

# MINIMUM PRICE VARIATIONS, TIME PRIORITY AND QUOTE DYNAMICS

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## ABSTRACT

### Minimum Price Variations, Time Priority and Quote Dynamics\*

This paper analyses the impact of a minimum price variation (tick) and time priority on the quote dynamics and on trading costs when competition for the order flow is dynamic. It finds that convergence to competitive prices can take time and that the speed of convergence is influenced by the tick size, the priority rule and the characteristics of the order arrival process. It also shows that a zero minimum price variation is never optimal when competition for the order flow is dynamic. The paper compares the trading outcomes with and without time priority. It shows that, for reasonable parameterizations, time priority reduces trading costs because it prevents equilibria in which uncompetitive spreads can be sustained. Finally, the paper relates (a) the trading costs to the speed with which liquidity suppliers react to their competitors' offers and (b) the dynamics of the best price in the market to the state of the book.

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## NON-TECHNICAL SUMMARY

The design of trading rules is a central issue in financial markets. The changes in the organization of security markets (for instance the development of electronic trading systems) and the accrued competition for the order flow between exchanges have called for a better understanding of the costs and benefits of different trading arrangements. This paper focuses on two specific trading rules, namely the mandatory minimum price variation (the so-called 'tick') and time priority.

A mandatory minimum price variation obliges liquidity suppliers (dealers or limit order traders) to post prices on a pre-specified grid. The tick size is the minimum price increment between two prices on the grid. For instance in the Paris Bourse, the tick size is at least Ff0.1 for stocks with prices between Ff100 and Ff500. Minimum price variations are very commonly used in financial markets. The tick size varies across the exchanges and depends, in general, on the price level.

The size and the desirability of minimum price variations have been hotly debated recently. A main argument against the enforcement of a minimum price variation is that it increases the trading costs for investors. First, it imposes a minimum size for the bid-ask spread and empirical studies have found that this size is frequently binding. Second, several authors have theoretically shown that, in presence of a tick, price competition does not drive spreads to competitive levels. According to these results, a decrease in the tick size must entail a decrease in the trading costs.

It is not clear, however, that those results apply to all types of market structure. The theoretical arguments, for instance, are based on the assumption that price competition between liquidity suppliers is static and that they can quote prices after liquidity demanders submit their market orders. In most markets, however, price competition is dynamic and liquidity suppliers post quotes before the arrival of market orders. Here again the Paris Bourse illustrates this claim. The contribution of this article is to show that, in this type of market structure, the optimal tick size is different from zero.

In a first step the paper provides a model of dynamic price competition between liquidity suppliers in which: i) a minimum price variation is enforced; and ii) orders arrive randomly to the market. It finds that when price competition is dynamic, it can take time for the quotes to adjust to the competitive level (liquidity suppliers' reservation prices rounded to the nearest

tick). As a consequence, some transactions will occur at uncompetitive prices and the trading costs will be determined in part by the speed with which quotes adjust to competitive levels. It turns out that this speed is dependent on the size of the tick. The intuition is very simple. Assume that the current best price to sell in the market is 140 and that traders' reservation price is 439. Liquidity suppliers can either undercut the current price by one tick or post a price equal to their reservation price plus one tick (the 'competitive quote'). Liquidity suppliers choosing the first option will be executed only if a market order arrives before a competitor has time to improve upon their offer. With the second option, liquidity suppliers are certain to be executed since their prices cannot be improved. The first option provides a larger profit in case of execution, but execution is uncertain. The second option provides, with certainty, a profit equal to the tick size. Now if the tick size is too small the first option dominates the second, i.e. a too-small tick size slows down the process of quote convergence to competitive quotes.

The paper formalizes this intuition. It shows that the cost of a small tick is that it increases the time it takes for the quotes to reach the competitive level. This cost counterbalances the benefit which is that, with a small tick, the wedge between liquidity suppliers' reservation price and the competitive quote is reduced. Building on this trade-off, the paper proves that, in its stylized setting, the optimal tick is always different from zero.

Because the paper models dynamic price competition between liquidity suppliers, it is natural to consider the role of time priority as well. Time priority is used to allocate a trade when several liquidity suppliers quote the best price. By this rule, at a given price, orders entered first are executed first. This priority rule is prevailing in many continuous markets, but there are some exceptions. For instance, time priority is generally not enforced in open-outcry markets, like the Chicago Board of Trade (CBOT). In this case, the trader who executes an order is chosen randomly among those who post the best price. The paper evaluates the performance of time priority against this alternative allocation rule by comparing the transaction prices and the size of trading costs under both allocation rules.

The paper highlights one advantage of time priority; it prevents liquidity suppliers from using bidding strategies leading to uncompetitive spreads which can be sustained over time. This is not the case for the other allocation rule. It then compares the expected trading costs obtained when time priority is enforced with those obtained when time priority is not enforced. Except for unrealistic parameterizations, it is shown that time priority minimizes the

trading costs. Overall the results strongly support the enforcement of time priority in security markets.

Finally, for empirical purposes, the paper relates the quote dynamics in equilibrium to the characteristics of the order arrival process and the state of the book characterized by the size of the inside spread. It finds that larger trading costs should be obtained when the order arrival rate is larger relative to the speed with which liquidity suppliers react to their competitors' offers. It also finds that different patterns for the evolution of the best quotes should be observed in equilibrium. In particular, equilibria are obtained in which initial bidders quote immediately small spreads, which are not improved afterwards and equilibria in which successive small price improvements (equal to the size of the tick) are observed. These patterns are found in recent empirical studies on limit order markets. It turns out that cases in which the best price evolves by successive small increments should be more frequent when the size of the inside spread is larger than when it is small.

## 1. Introduction.

The design of trading rules is a central issue in financial markets. The changes in the organization of security markets<sup>1</sup> and the accrued competition between those markets have called for a better understanding of the costs and benefits of the different possible trading arrangements. In this paper, we study two specific trading rules, namely a mandatory minimum price variation (the “tick”) and time priority. Our contribution is to analyze, theoretically, the impact of those rules on the trading costs and quote formation when competition for the order flow is dynamic.

In many Exchanges, liquidity suppliers must quote prices on a pre-specified grid. The tick size is equal to the increment between two prices on this grid<sup>2</sup>. The desirability of this trading rule is not clear. Actually minimum price variations can have a significant impact on the trading costs for at least two reasons. First they impose a minimum size for the bid-ask spread which can result in substantial transaction costs. Harris (1994) finds that the minimum price variation was binding for 48% of the quotation spreads for the NYSE and AMEX stocks in his sample<sup>3</sup>. Second it makes price improvements costly. In different frameworks. Anshuman and Kalay (1994), Bernhardt and Hughson (1995), Chordia and Subrahmanyam (1995) and Kandel and Marx (1996) show that this friction can lead to uncompetitive spreads. According to their results, there is no benefit (for liquidity demanders) of enforcing minimum price variations.

On the contrary, we show that a mandatory, non-zero minimum price variation minimizes the trading costs when liquidity suppliers compete sequentially in prices and when no trader is in position to observe the order flow before posting quotes. This is the case in electronic limit order markets such as, for instance, the Paris Bourse<sup>4</sup>. Biais, Hillion and Spatt (1995) show empirically that dynamic price competition between

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<sup>1</sup>For instance the recent development of electronic trading mechanisms.

<sup>2</sup>The tick size varies across Exchanges and can depend on the level of prices. For example, in the Paris Bourse, the minimum price variation is  $FF0.01$  for prices below  $FF5$ ,  $FF0.05$  for prices between  $FF5$  and  $FF100$ ,  $FF0.1$  for prices between  $FF100$  and  $FF500$  and  $FF1$  for prices greater than  $FF500$ . On the New-York Stock Exchange, the tick size is  $\$ \frac{1}{8}$  for prices greater than  $\$1$ ,  $\$ \frac{1}{16}$  for prices between  $\$0.25$  and  $\$1$  and  $\$ \frac{1}{32}$  for lower prices.

<sup>3</sup>In a related study, Ahn et al. estimate to  $\$12.8$  million the annualized savings in transaction costs due to the decrease in the tick size for 53 stocks quoted on the AMEX. Harris' result also suggests that minimum price variations can account for a large part of trading costs relative to other determinants (adverse selection, inventory effects, order processing costs). This is confirmed by Glosten and Harris (1988) who estimate to  $\$0.0133$  the adverse selection component of the spread for NYSE stocks. This is smaller than  $\$0.125$  which is the minimum price variation for those stocks.

<sup>4</sup>Other examples are the Madrid Stock Exchange, the Helsinki Stock Exchange, the Toronto Stock Exchange, the Stockholm

liquidity suppliers plays a major role in the quote dynamics in the Paris Bourse. In particular, they find that after an initial increase in the spread due to liquidity shocks, the spread decreases over time as liquidity providers compete for the order flow (See Figure 1 in the Appendix)<sup>5</sup>. They write (page 1657):

"[...] the flow of order placement is concentrated at and inside the bid-ask quote. A large fraction of the order placement improves upon the best bid or ask quote. Such improvements on one side of the quotes tend to occur in succession (undercutting) which reflects competition in the supply of liquidity"

Consequently it is important to analyze dynamic price competition between liquidity providers since it is a feature of price formation in continuous limit order markets.

Many models in the market microstructure literature assume that dealers post their prices simultaneously. Under this assumption, it is obtained that dealers choose quotes equal to their reservation prices (if they are risk-neutral then prices are just equal to conditional expectations)<sup>6</sup>. We show that bidding strategies with dynamic price competition have different properties. In particular we find that it might take time for the best price to adjust to the competitive level (the expected value of the asset rounded to the nearest tick in our setting) and that the dealers will earn extra-profits (even if the grid size is zero). Some transactions will occur at uncompetitive prices and the trading costs will depend in part on the determinants of the speed with which quotes adjust to the competitive level.

The size of the minimum price variation is one of these determinants. The intuition is simple. Suppose the ask price is at least two ticks above the competitive ask price and consider a liquidity supplier who wants to obtain priority of execution. He can choose either to undercut of one tick, at the risk of not being executed, or he can choose to quote immediately the competitive ask price in order to secure

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Stock Exchange. Domowitz (1993) reports that 35 financial markets have features of electronic limit order markets. The unobservability of the order flow is not a feature of all financial markets. For instance, in the NYSE, the specialist can determine his quotes knowing the size of the market order to be executed. This gives him market power relative to limit order traders as shown by Seppi (1996).

<sup>5</sup>See Hedvall and Niemver (1994) for similar findings for the Helsinki Stock Exchange.

<sup>6</sup>See for instance Glosten and Milgrom (1985) in the case of risk-neutrality and Ho and Stoll (1983) in the case of risk-aversion.



execution. If the tick is too small, the second option is less attractive than the first one, as long as a quote above the competitive ask price has a positive probability of execution. This effect suggests that a too small minimum price variation can hurt liquidity traders. Based on this intuition, we prove that the tick size which minimizes the expected trading costs is always different from zero in our stylised framework.

In dynamic trading environments, time priority is often used to allocate a trade between liquidity suppliers posting the best price. By this rule, at a given price, orders entered first are executed first<sup>7</sup>. We evaluate the performance of time priority against another allocation rule: the liquidity supplier who executes a trade is chosen randomly among the traders quoting the best price. We point out one nice property of time priority: spreads above the competitive spreads cannot be sustained over time. This is not the case with the other allocation rule. With time priority, dealers cannot share an order. This property prevents forms of implicit collusion which can occur in a dynamic setting with other allocation rules. Then we compare the expected trading costs obtained when time priority is enforced with those obtained when it is not enforced. We show that, for reasonable parameterizations, trading costs are lower with time priority.

For empirical investigations, we relate the quote dynamics to the characteristics of the order arrival process and the state of the book characterized by the size of the inside spread. We find that larger trading costs should be obtained when the ratio of the transaction frequency to the frequency of new limit orders increases<sup>8</sup>. We also show that different patterns for the evolution of the best quotes should be observed in equilibrium. In particular, we obtain equilibria in which initial bidders quote immediately small spreads, which are not improved afterwards and equilibria in which successive price improvements are observed. These patterns are found in the data by Bias, Hillion and Spatt (1995). As in their paper, it turns out that cases in which quotes evolve by successive improvements, starting from relatively uncompetitive quotes, should be more frequent when the size of the inside spread is large.

Our framework is related to the model of dynamic price competition introduced by Maskin and Tirole

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<sup>7</sup>Domowitz (1993) shows that this rule is prevailing in many electronic markets. There are important exceptions however. Time priority is generally not enforced in open-outcry markets like the CBOT. It is not enforced on the NASDAQ as well.

<sup>8</sup>The empirical literature has just begun to consider time between trades and order placements as a possible determinant of spreads. See for instance Hausman et al. (1992).

(1988). In contrast with Maskin and Tirole (1988), we assume that dealers just observe the best quotes in the market (i.e. the book is “closed” as it is the case in some financial markets). This assumption simplifies the characterization of the equilibria and makes tractable the analysis of the relationships between the quote dynamics, the trading rules and the order arrival process. By comparing the possible patterns for prices in equilibrium with those obtained in Maskin and Tirole (1988), we argue that one advantage of a closed limit order book is that it prevents the dealers from using strategies sustaining very uncompetitive prices<sup>9</sup>.

The role of the tick has been analyzed taking into account trading features of the NYSE (Bernhardt and Hughson (1995) or Seppi (1996)) or the NASDAQ (Kandel and Marx (1996)). We think our framework (in particular the assumptions of dynamic price competition combined with unpredictability of market orders arrival time) is more adequate to describe competition for the order flow in electronic limit order markets such as the Paris Bourse. Parlour (1996) has shown that time priority and price competition can induce systematic patterns for the order flow in electronic limit order markets. However she does not analyze how trading costs are related to the size of the tick and the enforcement of time priority. Recently, Dutta and Madhavan (1996) show that collusive outcomes can be sustained when dealers compete over time. They consider a repeated game setting in which dealers post simultaneously prices at different points in time. On the contrary, in our framework, liquidity suppliers post their quotes in sequence, reacting optimally to the quotes posted previously by their competitors. In this setting, there is a role for time priority and we show that this priority rule prevents (implicit) collusive pricing.

The next section spells out the model and the equilibrium concept used to solve the trading game. Section 3 characterizes the dynamics of quotes in equilibrium when time priority is enforced and when it is not. Section 4 analyzes the policy implications of the model. Section 5 studies the robustness of the results. Section 6 points some empirical implications of the model. Section 7 concludes.

## 2. The Model.

In this section, we describe the simplest version of the model used in this paper and we present the equi-

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<sup>9</sup>There are other differences with Maskin and Tirole (1988). For instance, we assume that the arrival time of an order is random and we consider the role of time priority.

librium concept used to solve the trading game.

## 2.1 The Traders and the Trading Process.

We consider the market for a risky asset. Time is continuous and is indexed by  $t \in [0, T]$ . Let  $\tilde{V}$  be the payoff of the asset at time  $T$ . The expected payoff will be denoted by  $\mu = E(\tilde{V})$ .

Two risk-neutral liquidity suppliers (dealers)<sup>10</sup> compete for the order flow by quoting prices *sequentially* at dates  $\{0, 1, \dots, \tau, \tau + 1, \dots\}$ . The duration of each period is fixed and is denoted  $\Delta$ . For brevity, we only focus on the equilibrium ask prices. Our results extend to the case in which dealers choose both ask and bid prices as long as the arrival date of a buy order is independent of the arrival date of a sell order. At date  $\tau$ , a dealer, say dealer 1, has the possibility to quote an ask price  $a_t^1$ . Then, at date  $\tau + 1$ , dealer 2 reacts with an ask price  $a_{\tau+1}^2$  and so on. The dealer who quotes the first price, at date 0, is chosen randomly. Let  $a_t^j$  be the quote of dealer  $j$  at time  $t$ . The best quote posted at time  $t$  is denoted  $a_t^m$ . We will follow the convention that a quote indexed by  $\tau$  is an offer just *after* the reaction of the dealer at date  $\tau$ . Because dealers alternate in quoting prices, we have  $a_t^j = a_t^j \quad \forall t \in \{\tau, \tau + 2\}$ , if  $\tau$  is the last date at which dealer  $j$  had the opportunity to make an offer. This model of alternating price competition aims at capturing the fact that, in a continuous limit order market, liquidity suppliers do not post their quotes simultaneously but rather sequentially. The interval of time between quote revisions ( $\Delta$ ) can be interpreted as the mean time between new offers (see Section 5). Our results hold as long as  $\Delta$  is strictly positive. This can be justified by the fact that it is costly for dealers to monitor the market continuously.

Trades occur when a trader wants to buy the asset for liquidity reasons. Following a well-established tradition in the market microstructure literature, we assume an exogenous arrival process for those traders. The number of buyers' arrivals in a given time interval follows a Poisson process with rate  $\lambda$ <sup>11</sup>. The waiting time between arrivals is therefore exponentially distributed with parameter  $\lambda$ . Consequently, in each interval  $[\tau, \tau + 1]$ , the probability that a buyer will arrive to the market is  $(1 - e^{-\lambda\Delta})$ . For simplicity, we

<sup>10</sup>We will refer to the liquidity suppliers as dealers. More generally they can be thought as limit order traders who monitor the market and who post limit orders opportunistically when profits opportunities exist. Biais, Hillion and Spatt (1995) provide evidences of the existence of such limit order traders.

<sup>11</sup>This assumption is used for instance in Garman (1976), Ho and Stoll (1981) or Easley, Kiefer and O'Hara (1995). Biais et al.(1995), Easley et al.(1995), Gouriéroux et al. (1997), Hedvall and Niemeyer (1996) provide estimations of the arrival rates.

assume that the buyers have rectangular demands<sup>12</sup>. Namely, a buyer purchases  $L$  units of the asset if the best ask price is lower than or equal to  $R_B$ . Consequently  $R_B$  is the maximum ask price which can be quoted by the dealers. Alternatively we could assume that there already exists an ask price in the book, just above  $R_B$ , when dealers start competing. This would give the same results. Liquidity traders and dealers have the same information.

It is convenient to assume that the payoff date  $T$  is random and exponentially distributed with parameter  $\gamma$ . This guarantees that the model is stationary and simplifies the derivations. The price competition game between the dealers stops when a trade occurs or when the payoff date is realized. We denote  $\bar{t}$  the date at which the trading process stops and  $\bar{o}$  the date of an order arrival. With these assumptions, the dealers just compete for the next order to arrive in the market. Consequently the model is best viewed as describing price competition and quote adjustments *between trades*. However, it is easy to see that our framework could be embedded in a sequential trade model (see, for instance, Easley and O'Hara (1992)). In this case, after a trade, (i) the dealers cancel their offers (ii) new public information arrives and (iii) the dealers enter into a new round of price competition. The game which is analyzed here can be viewed as one round of this trading process.

Risk-neutrality and the absence of asymmetric information between dealers and potential buyers imply that dealers' reservation prices are just equal to the expected value of the asset. The model could be modified in order to incorporate risk-aversion or asymmetric information. In our framework, risk-aversion or asymmetric information would create a wedge between the dealers' reservation prices and the expected value of the asset<sup>13</sup> but would not change qualitatively the dynamics of quotes to these reservation prices. This is the dynamic which is the focus of the paper. As the effects of risk-aversion and asymmetric information on the quotes are well-known and will not interact, in our framework, with the effects coming from dynamic price competition, we just ignore them for simplicity.

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<sup>12</sup>In a previous version of the paper, we analyzed the case in which the order flow was price dependent. The results are qualitatively the same as those obtained in this simpler framework.

<sup>13</sup>See, for instance, Glosten and Milgrom (1985) or Ho and Stoll (1981).

## 2.2 Market Structure.

The set of possible prices is discrete and is characterized by the size of the minimum price variation:  $g$ . We denote by  $\langle p \rangle^-$  and  $\langle p \rangle^+$  the prices on the grid respectively immediately strictly lower than  $p$  and greater than or equal to  $p$ . We assume that  $\mu - \langle \mu \rangle^- = \langle \mu \rangle^+ - \mu = \frac{g}{2}$ , i.e. the position of the expected value is half way between a tick. The set of possible prices is:  $\mathcal{P} = \{\dots, p(-i), \dots, p(0), \dots, p(i), \dots\}$  with  $p(i) = \langle \mu \rangle^- + ig, i \in \mathbb{N}$ . For technical reasons, we will assume that  $\mathcal{P}$  is finite. Liquidity traders' reservation price is on the grid:  $R_B = p(k)$  with  $k > 2$ , for the problem to be of interest. In Section 4, we consider the impact of shrinking the price grid on the trading costs.

A buy order is executed against the best quote at the time of his arrival (price priority rule). In case of ties, we will consider two alternative allocation rules. The first is the time priority rule (TPR) which is used in many electronic trading systems (e.g. the system CAC, in the Paris Bourse). In this case, the first dealer to quote the eligible price executes the trade. With the second allocation rule (RAR), the dealer who executes the order is chosen randomly among the dealers quoting the best price. Such a rule is implicitly<sup>14</sup> used in open outcry markets (e.g. futures markets such as the CBOT). In this case, the results serve as a benchmark against which we evaluate the performance of time priority. We note that, under our assumption of risk-neutrality, the random allocation rule and the allocation rule which splits equally a trade among the liquidity providers give the same outcomes in equilibrium<sup>15</sup>

For tractability, we assume that the book is "closed", i.e. only the best quotes are displayed. This implies that a dealer who has strict priority on the ask side, does not observe the quote posted by the other dealer on this side of the market. Some continuous markets (for instance, the Tokyo Stock Exchange or the NYSE) have features of a closed limit order book market<sup>16</sup>.

## 2.3 Two benchmarks.

<sup>14</sup>See Massumb and Phelps (1994) on this point.

<sup>15</sup>Biais et al. (1996) study the impact of priority rules based on quantities in a static model of price-quantity competition. They show that in case of risk-aversion the random allocation rule and the equal splitting rule yield different trading outcomes.

<sup>16</sup>This rule is also used in some laboratory studies of asset markets (see Friedman (1993)). In some markets, however, a larger set of bids and asks are displayed. In the Paris Bourse, participants observe the 5 best offers on each side of the market. In our model, this element of the market structure will be used to rule out some collusive equilibria.

It has often been assumed in the empirical literature on price discreteness that, in presence of a minimum price variation, the best quotes are determined by a rounding mechanism (for instance Gottlieb and Kalai (1985)). Namely the best quotes are equal to the reservation price of the dealers rounded to the nearest tick, i.e.  $p(1)$  in our framework. Therefore  $p(1)$  is a first benchmark to which we will compare our results. We will refer to  $p(1)$  as *the competitive price* and to  $p(1) - \mu$  as *the competitive spread*.

Another natural benchmark is the equilibrium which would be obtained if dealers were posting price simultaneously as it is generally assumed in models of market microstructure. In the case of the random allocation rule, it is straightforward that  $p(1)$  or  $p(2)$  are two possible Nash equilibria. Obviously the multiplicity is due to the existence of the minimum price variation. It is also due to the fact that we only consider, for the moment, two dealers. When the number of dealers is greater than or equal to 4, it is straightforward that  $p(1)$  is the unique equilibrium. In Section 5, we show that this last result does not hold any more when price competition is dynamic.

## 2.4. Pricing Strategies and Equilibrium Definition.

We turn now to the definition of the equilibrium concept used to solve the trading game. This definition is not completely straightforward because the game is dynamic and dealers' information on their competitor's quote can be imperfect. For this reason, we detail the different steps.

### 2.4.1 Expected Profits.

A dealer's expected profit upon execution depends on the position of his quote relative to his competitor's quote and in case of ties, on his priority status. In order to keep track of priority, we define the indicator variable  $q_t^j$  which takes the value 0 when  $j$  does not have strict priority of execution at the best quote and 1 if  $j$  has strict priority. If  $q_t^j = 0$ , it is the case that  $j$  has either equal priority (with RAR) or no priority. We denote by  $\Pi_j(a_t^h, q_t^h, a_t^j)$  dealer  $j$ 's expected profit, conditional on the arrival of a buy order at time  $t$  given the quote  $a_t^h$  of his competitor. We have for  $h, j \in \{1, 2\}$ ,  $h \neq j$ :

$$\Pi_j(a_i^h, q_i^j, a_i^j) = \begin{cases} 0 & \text{if } a_i^j > a_i^h \\ 0 & \text{if } a_i^j = a_i^h \text{ and } q_i^j = 0 \text{ under } TPR. \\ L(a_i^j - \mu) & \text{if } a_i^j = a_i^h \text{ and } q_i^j = 1. \\ \frac{L}{2}(a_i^j - \mu) & \text{if } a_i^j = a_i^h \text{ and } q_i^j = 0 \text{ under } RAR. \\ L(a_i^j - \mu) & \text{if } a_i^j < a_i^h \end{cases} \quad (1)$$

Of course, if the game ends before an order arrival, the dealers obtain a payoff equal to zero. We denote  $s_i^j = \{a_i^j, q_i^j\}$  the public information on the state of the book relevant for dealer  $j$  at time  $i$ . This information will be updated at dates  $\tau \in \{0, 1, \dots\}$  according to the actions of the different dealers. The following equations describe the possible values for  $s_i^j$  according to dealers' offers:

$$\begin{cases} s_i^j = s_i^j & \forall t \in \{\tau, \tau + 1\} \\ s_i^j = (a_i^j, 1) & \text{if } a_i^j < a_i^h \\ s_i^j = (a_i^j, 1) & \text{if } a_i^j = a_i^h \text{ and } j \text{ has strict priority at } a_i^j \\ s_i^j = (a_i^h, 0) & \text{if } a_i^j = a_i^h \text{ and } j \text{ does not have strict priority at } a_i^j \\ s_i^j = (a_i^h, 0) & \text{if } a_i^j < a_i^h \end{cases} \quad (2)$$

The first equation is a consequence of the assumption that traders' quotes are constant in the interval  $[\tau, \tau + 1)$ . Just before the quote revision at date  $\tau$ , the information for  $j$  is  $s_{\tau-1}^j$ . Consequently let  $H_{\tau-1}^j = \bigcup_{k=0}^{\tau-1} s_k^j$  be the trading history observed by  $j$  until date  $\tau$ . It is straightforward that a dealer can infer the trading history observed by his competitor using his own trading history.

A dealer's bidding strategy specifies his quote, each time he has to react, as a function of the trading history until that time. We will only consider pure strategies. For given strategies of dealer 1 and 2, the next lemma provides a simple expression for the dealers' objective functions<sup>17</sup>.

**Lemma 1** . *For given bidding strategies of the two dealers, a dealer's expected profit at date  $\tau_0$ , conditional on continuation of the trading process until this date and the trading history is:*

$$E(\Pi_j(a_i^h, q_i^j, a_i^j) \mid \bar{i} \geq \tau_0 \Delta, H_{\tau_0-1}^j) = \sum_{\tau=\tau_0}^{+\infty} \left( \frac{\lambda}{\lambda + \gamma} \right) \Phi (1 - \Phi)^{(\tau - \tau_0)} \Pi_j(a_i^h, q_i^j, a_i^j) \quad (3)$$

<sup>17</sup>We assume that dealers' discount factor is one since the dynamic interactions which are described here take place over very short periods of time in practice.

with  $j, h \in \{1, 2\}, j \neq h$ ,  $\Phi = 1 - e^{-(\tau+\lambda)\Delta}$  and  $\Pi_j(a_j^h, q_j^h, a_j^l)$  defined in Equation (1).

All the proofs are given in the Appendix. The intuition for this lemma is as follows. First remark that dealers' profits are constant over intervals  $[\tau, \tau + 1)$ . Second  $(\frac{\lambda}{\lambda+\tau})\Phi$  is the probability that an order arrives in an interval  $[\tau, \tau + 1)$  before the payoff date is realized. Finally  $(1 - \Phi)^{(\tau-\tau_0)}$  is the probability that the game will not stop before date  $\tau$ .

#### 2.4.2 Beliefs Formation.

Consider a dealer who is given the opportunity to make a new offer at date  $\tau$ . Certainly his offer will depend on the public information on the current state of the book:  $s_{\tau-1}^j$ . When the dealer does not observe his competitor's offer (a consequence of the fact that the book is closed), his offer will also be determined by his belief concerning the position of this offer. Let  $\hat{a}(H_{\tau-1}^j)$  be this belief which can be a function of the observed trading history. We call  $\{s_{\tau-1}^j, \hat{a}(H_{\tau-1}^j)\}$  the state of the market at date  $\tau$  for dealer  $j$ . We will restrict our attention to markov pure strategies<sup>18</sup>, i.e. strategies which are functions of the state of the market but not directly of the trading history. Moreover, we will consider only symmetric strategies: the two dealers behave in the same way in the same state of the market. Under these restrictions, we can represent the bidding strategy by a reaction function  $R(\cdot)$  which gives the price chosen by dealer  $j$ , when he has the opportunity to make an offer, for each possible state of the market<sup>19</sup>

Consider a potential candidate  $R(\cdot)$  for an equilibrium. When a dealer observes his competitor's quote (he has not strict priority) then  $\hat{a}(H_{\tau-1}^j) = a_{\tau-1}^m$ . When he does not observe his competitor's offer (strict priority), we distinguish two cases. First if  $s_{\tau-1}^j = \{a_{\tau-1}^m, 1\}$  is on the equilibrium path then a dealer's belief must be consistent with the equilibrium reaction of his competitor at the previous period, i.e.<sup>20</sup>  $\hat{a}(H_{\tau-1}^j) = R(s_{\tau-2}^h, \hat{a}(H_{\tau-2}^h))$ . If  $s_{\tau-1}^j = \{a_{\tau-1}^m, 1\}$  is out-of-the equilibrium path then the beliefs are arbitrary. We assume that  $\hat{a}(H_{\tau-1}^j) = a_{\tau-1}^m + g$  in this case. As discussed in Section 3 (Remark 1), this choice

<sup>18</sup>See Maskin and Tirole (1993) for a more formal discussion of markov strategies.

<sup>19</sup>Because the trading game is stationary, we take the reaction function to be stationary, i.e. it does not depend directly on time.

<sup>20</sup>Here we use the fact that dealers' trading history is public information. Consequently dealer  $j$  knows  $\{s_{\tau-2}^h, H_{\tau-2}^h\}$  and conjectures correctly  $\hat{a}(H_{\tau-2}^h)$ .



is without consequence for our results.

The dealers' bidding strategies form a Markov sequential equilibrium if i) the strategies are Markov, ii) the dealers form their beliefs on their competitors' quotes as explained above and iii) for each possible state of the market, the quote posted by a dealer maximizes his expected profit given the subsequent actions of his rival and himself.

**Lemma 2** : Consider a trading history  $H_{\tau-1}^j$  which leads at date  $\tau$  to the following observation for dealer  $j$ :  $s_{\tau-1}^j = \{p, 1\}$ . If this observation belongs to the equilibrium path then  $\hat{a}(H_{\tau-1}^j) = R(p, 0, p)$ .

This lemma means that dealers' beliefs on the equilibrium path have always a simple structure: a dealer, quoting the best price  $p$ , who does not observe his competitor's quote must believe that the latter has quoted the best response to the current best quote  $p$ . This implies that in all the cases, the only part of the trading history which is useful to form the beliefs is the current information on the state of the book. This allows us to simplify our notations. From now on, let  $R(p, 0)$  be the optimal reaction of a dealer when he observes the state of the book  $\{p, 0\}$  and let  $R(p, 1)$  be his optimal reaction when he observes the state of the book  $\{p, 1\}$ . It will be implicit that if  $\{p, 1\}$  is on the equilibrium path, the dealer believes  $R(p, 0)$  to be his competitor's quote (Lemma 2) while if  $\{p, 1\}$  is not on the equilibrium path, he believes  $p + g$  to be his competitor's quote.

#### 2.4.3 Equilibrium Definition.

Given a reaction function  $R(\cdot)$ , let  $V(s_{\tau-1}^j)$  be dealer  $j$ 's expected profit given that i) the state of the book is  $s_{\tau-1}^j$ , ii)  $j$  is about to react and iii) from date  $\tau + 1$  on, both dealers will behave according to the reaction function  $R(\cdot)$ . We also define  $W(s_{\tau-1}^j, a_{\tau-1}^j)$  dealer  $j$ 's expected profit at date  $\tau$  in state  $s_{\tau-1}^j$  given that his competitor is about to react and that  $j$  chose  $a_{\tau-1}^j$  at the previous date. Using Lemma 1,  $V(\cdot)$  can be expressed by the dynamic programming relationship<sup>21</sup>:

$$V(s_{\tau-1}^j) = \max_{a, \in \mathcal{P}} \left( \frac{\lambda}{\lambda + \gamma} \right) \Phi \Pi_j(\hat{a}_{\tau}^j, q_{\tau}^j, a_j) + (1 - \Phi) W(s_{\tau}^j, a_j) \quad (4)$$

<sup>21</sup>See Maskin and Tirole (1988). As in their paper, we can apply dynamic programming because the set of prices is finite so that the conditional expected profit functions are bounded. We follow closely their notations here.

and  $W(\cdot, \cdot)$  must be such that:

$$W(s'_\tau, a_j) = \left(\frac{\lambda}{\lambda + \gamma}\right)\Phi\Pi_j(R(s_\tau^h), q_{\tau+1}^j, a_j) + (1 - \Phi)V(s'_{\tau+1}) \quad (5)$$

where the evolution of the available information on the state of the book, between dates  $\tau$  and  $\tau + 1$ , is determined by dealer  $j$ 's quote at date  $\tau$  and henceforth by the actions prescribed by the reaction functions, according to Equation (2). A reaction function is a Markov equilibrium if  $R(s'_{\tau-1})$  is solution of Equation (4) for each state  $s'_{\tau-1}$  given that dealers' beliefs on their competitor's quote are specified as explained in 2.4.2.

### 3. Quote Dynamics in Equilibrium.

In this section, we analyze the possible patterns for the quotes in equilibrium, first when time priority is enforced and second when the random allocation rule is enforced. We relate these patterns to the size of the minimum price variation, the waiting time between new quotes and the frequency of liquidity traders' arrivals.

#### 3.1 Equilibrium with Time Priority.

When a dealer is about to react, he has basically two choices. He can improve the best price by one or several ticks. In this case he captures price and time priority. Or he can choose to match the best price or to quote a higher price, at the cost of losing price and time priority for the next period. The next proposition establishes that the second option is never optimal when a trader does not have time priority as long as the price is greater than the competitive price  $p(1)$ .

**Proposition 1** *Consider  $p$  a best quote on the equilibrium path and  $R(\cdot)$  an equilibrium reaction function.*

1. *A dealer always improves  $p$  if he does not have time priority at this price and if this price is greater than or equal to  $p(2)$ , i.e.:  $R(p, 0) < p$  if  $p \geq p(2)$ .*
2. *If  $p = p(1)$  then  $R(p, 0)$  is equal to  $p(1)$  or  $p(2)$ .*
3. *If  $p = p(1)$  then  $R(p, 1)$  is equal to  $p(1)$ .*

Proposition 1 restricts the possible patterns for the the best price in equilibrium. The first part implies that the best price will necessarily decrease as long as it is greater than the competitive price. The second and the third parts imply that it will not change any more once the competitive price is posted. One could argue that this is the prediction of the static model of Bertrand competition. This interpretation is misleading for two reasons. First, either one considers that the static model predicts an *instantaneous* adjustment of the best price to the competitive price or admits that it says nothing concerning the dynamics of the adjustment. The next results show that the speed and the dynamics of the adjustment can vary according to market conditions when price competition is dynamic. Second, in view of the results obtained by Maskin and Tirole (1988), convergence to the competitive outcome should not be expected. Indeed one should expect the prices to converge to a greater price than  $p(1)$  or to never settle down, cycling for instance, between  $R_B$  and  $p(1)$ .

There are two reasons, related to market structure, which explain that we do not obtain the same type of results as Maskin and Tirole (1988). First, dealers do not observe the price of their rivals but only the best price. This prevents a dealer from raising his quote above the best price in order to induce his rival to also raise his quote (a possible form of collusive behavior in equilibrium as shown by Maskin and Tirole). Actually a dealer, say 1, cannot make public in a credible way that he has raised his quote to a *given level*. If dealer 2 was to follow, raising his quote to the level immediately below the quote dealer 1 claims to have posted, then dealer 1 would have an incentive to undercut dealer 2 "secretely", in the first place. This is reminiscent of the mechanism at work in the static model of price competition. In fact, here, the dealers' information set is intermediate between the case in which the dealers have no information on their competitors' offers (as in static Bertrand competition) and the case in which they observe perfectly those offers (as in Maskin and Tirole). The result stresses the importance of the market rules concerning the information available on the quotes posted in the market for the outcome of price competition between liquidity suppliers. A second reason is that time priority prevents the dealers from sharing the market. Therefore, the dealers will improve upon their competitor 's offer until no further profitable price improvements are possible.

With Proposition 1 at hand, we can now derive the dealers' bidding strategy in equilibrium. This is the next proposition We will denote the greatest integer strictly lower than  $x$  by  $\lfloor x \rfloor$ .

**Proposition 2** : *The following reaction function is a sequential Markov equilibrium of the dynamic price competition between dealers when time priority is enforced:*

$$1. R(p, 0) = p - g \quad \text{if } p(i^* + 2) \leq p \leq R_B.$$

$$2. R(p, 0) = p(1) \quad \text{if } p(1) \leq p \leq p(i^* + 1).$$

$$3. R(p, 1) = p \quad \text{if } p(i^* + 1) \leq p \leq R_B.$$

$$4. R(p, 1) = p(1) \quad \text{if } p(1) \leq p \leq p(i^*).$$

$$\text{with } p(i^* + 1) = (\mu + \frac{g}{2\Phi})^+ = p(1) + \lfloor \lfloor \frac{\Phi+1}{2\Phi} \rfloor \rfloor g.$$

The intuition for Proposition 2 is as follows. Consider a dealer who is about to react to a quote  $p$  strictly greater than the competitive price. He faces the following trade-off. He can secure execution but at a low price,  $p(1)$ , relatively to the current quote. He can obtain a better execution price by slightly improving the current quote but in this case he runs the risk of being undercut and finally non-executed. For a given tick size and for a given frequency of arrivals, the solution to this trade-off is determined by the position of the best quote. When the difference between the best quote and the asset expected value is sufficiently large (i.e. when  $p - \mu > p(i^* + 1) - \mu$ ), the optimal decision is to improve the current quote by only one tick. For lower markups, the dealer will improve once for all the best price, quoting  $p(1)$ . For a given quote  $p$ , the dealer's bidding decision is determined by the probability that an order will arrive before his competitor reacts ( $\Phi$ ) and the tick size. When  $\Phi$  is large, the execution risk when the dealer just slightly improves his competitor's quote is small. When the tick size is small, the dealer must quote a very low price to secure execution while he can seize, temporarily, priority at the cost of a small price improvement. In those two cases, small price improvements, despite the execution risk they entail, are more attractive than a jump once for all to the competitive price  $p(1)$ . This explains that  $p(i^* + 1)$  is decreasing in  $\Phi$  and increasing in the tick size. Proposition 2 has an immediate corollary:

**Corollary 1** . *In equilibrium, the dynamics of the best ask price in the market is as follows:*

$$a_\tau^m = R_B - \tau g \quad \text{for } \tau \leq \tau_{TPR}^* \tag{6}$$

$$a_\tau^m = p(1) \quad \text{for } \tau > \tau_{TPR}^* \tag{7}$$

$$\text{with } \tau_{TPR}^* = \frac{R_B - p(i^* + 1)}{g} \quad (\text{See Figure 2}).$$

Corollary 1 describes the dynamics of the best price in equilibrium, given the buyer's reservation price or the initial state of the book. Let denote by  $T_{TPR}^*$ , the time it takes for the quotes to converge to the competitive level:

$$T_{TPR}^* = \text{Max}\{\tau_{TPR}^* + 1, 0\}\Delta \quad (8)$$

which gives:

$$T_{TPR}^* = \text{Max}\left\{\frac{R_B - \mu}{g} + \frac{1}{2} - \left\lceil \left\lfloor \frac{\Phi + 1}{2\Phi} \right\rfloor \right\rceil, 0\right\}\Delta \quad (9)$$

As shown in Section 4,  $T_{TPR}^*$  influences the size of the trading costs. When the markup between the asset expected value and the buyers' reservation price or the initial quote in the book is sufficiently large then prices will not adjust immediately. Moreover a decrease in the tick size increases the length of time during which the quotes are greater than  $p(1)$ . Actually a decrease in the tick lowers i) the price ( $p(i^* + 1)$ ) at which the dealers find it profitable to quote the competitive price and ii) the size of price improvements in each period.  $T^*$  is weakly increasing in the arrival frequency ( $\lambda$ ) and increasing in the waiting time between quote revisions ( $\Delta$ ). The execution probability of a dealer in a given period increases with those parameters and consequently such an increase reduces the competitive pressures.

The results of this section show that uncompetitive spreads can be quoted even when the dealers do not cooperate to set prices. This occurs because i) the dealers set prices sequentially and not simultaneously and ii) there is some uncertainty concerning the arrival date of an order. Interestingly, factors such as price discreteness and the waiting time between offers or order arrivals are shown to affect the possibility of uncompetitive spreads. However time priority and a closed book are sufficient conditions for the inside spread to converge to the competitive level<sup>22</sup>

**Remark 1.** The proof of Proposition 1 does not rely on the specification of dealers' beliefs out-of-the equilibrium path. This entails that the quote dynamics which will be observed in equilibrium is independent of this specification. In particular the price  $p(i^* + 1)$  at which a dealer chooses to "jump" to the competitive price would be the same for other choices for the beliefs out-of-the equilibrium path. However, the reaction

<sup>22</sup>Friedman (1993) finds experimentally that spreads are larger in open book environments compared to closed book environments. This is consistent with our results.

of the dealers for states out-of the equilibrium path in Proposition 2 depends on our specification. Consequently, strictly speaking, the equilibrium we have derived here is not the unique equilibrium. Nevertheless this is inconsequential for the measure of trading costs which are determined only by the evolution of the best quote on the equilibrium path.

### 3.2. Equilibrium without Time Priority.

In the last section, we proved that there was no price at which it was optimal for a dealer to post a price above the best quote (in order to induce his competitor to do the same). As shown in the next proposition, it is still true with the random allocation rule. This is a property of price formation in markets with closed order book rather than a consequence of the allocation rule.

**Proposition 3** *Let  $R(\cdot)$  be an equilibrium reaction function.*

1. *Consider  $p$  a price on the equilibrium path. It is necessarily the case that:  $R(p, 0) \leq p$ .*
2. *If  $\bar{p}$  is on the equilibrium path and such that  $R(\bar{p}, 0) = \bar{p}$  then  $\forall p \geq \bar{p}, R(p, 0) \in [\bar{p}, p]$ . Moreover it is necessarily the case that  $\bar{p} \geq p(2)$ .*

Proposition 3 (parts 1 and 2) shows that in equilibrium, the dynamics of prices without time priority is similar to the dynamics with time priority: the best quote converges to a price at which no further price improvements take place. We call a price with this property a *focal price*. The main difference is that this price is strictly above the competitive price and at least equal to  $p(2)$ . This is to be compared with the case in which dealers post prices simultaneously for which the competitive price is another possible outcome. Moreover, when time priority is enforced, we have shown previously that the competitive price was the only possible focal price. The next result shows that there is always an equilibrium in which  $p(2)$  is a focal price. Proposition 5 states that even more collusive prices can be sustained for some parameters values.

**Proposition 4** *For all the parameters values,  $p(2)$  is a focal price which is sustained in equilibrium by the following reaction function:*

1.  $R(p, 0) = p - g$  if  $p(4) < p \leq R_B$

$$2. R(p, 0) = p(2) \quad \text{if } p \leq p(4)$$

$$3. R(p, 1) = p \quad \text{if } p(4) \leq p \leq R_B$$

$$4. R(p, 1) = p(2) \quad \text{if } p < p(4)$$

As in the previous section, the next corollary gives the dynamics of the best price in this equilibrium.

**Corollary 2 :** *In the previous equilibrium, the dynamics of the best price is as follows:*

$$a_\tau^m = R_B - \tau g \quad \text{for } \tau \leq \tau_{RAR}^* \quad (10)$$

$$a_\tau^m = p(2) \quad \text{for } \tau > \tau_{RAR}^* \quad (11)$$

with  $\tau_{RAR}^* = \frac{R_B - p(4)}{g}$ .

As with time priority, the bidding strategies depend on the the size of the initial spread ( $R_B - \mu$ ) and the minimum price variation. Defining  $T_{RAR}^*$  as in Equation (8), it is direct that the time it takes for the quotes to reach the focal price is weakly increasing with the initial size of the spread ( $R_B - \mu$ ) and weakly decreasing with the size of the tick for the same reasons as with time priority. The main difference is that  $T_{RAR}^*$  is not influenced any more by  $\Phi$  which influences the execution risk. This reflects the fact that execution risk is much less a concern in absence of time priority. The intuition is straightforward. With time priority, a dealer runs the risk of not participating to the next trade if he does not quote the competitive price. This risk increases as  $\Phi$  decreases. With the random allocation rule, a dealer has always the possibility to avoid non-execution by matching the quote of his competitor if the latter quotes  $p(2)$ .

Can we have greater focal prices than  $p(2)$ ?

**Proposition 5 :** *For  $\Phi \in [\frac{2}{5}, \frac{2}{3}]$ , it is possible to sustain higher focal prices (namely  $p(4)$  or  $p(6)$ ) in equilibrium (if  $R_B$  is sufficiently large).*

For intermediate values of  $\Phi$ , we find that very uncompetitive prices can be sustained. These prices are greater than those which can be obtained when dealers quote their prices simultaneously. The intuition is as follows. First  $\Phi$  should not be too high, otherwise the temptation to undercut is strong at any candidate to be a focal price above  $p(2)$ . Moreover a price greater than  $p(2)$  can be a focal price only if

a dealer, say dealer 1, expects to be undercut if he undercuts the focal price. If dealer 2 just matches dealer 1's offer when the latter undercuts the focal price then it becomes optimal for dealer 1, i) to undercut the focal price and ii) to raise his quote in two periods to the focal price<sup>23</sup>. But in order to induce a dealer to undercut his competitor when the latter has undercut the focal price,  $\Phi$  should not be too small.

The results of this section show that the random allocation rule creates opportunities for collusive behaviors which are not present when dealers quote their prices simultaneously (Proposition 5) or when time priority is enforced (Proposition 4 and 5). This emphasizes one benefit of time priority: uncompetitive spreads cannot last with time priority while they can if it is not enforced. Time priority prevents the dealers from using bidding strategies by which uncompetitive prices can be sustained in equilibrium.

**Remark 2.** With respect to the specification of the beliefs out-of-the equilibrium path, the same comments as in Remark 1 apply here.

## 4. Policy Implications.

We examine now the implications of the model for the choices of the tick size and the priority rule in financial markets. For simplicity, we just consider the limit case in which  $\gamma = 0$ . This is because the expected trading costs are not defined if the game stops before the order arrival time. Our conclusions are still valid (qualitatively) if we assume  $\gamma > 0$  and if we specify, arbitrarily, the expected trading cost to be zero if the game stops before the arrival of a market order. When  $\gamma = 0$ ,  $\Phi$  is just the execution probability, in the next interval of time, of the limit order with execution priority.

### 4.1. Tick Size.

It has often been argued that, without a minimum price variation, price competition will drive down liquidity suppliers' quotes to their reservation prices. Consequently a zero minimum price variation minimizes the inside spread. This intuition is correct when liquidity suppliers quote their prices simultaneously<sup>24</sup>.

<sup>23</sup>This will bring back the best price permanently to the focal price according to our specification of the beliefs out-of-the equilibrium path.

<sup>24</sup>See Section 2.3. See also Anshuman and Kalay (1993) or Kandel and Marx (1996).



The previous results suggest that this intuition does not hold any more when the competition for the order flow is dynamic. Actually, a too small tick will lengthen the time it takes for the spreads to adjust to the competitive level and this can raise ultimately the expected trading cost.

In order to examine this point more formally, suppose that the tick size is  $g(0) = g$  and that one considers using a finer grid size:  $g(n) = \frac{g}{2n+1}$   $n \in \mathbb{N}$ . Shrinking the grid in this way makes sure that  $\mu$  is always half way between a tick and that  $R_B$  stays on the grid. Consequently, the equilibria obtained for a grid size  $g(n)$  are the same as in Section 3, replacing  $g$  by  $g(n)$ . In the case of the random allocation rule, we only consider the equilibrium described in Proposition 4, which is the less collusive. Consequently we give its best chance to this allocation rule against time priority.

Let  $\bar{T}C(n)$  be the trading cost obtained with a grid size  $g(n)$ . It is measured by the difference between the transaction price and the expected value of the asset. When time priority is enforced, Corollary 1 implies:

$$\bar{T}C(n) = \begin{cases} a^m - \mu & \text{if } \bar{\delta} \in (\tau, \tau + 1) \text{ and } \bar{\delta} \leq \tau_{TPR}^* \\ p(1) - \mu & \text{if } \bar{\delta} > \tau_{TPR}^* \end{cases} \quad (12)$$

The definition of the trading cost is the same when the random allocation rule is used except that  $\tau_{TPR}^*$  is replaced by  $\tau_{RAR}^*$  and  $p(1)$  is replaced by  $p(2)$ . The trading cost is random because the transaction price depends on the order arrival time which is random. Moreover it depends on the tick size because the minimum price variation influences both the difference between the transaction price (at a given point in time) and the asset expected value (in particular  $(p(1) - \mu)$ ) and  $\tau_{TPR}^*$  (or  $\tau_{RAR}^*$ ). We search  $n^*$  such that the grid size  $g(n^*)$  minimizes the expected trading costs  $E(\bar{T}C(n))$ . We suppose that  $g(0)$  is sufficiently large so that there is some room for a decrease in the expected trading cost with a finer grid<sup>25</sup>. We obtain the following result.

**Proposition 6** *For all  $\Delta > 0$ , the optimal tick size is always strictly greater than zero and depends on the priority rule:*

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<sup>25</sup>This condition is satisfied as soon as  $g(0)$  is sufficiently large for the associated  $\tau^*$  to be lower than 0 for both allocation rules. We impose this condition just to avoid the case in which the grid should be enlarged in order to minimize the expected trading costs so that  $n^* = 0$  would be the only solution in our framework.

(i) With time priority, the optimal tick size is  $g(n_{TPR}^*)$  with  $n_{TPR}^* = \lfloor \lfloor \frac{(1+2h(\Phi))g(0)}{4(R_B-\mu)} - 1 \rfloor \rfloor$  and  $h(\Phi) = \lfloor \lfloor \frac{\Phi+1}{2\Phi} \rfloor \rfloor$ .

(ii) With the random allocation rule, the optimal tick size is  $g(n_{RAR}^*)$  if  $\Phi \geq \bar{\Phi}$  and  $g(n_{RAR}^* + 1)$  if  $\Phi < \bar{\Phi}$  with  $n_{RAR}^* = \lfloor \lfloor (\frac{7g(0)}{2(R_B-\mu)} - 1) \frac{1}{2} \rfloor \rfloor$ . ( $\bar{\Phi}$  is defined in the appendix.)

When time priority is enforced, the intuition is as follows. On one hand, a small tick reduces the expected trading costs because it forces the dealers to quote a price very close to their reservation value if they want to capture time priority once for all (i.e. if they want to be executed with probability one). On the other hand, it reduces the dealers' incentives to quote the competitive price immediately. This second effect increases the length of time during which the inside spread will be greater than the competitive spread. This increases the probability of order execution at a relatively high price and the expected trading costs. A tick bounded away from zero balances optimally those effects. We note that the optimal tick size is increasing in  $\Phi$ . This reflects the fact that the option of quoting a large spread, at the risk of losing priority of execution, becomes more attractive in this case. The tick must increase to counterbalance this effect. The same intuition is valid when time priority is not enforced. The main difference is that the optimal tick is much less sensitive to  $\Phi$ . This is due to the fact that the dealers have no incentive to quote the lowest possible spread at the initial stage, even when  $\Phi$  is small, as long as  $R_B \geq p(4)$ .

The model establishes a rationale for the use of minimum price variations. However we recognize that it is too stylised to point what should be the determinants of the optimal tick size. Perhaps this question must be solved ultimately by empirical methods (see for instance Harris (1994) or Ahn *et al.* (1996)). We offer two remarks which could be useful for further empirical investigations on this issue. First whatever the allocation rule, the optimal tick size is weakly increasing with  $\Phi$ . This parameter is negatively related to the average waiting time between liquidity trader arrivals ( $\lambda^{-1}$ ) and positively related to the waiting time between dealers' arrivals ( $\Delta$ ). In the model, the tick size should be small when the offers arrive sufficiently rapidly relative to the frequency of liquidity trader arrivals. We conclude that the optimal tick size might be related to the transaction frequency (a proxy for  $\lambda$ ) relative to the frequency of new offers. Second the optimal tick size is not the same with time priority and with the random allocation rule. Since the allocation rule is another dimension of the market structure, this suggests that the optimal tick size might

be dependent on other aspects of the market structure<sup>26</sup>.

## 4.2 Priority Rule.

First we remark that the comparison of the equilibria with and without time priority shows that time priority precludes bidding strategies by which dealers can sustain uncompetitive spreads. In cases for which implicit collusion might be a concern and time priority is not enforced<sup>27</sup>, the model suggests that time priority could be used to prevent implicit collusion.

The next proposition compares the size of the expected trading costs for the two allocation rules and qualifies a bit the last statement.

**Proposition 7 :** *For a given minimum price variation and  $R_B > p(2)$ , the expected trading costs are minimized when time priority is enforced if  $\Phi \leq \Phi^*$  while they are minimized when the random allocation rule is enforced if  $\Phi > \Phi^*$  with  $\Phi^* = \frac{3-\sqrt{5}}{2}$ .*

Surprisingly, for sufficiently large values of  $\Phi$ , trading costs can be higher when time priority is enforced. The intuition is as follows. Consider a dealer who is about to move in state  $p(4)$ . With time priority, this dealer compares the expected gain if he undercuts with his expected gain if he quotes  $p(1)$ . It turns out that for  $\Phi > \Phi^*$ , the first alternative is better than the second (in fact it is better for all prices greater than  $p(2)$ ). With the random allocation rule, the dealer is always better off quoting directly  $p(2)$ . Consequently, trading might take place at quote  $p(3)$  with time priority while this will not occur with the random allocation rule. Combined with the fact that the probability of execution at a relatively high price is large when  $\Phi > \Phi^*$ , this gives the result. This is illustrative of one potential cost of time priority. The focal price with time priority (the competitive price) is lower than the focal price without time priority but it might take more time to reach the focal price when time priority is enforced. However, reasonable

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<sup>26</sup>In the same line, we have already emphasized that the difference of result with the previous literature concerning the optimal tick size comes from the fact that we model different market structures.

<sup>27</sup>Dutta and Madhavan (1995) and Kandel and Marx (1996) argue that the empirical results of Christie and Schultz (1995) concerning dealers' pricing strategies in the NASDAQ can be explained by implicit collusion among dealers. They do not consider time priority since NASDAQ does not enforce time precedence among dealers' quotes (except in the SOES system which represents a small fraction of the order flow).

parameterizations for  $\Phi$  are likely to be in the range in which time priority is optimal. Moreover, remind that we have considered the least collusive outcome for the Random Allocation Rule. Overall our results support the enforcement of time priority.

## 5. Extensions.

### 5.1 Random Waiting Time between Dealers' Offers.

In the previous sections, we assumed that dealers were moving in turn and were checking their quote at fixed points in time. A more palatable assumption is that the time interval between quote revisions is random.

Specifically, assume that the time between quote revisions by a given dealer is exponentially distributed with parameter  $r$ . A difficulty arises when the dealer who considers revising her quote has strict priority at the best quote since the last time she moved. Actually, in this case it can be that either her competitor has not revised his quote yet or that her competitor has not improved upon her quote. Although this will not change the results obtained previously, this uncertainty would make the presentation and the derivations of the results substantially more involved<sup>28</sup>. In order to avoid this problem, let simply assume that a dealer in this situation receives some information which allows to distinguish between the two cases mentioned above. Under this assumption, it is straightforward that a dealer does not change her quote as long as she does not observe or learn that her competitor has revised his quote since her last move.

We denote by  $\bar{\tau}(k)$  the date of the  $k^{th}$  quote revision such that the dealers acting at dates  $\bar{\tau}(k)$  and  $\bar{\tau}(k+1)$  are different. The last remark of the previous paragraph implies that the dealers' expected profits conditional on the arrival of an order in the interval  $[\bar{\tau}(k), \bar{\tau}(k+1)[$  are constant. Moreover they just depend on  $k$  and not on the date  $\tau(k)$  at which the order revision takes place. In this case, some algebra shows that the expected profit of a dealer revising his quote at date  $\tau(k)$  can be written as:

<sup>28</sup>The intuition that the results are unchanged is as follows. In this case, dealers must assign a probability to each of the two possible events. Then we can proceed as in Section 3 to show that the probability of a dealer not improving upon the best quote must be zero in equilibrium. As a consequence, the equilibria will be of the same type as the equilibria described in Section 3.

$$E(\Pi_j(a_\tau^h, q_\tau^j, a_\tau^j) \mid \bar{i} \geq \tau(k)\Delta, H_{\tau(k-1)}^j) = \sum_{l=k}^{+\infty} \left( \frac{\lambda}{\lambda + \gamma + \tau} \right) \left( \frac{\tau}{\lambda + \gamma + \tau} \right)^{l-k} \Pi_j(a_{\tau(l)}^h, q_{\tau(l)}^j, a_{\tau(l)}^j) \quad (13)$$

with  $j \neq h$  and  $\{j, h\} \in \{1, 2\}$ . Defining  $\Phi' = \frac{\lambda + \tau}{\lambda + \gamma + \tau}$ , we get:

$$E(\Pi_j(a_\tau^h, q_\tau^j, a_\tau^j) \mid \bar{i} \geq \tau(k)\Delta, H_{\tau(k-1)}^j) = \sum_{l=k}^{+\infty} \left( \frac{\lambda}{\lambda + \gamma} \right) \Phi'^l (1 - \Phi')^{l-k} \Pi_j(a_{\tau(l)}^h, q_{\tau(l)}^j, a_{\tau(l)}^j) \quad (14)$$

Equation (14) shows that formally the random waiting time model is the analogue of the model in which dealers were alternating in quoting prices at discrete points in time ( $\Phi'$  playing the role of  $\Phi$ ). Consequently, the results obtained are still valid in this case. For  $\gamma = 0$ , the execution probability ( $\Phi'$ ) of the best quote is increasing in the ratio  $\frac{\lambda}{\tau}$ . This remark is used when we discuss the empirical implications of the model.

## 5.2 A Large Number of Dealers.

Two properties of the equilibria obtained with two dealers are crucial for our policy implications. First, the adjustment of the best price to the competitive price is not immediate. Second, when time priority is not enforced, uncompetitive spreads can be sustained in equilibrium. We argue now that these properties still hold when more than two liquidity suppliers compete for the order flow.

### With time priority.

One might wonder if the the adjustment of quotes to competitive levels is not faster when one enlarges the number of traders. In order to deal with this concern, let consider a polar case in which the number of dealers is large so that each dealer has the opportunity to move only once. In this case, the analysis is simplified since it cannot be optimal for a dealer to post a price above the best price. In fact, since the dealers are offered only one opportunity to make an offer, we can dispense with our assumption that the book is closed. The choice of a dealer arriving at date  $\tau$  is either to quote  $p(1)$  or to quote  $a_{\tau-1}^m - g$ . A dealer will undercut the best quote by only one tick if:

$$\Phi(a_{\tau-1}^m - g - \mu) \geq (p(1) - \mu) \quad (15)$$

This inequality is satisfied as long as  $a_{\tau-1}^m \geq p(i^* + 2)$ . This implies that the best price will be improved by small increments as long as the best quote is greater than  $p(i^* + 1)$  and its dynamics will be exactly the

same as the one obtained when two dealers move in turn repeatedly.

### Without time priority.

We want to show that even when the number of traders is large,  $p(2)$  can still be a focal price when time priority is not used. Here again in order to focus directly on the issue in the simplest manner, we consider the particular case in which  $R_B = p(2)$ . In this case there are only two possible positions for the price of a dealer:  $p(1)$  or  $p(2)$ . We suppose that the number of dealers is  $N > 2$  and that the dealers choose their prices in turn. In this setting, there are always parameters values (for  $N$  and  $\Phi$ ) such that  $p(2)$  is a focal price in equilibrium. We just convey the intuition, skipping the formal proof for brevity. Consider a dealer, say  $n$ , who is about to revise his offer when all the dealers quote already  $p(2)$ . If dealer  $n$  matches the current offer, he obtains (taking into account that  $p(2)$  is a focal price):

$$\left(\frac{\lambda}{\lambda + \gamma}\right) \frac{(p(2) - \mu)}{N} \quad (16)$$

If he undercuts, his expected profit depends on the reaction of his rivals. It is clear that if, in equilibrium, none of his rivals matches his offer at  $p(1)$  then they do not behave optimally. Therefore, his offer will be matched by at least one competitor. But in this case, one can show that the best response of *all* the dealers is to quote  $p(1)$  when they will revise their quote. The expected profit of a dealer who undercuts  $p(2)$  is then:

$$\left(\frac{\lambda}{\lambda + \gamma}\right) (p(1) - \mu) \left[ \sum_{k=0}^{N-2} \Phi \frac{(1 - \Phi)^k}{k + 1} + \frac{(1 - \Phi)^{N-1}}{N} \right] \quad (17)$$

Comparison of (16) and (17) show that, for a given number of dealers, dealer  $n$  is better off quoting  $p(2)$  than undercutting if  $\Phi$  is sufficiently small. Now if it is optimal for one dealer to quote  $p(2)$  when all his  $N$  competitors quote this price, it is also the case when less than  $N$  dealers have already made an offer at  $p(2)$ . Consequently all the values for  $N$  and  $\Phi$  such that (16) is greater than (17) are such that  $p(2)$  is a focal price.

The result is in sharp contrast with the prediction of the static model of Bertrand competition. For  $N > 3$ ,  $p(1)$  is the unique outcome in this case. The reason of this difference is straightforward. In a dynamic setting, a dealer who undercuts  $p(2)$  takes into account that his competitors will react optimally by matching his offer. This threat can deter a dealer from undercutting the uncompetitive spread when  $\Phi$  is low.

Actually, in this case, the decrease in profit which occurs as competitors match his quote, dominates the immediate expected profit to capturing a large share of the order flow<sup>29</sup>.

To sum up, even when the number of dealers is large, non-competitive ask prices can still be obtained when time priority is not enforced. However uncompetitive quotes are more difficult to sustain as  $N$  or  $\Phi$  are large.

## 6. Testable Implications.

### 6.1 Trading Costs Determinants.

Many empirical studies analyze the links between the size of the inside spread and the transaction frequency. The model suggests that the frequency of new offers is also an important determinant of trading costs. Actually the expected trading costs are weakly increasing in  $\Phi$  under both allocation rules. With the random allocation rule, it takes some time for the quotes to adjust whatever the value of  $\Phi$ . As a consequence when  $\Phi$  increases the probability of a trade at a price greater than the lowest possible price increases and this gives the result. With time priority this effect is combined with the fact that  $\Phi$  needs to be sufficiently large in order to induce the dealers to post prices above the competitive level. The random waiting time version (Section 5.1) shows that  $\Phi$  is positively related to the frequency of liquidity trader arrivals in the market ( $\lambda$ ) and inversely related to the frequency of quote revisions by liquidity suppliers ( $r$ ). For empirical purposes, the transaction frequency could be taken as a proxy for  $\lambda$  while the frequency of new limit orders within the inside spread could be taken as a proxy<sup>30</sup> for  $r$ . The model predicts that larger spreads should be observed on average when the ratio between the transaction frequency and the frequency of new limit orders within the spread increases.

### 6.2 Quote Dynamics.

Our results (Propositions 2 and 4) show that price improvements should occur in sequence when the spread is large. On the contrary, a unique or no price improvement should be observed when the spread is small. This relationship between the dynamics of the best price and the state of the book (characterized by the

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<sup>29</sup>Although our framework is very different, this is reminiscent of the argument developed by Dutta and Madhavan (1995).

<sup>30</sup>Because in the model, in equilibrium, a quote revision necessarily leads to a price improvement if the ask price is not competitive.

size of the spread) is consistent with the empirical findings in Biais, Hillion and Spatt (1995). Moreover different price dynamics should be observed according to market conditions when time priority is enforced. Actually successions of small price improvements should be more frequent when  $\Phi$  is large than when  $\Phi$  is small. Other things equal, this sensitivity of the quote dynamics to  $\Phi$  should not be observed in markets in which time priority is not enforced. Finally the model predicts that spreads will depend on the time between transactions, with spreads being decreasing as this time increases. Easley and O'Hara (1992) obtain a similar result but for a very different reason. In their model, the absence of trade is a signal which induces dealers to revise downward their probability of trading with an informed agent. Here, the spread does not immediately adjust to its lowest possible level but is reduced over time because of price competition between liquidity suppliers.

## 7. Conclusions.

This paper analyzes the impact of two trading rules, namely a mandatory minimum price variation and time priority, on trading costs. We model explicitly the dynamic bidding process which characterizes competition between liquidity suppliers in continuous limit order markets. In this setting, we obtain that a zero minimum price variation never minimizes the trading costs and that time priority, contrary to another simple tie-breaking rule, prevents uncompetitive spreads from being sustained over time. We also investigate how the quote dynamics is related to the characteristics of the order arrival process.

Two important extensions could be considered. In our stylised framework, liquidity suppliers post prices but not quantities. We assume, in line with the Bertrand analysis, that at their posted quotes the dealers stand ready to accommodate the incoming market orders up to the maximum possible quantity for those orders<sup>31</sup>. As quantity is also a decision variable, future research should consider dynamic competition when liquidity suppliers can announce both a price and a maximum quantity. In our analysis, the time between liquidity suppliers' offers is exogenous. It would be interesting to endogenize this time in order to analyze what are the possible determinants of the waiting time between quotes. This would help to better understand the order arrival process in financial markets.

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<sup>31</sup>Note that this assumption is also present in most of the models of price formation in dealership markets. In those models, dealers are allowed to quote prices contingent on the total order flow but this is quite different from quoting prices and quantities contingent on the order flow. See Dornick (1993) and Biais *et al.* (1996) for the effects of allowing dealers to choose both prices and quantities.



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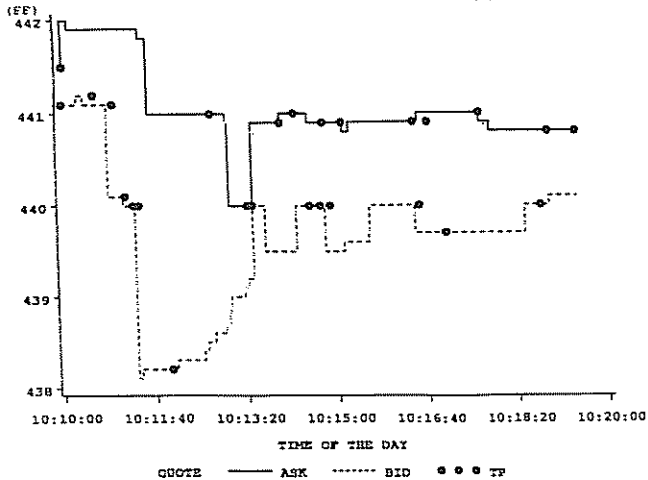
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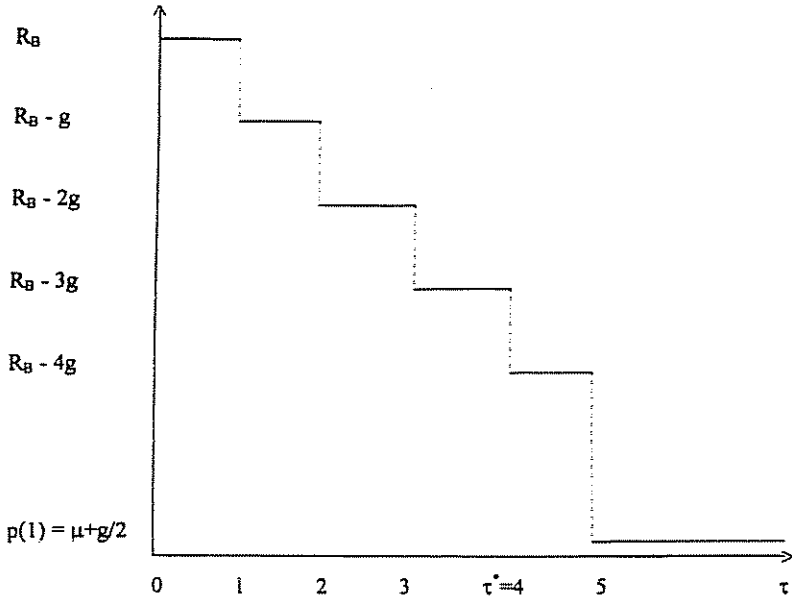
Appendix.

Figure 1.



This graphic is the same as Figure 3 page 1681 in Biais, Hillion and Spatt (1995). It represents the evolution of the best quotes (the full line is the ask quote and the dashed line the bid quote) for Elf-Aquitane, November 9, 1991. The dots represent transaction prices. This figure is chosen by the authors because it illustrates both mean-reversion in the spread and competition in the supply of liquidity. The initial sequence of orders widens the spread and triggers a decrease in the bid until FF438.1. Then new bid quotes are posted outbidding each other, bringing back the bid to FF440.5. The tick size for this stock is FF0.1.

Figure 2.



Quote dynamics with Time Priority.

## Proof of Lemma 1.

In order to simplify a bit the notations, let  $\Pi_i^j \equiv \Pi_j(a_i^j, q_i^j, a_i^j)$ . By definition, if  $\bar{o} \leq \bar{T}$  then the game stops because of an order arrival. Remark that i) the expected profit of a dealer is zero if the game stops before execution and ii) that the date at which the game stops is independent of the trading history. These remarks imply that:

$$E(\Pi_i^j \mid \bar{i} \geq \tau_0 \Delta, H_{\tau_0-1}^j) = \int_{\tau_0 \Delta}^{+\infty} E(\Pi_i^j \mid \bar{o} = o, \bar{T} \geq o, H_{\tau_0-1}^j) \text{Prob}(\bar{T} \geq o \mid \bar{T} \geq \tau_0 \Delta) \lambda e^{-\lambda(o - \tau_0 \Delta)} do \quad (18)$$

Now for a given history until time  $\tau_0$ ,  $E(\Pi_i^j \mid \bar{o} = o, \bar{T} \geq o, H_{\tau_0-1}^j) = \Pi_o^j$ . Using this remark and the fact that  $\bar{T}$  is exponentially distributed, we get:

$$E(\Pi_i^j \mid \bar{i} \geq \tau_0 \Delta, H_{\tau_0-1}^j) = \lambda \int_{\tau_0 \Delta}^{+\infty} \Pi_o^j e^{-(\lambda + \gamma)(o - \tau_0 \Delta)} do \quad (19)$$

Now  $\Pi_o^j = \Pi_{\tau}^j$ ,  $\forall o \in [\tau \Delta, (\tau + 1)\Delta)$ . This implies that we can rewrite (19):

$$E(\Pi_i^j \mid \bar{i} \geq \tau_0 \Delta, H_{\tau_0-1}^j) = \left( \frac{\lambda}{\lambda + \gamma} \right) \sum_{\tau=\tau_0}^{+\infty} \Pi_{\tau}^j \int_{\tau \Delta}^{(\tau+1)\Delta} (\lambda + \gamma) e^{-(\lambda + \gamma)(o - \tau_0 \Delta)} do \quad (20)$$

Then direct computations give the result announced in the lemma. *Q.E.D.*

## Proof of Lemma 2.

a) **If time priority is enforced. Initial remark.** In state  $\{p, 1\}$ , it is necessarily the case that  $\bar{a} \geq p$ . If  $\bar{a} > p$  then  $R(p, 1, \bar{a}) = R(\bar{a}, 0, \bar{a})$ . Actually a dealer in state:  $\{p, 1, \bar{a}\}$  has exactly the same opportunities as a dealer in state  $(\bar{a}, 0, \bar{a})$ . If  $\bar{a} = p$ , either  $R(p, 1, \bar{a}) = p$  or  $R(p, 1, \bar{a}) = R(p, 0, p)$  because a dealer with time priority at  $p$ , knowing that his competitor has quoted  $p$ , is not more constrained than a dealer without time priority at  $p$  and he has the additional possibility to keep priority at  $p$ .

Consider now a dealer, say  $i$ , who is about to move at date  $\tau$  and who has priority at the current price  $a_{\tau-1}^m = p$ . It is necessarily the case that he posted this price at time  $\tau - 2$ . Two cases can occur.

**Case 1.** Dealer  $i$  has captured strict priority at time  $\tau - 2$  and he has not been undercut at time  $\tau - 1$  by his competitor. In equilibrium, his competitor has reacted offering  $R(p, 0, p)$ . Therefore the belief of dealer  $i$  must be  $R(p, 0, p)$ .

**Case 2.** Dealer  $i$  has not seized strict priority at time  $\tau - 2$ . Let  $p'$  be dealer 2's price just before he reacts at time  $\tau - 1$ . If  $\{p', 1\}$  is on the equilibrium path, then dealer 2 must have inferred correctly that dealer  $i$  offered  $p$  at time  $\tau - 2$ . Therefore according to our initial remark, he must react with  $R(p', 1, p) = R(p, 0, p)$  if  $p' \neq p$ . If  $p' = p$  and if  $R(p, 1, p) = p$  then dealer  $i$  cannot have priority at  $\{p, 1\}$  at time  $\tau$ . Therefore  $R(p', 1, p) = R(p, 0, p)$  in this case also. If  $\{p', 1\}$  is not on the equilibrium path, then in equilibrium Case 1 must occur (otherwise  $\{p, 1\}$  would be reached only with out-of-equilibrium actions).

b) **If time priority is not enforced.**

In this case, it is necessarily the case that  $\bar{a} > p$  when a dealer does not observe his competitor's quote. Then the proof is exactly the same as before, except that we do not need to consider the case in which  $p' = p$ . *Q.E.D.*

The following two lemmata are useful to establish Proposition 1.

**Lemma 3** *in equilibrium, if  $R(p, 0) = p$  and if  $\{p, 1\}$  belongs to the equilibrium path then  $R(p, 1) = p$ .*

**Proof:** If  $\{p, 1\}$  belongs to the equilibrium path, then the dealer who observes state  $\{p, 1\}$  believes that his competitor has quoted  $R(p, 0) \geq p$  (Lemma 2). We have either  $R(p, 1) = R(R(p, 0), 0)$  or  $R(p, 1) = p$  (see remark in the proof of Lemma 2). If  $R(p, 0) = p$  then it is straightforward that, in all the cases,  $R(p, 1) = p$ . *Q.E.D.*

**Lemma 4** . In equilibrium, the set of prices such that  $R(p,0) > p$  and  $R(p,1) > p$  with  $p \geq p(1)$  is empty.

**Proof:**

Assume that this is not the case and denote by  $p^*$  the largest price such that  $R(p^*,0) > p^*$  and  $R(p^*,1) > p^*$ . Consider first the case of a dealer, say dealer 1, reacting to state  $\{p^*,0\}$ . Assume (to be contradicted) that dealer 1 reacts with:  $R(p^*,0) > R(p^*,1)$ . He will lose price priority for the next two rounds and dealer 2 will quote  $R(p^*,1)$  at the next period. Consequently dealer 1 obtains  $(1-\Phi)^2V(R(p^*,1),0)$ . Now consider the following deviation for dealer 1. He quotes (secretely)  $R(p^*,0) = R(p^*,1)$ , he obtains time priority over the next dealer in one period (because this dealer will quote  $R(p^*,1) > p^*$ ) and then (in two periods) he follows the pricing policy he would follow in state  $\{R(p^*,1),0\}$ . This deviation gives him  $(1-\Phi)\frac{1}{\lambda+1}\Phi\Pi_1(R(p^*,1),1,R(p^*,1)) + (1-\Phi)^2V(R(p^*,1),0)$  which is greater than  $(1-\Phi)^2V(R(p^*,1),0)$ . This implies that in equilibrium, it is necessarily the case that:

$$R(p^*,0) \leq R(p^*,1) \quad (21)$$

Now consider dealer 1 reacting in state  $\{p^*,1\}$ . According to Lemma 2, dealer 1 conjectures that dealer 2 has posted  $R(p^*,0)$ . Moreover  $R(p^*,1) = R(R(p^*,0),0)$  since the problem faced by dealer 1 is similar to the problem he would face in state  $\{R(p^*,0),0\}$  because  $R(p^*,0) > p^*$ . We prove now that  $R(R(p^*,0),0) < R(p^*,0)$ . By definition of  $p^*$ , it must be the case, since  $R(p^*,0) > p^*$  that either a)  $R(R(p^*,0),0) \leq R(p^*,0)$  or b)  $R(R(p^*,0),1) \leq R(p^*,0)$ .

If b) is satisfied then we can show that  $R(R(p^*,0),0) < R(p^*,0)$  as follows. Define  $p' \equiv R(p^*,0)$  in order to simplify the notation. There are 4 cases:

**Case 1.**  $R(p',1) = p'$ . In this case it is clear that  $R(p',0) \geq p$  is not optimal since this would give a zero expected profit to the dealer moving in state  $\{p',0\}$  while he could obtain a strictly positive expected profit by undercutting (because  $p' \geq p(2)$ ).

**Case 2.**  $R(p',1) \leq p' - 2g$ . Suppose that in equilibrium dealer 1 reacts to  $\{p',0\}$  with  $R(p',0) \geq p'$ . Then he loses priority for the next two periods and he obtains:  $(1-\Phi)^2V(R(p',1),0)$ . Now consider the deviation which consists in quoting  $p' - g$  for dealer 1. Dealer 2 will still quote  $R(p',1) < p' - g$  at the next period since he has lost priority over a set of prices that he would not have chosen anyway. Therefore dealer 1 obtains  $\frac{1-\Phi}{\lambda+1}\Phi\Pi_1(p',1,p'-g) + (1-\Phi)^2V(R(p',1),0)$  which is strictly better (because  $p' \geq p(2)$ ) than what he is supposed to obtain in equilibrium. A contradiction. Therefore  $R(p',0) < p'$  in this case also.

**Case 3.**  $R(p',1) = p' - g = p^*$ . Suppose that in equilibrium dealer 1 reacts to  $\{p',0\}$  with  $R(p',0) \geq p'$ . If  $p^* = p' - g$  then the expected profit of the dealer moving in state  $\{p',1\}$  is bounded by  $\frac{1-\Phi}{\lambda+1}\Pi(p',1,p')$ . But then  $R(p',1) = p'$  is optimal since it gives exactly this profit if  $R(p',0) \geq p'$ .

**Case 4.** If  $R(p',1) = p' - g > p^*$ . Suppose that in equilibrium dealer 1 reacts to  $\{p',0\}$  with  $R(p',0) \geq p'$  and that he optimally undercuts in state  $\{p' - g,0\}$  with  $R(p' - g,0)$ . Then there is a contradiction since dealer 1 would be better off quoting initially directly  $R(p' - g,0)$ , rather than losing the chance to trade during two periods (postponing profits two periods is suboptimal). If dealer 1 does not undercut in state  $\{p' - g,0\}$ , then we have necessarily  $R(p' - g,1) \leq p' - g$  by definition of  $p^*$ . Then it is clear that we can reiterate the arguments offered for Case 1, 2 and 3 until the contradiction is found.

If a) is satisfied as an equality then Lemma 3 implies that  $R(R(p^*,0),1) = R(p^*,0)$ . But in this case, the dealer who reacts to  $\{R(p^*,0),0\}$  will obtain a zero profit while he can obtain a strictly positive profit by undercutting of at least one tick. Therefore, in all the cases, we must have  $R(R(p^*,0),0) < R(p^*,0)$  which means:

$$R(p^*,1) < R(p^*,0) \quad (22)$$

but then we arrive to a contradiction between (21) and (22). Therefore  $p^*$  does not exist. *Q.E.D*

## Proof of Proposition 1.

**Step 1.** We know from Lemma 4 that, in equilibrium, it is not possible for the reaction function to have the following values in  $p(1)$ :  $R(p(1),1) > p(1)$  and  $R(p(1),0) > p(1)$ . Moreover Lemma 3 implies that  $R(p(1),1) > p(1)$  and  $R(p(1),0) = p(1)$  is impossible in equilibrium. Therefore the only possibilities are i)  $R(p(1),1) = p(1)$

and  $R(p(1), 0) = p(1)$  or  $R(p(1), 1) = p(1)$  and  $R(p(1), 0) = p(2)$ . This proves the second and the third claim of Proposition 1.

**Step 2.** Now consider any price such that  $p \geq p(2)$ . Lemma 4 implies that at least one of those two inequalities must be true a) either  $R(p, 0) \leq p$  or b)  $R(p, 1) \leq p$ . Now using Lemma 3 and the previous step, it is straightforward that  $R(p, 0) \geq p$  cannot be optimal since with such a reaction, a dealer would lose the chance to trade during two periods and would not trigger an increase in the quote of the other dealer. This implies that necessarily  $R(p, 0) < p$  for  $p \geq p(2)$  which is the first claim of Proposition 1. *Q.E.D.*

## Proof of Proposition 2.

Denote by  $i^*$  the first integer below  $\frac{1}{2}(\frac{\lambda}{\phi} + 1)$ . We have:  $p(i^* + 1) = p(1) + \lfloor \lfloor \frac{1}{2}(\frac{\lambda}{\phi} + 1) \rfloor \rfloor g$ . Remark that  $i^* \geq 1$  since  $\phi < 1$ .

**Case 1.** Consider a dealer in state  $\{p(i), 0\}$ ,  $p(i) \in [p(2), p(i^* + 1)]$ . If he quotes  $p(1)$ , he obtains strict priority until the end of the game and an expected gain of  $(\frac{\lambda}{\lambda + \tau})^{\frac{1}{2}} L$ . If he undercuts of one tick and if he is not executed before the arrival of his competitor, he will lose priority until the end of the game according to the conjectured equilibrium. Consequently this alternative gives him an expected profit of  $(\frac{\lambda}{\lambda + \tau})^{\frac{1}{2}} \phi \frac{1}{2} (2i - 3)L$ . This is lower than  $\frac{\lambda}{\lambda + \tau} \frac{1}{2} L$  since  $i \leq i^* + 1$ . Finally if he quotes a price greater than or equal to  $p$ , the other dealer will maintain his quote at the same level as long as he is not undercut or will quote  $p(1)$  at the next round. Consequently, the dealer must choose  $p(1)$ , which proves the second part of Proposition 2.

**Case 2.** Now consider a dealer in state  $\{p(i), 0\}$  with  $p(i) \in [p(i^* + 2), R_B]$ . First remark that in this case, if the dealer undercuts of one tick he obtains at least:  $(\frac{\lambda}{\lambda + \tau})^{\frac{1}{2}} \phi \frac{1}{2} (2i - 3)L$  while if he quotes  $p(1)$  he obtains:  $\frac{\lambda}{\lambda + \tau} \frac{1}{2} L$ . Since  $i > i^* + 1$ , the dealer is better off if he just undercuts. Moreover his competitor will not change his quote as long as this quote is not improved since  $p(i) > p(i^*)$ . Consequently in this case  $R(p(i), 0) = p(i) - g$  is a best response. This proves the first part of Proposition 2. The third part of the proposition has already been obtained in Proposition 1.

**Case 3.** Consider a dealer in state  $\{p(i), 1\}$  with  $p(i^* + 1) \leq p(i) \leq R_B$ . Proposition 1 implies that  $\{p, 1\}$  is never on the equilibrium path if  $p > p(1)$ . Consequently the dealer believes that his competitor has quoted  $p(i) + g$ . Since  $p(i) + g \geq p(i^* + 2)$ , using the argument developed in the previous case, it is straightforward that  $R(p(i), 1) = p(i)$  is a best response for the dealer. This proves the fourth part of Proposition 2.

**Case 4.** Finally consider a dealer in state  $\{p(i), 1\}$  with  $p(1) \leq p(i) \leq p(i^*)$ . For the same reason as before, the dealer believes that his competitor has quoted  $p(i) + g$ . Since  $p(i) + g \leq p(i^* + 1)$ , using the argument developed in the first case, it is straightforward that  $R(p(i), 1) = p(1)$  is a best response for the dealer. This proves the fifth part of Proposition 2. *Q.E.D.*

## Proof of Corollary 1.

Consider the dealer who is about to react at time 0. There is no quote posted in the book at this time. Quoting a price strictly greater than  $R_B$  is not optimal since he has no chance to be executed as long as his quote is not lower than  $R_B$ . Case 3 in the proof of Proposition 2 shows that the dealer must quote  $R_B$  if  $R_B \geq p(i^* + 1)$ . Then from Proposition 2, we deduce that the dealers will continue to undercut each other until time  $\tau_{\overline{P}PR}^*$  such that:  $R_B - \tau_{\overline{P}PR}^* g = p(i^* + 1)$ . The dealer who is about to move at time  $\tau_{\overline{P}PR}^* + 1$  will observe a best quote equal to  $p(i^* + 1)$  and consequently will quote  $p(1)$ . If  $R_B < p(i^* + 1)$  then  $\tau^* < 0$  and the first dealer quotes immediately  $p(1)$ . *Q.E.D.*

The following result is useful to prove Proposition 3:

**Lemma 5** *in equilibrium, with the random allocation rule, if  $p^*$  is the greatest price such that  $R(p^*, 0) > p^*$  and  $R(p^*, 1) > p^*$  then  $R(p^*, 0) = R(p^*, 1)$  and  $R(R(p^*, 0), 0) = R(R(p^*, 0), 1) = R(p^*, 0)$ .*

**Proof:** The proof that  $R(p^*, 0) = R(p^*, 1)$  follows the steps of the proof of Lemma 4. The only difference is that in Case 1, we cannot exclude that  $R(R(p^*, 0), 0) = R(p^*, 0)$  because with the random allocation rule, a trader can



still hope some gains if he matches his competitor's quote. The arguments used in the proof of Lemma 4 show that  $p^*$  can exist only if this equality is satisfied. Therefore it must be the case that:  $R(R(p^*, 0), 0) = R(p^*, 0)$ . The fact that  $R(R(p^*, 0), 1) = R(p^*, 0)$  comes from Lemma 3. *Q.E.D.*

### Proof of Proposition 3.

**Part 1).** Suppose that it exists  $p^*$  on the equilibrium path such that  $R(p^*, 0) > p^*$  and  $R(p^*, 1) > p^*$ . Then Lemma 5 implies that  $R(p^*, 0) = R(p^*, 1)$ . Define  $\bar{p} \equiv R(p^*, 0)$ . From Lemma 5, we have:  $R(\bar{p}, 0) = R(\bar{p}, 1) = \bar{p}$ . Let call a price with this property a focal price. For  $p^*$  to be on the equilibrium path, it must exist:  $\bar{p} > p^*$  such that  $R(\bar{p}, 0) = p^*$  or  $R(\bar{p}, 1) = p^*$ . With this reaction at  $\bar{p}$ , a dealer obtains:

$$\frac{\lambda}{\lambda + \gamma} L[\Phi(p^* - \mu) + \Phi(1 - \Phi)(p^* - \mu) + (1 - \Phi)^2 \frac{(\bar{p} - \mu)}{2}] \quad (23)$$

Moreover since  $R(p^*, 0) = \bar{p}$ , it must be the case that:

$$\frac{\lambda}{\lambda + \gamma} (1 - \Phi) L \frac{(\bar{p} - \mu)}{2} \geq L \frac{\lambda}{\lambda + \gamma} [\Phi \frac{(p^* - \mu)}{2} + \Phi(1 - \Phi)(p^* - \mu) + (1 - \Phi)^2 \frac{(\bar{p} - \mu)}{2}] \quad (24)$$

Now at price  $\bar{p}$ , quoting directly  $\bar{p}$ , a dealer could obtain:

$$\frac{\lambda}{\lambda + \gamma} L[\Phi(\bar{p} - \mu) + \Phi(1 - \Phi) \frac{(\bar{p} - \mu)}{2} + (1 - \Phi)^2 \frac{(\bar{p} - \mu)}{2}] \quad (25)$$

Using Equation (24), it is straightforward that (25) is greater than (23). Therefore  $p^*$  cannot be on the equilibrium path. This implies that  $R(p, 0) \leq p$  or  $R(p, 1) \leq p \quad \forall p$  on the equilibrium path. Now it is clear that  $R(p, 0) > p$  is suboptimal since it will never induce a competitor to raise his quote above the best quote. Therefore it is always the case that:  $R(p, 0) \leq p$ .

**Part 2).** Consider a focal price  $\bar{p}$ , i.e. such that  $R(\bar{p}, 0) = R(\bar{p}, 1) = \bar{p}$ , on the equilibrium path. The previous part implies that  $R(p, 0) \leq p \quad \forall p \geq \bar{p}$ . If  $R(p, 0) = p$  then  $p$  is another focal point above  $\bar{p}$  but then it is easy to show that  $\bar{p}$  cannot be on the equilibrium path. Moreover  $R(p, 0) < \bar{p}$  cannot be optimal in equilibrium. Actually this would imply that:

$$\frac{\lambda}{\lambda + \gamma} \Phi L(R(p, 0) - \mu) + (1 - \Phi)W(p, 1, R(p, 0)) > \frac{\lambda}{\lambda + \gamma} [\Phi L(\bar{p} - \mu) + (1 - \Phi)L \frac{(\bar{p} - \mu)}{2}] \quad (26)$$

But since  $R(\bar{p}, 0) = \bar{p}$  then:

$$\frac{\lambda}{\lambda + \gamma} \Phi L(R(p, 0) - \mu) + (1 - \Phi)W(\bar{p}, 1, R(p, 0)) \leq \frac{\lambda}{\lambda + \gamma} [L \frac{\Phi(\bar{p} - \mu)}{2} + (1 - \Phi)L \frac{(\bar{p} - \mu)}{2}] \quad (27)$$

Since  $R(p, 0) < \bar{p}$ ,  $W(\bar{p}, 1, R(p, 0)) = W(p, 1, R(p, 0))$  and the two previous inequalities cannot hold simultaneously. This proves that  $R(p, 0) \in [\bar{p}, p]$  for  $p > \bar{p}$ .

We prove now that  $\bar{p} \geq p(2)$ . Suppose that this is not the case and that there exists an equilibrium with  $\bar{p} = p(1)$ . Consider a dealer in state  $\{p(2), 0\}$ . If he quotes  $p(2)$ , he obtains at least (assuming that the other dealer undercuts, choosing  $p(1)$  until the end of the game, which is the worst possibility):

$$\frac{\lambda}{\lambda + \gamma} [\Phi \frac{p(2) - \mu}{2} + (1 - \Phi)^2 \frac{p(1) - \mu}{2}] L \quad (28)$$

If he undercuts he gets:

$$\frac{\lambda}{\lambda + \gamma} [\Phi p(1) - \mu + (1 - \Phi) \frac{p(1) - \mu}{2}] L \quad (29)$$

It is straightforward that Equation (28) is always greater than Equation (29). Consequently,  $R(p(2), 0) = p(2)$ . But then this implies that at any price above  $p(2)$ , a dealer is better off quoting  $p(2)$  instead of  $p(1)$ . Consequently  $p(1)$

cannot be on the equilibrium path. *Q.E.D.*

## Proof of Proposition 4.

We can proceed as in the proof of Proposition 2 to show that the reaction function of Proposition 4 is an equilibrium. The computations are routine.

## Proof of Proposition 5.

This proof is long and is skipped for the sake of brevity. It can be obtained upon request.

## Proof of Proposition 6.

### a) With time priority.

We remark first that  $\tau_{TPR}^*$  is decreasing in  $g(n)$ , i.e. increasing in  $n$  and that it is positive as soon as  $n$  is sufficiently large for  $R_B - p(i^* + 1)$  to be positive. Let  $n_{TPR}^*$  be the greatest value of  $n$  such that  $\tau_{TPR}^*$  is negative (i.e.  $p(i^* + 1) > R_B$ ). Using the definition of  $\tau_{TPR}^*$ , one obtains:

$$n_{TPR}^* = \lfloor \lfloor \frac{(1 + 2h(\Phi))g(0) - 1}{4(R_B - \mu)} \rfloor \rfloor \quad (30)$$

with  $h(\Phi) = \lfloor \lfloor \frac{\Phi + 1}{2\Phi} \rfloor \rfloor$ .

First we consider the difference in expected trading cost for a grid of size  $g(n + 1)$  and for a grid of size  $g(n)$  for  $n \geq n_{TPR}^* + 1$ . Denote  $E(\Delta TC(n + 1))$  this difference. Using Equation (12), Corollary 1 and the fact that  $\tau^*$  is decreasing with the tick size, some algebra yields:

$$E(\Delta TC(n + 1)) \geq \frac{g(n) - g(n + 1)}{\Phi} K(\Phi, \tau^*(n)) \quad (31)$$

with  $K(\Phi, \tau^*(n)) = (1 - \Phi)(1 - \Phi)^{(\tau^*(n)+1)} - (\tau^*(n) + 1)(1 - \Phi)^{(\tau^*(n)+1)} + \tau^*(n)(1 - \Phi)^{(\tau^*(n)+2)}$  and  $\tau^*(n) = \tau_{TPR}^*(n)$ . As  $\tau^*(n) \geq 0$  for  $n \geq n_{TPR}^*$ , it is straightforward that the R.H.S of (31) is positive. This implies that the expected trading costs increase as the tick decreases for  $n \geq n_{TPR}^* + 1$ . Now suppose that  $g(n)$  is large enough ( $n \leq n_{TPR}^*$ ) so that  $\tau_{TPR}^*$  is negative, the quotes adjust immediately to  $p(1)$ . The expected trading cost in this case is:

$$E(\tilde{TC}(n)) = \frac{g(n)}{2} \quad (32)$$

In this case the expected trading cost decreases with  $g(n)$ . This shows that the optimal value for  $n$  is either  $n_{TPR}^*$  or  $n_{TPR}^* + 1$ . Now if  $g(n_{TPR}^* + 1)$  is chosen, the expected trading cost is at least:

$$E(\tilde{TC}(n)) = \Phi(R_B - \mu) + (1 - \Phi) \frac{g(n_{TPR}^* + 1)}{2} \quad (33)$$

This is to be compared with  $g(n_{TPR}^*)/2$ . Equation (33) is greater than  $g(n_{TPR}^*)/2$  iff:

$$\Phi(R_B - \mu - \frac{g(n_{TPR}^* + 1)}{2}) \geq \frac{1}{2}(g(n_{TPR}^*) - g(n_{TPR}^* + 1)) \quad (34)$$

By definition of  $n_{TPR}^*$ ,  $R_B - \mu - \frac{g(n_{TPR}^* + 1)}{2} \geq h(\Phi)g(n_{TPR}^* + 1)$  while the R.H.S of (34) is equal to

$$\frac{g(0)}{(2n_{TPR}^* + 1)(2n_{TPR}^* + 3)} \quad (35)$$

Therefore a sufficient condition for the last inequality to be satisfied is:

$$\Phi h(\Phi) \geq \frac{1}{2n_{TPR}^* + 1} \quad \text{for } n_{TPR}^* \geq 1 \quad (36)$$

Using the definition of  $h(\Phi)$ , this is always true. For all strictly positive values of  $\Delta$ ,  $h(\Phi)$  is finite and consequently the optimal tick size is strictly positive.

**b) Without time priority.**

The proof follows the same lines. Now  $n_{RAR}^*$  is given by

$$n_{RAR}^* = \lfloor \lfloor \frac{7}{4(R_B - \mu)} g(0) - \frac{1}{2} \rfloor \rfloor \quad (37)$$

The main difference is that this is independent of  $\Phi$  and that we cannot rule out as before that  $n_{RAR}^* + 1$  is a solution. In this case the expected trading cost is:

$$\Phi(R_B - \mu) + (1 - \Phi) \frac{3}{2} g(n_{RAR}^* + 1) \quad (38)$$

This is decreasing in  $\Phi$ . If  $n_{RAR}^*$  is chosen then the expected trading cost is  $p(2) - \mu = \frac{3g}{2}$ . Therefore  $n_{RAR}^*$  is the solution if  $\Phi$  is greater than or equal to  $\bar{\Phi}$  such that:

$$\bar{\Phi}(R_B - \mu) + (1 - \bar{\Phi}) \frac{3}{2} g(n_{RAR}^* + 1) = \frac{3}{2} g(n_{RAR}^*) \quad (39)$$

Otherwise  $n_{RAR}^* + 1$  is the solution. It is direct that for all values of  $\Delta$  strictly positive, the optimal tick is strictly positive here again. *Q.E.D.*

**Proof of Proposition 7.**

When the adjustment is faster with time priority than with the random allocation rule, i.e.  $p(i^* + 1) > p(4)$  (which implies  $\Phi < \frac{1}{5}$ ) or  $p(i^* + 1) > R_B$ , expected trading costs are obviously lower with time priority. When  $p(i^* + 1) < p(4)$ , the ranking of the two allocation rules is less straightforward. We detail the computations only in the case  $p(i^* + 1) < p(4) \leq R_B$ . They are similar for the other possible positions of  $R_B$ . Consider first the case in which  $p(i^* + 1) = p(3)$ , i.e.  $\Phi \in [1/5, 1/3)$ . The trading costs are the same if the trade occurs at a time such that the price is greater than or equal to  $p(4)$ . Consequently, the difference between the expected trading cost with the random allocation rule and the expected trading cost with time priority will be:

$$(1 - \Phi)^{(r_{\lambda n} + 1)} \{ (p(2) - \mu) - \Phi(p(3) - \mu) - (1 - \Phi)(p(1) - \mu) \}. \quad (40)$$

It is direct to check that this is always positive for  $\Phi < \frac{1}{3}$ . Consider now the case in which  $p(i^* + 1) = p(2)$ , i.e.  $\Phi \in [\frac{1}{3}, 1]$ . Then the difference between the expected trading costs with RAR and TPR respectively is:

$$(1 - \Phi)^{(r_{\lambda n} + 1)} \{ (p(2) - \mu) - \Phi(p(3) - \mu) - \Phi(1 - \Phi)(p(2) - \mu) - (1 - \Phi)^2(p(1) - \mu) \}. \quad (41)$$

This is positive iff:

$$3 \geq 5\Phi + 3(1 - \Phi)\Phi + (1 - \Phi)^2 \quad (42)$$

which is never satisfied for  $\Phi \geq \frac{3 - \sqrt{5}}{2} > \frac{1}{3}$ . *Q.E.D.*