_t, due to $q \in (0, 1)$. In each equilibrium, the condition $w_t^R > w_t^A$ holds if

$$w_t \theta_R \phi_R R_{t-1} > \frac{(\sigma - 1)}{N\sigma} \theta_A \phi_A w_t R_{t-1}^q.$$

Hence, a transition of high-skilled workers to the AR sector is preferable, starting from a certain level of AR blueprints. Thus, we define the third tipping point, when high-skilled agents move away from AI to AR. It is more attractive to work in the AR sector if

$$R_{t_3}^* > \left(\frac{(\sigma - 1)}{N\sigma} \frac{\phi_A \theta_A}{\phi_R \theta_R} \right)^{\frac{1}{1-q}}. \tag{43}$$

This level is reached in period t_3, with the required stock of AR blueprints being $R_{t_3}^*$. At this point, we want to highlight that the *Wage Effect* now has an opposite impact on the transitions of high-skilled agents from AI to AR. The *Wage Effect* leads to earlier transitions of high-skilled agents from AI to AR. The higher the *Wage Effect*, the lower the markdown on wages in AI, which, in turn, means that AR is more attractive than AI at a later date. The decision of high-skilled agents is driven by the wage difference which, in turn is determined by technological variables, the *Wage Effect* and the scarcity of entrepreneurs, given by $\frac{1}{N}$.

After passing this tipping point, the economy reaches the fourth labor market constellation, given by Condition (20). Entrepreneurs still develop new software in the AI sector, high-skilled workers are employed in the AR sector, and low-skilled workers remain in the final good production. Comparing $R_{t_3}^*$ and $R_{t_2}^*$ it is a priori not clear which value is larger. Subsequently and for the rest of the paper, we assume that for small values of

R,— e.g. for values below one—$R^*_{t_2}$ is smaller than $R^*_{t_3}$, so that the second tipping point does occur before the third, and we have $t_2 < t_3$.[16]

We note that high-skilled workers will leave the AI sector only to work in the AR sector but not in the final good sector. The respective condition for high-skilled workers not choosing the final good sector is $w^R_{t_3} > w_{t_3}$, which is exactly the condition for the second tipping point. This means that once the economy passes through the second tipping point and R_{t-1} is such that wages w^A_t in the AI sector are larger than wages in the final good sector w_t, the wage gap increases with R_t. Therefore, it is never optimal for high-skilled individuals to return to the final good sector. This logic carries over to entrepreneurs who also never go back to the final good firm. However, they can benefit from moving to the AR sector, as we show in the next section.

5.4 Fourth Tipping Point

The final tipping point stipulates when entrepreneurs supply their labor to AR and stop actively developing new AI. Their decision is governed by the following inequality:

$$w^R_t l^E + \underbrace{w_t \phi_A R^q_{t-1}}_{\sum_{j=1}^N \Pi^A_{t,j}} > w^A_t l^E + \underbrace{w_t \phi_A R^q_{t-1}\left[1 + \frac{\theta_A l^E}{N\sigma}\right]}_{\sum_{j=1}^N \hat{\Pi}^A_{t,j}},$$

where the left-hand-side displays the total income of entrepreneurs after changing to the AR sector, where they receive the wage payment w^R_t and earn the AI profit share $N\Pi^A_t$, even without an employment in the AI sector. The right-hand-side shows their income if they remain in the AI sector and continue to increase the AI stock by supplying their labor to develop new algorithms. Plugging in w^R_t and w^A_t from Table 1 yields

$$R^*_{t_4} > \left(\frac{1}{N}\left(\frac{1}{\sigma} + \frac{\sigma - 1}{\sigma}\right)\frac{\phi_A \theta_A}{\phi_R \theta_R}\right)^{\frac{1}{1-q}}, \tag{44}$$

denoting the fourth tipping point $R^*_{t_4}$ at period t_4. Again, as $\frac{1}{\sigma} + \frac{\sigma-1}{\sigma} = 1$, the combined effect of the *Wage Effect* and *Profit Effect* has no influence on the transition timing of entrepreneurs from AI to AR.

Thus, we find a condition under which entrepreneurs finally move from AI to AR such that the economy reaches the fifth labor market constellation, given by Condition (21). All entrepreneurs and high-skilled workers are employed in the AR sector, and low-skilled agents work in the final good firm. It is easy to see that $R^*_{t_3} < R^*_{t_4}$, so that the third tipping point occurs before the fourth. An economic explanation for this is that since entrepreneurs obtain profits from the AI sector, they stay in the AI sector longer than

[16] This can be guaranteed by assuming that $\left(\frac{(\sigma-1)}{N\sigma}\frac{\phi_A\theta_A}{\phi_R\theta_R}\right)^{\frac{1}{1-q}} > \left(\frac{N}{\phi_A\theta_A}\left(\frac{\sigma-1}{\sigma}\right)^{-1}\right)^{\frac{1}{q}}$.

high-skilled agents who do not benefit from AI profits. When entrepreneurs leave the AI sector in period t_4, they supply their labor to the AR sector, but they remain the owners of the AI firms. From then on, the level of AI increases autonomously, solely via a growing stock of AR, as defined in (1), but not because of human effort, as $l_t^A = 0$. This demonstrates that after passing the fourth tipping point, the self-learning nature of AI is particularly pronounced. Still, the resulting stock of AI algorithms continues to be used as an intermediate in the final good production. Let us define the value of R that fulfills (38) with equality as R^{crit}. We summarize our findings in the following proposition and Table 2:

Proposition 4

If the initial condition that $R_0 \geq R^{crit}$ is fulfilled, we can identify the following tipping points, where we claim that it holds that $t_1 < t_2 < t_3 < t_4$:

(i) Entrepreneurs move to the AI sector in t_1, defined in (38)*, increase the stock of AI and set the economy on a growth path.*

(ii) High-skilled workers move from the final good firm to the AI sector in period t_2, which is defined in (41)*.*

(iii) High-skilled workers move from the AI sector to the AR sector in period t_3, which is defined in (43)*.*

(iv) Entrepreneurs move from the AI sector to the AR sector in period t_4, which is defined in (44)*.*

Tipping Point	Decentralized Solution
Production to AI (l^E)	$R^*_{t_1} > \left(\frac{N}{\phi_A \theta_A} \left(\frac{1}{\sigma} + \frac{\sigma-1}{\sigma} \right)^{-1} \right)^{\frac{1}{q}}$
Production to AI (l^H)	$R^*_{t_2} > \left(\frac{N}{\phi_A \theta_A} \left(\frac{\sigma-1}{\sigma} \right)^{-1} \right)^{\frac{1}{q}}$
AI to AR (l^H)	$R^*_{t_3} > \left(\frac{1}{N} \frac{(\sigma-1)}{\sigma} \frac{\phi_A \theta_A}{\phi_R \theta_R} \right)^{\frac{1}{1-q}}$
AI to AR (l^E)	$R^*_{t_4} > \left(\frac{1}{N} \left(\frac{1}{\sigma} + \frac{\sigma-1}{\sigma} \right) \frac{\phi_A \theta_A}{\phi_R \theta_R} \right)^{\frac{1}{1-q}}$

Table 2: Tipping Points in a Decentralized Economy.

We note that the tipping points arise due the the relative productivity differences between the three sectors and due to the fact that AI firms are owned by entrepreneurs. This leads to the described transitions between the sectors. After the last tipping point, we arrive at a situation in which no agent chooses to work in the AI sector. This result is quite stark and arises due to our model assumptions which we made for analytical tractability. In order to guarantee that AI firms continue to employ some labor, we could assume that AI requires some minimal amount of human labor to improve itself after the last tipping point.

6 Steady State

In Section 4, we showed which wage constellations can occur and defined equilibrium conditions. Then, we highlighted the resulting transitions between the sectors in Section 5 and showed which employment pattern the economy will approach in the long run. We now start from the point where all tipping points have been passed. We see that we have constant employment in all three sectors in the long run—no employment in the AI sector, while low-skilled workers are employed in final good production and high-skilled and entrepreneurial-skilled workers are employed in AR. Recall that this constellation is characterized by Condition (21). Such constant employment suggests the existence of a path for the economy along which output grows at a constant rate. This section is devoted to the study of such a path, which will be called a Balanced Growth Path (BGP). Therefore, we propose the following:

Proposition 5
For the initial conditions $k_0 > 0$, $R_0 > 0$, $A_0 > 0$ and if the initial endowment satisfies $R_0 \geq R^{crit}$, the economy will pass the described tipping points and reach a unique steady state.

We know that in the long run, Constellation (21) is reached and all entrepreneurial-skilled and high-skilled workers are employed in AR and none in AI. Thus, we have $l_t^R = l^H + l^E$ in the long run. Since the spillovers from AR to AI are described by a concave function (1), growth will be dominated by the increase in AR in the long run. Therefore, along a BGP, output growth is driven by the increase of the effective labor force $l_t + \phi_A A_t + \phi_R R_t$ through the growing stock of blueprints R_t. From the equation of motion of AR, specified by Equation (3), we derive that

$$\Delta R_t = R_{t+1} - R_t = R_t \left(\theta_R l_{t+1}^{R,D} + \psi_A l_{t+1}^{A,D} \right).$$

As nobody is employed in AI ($l_t^A = 0$) and all entrepreneurs and high-skilled individuals work in AR ($l_t^R = l^H + l^E$), we obtain the following equations:

$$R_t(1 + g_R) - R_t = R_t \left(\theta_R l_{t+1}^{R,D} + \psi_A l_{t+1}^{A,D} \right) = R_t \theta_R l^H$$

Thus, the growth rate of AR can be defined as $g_R = \theta_R(l^H + l^E)$. After entrepreneurs have transitioned to the AR sector, no one is employed in the AI sector and the stock of AI only grows due to spillovers. Due to the concave function determining the spillovers from AR to AI, as defined by (1), the growth rate of AI on a BGP is defined as[17]

$$g_A = (1 + g_R)^q - 1.$$

[17] The derivation of g_A is provided in Appendix A.

We see that even without agents employed in the AI sector, the level of AI and the profits in this sector are growing at rate g_A, where $g_A < g_R$, as $q \in (0, 1)$. Our Euler Equation (13) allows for a steady state along which consumption grows at a constant rate, and the return to capital is constant. As the return depends on the ratio of output to physical capital, output and capital must grow at the same rate, i.e. $g_C = g_Y = g_K$. Hence, taking the logarithm of our Cobb-Douglas production function we obtain

$$g_Y = \alpha g_R + (1-\alpha) g_K \quad \text{and thus} \quad g_Y = g_K = g_R.$$

The previous findings have an impact on the growth rate of consumption. We summarize our findings in the following proposition:

Proposition 6
The economy has an asymptotic steady state along which the following holds:

(i) Employment in final good production $l = l^U$ and employment in the AR sector $l^R = l^H + l^E$, as well as the rental rate of capital r_t and the wage in the productive sector w_t, remain constant.

(ii) Output Y_t, capital K_t, aggregate consumption C_t and the wage in the AR sector ${w_t}^R$ grow at the constant rate $g_R = \theta_R(l^H + l^E)$.

(iii) Noone is employed in the AI sector, $l^A = 0$, entailing ${w_t}^A = 0$. The level of AI and the profits in the AI sector Π_t^A grow at rate $g_A = (1 + g_R)^q - 1$, due to spillovers from AR.

In Appendix B, we provide a detailed analysis of the convergence of the model to the steady state.

7 The Social Planner's Problem

We now identify the dynamics in the social planner's solution and compare it to the decentralized solution. We suppose that A_0, R_0 and k_0 are given and consider the problem under the general conditions given in Section 3. It has to hold in the social planner's solution that $c_{t,\eta} = c_t$ and $A_{t,j} = A_t$. Hence, the optimization problem reads as follows:

$$
\begin{aligned}
&\max_{\{c_t, K_{t+1}, A_t, l_t^A, R_t, l_t^R\}} \sum_{t=0}^{\infty} \beta^t u(c_t) \quad \text{s.t.} \\
Y_t &= B(l + l^H - l_t^A - l_t^R + \phi_A A_t + \phi_R R_t)^{1-\alpha} K_t^{\alpha}, \\
K_{t+1} &= Y_t - c_t + (1-\delta)K_t, \\
A_t &= R_{t-1}^q (1 + \theta_A \frac{l_t^A}{N}), \\
R_t &= R_{t-1}(1 + \theta_R l_t^R + \psi_A l_t^A), \\
l^H &\geq l_t^A + l_t^R.
\end{aligned}
$$

The planner allocates the entire labor force to the three sectors. Since no profit shares are distributed to any agent in the social planner's solution, the planner makes no distinction between entrepreneurial-skilled and high-skilled workers. In spite of that, it still holds that low-skilled workers l^U can only be employed in final good production.

The social planner aims at finding the optimal allocation of high-skilled workers and entrepreneurs to AI and AR. Agents in the AI sector have to be equally distributed among all symmetric AI firms in the social planner solution. In this way, each AI firm produces AI intermediates with the production function $A_t = R_{t-1}^q(1 + \theta_A \frac{l_t^A}{N})$. Moreover, the social planner takes into account the agents' savings $s_t = Y_t - c_t$ and the stock functions for A_t, R_t and K_t, given by (1), (3) and (9), respectively. Therefore, we define the following Lagrangian:

$$
\begin{aligned}
\mathcal{L} = \sum_{t=0}^{\infty} \beta^t \Bigg\{ & u(c_t) - \lambda_t \left[A_t - (1 + \theta_A \frac{l_t^A}{N}) R_{t-1}^q \right] - \zeta_t [R_t - (1 + \theta_R l_t^R + \psi_A l_t^A) R_{t-1}] \\
& - \mu_t \left[K_{t+1} - B(l + l^H - l_t^A - l_t^R + \phi_A A_t + \phi_R R_t)^{1-\alpha} K_t^{\alpha} + c_t - (1-\delta) K_t \right] \\
& - \xi_t \left[l_t^A + l_t^R - l^H \right] \Bigg\} .
\end{aligned} \tag{45}
$$

For notational ease, we write $l+l^H-l_t^A-l_t^R+\phi_A A_t+\phi_R R_t=V_t$. We obtain the following derivatives:

$$\frac{\partial \mathcal{L}}{\partial c_t}=\beta^t\left[u'(c_t)-\mu_t\right]=0, \tag{46}$$
$$\frac{\partial \mathcal{L}}{\partial l_t^A}=\beta^t\left[\frac{1}{N}\lambda_t\theta_A R_{t-1}^q+\zeta_t\psi_A R_{t-1}-\mu_t\frac{(1-\alpha)Y_t}{V_t}-\xi_t\right]\leq 0, \tag{47}$$
$$\frac{\partial \mathcal{L}}{\partial K_{t+1}}=\beta^t\mu_t+\beta^{t+1}\mu_{t+1}\left[\frac{\alpha Y_{t+1}}{K_{t+1}}+1-\delta\right]=0, \tag{48}$$
$$\frac{\partial \mathcal{L}}{\partial A_t}=\beta^t\left[-\lambda_t+\mu_t\frac{(1-\alpha)\phi_A Y_t}{V_t}\right]=0, \tag{49}$$
$$\frac{\partial \mathcal{L}}{\partial l_t^R}=\beta^t\left[\zeta_t\theta_R R_{t-1}-\mu_t\frac{(1-\alpha)Y_t}{V_t}-\xi_t\right]\leq 0, \tag{50}$$
$$\frac{\partial \mathcal{L}}{\partial R_t}=\beta^t\left[-\zeta_t+\mu_t\frac{(1-\alpha)\phi_R Y_t}{V_t}\right]+\beta^{t+1}\left[\lambda_{t+1}(1+\theta_A\frac{l_{t+1}^A}{N})qR_t^{q-1}\right]+$$
$$\beta^{t+1}\left[\zeta_{t+1}(1+\theta_R l_{t+1}^R+\psi_A l_{t+1}^A)\right]=0. \tag{51}$$

Additionally, we have two complementary slackness conditions referring to the labor market clearing for high-skilled workers:

$$\left(\frac{1}{N}\lambda_t\theta_A R_{t-1}^q+\zeta_t\psi_A R_{t-1}-\mu_t\frac{(1-\alpha)Y_t}{V_t}-\xi_t\right)(l_t^A+l_t^R-l^H)=0 \quad \text{and} \tag{52}$$
$$\left(\zeta_t\theta_R R_{t-1}-\mu_t\frac{(1-\alpha)Y_t}{V_t}-\xi_t\right)(l_t^A+l_t^R-l^H)=0. \tag{53}$$

We have to consider whether $\xi_t=0$ or $l_t^A+l_t^R=l^H$ holds. Let us look at the first case.[18] We can use (46) and (48) to obtain the following Euler equation in the social planner's optimization:

$$\frac{u'(c_{t,\eta})}{u'(c_{t+1,\eta})}=\beta\left[\frac{\alpha Y_{t+1}}{K_{t+1}}+1-\delta\right]=\beta\left(1-\delta+r_{t+1}\right). \tag{54}$$

The social planner allocates consumption equally across individuals and we note that the Euler equation is identical to the one in the decentralized solution (see Equation 13).

7.1 Allocation to AI

As in the analysis of the decentralized economy, we first examine transitions of entrepreneurial-skilled and high-skilled agents from the final good firm to the AI sector.

[18] We distinguish between both cases, $\xi_t=0$ and $l_t^A+l_t^R=l^H$, in more detail in Appendix B and show that we can determine the same tipping points.

Substituting (46) into (49) and (50), we obtain

$$\lambda_t = \frac{u'(c_t)(1-\alpha)\phi_A Y_t}{V_t} \quad \text{and} \quad \zeta_t = \frac{u'(c_t)(1-\alpha)Y_t}{\theta_R R_{t-1} V_t}.$$

Combining these findings with (47), we see that

$$\frac{1}{N}\frac{u'(c_t)(1-\alpha)\phi_A Y_t}{V_t}\theta_A R_{t-1}^q + \frac{u'(c_t)(1-\alpha)Y_t}{\theta_R R_{t-1} V_t}\psi_A R_{t-1} - \frac{u'(c_t)(1-\alpha)Y_t}{V_t} = 0.$$

Simplifying and rearranging yields

$$\frac{R_{t-1}^q \theta_A \phi_A}{N} + \underbrace{\frac{\psi_A}{\theta_R}}_{\text{Knowledge Spillovers}} = 1. \tag{55}$$

The left-hand-side of the equation can be interpreted as the social benefits of AI, as workers of mass 1 that move from the final good firm to the AI sector increase the AI stock by $\frac{R_{t-1}^q \theta_A}{N}$. The additional AI stock is more productive than low-skilled workers by the factor ϕ_A leading to the overall benefit of AI in final good production. In addition, the term called *Knowledge Spillovers* shows that the social planner takes into account the spillovers from the AI sector to the AR sector, which are given by $\frac{\psi_A}{\theta_R} > 0$. An additional unit of labor in the AI sector has the productivity ψ_A in the AR sector, while the labor that is actually employed in the AR sector has the productivity θ_R. Hence, the net effect of the spillovers is given by the ratio of the two terms. We note that the benefits outweigh the marginal costs of one additional unit of labor in AI if R_{t-1} is sufficiently large. For this reason, the social planner prefers to allocate all high-skilled agents from the final good firm to the AI sector if

$$R_{P_1}^* = R_{P_2}^* > \left((1 - \frac{\psi_A}{\theta_R})\frac{N}{\phi_A \theta_A}\right)^{\frac{1}{q}}. \tag{56}$$

As the social planner does not distinguish between entrepreneurs and high-skilled agents, their corresponding tipping points for a transition from the final good firm to the AI sector are identical and $R_{P_1}^* = R_{P_2}^*$.

After deriving the tipping points in both the decentralized solution and the social planner's solution, we now assess the differences concerning the timing of the tipping points for transitions from final good production to the AI sector. Recall that transitions of workers with skills $\eta = \{H\}$ from the final good firm to the AI sector take place if $w_t^A > w_t$ in a symmetric decentralized economy.

As defined by Equation (41), we write the corresponding threshold as

$$\frac{R^q_{t-1}\theta_A\phi_A}{N} = \underbrace{\frac{\sigma}{\sigma-1}}_{\text{Wage Effect}} . \tag{57}$$

As explained in Section 5, the markdown on wages in the monopolistic AI sector, given by the *Wage Effect*, delays the transition of high-skilled agents from the final good firm to AI, since $\frac{\sigma-1}{\sigma} < 1$. On the opposite, wages do not exist in the social planner's problem and all agents receive the same level of consumption, so that we can think of w^R_t, w^A_t and w_t as being the same. The social planner is indifferent between an allocation of high-skilled agents to the final good firm or the AI sector if

$$\frac{R^q_{t-1}\theta_A\phi_A}{N} + \underbrace{\frac{\psi_A}{\theta_R}}_{\text{Knowledge Spillovers}} = 1. \tag{58}$$

The right-hand-side of (58) can be interpreted as the marginal costs of AI, whereas the left-hand-side represents the marginal benefits. We note that the difference in the tipping points of high-skilled workers between the decentralized and the social planner's solution arises because of the *Wage Effect* in (57) and the *Knowledge Spillovers* in (58). Due to the disregard of *Knowledge Spillovers* and the markdown on wages, expressed by the *Wage Effect*, transitions of high-skilled workers from final good production to AI are delayed in a decentralized economy, compared to the social planner's solution.
Now, we focus on the transition of entrepreneurs from the final good firm to the AI sector. As shown in (38), entrepreneurs prefer to work in AI if their total income is higher when they move to the AI sector than if they stay in the final good production. Recall that this is the case above the threshold defined by

$$\frac{R^q_{t-1}\theta_A\phi_A}{N}\left(\underbrace{\frac{1}{\sigma}}_{\text{Profit Effect}} + \underbrace{\frac{\sigma-1}{\sigma}}_{\text{Wage Effect}}\right) = 1.$$

As explained in Section 5, transitions of entrepreneurs from final good production to AI are not affected by monopolistic distortions in the AI sector in a decentralized economy, as $\frac{1}{\sigma} + \frac{\sigma-1}{\sigma} = 1$, since entrepreneurs receive all the revenues of AI and do not care about market distortions. Since the social planner does not distinguish between entrepreneurs and high-skilled workers, and considers perfect competition in the AI sector, the respective condition for the allocation of entrepreneurs to AI is the same as for high-skilled agents in the social planner's solution, namely Condition (58). We see that the difference between the tipping points for entrepreneurs in the decentralized and the social planner's solution depends solely on the *Knowledge Spillovers*, since the *Wage Effect* and *Profit Effect* cancel each other out. Due to the disregard of *Knowledge Spillovers*, transitions of entrepreneurs from final good production to the AI sector are generally delayed in a decentralized economy, compared to the social planner's solution.

7.2 Allocation to AR

In a decentralized economy, high-skilled agents are indifferent between being employed in the final good sector or in the AR sector if $w_t^R = w_t$, which eventuates if

$$\theta_R \phi_R R_{t-1} = 1.$$

For investigating the allocation of agents from the final good firm to the AR sector in the social planner's solution, we substitute ζ_t, μ_t, λ_{t+1}, and ζ_{t+1} into (51) to obtain the following equation:

$$\beta^t \left[-\frac{u'(c_t)(1-\alpha)Y_t}{\theta_R R_{t-1} V_t} + \frac{u'(c_t)(1-\alpha)\phi_R Y_t}{V_t} \right] + \beta^{t+1} \frac{u'(c_{t+1})(1-\alpha)\phi_A Y_{t+1}}{V_{t+1}} (1 + \theta_A \frac{l_{t+1}^A}{N}) q R_t^{q-1}$$
$$+ \beta^{t+1} \frac{u'(c_{t+1})(1-\alpha)Y_{t+1}}{\theta_R R_t V_{t+1}} (1 + \theta_R l_{t+1}^R + \psi_A l_{t+1}^A) = 0.$$

Simplifying and rearranging yields

$$\theta_R \phi_R R_{t-1} + \overbrace{\frac{\beta u'(c_{t+1})}{u'(c_t)} \frac{V_t}{V_{t+1}} \frac{Y_{t+1}}{Y_t} \frac{R_{t-1}}{R_t}}^{\text{(I)}} \left[\overbrace{(1 + \theta_R l_{t+1}^R + \psi_A l_{t+1}^A)}^{\text{(II)}} + \overbrace{\phi_A \theta_R q R_t^q (1 + \theta_A \frac{l_{t+1}^A}{N})}^{\text{(III)}} \right]$$
$$= 1. \quad (59)$$

The right-hand-side of (59) can be interpreted as the marginal costs of AR, whereas the left-hand-side represents the marginal benefits. AR benefits today amount to $\theta_R \phi_R R_{t-1}$. A worker who moves from final good production to the AR sector increases the AR stock by $\theta_R R_{t-1}$. The additional AR stock is more productive than low-skilled workers by the factor ϕ_R leading to the overall benefit. Furthermore, there are benefits of AR that occur tomorrow and are given by (II) and (III), with (I) being an effective social discount rate. We describe these benefits in turn:
The social planner considers the *Inter-temporal Spillovers*, as he considers the effect of a higher AR stock today on the AR stock tomorrow, given by (II). Moreover, he takes into account the effect of a higher AR stock today on the AI stock tomorrow. If the stock of AR increases, AI algorithms can train and develop on a wider application area. We thus define the term (III) as the *Application Gains* of AI development which arise due to a

greater application area of AI in AR if the stock of AR grows. *Application Gains* increase in the productivity ϕ_A of AI in production, the value of returns from AR to AI, given by q, and the productivity θ_R of workers in AR. We observe that the multiplicative effect of (I), (II) and (III) yields an increase of the marginal benefits of AR. From the period on where (59) holds and which we label as $R^*_{P_3}$, the social planner favors an allocation of all high-skilled agents from the final good firm to the AR sector. As a general closed form solution for $R^*_{P_3}$ cannot be derived, we will focus on the graphical illustration of this tipping point in a numerical example in Section 9.

7.3 Allocation to AI or AR?

In the two previous subsections, we separately compared an allocation of high-skilled workers and entrepreneurs to the final good firm with an allocation to the AI sector or AR sector, respectively. The final question is whether the social planner favors an allocation of those agents to the AI or the AR sector. In a joint comparison, the sector with a larger net gain will employ all high-skilled workers and entrepreneurs. The net gain is the difference of marginal benefits minus marginal costs in a given sector. The difference in net gains between the two sectors can be studied by comparing (59) with (55). The social planner is indifferent between allocating all high-skilled agents to the AR or AI sector if the following condition holds:

$$\phi_R\theta_R R_{t-1} + \frac{\beta u'(c_{t+1})}{u'(c_t)}\frac{V_t}{V_{t+1}}\frac{Y_{t+1}}{Y_t}\frac{R_{t-1}}{R_t}\left[(1+\theta_R l^R_{t+1} + \psi_A l^A_{t+1}) + \phi_A\theta_R q R^q_t(1+\theta_A\frac{l^A_{t+1}}{N})\right]$$
$$= \frac{R^q_{t-1}\theta_A\phi_A}{N} + \frac{\psi_A}{\theta_R}. \tag{60}$$

The right-hand-side of the equation denotes the benefits of AI, whereas the left-hand-side characterizes the benefits of AR. The equation implies the threshold $R^*_{P_4}$, from which on the benefits of AR are higher than of AI, such that the social planner wants to employ all workers in AR. We explained the equation on the left side after (59), showing that the social planner considers the *Application Gains* of AR to AI and the *Inter-temporal Spillovers* of AR which play no role in the agents' decisions in a decentralized economy. Moreover, we see on the right-hand-side that the social planner takes the *Knowledge Spillovers* from AI to AR into account, given by $\frac{\psi_A}{\theta_R}$. We see that the greater the *Application Gains* of AR to AI and the *Inter-temporal Spillovers* of AR compared to the *Knowledge Spillovers* of AI, the earlier an allocation from AI to AR is being preferred in the social planner's solution compared to a decentralized economy. Nonetheless, it is not possible to derive a closed form solution for $R^*_{P_4}$. This is the reason why no general sequence of the tipping points can be determined.

We summarize the five causes leading to differences between the decentralized economy and the social planner's solution in Summary 1. Moreover, we illustrate in Table 3 how each cause—in an isolated assessment—impacts the inefficiency of a decentralized economy.

Summary 1

There are two forms of Monopolistic Distortions:

1. *A slow-fast effect through wages (abbreviated as **Wage Effect**). It arises due to markdowns on wages in the monopolistic AI sector and affects entrepreneurs and high-skilled workers, and*

2. *the countervailing **Profit Effect** that arises due to the distribution of profits to entrepreneurs in the monopolistic AI sector.*

In addition, three effects characterize the intertwining of AR and AI.

3. ***Knowledge Spillovers** from AI to AR. Due to static knowledge exchanges, AR benefits from spillovers via AI software developers. The more agents are developing AI today, the higher the knowledge spillovers on AR today.*

4. ***Application Gains** of AR to AI. There are inter-temporal effects of the stock of AR on the learning potential of AI. The higher the existing stock of AR today, the larger the application area for AI and the better AI algorithms can apply their self-learning characteristics, which increases the level of AI tomorrow.*

5. ***Inter-temporal Spillovers** of AR. There are inter-temporal effects of the stock of AR today on the stock of AR tomorrow. The higher the existing stock of AR today, the larger the stock of AR tomorrow.*

	Transition Timing			
	l^E **to AI**	l^H **to AI**	l^H **to AR**	l^E **to AR**
Wage Effect	$\otimes$	$\rightarrow$	$\leftarrow$	$\otimes$
Profit Effect				
Knowledge Spillovers	$\rightarrow$	$\rightarrow$	$\leftarrow$	$\leftarrow$
Application Gains			$\rightarrow$	$\rightarrow$
Inter-temporal Spillovers			$\rightarrow$	$\rightarrow$
Overall Effect	$\rightarrow$	$\rightarrow$	$(\rightarrow)$	$(\rightarrow)$

Table 3: Tipping Points in a Decentralized Economy Compared to the Social Planner's Solution

In short, the slow-fast *Wage Effect*, resulting from monopolistic distortions in the AI sector in a decentralized economy, *decelerates* the entry of high-skilled workers to the AI sector, but *accelerates* transitions from the AI sector to the AR sector. For entrepreneurs,

the slow-fast *Wage Effect* and *Profit Effect* balance each other and do not affect their transition timing. A social planner starts from the premise of a competitive AI sector and takes into account *Knowledge Spillovers*, *Inter-temporal Spillovers* and *Application Gains*. In a decentralized economy, agents do not consider *Knowledge Spillovers*, which leads to delayed entries in AI, and premature transitions from AI to AR if other externalities and market distortions are absent. Finally, as agents do not consider the *Application Gains* of AI due to an increasing stock of AR and *Inter-temporal Spillovers* of AR, they transition later from AI to AR in a decentralized economy than in the social planner's solution.
We summarize our findings on the tipping points in the social planner's solution in the following manner:

Summary 2
Let R_0 be the economy's initial endowment with blueprints and $R^*_{P_1}, R^*_{P_2}, R^*_{P_3}$ and $R^*_{P_4}$ be the tipping points in the social planner's solution. We have the following possible constellations in the social planner's optimum:

(i) If $R_0 < R^*_{P_1}$ and $R_0 < R^*_{P_3}$, Constellation (17) is the labor market equilibrium and the economy will not grow.

(ii) If $R^*_{P_3} \leq R_t < R^*_{P_1}$, the social planner will employ all high-skilled workers in AR but not in AI, as $R^*_{P_3} \leq R_t < R^*_{P_1}$ and $R^*_{P_4} \geq R_t$. The economy will grow and converge to the steady state described in Section 6.

(iii) If $R^*_{P_1} \leq R_t < R^*_{P_3}$, the social planner will employ all skilled workers in AI but not in AR, as $R^*_{P_1} \leq R_t < R^*_{P_3}$ and $R^*_{P_4} < R_t$. The economy will grow and converge to the steady state described in Section 6.

(iv) As $R^*_{P_1} < R^*_{t_1}$, the social planner promotes the development of AI earlier than in a decentralized economy.

The tipping points in the social planner's solution differ from the ones in the decentralized solution. It holds that the development of AI starts and ends earlier in the social planner's solution than in a decentralized economy. In addition, in both the decentralized economy or the social planner's solution—after passing all tipping points, constant employment in all sectors is attained. Therefore, following the same arguments as in Section 6, the economy in the social planner's solution will reach the same BGP as a decentralized economy.
At this point, we note that the overall effect of the abovementioned causes for differences between the decentralized solution and social planner solution regarding the transitions of entrepreneurs and high-skilled agents from AI to AR is not obvious, since there is no closed form solution for $R^*_{P_3}$ and $R^*_{P_4}$. However, we observe that the larger the *Application Gains* and the *Inter-temporal Spillovers*, and the lower the *Knowledge Spillovers* of AI,

the earlier the social planner will reallocate all agents from AI to AR. The first two forces increase the marginal benefits of AR and thus the first line of Equation (60) while the third force decreases the marginal benefits of AI and thus the second line of (60).

8 Policy Implementation

In this section we explore whether and how the social planner's solution can be achieved by tax and subsidy policies in the decentralized solution. We showed that an initial research stock—higher than in the social planner's solution—is necessary to encourage individuals to move from the final good firm to the AI sector in a decentralized economy. Furthermore, workers remain in the AI sector for too long before they opt for employment in AR. We now examine how to reduce these inefficiencies through policy taxes and subsidies, namely a tax on the profits of AI firms, a subsidy on AI intermediates and an AI-tax on the price for each AI intermediate. We assume that policy measures, if they create a net burden for the government, will be financed through a lump sum tax on all agents in order to avoid distortionary tax effects.

First, we consider a tax τ_t on the profits of AI firms, in combination with a subsidy z_t on the price for AI intermediates. The goal is to enforce a socially optimal timing of the transition of entrepreneurial-skilled and high-skilled workers from final good production to the AI sector in a decentralized economy. We assume that the government subsidizes each AI intermediate with $z_t \in (0, 1)$. Hence, the effective price for the final good firm to buy an AI intermediate is $(1 - z_t)p_{t,j}$. This subsidy shall be paid as long as not all entrepreneurial-skilled and high-skilled workers have moved to the AI sector, that is as long as $l_t^A < l^H + l^E$. Such a subsidy fosters transitions of workers from final good production to AI and will be suspended when all entrepreneurial-skilled and high-skilled individuals are employed in AI. As long as the subsidy is paid, the profits of the final good firm, given by (7), can be rewritten as

$$\Pi_t = Y_t - w_t l_t^D - r_t K_t^D - \sum_{j=1}^{N} (1 - z_t) p_{t,j} A_{t,j}^D - \gamma_t R_t^D. \tag{61}$$

Maximizing (61) with respect to $A_{t,j}^D$ yields the following inverse demand function for an AI intermediate of type j:

$$p_{t,j} = \left(\frac{A_{t,j}^D}{\pi_j A_t^D} \right)^{\frac{-1}{\sigma}} \frac{\phi_A w_t}{1 - z_t}. \tag{62}$$

Equating this inverse demand with the monopolistic price for AI, given by Equation (30), we obtain the following wage in the AI sector when the subsidy z_t is applied:

$$\overline{w}_t^A = \frac{(\sigma-1)\theta_A\phi_A w_t R_{t-1}^q}{(1-z_t)\sigma}\left(\frac{A_{t,j}^D}{\pi_j A_t^D}\right)^{\frac{-1}{\sigma}}. \tag{63}$$

We observe that due to the subsidy on the price for the AI intermediates, AI firms pay higher wages to their employees.

As an additional policy instrument, we assume that a tax τ_t is applied to the profits of the AI firms. The tax τ_t is always deducted, irrespective of the labor market constellation. It eliminates the distortion in the monopolistic AI sector, i.e. it eliminates the *Profit Effect*. Consequently, we intend to find the optimal tax and subsidy rate, such that the transitions of entrepreneurial-skilled and high-skilled employees to AI occur at the social planner's optimum. In line with the arguments provided in Section 5, we next show that the condition for the transition of entrepreneurs from final good production to AI can be derived in the following way: We rewrite Condition (37) in a setting with a profit tax τ_t and an AI subsidy z_t to $(1-\tau_t)\sum_{j=1}^N \Pi_{t,j}^A + w_t l^E < (1-\tau_t)\sum_{j=1}^N \hat{\Pi}_{t,j}^A + \overline{w}_t^A l^E$, which entails

$$\left(1-\frac{\sigma-1}{(1-z_t)N\sigma}\phi_A\theta_A R_{t-1}^q\right) < R_{t-1}^q\frac{\phi_A\theta_A(1-\tau_t)}{N\sigma}, \quad \text{or simply}$$

$$R_{t_1}^{tax} > \left(\frac{N}{\phi_A\theta_A}\left[\frac{1-\tau_t}{\sigma}+\frac{\sigma-1}{(1-z_t)\sigma}\right]^{-1}\right)^{\frac{1}{q}}. \tag{64}$$

Besides, high-skilled agents prefer to work in the AI sector if it holds that $\overline{w}_t^A > w_t$. In a setting with a subsidy z_t on the price for the AI intermediates and a profit tax τ_t, this holds if

$$\frac{(\sigma-1)\theta_A\phi_A w_t R_{t-1}^q}{(1-z_t)N\sigma} > w_t, \quad \text{or simply}$$

$$R_{t_2}^{tax} > \left(\frac{N(1-z_t)\sigma}{\phi_A\theta_A(\sigma-1)}\right)^{\frac{1}{q}}. \tag{65}$$

We solve for the tax τ_t and subsidy z_t that guarantee that transitions to AI in a decentralized solution happen at the same level of AR as in the social planner's optimum. After equating (65) with (56), we find that

$$z^* = 1-\frac{(\theta_R-\psi_A)(\sigma-1)}{\theta_R\sigma}, \quad \text{if} \quad l_t^A < l^H + l^E.$$

We note that the subsidy z^* paid on the price for the AI intermediates connects the inefficiencies that arise from *Knowledge Spillovers* that depend on ψ_A and θ_R, and markdowns on wages due to monopolistic competition in AI, expressed in the *Wage Effect* $\frac{\sigma-1}{\sigma}$. After solving for the optimal subsidization rate z^*, we derive τ^* by equating (64) with (56) and obtain

$$\tau^* = 1.$$

This finding can be interpreted as follows: By introducing $\tau^* = 1$, entrepreneurs do not obtain any share of the profits that AI firms reap, as profits are taxed away completely. Therefore, the difference between high-skilled workers and entrepreneurs, namely the AI profit share for entrepreneurs, is eliminated. Consequently, both groups solely make their decision to change the sector based on wages. Thus, through the combination of z^* and τ^*, all entrepreneurial-skilled and high-skilled individuals move from final good production to AI at the socially optimum time even in a decentralized economy, entailing $R_{t_1}^{tax} = R_{t_2}^{tax} = R_{p_1}^* = R_{p_2}^*$.

Having introduced ways to enforce agents' transitions to AI at the socially optimal time, we now introduce a tax that ensures that employees do not stay in AI for too long, but move on to AR in a timely manner. If all skilled agents have moved to the AI sector, the subsidy z_t is suspended, the tax τ_t on profits is kept, and a tax x_t on the price for each AI intermediate sold is introduced which we call AI-tax. The tax x_t has to be deducted as long as all entrepreneurs and high-skilled agents work in AI, given by $l_t^A = l^H + l^E$. With an AI-tax x_t, each AI firm has to pay a levy, deducted via each intermediate sold. Thus, the profit function of a monopolistic AI firm j, taking the inverse demand of the final good firm (27) as given, is denoted by

$$\Pi_{t,j}^A = (1 - x_t) p_{t,j} A_{t,j}^D - w_t^A l_{t,j}^{A,D}. \tag{66}$$

It is straightforward to verify that maximizing (66) with respect to $p_{t,j}$ yields the following monopolistic price for AI:[19]

$$p_{t,j} = \frac{\sigma w_t^A}{\theta_A(\sigma - 1) R_{t-1}^q (1 - x_t)}. \tag{67}$$

By equating $p_{t,j}$ from (67) with (27), we deduce that wages in the AI sector are given by

$$\underline{w}_t^A = \frac{(\sigma - 1)\theta_A \phi_A w_t R_{t-1}^q (1 - x_t)}{\sigma} \left(\frac{A_{t,j}^D}{\pi_j A_t^D} \right)^{\frac{-1}{\sigma}}.$$

As entrepreneurs continue to have zero profits, the employment decision of all entrepreneurial-skilled and high-skilled workers between AI and AR depends solely on the wage relation

[19] We substitute $l_{t,j}^{A,D}$, using the AI production function, given by (1), and the price for an AI intermediate, defined by (27).

between w_t^R and w_t^A. In an equilibrium where $A_{t,j} = A_t$, the AR sector is preferred over the AI sector if $w_t^R > \underline{w}_t^A$. This holds if

$$\theta_R \phi_R R_{t-1} > \frac{(\sigma-1)\theta_A \phi_A R_{t-1}^q (1-x_t)}{N\sigma}, \quad \text{or simply} \tag{68}$$

$$R_{t_3}^{tax} > \left(\frac{(\sigma-1)(1-x_t)}{N\sigma} \frac{\phi_A \theta_A}{\phi_R \theta_R} \right)^{\frac{1}{1-q}}. \tag{69}$$

We note that the optimal tax x^* is implicitly characterized by setting $R_{P_3}^* = R_{t_3}^{tax}$. In a set-up with a AI-tax $x_t > 0$, a lower stock of blueprints is sufficient to make AR attractive. This AI-tax is necessary to internalize and to price the *Application Gains* of AI via a growing stock of AR and the *Inter-temporal Spillovers* of AR and is applied if $l_t^A = l^H + l^E$. Thus, with the help of this tax, a transition from AI to AR takes place earlier than in the unregulated decentralized solution. However, the AI-tax only has to be imposed as long as all entrepreneurial-skilled and high-skilled agents are employed in the AI sector, i.e. as long as $l_t^A > 0$, to encourage high-skilled workers and entrepreneurs to timely transition from AI to AR.

To sum up, we have to correct for the five causes that are responsible for inefficiencies in a decentralized economy. Accordingly, we propose the implementation of the following policy instruments:

Proposition 7

1. *To counteract the* ***Wage Effect*** *and* ***Knowledge Spillovers****, a subsidy z^* has to be paid on the price for AI in the time frame where $l_t^A < l^H + l^E$ holds to induce transitions of entrepreneurial-skilled and high-skilled agents from final good production to the AI sector at the socially optimal time. It has to be applied during the labor market Constellation (17) and Constellation (18)*

2. *The* ***Profit Effect****, as a result of monopolistic competition in AI, is corrected through a profit tax τ^* to equalize the income of entrepreneurial-skilled and high-skilled agents. It has to be deducted in each labor market Constellation (17) - (21).*

3. *An AI-tax x^* internalizes the inter-temporal* ***Application Gains*** *AI can exploit with an increasing stock of AR and the* ***Inter-temporal Spillovers*** *of AR, and motivates entrepreneurial-skilled and high-skilled agents to transition from AI to AR in a timely manner. The AI-tax, that needs to be paid if $l_t^A > 0$ and is thus applied in labor market Constellation (18) - (20)*

Proposition 7 describes how the socially optimal path of AI and AR development can be replicated by the use of three policy instruments. To sum up, to induce optimal economic growth, AI must first be promoted through tax and subsidy policies when its level of development is low. When AI has reached a sufficient level due to its self-learning characteristics, an AI-tax on the price for AI can be justified.

9 Numerical Example

In this section, we provide a numerical example for our theoretical model and examine when the steady state is reached. Moreover, a numerical analysis of the tipping points facilitates a comparison of the decentralized solution with the social planner's solution. The choice of our parameters is shown in Table 4.[20]

Production	$\alpha = 0.3$	$\phi_R = 1.5$	$\phi_A = 9$	B =1	$\delta = 0.05$
AI	$\theta_A = 6$	q = 0.2			
AR	$\theta_R = 3$	$\psi_A = 0.3$			
Labor	L = 1	$l^U = 0.6$	$l^H = 0.3$	$l^E = 0.1$	
Firms	$\sigma = 1.8$	N = 5			
Starting Values	$k_0 = 20000$	$R_0 = 0.05$	T = 50	$\zeta = 2$	$\beta = 0.96$

Table 4: Parameter Choice for the Numerical Example.

We note that for $R_0 = 0$ and $A_0 = 0$, entailing that AI and AR do not exist, our model would be a Ramsey–Cass–Koopmans Model (Ramsey, 1928; Cass, 1965; Koopmans, 1963) with heterogeneously-skilled agents. We assume that 60% of the labor force are low-skilled workers. Moreover, 30% are high-skilled workers that could work in the AI sector or in the AR sector. The remaining 10% are entrepreneurial-skilled agents. The substitutability between the AI intermediates is captured by $\sigma = 1.8$. In the AI sector, five firms operate under monopolistic competition. We assume low initial levels of AI and AR to incentivize investments into these sectors from the start: $R_0 = 0.05$ and $A_0 = \sqrt{0.05}$. Following Mankiw et al. (1992), we adopt a value of $\alpha = \frac{1}{3}$ in our numerical example. The remaining parameters are in Table 4. We depict the tipping points in the decentralized solution for the first 50 periods in Figure 3. Using a starting point, represented by $t_0 = 1$, where all agents are employed in the final good firm, Condition (38) defines the minimum amount of initial blueprints required for the development of AI to take place. In line with our parameter specifications, our initial level of blueprints $R_0 > R^*_{t_1} = R_{crit} = 0.000006$ fulfills this condition.

At $t_0 = 1$, entrepreneurs observe that they can earn a higher total income by moving from final good production to the AI sector as $R_0 > R_{crit}$. Therefore, entrepreneurs move to the AI sector in $t^*_1 = 2$. Consequently, they promote the development of AI—and indirectly, the number of new blueprints in the AR sector. Starting from $t^*_2 = 4$, the wage in the AI sector is higher than in the final good firm and high-skilled agents move to the AI sector. Later, at the first period where AR wages exceed AI wages, high-skilled agents move from the AI sector to the AR sector at $t^*_3 = 33$. Finally, to work in AR will also be attractive for entrepreneurs, as they can obtain the highest total income in this sector from $t^*_4 = 37$ onwards.

[20] Recall from our theoretical model the condition that $\theta_R > \psi_A > 0$. Moreover, the choice of the parameter values in the numerical example guarantee that Inequality (39) is satisfied.

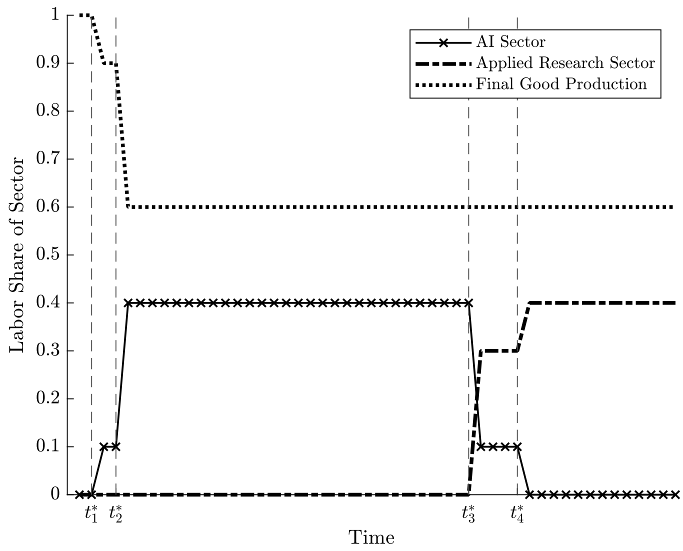

Figure 3: Tipping Points and Transitions between Sectors in a Decentralized Economy.

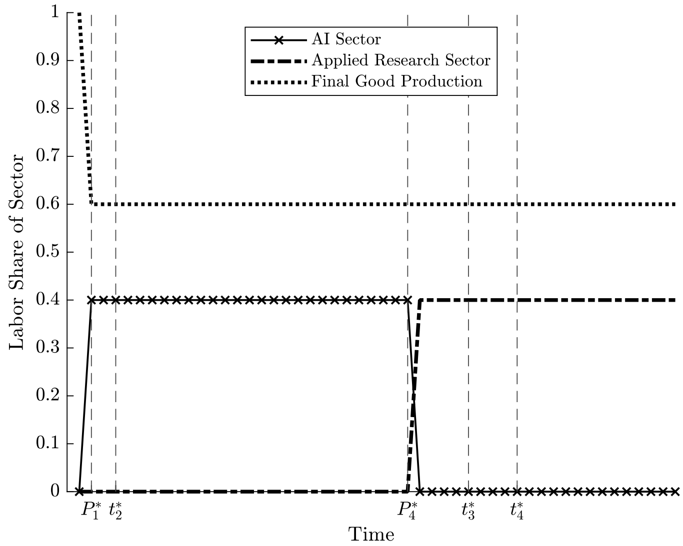

Figure 4: Tipping Points and Transitions between Sectors in the Social Planner's Solution.

To illustrate the timing inefficiencies of a decentralized economy, we compare our findings with the social planner's optimum which is depicted in Figure 4. For the decentralized solution, an initial value of $R_0 \geq R^*_{t_1}$ is necessary, such that investments in AI take place. However, the social planner requires a smaller initial amount of blueprints $R_0 \geq R^*_{t_1} \geq R^*_{P_1} = 0.000004$ to start developing AI. As the initial blueprint level is sufficiently high, the social planner allocates all skilled individuals to the AI sector from the beginning at $P^*_1 = 1$. Equation (60) shows at which period the net benefit of AR is higher than in AI production, which is the case at $P^*_4 = 28$. From then on, the social planner allocates all entrepreneurial-skilled and high-skilled workers to the AR sector. Analogously to the decentralized solution, all entrepreneurial-skilled and high-skilled workers are employed in the AR sector and low-skilled workers are employed in final good production in the long run. Nonetheless, we observe that the steady state in the social planner' solution is

reached earlier, at $P_4^* = 28$, compared to $t_4^* = 37$ in a decentralized economy.
We now illustrate in Figure 5 how the growth rates of AI and AR increase due to the socially optimal allocation of agents to the AI and AR sector. Since the social planner takes the spillovers between the AI and AR sector into account and does not face monopolistic distortions, the growth rates of AI and AR are generally higher in the social planner's solution than in a decentralized economy. Nonetheless, it can be noticed that in both, the decentralized solution and the social planner's solution, the AR stock grows at rate g_R and the AI stock at rate g_A after reaching the steady state. On this BGP, the self-learning feature of AI is particularly easy to assess, because—although no one is employed in the AI sector in the long run—the level of AI grows autonomously with rate g_A.

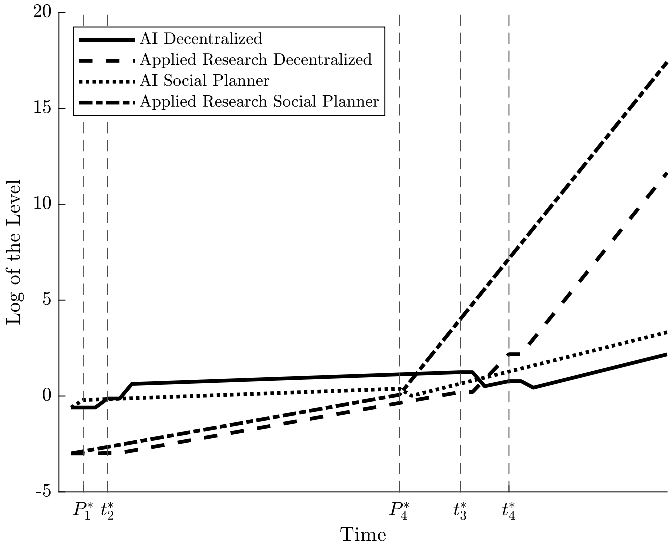

Figure 5: Log of the level of AI and AR.

To sum up, we note that in a decentralized economy, several timing inefficiencies can be documented in our numerical example, as predicted by our theoretical model. First, a higher initial blueprint stock would be needed to make AI attractive and put the economy on a growth path. Second, high-skilled workers and entrepreneurs transition too late to the AI sector, due to the distorted wage scheme resulting from monopolistic competition in the AI sector. Third, high-skilled workers and entrepreneurs transition too late to AR, as they do not internalize the positive spillovers between the AI sector and AR sector and do not consider the dynamic advantages of the AR sector.
In our numerical example, besides a 100 per cent taxation of the profits of the AI firms, a 92.00 per cent subsidy on the price for AI has to be applied in the period where AI is underdeveloped. When AI has sufficiently benefited from its self-learning characteristics due to a high AR stock, an AI-tax of 9.87 per cent is implemented to enforce socially optimal transitions of agents from AI to AR.

10 Extensions

We presented a model of AI as self-learning capital that displays sharp transitions of workers between sectors and allows for AI that can develop entirely autonomously in the long run. Of course, extensions of the model will provide a more nuanced perspective while retaining the self-learning characteristics of AI.

Patents on AR Blueprints

Our model is based on the assumption that acquired knowledge from previous periods is publicly accessible for all AR firms in the AR sector. Still, e.g. Gersbach et al. (2018) point out that firms protect their innovations over many years with the help of patent registration. If we extend our model and assume that M symmetric AR firms operate in a perfectly competitive AR sector instead of only one representative firm, we can extend our model by assuming patent protection for AR blueprints. We thus consider a more limited use of the knowledge stock of AR. Suppose that a share $\rho \in (0, 1)$ of the knowledge stock of AR is protected. Then, firms can always re-use their own AR blueprints from the last period, but only have access to the non-patented share $(1 - \rho)$ of blueprints from competing firms. Accordingly, the stock function for AR blueprints of a single firm k can be rewritten as follows:

$$R^S_{t,k} = \left[R^S_{t-1,k} + (1 - \rho) \sum_{v \neq k}^{M} R^S_{t-1,v} \right] (1 + \theta_R l^{R,D}_{t,k} + \psi_A l^{A,D}_t).$$

This entails the following wage in the AR sector:

$$w^R_t = \gamma_t \left[R^S_{t-1,k} + (1 - \rho) \sum_{v \neq k}^{M} R^S_{t-1,v} \right] \theta_R.$$

We note that the higher the share ρ of non-publicly accessible AR blueprints, the lower the wage in the AR sector. If an increasing quantity of knowledge is "closed source", the development of AR blueprints is hampered by a smaller knowledge base in AR, which negatively affects the wage in the AR sector. Lower wages in AR shift the tipping points at which employment in the AR sector becomes profitable for entrepreneurs and high-skilled workers. This means that agents are working in AI development for a longer period and that the steady state, in which nobody is employed in AI anymore, is reached at a later stage.

Basic Research on AI

We showed that the initial stock of AR is decisive for the path the economy takes. Only if the stock of AR is large enough, entrepreneurs will find it optimal to start running AI firms and develop new AI. If the stock is not large enough, entrepreneurs remain in the final good firm and no economic growth takes place. The same holds if the productivity of AI in final good production is small, since the smaller the AI productivity ϕ_A, the higher the initial stock of AR required to set the economy on a growth path. In such a situation, a publicly funded basic research sector may expand AI productivity by acquiring new ideas, theories and prototypes (Gersbach et al., 2018). Balconi and Laboranti (2006) highlight the importance of the link between universities and industries for knowledge exchanges.

In general, there has been a strong increase in the productivity of classical AI techniques over the last few years, as revealed in the AI Index Report (Zhang et al., 2021). For instance, the precision of image recognition has increased from 85% in 2013 to 99% in 2020, whereas humans perform with an accuracy of 94%. Also other measurands, such as training time, training costs or hardware costs have sunk noticeably within the same period. This advocates investments into basic AI research in order to i.e. expand the application possibilities of AI in final good production. In this sense, basic research on AI could promote innovative activity that positively affects long-term growth.

In addition, publicly-funded investments into basic research on AI would induce a different timing of the tipping points, due to new relative productivity differences between AI and AR. However, such basic research would have to be financed by taxes, of course. We propose to model the productivity of AI in the production of the final good, given by $\phi_{A,t}$, as a function of the basic research activity on AI. A simplification for the law of motion of the productivity of AI could be as follows:

$$\phi_{A,t} = \phi_{A,t-1}(1 + \theta_B l_t^B),$$

where we assume that l_t^B are workers in basic research, with a skill index $\eta \in \{H, E\}$, who search for possibilities to enhance the productivity of AI by a specific factor θ_B. However, since high-skilled workers in basic research need to be paid adequately to have an incentive to work in basic research and to leave AR or AI, the government would have to provide public funds to pay these agents. In addition, employees in basic research would have to quit their former jobs, so that production would decline at their previous employer. Thus, some employees who were previously responsible for the development of AI and AR would move to basic research, which leads to different growth dynamics in both the AI sector and the AR sector, as well as in the economy as a whole. Moreover, with a time-dependent, increasing AI productivity, the timing of the tipping points is altered over time. In addition to the policy interventions mentioned in Section 8, the promotion of basic research activities to an adequate extent offers a further possibility to enable an earlier development of AI algorithms. The higher the productivity parameter $\phi_{A,t}$, the easier to incentivize transitions of workers from final good production to AI. On

the reverse, when considering transitions from AI to AR, a higher $\phi_{A,t}$ leads to delayed transitions of agents from AI to AR.
In conclusion, basic research on AI and the patenting of AR blueprints are two factors that play a key role in the economic analysis of innovations and technological progress. If we extend our basic model by these factors, innovations in AI are made earlier and workers remain employed in the AI sector longer before they move to the AR sector. However, a more detailed analysis of the interplay of these two factors is beyond the scope of this paper. Yet, it seems to be an attractive avenue for future research.

11 Discussion

Finally, we would like to address some issues that connect our results to other issues discussed in the literature.

Market Power of AI Firms
Korinek and Stiglitz (2021) and Autor et al. (2020) assess that the production of upcoming information technologies—such as AI—involves the rise of natural monopolies or so-called "superstar firms". Today, Tech Giants are gaining market power and may even influence important election processes (Rathi, 2019). In general, market distortions due to a concentration of market power in a handful of firms may offset certain benefits of innovation. This could justify political interventions. For example, some firms' great bargaining power may have an influence on wage negotiations and may increase fluctuations in unemployment (Lu, 2020). Moreover, the huge market capitalization and economic power of Tech Giants facilitates the acquisition of promising start-ups (Makridakis, 2017; Gersbach, 2020), enhances the growth potential of these companies and may reinforce their monopolistic position. In our model, we assume from the beginning that AI firms operate under monopolistic competition. However, the monopoly position that some Tech Giants hold today is relatively new. Therefore, it would be useful to set the market power of AI firms in relation to the state of development of AI, as an extension to our model.

Technological Unemployment
As soon as AI is invented, it is probable that firms adjust their production and start substituting human labor by AI, leading to potential labor market frictions. For instance, Autor and Salomons (2018) refer to technological process as "employment-augmenting but labor-displacing". However, we do not examine the potential replacement of human labor by AI. Moreover, we do not consider labor market frictions or technological unemployment, issues that are addressed by e.g. Hémous and Olsen (2014); Korinek and Stiglitz (2017); Acemoglu and Restrepo (2018a,b), who analyze the effects of modern

technology—not only of AI—on the labor market. Nonetheless, even if full employment was preserved, our theoretical model shows an increasing wage divergence between high-skilled and low-skilled workers, due to different wage growth rates in the sectors, leading to greater inequality (Furman and Seamans, 2019). Especially developing countries may lack the institutional set-up to counteract the rise in inequality induced by unequally-distributed skill levels and technological advances (Korinek and Stiglitz, 2021). This should not be ignored in future research.

Industry-specific Effects of AI on the Factor Shares in Production
Jones and Romer (2010) point out that already during the 20^{th} century, there were concerns that increasing technological progress would render one of the prominent Kaldor facts untrue (Kaldor, 1961)—a *constant* labor share in national income. The coming decades will show whether this statement stays relevant for economies with growing levels of AI. Indeed, Autor and Salomons (2018) reveal that the trend of a decreasing labor share was apparent in many countries in the last decades. The explanation of Karabarbounis and Neiman (2014) for this decline is that especially advances in information technology that affect the price of investment goods led to a factor shift from labor to capital. Yet, AI will have disparate effects on factor shares in specific industries, as technology-induced effects on employment are indubitably industry-dependent (Bessen, 2019). In particular, the link between capital and labor (Korinek and Stiglitz, 2021) and the substitution elasticity of human labor, capital and AI in a specific industry play an essential role in the analysis of the effects of AI on the factor shares in production.
Our model is characterized by the following elasticities of substitution: $\sigma_{K,L} = \sigma_{K,A} = \sigma_{K,R} = 1$ and $\sigma_{A,L} = \sigma_{R,L} = \infty$. In addition, we do not include an industry-specific or time-dependent substitution elasticity of the input factors in production. It could be particularly interesting to explore whether AI is more likely to substitute with labor or capital by using more flexible ways of modeling the elasticity of substitution between the input factors. For example, it might be useful to extend our model with (i) a variable elasticity of substitution (VES)—where input factors have a flexible elasticity of substitution (Lu, 2020) that may change over time (Paul, 2019), depending on the development of AI or (ii) an industry-specific AI productivity. In this sense, the effects of AI on the factor shares in production, depending on the elasticity of input factors and the industry-specific AI productivity, could be examined in more detail.

12 Conclusion

This paper focuses on the effect of the emergence of AI and evolution of AI. With a three-sector model with a final good sector that uses AR blueprints and AI algorithms for production, transition dynamics of workers between the sectors are examined which drive the dynamics of AI, AR and final output. The novelty of our approach is that the self-learning feature of AI which allows that AI can grow in the long-run even if little or no labor is employed.

Due to (i) monopolistic market distortions affecting both the wages and profits in the AI sector, (ii) positive knowledge spillovers of workers employed in AI on the development of AR, (iii) application gains of AI benefiting from an increasing stock of AR, and (iv) inter-temporal spillovers of AR, a decentralized economy does not yield a socially optimal development of AI—a finding we also illustrate in a numerical example.

We provide a macroeconomic rationale for several policy interventions and show how a mix of taxes and subsidies can promote the optimal evolution of AI into an economy. When the level of AI is low, subsidization of AI is justified. In addition, taxing AI profits can promote the early development of AI and reduce the monopolistic distortions in the AI sector. When AI is more developed and sufficiently benefits from its self-learning ability by application in AR, the introduction of an AI-tax on the price for AI prevents agents from staying in AI development for too long which fosters growth enhancing AR development. The introduction of the abovementioned policy instruments can ensure that the balanced growth path replicates the socially optimal path. Finally, we describe how basic research on AI and patenting of AR blueprints might affect the development of AI and the labor market transitions of workers, and we discuss the shortcomings of our model. Our discussion in Section 10 and Section 11 points at several useful avenues for future research.

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List of Symbols and Variables

Abbreviation	Description
$A_{t,j}$	Effective AI stock of a firm j in the AI sector at time t
B	Index of Total factor productivity
$c_{t,\eta}$	Agent η's consumption in period t
^{D}	Index for the demand of a good
E	Indicator for an Entrepreneur
g_A	Growth rate of AI
g_C	Growth rate of Consumption
g_K	Growth rate of Capital
g_R	Growth rate of AR
g_Y	Growth rate of Production
H	Indicator for a High-Skilled Worker
$K_{t,\eta}$	Physical capital in period t of a worker group $\eta \in \{U, H, E\}$
L	Fixed labor supply
l_t	Workers in Final Good production
l_t^A	Workers in the AI sector
l_t^B	Workers in Basic Research
l^E	Entrepreneurs in the labor force
l^H	High-skilled agents in the labor force
l_t^R	Workers in the AR sector
l^U	Low-skilled agents in the labor force
N	Number of firms in the monopolistic AI sector
$p_{t,j}$	Inverse demand of an AI intermediate of type j at period t
q	Marginal benefits of a higher AR stock on AI development
r_t	Interest Rate in period t
R_t	Stock of AR produced until period t
$R^*_{t_b}$	Tipping Point in the Social Planner's solution, $b \in \{p_1, p_2, p_3, p_4\}$
R_{crit}	Required amount of AR blueprints to set the economy on a growth path
$R^*_{t_f}$	Tipping Point in the decentralized solution, $f \in \{t_1, t_2, t_3, t_4\}$
$R^{tax}_{t_f}$	Tipping Point in the decentralized solution with policy instruments, $f \in \{t_1, t_2, t_3, t_4\}$
$s_{t,\eta}$	Savings made by an agent with skill level η in period t
^{S}	Index for the supply of a good
^{ss}	Index indicating a steady state value
t	Subscript indicating the time period

$u(c_{t,\eta})$	instantaneous utility function, depending on agent η's consumption
U_η	Life-time utility of an agent with skill η
U	Indicator for a Low-Skilled Worker
w_t	Wage for a worker in Final Good production at period t
w_t^A	Wage for a worker in AI at period t
$\overline{w}_t^A$	Wage for a worker in AI at period t if subsidy z_t is applied
$\underline{w}_t^A$	Wage for a worker in AI at period t if tax x_t is applied
w_t^R	Wage for a worker in the AR sector at period t
x_t	AI-tax on the price for each AI intermediate sold
Y_t	Production of a consumption good
z_t	Subsidy on AI intermediates in period t
α	Share of capital in production
β	Discount factor of individual consumption
γ_t	Inverse demand for an AR intermediate at period t
δ	Capital depreciation rate
ζ_t	Lagrange Parameter for period t in Social Planner Solution
η	Skill-level of an agent, where $\eta \in \{U, H, E\}$
θ_A	Worker Productivity in the AI sector
θ_B	Worker Productivity in the Basic Research sector
θ_R	Worker Productivity in the AR sector
κ	Arrow-Pratt Parameter for Risk aversion in individual utility
λ_t	Lagrange Parameter for period t in Social Planner Solution
μ_t	Lagrange Parameter for period t in Social Planner Solution
ξ_t	Lagrange Parameter for period t in Social Planner Solution
π_j	Share of an AI variant j in the composite AI supply
Π_t	Profit of the final good firm in period t
$\Pi_{t,j}^A$	Profit of a firm j at period t in the AI sector
$\hat{\Pi}_{t,j}^A$	Profit of a firm j at period t in the AI sector, if all entrepreneurs are employed in the AI sector
Π_t^R	Profit of the representative AR firm at period t
ρ	Share of protected AR blueprints
σ	Elasticity of Substitution between AI variants from different AI firms
τ_t	Tax on AI profits in period t
ψ_A	Knowledge exchange effect of workers in AI on AR development

A Appendix

A.1 Derivation of the Euler Equation in a Decentralized Economy

In a decentralized economy, each individual optimizes his consumption under the conditions on the evolution of the capital stock (9) and an individual budget constraint. Recall that depending on the sector of employment, the individuals' budget constraints differ. For instance, a constellation where all entrepreneurs are employed in AI, high-skilled agents are employed in AR and low-skilled agents work in the final good production, is characterized by (21). In such a constellation, the individual budget constraints are specified by (10), (11) and (12) and the optimization problem reads as follows:

$$
\begin{aligned}
&\text{argmax} \quad U_\eta = \sum_{t=0}^{\infty} \beta^t u(c_{t,\eta}), \\
&\text{s.t.} \quad K_{t+1,\eta} = (1-\delta)K_{t,\eta} + s_{t,\eta}, \\
&c_{t,\eta} + s_{t,\eta} = w_t + r_t K_{t,\eta} + \Pi_t^R \quad \text{for} \quad \eta \in \{U\} \quad \text{in final good production}, \\
&c_{t,\eta} + s_{t,\eta} = w_t^R + r_t K_{t,\eta} + \Pi_t^R \quad \text{for} \quad \eta \in \{H\} \quad \text{in AR}, \\
&c_{t,\eta} + s_{t,\eta} = w_t^A + r_t K_{t,\eta} + \Pi_t^R + \sum_{j=1}^{N} \Pi_{t,j}^A \quad \text{for} \quad \eta \in \{E\} \quad \text{in AI}, \\
&\text{with } K_{0,\eta} \text{ given.}
\end{aligned}
$$

By setting up a Lagrange Equation for an individual with skill level η with a Lagrange multiplier $\lambda_{t,\eta}$ and taking into account the evolution of the capital stock, we obtain the following first order conditions:

$$
\begin{aligned}
FOC_{c_{t,\eta}}: &\quad \beta^t u'(c_{t,\eta}) - \lambda_{t,\eta} \overset{!}{=} 0, \\
FOC_{K_{t+1,\eta}}: &\quad -\lambda_{t,\eta} + \lambda_{t+1,\eta}(1-\delta) + \lambda_{t+1,\eta} r_{t+1} \overset{!}{=} 0.
\end{aligned}
$$

Combining the first order conditions translates into the following Euler Equation:

$$
\frac{u'(c_{t,\eta})}{u'(c_{t+1,\eta})} = \beta\left(1 - \delta + r_{t+1}\right).
$$

We note that the Euler Equation is independent of the individual skill level η. Moreover, it is easy to verify that the same Euler Equation holds in all other possible labor market constellations.

A.2 Growth of AI on a BGP

On a BGP, where $l_t^A = 0$ and the growth rate of AR blueprints is given by g_R, we write

$$\begin{aligned} A_{t+1} &= A_t(1+g_A) = R_{t-1}^q(1+g_A) \rightarrow \\ (1+g_A)^{\frac{1}{q}} &= \frac{R_t}{R_{t-1}} = \frac{R_{t-1}(1+g_R)}{R_{t-1}} \rightarrow \\ (1+g_A) &= \left(\frac{R_{t-1}(1+g_R)}{R_{t-1}}\right)^q = (1+g_R)^q \rightarrow \\ g_A &= (1+g_R)^q - 1. \end{aligned}$$

B Appendix

B.1 Convergence to the Steady State

Recall that after passing all tipping points, labor market Constellation (21) is in place, in which all entrepreneurs and high-skilled workers are employed in the AR sector, but the low-skilled agents work in final good production. We use $c_{t,H}$ for the consumption of a high-skilled agent and $c_{t,U}$ for the consumption of a low-skilled agent. The corresponding value for an entrepreneur is denoted by $c_{t,E}$. Consumption of high-skilled workers employed in AR grows due to increasing wages, growing profits in the AR sector, and the accumulation of capital. On top of that, entrepreneurs additionally obtain their share of the profits from the AI sector. The consumption of low-skilled workers in the final good firm increases only due to the accumulation of capital over time and growing profits in the AR sector. Considering the individual savings denoted by (9), the budget constraint of a low-skilled worker U, employed in the final good firm, can be rewritten as

$$K_{t+1,U} = w_t + (1-\delta+r_t)K_{t,U} - c_{t,U} + \Pi_t + \Pi_t^R,$$

whereas for a high-skilled worker H in AR, we have

$$K_{t+1,H} = w_t^R + (1-\delta+r_t)K_{t,H} - c_{t,H} + \Pi_t + \Pi_t^R,$$

and for an entrepreneur E in AR

$$K_{t+1,E} = w_t^R + (1-\delta+r_t)K_{t,E} - c_{t,E} + \Pi_t + \Pi_t^R + \sum_{j=1}^{N} \Pi_{t,j}^A.$$

We add the three equations and integrate over all agents, while assuming that all agents of a specific type act in the same way, to obtain the aggregate capital stock.[21] This yields

$$
\begin{aligned}
K_{t+1} =& l^U K_{t+1,U} + l^H K_{t+1,H} + l^E K_{t+1,E} \\
=& (1-\delta+r_t)K_t - C_t + l^U w_t + (l^H + l^E) w_t^R + \Pi_t + \Pi_t^R + \sum_{j=1}^{N} \Pi_{t,j}^A.
\end{aligned}
$$

As demonstrated in Section 5, the economy will undergo a process with several tipping points if $R_0 \geq R^{crit}$, finally leading to an equilibrium where Condition (21) holds. We replace the interest rate and the wages in the final good firm and AR sector by the equilibrium values, specified in Table 1. Accordingly, we can rewrite the aggregate profits in the sectors based on the equilibrium wages and given the labor force in the specific sectors after reaching the steady state to obtain

$$
\sum_{j=1}^{N} \Pi_{t,j}^A = \phi_A w_t R_{t-1}^q, \quad \Pi_t^R = \phi_R w_t R_{t-1} \quad \text{and} \quad \Pi_t = 0,
$$

where we use the evolution of AR blueprints given by (3) and the fact that $l_t = l^U$, $l_t^A = 0$, and $l_t^R = l^H + l^E$ after passing all tipping points. It follows that

$$
\begin{aligned}
K_{t+1} =& \left(1-\delta+\alpha B\left(l_t + \phi_A A_t + \phi_R R_t\right)^{1-\alpha} K_t^{\alpha-1}\right) K_t - C_t + \\
& (1-\alpha) B \left(l_t + \phi_A A_t + \phi_R R_t\right)^{-\alpha} K_t^{\alpha} \left\{ l_t + \phi_A R_{t-1}^q + \phi_R R_{t-1}(1 + \frac{\theta_R l_t^R}{M}) \right\}.
\end{aligned}
$$

In the long run, R_t becomes arbitrarily large, and when we divide the above expression by R_t, we can neglect the terms l_t/R_t, l_t^R/R_t, R_{t-1}^q/R_t and A_t/R_t. Defining $k_t \equiv K_t/R_t$ and $c_t \equiv C_t/R_t$, we obtain[22]

$$
\begin{aligned}
k_{t+1}(1+\theta_R(l^H+l^E)) &= \left(1-\delta+\alpha B\left(\frac{\phi_R}{k_t}\right)^{1-\alpha}\right) k_t - c_t + (1-\alpha)B\left(\frac{\phi_R}{k_t}\right)^{-\alpha}\phi_R \\
&= (1-\delta)k_t + \alpha B \phi_R \left(\frac{\phi_R}{k_t}\right)^{-\alpha} + (1-\alpha) B \phi_R \left(\frac{\phi_R}{k_t}\right)^{-\alpha} - c_t \\
&= (1-\delta)k_t + k_t^{\alpha} \phi_R^{1-\alpha} B - c_t. \qquad (70)
\end{aligned}
$$

Next, we consider the Euler equation given by (13), which holds for all individuals.

[21]The aggregate capital stock and aggregate consumption are defined by $K_t = l^U K_{t,U} + l^H K_{t,H} + l^E K_{t,E}$ and $C_t = l^U c_{t,U} + l^H c_{t,H} + l^E c_{t,E}$.

[22]Assuming in a steady state that $R_{t+1} = R_t(1+\theta_R(l^H+l^E))$, we obtain $k_{t+1} = \frac{K_{t+1}}{R_t} = \frac{K_{t+1}(1+\theta_R(l^H+l^E))}{R_{t+1}}$ and $c_{t+1} = \frac{C_{t+1}}{R_t} = \frac{C_{t+1}(1+\theta_R(l^H+l^E))}{R_{t+1}}$.

Combined with an iso-elastic utility function $u(C) = \frac{C^{1-\kappa}}{1-\kappa}$, with $\kappa < \infty$ and $\kappa \neq 1$, it follows that

$$\left(\frac{C_{t+1,m}}{C_{t,m}}\right)^{\kappa} = \beta\left(1 - \delta + \alpha B\left(l_t + \phi_A A_t + \phi_R R_t\right)^{1-\alpha} K_t^{\alpha-1}\right).$$

Rewriting K_t/R_t as k_t and C_t/R_t as c_t, and neglecting the arbitrarily small terms, we obtain

$$\left(\frac{c_{t+1}(1+\theta_R(l^H+l^E))}{c_t}\right)^{\kappa} = \beta\left(1 - \delta + \alpha B\left(\frac{\phi_R}{k_t}\right)^{1-\alpha}\right). \tag{71}$$

Along a steady state, we have $k = k_{t+1} = k_t$ and $c = c_{t+1} = c_t$, such that (70) and (71) yield the following steady state values:

$$c^{ss} = (k^{ss})^{\alpha}\phi_R^{1-\alpha}B - (\delta + \theta_R(l^H + l^E))k^{ss} \text{ and}$$
$$k^{ss} = \left(\left(\frac{(1+\theta_R(l^H+l^E))^{\kappa}}{\beta} + \delta - 1\right)\frac{1}{\alpha B}\right)^{\frac{1}{\alpha-1}}\phi_R.$$

Given Equation (71) which depends on the values for c_t and k_t, we note that for $k > k_t$, the fraction on the left-hand-side has to diminish, entailing that $c_{t+1} < c_t$, and vice versa, for $k < k_t$. Accordingly, as shown in (70), we can see that, for $c_t > c$, it has to hold that $k_t < k_{t+1}$ and $k_{t+1} < k_t$, for $c_t < c$. These links between c and k allow us to depict the steady state values and the described dynamics in a phase diagram in Figure 6. We find saddle-path stability that can occur either from the left lower sector or the right upper sector. Hence, Figure 6 shows that given some initial conditions on k_0, R_0 and A_0, there exists a unique path of the economy and that this BGP converges to the steady state.

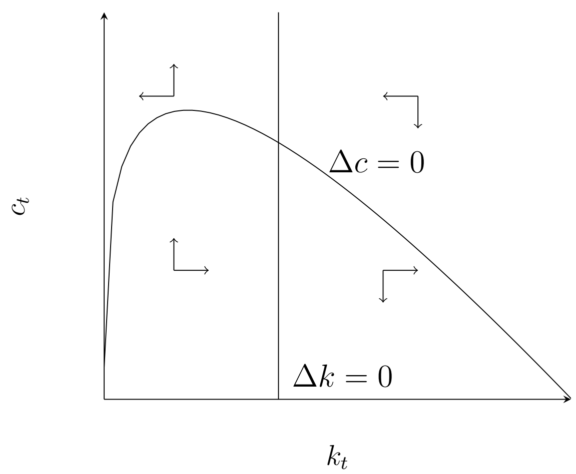

Figure 6: Phase Diagram for Consumption and Capital per Level of AR.

B.2 Complementary Slackness Condition in the Social Planner's Problem

In the social planner's solution, we have to take into account whether $\xi_t = 0$ or $l_t^A + l_t^R = l^H + l^E$ holds. The complementary slackness condition requires that $\xi_t(l_t^A + l_t^R - l^H - l^E) = 0$. Thus, we distinguish between the following two cases:

1. $l_t^A + l_t^R < l^H + l^E$: The constraint on the maximum amount of workers in AI and AR is not binding and is ineffective. For the complementary slackness condition to hold, we thus have to show that $\xi_t = 0$.
2. $l_t^A + l_t^R = l^H + l^E$: The social planner favors an allocation where all high-skilled individuals and entrepreneurs are employed in the AR sector or the AI sector. For the complementary slackness condition to hold, we thus have to show that $\xi_t \geq 0$.

In the following, we show that the value for $\xi_t \in \mathbb{R}_{\geq 0}$ depends on the allocation of entrepreneurial-skilled and high-skilled workers to the sectors and thus on the tipping points, as shown in Section 7. We have to distinguish between the case when the social planner favors an allocation of the agents to the (i) AI sector, or to the (ii) AR sector.[23]

(i) Substituting (46) into (49) and (50), we obtain

$$\lambda_t = \frac{u'(c_t)(1-\alpha)\phi_A Y_t}{V_t} \quad \text{and} \quad \zeta_t = \frac{u'(c_t)(1-\alpha)Y_t}{\theta_R R_{t-1} V_t} + \frac{\xi_t}{\theta_R R_{t-1}}.$$

Combining these findings with (47), we see that

$$\frac{1}{N}\frac{u'(c_t)(1-\alpha)\phi_A Y_t}{V_t}\theta_A R_{t-1}^q + \zeta_t \psi_A R_{t-1} - \frac{u'(c_t)(1-\alpha)Y_t}{V_t} - \xi_t = 0.$$

After inserting ζ_t into this equation, we are in a position to show that this entails

$$\xi_t = \frac{\theta_R}{\theta_R + \psi_A}\frac{u'(c_t)(1-\alpha)Y_t}{V_t}\left(\frac{1}{N}\theta_A R_{t-1}^q \phi_A + \frac{\psi_A}{\theta_R} - 1\right).$$

As $\frac{\theta_R}{\theta_R+\psi_A}\frac{u'(c_t)(1-\alpha)Y_t}{V_t} > 0$ by construction, the value of ξ_t depends on the term $\frac{1}{N}\theta_A R_{t-1}^q \phi_A + \frac{\psi_A}{\theta_R} - 1$. We note that this term is equivalent to the tipping point for transitions of agents from the final good firm to the AI sector, given by Condition (55).

[23] We separately compare the allocation of high-skilled workers and entrepreneurs to the final good firm with an allocation to the AI sector or the AR sector, respectively. We neglect transitions of agents between the AI sector and the AR sector, where the sector with a larger net gain will employ all high-skilled workers and entrepreneurs as $l_t^A + l_t^R = l^H + l^E$ applies in any case.

We can thus determine how ξ_t is specified, depending on the employment in AI

$$
\begin{aligned}
&\text{(I)} \quad \xi_t > 0 \quad \text{and} \quad l_t^A + l_t^R = l^H + l^E \quad \text{if} \quad \frac{1}{N}\theta_A R_{t-1}^q \phi_A > \frac{\psi_A}{\theta_R} - 1, \\
&\text{(II)} \quad \xi_t = 0 \quad \text{and} \quad l_t^A + l_t^R < l^H + l^E \quad \text{if} \quad \frac{1}{N}\theta_A R_{t-1}^q \phi_A \leq \frac{\psi_A}{\theta_R} - 1.
\end{aligned}
$$

In case (II), entrepreneurs and high-skilled workers prefer an employment in the final good firm compared to the AI sector. This implies that they stay in the final good firm and that $l_t^A + l_t^R < l^H + l^E$. The constraint on the maximum amount of workers in AI and AR is not binding and is ineffective. Therefore, the marginal utility of relaxing the constraint is zero and $\xi_t = 0$. In case (I), entrepreneurs and high-skilled workers prefer an employment in the AI sector, compared to the final good production. This implies that they leave the final good firm, work in the AI sector and that $l_t^A + l_t^R = l^H + l^E$. The constraint on the maximum amount of workers in AI and AR is binding and as a result, $\xi_t \geq 0$. To sum up, we have $\xi_t = 0$ before the tipping point defined by Equation (55) and $\xi_t \geq 0$ afterwards.

(ii) Substituting λ_t, ζ_{t+1}, ζ_t and μ_t into (51), we obtain

$$
\begin{aligned}
&\beta^t \left[-\frac{u'(c_t)(1-\alpha)Y_t}{\theta_R R_{t-1} V_t} - \frac{\xi_t}{\theta_R R_{t-1}} + \frac{u'(c_t)(1-\alpha)\phi_R Y_t}{V_t} \right] \\
&+ \beta^{t+1} \left(\frac{u'(c_{t+1})(1-\alpha)\phi_A Y_{t+1}}{V_{t+1}} \right) (1 + \theta_A \frac{l_{t+1}^A}{N}) q R_t^{q-1} \\
&+ \beta^{t+1} \left(\frac{u'(c_{t+1})(1-\alpha)Y_{t+1}}{\theta_R R_t V_{t+1}} + \frac{\xi_{t+1}}{\theta_R R_t} \right) (1 + \theta_R l_{t+1}^R + \psi_A l_{t+1}^A) = 0.
\end{aligned}
$$

We proceed by showing that

$$
\begin{aligned}
\theta_R \phi_R R_{t-1} = 1 &+ \frac{V_t}{u'(c_t)(1-\alpha)Y_t} (\xi_t - \beta \xi_{t+1}) - \\
&\underbrace{\frac{\beta u'(c_{t+1})}{u'(c_t)} \frac{V_t}{V_{t+1}} \frac{Y_{t+1}}{Y_t} \frac{R_{t-1}}{R_t} \left[(1 + \theta_R l_{t+1}^R + \psi_A l_{t+1}^A) + \phi_A \theta_R q R_t^q (1 + \theta_A \frac{l_{t+1}^A}{N}) \right]}_{\text{D}}.
\end{aligned}
$$

Without a loss of generality, we now turn to the case where $\xi_t = \xi_{t+1}$ and write

$$
\xi_t = \frac{u'(c_t)(1-\alpha)Y_t}{V_t(1-\beta)} \left(\theta_R \phi_R R_{t-1} - 1 + D \right).
$$

As is holds that $\frac{u'(c_t)(1-\alpha)Y_t}{V_t(1-\beta)} > 0$ by construction, the value of ξ_t depends on the term $(\theta_R \phi_R R_{t-1} - 1 + D)$. We note that this term is equivalent to the tipping point for transitions of agents from the final good firm to the AR sector, given by Condition (59).

We can thus determine how ξ_t is specified, depending on the employment in AR,

$$
\begin{aligned}
&\text{(I)} \quad \xi_t > 0 \quad \text{and} \quad l_t^A + l_t^R = l^H + l^E \quad \text{if} \quad \theta_R \phi_R R_{t-1} > 1 + D, \\
&\text{(II)} \quad \xi_t = 0 \quad \text{and} \quad l_t^A + l_t^R < l^H + l^E \quad \text{if} \quad \theta_R \phi_R R_{t-1} \leq 1 + D.
\end{aligned}
$$

In case (II), entrepreneurs and high-skilled workers prefer an employment in the final good firm, compared to the AR sector. This implies that they stay in the final good firm and that $l_t^A + l_t^R < l^H + l^E$. The constraint on the maximum amount of workers in AI and AR is not binding and is ineffective. Therefore, the marginal utility of relaxing the constraint is zero and $\xi_t = 0$. In case (I), entrepreneurs and high-skilled workers prefer an employment in the AR sector, compared to the final good production. This implies that they leave the final good firm, work in the AR sector and that $l_t^A + l_t^R = l^H + l^E$. The constraint on the maximum amount of workers in AI and AR is binding and as a result, $\xi_t \geq 0$. Again, $\xi_t = 0$ before the tipping point defined by Equation (59) and $\xi_t \geq 0$ afterwards.

To sum up, we have shown how the value for $\xi_t \in \mathbb{R}_{\geq 0}$ depends on the distribution of high-skilled workers and entrepreneurs to the sectors, as defined in Section 7. We see that the value of ξ_t is determined by those same conditions that also define the tipping points, and we note that the complementary slackness condition $\xi_t \left(l_t^A + l_t^R - l^H - l^E\right) = 0$ always holds.

C Special Case q=1

For the special case of $q = 1$, the self-learning of AI has constant returns. We show how this assumption affects the tipping points the possible long-run growth rate of the economy:
The first and the second tipping point in a decentralized economy are given by (38) and (41) so that we have $R_{t_1}^* > \left(\frac{N}{\phi_A \theta_A}\right)$ and $R_{t_2}^* > \left(\frac{N}{\phi_A \theta_A}\left(\frac{\sigma-1}{\sigma}\right)^{-1}\right)$ when $q = 1$. The respective conditions on parameter that we obtain from (39) and (42) are

$$
\frac{1}{\phi_R \theta_R} > \frac{N}{\phi_A \theta_A} \quad \text{and} \quad \frac{1}{\phi_R \theta_R} > \frac{N\sigma}{(\sigma - 1)\phi_A \theta_A}. \tag{72}
$$

Recall that the third tipping point which is characterized by $w_t^R > w_t^A$ reads

$$
w_t \theta_R \phi_R R_{t-1} > \frac{(\sigma - 1)}{N\sigma} \theta_A \phi_A w_t R_{t-1}^q
$$

and hence the AR sector becomes favorable for high-skilled workers if

$$\theta_R \phi_R > \frac{(\sigma - 1)}{N\sigma} \theta_A \phi_A. \tag{73}$$

The fourth tipping point is determined by

$$w_t^R l^E + \sum_{j=1}^{N} \Pi_{t,j}^A > w_t^A l^E + \sum_{j=1}^{N} \hat{\Pi}_{t,j}^A,$$

and it follows that the AR sector becomes favorable for entrepreneurial-skilled workers if

$$\theta_R \phi_R > \frac{1}{N} \theta_A \phi_A. \tag{74}$$

We find that Condition (73) and Condition (74) contradict (72). This means that we can either observe the first two or the last two tipping points but not all four, implying that once an agent moves from the final good sector to either the AR or the AI sector he will remain there and not change the sector for a second time.
Thus, three different constellations can arise

(I) Condition (73) and Condition (74) are fulfilled. Entrepreneurs and high-skilled workers supply their labor to the AR sector and the growth rate of AR is $g_R = \phi_A(l^H + l^E)$, while AI grows at rate $g_A = g_R$.

(II) Only Condition (73) is fulfilled. Entrepreneurs work in the AI sector, while high-skilled workers supply their labor to the AR sector and the growth rate of AR is $g_R = \phi_A l^H + \psi l^E$ while AI grows at rate $g_A = g_R$.

(III) Condition (73) is not fulfilled, which implies that Condition (74) is not fulfilled. Entrepreneurs and high-skilled workers supply their labor to the AI sector and the growth rate of AR is $g_R = \psi_A(l^H + l^E)$, while AR grows at rate $g_A = g_R$.

We find that the economy can find itself in one of three steady states in the long-run which imply a different growth rate, depending on the model's parameters.