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JEL Classification: G2, D82, D83

Keywords: stress test, Information, Bank Regulation, Filtering, learning

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Designing Stress Scenarios*

Cecilia Parlatore† Thomas Philippon‡

March 22, 2022

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1 Introduction

Stress tests are ubiquitous in risk management and financial supervision. Risk officers use stress tests to set and monitor risk limits within their organizations, and financial regulators around the world use stress tests to assess the health of financial institutions. For example, financial firms use stress tests to complement their statistical risk management tools (e.g., Value at Risk); asset managers stress test their portfolios; trading venues stress tests their counter-party exposures; regulators mandate large scale stress tests for banks and insurance companies and use the results to enforce capital requirements and validate dividend policies.[1]

Despite the growing importance of stress testing, and the amount of resources devoted to them, there is little theoretical guidance on exactly how one should design stress scenarios. A theoretical literature has focused on the trade-offs involved in the *disclosure* of supervisory information (see Goldstein and Sapra, 2014 for a review), which range from concerns about the reputation of the regulator (Shapiro and Skeie, 2015) to the importance of having a fiscal backstop (Faria-e-Castro et al., 2017). These papers provide insights about disclosure and regulatory actions but they are silent about the design of forward-looking hypothetical scenarios. In that sense, existing models are models of asset quality reviews (and their disclosure) more than models of *stress testing*.

The goal of our paper is to start filling this void. Stress tests are used for risk management. Risk management is a two-tier process involving risk discovery (learning) and risk mitigation (intervention). Stress tests belong to the risk discovery phase but one cannot analyze the design of a test without understanding the remedial actions that can be taken once the results are known. We therefore model both the risk discovery stage and the risk mitigation stage.

We consider a principal and a potentially large number of agents. The agents can be traders within a financial firm, or they can be financial firms within a financial system. The principal can be a regulator designing supervisory tests, or a risk officer running an internal stress test. For concreteness we will use the supervisory stress testing analogy in much of the paper. Banks are exposed to a set of risk factors, but their exposures to these factors are unknown. By exposure we mean the relevant elasticity that determines the loss of a position under a given scenario. An exposure is therefore not the same as the book or market value of a position. Banks and

[1] Central banks in the United States, Europe, England, Brazil, Chile, Singapore, China, Australia, and New Zealand, as well as the International Monetary Fund in Japan, have recently used stress tests to evaluate the banking sector's solvency and guide banking regulation.

regulators usually agree on the nominal size of positions and on the market value, at least for liquid portfolios. They can disagree about the value of illiquid positions, and in all cases, liquid or not, the impact of a scenario on the loss on that position needs to be estimated. What we call "exposure" combines the position (measured with near certainty in some cases) with its value under stress scenarios (estimated with error).

The regulator is risk averse and worries about the financial system experiencing large losses in some states of the world. The regulator then designs a set of hypothetical scenarios and asks the banks to report their losses under these scenarios. The regulator uses reported losses across all banks and scenarios to extract information about underlying exposures. Based on this information, the regulator decides how to intervene, i.e., she can ask a set of banks to reduce their exposures to some factors. Interventions are costly, either directly – by drawing on limited regulatory resources, creating disruptions – or indirectly – by preventing banks from engaging in valuable activities.

Our main insight comes from writing the learning problem as a Kalman filter. The filter gives us a mapping from prior beliefs and test results into posterior beliefs. The precision of the mapping depends on the scenarios in the stress test. We can then formulate the regulator's problem as an information acquisition problem in which the regulator chooses the precision of her signals about risk exposures. Formally, we map the primitive parameters of the model, such as the priors of the regulator regarding the banks' exposures, to the feasible set of posteriors beliefs. If, for instance, the regulator is worried about a particular risk factor, we can derive the stress test that maximizes learning about exposures to that factor.

Will the regulator focus a particular risk factor? Or will she try and learn about several factors at the same time? We show how the answers depend on her prior beliefs about the banks' risk exposures and on on the information-sensitivity of her interventions. The regulator can always mandate a broad risk reduction, such as an increase in overall capital requirements, which does not require much information but is likely to involve unnecessary changes and disruptions. With more accurate information the regulator can better target her interventions and reduce the associated costs. The regulator therefore values information insofar as it enables accurate and parsimonious interventions.

Our model sheds light on broad versus specialized learning. The regulator can increase her learning about exposures to a risk factor by choosing a more extreme scenario for that factor, but extreme scenarios lead to noisy answers. Whether or not the optimal scenario implies specialization in learning depends on the sensitivity of targeted interventions to stress test information, and on the trade off between noise and information quality along different different dimensions of risk. The reduction in overall information quality depends on the prior distribution of the risk exposures through the Kalman filter.

More generally, the *costs of intervention* and the *prior beliefs* of the regulator are central in determining the optimal scenario design. The effect of intervention costs on the optimal scenario is not monotone. On the one hand, a higher intervention cost makes accurate interventions more important, which pushes the regulator to acquire more information relevant to that intervention. On the other hand, an intervention that is very costly is rarely used and there is no point to learn about its associated risk factors.

The regulator's priors about average exposures – holding constant her uncertainty – also have two effects on the optimal stress scenario. A higher expected exposure increases the likelihood of intervention, which makes accurate information more valuable. This effect pushes the regulator to learn about factors with high expected risk exposures. On the other hand, when the regulator's prior mean is high, her belief about true exposure is less sensitive to new information, which discourages learning along that dimension. This second effect dominates when the expected risk exposure is high. Hence, the weight of a factor in the stress scenario is hump-shaped with respect to the regulator's prior. With uncorrelated factors, we find optimal scenarios with zero weight on factors with high expected risk exposures.

The regulator's prior uncertainty about risk exposures or risk factors also shapes the optimal stress scenario design. A higher expected exposure to a particular factor increases the likelihood of an intervention and therefore the value of information. The regulator thus wants to learn more about uncertain exposures. On the other hand high uncertainty about a risk factor makes the regulator's intervention policy less sensitive to new information. In this case, the regulator puts less weight in the stress scenario on risk factors about which she is more uncertain.

Correlated risk exposures, within or across banks, play an important role in our analysis. When exposures are correlated, learning about one provides information about the others. The regulator therefore stresses more the factors with correlated exposures. This is true for correlated exposures within a bank as well as correlated exposures across banks. Correlated factors are more systemic and our model predicts that they play an outsize role in scenario design. The regulator may focus mostly on these factors if the correlation is high enough, but, due to the convexity of information sets, specialization is usually incomplete and the design tends to put some weight on all factors. Our results on the impact of priors – means, volatilities, correlations of factors and exposures – provide insights about the design of stress tests during crises and in normal times.

Our model allows for two types of interventions: minimum capital requirements and targeted risk reductions that can be interpreted as limits (e.g. LTV ratios) linked to specific asset classes. It is important to emphasize, however, that we model stress testing as a optimal learning mechanism. Actual stress tests used for capital adequacy in banking often impose a mechanical link between tests results and capital requirements. A particular scenario – usually called the adverse scenario – is used to deliver "pass/fail" grades. To pass the test the banks must show that their capital ratio does not fall below a pre-specified level under the adverse scenario. This mechanical link conflates two conceptually separate issues – learning and intervention – and is not appropriate for a theoretical model.[2] In our baseline analysis we therefore do not assume such a mechanical mapping. We assume instead that regulators choose optimal actions conditional on the results of the test. This gives them complete freedom to design the most informative scenarios.

The optimal design approach in our paper allows us to shed new light on actual stress tests. First, as a matter of implementation, we can always recast our model in terms of pass/fail outcomes based on pre-specified rules since optimal actions are predictable *functions* of stress test results. Second, and more importantly, we can quantify the welfare losses from using a constrained approach where a plausible adverse scenario must be used to set capital requirements. We find that the welfare losses are relatively small when the regulator retains one free scenario for optimal learning.

Literature Review

Most of the literature on stress tests focuses on banking. Several recent papers study specifically the trade-offs involved in disclosing stress test results. Goldstein and Leitner (2018) focus on the Hirshleifer (1971) effect: revealing too much information destroys risk-sharing opportunities between risk neutral investors and (effectively) risk averse bankers. These risk-sharing arrangements also play an important role in Allen and Gale (2000). Shapiro and Skeie (2015) study the reputation concerns of a regulator when there is a trade-off between moral hazard and runs. Faria-e-Castro et al. (2017) study a model of optimal disclosure where the

[2]For instance, imagine that a bank needs the same level of ex-ante equity to satisfy a 9% capital requirement after scenario 1 or a 7% requirement after scenario 2 (presumably because scenario 2 embodies a higher degree of stress). As far as ex-ante capital adequacy is concerned, these two regulations are equivalent.

government trades off Lemon market costs with bank run costs, and show that a fiscal backstop allows government to run more informative stress tests. Schuermann (2014) analyzes the design and governance (scenario design, models and projection, and disclosure) for more effective stress test exercises. Schuermann (2016) particularly determines how stress testing in crisis times can be adapted to normal times in order to insure adequate lending capacity and other key financial services. Orlov et al. (2017) look at the optimal disclosure policy when it is jointly determined with capital requirements, while Gick and Pausch (2014), Inostroza and Pavan (2017), and Williams (2017) do so in the context of Bayesian persuasion. Our model's predictions are consistent with the results in Orlov et al. (2017) that the optimal sequential capital requirements involve a precautionary recapitalization of banks followed by a recapitalization contingent on stress test results. Huang (2021) studies the optimal disclosure in banking networks with potential spillovers and contagion among banks. As argued by Goldstein and Leitner (2020), stress test design and disclosure policy are connected. We complement this strand of papers by explicitly modeling the stress scenario design, which allows us to study the kind of information in the optimal stress test–the relative weight of each factor in the optimal scenario–and not only on how much information it contains.

While most of the existing literature on stress testing, theoretical and empirical, analyzes the disclosure of stress test results, some papers have focused on the risk modeling part of stress testing. For example, Leitner and Williams (2018) focus on the disclosure of the regulator's risk modeling. They examine the trade-offs involved in disclosing the model the regulator uses to perform the stress test to banks. Relatedly, Cambou and Filipovic (2017) focus on how scenarios translate into losses when the regulator and the banks face model uncertainty. However, none of these papers consider the optimal scenario design, which is the focus of our paper.

Most empirical papers on stress tests focus on the information content at the time of disclosure, using an event study methodology to determine whether stress tests provide valuable information to investors. Petrella and Resti (2013) assess the impact of the 2011 European stress test exercise. For the 51 banks with publicly traded equity, they find that the publication of the detailed results provided valuable information to market participants. Similarly, Donald et al. (2014) evaluate the 2009 U.S. stress test conducted on 19 bank holding companies and find significant abnormal stock returns for banks with capital shortfalls. Candelon and Sy (2015), Bird et al. (2015), and Fernandes et al. (2015) also find significant average cumulative abnormal returns for stress tested BHCs around many of the stress test disclosure dates. Flannery et al. (2017) find that U.S.

stress tests contain significant new information about assessed BHCs. Using a sample of large banks with publicly traded equity, the authors find significant average abnormal returns around many of the stress test disclosures dates. They also find that stress tests provide relatively more information about riskier and more highly leveraged bank holding companies. Glasserman and Tangirala (2016) evaluate one aspect of the relevance of scenario choices. They show that the results of U.S. stress tests are somewhat predictable, in the sense that rankings according to projected stress losses in 2013 and 2014 are correlated. Similarly, the rankings across scenarios in a given year are also correlated. They argue that regulators should experiment with more diverse scenarios, so that it is not always the same banks that project the higher losses. Acharya et al. (2014) compare the capital shortfalls from stress tests with the capital shortfalls predicted using the systemic risk model of Acharya et al. (2016) based on equity market data. Camara et al. (2016) study the quality of the 2014 EBA stress tests using the actual micro data from the tests.

Finally, our paper is related to the large theoretical literature on information acquisition following Verrecchia (1982), Kyle (1989), and especially Van Nieuwerburgh and Veldkamp (2010). In this class of models, the cost of acquiring information pins down the set of feasible precisions and determines whether the signals are complement or substitutes. Vives (2008) and Veldkamp (2009) provide a comprehensive review of this literature. These papers take the information processing constraint on the signal precisions as given. In contrast, our paper focuses on the design of the signals that the regulator receives and endogenizes the information processing constraint.

The rest of the paper is organized as follows. Section 2 describes the environment. Section 3 describes how the regulator learns from stress test. Sections 4 and 5 characterize the optimal intervention policy and the optimal stress scenarios, respectively. Section 7 discusses the practical implications of our analysis and concludes.

2 Technology and Preferences

We consider the problem of a principal who wants to manage the risk exposures of a set of agents. The model has several natural interpretations. The principal could be a chief risk officer and the agents could be traders in her firm. The remedial actions could be hedging or downsizing the traders' positions. Alternatively, the principal could be a regulator and the agents could be a set of banks. The remedial actions could be hedging, reducing new deal flows, selling non-performing assets, or raising capital.

To be concrete we use the regulator/banks metaphor when describing the model. The regulator elicits information from the banks in the form of stress tests. In our model, a stress test is a technology used by regulators to ask questions about profits and losses under hypothetical scenarios. The banks cannot evade the questions and have to answer to the best of their abilities. Banks in our model can only lie by omission: they do not have to volunteer information, but they have to provide estimates of their losses under various scenarios.

2.1 Banks and Risks

There is one regulator overseeing N banks indexed by $i \in [1, .., N]$ exposed to systematic and idiosyncratic risks. The macro-economy is described by a vector of J systematic factors. We denote by s_j the value of factor j. The macroeconomic state of the economy is

$$\mathrm{s} = \begin{bmatrix} s_1 \\ \vdots \\ s_J \end{bmatrix}.$$

The risks of bank i are captured by a vector of J exposures

$$\mathrm{x}_i = \begin{bmatrix} x_{i,1} \\ \vdots \\ x_{i,J} \end{bmatrix},$$

where $x_{i,j}$ represents the exposure of bank i to factor j. We use the term "exposure" to denote the relevant elasticity that determines losses under a given realization of the macroeconomic state. An exposure is therefore not the same as the nominal value of a position. In many cases (e.g., a commercial loan) the size of the position is unambiguous but the impact of a realization of the macro state on the loss on that position needs to be estimated. What we call "exposure" combines the position (measured with near certainty) with its value in a particular macro state (computed with error).

The losses of bank i in state s are given by

$$y_i\left(\mathrm{s}\right) = \mathrm{s} \cdot \mathrm{x_i} + \eta_i = \sum_{j=1}^{J} x_{i,j} s_j + \eta_i, \tag{1}$$

where η_i is a random idiosyncratic (i.e., bank specific) shock. Our model has only one period so $y_i\left(\mathrm{s}\right)$ should be interpreted as the cumulative losses in state s. We will assume that the exposures are normally distributed in order to apply the Kalman filter. Technically, therefore, it can happen that $x < 0$ but, as usual, we choose parameters to ensure that this is a negligible possibility.

The net worth of bank i is then given by

$$w_i\left(\mathrm{s}\right) = \bar{w}_i - y_i\left(\mathrm{s}\right), \tag{2}$$

where $\bar{w}_i$ is the mean level of net worth. Given Equation (1) and Equation (2), the aggregate net worth of the banking system is

$$W\left(\mathrm{s}\right) \equiv \sum_{i=1}^{N} w_i = \bar{W} - \bar{\eta} - \mathrm{s}{\cdot}X, \tag{3}$$

where $\bar{W}$, X and $\bar{\eta}$ are the sum of the corresponding variables across the N banks in the economy, e.g., $\bar{\eta} \equiv \sum_{i=1}^{N} \eta_i$.

Interpretation Regulators specify stress scenarios in terms of traditional macroeconomic variables such as GDP, unemployment, and house prices. In DSGE models, on the other hand, these macro variables would themselves be functions of underlying structural shocks such as productivity, beliefs, risk aversion, etc.[3] Formally, let ϵ^s be the structural shocks and H the solution matrix of the DSGE model, so that $\mathrm{s} = H\epsilon^s$. In a fully specified model, banks' losses would also be functions of the structural shocks: $y_i\left(\epsilon^{\mathrm{s}}\right) = \tilde{\mathrm{x}}_{\mathrm{i}}'\epsilon^{\mathrm{s}} + \eta_i$, where $\tilde{\mathrm{x}}_i$ are structural exposures. This equation is equivalent to (1) when H is invertible. In that case we can write $\epsilon^s = H^{-1}\mathrm{s}$ and define $\mathrm{x}_i = H'^{-1}\tilde{\mathrm{x}}_i$, and we obtain $y_i\left(\mathrm{s}\right) = \mathrm{x}_i'\mathrm{s} + \eta_i$.

In theory the regulator could supply the structural shocks ϵ^s and ask for estimated losses. In practice regulators supply directly the macro variables s. This reflects the fundamental issue of model ambiguity. Even if H is invertible, models for H would likely differ across banks as well as between banks and regulators. By contrast, a handful of macro-economic variables (GDP, credit

[3] The typical DSGE model contains Euler equations, production functions and resource constraints that lead to a set of equations such as

$$\mathbf{A}\mathbf{s}_t = \mathbf{C} + \mathbf{B}\mathbf{s}_{t-1} + \mathbf{D}\mathbb{E}_t\mathbf{s}_{t+1} + \mathbf{F}\epsilon_t^s.$$

When we solve the model we invert the mapping to obtain a VAR representation

$$\mathbf{s}_t = \mathbf{J} + \mathbf{Q}\mathbf{s}_{t-1} + \mathbf{H}\epsilon_t^s.$$

In our simple framework we have $\mathbf{J} = \mathbf{Q} = 0$ since we normalize the baseline scenario to 0 and we have only one period.

spreads, house and stock prices, etc.) are well-understood by all participants and capture much of the macro-economic dynamics that matter for expected losses. This is why stress tests are written in terms of s and not ϵ^s. In most of our applications we will assume that H is invertible and that the regulator feels confident about estimating H^{-1}. In that case there is no real difference between estimating x_i or $\tilde{\mathrm{x}}_i$ and we can assume that the factors are independently distributed.

2.2 Regulator's Preferences and Interventions

Following Acharya et al. (2016) we assume that the regulator has preferences $U(W)$ over the total net worth of the banking system W. Philippon and Wang (2021) show that this specification arises generically when there is an effective way to relocate assets and liabilities across banks, e.g. when healthy banks can take over failed ones.[4] If the regulator believes that the risks in the system are too high, she can intervene to force the banks to increase their capital or lower their exposures. We denote by $\mathcal{K}\left(\bar{W}\right)$ the cost of requiring banks to increase their capital. Targeted interventions include capital and collateral requirements against specific types of loans or specific borrowers (e.g., LTV ratios in commercial real estate), as well as assets sales and divestitures. The most granular description of interventions is at the bank×factor level. In some cases, however, a targeted intervention would affect exposures to several factors. We will discuss in details how we model these constraints in Section 4. For now we denote the action as a (large) vector $\mathbf{a}$ in some feasible set $\mathcal{A}$ with the understanding that higher actions reduces exposure more: the vector x_i becomes $(\mathbf{1}_{NJ\times 1} - \mathbf{a}_i)\,\mathrm{x}_i$ where $\mathbf{a}_i$ are the set of actions taken on bank i. Interventions are costly. There are direct costs born by the regulators and the banks, as well as indirect costs from the disruption of valuable activities. We let $\mathcal{C}(\mathbf{a})$ denote the cost of action $\mathbf{a}$.

Let $\mathscr{S}$ denote the information set of the regulator at the time when she chooses her intervention policy. The regulator's problem is then to choose an intervention policy $\left(\bar{W}, \mathbf{a}\right)$ to maximize her expected utility given by

$$\mathbb{E}\left[U\left(\bar{W} - \bar{\eta} - \mathrm{s}\cdot\left(\sum_{i=1}^{N}(\mathbf{1}_{NJ\times 1} - \mathbf{a}_i)\,\mathrm{x}_i,\right)\right)\middle|\,\mathscr{S}\right] - \mathcal{C}(\mathbf{a}) - \mathcal{K}\left(\bar{W}\right).$$

[4] The general case is $U\left([w_i]_{1..N}\right)$, where the idiosyncratic failure of bank i matters regardless of the health of the banking sector as a whole. As in the systemic risk literature, we assume here that only $W = \sum_{i=1}^{N} w_i$ matters. As a result, a financial crisis only happens when the financial system as a whole is under-capitalized. See Philippon and Wang (2021) for a proof of how transfers of assets from under- to well-capitalized banks transform the value function $U\left([w_i]_{1..N}\right)$ into $U(W)$.

An important point of our analysis is the targeted interventions require more information than non-targeted ones. We think of stress tests as a way of eliciting information to determine the best interventions across banks and activities.

2.3 Prior beliefs and stress tests

The banks' risk exposures to the macro factors are unknown to the regulator and the banks. The regulator has prior beliefs over the distribution of exposures within banks and across banks. These prior beliefs come from historical experiences and the regulator's own risk models. We stack the banks' exposures in one large $NJ \times 1$ vector as follows

$$\mathbf{x} \equiv \begin{bmatrix} \mathrm{x}_1 \\ \vdots \\ \mathrm{x}_N \end{bmatrix},$$

and we summarize the regulator's prior over the vector of exposures $\mathbf{x}$ as

$$\mathbf{x} \sim N\left(\overline{\mathbf{x}}, \Sigma_{\boldsymbol{x}}\right),$$

where the $NJ \times 1$ vector of unconditional means and the $NJ \times NJ$ covariance matrix are, respectively,

$$\overline{\mathbf{x}} = \begin{pmatrix} \overline{\mathrm{x}}_1 \\ \vdots \\ \overline{\mathrm{x}}_N \end{pmatrix} \quad \text{and } \Sigma_{\boldsymbol{x}} = \begin{bmatrix} \Sigma_{\boldsymbol{x}}^1 & \Sigma_{\boldsymbol{x}}^{1,2} & \cdots & \Sigma_{\boldsymbol{x}}^{1,N} \\ \Sigma_{\boldsymbol{x}}^{1,2} & \Sigma_{\boldsymbol{x}}^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \Sigma_{\boldsymbol{x}}^{(N-1),N} \\ \Sigma_{\boldsymbol{x}}^{1,N} & \cdots & \Sigma_{\boldsymbol{x}}^{(N-1),N} & \Sigma_{\boldsymbol{x}}^N \end{bmatrix}$$

with $\Sigma_{\boldsymbol{x}}^i = \mathbb{V}\mathrm{ar}\left(\mathrm{x}_i\right)$ is the $J \times J$ covariance of exposures of bank i, and $\Sigma_{\boldsymbol{x}}^{i,h} = \mathbb{C}\mathrm{ov}\left(\mathrm{x}_i, \mathrm{x}_h\right)$ for all $i \neq h$ is the covariance of exposures across banks. If $\Sigma_{\boldsymbol{x}}^i$ is diagonal the regulator expects the exposures of bank i to the different factors to be independent of each other. If $\Sigma_{\boldsymbol{x}}^{i,h} = 0$, the regulator's prior is that the risk exposures of banks i and h are independent. In almost all empirically relevant cases the covariance matrices are not diagonal.

To learn about the banks' risk exposures, the regulator asks the banks to estimate and report their losses under a particular realization of the macroeconomic state. This choice of macroeconomic state is a *scenario* $\hat{s}$.

Definition 1. **(Scenario)** A scenario $\hat{\mathrm{s}} = (\hat{s}_1, \ldots, \hat{s}_J)'$ is a realization of the vector of states s.

A scenario $\hat{s}$ is a row-vector of size J that represents an aggregate state of the economy. We entertain two interpretations of the size of the state space, J. The simplest way is to think of J as exogenously given. There might be a limited number of macroeconomic variables (GDP, unemployment, house prices) that everyone agrees need to be included in the test. The other way to think about J is as a large number capturing the set of all possible risk factors and in any given tests many have zero loadings. A non-zero weight is then a statement about whether that risk factor is included in the particular stress test. Our model can then shed light on which risk factors should be used.

Given our normalization of the baseline state to $s = 0$, a scenario close to 0 is a scenario close to the baseline of the economy. A scenario $\hat{s}$ in which element $\hat{s}_j$ is large, represents a large deviation from the baseline along the dimension of factor j. The larger $|\hat{s}_j|$, the more extreme the scenario along dimension j. When designing a stress test, the regulator specifies a *set* of scenarios for which the banks need to report their losses.

Definition 2. (Stress test) A stress test is a collection of M scenarios $\{\hat{\mathrm{s}}^m\}_{m=1}^{M}$ presented by the regulator, and a collection of estimated losses $\{\hat{y}_i^m\}_{i=1..N}^{m=1..M}$ reported by the banks.

For each scenario m, each bank i estimates and reports its net losses $\hat{y}_i^m$ given the input parameters in scenario $\hat{\mathrm{s}}^m$.

2.4 Stress test results

Banks use imperfect models to predict their losses under the stress test scenarios. Bank i estimates its losses under scenario $\hat{\mathrm{s}}^m$ as

$$\hat{y}_i\left(\hat{\mathrm{s}}^m, M\right) = \hat{\mathrm{s}}^m \cdot \mathrm{x}_i + \hat{\epsilon}_{i,m}\left(\|\hat{\mathrm{s}}^m\|, M\right), \tag{4}$$

where the error term $\hat{\epsilon}_i\left(\|\hat{\mathrm{s}}\|, M\right)$ captures measurement error and model uncertainty and is increasing in the norm of the scenario $\|\hat{\mathrm{s}}\|$ and in the number of scenarios M. The results of the stress test for one bank are summarized in the $M \times 1$ vector

$$\hat{\mathrm{y}}_i\left(\hat{S}\right) = \hat{S}\,\mathrm{x}_i + \hat{\varepsilon}_i,$$

where $\hat{\mathrm{y}}_i\left(\hat{S}\right)$ represents the results of the test for bank i, the $M \times J$ matrix $\hat{S}$ gathers the scenarios in the stress test, and the errors in bank i's reported losses are gathered in the $M \times 1$ vector $\hat{\varepsilon}_i$,

Scenario Design	**Stress Testing**	**Intervention**
Regulator chooses stress scenarios	Banks report stress test results	Regulator chooses interventions to reduce banks' risk exposures
$\hat{S}$	$\hat{\mathbf{y}}\left(\hat{S}\right)$	$\bar{W}^{\star}, \mathbf{a}^{\star}\left(\hat{\mathbf{y}}\right)$

Figure 1: Timeline

i.e.,

$$\hat{\mathrm{y}}_i\left(\hat{S}\right)=\begin{bmatrix}\hat{y}_i\left(\hat{\mathrm{s}}^1,M\right)\\ \vdots\\ \hat{y}_i\left(\hat{\mathrm{s}}^M,M\right)\end{bmatrix},\quad \hat{S}\equiv\begin{bmatrix}\left(\hat{\mathrm{s}}^1\right)'\\ \vdots\\ \left(\hat{\mathrm{s}}^M\right)'\end{bmatrix},\quad \text{and}\quad \hat{\varepsilon}_i=\begin{bmatrix}\hat{\epsilon}_{i,1}\left(\left\|\hat{\mathrm{s}}^1\right\|,M\right)\\ \vdots\\ \hat{\epsilon}_{i,M}\left(\left\|\hat{\mathrm{s}}^M\right\|,M\right)\end{bmatrix}.$$

Banks build their internal risk models using historical data therefore the mistakes are likely to be correlated *across scenarios.* The variance-covariance matrix of the errors made by bank i in computing its stress test results is given by

$$\mathbf{\Sigma}^i_{\hat{\varepsilon}}\equiv\mathbb{V}ar\left[\hat{\varepsilon}_i\right].$$

Differences in $\mathbf{\Sigma}^i_{\hat{\epsilon}}$ across banks reflect differences in information (priors), in the amount or quality of data available to each bank, or in the bank's information processing capacity. We assume that x_i and $\hat{\varepsilon}_i$ are independent, but we allow banks to make correlated mistakes. The error term in the estimated losses captures various kinds of measurement error and model uncertainty. To guarantee an interior solution, we assume that $\mathbb{V}ar\left(\hat{\epsilon}_i\left(\|\hat{\mathrm{s}}\|,M\right)\right)$ is continuous at $\|\hat{\mathrm{s}}\|=0$ and $\lim_{\hat{s}_j\to\infty}\frac{\left(\hat{s}_j\right)^2}{\mathbb{V}ar\left(\hat{\epsilon}_i\left(\|\hat{\mathrm{s}}\|,M\right)\right)}=0$ for all m and for all j.

2.5 Timing

To summarize, there are three stages in our model: the scenario design stage, the stress testing stage, and the intervention stage. First, the regulator chooses stress scenarios taking into account that the scenario choices will affect the information in the stress test results submitted by the banks. Then, the regulator elicits information from the banks in the form on stress tests. The banks' stress test results consist of projected losses for each bank under each scenario chosen by the regulator. Finally, the regulator chooses her targeted interventions and capital requirements after observing the stress test results. Figure (1) shows the timeline of the model.

3 Learning

The information set of the regulator depends on the regulator's prior beliefs and on the information she acquires from the stress tests. Neither the banks nor the regulator know the true exposures, but banks have imperfect (noisy) models to project their losses in a given state. We model stress tests as a mechanism for the regulator to elicit this information, which the regulator can then use to design its optimal intervention policy.

3.1 A Kalman Filter

A key insight of our paper is that stress test results can be interpreted as signals about the banks' risk exposures by defining the error terms and the signals appropriately. Let us briefly summarize all the sources of risks and information in our model:

1. For each bank i, $\hat{y}_i\left(\hat{S}\right)$ is an $M \times 1$ vector that summarizes the estimated losses in the various scenarios. This is the key source of information for the regulator.

2. For each bank i, the $M \times M$ matrix $\boldsymbol{\Sigma}_{\hat{\varepsilon}}^i \equiv \mathbb{V}ar\left[\hat{\varepsilon}_i\right]$ contains the size and correlations of estimation errors across the M scenarios. Element m of the $M \times 1$ vector of errors $\hat{\boldsymbol{\varepsilon}}_i$ is given by $\hat{\epsilon}_{i,m}\left(\left\|\hat{\mathrm{s}}^m\right\|, M\right)$.

3. For all banks and risk factors, the $NJ \times NJ$ covariance matrix $\Sigma_{\boldsymbol{x}}$ contains the priors of the regulators regarding exposures within and across banks. The covariance matrix $\Sigma_{\boldsymbol{x}}$ is predetermined and unaffected by the scenarios.

For N banks, we stack the reported losses and error terms in the $NM \times 1$ vectors

$$\hat{\mathbf{y}} \equiv \begin{bmatrix} \left[\hat{y}_1\left(\hat{S}\right)\right] \\ \vdots \\ \left[\hat{y}_N\left(\hat{S}\right)\right] \end{bmatrix} \quad \text{and} \quad \hat{\boldsymbol{\varepsilon}} = \begin{bmatrix} \left[\hat{\varepsilon}_1\right] \\ \vdots \\ \left[\hat{\varepsilon}_N\right] \end{bmatrix}.$$

The state-space representation of the stress test is then

$$\hat{\mathbf{y}} = \hat{\mathbf{S}}\mathbf{x} + \hat{\boldsymbol{\varepsilon}}, \tag{5}$$

where $\hat{\mathbf{S}} \equiv \left(\mathbf{I}_N \otimes \hat{S}\right)$ simply repeats $\hat{S}$ on its diagonal, and $\hat{\boldsymbol{\varepsilon}} \sim N\left(0, \boldsymbol{\Sigma}_{\hat{\varepsilon}}\right)$. Remember that the regulator observes $\hat{\mathbf{y}}$ and wants to learn about $\mathbf{x}$. Expressing the stress test as in equation (5) allows us to apply the Kalman filter and to obtain a full characterization of the posterior beliefs of the regulator.

Proposition 1. ***(Posterior beliefs)*** *After observing the results* $\hat{\mathbf{y}}$ *of the stress test, the posterior beliefs of the regulator regarding the banks' risk exposures are*

$$\mathbf{x}|\,\hat{\mathbf{y}} \sim N\left(\hat{\mathbf{x}}, \hat{\Sigma}_{\mathbf{x}}\right) \ ,$$

where the posterior mean $\hat{\mathbf{x}}$*, the Kalman gain* K*, and the residual covariance matrix* $\hat{\Sigma}_{\mathbf{x}}$ *are given by*

$$\hat{\mathbf{x}} = \left(\mathbf{I}_{NJ} - K\hat{\mathbf{S}}\right)\bar{\mathbf{x}} + K\hat{\mathbf{y}}\,, \tag{6}$$

$$K = \Sigma_{\mathbf{x}}\hat{\mathbf{S}}'\left(\hat{\mathbf{S}}\Sigma_{\mathbf{x}}\hat{\mathbf{S}}' + \boldsymbol{\Sigma}_{\hat{\varepsilon}}\right)^{-1} \ , \tag{7}$$

$$\hat{\Sigma}_{\mathbf{x}} = \Sigma_{\mathbf{x}} - K\hat{\mathbf{S}}\Sigma_{\mathbf{x}}\,. \tag{8}$$

The proof of Proposition 1 is a direct application of the Kalman filter. The Kalman gain K is an $NJ \times MN$ matrix. A few special cases can give some intuition. With one bank ($N = 1$), then $K_{j,m}$ is a measure of the amount of information about the exposure to risk factor j contained in the results from scenario m. With one scenario ($J = 1$) and uncorrelated exposures among banks, $K_{j,m}$ also measures the reduction in uncertainty about bank j's exposure to the risk factor.

The posterior covariance matrix $\hat{\Sigma}_{\mathbf{x}}$ plays a critical role in our analysis. The true exposures are distributed around $\hat{\mathbf{x}}$ with covariance $\hat{\Sigma}_{\mathbf{x}}$. Thus, $\hat{\Sigma}_{\mathbf{x}}$ measures the residual uncertainty that persists after observing the results of the stress test. The goal of the stress test is to reduce this residual uncertainty as much as possible, along dimensions that depend on the objective function and on the priors of the regulator.

In the standard state-space representation in Equation (5), the stress test scenarios determine the structure of the signals observed by the regulator by controlling the weight of each exposure in the reported losses. The scenarios also determine the precision of the banks' reported losses in

Equation (4). Increasing $|s_j|$ in a scenario makes the results more informative about exposures to factor j, but extreme scenarios reduce the precision of the banks' estimates and the noise might spill over to the measurement of other exposures. On the other hand, the regulator can improve her learning by taking into account the fact that true exposures are correlated across positions and across banks.

When designing the scenarios, the regulator must anticipate how she will interpret and use the results of the test. The extent to which learning takes place is captured *ex ante* by the *distribution of the posterior mean*, given by

$$\hat{\mathbf{x}} \sim N\left(\overline{\mathbf{x}}, \Sigma_{\hat{\mathbf{x}}}\right), \tag{9}$$

where the *expected variance of the posterior mean*, $\Sigma_{\hat{\mathbf{x}}}$, is given by

$$\Sigma_{\hat{\mathbf{x}}} \equiv \Sigma_{\mathbf{x}} - \hat{\Sigma}_{\mathbf{x}} = K\hat{\mathbf{S}}\Sigma_{\mathbf{x}}.$$

The matrix $\Sigma_{\hat{\mathbf{x}}}$ represents the expected amount of learning from the stress test. If the stress test is pure noise, $K = 0$, the regulator learns nothing, $\Sigma_{\hat{\mathbf{x}}} = 0$, and $\hat{\Sigma}_{\mathbf{x}} = \Sigma_{\mathbf{x}}$. If the test is fully informative, then $\hat{\Sigma}_{\mathbf{x}} = 0$ and the regulator learns exactly all the exposures, i.e., $\Sigma_{\hat{\mathbf{x}}} = \Sigma_{\mathbf{x}}$. One goal of the regulator is to maximize $\Sigma_{\hat{\mathbf{x}}} = K\hat{\mathbf{S}}\Sigma_{\mathbf{x}} = K\left(\mathbf{I}_N \otimes \hat{S}\right)\Sigma_{\mathbf{x}}$ – or equivalently to minimize the residual uncertainty $\hat{\Sigma}_{\mathbf{x}}$ – so as to be able to design an accurate policy intervention. When choosing what to learn, the regulator takes into account that the Kalman gain K is itself a function of the scenarios, given by equation (7).

Remark 1. **(Learning and interventions)** A goal of the regulator is to maximize the amount of learning $\Sigma_{\hat{\mathbf{x}}} = K\left(\mathbf{I}_N \otimes \hat{S}\right)\Sigma_{\mathbf{x}}$ to intervene more accurately.

3.2 Example

Consider the case of one bank ($N = 1$), one scenario ($M = 1$), and two risk factors ($J = 2$). To simplify the notation, we omit the argument M and the bank-specific subscript i, and we denote $\sigma_1^2 \equiv \Sigma_{x,11}$, $\rho\sigma_1\sigma_2 \equiv \Sigma_{x,12}$ and $\sigma_\epsilon^2(\hat{\mathrm{s}}) \equiv \mathbb{V}\mathrm{ar}\left[\hat{\epsilon}\left(\|\hat{\mathrm{s}}\|, 1\right)\right]$. The stress test result under scenario $\hat{s}$ is

$$\hat{y} = \hat{s}_1 x_1 + \hat{s}_2 x_2 + \hat{\epsilon}.$$

The Kalman gain in this case is a 2×1 vector $K = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$ where

$$k_1 = \frac{\sigma_1^2\hat{s}_1 + \rho\sigma_1\sigma_2\hat{s}_2}{\sigma_1^2\hat{s}_1^2 + 2\rho\sigma_1\sigma_2\hat{s}_1\hat{s}_2 + \sigma_2^2\hat{s}_2^2 + \sigma_\epsilon^2} \quad \text{and} \quad k_2 = \frac{\sigma_2^2\hat{s}_2 + \rho\sigma_1\sigma_2\hat{s}_1}{\sigma_1^2\hat{s}_1^2 + 2\rho\sigma_1\sigma_2\hat{s}_1\hat{s}_2 + \sigma_2^2\hat{s}_2^2 + \sigma_\epsilon^2}.$$

The posterior mean is then

$$\hat{\mathrm{x}} = \begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix} = \overline{\mathrm{x}} + K\left(\hat{y} - \hat{s}'\overline{\mathrm{x}}\right)$$

and the learning matrix is

$$\Sigma_{\hat{\mathbf{x}}} = \begin{bmatrix} k_1\hat{s}_1 & k_1\hat{s}_2 \\ k_2\hat{s}_1 & k_2\hat{s}_2 \end{bmatrix} \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix},$$

The chosen scenario $\hat{s}$ affects the Kalman gain in two ways: directly via the correlation between the results of the stress test and the unknown risk exposures; and indirectly, through the amount of noise in the stress test result, $\sigma_{\hat{\epsilon}}^2(\hat{\mathrm{s}})$. Using the above formula we see that the top left term of the learning matrix is

$$\frac{\Sigma_{\hat{\mathbf{x}}}(1,1)}{\sigma_1^2} = \frac{\left(\sigma_1\hat{s}_1 + \rho\sigma_2\hat{s}_2\right)^2}{\left(\sigma_1\hat{s}_1 + \rho\sigma_2\hat{s}_2\right)^2 + \left(1-\rho^2\right)\left(\sigma_2\hat{s}_2\right)^2 + \sigma_{\hat{\epsilon}}^2}. \tag{10}$$

The upper bound $\frac{\Sigma_{\hat{\mathbf{x}}}(1,1)}{\sigma_1^2} = 1$ corresponds to learning everything about exposure x_1. When x_1 and x_2 are correlated we can learn about x_1 by increasing $\hat{s}_2$, but the potential to learn is bounded by this correlation. Holding constant $\sigma_{\hat{\epsilon}}$, if we let $\hat{s}_2 \to \infty$ the limit is $\frac{\Sigma_{\hat{\mathbf{x}}}(1,1)}{\sigma_1^2} \to \rho^2$.

Equation (10) shows the relevance of the *endogenous* noise term. If $\sigma_{\hat{\epsilon}}$ is exogenous, then learning is trivially maximized by sending $\hat{s}_1 \to \infty$. In reality extreme scenarios are more difficult to estimate and $\sigma_{\hat{\epsilon}}$ is increasing in the size of the deviation of the stress scenario from the baseline. We obtain an interior solution for the regulator's scenario choice problem as long as $\sigma_{\hat{\epsilon}}$ is convex enough in $\|\hat{s}\|$. To make this idea more precise consider the case where σ_2 is small, and, therefore, the planner only wishes to learn about x_1. Normalizing $\sigma_1 = 1$ and defining $\sigma_{\hat{\epsilon}}^2 = z\left(s^2\right)$ we see that the planner solves max $\frac{s^2}{s^2+z(s^2)}$ and the FOC is $s^2 z' = z$. The class of noise models $z = \alpha s_1^2 + \beta e^{\theta s_1^2}$ is then particularly useful because the uni-dimensional learning solution is simply $\theta\hat{s}_1^2 = 1$, which does not depend on σ_1, α, β. This motivates our functional form

$$\sigma_{\hat{\epsilon}}^2 = \alpha\left\|\hat{s}\right\|^2 + \beta e^{\theta\|\hat{s}\|^2}$$

in our application. In this case, when $\rho = 0$, the first learning coefficient is

$$\frac{\Sigma_{\hat{\mathbf{x}}}(1,1)}{\sigma_1^2} = \frac{\sigma_1^2\hat{s}_1^2}{\left(\alpha + \sigma_1^2\right)\hat{s}_1^2 + \left(\alpha + \sigma_2^2\right)\hat{s}_2^2 + \beta e^{\theta\left(\hat{s}_1^2 + \hat{s}_2^2\right)}}. \tag{11}$$

Lemma 1. *When $\rho = 0$, $\Sigma_{\hat{\mathbf{x}}}(1,1)$ is increasing in $\hat{s}_1^2$ when $\beta e^{\theta\left(\hat{s}_1^2+\hat{s}_2^2\right)}\left(\theta \hat{s}_1^2 - 1\right) < \left(\alpha + \sigma_2^2\right)\hat{s}_2^2$ and decreasing afterwards.*

The overall effect of an increase in $\hat{s}_1^2$ on the amount of learning about x_1 depends on the trade off between the salience of the stress on factor 1 and the increase in noise associated with more extreme scenarios. When the departure from the baseline is small the direct effect dominates and stressing factor 1 leads to more learning about factor 1. When the scenario is more extreme the second effect dominates. Note that the presence of the second factor expands the range where $\Sigma_{\hat{\mathbf{x}}}(1,1)$ is increasing in $\hat{s}_1^2$. When $\hat{s}_2^2 = 0$ the range is simply $\left[0, \theta^{-1/2}\right]$. When $\hat{s}_2^2 > 0$ it expands beyond $\theta^{-1/2}$ because of the baseline noise from factor 2 in the denominator of Equation (11).

The effect of the scenario choice on learning is more complex when the risk exposures are correlated because the planner can learn about x_1 by increasing $\hat{s}_2$ instead of $\hat{s}_1$. In addition, in some risk settings the planner might care a lot about the covariance term

$$\frac{\Sigma_{\hat{\mathbf{x}}}(1,2)}{\sigma_1\sigma_2} = \frac{\rho\left(\left(\sigma_1\hat{s}_1 + \sigma_2\hat{s}_2\right)^2\right) + (1-\rho)^2\sigma_1\sigma_2\hat{s}_1\hat{s}_2}{\left(\sigma_1\hat{s}_1 + \sigma_2\hat{s}_2\right)^2 - 2(1-\rho)\sigma_1\sigma_2\hat{s}_1\hat{s}_2 + \sigma_{\hat{\epsilon}}^2}.$$

When $\rho = 0$ we get $\frac{\Sigma_{\hat{\mathbf{x}}}(1,2)}{\sigma_1\sigma_2} = \frac{\sigma_1\sigma_2\hat{s}_1\hat{s}_2}{\sigma_1^2\hat{s}_{11}^2 + \sigma_2^2\hat{s}_2^2 + \sigma_{\hat{\epsilon}}^2}$, when $\rho = 1$ we get $\frac{\Sigma_{\hat{\mathbf{x}}}(1,2)}{\sigma_1\sigma_2} = \frac{(\sigma_1\hat{s}_1+\sigma_2\hat{s}_2)^2}{(\sigma_1\hat{s}_1+\sigma_2\hat{s}_2)^2+\sigma_{\hat{\epsilon}}^2}$, and we can show that $\frac{\partial\Sigma_{\hat{\mathbf{x}}}(1,2)}{\partial\rho} > 0$ so learning about the posterior mean is easier when the exposures are correlated.

3.3 Scenario choice as information precision choice

Every set of stress scenarios $\hat{S}$ has a unique posterior covariance matrix associated with it. Choosing a set of scenarios $\hat{S}$ is therefore equivalent to choosing a posterior covariance matrix $\hat{\Sigma}_{\mathbf{x}} \in \mathbf{\Sigma}$ (or alternatively, choosing how much to learn $\Sigma_{\hat{\mathbf{x}}}$), where the set $\mathbf{\Sigma}$ is given by Equations (7) and (8) for all possible scenario choices. The shape of the feasibility set $\mathbf{\Sigma}$ is determined by the regulator's priors $\Sigma_{\boldsymbol{x}}$ and the errors in banks' models, $\mathbf{\Sigma}_{\hat{\boldsymbol{\epsilon}}}$.

Proposition 2. ***(Scenario choice and information)*** *Choosing stress scenarios $\hat{S}$ is equivalent to choosing a residual covariance matrix $\hat{\Sigma}_{\mathbf{x}} \in \mathbf{\Sigma}$.*

The Kalman filter maps scenarios M to the elements in the posterior covariance matrix $\Sigma_{\hat{\mathbf{x}}}$, which is ultimately what the regulator cares about. More specifically, the Kalman filter implies $\frac{JN(JN-1)}{2}$ equations mapping the $J \times M$ elements in the scenario matrix $\hat{S}$ to the elements of $\Sigma_{\hat{\mathbf{x}}}$.

Using Proposition 2 we can write the regulator's scenario choice problem as an information precision choice problem, where the feasible set from which the regulator chooses is determined by the Kalman filter. This set plays the role of a capacity constraints in models of information acquisition. Figures (2) shows the feasible set of posterior variances $\left\{\hat{\Sigma}_{\mathbf{x},11}, \hat{\Sigma}_{\mathbf{x},22}\right\}$ in a model with one representative bank and two risk factors, and for different values of prior correlations among risk exposures.

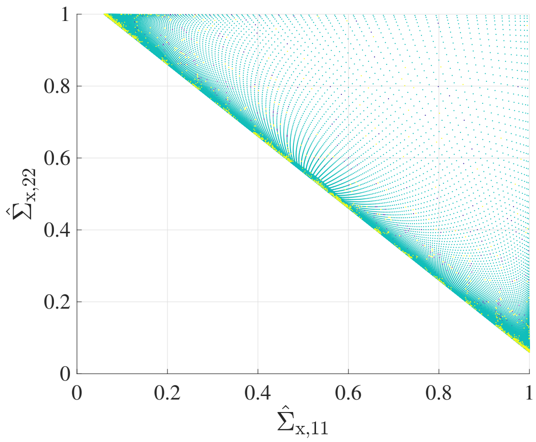
Remark 2. When the number of banks is higher than the number of scenarios ($N > M$), the design problem boils down to choosing residual variances about the risk exposures of any M banks since it is equivalent for the regulator to choose the stress scenarios or to choose $J \times M$ elements of the residual covariance matrix.

Perfect learning is not feasible, as can be seen in Figures 2 where the budget does not include a variance of zero. As discussed above, choosing a more extreme scenario has two effects on the amount of information that the regulator can acquire. On the one hand, a higher value of $\hat{s}_i$ increases the weight the bank's stress test results put on the bank's exposure to factor i. On the other hand, more extreme scenarios increase the noise $\mathbf{\Sigma}_{\hat{\varepsilon}}$ in the stress tests result.

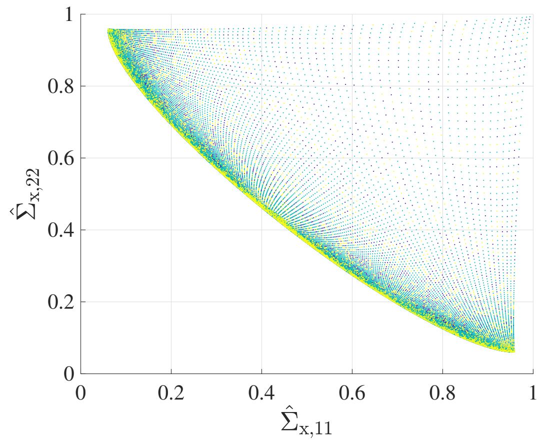
Prior correlation among risk exposures makes it easier to learn and reduces the posterior variances. Moreover, the regulator cannot learn about the bank's exposure to factor 1 without learning about the bank's exposure to factor 2. Hence, as it can be seen from panels *a, b, c* and *d* in Figure 2, the boundary of set of feasible posterior precisions, $\mathbf{\Sigma}$, becomes more convex. When the correlation is high, the boundary of the feasible set slopes up in the tails. This is in line with the discussion in the example: when s_1 is already large, it becomes more efficient to learn about x_1 by increasing s_2 instead of s_1.

4 Taking Action

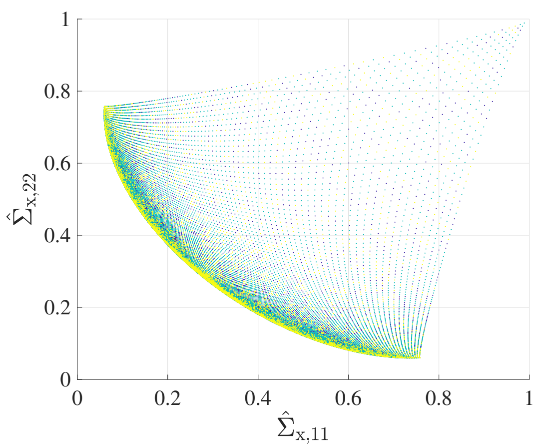
The regulator values information from the stress test because it allows her to intervene more accurately. In return, the design of optimal scenarios depends on the actions that the regulator expects to take. Regulators typically have two ways of intervening in the banking sector. They can mandate a broad increase in capital, or they can restrict specific activities, for instance by imposing loan-to-value ratios or collateral requirements.



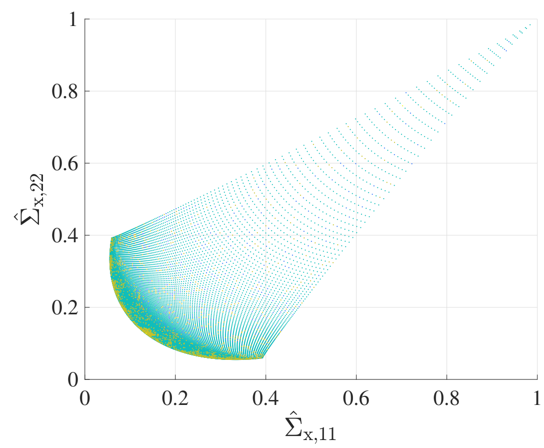
(a) Prior correlation in exposures $\Sigma_{x,12} = 0$



(b) Prior correlation in exposures $\Sigma_{x,12} = 0.2$



(c) Prior correlation in exposures $\Sigma_{x,12} = 0.5$



(d) Prior correlation in exposures $\Sigma_{x,12} = 0.8$

Figure 2: Feasible set of residual variances, $\left(\hat{\Sigma}_{\mathbf{x},11}, \hat{\Sigma}_{\mathbf{x},22}\right)$ when there are two factors and one representative bank for different values of the regulator's prior correlation among the bank's risk exposures to factors 1 and 2.

Note: Figures 1 illustrates the set of feasible posterior variances, $\mathbf{\Sigma}$ for different values of prior correlations among risk exposures when $\sigma_{\hat{\epsilon}}^2 = \alpha \left\|\hat{s}\right\|^2 + \beta e^{\theta\|\hat{s}\|^2}$. The parameters used are $M = 1$, $N = 1$, $J = 2$, $\gamma = 0.5$, $\phi^0 = [1, 1]'$, $\phi^1 = [0.1, 0.1]'$, $\bar{x} = [1, 1]'$, $\Sigma_{\boldsymbol{x}} = \boldsymbol{I}_J$, $\alpha = 0.01$, $\beta = 0.02$, $\theta = 1$, and $\mathbb{E}\left[\epsilon^2\right] = 1$.

4.1 General Case

The most granular description of intervention is at the bank×exposure level. If the regulator takes action $\mathrm{a}_i = \{a_{i,j}\}_{j=1..J}$ on bank i, the exposure of bank i to factor j becomes $(1 - a_{i,j})\, x_{i,j}$. Aggregate banking wealth is then given by

$$W\left(s, x; \mathbf{a}, \bar{W}\right) = \bar{W} - \bar{\eta} - \sum_{i=1}^{N} \sum_{j=1}^{J} (1 - a_{i,j})\, x_{i,j} s_j, \tag{12}$$

where $\bar{W}$ is the capital required by the regulator. The regulator takes action after observing the results of the test, therefore her expected utility in the intervention stage is a function of her information set

$$V(\mathscr{S}) = \max_{\bar{W}, \mathbf{a} \in \mathcal{A}} \mathbb{E}\left[U\left(W\left(s, x; \mathbf{a}, \bar{W}\right)\right) \mid \mathscr{S}\right] - \mathcal{C}(\mathbf{a}) - \mathcal{K}\left(\bar{W}\right),$$

where $\mathscr{S}$ denotes the information set after the stress test is conducted.

In an interior solution, the first order conditions equate the marginal cost of an intervention to its expected marginal benefit. For capital requirements we obtain

$$\mathcal{K}'\left(\bar{W}\right) = \mathbb{E}\left[U'(W) \mid \mathscr{S}\right]. \tag{13}$$

Since capital is useful in all states of the world the optimality condition simply states that the marginal cost of banking capital be equal to the expected marginal utility of banking net worth. Similarly, the optimal targeted intervention on the exposure of bank i to factor j is

$$\frac{\partial \mathcal{C}(\mathbf{a})}{\partial a_{i,j}} = \mathbb{E}\left[x_{i,j} s_j U'(W) \mid \mathscr{S}\right]. \tag{14}$$

The expected marginal benefit of reducing the risk exposure to factor j in bank i depends on the covariance between the marginal social utility $U'(W)$ and the contribution of factor j to bank i's losses, $x_{i,j} s_j$. Risk reduction is more valuable when the planner expects high losses in states of the world where U' is also large.

4.2 Pseudo Mean-Variance Preferences

In our application we assume the following utility function:

$$U(W) = \begin{cases} W & \text{if} \quad \mathscr{D} = 0, \\ W - \frac{\gamma}{2}(\mathcal{W} - W)^2 & \text{if} \quad \mathscr{D} = 1, \end{cases} \tag{15}$$

where $\mathscr{D}$ is a dummy variable for (potential) economy-wide financial distress and $\mathcal{W}$ is a satiated level of capital. We assume that $\mathcal{W}$ is high enough that $W < \mathcal{W}$ when $\mathscr{D} = 1$ and, with a slight abuse, we say that these preferences are linear quadratic. The utility function implies that the regulator's absolute risk aversion is higher in bad states of the world – as it would be under CRRA preferences for example – while retaining the tractability of mean-variance analysis.

We now need to discuss the physical distribution of the states s. Risk management focuses on downside risk, and the relevant states are the ones where $\mathscr{D} = 1$. We normalize the unconditional expectation of s to zero $\mathbb{E}[\mathrm{s}] = 0$, and we define the probability of (potential) distress as

$$p \equiv Pr\left(\mathscr{D} = 1\right).$$

Conditional on $\mathscr{D} = 1$ we assume that $\mathrm{s}|\,\mathscr{D} = 1 \in [0, \infty)$, $\mathbb{E}[\mathrm{s}|\,\mathscr{D} = 1] = \tilde{s} > 0$ and that $\mathbb{V}ar\,[\mathrm{s}|\,\mathscr{D} = 1] = \tilde{\Sigma}_{\mathrm{s}}$. We can then write the expected utility of the regulator as

$$\mathbb{E}\left[U\left(W\right)\right] = \mathbb{E}\left[W\right] - \frac{p\gamma}{2}\tilde{\mathbb{E}}\left[\left(\mathcal{W} - W\right)^2\right], \tag{16}$$

where $\tilde{\mathbb{E}}$ is the expectation conditional on the economy being in distress, i.e., $\tilde{\mathbb{E}}[.] \equiv \mathbb{E}[.\mid \mathscr{D} = 1]$. Since $\mathbb{E}[\mathrm{s}] = \mathbb{E}[\bar{\eta}] = 0$, we see from (12) that $\mathbb{E}[W] = \bar{W}$ and obtain the following Lemma.

Lemma 2. *With linear quadratic preferences the planner's intervention problem is*

$$\max_{\bar{W}, \mathbf{a} \in \mathcal{A}} \bar{W} - \mathcal{K}\left(\bar{W}\right) - \frac{p\gamma}{2}\tilde{\mathbb{E}}\left[\left(\mathcal{W} + \sum_{i=1}^{N}\sum_{j=1}^{J}\left(1 - a_{i,j}\right)x_{i,j}s_j - \bar{W}\right)^2 \mid \mathscr{S}\right] - \mathcal{C}\left(\mathbf{a}\right).$$

4.3 Standard Capital Requirement

Under linear-quadratic preferences the first order condition for optimal capital is $\mathcal{K}'\left(\bar{W}^\star\right) = 1 + p\gamma\left(\mathcal{W} - \bar{W}^\star + \sum_{i=1}^{N}\sum_{j=1}^{J}\left(1 - a_{i,j}\right)\hat{x}_{i,j}\tilde{\mathbb{E}}\left[s_j\right]\right)$. Capital requirements are increasing in risk aversion γ, in probability of distress p, and in estimated risk exposures $\hat{x}$. Capital requirements do not depend directly on the variance-covariance matrices of states or exposures, but, in general, the requirements are mitigated by targeted interventions. Suppose, for example, that a stress test uncovers excessive exposures to commercial real estate lending. Without targeted interventions the regulator would have to increase overall requirements. With targeted interventions, on the other hand, the regulator might instead mandate lower LTV ratios for that specific class of loans.

To understand the connection between our optimal capital requirement and actual stress tests it is useful to consider a model *without* targeted interventions since it is the way actual stress tests are framed: they are conducted "all else equal", i.e., assuming no actions from the regulator. Let us then assume $\mathbf{a} = 0$. We obtain the following proposition.

Proposition 3. ***(Standard capital requirements)*** *Standard capital requirements (assuming* $\mathbf{a} = 0$*) are a function of expected losses under the (physical) adverse scenario:*

$$\bar{W}^{\star} + \frac{\mathcal{K}'\left(\bar{W}^{\star}\right) - 1}{p\gamma} = \mathcal{W} + \sum_{i=1}^{N} \mathbb{E}\left[y_i \mid \mathscr{D} = 1, \mathscr{S}\right], \tag{17}$$

where expected losses, from equation (1), satisfy

$$\mathbb{E}\left[y_i \mid \mathscr{D} = 1, \mathscr{S}\right] = \tilde{\mathrm{s}} \cdot \hat{\mathrm{x}}_{\mathrm{i}},$$

with $\tilde{\mathrm{s}} = \mathbb{E}\left[\mathrm{s} \mid \mathscr{D} = 1\right]$ *and* $\hat{\mathrm{x}}_{\mathrm{i}} = \mathbb{E}\left[\mathrm{x}_{\mathrm{i}} \mid \mathscr{S}\right]$*.*

The net marginal cost of bank equity $\mathcal{K}'\left(\bar{W}^{\star}\right) - 1$ is compared to the adjusted risk of distress $p\gamma$.[5] Net of this cost, capital requirements are set to cover losses in the adverse scenario. It is important to understand the similarities and the differences between our Proposition 3 and what regulators do in practice. Exactly as in actual stress tests, our model says that the requirements should be set to cover losses under an adverse scenario. The adverse scenario in our model is the (physical) expectation of the state conditional on potential distress, $\tilde{\mathrm{s}} = \tilde{\mathbb{E}}\left[\mathrm{s}\right]$.

The main difference is that, in our model, the regulator uses expected exposures from the Kalman filter, $\hat{\mathrm{x}}_{\mathrm{i}} = \mathbb{E}\left[\mathrm{x}_{\mathrm{i}} \mid \mathscr{S}\right]$. In general, therefore $\mathbb{E}\left[y_i \mid \mathscr{D} = 1, \mathscr{S}\right] \neq \hat{y}_i\left(\tilde{\mathrm{s}}\right) = \tilde{\mathrm{s}} \cdot \mathrm{x}_i + \hat{\epsilon}_i\left(\|\tilde{\mathrm{s}}\|\right)$: the optimal forecast of losses under the adverse scenario differ from the losses the bank would report under the adverse scenario for several reasons. The first is noise in the bank's report $\hat{\epsilon}_{i,}$: if this noise is zero then $\mathbb{E}\left[y_i \mid \mathscr{D} = 1, \mathscr{S}\right] = \hat{y}_i\left(\tilde{\mathrm{s}}\right) = \tilde{\mathrm{s}} \cdot \mathrm{x}_i$ would be exact. As argued before, however, the bank itself must estimate its exposures. Given this unavoidable measurement error the regulator uses two other sources of information to form an optimal forecast $\mathbb{E}\left[\mathrm{x}_{\mathrm{i}} \mid \mathscr{S}\right]$: reported losses from other scenarios ($\hat{\mathrm{s}} \neq \tilde{\mathrm{s}}$) and from other banks ($j \neq i$).

[5] Without any cost of raising bank equity, it would trivially be optimal to set requirements at the satiated at the level $\mathcal{W}$ that brings the marginal utility of net worth back to 1 in all states of the world. As explained earlier, the formula above assumes an interior solution where $W < \mathcal{W}$ when $\mathscr{D} = 1$, which is the empirically realistic case.

4.4 Optimal Interventions and the Distress Uncertainty Matrix

To set capital requirements, the regulator essentially asks: what is the expected stress, and what is the average exposure? Targeted interventions, on the other hand require more information. Under linear quadratic preferences the first order condition for optimal action is $\frac{\partial \mathcal{C}(\mathbf{a})}{\partial a_{i,j}} = \hat{x}_{i,j}\mathbb{E}[s_j] + p\gamma\tilde{\mathbb{E}}[x_{i,j}s_j(\mathcal{W} - W) \mid \mathscr{S}]$. Given our normalization of the unconditional mean $\mathbb{E}[s_j] = 0$, we obtain

$$\frac{\partial \mathcal{C}(\mathbf{a})}{\partial a_{i,j}} = p\gamma\hat{x}_{i,j}\tilde{s}_j\tilde{\mathbb{E}}[\mathcal{W} - W \mid \mathscr{S}] + p\gamma\sum_{h=1}^{N}\sum_{l=1}^{J}(1 - a_{h,l})\,\mathbb{C}ov[x_{i,j}s_j, x_{h,l}s_l \mid \mathscr{S}, \mathscr{D} = 1]$$

For the remainder of the paper we assume the following functional forms.

Assumption LQ. *The cost of bank capital is linear and the cost of targeted actions is a quadratic form*

$$\mathcal{K}\left(\bar{W}\right) = (1 + \kappa)\,\bar{W},$$
$$\mathcal{C}(\mathbf{a}) = \frac{1}{2}\mathbf{a}'\Phi\mathbf{a}.$$

We assume that the cost of bank equity is linear for simplicity but also because it shows that our results do not hinge on large or very convex costs of equity. For the quadratic form we often write the special case $\mathcal{C}(\mathbf{a}) = \frac{1}{2}\sum_{i=1}^{N}\phi_i^0\|a_i\|^2 + \sum_{j=1}^{J}\phi_j^1\left(\sum_{i=1}^{N}a_{ij}\right)^2$. This cost function captures increasing marginal costs at the bank level $\|a_i\|^2$ as well as congestion effects (e.g., fire sales) at the aggregate level, $\left(\sum_{i=1}^{N}a_{ij}\right)^2$. Under **LQ** the optimal capital in Equation (17) is[6]

$$\bar{W}^{\star} = \mathcal{W} - \frac{\kappa}{p\gamma} + \sum_{i=1}^{N}\sum_{j=1}^{J}(1 - a_{i,j})\,\hat{x}_{i,j}\tilde{s}_j, \tag{18}$$

and the planner's objective function in Lemma 2 simplifies to

$$\min_{\mathbf{a}\in\mathcal{A}} \kappa\bar{W}^{\star}(\mathbf{a}) + \frac{p\gamma}{2}\tilde{\mathbb{E}}\left[\left(\frac{\kappa}{p\gamma} + \sum_{i=1}^{N}\sum_{j=1}^{J}(1 - a_{i,j})\,(x_{i,j}s_j - \hat{x}_{i,j}\tilde{s}_j)\right)^2\right] + \mathcal{C}(\mathbf{a}).$$

Under these assumptions we can show that optimal interventions depend on a specific covariance matrix.

[6]In vector notations $\bar{W}^{\star} = \mathcal{W} - \frac{\kappa}{p\gamma} + (\mathbf{1}_N \otimes \tilde{\mathrm{s}})'(\hat{\mathbf{x}} \circ (\mathbf{1}_{NJ} - \mathbf{a}^{\star}))$

Proposition 4. ***(Optimal interventions)*** *In the linear quadratic model, optimal interventions depend on the* ***Distress Uncertainty Matrix***

$$\tilde{\mathbb{V}} \equiv \mathbb{COV}\left[(\mathbf{1}_N \otimes \mathrm{s}) \circ \mathbf{x} \mid \mathscr{S}, \mathscr{D} = 1\right].$$

Optimal targeted actions are given by

$$\mathbf{a}^{\star} = \left(\Phi + p\gamma\tilde{\mathbb{V}}\right)^{-1}\left(\kappa\left(\mathbf{1}_N \otimes \tilde{\mathrm{s}}\right) \circ \hat{\mathbf{x}} + p\gamma\tilde{\mathbb{V}}\mathbf{1}_{NJ}\right), \tag{19}$$

and optimal capital requirements are given by Equation (18).

The optimal targeted intervention policy of the regulator depends on the combined covariance matrix $\tilde{\mathbb{V}}$ of the NJ vector $(\mathbf{1}_N \otimes \mathrm{s}) \circ \mathbf{x} = (x_{i,j}s_j)_{i=1:N}^{j-1:J} = [s_1x_{1,1}, .., s_Jx_{1,J}, s_1x_{2,1} .., s_Jx_{N,J}]$. The covariance is conditional on the stress test results – ex-post mean $\hat{x}$ and residual variance $\hat{\Sigma}_{\mathbf{x}}$ – and on the physical distribution of stresses – $\tilde{\mathrm{s}}$ and $\tilde{\Sigma}_{\mathrm{s}}$.[7] We have assumed that the error term ε in stress results is independent of future realization of physical stress s.[8] Since s and $\hat{\mathbf{x}}$ are independent we can write the covariance matrix as[9]

$$\tilde{\mathbb{V}} = \left(\mathbf{1}_{\boldsymbol{N}\times\boldsymbol{N}} \otimes \tilde{\Sigma}_{\mathrm{s}}\right) \circ \left(\hat{\mathbf{x}}\hat{\mathbf{x}}'\right) + \left(\mathbf{1}_{\boldsymbol{N}\times\boldsymbol{N}} \otimes \left(\tilde{\Sigma}_{\mathrm{s}} + \tilde{\mathrm{s}}\tilde{\mathrm{s}}'\right)\right) \circ \hat{\Sigma}_{\mathbf{x}}.$$

$\tilde{\mathbb{V}}$ therefore combines uncertainty about the macro state under distress $\tilde{\Sigma}_{\mathrm{s}}$ with residual uncertainty about banks' exposures $\hat{\Sigma}_{\mathbf{x}}$. For example the matrix is large when estimated exposures $\hat{\mathbf{x}}\hat{\mathbf{x}}'$ are high in states where conditional risk $\tilde{\Sigma}_{\mathrm{s}}$ is also high.

Equation (19) says that the regulator intervenes more against high and uncertain exposures to bad and uncertain states. Her interventions are limited by the cost of targeted interventions and the uncertainty itself. High residual uncertainty limits the responsiveness of the targeted interventions to the expected exposures $\hat{\mathbf{x}}$. The regulator cares about information quality to

[7]The residual uncertainty is known in advance since the evolution of the covariance matrix is deterministic, but the posterior mean depends on the random realization the test itself, since $\hat{\mathbf{x}} = \bar{\mathbf{x}} + K\left(\hat{\mathrm{y}} - \hat{\mathrm{s}}'\bar{\mathbf{x}}\right)$.

[8]While this is an obvious assumption to make at this point, we note that it is not without loss of generality if we consider endogenous financial crises. Suppose, for example, that banks are too optimistic about mortgage risk: ε is negative and their perceived exposures are lower than their true exposures. This might lead to excessive lending, real estate price appreciation, and this might increase the probability of a future decrease in real estate prices. This would violate the assumption of correlation between ε and $\tilde{\mathrm{s}}$.

[9]The matrix notations are somewhat complicated but in the one dimensional case the formula is simply the variance of a product of independent variables: $\tilde{\mathbb{V}}(xs) = \tilde{\mathbb{V}}(x)\,\tilde{\mathbb{V}}(s) + \tilde{\mathbb{V}}(x)\left(\tilde{\mathbb{E}}[s]\right)^2 + \tilde{\mathbb{V}}(s)\left(\tilde{\mathbb{E}}[x]\right)^2$.

improve the accuracy of both capital requirements and targeted interventions. More accurate interventions translate into higher variation in ex post net exposure $(\mathbf{1}_{NJ\times 1} - \mathbf{a}^{\star})$ since the regulator intervenes more when it is needed, and less when exposures are low.

5 Designing the Optimal Scenario

Once we account for optimal actions the interim utility of the regulator $V(\mathscr{S})$ depends on the scenarios chosen and on the stress test results $\hat{\boldsymbol{y}}$. In the design stage, the regulator chooses the stress scenarios $\hat{S}$ to maximize the expected value of her information. The scenario design problem is therefore

$$\hat{S}^{*} = \arg\max_{\hat{S}} \mathbb{E}\left[V(\mathscr{S})\mid \hat{S}\right]. \tag{20}$$

We could incorporate a cost of creating additional scenarios for the regulator: choosing M scenarios for the stress test could have a cost $\mathscr{C}(M)$. In that case the objective function would simply be $\mathbb{E}_{\hat{\boldsymbol{y}}}\left[V(\mathscr{S})\mid\hat{S}\right] - \mathscr{C}(M)$ and the regulator would also choose the number of scenarios to include in the stress test. The relevant cost function depends on institutional details (e.g., stress testing insurance portfolios) and we leave this for future applied work.

5.1 Scenario Design for Standard Capital Requirements

In the case of standard capital requirements we obtain a particularly simple result.

Proposition 5. ***Scenario Design for Standard Capital Requirements.*** *Under* ***LQ*** *the planner designs standard capital stress scenarios* ($\mathbf{a} = 0$) *to minimize the Distress Uncertainty Matrix*

$$\min_{\hat{\Sigma}_{\mathbf{x}}\in\boldsymbol{\Sigma}} \mathbf{1}_{1\times NJ}\mathbb{E}\left[\tilde{\mathbb{V}}\right]\mathbf{1}_{NJ\times 1}, \tag{21}$$

where $\boldsymbol{\Sigma}$ *is the set of feasible residual uncertainty implied by the Kalman filter.*

The key simplification comes from the FOC for capital: $\bar{W}^{\star} = \mathcal{W} - \frac{\kappa}{p\gamma} + \sum_{i=1}^{N}\mathbb{E}\left[y_i \mid \mathscr{D}=1,\mathscr{S}\right]$. This FOC is linear in y, and therefore, the expected capital cost $\kappa\mathbb{E}\left[\bar{W}^{\star}\right]$ is independent of $\hat{\Sigma}_{\mathbf{x}}$. As a result the regulator only cares about minimizing uncertainty. The program is equivalent to $\min_{\hat{\Sigma}_{\mathbf{x}}\in\boldsymbol{\Sigma}} \mathbf{1}_{1\times NJ}\left(\mathbf{1}_{N\times N}\otimes\left(\tilde{\Sigma}_s + \tilde{\mathrm{s}}\tilde{\mathrm{s}}'\right)\right)\circ\hat{\Sigma}_{\mathbf{x}}\mathbf{1}_{NJ\times 1}$. The value of learning about factor j depends on the unit cost of exposure $\left(\tilde{\Sigma}_s + \tilde{\mathrm{s}}\tilde{\mathrm{s}}'\right)$ and on residual exposure uncertainty $\hat{\Sigma}_{\mathbf{x}}$.

Corollary 1. *Optimal scenarios for standard capital requirements depends only on the physical distribution of risk factors – $\tilde{s}$ and $\tilde{\Sigma}_s$ – on prior uncertainty $\Sigma_{\mathbf{x}}$, and on stress test noise $\boldsymbol{\Sigma}_{\hat{\boldsymbol{\varepsilon}}}$.*

5.2 General Scenario Design

Given that $\hat{\mathbf{y}}$ is normally distributed and that macro factors are independent from risk exposures, we can integrate the indirect value function $\mathbb{E}\left[V\left(\mathscr{S}\right)|\hat{S}\right]$ and express it as function of the covariance matrices $\hat{\Sigma}_{\mathbf{x}}$ and $\tilde{\Sigma}_s$. To see this, note that the regulator's objective depends on the result of the stress test $\hat{\mathbf{y}}$ only through $\hat{\mathbf{x}}$. Moreover, the optimal targeted interventions only depend on $\hat{\mathbf{x}}$ and on the posterior variance. The regulator then solves

$$\min_{\hat{\Sigma}_{\mathbf{x}}\in\boldsymbol{\Sigma}} \mathbb{E}\left[\kappa\bar{W}^{\star} + \frac{1}{2}\left(\left(1_{NJ\times 1} - \mathbf{a}^{\star}\right)' p\gamma\tilde{\mathbb{V}}\left(1_{NJ\times 1} - \mathbf{a}^{\star}\right) + \mathbf{a}^{*\prime}\Phi\mathbf{a}^{\star}\right)\right]. \tag{22}$$

It is useful to compare (22) with (21). If we force $\mathbf{a}^{\star} = 0$ in (22) we obtain (21) since, when $\mathbf{a}^{\star} = 0$, $\mathbb{E}\left[\bar{W}^{\star}\right]$ is independent of $\hat{\Sigma}_{\mathbf{x}}$. Three changes occur when $\mathbf{a}^{\star}$ is optimally chosen. First $\mathbb{E}\left[\bar{W}^{\star}\right]$ now depends on $\hat{\Sigma}_{\mathbf{x}}$ via $\mathbf{a}^{\star}$. Second, $\mathbf{a}^{\star}$ mitigates the cost of uncertainty, as seen in the middle term by limiting the ex-post exposures to the macro factors. Finally, the cost of action appears as $\mathbf{a}^{*\prime}\Phi\mathbf{a}^{\star}$. If we replace the optimal actions and optimal capital requirements we obtain the following proposition.

Proposition 6. ***(Regulator's problem)*** *Under **LQ** the stress scenario design problem is equivalent to*

$$\min_{\hat{\Sigma}_{\mathbf{x}}\in\boldsymbol{\Sigma}} \mathbb{E}_{\hat{\mathbf{x}}}\left[\kappa\bar{W}^{\star} + \Phi\mathbf{a}^{\star}\right], \tag{23}$$

where $\mathbf{a}^{\star}$ is given by (19), $\bar{W}^{\star}$ by (18), and $\boldsymbol{\Sigma}$ is the set of feasible residual uncertainty implied by the Kalman filter.

The simplicity of Equation (23) comes from the linear cost of capital and the quadratic costs and benefits of targeted actions. Consider for simplicity the one dimensional case, $NJ = 1$. Then we have $\bar{W}^{\star} = \mathcal{W} - \frac{\kappa}{p\gamma} + \kappa\hat{x}\tilde{s}\left(1-a^{*}\right)$ and the program is $\min\mathbb{E}\left[\kappa\bar{W}^{\star} + \frac{1}{2}\left(p\gamma\tilde{\mathbb{V}}\left(1-a^{*}\right)^{2} + \phi\left(a^{*}\right)^{2}\right)\right]$. The optimal action $a^{*} = \frac{p\gamma\tilde{\mathbb{V}}+\kappa\hat{x}\tilde{s}}{\phi+p\gamma\tilde{\mathbb{V}}}$ implies $p\gamma\tilde{\mathbb{V}}\left(1-a^{*}\right)^{2} + \phi\left(a^{*}\right)^{2} = \phi a^{*} - \left(1-a^{*}\right)\kappa\hat{x}\tilde{s}$ and therefore the program is equivalent to $\min\mathbb{E}\left[\kappa\hat{x}\tilde{s}\left(1-a^{*}\right) + \phi a^{*}\right]$

The regulator anticipates that she will intervene optimally after observing the results of the stress test and she chooses a posterior covariance matrix $\hat{\Sigma}_{\mathbf{x}}$ to maximize the accuracy of her actions weighted by the relevant costs. The set $\boldsymbol{\Sigma}$ restricts the amount of information about different exposures the regulator can choose. To highlight the intuition behind the regulator's scenario choice, consider the case with one bank and one scenario. The benefit of learning more about exposure j, i.e. of decreasing $\hat{\Sigma}^j_{\mathbf{x}}$, is given by

$$\mathbb{E}\left[\left(\mathbf{1}_{NJ\times 1}-\mathbf{a}^{\star\prime}\right)\left(\kappa\left(\mathbf{1}_{N\times 1}\otimes\tilde{\mathrm{s}}\right)\circ\frac{\partial\hat{\mathbf{x}}}{\partial\hat{\Sigma}^j_{\mathbf{x}}}\right)+\left(\Phi-\kappa\left(\mathbf{1}_{N\times 1}\otimes\tilde{\mathrm{s}}\right)\circ\hat{\mathbf{x}}\right)\frac{d\mathbf{a}^{\star\prime}}{d\hat{\Sigma}^j_{\mathbf{x}}}\right]. \tag{24}$$

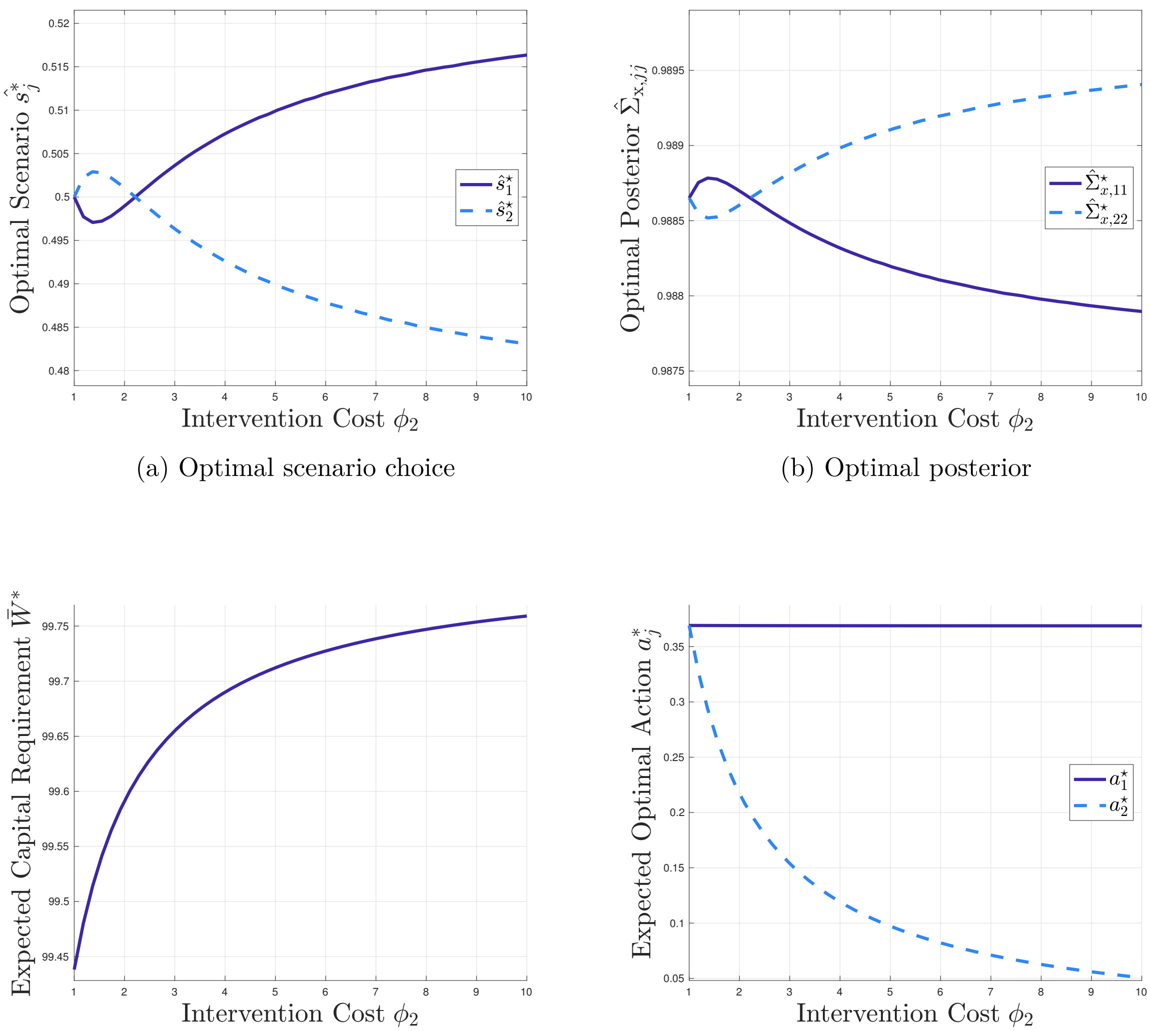
The value of learning depends on how valuable it is to intervene and on how responsive interventions are to the new information. The first term in Equation (24) represents the impact of information on the posterior expected risk exposure. The more precise this information, the more sensitive $\hat{\mathbf{x}}$ is to the new information in the stress test. Note that

$$\frac{\partial\bar{W}^{\star}}{\partial\hat{\mathbf{x}}}=\left(\mathbf{1}_{NJ\times 1}-\mathbf{a}^{\star\prime}\right)\left(\mathbf{1}_{N\times 1}\otimes\tilde{\mathrm{s}}\right).$$

Therefore, the first term in Equation (24) represents the reduction in the cost of the capital requirements when information in the stress tests is more precise along dimension j. The second term in Equation (24) captures the benefit of changing the targeted interventions when $\hat{\Sigma}^j_{\mathbf{x}}$ is lower. The effect of $\hat{\Sigma}^j_{\mathbf{x}}$ on $\mathbf{a}^{\star}$ deserves further attention. Since $\mathbf{a}^{\star}\left(\hat{\mathbf{x}},\tilde{\mathbb{V}}\right)$ we have

$$\frac{d\mathbf{a}^{\star}}{d\hat{\Sigma}^j_{\mathbf{x}}}=\frac{\partial\mathbf{a}^{\star}}{\partial\tilde{\mathbb{V}}}\frac{d\tilde{\mathbb{V}}}{d\hat{\Sigma}^j_{\mathbf{x}}}+\frac{\partial\mathbf{a}^{\star}}{\partial\hat{\mathbf{x}}}\frac{d\hat{\mathbf{x}}}{d\hat{\Sigma}^j_{\mathbf{x}}} \tag{25}$$

The first term in Equation (25) captures the change in the targeted intervention when the residual uncertainty faced by the regulator changes. The second term measures the value of intervening more accurately and it depends on the sensitivity of the targeted intervention to the ex-post expected exposures $\frac{\partial\mathbf{a}^{\star}}{\partial\hat{\mathbf{x}}}$, and on how the new information from the stress test changes this posterior mean, which is determined by $\frac{d\hat{\mathbf{x}}}{d\hat{\Sigma}^j_{\mathbf{x}}}$. Suppose that the regulator's prior beliefs were very precise. In this case, only extreme realizations of $\hat{\mathbf{y}}$ would move the regulator's priors and the sensitivity of the intervention policy to new information would be low, i.e., $\frac{d\hat{\mathbf{x}}}{d\hat{\Sigma}^j_{\mathbf{x}}}$ would be small. In this case, increasing the precision of the stress test along dimension j would not improve the accuracy of the intervention much and the value of learning along dimension j would be low. Similarly, if the intervention policy was not very responsive to information, reducing the residual uncertainty of the regulator along dimension j would not have a large effect on the intervention policy and the value of information would be low.



(a) Optimal scenario choice

(b) Optimal posterior

(c) Expected optimal capital requirement

(d) Expected optimal targeted interventions

Figure 3: Optimal scenario, information choice and expected interventions as a function the intervention cost to reduce the exposure to factor 2, ϕ_2.

Note: The parameters used are $M = 1$, $N = 1$, $J = 2$, $\gamma = 1.1$, $\phi^0 = [1, 1]'$, $\phi^1 = [0.1, 0.1]'$, $\bar{x} = [1, 1]'$, $\Sigma_{\boldsymbol{x}} = \boldsymbol{I}_J$, $\alpha = 5$, $\beta = 7$, $\theta = 2$, $\mathbb{E}[s] = [0, 0]'$, $\tilde{\mathbb{E}}[s] = [1, 1]'$, $\Sigma_s = \boldsymbol{I}_J$, $\tilde{\mathbb{E}}[s] = \boldsymbol{I}_J$, $\kappa = 2$, $\mathcal{W} = 100$, $p = 0.1$, $\mathbb{E}[\eta] = 0$, $\mathbb{E}\left[\eta^2\right] = 1$, and $\mathbb{E}[\epsilon\epsilon'] = \boldsymbol{I}_N$.

6 Comparative Statics

The weights of the different risk factors in the optimal scenario chosen by the regulator depend on how much her targeted interventions will respond to the new information in the stress tests. This, in turn, depends on the intervention costs and the prior beliefs of the regulator. In this section, we provide comparative statics to illustrate the determinants of the optimal stress scenarios. Most examples below use two factors and one bank and, unless explicitly stated otherwise, we assume that the variance of the variance or the error term in the stress test results is of the form $\sigma_{\hat{\epsilon}}^2 = \alpha \|\hat{s}\|^2 + \beta e^{\theta\|\hat{s}\|^2}$ and that the regulator's priors about exposures are uncorrelated across factors.[10]

6.1 Intervention costs

The first important point is that intervention costs have a non monotone impact on scenario design. When intervention costs are low, the regulator can intervene preemptively to reduce exposures. Inaccurate interventions are not too costly and the regulator cares little about learning about that factor. When the intervention costs are intermediate, interventions are sensitive to the information produced by the stress tests and the regulator values learning to avoid wasteful interventions. Finally, when the intervention costs are high, the ex-post interventions are small irrespective of stress test results and learning is less valuable for the regulator.

The first two panels in Figure 3 illustrates the optimal scenario design and the implied posterior precisions as we vary ϕ_2, the cost of reducing the bank's exposure to factor 2. The first panel shows that the stress on factor 2 increases and then decreases with ϕ_2. When ϕ_2 increases from a low value the regulator finds it optimal to learn more about factor 2 at the cost of learning less about factor 1 in order to minimize her total intervention costs. As ϕ_2 increases, however, the informational sensitivity of $a_2^\star$ decreases. This leads to a decrease in the weight of factor 2 and an increase in the weight of factor 1 in the stress test. In the limit as ϕ_2 goes to infinity the regulator only learns about factor 2 to properly set capital requirements, not to perform a targeted intervention. In the absence of capital requirements, the regulator would choose not to

[10]If the regulator's prior expectation is that exposures are correlated, it may still be beneficial for her to stress factor j even if doing so does not improve the accuracy of her intervention along dimension j. In this case, stressing factor j would only be valuable to learn about the exposures to other factors and improve the accuracy of the targeted interventions along these other dimensions.

learn about the exposure to factor 2 at all for large enough values of ϕ_2.[11]

Figures 3c and 3d show the expected intervention policy as a function of the intervention cost for factor 2, respectively. The expected intervention is decreasing and convex in the intervention cost. When the intervention cost ϕ_2 is small, the expected intervention decreases quickly for two reasons. First, the regulator intervenes less because interventions are costly. Second, more precise information about the bank's risk exposures allows the regulator to intervene more accurately. As the cost of targeted interventions increases, the regulator optimally compensates by increasing capital requirements.

6.2 Prior Exposures

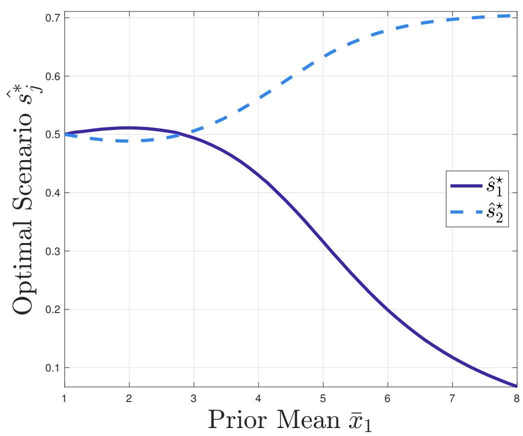
Prior mean of exposures imply relatively similar comparative to those of intervention costs, as it can be seen from Panels (a) and (b) in Figure 4. The optimal scenario is non monotonic because there are two opposing forces. Targeted intervetions increase with expected exposure which increases the value of learning about high exposures. On the other hand, a tight prior reduces the value of information. When the prior mean exposure to factor 1 is high, the regulator's posterior mean is anchored around this value and the posterior mean is likely to be large regardless of the information produced by the tests. New information is not very valuable and the weight of factor 1 decreases. When the prior is high enough, the regulator finds it optimal not to stress that factor at all.

Figures 4c and 4d respectively show optimal capital requirements and targeted interventions as a function of the prior mean exposure to factor 1. As expected, targeted interventions increase with the prior mean. The effect on capital requirements is non monotonic, however, because there are two opposing forces. The direct effect of high exposure is to increase capital requirements. The indirect effect comes from the increase in targeted interventions which reduce the ex-post expected exposure.

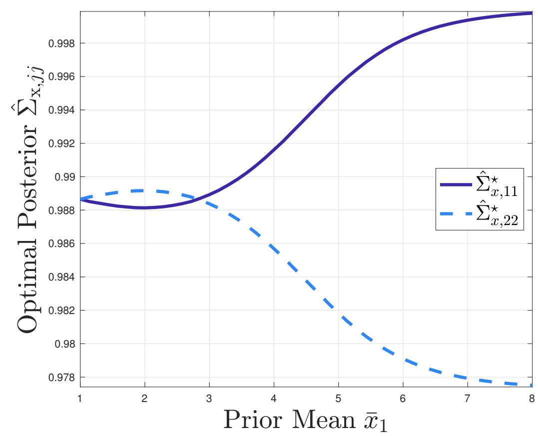
6.3 Uncertainty

Two dimensions of uncertainty shape the regulator's choice of stress scenarios: uncertainty about risk exposures and uncertainty about risk factors. The regulator intervenes more along dimensions

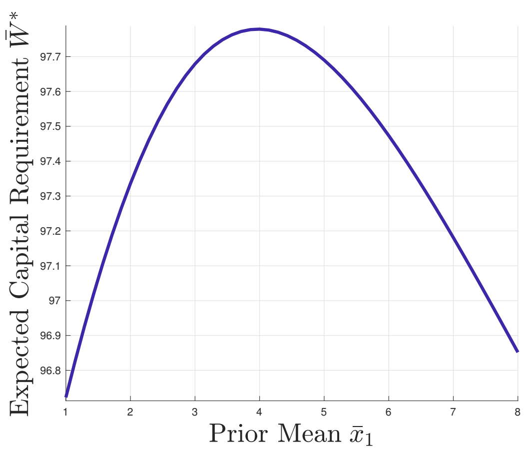
[11]In previous versions of this paper, we showed this result formally.



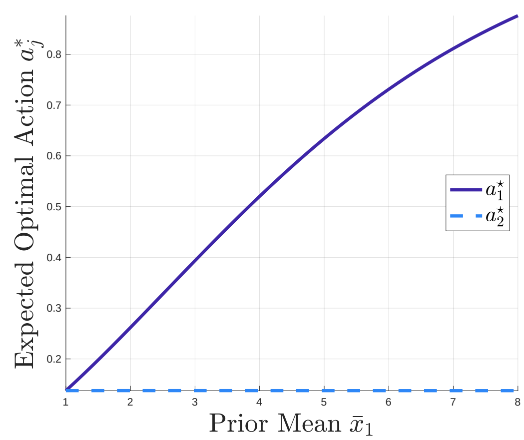
(a) Optimal scenario choice



(b) Optimal posterior variances



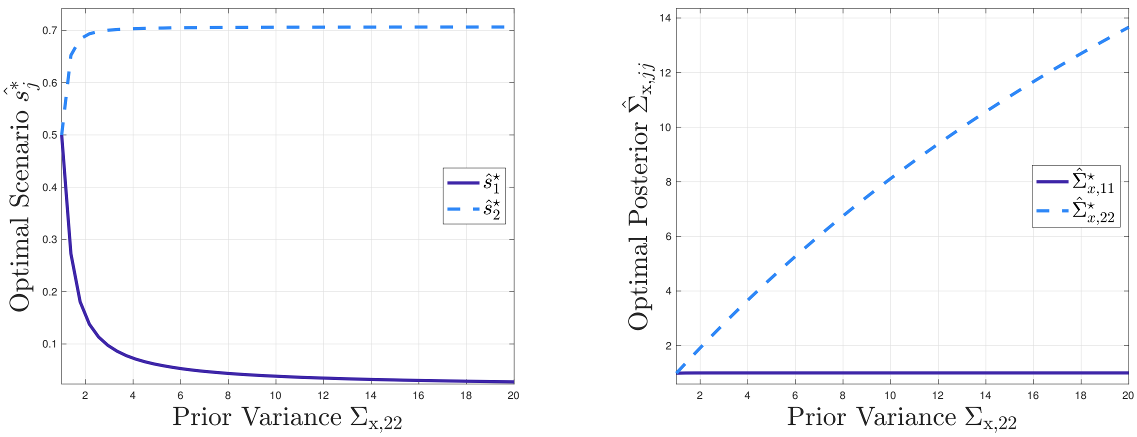
(c) Expected optimal capital requirements



(d) Expected optimal targeted action

Figure 4: Optimal scenario, information choice and expected interventions as a function of the prior mean $\bar{x}_1$.

Note: The parameters used are $M = 1$, $N = 1$, $J = 2$, $\gamma = 0.2$, $\phi^0 = [1, 1]'$, $\phi^1 = [0.1, 0.1]'$, $\bar{x} = [1, 1]'$, $\Sigma_{\boldsymbol{x}} = \boldsymbol{I}_J$, $\alpha = 5$, $\beta = 7$, $\theta = 2$, $\mathbb{E}[s] = [0, 0]'$, $\tilde{\mathbb{E}}[s] = [1, 1]'$, $\Sigma_s = \boldsymbol{I}_J$, $\tilde{\mathbb{E}}[s] = \boldsymbol{I}_J$, $\kappa = 0.1$, $\mathcal{W} = 100$, $p = 0.1$, $\mathbb{E}[\eta] = 0$, $\mathbb{E}\left[\eta^2\right] = 1$, and $\mathbb{E}[\epsilon\epsilon'] = \boldsymbol{I}_N$ . .



(a) Optimal scenario choice

(b) Optimal posterior variances

Figure 5: Optimal scenario and information choice as a function of the regulator's prior uncertainty of the exposure to factor 2, $\Sigma_{\mathbf{x},22}$.

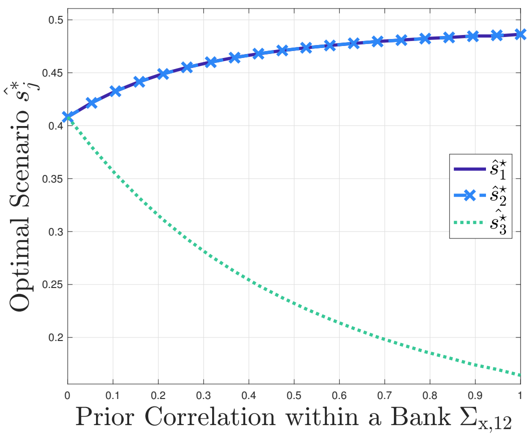
Note: Figure 5 illustrates the regulator's optimal choice of scenario and the implied posterior variance as a function of the regulator's prior variance of the exposure to factor 2, $\Sigma_{\mathbf{x},22}$. The parameters used are $N = 1$, $J = 2$, $\gamma = 0.5$, $\phi = [1, 1]'$, $\bar{x} = [1, 1]'$, $\Sigma_{\boldsymbol{x}} = \boldsymbol{I}_J$, $\alpha(M) = \sqrt{M}$, $\beta(M) = \sqrt{M}$, $M = 1$, $\theta = 1.1$, $\mathbb{E}\left[s_k^2\right] = 1$, $\mathbb{E}\left[\epsilon_{1,0}^2\right] = 1$ and $\mathbb{E}\left[\epsilon_{1,1}^2\right] = 1$.

about which she is more uncertain. When the regulator is more uncertain about exposures to risk factor j, her targeted intervention along dimension j is more responsive to the information contained in the stress test results and information is more valuable.

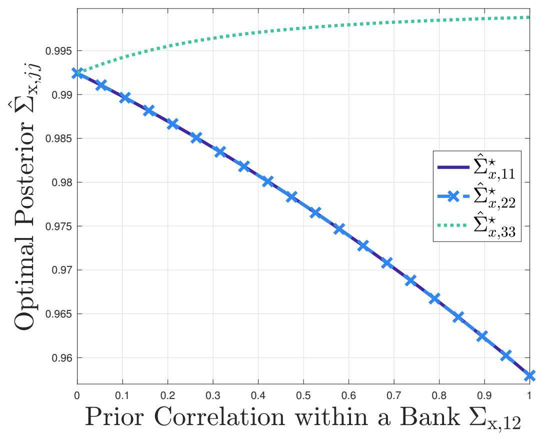
Figure 5 shows the effect of prior uncertainty regarding exposures to factor 2, $\Sigma_{\mathbf{x},22}$, on the optimal stress on factors 1 and 2. When $\Sigma_{\mathbf{x},22}$ is high the regulator stresses factor 2 to improve the efficiency of her expected interventions. The consequences of uncertainty about the risk factors themselves are similar, as shown in Figure B.1 in the Appendix.

6.4 Correlated risks

Let us now consider the role of correlations among risk exposures, within and across banks. We saw earlier in Figure (2) that correlations affect the shape of $\mathbf{\Sigma}$, the feasible set of posterior precisions. When correlations are low, stressing one risk factor convey little information about the



(a) Optimal scenario choice



(b) Optimal posterior variance

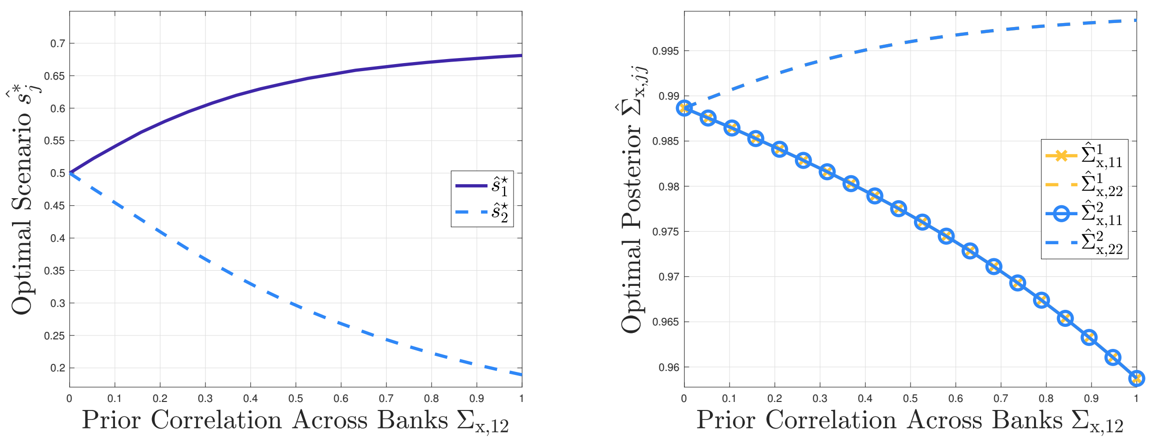
Figure 6: One bank, Three Risk Factors. Optimal scenario and information as a function of $\Sigma_{\mathbf{x},12}$, the prior correlation among exposures 1 and 2.

Note: The parameters used are $M = 1$, $N = 1$, $J = 3$, $\gamma = 1$, $\phi^0 = [1, 1]'$, $\phi^1 = [0.1, 0.1]'$, $\bar{x} = [1, 1]'$, $\Sigma_{\boldsymbol{x}} = \boldsymbol{I}_J$, $\alpha = 5$, $\beta = 7$, $\theta = 2$, $\mathbb{E}[s] = [0, 0]'$, $\tilde{\mathbb{E}}[s] = [1, 1]'$, $\Sigma_s = \boldsymbol{I}_J$, $\tilde{\mathbb{E}}[s] = \boldsymbol{I}_J$, $\kappa = 2$, $\mathcal{W} = 100$, $p = 0.1$, $\mathbb{E}[\eta] = 0$, $\mathbb{E}\left[\eta^2\right] = 1$, and $\mathbb{E}[\epsilon\epsilon'] = \boldsymbol{I}_N$.

banks other risk exposures. Correlation between risk exposures attenuates the trade-off as signals about one exposure contain some information about the others.

Panel (a) in Figure 6 plots the optimal stresses among 3 factors as a function of prior correlation among the first two exposures. Panel (b) shows that the amount of information that the regulator can learn increase with the prior correlation.

Risk exposures are also likely to be correlated among banks. Figure 7 shows the optimal stresses as a function of the prior correlation between exposures to factor 1 among two banks. When the exposures to factor 1 are correlated across banks, reported losses from one bank contain information about that bank's exposures but also about the other bank's exposure to the correlated factor. Learning about factor 1 becomes more valuable. When the correlation is strong the regulator barely learns about the other factor. The posterior variance of risk exposures to factor 2 tends towards the prior variance.



(a) Optimal scenario choice

(b) Optimal posterior variance

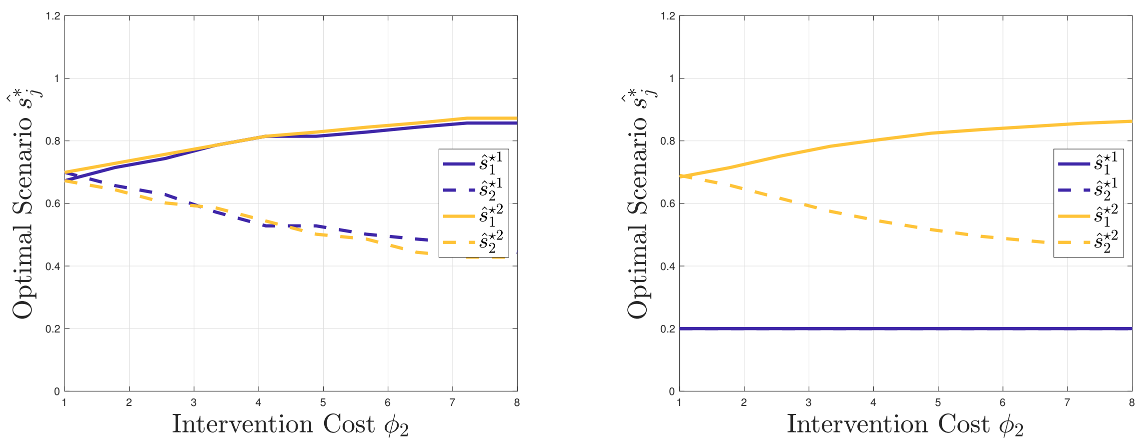
Figure 7: Two factors, Two banks. Optimal scenario and information choice as a function of the prior correlation between the bank's risk exposures to factor 1, $\Sigma^{11}_{\mathbf{x},12}$.

Note: Figure 7 illustrates the regulator's optimal choice of scenario and the implied posterior variance as a function of prior correlation in risk exposures. The parameters used are $N = 2$, $J = 2$, $\gamma = 0.5$, $\phi = [1, 1]'$, $\bar{\mathrm{x}} = \mathbf{1}_{NJ\times 1}$, $\Sigma_{\boldsymbol{x}} = \boldsymbol{I}_{NJ}$, $\alpha(M) = \sqrt{M}$, $\beta(M) = \sqrt{M}$, $M = 1$, $\theta = 3$, $\mathbb{E}\left[s_k^2\right] = 1$, $\mathbb{E}\left[\epsilon_{j,0}^2\right] = \mathbb{E}\left[\epsilon_{j,1}^2\right] = 1$ and $\mathbb{E}\left[\epsilon_{1,0}\epsilon_{2,0}\right] = \mathbb{E}\left[\epsilon_{1,1}\epsilon_{2,1}\right] = 1$.

6.5 Multiple Scenarios and Constrained Tests

The qualitative patterns described above do not change when we add more scenarios because the number of scenarios only affects the set of feasible precisions $\mathbf{\Sigma}$ and not the value of information. Considering multiple scenarios is important, however, because it allows us to connect our model to actual stress tests where one scenario is used to set minimal capital requirements under *physically plausible* adverse circumstances.

Consider, then, a model with two risk factors and two scenarios. We analyze and compare two regulatory problems. The first problem (panel a in Figure 8) is the unconstrained scenario design studied earlier where the regulator freely chooses both scenarios to optimize her learning. The second problem (panel b in Figure 8) captures an important feature of actual stress tests. In practice one scenario is almost always used to set capital requirements under a *plausible* adverse scenario. In the second problem, we therefore force one scenario to be the adverse scenario $\hat{\mathrm{s}}^1 = \tilde{\mathrm{s}}$,



(a) Unconstrained Optimal Scenarios

(b) One Scenario Constrained to $\hat{\mathrm{s}}^1 = \tilde{\mathrm{s}}$

Figure 8: Scenario Design With One Scenario Used for Capital Requirements

Note: We assume $\sigma_\epsilon^2 = \alpha^2 + \beta^2 ||s||^{1+\theta}$. The parameters used are $M = 2$, $N = 1$, $J = 2$, $\gamma = 3$, $\phi^0 = [1, 1]'$, $\phi^1 = [0, 0]'$, $\bar{x} = [1, 1]'$, $\Sigma_{\boldsymbol{x}} = \boldsymbol{I}_J$, $\alpha = 1.3$, $\beta = 1$, $\theta = 2$, $\mathbb{E}[s] = [0, 0]'$, $\tilde{\mathbb{E}}[s] = [0.2, 0.2]'$, $\Sigma_s = \boldsymbol{I}_J$, $\tilde{\mathbb{E}}[s] = \boldsymbol{I}_J$, $\kappa = 0.1$, $\mathcal{W} = 100$, $p = 0.1$, $\mathbb{E}[\eta] = 0$, $\mathbb{E}\left[\eta^2\right] = 1$, $\Sigma_{x,12} = 0.3$,and $\mathbb{E}[\epsilon\epsilon'] = \boldsymbol{I}_N$.

and we let the regulator optimize over the second scenario.

Panel (a) shows the unconstrained design. We keep one intervention cost at its baseline value $\phi_1 = 1$ and we vary ϕ_2. When $\phi_2 = 1$ the solution is essentially symmetric: $\hat{s}_j^{*,m} = s$ for $m = 1, 2$ and $j = 1, 2$. The planner wants to learn equally about all risk exposures and stresses equally all the factors. As ϕ_2 increases, the planner specializes her learning to focus on factor 2. She is less worried about factor 1 because she can use a targeted intervention at a relatively low cost in that dimension.

In Panel (b), the planner is constrained to set $\hat{s}_j^{*,1} = \tilde{s}_j$. This has several implications. First, the stress value is constrained to be "plausible", which here means $\|s\| = \|\tilde{s}\|$. This reflects the fact that the plausible scenario does not optimize the signal to noise ratio but rather must deliver a plausible capital requirement. Second, specialization happens somewhat more slowly.

Figure 9 shows the welfare costs of fixing one scenario to be "plausible". Welfare decreases with intervention costs, as expected. What is more interesting is the welfare cost of using up one scenario to set capital requirements. The punchline of Figure 9 is that the welfare losses are

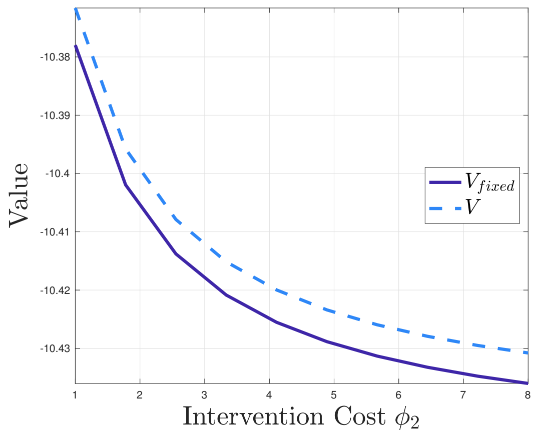


Figure 9: Welfare under Constrained and Unconstrained Scenario Designs

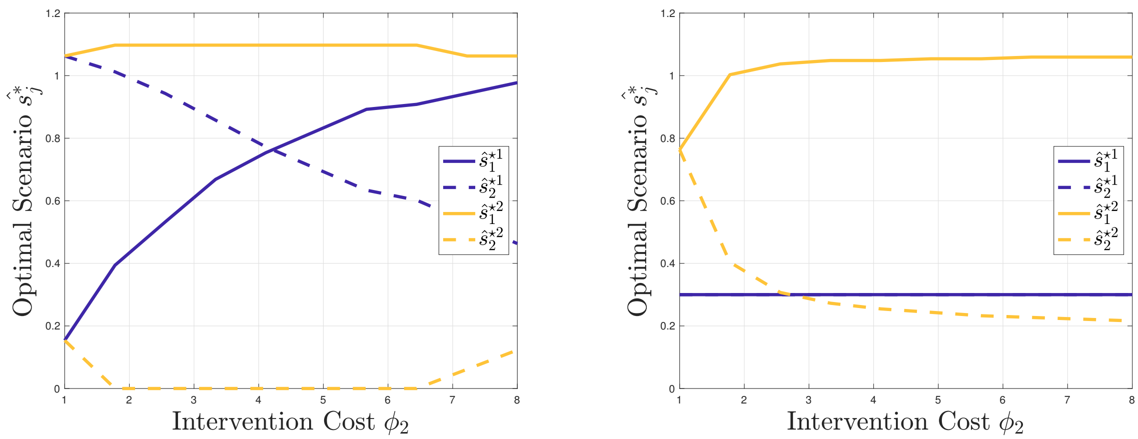
Note: The parameters used are $M = 2$, $N = 1$, $J = 2$, $\gamma = 1.3$, $\phi^0 = [1, 1]'$, $\phi^1 = [0, 0]'$, $\bar{x} = [1, 1]'$, $\Sigma_{\boldsymbol{x}} = \boldsymbol{I}_J$, $\alpha = 1.3$, $\beta = 1$, $\theta = 2$, $\mathbb{E}[s] = [0, 0]'$, $\tilde{\mathbb{E}}[s] = [0.2, 0.2]'$, $\Sigma_s = \boldsymbol{I}_J$, $\tilde{\mathbb{E}}[s] = \boldsymbol{I}_J$, $\kappa = 0.1$, $\mathcal{W} = 100$, $p = 0.1$, $\mathbb{E}[\eta] = 0$, $\mathbb{E}\left[\eta^2\right] = 1$, $\Sigma_{x,12} = 0.3$, and $\mathbb{E}[\epsilon\epsilon'] = \boldsymbol{I}_N$.

moderate as long as the interventions costs are not very high. This suggests that, in practice, a stress test design where only one scenario is used for exploring and extracting information is not far from the optimum.

Figures 10 and 11 perform the same analysis in an economy where the optimal scenarios are asymmetric. One scenario is used to explore factor 1 and the other to explore factor 2. In that case the constrained design is rather different from the unconstrained one. The welfare losses are still moderate compared to the losses stemming from higher intervention costs.

7 Discussion and Conclusion

Despite the growing importance of stress testing for financial regulation and risk management, economists still lack a theory of the design of stress scenarios. We model stress testing as a learning mechanism and show how to map the scenario choice problem into an information acquisition problem. In this framework, we derive optimal scenarios and characterize how their design depends on the cost of interventions, the prior beliefs of the regulator, the precision of regulatory information, the uncertainty about the risk factors, and the presence of systemic risk



(a) Optimal scenario choice with both scenarios chosen freely (b) Optimal scenario choice with one fixed scenario

Figure 10: Optimal scenario choices as a function of the intervention cost to reduce the exposure to factor 2 and the expected exposure to factor 1, ϕ_2. Panel a shows the case in which both scenarios are chosen freely and panel b shows the case in which one scenario is fixed to the expected macro state in distress.

Note: We assume $\sigma_\epsilon^2 = \alpha^2 + \beta^2 ||s||^{1+\theta}$. The parameters used are $M = 2$, $N = 1$, $J = 2$, $\gamma = 1.3$, $\phi^0 = [1, 1]'$, $\phi^1 = [0, 0]'$, $\bar{x} = [1, 1]'$, $\Sigma_{\boldsymbol{x}} = \boldsymbol{I}_J$, $\alpha = 1.3$, $\beta = 1$, $\theta = 1.2$, $\mathbb{E}[s] = [0, 0]'$, $\tilde{\mathbb{E}}[s] = [0.3, 0.3]'$, $\Sigma_s = \boldsymbol{I}_J$, $\tilde{\mathbb{E}}[s] = \boldsymbol{I}_J$, $\kappa = 0.1$, $\mathcal{W} = 100$, $p = 0.1$, $\mathbb{E}[\eta] = 0$, $\mathbb{E}\left[\eta^2\right] = 1$, and $\mathbb{E}[\epsilon\epsilon'] = \boldsymbol{I}_N$.

factors.

Our approach is consistent with the general principles of current policies implement in various jurisdiction, but it has the advantage that our optimal scenarios are not arbitrary. For example, the current policy on stress scenario design in the U.S. allows for the stress scenarios to "follow either a recession approach, a probabilistic approach, or an approach based on historical experiences."[12] These concepts are somewhat vague and have generated much discussion among banks and regulators. Some commentators argue that scenarios should be predictable while others advocate a flexible design to accommodate emerging risks and changing exposures. Our learning approach shows how to incorporate this goals in the design of the stress scenarios.

[12]See 12 CFR Part 252 Appendix A.

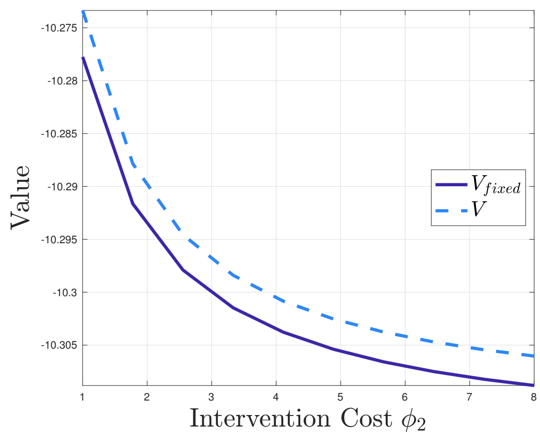


Figure 11: Welfare under Constrained and Unconstrained Scenario Designs

Note: The parameters used are $M = 2$, $N = 1$, $J = 2$, $\gamma = 1.3$, $\phi^0 = [1, 1]'$, $\phi^1 = [0, 0]'$, $\bar{x} = [1, 1]'$, $\Sigma_{\boldsymbol{x}} = \boldsymbol{I}_J$, $\alpha = 1.3$, $\beta = 1$, $\theta = 21.$, $\mathbb{E}[s] = [0, 0]'$, $\tilde{\mathbb{E}}[s] = [0.3, 0.3]'$, $\Sigma_s = \boldsymbol{I}_J$, $\tilde{\mathbb{E}}[s] = \boldsymbol{I}_J$, $\kappa = 0.1$, $\mathcal{W} = 100$, $p = 0.1$, $\mathbb{E}[\eta] = 0$, $\mathbb{E}\left[\eta^2\right] = 1$, and $\mathbb{E}[\epsilon\epsilon'] = \boldsymbol{I}_N$.

Our comparative static exercises above shed light on the optimal stress scenario design in the presence of systemic factor, in times of distress, over time, and its relation to capital requirements.

Correlated Exposures and Systemic Factors Stress tests are widely used as a risk management tool. Regulatory stress testing in particular focuses on assessing the resilience of the financial system as a whole. In this context, risk factors that lead to correlated losses among banks are of particular interest. Our analysis on correlated exposures among banks suggests that the optimal stress scenario would put relatively more weight on these factors. Moreover, if the correlation of the banks' exposures to some factor is high enough, the optimal stress scenario may put weight only on these systemic factors.

Scenario design in times of uncertainty Uncertainty about the evolution of macroeconomic variables and the regulator's attitude towards this uncertainty are crucial in determining the optimal stress scenario. From our analysis above, it follows that the optimal stress test design calls for stressing more uncertain factors more. Moreover, our model implies that an across the board increase in uncertainty or in the regulator's risk aversion can lead to the optimal stress scenario putting more weight on fewer factors. This implies a positive correlation between the severity of the stress scenarios and the uncertainty in the economy.

Evolution of stress scenarios Changes in the information set of the regulator will lead to different scenario choices. The more the regulator knows about the banks' exposures to a particular factor, the less she will choose to stress that factor in the optimal stress scenario. Since the regulator learns more about the exposures to factors that are stressed more in the optimal scenario, sequences of stress tests may optimally put weight on different factors each period. These changes resemble experimentation but they are not driven by changes in the regulator's objective nor by changes in the expected evolution of risk factors. Instead they may simply reflect the evolution of the regulator's information about banks' losses.

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Appendix

The Appendix contains some auxiliary calculation for formulas in the text. It needs to be completed.

A Proofs

A.1 Learning from stress tests

Proof of Lemma 1

When $N = 1$, $M = 1$, and $J = 2$, the Kalman gain is given by

$$K = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \begin{bmatrix} \hat{s}_1 \\ \hat{s}_2 \end{bmatrix} \left(\begin{bmatrix} \hat{s}_1 & \hat{s}_2 \end{bmatrix} \left(\begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \right) \begin{bmatrix} \hat{s}_1 \\ \hat{s}_2 \end{bmatrix} + \sigma_{\hat{\varepsilon}}^2 (\hat{\mathrm{s}}) \right)^{-1},$$

which implies

$$k_1 = \frac{\sigma_1^2 \hat{s}_1 + \rho\sigma_1\sigma_2\hat{s}_2}{\sigma_1^2\hat{s}_1^2 + 2\rho\sigma_1\sigma_2\hat{s}_1\hat{s}_2 + \sigma_2^2\hat{s}_2^2 + \sigma_{\hat{\epsilon}}^2(\hat{\mathrm{s}})}, \quad \text{and}$$
$$k_2 = \frac{\sigma_2^2 \hat{s}_2 + \rho\sigma_1\sigma_2\hat{s}_1}{\sigma_1^2\hat{s}_1^2 + 2\rho\sigma_1\sigma_2\hat{s}_1\hat{s}_2 + \sigma_2^2\hat{s}_2^2 + \sigma_{\hat{\epsilon}}^2(\hat{\mathrm{s}})}.$$

Moreover,

$$\Sigma_{\hat{\mathbf{x}}}(j,j) = k_j \hat{s}_j \left(\sigma_j^2 + \frac{\hat{s}_h}{\hat{s}_j} \rho\sigma_1\sigma_2 \right),$$

where

$$k_j\hat{s}_j = \frac{\sigma_j^2\hat{s}_j^2 + \rho\sigma_1\sigma_2\hat{s}_2\hat{s}_1}{\sigma_1^2\hat{s}_1^2 + 2\rho\sigma_1\sigma_2\hat{s}_1\hat{s}_2 + \sigma_2^2\hat{s}_2^2 + \sigma_{\hat{\epsilon}}^2(\hat{\mathrm{s}})} = 1 - \frac{\rho\sigma_1\sigma_2\hat{s}_1\hat{s}_2 + \sigma_h^2\hat{s}_h^2 + \sigma_{\hat{\epsilon}}^2}{\sigma_1^2\hat{s}_1^2 + 2\rho\sigma_1\sigma_2\hat{s}_1\hat{s}_2 + \sigma_2^2\hat{s}_2^2 + \sigma_{\hat{\epsilon}}^2(\hat{\mathrm{s}})}.$$

Note that

$$\frac{\partial k_j}{\partial \sigma_{\hat{\epsilon}}^2(\hat{\mathrm{s}})} = -\frac{k_j}{\sigma_1^2\hat{s}_1^2 + 2\rho\sigma_1\sigma_2\hat{s}_1\hat{s}_2 + \sigma_2^2\hat{s}_2^2 + \sigma_{\hat{\epsilon}}^2(v)} \leq 0$$

and

$$\frac{\partial \Sigma_{\hat{\mathbf{x}}}(j,j)}{\partial \sigma_{\hat{\epsilon}}^2(\hat{\mathrm{s}})} = -\frac{\Sigma_{\hat{x}}(j,j)}{\sigma_1^2\hat{s}_1^2 + 2\rho\sigma_1\sigma_2\hat{s}_1\hat{s}_2 + \sigma_2^2\hat{s}_2^2 + \sigma_{\hat{\epsilon}}^2(\hat{\mathrm{s}})} \leq 0$$

Since

$$\hat{\epsilon}(\hat{\mathrm{s}}) = \alpha\epsilon_0 + \beta\left(\|\hat{\mathrm{s}}\|^{\frac{1}{2}} + \|\hat{\mathrm{s}}\|^{1+\theta} \right)\epsilon_1,$$

we have

$$\sigma_{\hat{\epsilon}}^2(\hat{\mathrm{s}}) = \alpha^2 + \beta^2\left(\left(\hat{s}_1^2 + \hat{s}_2^2\right)^{\frac{1}{2}} + \left(\hat{s}_1^2 + \hat{s}_2^2\right)^{1+\theta} \right),$$

which is increasing in $|s_j|$ for $j = 1, 2$. Therefore, more extreme scenarios decrease the amount of learning. The effect of an increase in noise on the amount of learning is negligible close to the baseline, i.e.,

$$\lim_{|\hat{s}_j| \to 0} \frac{\partial \Sigma_{\hat{\mathbf{x}}}(j,j)}{\partial \sigma_{\hat{\epsilon}}^2(\hat{s})} \frac{\partial \sigma_{\hat{\epsilon}}^2(\hat{\mathrm{s}})}{\partial |\hat{s}_j|} = -\frac{\Sigma_{\hat{\mathbf{x}}}(j,j)\, \beta^2 \left(\left(\hat{s}_1^2 + \hat{s}_2^2\right)^{-\frac{1}{2}} + (1+\theta)\left(\hat{s}_1^2 + \hat{s}_2^2\right)^{\theta} 2 \right) |\hat{s}_j|}{\sigma_1^2 \hat{s}_1^2 + 2\rho\sigma_1\sigma_2\hat{s}_1\hat{s}_2 + \sigma_2^2\hat{s}_2^2 + \sigma_{\hat{\epsilon}}^2(\hat{\mathrm{s}})} = 0 \quad \forall j = 1, 2. \quad \text{(A.1)}$$

Moreover, the direct effect of a more extreme scenario on the amount of learning is given by

$$\frac{\partial \Sigma_{\hat{\mathbf{x}}}(j,j)}{\partial |\hat{s}_j|} = \frac{\partial (k_j \hat{s}_j)}{\partial |\hat{s}_j|} \sigma_j^2 + \frac{\partial k_j}{\partial |\hat{s}_j|} \hat{s}_h \rho \sigma_1 \sigma_2 \quad \forall j, h = 1, 2,\ j \neq h.$$

When $\rho = 0$,we have

$$\frac{\partial \Sigma_{\hat{\mathbf{x}}}(j,j)}{\partial |\hat{s}_j|} = \frac{\partial (k_j \hat{s}_j)}{\partial |\hat{s}_j|} \sigma_j^2 = \frac{\sigma_h^2 \hat{s}_h^2 + \sigma_{\hat{\epsilon}}^2}{\left(\sigma_1^2 \hat{s}_1^2 + \sigma_2^2 \hat{s}_2^2 + \sigma_{\hat{\epsilon}}^2(\hat{\mathrm{s}})\right)^2} \sigma_j^4 2 |\hat{s}_j| \geq 0.$$

Then, using Equation A.1 we have that

$$\Sigma_{\hat{\mathbf{x}}}(j,j) = \frac{\sigma_j^2 \hat{s}_j^2}{\sigma_1^2 \hat{s}_1^2 + \sigma_2^2 \hat{s}_2^2 + \alpha^2 + \beta^2 \left(\hat{s}_1^2 + \hat{s}_2^2\right)^{1+\theta}}$$

$$\begin{aligned}
\frac{d\Sigma_{\hat{\mathbf{x}}}(j,j)}{d\hat{s}_j^2} &= \frac{\sigma_j^2 \left(\sigma_h^2 \hat{s}_h^2 + \alpha^2 + \beta^2 \left(\hat{s}_1^2 + \hat{s}_2^2\right)^{1+\theta}\right) - \sigma_j^2 \hat{s}_j^2 \beta^2 (1+\theta) \left(\hat{s}_1^2 + \hat{s}_2^2\right)^{\theta}}{\left(\sigma_1^2 \hat{s}_1^2 + \sigma_2^2 \hat{s}_2^2 + \alpha^2 + \beta^2 \left(\hat{s}_1^2 + \hat{s}_2^2\right)^{1+\theta}\right)^2} \\
&= \frac{\sigma_j^2 \left(\frac{\sigma_h^2 \hat{s}_h^2 + \alpha^2}{\beta^2 \left(\hat{s}_1^2 + \hat{s}_2^2\right)^{1+\theta}} + 1 \right) - \frac{\sigma_j^2 \hat{s}_j^2 (1+\theta)}{\left(\hat{s}_1^2 + \hat{s}_2^2\right)}}{\left(\frac{\sigma_1^2 \hat{s}_1^2 + \sigma_2^2 \hat{s}_2^2 + \alpha^2}{\beta^2 \left(\hat{s}_1^2 + \hat{s}_2^2\right)^{1+\theta}} + 1 \right)^2}
\end{aligned}$$

$$\lim_{\hat{s}_j \to 0} \frac{d\Sigma_{\hat{\mathbf{x}}}(j,j)}{d\hat{s}_j^2} > 0 \quad \text{and} \quad \lim_{\hat{s}_j \to \infty} \frac{d\Sigma_{\hat{\mathbf{x}}}(j,j)}{d\hat{s}_j^2} = -\theta \sigma_j^2 < 0$$

When $\rho > 0$, using Equation (**??**) and the definition of $\sigma_{\hat{\epsilon}}^2(\hat{s})$, we have that

$$k_1 \hat{s}_1 = \frac{\sigma_1^2 \hat{s}_1^2 + \rho\sigma_1\sigma_2\hat{s}_2\hat{s}_1}{\sigma_1^2 \hat{s}_1^2 + 2\rho\sigma_1\sigma_2\hat{s}_1\hat{s}_2 + \sigma_2^2\hat{s}_2^2 + \alpha^2 + \beta^2 \left(\hat{s}_1^2 + \hat{s}_2^2\right)^{1+\theta}}$$

and

$$\Sigma_{\hat{\mathbf{x}}}(j,j) = \frac{\sigma_j^2 \hat{s}_j^2 + \rho\sigma_1\sigma_2\hat{s}_2\hat{s}_1 + \rho\sigma_j^2 \hat{s}_j \hat{s}_h + (\rho\sigma_1\sigma_2)^2 \hat{s}_h^2}{\sigma_1^2 \hat{s}_1^2 + 2\rho\sigma_1\sigma_2\hat{s}_1\hat{s}_2 + \sigma_2^2\hat{s}_2^2 + \alpha^2 + \beta^2 \left(\hat{s}_1^2 + \hat{s}_2^2\right)^{1+\theta}}.$$

Proof of Proposition 2

The Kalman filter in Equations (7) and (8) imposes restrictions on the set of posterior variances that can be attained. More specifically, the Kalman filter maps the $J \times M$ elements in the stress scenarios $\hat{S}$ to the the $\frac{NJ(NJ+1)}{2}$ elements of the posterior precision $\hat{\Sigma}_{\mathbf{x}}$ from the set $\overline{\mathbf{\Sigma}}$. Moreover, the regulator's choice will always be on the frontier of the feasibility set, given by $\overline{\mathbf{\Sigma}} \equiv \left\{K\left(\hat{S}_{\omega}\right)\hat{S}'_{\omega}\Sigma_{\boldsymbol{x}} \quad \text{for all} \quad \omega \in [0,1]^{N\times J} \quad \text{with} \quad \sum_{h=1}^{N\times J}\omega_h\right\}$, where

$$\hat{S}_{\omega} \equiv \arg\max_{\hat{S}} \omega' K\left(\hat{S}\right)\hat{S}'\Sigma_{\boldsymbol{x}}\omega.$$

Given our assumptions on $\hat{\epsilon}_i\left(\hat{\mathrm{s}}, M\right)$, $\hat{S}_{\omega}$ is unique. Hence, since the objective function of the regulator depends on $\hat{S}$ only through the posterior variance, the regulator's scenario choice problem can be cast in term of choosing $\hat{\Sigma}_{\mathbf{x}}$.

When $M < N$,as long as all risk dimensions are spanned, choosing the $J \times M$ elements in $\hat{S}$ is equivalent to choosing $J \times M$ elements of the posterior precision $\hat{\Sigma}_{\boldsymbol{x}}$. Without loss of generality, one can focus on the posterior variances of the risk exposures of M banks from the set $\overline{\mathbf{\Sigma}}_M \equiv \left\{K\left(\hat{S}_{\omega}\right)\hat{S}'_{\omega}\Sigma_{\boldsymbol{x}} \quad \text{for all} \quad \omega \in [0,1]^{N\times J} \quad \text{with} \quad \sum_{m=1}^{M}\sum_{j=1}^{J}\omega_{(j-1)J+m}\right\}$.

A.2 Taking action

Proof of Lemma 2

Under linear quadratic preferences, the first order condition that characterizes the optimal capital requirement is

$$K'\left(\bar{W}^{\star}\right) = 1 + p\gamma\left(\mathcal{W} - \bar{W}^{\star} + \tilde{\mathbb{E}}\left[\mathrm{s}\right]\cdot\left(\sum_{i=1}^{N}\left(\mathbf{1} - \mathrm{a}_i\right)\circ\hat{\mathrm{x}}_i\right)\right).$$

Proof of Lemma 4

Under linear quadratic preferences, the first order condition that characterizes the regulator's optimal targeted intervention policy is

$$\begin{aligned}
\frac{\partial\mathcal{C}\left(\mathbf{a}^{\star}\right)}{\partial a_{i,j}} &= \hat{x}_{i,j}\mathbb{E}\left[s_j\right] - p\gamma\tilde{\mathbb{E}}\left[x_{i,j}s_j\left(W - \mathcal{W}\right) \mid \mathscr{S}\right] \\
&= \hat{x}_{i,j}\mathbb{E}\left[s_j\right]\mathbb{E}\left[U'\left(W\right) \mid \mathscr{S}\right] - p\gamma\tilde{\mathbb{E}}\left[\left(x_{i,j}s_j - \hat{x}_{i,j}\mathbb{E}\left[s_j\right]\right)\left(W - \mathcal{W}\right) \mid \mathscr{S}\right] \\
&= \hat{x}_{i,j}\mathbb{E}\left[s_j\right]\mathcal{K}'\left(\bar{W}\right) - p\gamma\hat{x}_{i,j}\left(\tilde{\mathbb{E}}\left[s_j\right] - \mathbb{E}\left[s_j\right]\right)\tilde{\mathbb{E}}\left[\left(W - \mathcal{W}\right) \mid \mathscr{S}\right] \\
&+ p\gamma\tilde{\mathbb{E}}\left[\left(x_{i,j}s_j - \hat{x}_{i,j}\tilde{\mathbb{E}}\left[s_j\right]\right)\sum_{h=1}^{N}\sum_{l=1}^{J}\left(1 - a_{h,l}\right)x_{h,l}s_l \mid \mathscr{S}\right] \\
&= \hat{x}_{i,j}\mathbb{E}\left[s_j\right]\mathcal{K}'\left(\bar{W}\right) - p\gamma\hat{x}_{i,j}\left(\tilde{\mathbb{E}}\left[s_j\right] - \mathbb{E}\left[s_j\right]\right)\tilde{\mathbb{E}}\left[\left(W - \mathcal{W}\right) \mid \mathscr{S}\right] + p\gamma\tilde{\mathbb{C}ov}\left(x_{i,j}s_j, \sum_{h=1}^{N}\sum_{l=1}^{J}\left(1 - a_{h,l}\right)x_{h,l}s_l\right) \\
&= \hat{x}_{i,j}\mathbb{E}\left[s_j\right]\mathcal{K}'\left(\bar{W}\right) - p\gamma\hat{x}_{i,j}\left(\tilde{\mathbb{E}}\left[s_j\right] - \mathbb{E}\left[s_j\right]\right)\tilde{\mathbb{E}}\left[\left(W - \mathcal{W}\right) \mid \mathscr{S}\right] + p\gamma\sum_{h=1}^{N}\sum_{l=1}^{J}\left(1 - a_{h,l}\right)\tilde{\mathbb{C}ov}\left(x_{i,j}s_j, x_{h,l}s_l\right).
\end{aligned}$$

Using the FOC for $\bar{W}$ we have

$$\bar{W}^{\star} - \mathcal{W} = -\frac{\kappa}{p\gamma} + \sum_{i=1}^{N}\sum_{j=1}^{J}\left(1-a_{i,j}\right)\hat{x}_{i,j}\tilde{\mathbb{E}}\left[s_j\right]$$

which implies

$$\begin{aligned}
\tilde{\mathbb{E}}\left[\left(W-\mathcal{W}\right) \mid \mathscr{S}\right] &= \tilde{\mathbb{E}}\left[\left(\bar{W}^{\star} - \bar{\eta} - \sum_{i=1}^{N}\sum_{j=1}^{J}\left(1-a_{i,j}\right)x_{i,j}s_j - \mathcal{W}\right) \mid \mathscr{S}\right] \\
&= \tilde{\mathbb{E}}\left[-\frac{\kappa}{p\gamma} + \sum_{i=1}^{N}\sum_{j=1}^{J}\left(1-a_{i,j}\right)\hat{x}_{i,j}\tilde{\mathbb{E}}\left[s_j\right] - \sum_{i=1}^{N}\sum_{j=1}^{J}\left(1-a_{i,j}\right)x_{i,j}s_j \mid \mathscr{S}\right] \\
&= -\frac{\kappa}{p\gamma}.
\end{aligned}$$

Moreover, using that

$$\mathbb{E}\left[\mathrm{s}\right] = 0 \quad \text{and} \quad \mathrm{s}|\,\mathscr{D} = 1 \sim N\left(\tilde{\mathrm{s}}, \tilde{\Sigma}_{\mathrm{s}}\right),$$

the first-order conditions for the targeted actions become

$$\frac{\partial \mathcal{C}\left(\mathbf{a}^{\star}\right)}{\partial a_{i,j}} = \kappa\gamma\hat{x}_{i,j}\tilde{\mathbb{E}}\left[s_j\right] + p\gamma\sum_{h=1}^{N}\sum_{l=1}^{J}\left(1-a_{h,l}\right)\tilde{\mathbb{C}ov}\left(x_{i,j}s_j, x_{h,l}s_l\right) \quad \forall i,j. \tag{A.2}$$

Proof of Proposition 3

Rewriting the system in Equations (A.2) in vector form for $\mathbf{a}^{\star}$, gives

$$\mathbf{a}^{\star} = \left(\overline{\Phi} + p\gamma\tilde{\mathbb{V}}\right)^{-1}\left(\kappa\left(\mathbf{1}_{N\times 1}\otimes\tilde{\mathrm{s}}\right)\circ\hat{\mathbf{x}} + p\gamma\tilde{\mathbb{V}}\mathbf{1}_{NJ\times 1}\right),$$

where

$$\tilde{\mathbb{V}} = \tilde{\mathbb{C}ov}\left[\left(\mathbf{1}_{N\times 1}\otimes\mathrm{s}\right)\circ\mathbf{x} \mid \mathscr{S}\right] = \left(\left(\mathbf{1}_{\boldsymbol{N}\times\boldsymbol{N}}\otimes\tilde{\Sigma}_{\mathrm{s}}\right)\circ\hat{\Sigma}_{\mathbf{x}}\right) + \left(\mathbf{1}_{\boldsymbol{N}\times\boldsymbol{N}}\otimes\tilde{\Sigma}_{\mathrm{s}}\right)\circ\left(\hat{\mathbf{x}}\hat{\mathbf{x}}'\right) + \left(\mathbf{1}_{N\times N}\otimes\tilde{\mathrm{s}}\tilde{\mathrm{s}}'\right)\circ\hat{\Sigma}_{\mathbf{x}},$$

which proves the result.

A.3 Optimal scenario choice

The objective function of the regulator is given by

$$\begin{aligned}
\mathbb{E}_{\hat{\mathbf{x}}}\left[O\right] &= \mathbb{E}_{\hat{\mathbf{x}}}\left[\mathbb{E}_{\mathrm{s},\eta,\mathrm{x}}\left[U\left(W\left(\mathbf{a}^{\star},\bar{W}^{\star}\right)\right)|\mathscr{S}\right] - \mathcal{C}\left(\mathbf{a}^{\star}\right) - \mathcal{K}\left(\bar{W}^{\star}\right)\right] \\
&= \mathbb{E}_{\hat{\mathbf{x}}}\left[\mathbb{E}_{\mathrm{s},\eta,\mathrm{x}}\left[W\left(\mathbf{a}^{\star},\bar{W}^{\star}\right)|\mathscr{S}\right] - \frac{p\gamma}{2}\tilde{\mathbb{E}}_{\mathrm{s},\eta,\mathrm{x}}\left[\left(W\left(\mathbf{a}^{\star},\bar{W}^{\star}\right) - \mathcal{W}\right)^2|\mathscr{S}\right] - \mathcal{C}\left(\mathbf{a}^{\star}\right) - \left(1+\kappa\right)\bar{W}^{\star}\right].
\end{aligned}$$

Proof of Proposition 5

The optimal capital requirement in the absence of targeted actions is

$$\bar{W}^{\star} = \mathcal{W} - \frac{\kappa}{p\gamma} + \left(\mathbf{1}_{N\times 1} \otimes \tilde{\mathbb{E}}\left[\mathrm{s}\right]\right)' \hat{\mathbf{x}}.$$

The total wealth of the regulator is then

$$W = \mathcal{W} - \bar{\eta} - \frac{\kappa}{p\gamma} - \left(\left(\mathbf{1}_{N\times 1} \otimes \mathrm{s}\right)' \mathbf{x} - \left(\mathbf{1}_{N\times 1} \otimes \tilde{\mathbb{E}}\left[\mathrm{s}\right]\right)' \hat{\mathbf{x}} \right),$$

which implies

$$\mathbb{E}\left[W\left(\mathbf{a}^{\star}, \bar{W}^{\star}\right) | \mathscr{S}\right] = \bar{W}^{\star}$$

and

$$\tilde{\mathbb{E}}\left[\left(W\left(\mathbf{a}^{\star}, \bar{W}^{\star}\right) - \mathcal{W}\right)^2 | \mathscr{S}\right] = \mathbb{V}ar\left[\bar{\eta}\right] + \mathbf{1}_{1\times NJ} \tilde{\mathbb{V}} \mathbf{1}_{NJ\times 1},$$

where $\tilde{\mathbb{V}} \equiv \tilde{\mathbb{C}ov}\left[\left(\mathbf{1}_{N\times 1} \otimes \mathrm{s}\right)' \mathbf{x} \,|\mathscr{S}\right]$. Then, the objective function of the regulator becomes

$$\begin{aligned}
\mathbb{E}\left[O\right] &= \mathbb{E}\left[\mathbb{E}\left[W\left(\mathbf{a}^{\star}, \bar{W}^{\star}\right) | \mathscr{S}\right] - \frac{p\gamma}{2} \tilde{\mathbb{E}}\left[\left(W\left(\mathbf{a}^{\star}, \bar{W}^{\star}\right) - \mathcal{W}\right)^2 | \mathscr{S}\right] - (1+\kappa)\, \bar{W}^{\star}\right] \\
&= -\frac{p\gamma}{2} \mathbb{V}ar\left[\bar{\eta}\right] - \kappa \left(\mathcal{W} - \frac{\kappa}{p\gamma} + \left(\mathbf{1}_{N\times 1} \otimes \tilde{\mathbb{E}}\left[\mathrm{s}\right]\right)' \bar{\mathbf{x}}\right) - \frac{p\gamma}{2} \mathbf{1}_{1\times NJ} \mathbb{E}\left[\tilde{\mathbb{V}}\right] \mathbf{1}_{NJ\times 1}.
\end{aligned}$$

Note that the only term that the regulator can affect by choosing a stress scenario, or alternatively a posterior covariance matrix, is $\tilde{\mathbb{V}}$. Therefore, the regulator's objective in the design problem is to minimize her residual uncertainty $\mathbb{E}\left[\tilde{\mathbb{V}}\right]$.

Proof of Corollary ??

The proof follows by noticing that the residual uncertainty of the regulator can be written as

$$\begin{aligned}
\mathbb{E}\left[\tilde{\mathbb{V}}\right] \equiv \mathbb{E}\left[\tilde{\mathbb{C}ov}\left[\left(\mathbf{1}_{N\times 1} \otimes \mathrm{s}\right)' \mathbf{x} \,|\mathscr{S}\right]\right] &= \left(\mathbf{1}_{N\times N} \otimes \tilde{\Sigma}_s\right) \circ \mathbb{E}\left[\mathbb{E}\left[\mathbf{x}\mathbf{x}' \mid \mathscr{S}\right]\right] + \left(\mathbf{1}_{N\times N} \otimes \left(\tilde{\Sigma}_s + \tilde{\mathrm{s}}\tilde{\mathrm{s}}'\right)\right) \circ \hat{\Sigma}_{\mathbf{x}}, \\
&= \left(\mathbf{1}_{N\times N} \otimes \tilde{\Sigma}_s\right) \circ \left(\Sigma_{\mathbf{x}} + \bar{\mathbf{x}}\bar{\mathbf{x}}'\right) + \left(\mathbf{1}_{N\times N} \otimes \tilde{\mathrm{s}}\tilde{\mathrm{s}}'\right) \circ \hat{\Sigma}_{\mathbf{x}}
\end{aligned}$$

which is separable in $\bar{\mathbf{x}}\bar{\mathbf{x}}'$ and $\hat{\Sigma}_{\mathbf{x}}$ because

$$\mathbb{E}\left[\mathbf{x}\mathbf{x}'\right] = \Sigma_{\mathbf{x}} + \bar{\mathbf{x}}\bar{\mathbf{x}}'.$$

Proof of Proposition 6

When utility is mean-variance, targeted intervention costs are quadratic and capital requirement costs are linear, the optimal capital requirement is given by

$$\bar{W}^{\star} = \mathcal{W} - \frac{\kappa}{p\gamma} + \left(\mathbf{1}_{N\times 1} \otimes \tilde{\mathbb{E}}\left[s\right]\right)' \left(\hat{\mathbf{x}} \circ \left(1_{NJ\times 1} - \mathbf{a}^{\star}\right)\right),$$

which implies that the wealth of the financial system is given by

$$W = \mathcal{W} - \bar{\eta} - \frac{\kappa}{p\gamma} - \left((\mathbf{1}_{N\times 1} \otimes \mathrm{s})' (\mathbf{x} \circ (1_{NJ\times 1} - \mathbf{a}^{\star})) - \left(\mathbf{1}_{N\times 1} \otimes \tilde{\mathbb{E}}[\mathrm{s}]\right)' (\hat{\mathbf{x}} \circ (1_{NJ\times 1} - \mathbf{a}^{\star})) \right),$$

which implies

$$\mathbb{E}\left[W\left(\mathbf{a}^{\star}, \bar{W}^{\star}\right) | \mathscr{S} \right] = \bar{W}^{\star}$$

and

$$\tilde{\mathbb{E}}\left[\left(W\left(\mathbf{a}^{\star}, \bar{W}^{\star}\right) - \mathcal{W} \right)^2 | \mathscr{S} \right] = \mathbb{V}ar[\bar{\eta}] + \tilde{\mathbb{C}ov}\left[(\mathbf{1}_{N\times 1} \otimes \mathrm{s})' (\mathbf{x} \circ (1_{NJ\times 1} - \mathbf{a}^{\star})) | \mathscr{S} \right],$$

where

$$\tilde{\mathbb{C}ov}\left[(\mathbf{1}_{N\times 1} \otimes \mathrm{s})' (\mathbf{x} \circ (1_{NJ\times 1} - \mathbf{a}^{\star})) | \mathscr{S} \right] = (1_{NJ\times 1} - \mathbf{a}^{\star})' \tilde{\mathbb{V}} (1_{NJ\times 1} - \mathbf{a}^{\star})$$

and

$$\tilde{\mathbb{V}} = \tilde{\mathbb{C}ov}\left[(\mathbf{1}_{N\times 1} \otimes \mathrm{s}) \circ \mathbf{x} \, | \mathscr{S} \right]$$

Using these expressions, we have that the regulator maximizes the expected value of

$$\begin{aligned} O &= \mathbb{E}\left[W\left(\mathbf{a}^{\star}, \bar{W}^{\star}\right) | \mathscr{S} \right] - \frac{p\gamma}{2} \tilde{\mathbb{E}}\left[\left(W\left(\mathbf{a}^{\star}, \bar{W}^{\star}\right) - \mathcal{W} \right)^2 | \mathscr{S} \right] - \frac{1}{2} \mathbf{a}^{*\prime} \Phi \mathbf{a}^{\star} - (1+\kappa) \bar{W}^{\star} \\ &= -\frac{p\gamma}{2} \mathbb{V}ar[\bar{\eta}] - \kappa \bar{W}^{\star} - \frac{1}{2} \left(p\gamma (1_{NJ\times 1} - \mathbf{a}^{\star})' \tilde{\mathbb{V}} (1_{NJ\times 1} - \mathbf{a}^{\star}) + \mathbf{a}^{*\prime} \Phi \mathbf{a}^{\star} \right). \end{aligned}$$

In an interior solution for $\mathbf{a}^{\star}$, we can use the FOC for $\mathbf{a}^{\star}$ and write $p\gamma (1_{NJ\times 1} - \mathbf{a}^{\star})' \tilde{\mathbb{V}} (1_{NJ\times 1} - \mathbf{a}^{\star}) + \mathbf{a}^{*\prime} \Phi \mathbf{a}^{\star}$

$$\begin{aligned} &= 1_{1\times NJ} p\gamma \tilde{\mathbb{V}} (1_{NJ\times 1} - 2\mathbf{a}^{\star}) + \mathbf{a}^{*\prime} \left(\Phi + p\gamma \tilde{\mathbb{V}} \right) \mathbf{a}^{\star} \\ &= 1_{1\times NJ} p\gamma \tilde{\mathbb{V}} (1_{NJ\times 1} - 2\mathbf{a}^{\star}) + \mathbf{a}^{*\prime} \left(\kappa \left(\mathbf{1}_{N\times 1} \otimes \tilde{\mathbb{E}}[\mathrm{s}] \right) \circ \hat{\mathbf{x}} + p\gamma \tilde{\mathbb{V}} \mathbf{1}_{NJ\times 1} \right) \\ &= 1_{1\times NJ} p\gamma \tilde{\mathbb{V}} (1_{NJ\times 1} - \mathbf{a}^{\star}) + \mathbf{a}^{*\prime} \left(\kappa \left(\mathbf{1}_{N\times 1} \otimes \tilde{\mathbb{E}}[\mathrm{s}] \right) \circ \hat{\mathbf{x}} \right) \\ &= 1_{1\times NJ} \Phi \mathbf{a}^{\star} - (\mathbf{1}_{NJ\times 1} - \mathbf{a}^{\star})' \kappa \left(\mathbf{1}_{N\times 1} \otimes \tilde{\mathbb{E}}[\mathrm{s}] \right) \circ \hat{\mathbf{x}}. \\ &= 1_{1\times NJ} \Phi \mathbf{a}^{\star} - \kappa \left(\mathbf{1}_{N\times 1} \otimes \tilde{\mathbb{E}}[\mathrm{s}] \right)' (\hat{\mathbf{x}} \circ (1_{NJ\times 1} - \mathbf{a}^{\star})) \end{aligned}$$

Then, using the optimal capital requirement we have

$$\begin{aligned} O = &-\frac{p\gamma}{2} \mathbb{V}ar[\bar{\eta}] + \left(\mathbf{1}_{N\times 1} \otimes \tilde{\mathbb{E}}[s] \right)' (\hat{\mathbf{x}} \circ (1_{NJ\times 1} - \mathbf{a}^{\star})) - \kappa \bar{W}^{\star} \\ &- \frac{1}{2} 1_{1\times NJ} \Phi \mathbf{a}^{\star} + \frac{1}{2} \kappa \left(\mathbf{1}_{N\times 1} \otimes \tilde{\mathbb{E}}[\mathrm{s}] \right)' (\hat{\mathbf{x}} \circ (1_{NJ\times 1} - \mathbf{a}^{\star})) \\ &= -\kappa \left(\mathcal{W} - \frac{\kappa}{p\gamma} \right) - \frac{p\gamma}{2} \mathbb{V}ar[\bar{\eta}] - \frac{\kappa}{2} \left(\mathbf{1}_{N\times 1} \otimes \tilde{\mathbb{E}}[s] \right)' (\hat{\mathbf{x}} \circ (1_{NJ\times 1} - \mathbf{a}^{\star})) - \frac{1}{2} 1_{1\times NJ} \Phi \mathbf{a}^{\star} \\ &= -\kappa \left(\mathcal{W} - \frac{\kappa}{p\gamma} \right) - \frac{p\gamma}{2} \mathbb{V}ar[\bar{\eta}] - \frac{\kappa}{2} \left(\bar{W}^{\star} - \mathcal{W} - \frac{\kappa}{p\gamma} \right) - \frac{1}{2} 1_{1\times NJ} \Phi \mathbf{a}^{\star} \\ &= -\frac{\kappa}{2} \left(\mathcal{W} - \frac{\kappa}{p\gamma} \right) - \frac{p\gamma}{2} \mathbb{V}ar[\bar{\eta}] - \frac{\kappa}{2} \bar{W}^{\star} - \frac{1}{2} 1_{1\times NJ} \Phi \mathbf{a}^{\star} \end{aligned}$$

Therefore,

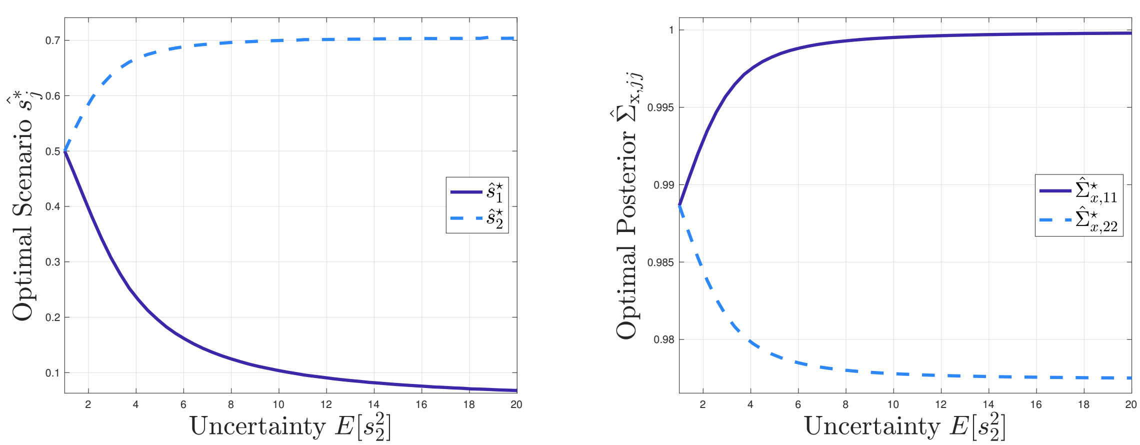
$$\mathbb{E}_{\hat{\mathbf{x}}}\left[O\right] = -\frac{\kappa}{2}\left(\mathcal{W} - \frac{\kappa}{p\gamma}\right) - \frac{p\gamma}{2}\mathbb{V}ar\left[\bar{\eta}\right] - \frac{\kappa}{2}\mathbb{E}_{\hat{\mathbf{x}}}\left[\bar{W}^{\star}\right] - \frac{1}{2}1_{1\times NJ}\Phi\mathbb{E}_{\hat{\mathbf{x}}}\left[\mathbf{a}^{\star}\right]$$

and the objective of the regulator can be written as

$$\min_{s} \kappa\mathbb{E}_{\hat{\mathbf{x}}}\left[\bar{W}^{\star}\right] + 1_{1\times NJ}\Phi\mathbb{E}_{\hat{\mathbf{x}}}\left[\mathbf{a}^{\star}\right].$$

B Additional comparative statics

B.1 Uncertainty about risk factors



(a) Optimal scenario choice (b) Optimal posterior

Figure B.1: Optimal scenario and information choice as a function of the uncertainty about risk factor 2, $\mathbb{E}\left[s_2^2\right]$.

Note: Figure B.1 illustrates the regulator's optimal choice of scenario and the implied posterior variance as a function of the uncertainty about risk factor 2, $\mathbb{E}\left[s_2^2\right]$. The parameters used are $N = 1$, $J = 2$, $\gamma = 0.3$, $\phi = [1, 1]'$, $\bar{\mathrm{x}} = [1, 1]'$, $\Sigma_{\boldsymbol{x}} = \boldsymbol{I}_J$, $\alpha(M) = \sqrt{M}$, $\beta(M) = \sqrt{M}$, $M = 1$, $\theta = 1$, $\mathbb{E}\left[s_1^2\right] = 1$, $\mathbb{E}\left[\epsilon_{1,0}^2\right] = 1$ and $\mathbb{E}\left[\epsilon_{1,1}^2\right] = 1$.

If one risk factor has a very low variance and will stay close to the the baseline, then it is less valuable to learn about the exposures to it and to intervene to reduce them. In this case, the factor's weight on the expected losses will be small and uncertainty about the exposure to it is less costly. However, if the variance of a risk factor is large it has the potential to be an important driver of bank losses depending on the risk exposures to it. In this case, the regulator has more incentives to learn and intervene along the dimension of this factor to curve its potential impact on losses. Therefore, the regulator will stress a risk factor more in the optimal scenario the highest the uncertainty about it. Figures B.1 show the optimal scenario choice as a function of the uncertainty about risk factor 2, $\mathbb{E}\left[s_2^2\right]$.