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JEL Classification: C61, C73, D81, G11, G20

Keywords: Strategic Portfolio Allocation, Incentive fees, Managed fund industry, tournaments, Peer-comparison Benchmarks

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February 10, 2022

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In the managed fund industry, compensation is performance-based and is evaluated with respect to a benchmark. The benchmarks can be an exogenous absolute index or the performance of comparable funds. We analyze the impact of a convex compensation scheme based on peer-group benchmarks. We develop a model of tournament between risk-averse fund managers who receive a fee proportionally to the return differential between their fund and the benchmark, provided that they beat the benchmark. We find that a more competitive benchmark leads to more risk-taking and more differentiated investment strategies. A more competitive (larger) industry provides similar incentives.

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1 Introduction

The institutionalization of financial markets has accelerated dramatically over the last four decades. Institutional investors now account for nearly 70 % of stock trading volume. As of the fourth quarter of 2019, investment funds total assets under management (AUM) were valued at more than $22.152 trillion in the US. In addition, a staggering record of $32.669 trillion was in the hands of insurance corporation and pension funds.

In absence of delegation, individual investors hold a diversified portfolio; its composition depends on their degree of risk aversion. When investment is delegated, the objectives of fund managers usually differ from those of their clients, and aligning these interests may prove difficult in practice. Fund managers' compensation is indeed complex and takes many forms (see for instance Ma, Tang and Gomez 2018). How to provide the right incentives to managers is a key question for academics and practitioners. The recent debate has focused on two aspects of the structure of compensation: the use of incentive fees (option-like contract) versus fulcrum fees (symmetric penalty and rewards), and the choice of the adequate benchmark to provide incentives for performance.

The most common compensation structure consists of management fees and performance fees. Management fees are paid regardless of performance as a fixed percentage of the AUM. Performance fees create explicit incentives for fund managers and are based on the fund's return relative to a benchmark. Symmetric performance fees (also called fulcrum fees) impose a penalty for under-performance equal to the gain for over-performance. Asymmetric performance fees reward the fund manager for outperformance relative to such a benchmark but do not penalize poor performance. This asymmetry around the benchmark can lead to excessive risk-taking. The objective of outperforming other funds creates a tournament between fund managers, which can lead to incentives for funds to herd with other funds or, on the contrary, to differentiate their investment strategies.

The choice of an appropriate benchmark is also key. If many contracts use an index composed of underlying securities (such as the S&P500), investors are increasingly relying on peer-group benchmarks, constructed from the performance of competing peer funds. Peer-group benchmarks can vary in their composition (what funds belong to the peer group) but also in their competitiveness (the average return of the funds in the group, the average return of the 10% best performing funds, etc.).

If management fees do not provide explicit incentives to perform, they can create implicit (reputational) incentives to outperform competitors as money flows to funds that perform better than the rest of the industry. The empirical literature has documented that funds that perform best indeed experience the highest inflow of money (see for instance Sirri and Tufano 1998). Investors rely on fund rankings like the Morningstar or NY Times rating, which are based on past relative performance (see Del Guercio and Tkac 2008, Reuter and Zitzewitz 2010). Chevalier and Ellison (1997) document that for top-performing mutual funds the relation between the flow of new funds and the performance is flat until a threshold and then increases sharply. Similarly, Ding, Getmansky, Liang, and Wermers (2008) find that the flow-performance relation is convex as money flows shoot up in the region of high past performance.

In this paper, we analyze the impact of the competitiveness of the benchmark on investment strategies in terms of risk-taking and differentiation. To do so, we develop a novel model in which fund managers face convex incentive fees based on a peer-group benchmark. Empirical evidence drives our modelling assumptions. Payoffs exhibit a tournament like structure: managers get higher payoff when their performance is better than the other funds' performance. However, compensation depends not only on the ranking of the fund in its peer-group but also on the performance differential. On the downside, when a fund performs less well than the benchmark, we assume a payoff of zero. The payoff function is thus convex, and is consistent both with asymmetric performances fees based on a peer-group benchmark and with management fees and asymmetric money flows based on funds' relative performances. Managers' payoff is similar to that of a call option, with a strike price determined endogenously by the benchmark. Carpenter (2000) and Cuoco and Kaniel (2011) use a comparable payoff structure but based on an exogenous benchmark. In our model, the benchmark is endogenous and is based on the returns of competing portfolios.

We use a standard binomial model of the financial markets, in which a risky asset and a safe assets are traded. Risk-averse fund managers are given some AUM to invest and can leverage their investments. We impose some bounds on the leverage strategies so that the AUM never become negative. The choice of investment translates into returns in the two states of the world. The strategy of an investor boils down to a choice of return in the upper state, going from 0 to the maximal return. These two extremes returns correspond to the manager betting on one of the two states, which we call extreme investment strategies.

As managers' payoffs are proportional to the difference in returns, they have incentives to differentiate their investment strategies. This need for differentiation creates a coordination game in which managers would like to choose opposite investment strategies. A peer-group benchmark thus creates incentives to anti-herd. This result is in sharp contrast with Maug and Naik (2011) and Arora and Ou-Yang (2001) who highlight that benchmarks lead to herding behavior by fund managers.

We analyze first the two-manager game. We characterize equilibria in pure strategies in which fund managers coordinate on the two extreme investment strategies. One manager gets a high return when the upper state happens and gets the maximum compensation as the other fund gets the minimum return in this state. When the lower state happens, roles are reversed. Despite their risk aversion, fund managers have no incentive to deviate as other strategies lead to lower compensation in the state they bet on and they still get zero compensation in the state that the other fund is betting on.

We then derive a full support mixed-strategy equilibrium in which fund-managers randomize on the whole space of investment strategies - the whole interval of potential returns in the upper state. Managers use atoms on extreme investment strategies and use a continuous distribution on the interior. The size of the atoms and the exact form of the probability distribution depends on the risk aversion of the managers and other parameters of the model. When risk-aversion decreases, the size of the atoms increases. In the limit, when managers are risk-neutral, at least one of the managers uses only extreme investment strategies. The same result applies when the AUM or incentive fee rate increase.

We then investigate the impact of peer-group benchmarks on investment strategies. In particular, we analyze how the competitiveness of the benchmark influences the behavior of managers. We first consider a change in the number n of funds competing keeping the benchmark fixed. Only the manager of the best-performing fund gets a performance fee that depends on the difference of performance with the second-best performing fund. The number of funds n thus parametrizes the degree of competition in the industry. We extend our characterization of equilibria to the case of n funds and we show that more competition leads managers to choose more extreme and thus more risky investment strategies. Second, we consider the impact of the degree of competitiveness of the benchmark. We allow for contracts that pay managers a fee if they do better than the kth better performing fund. The competitiveness of the benchmark is thus parametrized by k, with higher values of k corresponding to a more competitive benchmark. We extend the equilibrium characterization and show that more competitive benchmarkss lead to more extreme and thus more risky strategies by fund managers. Our results thus speak to the debate on peer-group benchmarks. We show that the competitiveness of the industry and the competitiveness of the benchmark are key parameters in understanding the impact of using such tools in managers' compensation contracts. To the best of our knowledge, these dimensions have not been investigated empirically.

Related literature

An extensive theoretical research focusses on delegated portfolio management[1]. A first strand of papers analyzes the effect of incentive fees on managers's behavior. Grinblatt and Titman (1989) study the impact of incentives fees in the form of a base fee and a performance fee using a benchmark based on a given return. Admati and Pfleiderer (1997) analyze the role of benchmarks and argue that benchmark-adjusted compensation is not consistent with optimal risk-sharing. Starks (1987) investigates the relative advantage of a symmetric, fulcrum performance fee compared with an asymmetric bonus contract. Das and Sundaram (2002) extends the model to allow for heterogeneity in fund managers' information. Buraschi, Kosowski and Sritrakul (2014), Cuoco and Kaniel (2011) and Carpenter (2000) look at the impact of asymmetric convex payoffs.

A vast literature explores the agency problem between investors and portfolio managers and studies how compensation contracts should be structured (see Bhattacharya and Pfleiderer (1985), Stoughton (1993), Heinkel and Stoughton(1994), Palomino and Prat (2003), Ou Yang (2003), Larsen (2005), Dybvig, Farnsworth, and Carpenter (2010), Cadenillas, Cvitanic, and

[1]See Stracca (2006) for a detailed survey.

Zapatero (2007), Cvitanic, Wan, and Zhang (2009), and Li and Tiwari (2009)). Other papers examine how commonly observed incentive contracts impact managers' decisions: it includes, Roll (1992), Carpenter (2000), Chen and Pennacchi (2005), Hugonnier and Kaniel (2010), and Basak, Pavlova, and Shapiro (2007).

Few papers consider investment behavior in a tournament setting, thus using peer-group comparison as the benchmark. Basak and Makarov (2014) analyze the competition between two (potentially asymmetric) fund managers in a continuous time setting in which the compensation depends on the ratio of both fund managers returns. Strack (2016) considers a contest between n fund managers with a rank-based prize structure. Lagziel and Lehrer (2018) study a contest in which managers' payoff depends on the difference between returns. These papers are the closest to our approach. Our contribution is to analyze the effect of convex contracts when the benchmark is based on competing funds' performance. In Strack (2016), payoffs depend on the rank in the tournament but not on the performance differential. In Lagziel and Lehrer (2018) payoffs are symmetric with respect to the benchmark.

Brown, Harlow, and Starks (1996) provide empirical evidence that when managers' compensation is linked to a peer group benchmark, funds with a lagging performance at mid year, experience an increase in volatility in an attempt to catch up by the end of the year. Evans, et al. (2020) examine the role of peer benchmarks versus pure benchmarks in mutual fund compensation.

The paper is organized as follows. Section 2 presents the model. In section 3, we analyze the two-player game and derive the Nash equilibria (in pure strategy and and the unique full support mixed strategy). Section 4 extends the analysis to the case of n funds evaluated under different benchmarks. In section 5, we discuss the interplay between coordination, differentiation and risk taking and presents some numerical examples that allow us to draw comparisons between the different types of equilibria. Section 6 concludes. Proofs are in the appendix.

2 The Model

2.1 Financial Markets

We consider a one-period binomial model[2], in which a risk-free bond and a risky stock are traded. Let r denote the risk free interest rate. At time 0, the stock price is S_0 and let p (resp. $1-p$) denote the (true) probability to reach at time 1 the upper state H (resp. lower state L) in which the stock price is $S_H = uS_0$ (resp. $S_L = dS_0$). We assume that $d < 1+r < u$. Markets are complete and the (unique) risk neutral probability p^* of the upper state H is given by

$$p^* = \frac{1+r-d}{u-d}.$$

The state density process $\overline{\pi}_1$ takes two values:

$$\begin{aligned}
\overline{\pi}_H &= \frac{1}{1+r}\frac{p^*}{p} \text{ (state } H)\\
\overline{\pi}_L &= \frac{1}{1+r}\frac{1-p^*}{1-p} \text{ (state } L).
\end{aligned}$$

2.2 Preferences, Investment Strategies and Compensation Contracts

2.2.1 Preferences

We assume that manager i has an exponential utility function, which displays constant absolute risk-aversion[3]:

$$u_i(W) = \begin{cases} \frac{1-e^{-b_i W}}{b_i}, \text{ if } b_i > 0 \\ W, \text{ if } b_i = 0. \end{cases}$$

When $b_i = 0$, a manager is risk neutral. We denote $W_{i0} > 0$ the initial assets AUM of fund i.

[2]In the on-line appendix, we analyze the extension to a multinomial model in which the return of the risky asset depends on more than two states of the world.

[3]We use CARA utility functions for the sake of exposition. In the online supplement appendix, we extend the model to the class of mixed risk aversion utility functions (see for instance Caballé and Pomansky [1996]), which includes most of the utility functions commonly used in the financial economics literature.

2.2.2 Investment Strategies

Let α_i (resp. θ_i) denotes the dollar amount invested in the bond (resp. in the stock) at time 0 so that $W_{i0} = \alpha_i + \theta_i$. At time 1, the value of the AUM is given by

$$W_{iH} = \alpha_i(1 + r) + u\theta_i \text{ (state } H\text{)}$$
$$W_{iL} = \alpha_i(1 + r) + d\theta_i \text{ (state } L\text{)}.$$

As markets are complete, the budget constraint can be written $W_{i0} = \frac{1}{\overline{\pi}_0} E\left[\overline{\pi}_1 W_{i1}\right]$. Normalizing $\overline{\pi}_0 = 1$ and denoting $R_{i1} = \frac{W_{i1}}{W_{i0}}$ the (gross) return, we obtain

$$1 = \pi_H R_{iH} + \pi_L R_{iL},$$

where $\pi_H = p\overline{\pi}_H$ and $\pi_L = (1 - p)\overline{\pi}_L$. Thus, an investment strategy is fully characterized by a pair of returns (R_{iH}, R_{iL}), which is the most convenient notation.

Finally, we assume that the AUM cannot become negative at time 1 so that returns (R_{iH}, R_{iL}) must be in $[0, \frac{1}{\pi_H}] \times [0, \frac{1}{\pi_L}]$. This is equivalent to imposing the following leverage constraint

$$-\frac{1 + r}{u - (1 + r)} \leq \frac{\theta_i}{W_{i0}} \leq \frac{1 + r}{1 + r - d}. \tag{1}$$

We call extreme strategies investment strategies that lead to extreme returns, $R_{iH} = 0$ and $R_{iH} = \frac{1}{\pi_H}$.

2.2.3 Manager's Compensation Contract

Two fund managers are competing in a tournament. Manager i's contract takes the form of an incentive fee. At time 1, manager i's payoff is given by

$$k_{i0} W_{i0} + k_i W_{i0} \left[R_{i1} - R_{j1}\right]^+,$$

with $(k_{i0}, k_i) \in (0, 1)^2$.

Starks (1987) uses similar contracts based on an exogenous benchmark. The manager receives a fixed compensation proportional to the initial AUM. Then, if he wins the tournament, his payoff increases linearly in the difference between the returns. The payoff of the managers is convex. Without loss of generality, we set $k_{i0} = 0$.

For convenience, we denote $\overline{b}_i = b_i k_i W_{i0}$ the effective degree of risk aversion. In what follows, we refer to the completely symmetric game when parameters take the values: $p = \frac{1}{2}$, $\pi_H = \pi_L$ and $\overline{b}_i = \overline{b}_j = \overline{b}$.

Before exploring portfolio allocations in presence of strategic motives, recall that in our setting, ignoring the leverage constraint, a CARA investor who maximizes his expected utility shall optimally hold an amount $\overline{\theta}_i = \frac{1}{\overline{b}_i} \frac{\ln[\overline{\pi}_L/\overline{\pi}_H]}{u-d}$ in the stock. This amount is independent of the initial AUM W_{i0} (no wealth effect with CARA preferences) and is decreasing in the degree of risk aversion.

3 Equilibrium Strategies with Two Funds

3.1 Pure Strategy Nash Equilibria

We first characterize pure strategy Nash equilibria. These equilibria feature coordination between the two managers on opposite extreme strategies.

Set $\Delta = [0, \frac{1}{\pi_H}]$. A pure strategy for manager j is a choice of return j in the high state $x_j \in \Delta$.

Proposition 1 *There are two pure strategy Nash equilibria:*

$$(x^*, y^*) = (0, \frac{1}{\pi_H})$$
$$(x^*, y^*) = (\frac{1}{\pi_H}, 0),$$

where x^* (resp. y^*) denote manager i (resp. manger j) strategy in Δ. In each of the equilibria, the level of utility derived by one of the two managers is the highest possible he can get among all Nash Equilibria.

Proof. See appendix. ■

Managers differentiate their investment strategies as much as the leverage constraint allows them to (1) and coordinate on opposite investment strategies. When one funds does well, the other is doing badly, which maximizes the compensation of the manager who wins the tournament. This is an example of an anti-coordination game that generates anti-herding incentives. In general, a manager is better off in one of the two equilibria. Depending on the parameters of the model, managers may both prefer the same equilibrium or prefer different equilibria. In his preferred equilibrium, a manager gets his highest possible equilibrium level of utility; however, in the other equilibrium, the manager does not receive his lowest utility level among all Nash equilibria (pure or mixed).

We show, in the appendix, that if the extreme strategy 0 is in the support of the equilibrium distribution, then manager j' s utility level is strictly smaller than in a pure strategy Nash equilibrium in which he assigns full mass at 0. A similar result holds if $\frac{1}{\pi_H}$ is in the support of the distribution of a manager. Finally, when $p \neq 1/2$ as a manager becomes more and more risk averse, his best response is getting closer to the edge that corresponds to the state that is most likely to occur.

3.2 Mixed Strategy Nash Equilibria

A mixed strategy consists in a pair of distributions (Γ_i, Γ_j) with cdf (F_i, F_j) with supports $\Delta_i \times \Delta_j \sqsubseteq \Delta \times \Delta$. The pair (Γ_i, Γ_j) is a mixed strategy Nash equilibrium if for all given $x \in \Delta_i$ (resp. $y \in \Delta_j$) chosen by manager i (resp. manager j), the (expected) utility level U_i (resp. U_j) is independent from the choice of x (resp. y).

Given F_j, manager i's utility U_i at return $x \in \Delta_i$ is

$$U_i(x) = p \int_{\Delta_j} \frac{1 - e^{-\bar{b}_i(x-y)}}{b_i} H(x-y) dF_j(y) + (1-p) \int_{\Delta_j} \frac{1 - e^{-\bar{b}_i \frac{\pi_H}{\pi_L}(x-y)}}{b_i} H(y-x) dF_j(y), \quad (2)$$

where H denote the Heaviside function[4]. It is convenient to define $\overline{U}_i = \frac{\widehat{U}_i}{k_i W_{i0}}$ where $\widehat{U}_i$ is a fixed utility level.

We start with in the simple case of risk-neutral fund managers. For convenience, set $\alpha = \frac{(1-p)\pi_H}{p\pi_L + (1-p)\pi_H} = \frac{\overline{\pi}_H}{\overline{\pi}_H + \overline{\pi}_L}$. Parameter α is a measure of attractiveness of a state for a risk neutral

[4]As mixed strategies may involve multiple (possibly infinitely many) atoms, to circumvent non differentiability (in the classical sense) issues, we formulate the problem using the Dirac distribution function and Heaviside step function. The Laplace transform is a natural tool to deal with these distributions. We compute the Laplace transform of the mixed strategy on a continuous support. We then use the injectivity of the Laplace Transform to prove uniqueness and the inverse Laplace operator to recover the equilibrium distribution. To the best of our knowledge, these tools have not been used before to study mixed-strategy equilibria in contests.

manager: if $\alpha < 1/2$ (resp. $> 1/2$), then $\overline{\pi}_H < \overline{\pi}_L$ (resp. $> \overline{\pi}_L$): a given dollar amount is more (resp. less) valuable in state H than in state L.

3.2.1 Risk Neutral Fund Managers

Proposition 2 *When managers are risk neutral, there exist multiple mixed-strategy Nash equilibria. They all consist in one manager randomizing between extreme returns $x = 0$ and $x = \frac{1}{\pi_H}$ with weights:*

$$\phi_0 = \alpha$$

$$\phi_1 = 1 - \alpha,$$

while the other manager is using a distribution with a mean equal to $\frac{1-\alpha}{\pi_H}$. In all equilibria, both managers get the same utility $\frac{p\alpha}{\pi_H} kW_0$.

Proof. See appendix. ■

Not surprisingly, due to the (piecewise) linearity of utility functions, there exist multiple equilibria. In any mixed strategy equilibrium, players' utility is always the same, so no mixed strategy equilibrium is preferred by the managers. However, both managers end up worse off than in a pure strategy equilibrium. Note that the manager who is randomizing between the two extreme returns assigns a larger atom to the more attractive state, the state with the lower price of consumption.

The expected return $\overline{R}$ is the same for both funds, is constant across equilibria and is equal to:

$$\overline{R} = \frac{p(1-\alpha)}{\pi_H} + \frac{(1-p)\alpha}{\pi_L}.$$

Regarding the riskiness of equilibrium investment strategies, we prove (see appendix) that assigning full mass at interior point $x = \frac{1-\alpha}{\pi_H}$ is the equilibrium investment strategy with the smallest variance given by

$$\text{var}_{\min}(R) = p(1-p)\left[\frac{1-\alpha}{\pi_H} - \frac{\alpha}{\pi_L}\right]^2.$$

This strategy is the most preferred by investors. Conversely, randomizing between extreme returns leads to the largest variance:

$$\mathrm{var}_{\max}(R) = \frac{(1-p)\alpha}{\pi_L^2} + \frac{p(1-\alpha)}{\pi_H^2} - (\frac{(1-p)\alpha}{\pi_L} + \frac{p(1-\alpha)}{\pi_H})^2.$$

As this strategy is played in any equilibrium, at least one fund is not acting in the best interests of its clients.

3.2.2 Risk Averse Fund Managers

When managers are risk averse, the strategies derived for risk-neutral managers no longer constitute an equilibrium. Managers find it more valuable to use interior investment strategies to decrease the risk of their compensation. To see this, note that if one manager randomizes between the two extreme returns, then the (expected) utility of his competitor is strictly concave. Randomizing between the two extreme returns is not optimal, and choosing a return in the interior of the support is better to maximize the trade off between risk and return. We first focus on a mixed-strategy equilibrium with full support. We examine later asymmetric equilibria in which identical managers chose different strategies.

Proposition 3 *There exists a unique full-support mixed strategy Nash equilibrium. When* $p \neq 1/2$, *managers assign atoms at the boundaries of* Δ*:*

$$\phi_{0j} = \frac{2p-1}{pe^{\frac{\lambda_i}{\pi_H}} - (1-p)}\alpha \text{ and } \phi_{1j} = \frac{2p-1}{p-(1-p)e^{-\frac{\lambda_i}{\pi_H}}}(1-\alpha),$$

and randomize over the interior of Δ *using an exponential distribution with density*

$$f_j(x) = \phi_{0j}(\lambda_i + \bar{b}_i)e^{\lambda_i x},$$

where $\lambda_i = \frac{(2p-1)\bar{b}_i\pi_H}{p\pi_L+(1-p)\pi_H}$. *When* $p = 1/2$, *managers assign atoms at the boundaries of* Δ*:*

$$\phi_{0j} = \frac{\pi_H}{\bar{b}_i + \pi_H + \pi_L} \text{ and } \phi_{1j} = \frac{\pi_L}{\bar{b}_i + \pi_H + \pi_L},$$

and randomize over the interior of Δ *using a uniform distribution with density*

$$f_j(x) = \phi_{0j}\bar{b}_i.$$

The levels of utility derived by the managers are the lowest possible among all Nash equilibria.

Proof. See appendix. ■

In the appendix, we show that if one manager uses a full support strategy in an equilibrium, then the other manager must also use a full support strategy.

3.2.3 Properties of the Investment Strategies

We now analyze how equilibrium investment strategies vary when parameter $\bar{b}_i$ varies. Recall that $\bar{b}_i = b_i k_i W_{i0}$; an increase in $\bar{b}_i$ thus represents an increase in the effective risk-aversion of the manager, due to either an increase in risk aversion, or an increase in the wealth under management or/and an increase in the manager's incentive fee rate.

Proposition 4 *The density function f_j is increasing (resp. decreasing) if $p > 1/2$ (resp. $p < 1/2$). As manager i's risk aversion rises, the size of the atoms of distribution Γ_j decreases and at the limit for $p \neq 1/2$, as manager i becomes infinity risk averse, distribution Γ_j converges to the Dirac distribution at point $x = 0$ (resp. $x = \frac{1}{\pi_H}$) if $p < 1/2$ (resp. $p > 1/2$). When $p = 1/2$, manager i becomes infinity risk averse, distribution Γ_j converges to the uniform distribution $U(0, \pi_H)$.*

Proof. See appendix. ■

Proposition 3 reveals that fund managers assign a larger probability to an investment strategy that leads to a higher a return in the interior of the support in the more likely state. Both fund managers give more weight to the more likely state. In contrast to what happens in pure-strategy equilibria, managers fail to coordinate in the full-support equilibrium. As managers are indifferent between any return they may choose, they are not concerned with differentiating their investment strategies from that of other funds. This absence of differentiation decreases the expected utility obtained in the full-support equilibrium. As the effective risk aversion of the funds increases, the skewness of the equilibrium distributions increases as both fund managers champion the more likely state, which leads to a more extreme lack of differentiation.

4 Peer-Group Benchmarks

We now analyze the effects of competition in an industry with many funds. We need to address the additional complexities of a tournament with more than two managers. One challenge is technical as deriving mixed strategy equilibria with managers using atoms becomes more complicated. Another complexity is conceptual as we need to define what the right benchmark is in this setting.

4.1 Ranking Funds

With only two funds, the performance of the other fund is the obvious benchmark. With more funds, there are several possible ways to define the benchmark. Several recent papers (for instance Espinosa and Touzi 2015, Lacker and Zariphopoulou 2017) analyze a model in which the benchmark corresponds to the average return of funds in the industry. Another avenue is to consider winner take-all contracts. In that case, the manager of the best performing fund is compensated proportionally to the difference with the second best-performing fund.

Alternatively, the contract may not use the second best performing fund as a benchmark, but the kth best performing fund. This practice is common in the industry where funds are ranked as a function of their performance. For instance, Morningstar Inc[5]. is an American financial services firm, with great influence on fund's assets under management and investors' mutual fund selection. Morningstar defines fund categories based on asset holdings. In the United States, Morningstar supports 64 categories, which map into four broad asset classes (U.S. Stock, International Stock, Taxable Bond, and Municipal Bond). The categories help investors identify the top-performing funds and assess potential risk. Then, mutual funds are ranked on a scale of one to five stars based on past performance relative to peer funds. The top 10% of funds receive five stars, the next 22.5% receive four stars, the middle 35% receive three stars, the next 22.5% receive two stars and the bottom 10% get one star. As documented in Del Guerco and Tkac (2008), a fund gaining (loosing) a star will experience an abnormal inflow (outflow) of money in the near future but effects may not be linear: investors praise funds that make it to the elite (four or five stars) but seem to punish funds that slip under the three star

[5]Other security research corporations such as Thompston Reuters Lipper and Zacks use similar rating metrics.

rating.

Analyzing a contest based on order statistics allows us to understand better the role of these 5 star ranking. In particular, we show that more intense competition where only the manager of the best performing fund get rewarded leads to more risk taking than compensation systems in which being among the 10% or 20% best performing funds is enough to receive a bonus. We first analyze a winner-takes-all contract, which corresponds the most competitive benchmark.

4.2 Competition under a Winner-Takes-All Benchmark

We consider an industry in which n funds are competing. Parameter n represents the level of competition in the industry. The compensation contract of manager i's takes the form:

$$k_{i0}W_{i0} + k_i W_{i0}[R_{i1} - \max_{k \neq i} R_{k1}]^+,$$

and we normalize k_{i0} to be equal to zero. Only the manager of the best performing fund receives incentive fees. The special case $n = 2$ corresponds to the analysis in section 3. Note that if x_i (resp. $\widehat{x}_i$) denotes the return delivered by manager i in the upper (resp. lower) state, then we have $\widehat{x}_i = \frac{1-\pi_H x_i}{\pi_L}$ and

$$\widehat{x}_i - \max_{j \neq i} \{\widehat{x}_j\} = \frac{\pi_H}{\pi_L}[\min_{j \neq i} \{x_j\} - x_i].$$

Thus the payoff the manager i in the lower state is given by:

$$k_i W_{i0} \frac{\pi_H}{\pi_L}[\min_{j \neq i} \{x_j\} - x_i]^+.$$

We restrict our attention to the case of symmetric equilibria so that $\overline{b}_i = \overline{b}$ and $\lambda_i = \lambda$ for all $i = 1, 2, ..., n$.

Pure Strategy Nash Equilibria. We look at the case where three funds compete ($n = 3$). Let x, y and z be the returns chosen by managers i, j and k respectively. Without loss of generality, we assume that $0 \leq x \leq y \leq z \leq \frac{1}{\pi_H}$. Manager j underperforms in both states so his utility level U_j is equal to 0. Manager k (resp. i) delivers the highest return in the high (resp. low) state so his utility level U_k (resp. U_i) is equal to $pu_k(z - y)$ (resp. $(1 - p)u_i(\frac{\pi_H}{\pi_L}(y - x))$. Optimally manager k (resp. i) chooses $z^* = \frac{1}{\pi_H}$ (resp. $x^* = 0$). For $n > 3$, if one manager chooses $x = 0$ and another $x = \frac{1}{\pi_H}$ and all the others are indifferent over any strategy. We deduce that in any pure Nash equilibrium, two managers assign full mass at the opposite edges and all the other managers are indifferent between their investment strategies.

Full Support Mixed Strategy Nash Equilibria. Consider $n-1$ iid random variables with cdf F; then the cdfs of the distributions of the maximum F_M and the minimum F_m are given by

$$
\begin{aligned}
F_M(z) &= F^{n-1}(z) \\
F_m(z) &= 1-(1-F(z))^{n-1}.
\end{aligned}
$$

For convenience, set

$$
\begin{aligned}
T_n(\phi_0) &= \bar{b}\left[(1-\alpha)\phi_0^{n-1} - \alpha(1-\phi_0)^{n-1}\right] \\
\widehat{T}_n(\phi_1) &= \bar{b}\frac{\pi_H}{\pi_L}\left[\alpha\phi_1^{n-1} - (1-\alpha)(1-\phi_1)^{n-1}\right].
\end{aligned}
$$

Proposition 5 *There exists a unique full support mixed strategy equilibrium in which the n fund managers use a distribution Γ with cdf F such that for all x in Δ*

$$
p\int_\Delta \frac{1-e^{-\bar{b}(x-z)}}{\bar{b}} H(x-z)dF_M(z) + (1-p)\int_\Delta \frac{1-e^{-\bar{b}\frac{\pi_H}{\pi_L}(\widehat{z}-x)}}{\bar{b}} H(\widehat{z}-x)dF_m(\widehat{z}) = \overline{U}_n,
$$

where $\overline{U}_n$ is a constant. Players assign atoms ϕ_0 and ϕ_1 at the lower and upper edge of the interval Δ (respectively) that satisfy $T_n(\phi_0) = \widehat{T}_n(\phi_1)$. In the interior of Δ, F is the solution of the following ODE

$$
(n-1)\left[(1-\alpha)F^{n-2}(x) + \alpha\left[1-F(x)\right]^{n-2}\right]F'(x) = T_n(\phi_0) + \bar{b}\alpha\left[1-F(x)\right]^{n-1} + (\bar{b}\alpha+\lambda)F^{n-1}(x),
$$

with initial condition $F(0) = \phi_0$ and normalization condition $\int_\Delta dF(z) = 1$.

Proof. See appendix. ■

Risk Neutral Managers. In this case, the (symmetric) equilibrium distribution F only consists in assigning atoms on the edges of Δ and we obtain that

$$
\begin{aligned}
\phi_0 &= \frac{1}{1+\left(\frac{1-\alpha}{\alpha}\right)^{1/(n-1)}} \\
\phi_1 &= \frac{1}{1+\left(\frac{\alpha}{1-\alpha}\right)^{1/(n-1)}}.
\end{aligned}
$$

Recall that $\alpha = \frac{\overline{\pi}_H}{\overline{\pi}_H + \overline{\pi}_L}$: As in the two players' case, players assign a larger atom in the state with the lower state price density. Furthermore, as competition intensifies (larger n), the size of the larger (smaller) atom shrinks (raises) until eventually converges to $\frac{1}{2}$. If $\overline{\pi}_H = \overline{\pi}_L$, the size of the atoms is always $\frac{1}{2}$, independently of the number of funds. In the appendix, we show that more competition induces a higher expected return and if $\pi_H = \pi_L$ while $\alpha \neq \frac{1}{2}$ i.e. the support of the returns in both states are equal, as the number of funds n increases the average return converges to $\frac{1}{2\pi_H}$ and the variance of the returns is increasing up $\frac{1}{4\pi_H^2}$. This indicates that investment strategies are getting riskier as the degree of competition increases.

We now derive a similar result on the properties of the investment strategies of risk averse fund managers as the size of the industry grows.

Proposition 6 *When the number of funds n increases the size of the atoms in the mixed strategy equilibrium converges to:*

$$\lim_{n\to\infty} \phi_0 = \frac{1}{2}$$
$$\lim_{n\to\infty} \phi_1 = \frac{1}{2}.$$

Proof. See appendix. ■

Proposition 6 indicates that fund managers asymptotically randomize between the edges of the support with equal probability. It holds independently of the risk aversion of the fund managers and the attractiveness of the states.

Recall that in the upper (resp. lower) state, an individual manager's benchmark is the maximum (resp. minimum) return delivered by his competitors. When $n = 2$, the maximum and the minimum return distributions are the same: if one manager chooses a large return in one state, it will be easy for his opponent to beat a small return in the other state. However, when $n > 2$, this is no longer the case. In the appendix, we show that if an individual distribution has an atom ϕ_0 at $x = \frac{1}{\pi_H}$ (resp. ϕ_1 at $x = 0$), then the maximum (resp. minimum) distribution has an atom at $x = \frac{1}{\pi_H}$ (resp. $x = 0$) of size $1 - (1 - \phi_1)^{n-1}$ (resp. $1 - (1 - \phi_0)^{n-1}$). As the number of funds raises, managers have to beat returns whose distributions assign increasing probabilities at one edge of Δ in each state. Thus, choosing a return in the interior of Δ is dominated in both states: More competition induces fund managers to choose more extreme (thus risky) investment strategies.

Numerical Simulations about the Impact of the Competitiveness of Industry.

We perform numerical simulations to analyze the impact of the number of funds in the industry on the investment strategy. We increase the number n of funds while keeping the benchmark constant. Only the manager of the best performing fund gets compensated on the basis of the differential with the second best performing fund. Simulations rely on a shooting method. Starting with an initial guess for $\phi_0 \in (0,1)$, we numerically solve the ODE satisfied by F on $(0, \frac{1}{\pi_H})$ and use the relationship that links ϕ_0 and ϕ_1 to check if the condition $\int_\Delta dF(z) = 1$ is satisfied. We repeat the operation by altering the choice of ϕ_0 until the condition $\int_\Delta dF(z) = 1$ is satisfied. For simplicity, we consider parameters that leads to a completely symmetric setting: $p = \frac{1}{2}$, $\pi_H = \pi_L = 0.3$, $\overline{b} = 2$. The equilibrium distribution is symmetric with respect to point $(\frac{1}{2\pi_H}, \frac{1}{2})$ and the midpoint $\frac{1}{2\pi_H}$ of the support Δ corresponds to the riskless portfolio.

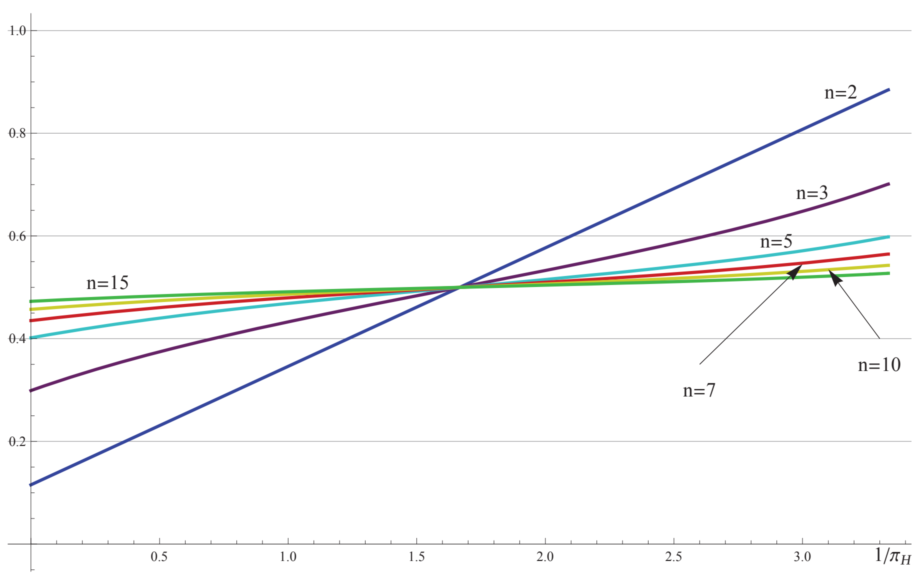

Figure 1 : Equilibrium distributions and industry size under the "winner takes all" benchmark

Figure 1 displays the equilibrium distribution F for several values of the number n of funds in the industry. When $n = 2$, the equilibrium distribution is uniform and F is linear. For $n > 2$, observe that the distribution F becomes concave (resp. convex) on the interval $(0, \frac{1}{2\pi_H})$ (resp.

$(\frac{1}{2\pi_H}, \frac{1}{\pi_H})$). The pdf is decreasing as we move away from the edges of the support and reaches its minimum at the middle of the support. As competition increases (larger value of n), the size of the atoms increases and consequently the curves flatten, which indicates that managers are reducing the probability of choosing returns in the interior of the support. Measured in terms of return delivered in the upper state, an individual manager's benchmark is the maximum (resp. minimum) return delivered by the industry in the upper (lower) state. When $n = 2$, the maximum and the minimum return distributions are the same: each manager wins the contest in a different state. When $n > 2$, this is no longer the case. As the numbers of funds n increases, the maximum (resp. minimum) return distribution converges to the Dirac distribution at $x = \frac{1}{\pi_H}$ (resp. $x = 0$): Individual investment strategies become riskier as it is the only way for a fund manager to be the best performer of the industry.

4.3 Competitiveness of the Benchmark

We now assume that n funds compete and that fund manager i receives compensation only if the return of his fund is among the k best returns. Parameter k measures the competitiveness of the benchmark. In the previous section, we analyzed the case $k = 1$ when the manager receives an incentive fee only if he delivers the best return of the industry. The payoff of fund manager i is:

$$k_{i0}W_{i0} + k_i W_{i0}\left[R_i - R_{(n-k)}\right]^+,$$

where $R_{(k)}$ is the kth order statistic return of the $n-1$ returns delivered by the industry (except manager i) so that we have

$$R_{(1)} \leq R_{(2)} \leq ... \leq R_{(n-1)}.$$

$R_{(1)} = \min\limits_{j \neq i} \{R_j\}$ and $R_{(n-1)} = \max\limits_{j \neq i} \{R_j\}$ and $R_{(n-k)}$ is the smallest return delivered by the k best fund managers (other than fund i). As before, we normalize k_{i0} to be equal to 0. Let x_i be the return delivered by manager i in the upper state; manager i's payoffs is given by

$$\begin{cases} k_i W_{i0}\left[x_i - R_{(n-k)}\right]^+ & \text{(state } H\text{)} \\ k_i W_{i0}\frac{\pi_H}{\pi_L}\left[R_{(k)} - x_i\right]^+ & \text{(state } L\text{).} \end{cases}$$

Pure Nash Equilibria. Let us look at the case where $n = 3$ and $k = 2$. The worst performing fund of the industry is the only one to receive no compensation. Let x, y and z be the returns chosen by managers i, j and k respectively. Without loss of generality, we assume that $0 \leq y \leq z \leq \frac{1}{\pi_H}$. Manager i's utility is given by

$$U_i = pu_i(x-y)H(x-y) + (1-p)pu_i(\frac{\pi_H}{\pi_L}(z-x))H(z-x).$$

Too keep things simple, we assume that $p = \frac{1}{2}$, $\pi_H = \pi_L$ and $\bar{b}_i = \bar{b}_j = \bar{b}_k$.

Proposition 7 *There exists no pure-strategy Nash equilibrium.*

Proof. See appendix. ■

Managers may be facing different benchmarks in the two states. This creates incentives to select a return in the interior of Δ. Due to the symmetry of the game, managers choose the middle point of the interval Δ which leads to identical benchmarks in both states. But this creates incentives to choose extreme strategies. Contrary to the two-player game, no coordination can be reached. This example illustrates why a pure strategy equilibrium may fail to exist. We thus focus on mixed-strategy Nash equilibria.

Full Support Mixed Strategy Nash Equilibria. We restrict attention to symmetric equilibria. Let F_k denote the cdf of the distribution $R_{(k)}$; in the symmetric game, we have

$$F_k(x) = L_{n,k}\left[F(x)\right],$$

where $L_{n,k}(y) = \sum_{i=k}^{n-1} \binom{n-1}{i} y^i (1-y)^{n-1-i}$. For convenience, set

$$\begin{aligned}
T_{n,k}(\phi_0) &= \bar{b}\left[(1-\alpha)L_{n,n-k}(\phi_0) - \alpha\left[1 - L_{n,k}(\phi_0)\right]\right] \\
\widehat{T}_{n,k}(\phi_1) &= \bar{b}\frac{\pi_H}{\pi_L}\left[\alpha L_{n,n-k}(\phi_1) - (1-\alpha)\left[1 - L_{n,k}(\phi_1)\right]\right].
\end{aligned}$$

Proposition 8 *There exists a unique full support mixed strategy equilibrium in which the n fund managers use a distribution Γ with cdf F such that for all x in Δ*

$$p\int_\Delta \frac{1-e^{-\bar{b}(x-z)}}{\bar{b}} H(x-z)dF_{n-k}(z)dz + (1-p)\int_\Delta \frac{1-e^{-\bar{b}\frac{\pi_H}{\pi_L}(\widehat{z}-x)}}{\bar{b}} H(\widehat{z}-x)dF_k(\widehat{z})d\widehat{z} = \overline{U}_{n,k},$$

where $\overline{U}_{n,k}$ is a constant. Players assign atoms ϕ_0 and ϕ_1 at the lower and upper edge of the interval Δ (respectively) that satisfy $T_{n,k}(\phi_0) = \widehat{T}_{n,k}(\phi_1)$. In the interior of Δ, F is the solution *of the following ODE*

$$(n-1)\binom{n-2}{k-1}\left[(1-\alpha)F^{n-k-1}(x)\left[1-F(x)\right]^{k-1}+\alpha F^{k-1}(x)\left[1-F(x)\right]^{n-1-k}\right]F'(x)$$

$$=T_{n,k}(\phi_0)+\bar{b}\alpha+(\bar{b}\alpha+\lambda)L_{n,n-k}[F(x)]-\bar{b}\alpha L_{n,k}[F(x)],$$

with initial condition $F(0)=\phi_0$ *and normalization condition* $\int_\Delta dF(z)=1$.

Proof. See appendix. ■

Risk Neutral Managers. When managers are risk-neutral, the (symmetric) equilibrium distribution F assigns atoms at the edges of Δ and atom ϕ_0 satisfies

$$(1-\alpha)L_{n,n-k}(\phi_0)=\alpha L_{n,n-k}(1-\phi_0).$$

As function $L_{n,n-k}$ is increasing from 0 to 1, ϕ_0 exists and is unique in $(0,1)$ with $\phi_0\geq\frac{1}{2}$ ($\phi_0\leq\frac{1}{2}$) if and only if $\alpha\geq\frac{1}{2}$ ($\alpha\leq\frac{1}{2}$). Fund managers' expected utility is given by

$$\overline{U}_{n,k}(x)=p\phi_{0,n-k}x+(1-p)\frac{\pi_H}{\pi_L}\phi_{1,k}[\frac{1}{\pi_H}-x],$$

where $x\in\Delta$ with $\phi_{0,n-k}=L_{n,n-k}(\phi_0)$ and $\phi_{1,k}=L_{n,n-k}(\phi_1)$ (see appendix).

We now examine the special case $k=n-1$: only the worst performing fund gets no compensation. Atom ϕ_0 is the solution of the equation

$$\alpha[1-\phi_0^{n-1}]=(1-\alpha)[1-(1-\phi_0)^{n-1}]. \tag{3}$$

The benchmark return distribution assigns atoms $\phi_{0,1}=1-(1-\phi_0)^{n-1}$ and $\phi_{1,n-1}=1-(1-\phi_1)^{n-1}$ at $x=0$ and $x=\frac{1}{\pi_H}$ respectively. When $\alpha=\frac{1}{2}$ the manager is indifferent between receiving a given compensation in state H or in state L. We have $\phi_0=\phi_1=\frac{1}{2}$, and as n increases, a manager is facing benchmark return distributions that assign increasingly full mass at one edge of Δ in each state and asymptotically gets an incentive fee equal to the maximum allowed by the leverage constraint with probability $\frac{1}{2}$, half of the time in state H and half of the time in state L. Now assume that $\alpha<\frac{1}{2}$ so that risk neutral managers shall prioritize high returns in state H. Using relationship (3) and the fact that $\phi_1+\phi_0=1$ we get:

$$\phi_1\underset{n\to\infty}{\sim}1-\frac{a_1}{n-1}$$
$$\phi_{0,1}\underset{n\to\infty}{\sim}1-e^{-a_1}$$
$$\phi_{1,n-1}\underset{n\to\infty}{\sim}1,$$

with $a_1 = \ln \frac{1-\alpha}{1-2\alpha} > 0$. For n sufficiently large, even if all managers assign almost full mass at point $x = \frac{1}{\pi_H}$, they still assign a remaining small probability to point $x = 0$, which implies that the minimum return distribution does assign some positive mass at point $x = 0$. If state H occurs, fund managers earn an incentive fee that is equal to the maximum allowed by the leverage constraint. In other words, as competition intensifies, fund managers choose an investment strategy that offers the largest differentiation with respect to the benchmark in the favored state. A similar analysis holds when $\alpha > \frac{1}{2}$ in which case managers prioritize state L.

Numerical Simulations about the Impact of the Competitiveness of the Benchmark.

We illustrate numerically the impact of the benchmark on the equilibrium investment strategies for an industry with $n = 10$ funds when the managers are risk adverse. We use the same parameter values as in the previous section and vary the parameter k of the competitiveness of the benchmark.

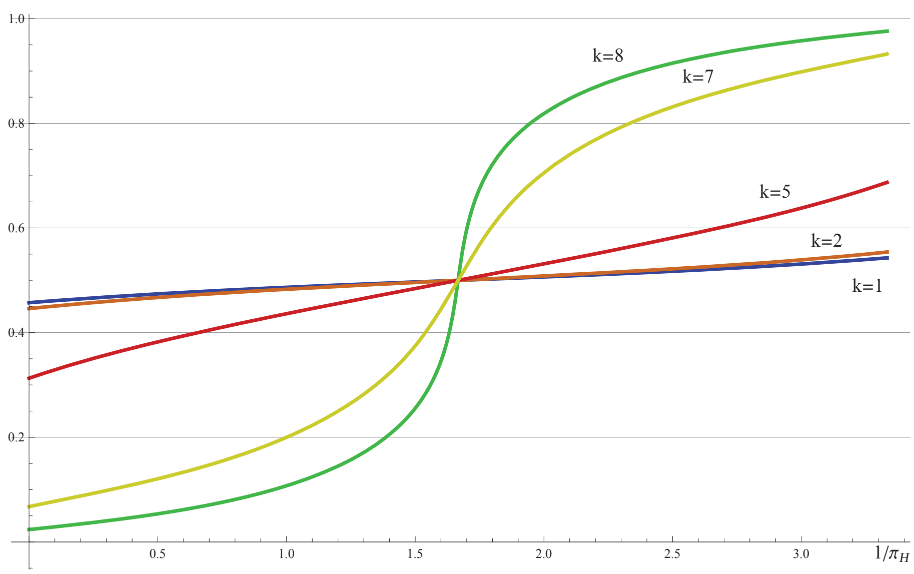

Figure 2 : Equilibrium distributions and competitiveness of the benchmark

In Figure 2, we plot the equilibrium distribution F for several benchmarks. For a sufficiently competitive benchmarks (low value of k, namely $k \leq 5$) equilibrium investment strategies are similar to those used in the winner takes all case. As the benchmark gets less competitive (increase in k), the size of the atoms shrinks even though the pdf distribution remains bimodal, which suggests that managers use less risky strategies. This last feature gets exacerbated as the benchmark becomes loose.

Conversely, for sufficient large values of k (namely k above 7), when the benchmark becomes loose, the equilibrium distribution F becomes convex (resp. concave) on the interval $(0, \frac{1}{2\pi_H})$ (resp. $(\frac{1}{2\pi_H}, \frac{1}{\pi_H})$), so the pdf distribution is unimodal and reaches its maximum in the middle of the support. Fund managers assign increasing probabilities to the returns away from the edges while reducing the size of the atoms on the edges. Managers are more concerned with the trade-off between risk and return. Contrary to the two-fund case and the winner-takes-all case, managers may receive a compensation in both states (for large values of k), when they rely on investment strategies with limited risk-exposure.

Finally, we investigate the impact of the competition in the industry n under the less demanding benchmark, $k = n - 1$.

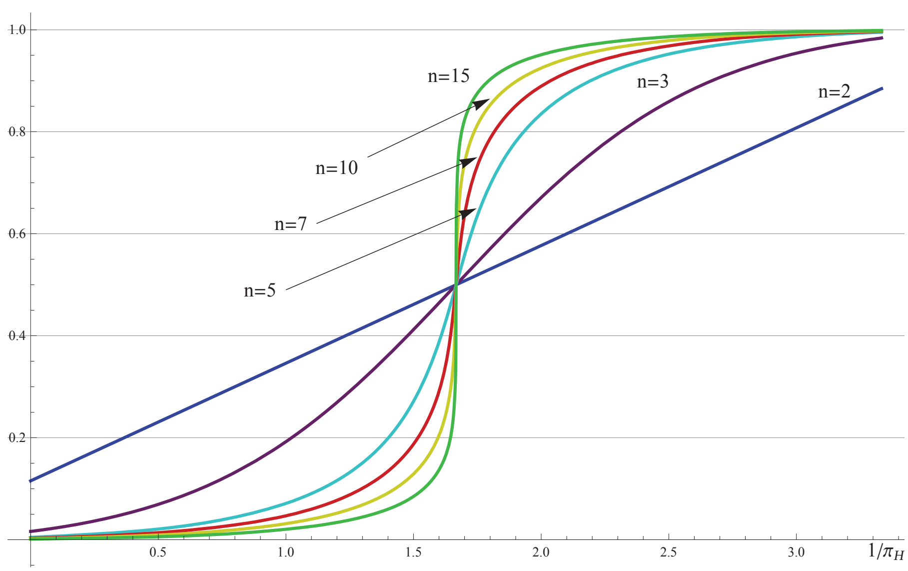

Figure 3 : Equilibrium distributions and industry size under less competitive benchmark

Figure 3 plots the equilibrium distribution F for several industry sizes under the less competitive benchmark. For $n > 2$, the distribution is convex (resp. concave) on the first (second) part of the support Δ. The density function is thus bell-shaped. As the number of funds increases, the size of the atoms at the edges of the support shrinks whereas the distribution become more convex (resp. concave) on the first (resp. second) part of Δ. In the fully symmetric case, the return $\frac{1}{2\pi_H}$ corresponds to the safe return so we conclude that managers are choosing more conservative strategies with a higher probability. The intuition is straightforward: as the number of funds increases, the probability to deliver the worse return of the industry is reduced and it becomes easier to earn an incentive fee. As the two states occur with equal probability, conservative portfolios maximize the likelihood to collect a compensation in both states.

Overall, the simulations reveal that more competitive benchmarks lead to riskier strategies and competition magnifies this effect.

5 Coordination, Differentiation and Risk Taking

In our model, fund managers' investment strategies are driven by two forces. The first force corresponds to the well-known trade-off between risk and return: Fund managers choose their investment strategy to limit the risk in their payoff.[6] If the compensation contract was paying the manager a fraction of the return of the fund, the optimal allocation between the risky asset and the risk-free bond would only depend on the level of risk aversion of the manager. However, as the compensation contract is benchmarked against the return of competing funds, a second force based on differentiation, or anti-herding, arises. This force is better understood when only two funds compete. When two managers use opposite investment strategies, their portfolios' return differ most of the time, which leads to a large payoff for one of the managers. If two managers were to use exactly the same investment strategies, their payoff would always be zero. Differentiation may not always be easy to achieve. The strategic nature of the managers' tournament is similar to that of anti-coordination games in which agents must choose opposite actions.[7] The trade-offs between these two forces govern equilibrium strategies. As we have shown, there exists multiple equilibria in which the degree of differentiation and risk-taking

[6]In a model with more assets, risk management would also take the form of a diversified portfolio.

[7]Traditional examples of anti-coordination games discussed in the game-theory literature are the entry games, the game of chicken and the hawk-dove game.

differ. This has important consequences both for the payoff of the managers and the welfare of their customers.

When managers are risk-neutral, there is no need for risk reduction. The trade-off between portfolio performance and differentiation is simpler. As shown in Proposition 2, in all mixed-strategy equilibria, the investment strategy average returns and the managers' payoffs are equal. One fund manager randomizes between the two extreme strategies while the other can pick many investment strategies. The riskiness of investment strategies depends on what equilibrium is played. In particular, in the symmetric equilibrium, in which both managers randomize between extreme return strategies, the riskiness of the investment strategy is maximized. The need for differentiation can lead managers to choose risky strategies that do not yield additional payoffs.

With more funds, in the full-support mixed strategy equilibrium, managers increase the size of the atoms linked with extreme strategies, thus increasing the risk involved. Interior strategies are chosen to reduce the risk in their compensation. However, with many funds, the probability to be the best performing fund is small and interior strategies most likely lead to zero payoff. Managers prefer to play extreme strategies that give them a chance to win the tournament. In a way, it becomes easier to coordinate as the value of coordination decreases. When the benchmark becomes less competitive, (high k and high n), interior strategies become more valuable as it is easier to be in the first k best performing funds. The risk reduction motive becomes more important and the overall investment strategies are less risky.

5.1 Coordination across Equilibria

Coordination becomes even more important when fund managers are risk-averse as payoffs are not constant across mixed-strategy equilibria. As discussed after Proposition 1, the payoff of a manager when he plays an extreme investment strategy (0 or $\frac{1}{\pi_H}$) is maximized when he plays the extreme strategy with probability one. A lot of coordination is required for both managers to play extreme strategies. In particular, pure strategy equilibria require to choose independently completely opposite strategies. These equilibria are not very natural and mixed strategy equilibria appear to be a better representation of agents' behavior. Furthermore, with more than two managers, no equilibrium exists in this simple form. In section 4, we solve for the pure strategy Nash equilibria and the full-support mixed strategy equilibrium of the game.

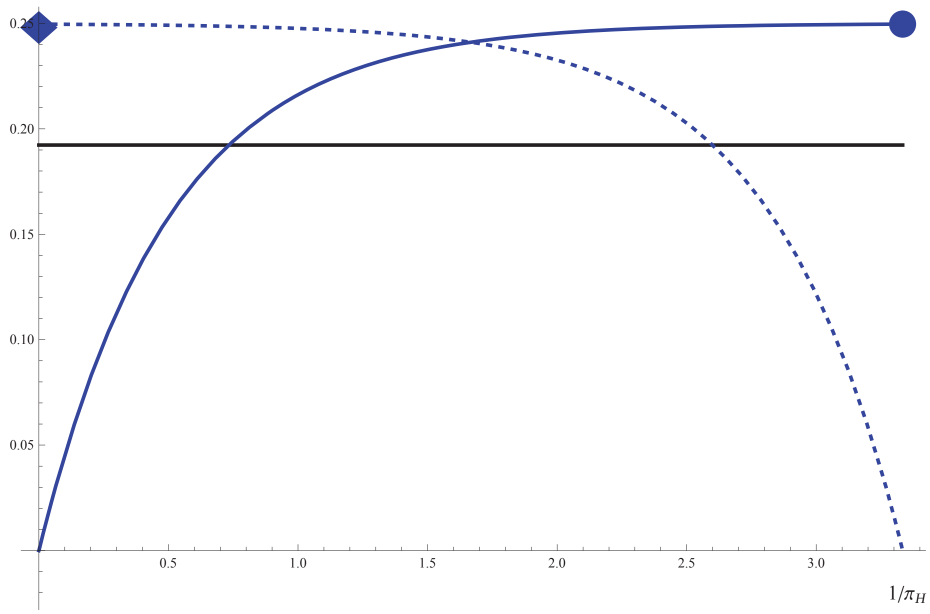

Figure 4 : Pure strategy and full support mixed strategy Nash Equilibria

Figure 4 compares the payoffs in the pure-strategy equilibrium with those in the full-support mixed-strategy equilibrium for the completely symmetric game using the same parameter values as in section 4.2.

The horizontal line represents the level of payoff received by the managers in the full support equilibrium; the payoff is constant which makes playing any strategy in the support optimal. The two curves represent the (possible) payoffs of the managers when the other manager is using a pure-strategy extreme investment strategy. The payoff is maximized when the managers differentiate their strategy to the maximum possible extent; the diamond and the circle represent the optimal strategy. Clearly, coordination leads to higher payoff as it allows managers to fully take advantage of the convexity of their compensation contracts.

5.2 Other Mixed-Strategy Equilibria

There exist other mixed strategy Nash equilibria in which managers use a finite number of investments. We characterize these equilibria for the two-player case in the supplement appendix. We show that in any mixed strategy Nash equilibrium that is not full support, the support of investments must be discrete. Managers use atoms that are disjoint and intertwined. If manager i assigns an atom at point $x_i \in (0, \frac{1}{\pi_H})$, manager j must assign atoms at points $\underline{x}_i$ and $\overline{x}_i$ with $0 \leq \underline{x}_i < x_i < \overline{x}_i \leq \frac{1}{\pi_H}$ and manager i assigns no atom on $[\underline{x}_i, x_i) \cup (x_i, \overline{x}_i]$. Finally, boundary points $\{0\}$ and $\{\frac{1}{\pi_H}\}$ belong to at least one equilibrium distribution support.

Equilibria may be classified by the number of atoms played by the fund managers. We call equilibrium of type 1 an equilibrium in which manager j is playing the two extreme strategies, while manager i is playing a pure strategy in the interior of the support, i.e., wants to hold a balanced portfolio. An equilibrium of type 2 is characterized by manager i playing two atoms while manager j plays two or three atoms. More generally, in a type N equilibrium, manager i plays N atoms while manager j plays N or $N+1$ atoms. More atoms means less coordination between the managers. In general, it is difficult to compare the payoffs across equilibria.

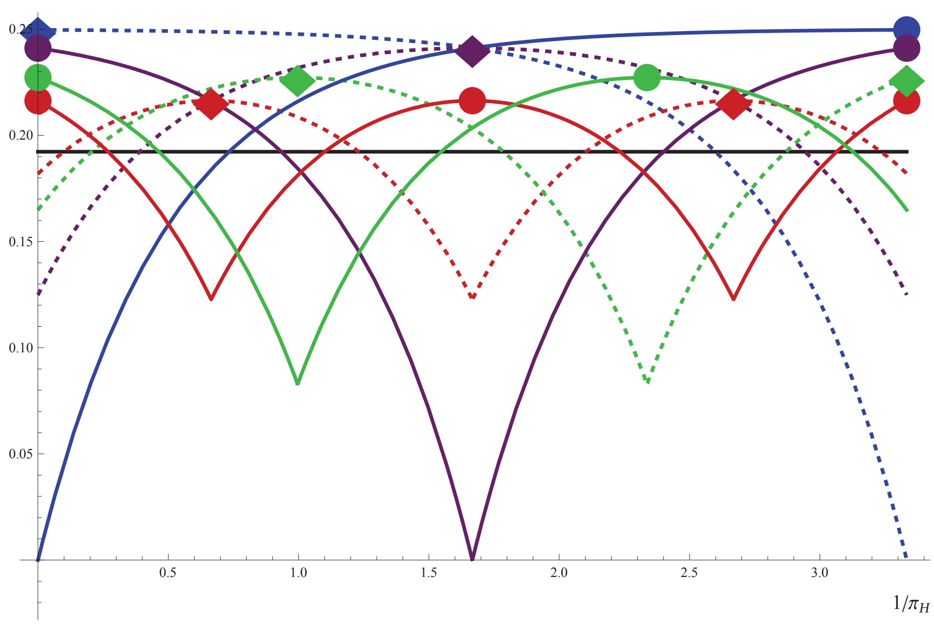

Figure 5 : Pure strategy, asymmetric and full support mixed strategy Nash Equilibria

Figure 5 depicts the payoffs of the managers in three mixed-strategy equilibria along with the two other equilibria already plotted in Figure 4. We use the same parameter values as in section 4.2 so that equilibrium distributions are symmetric with respect to point $\frac{1}{2\pi_H}$ and as before, the circles and the diamonds corresponds to the equilibrium payoffs (at investment strategies that are played).

These strategies indeed are optimal as the atoms are used at points that maximize the managers' payoff. Worth noticing is the fact that the more atoms used in an equilibrium, the lower the payoffs of the managers in this equilibrium. Equilibrium payoffs take values between the pure-strategy equilibrium payoff and the full-support mixed strategy payoff: This tends to show that the better the managers are able to coordinate, the higher their payoffs.

These examples of equilibria deal with the completely symmetric game for two funds. This is a special case for which a clean comparison between payoffs across equilibria is possible. Even though the study of asymmetric equilibria for more than two funds is beyond the scope of this paper, in the case of three funds under the winner takes all benchmark, one can verify that for the completely symmetric game, managers 1 and 3 randomizing with probability 1/2 between the edges of support Δ and manager 2 assigning full mass at the midpoint of Δ as well as managers 1 and 3 assigning full mass at the midpoint of Δ and manager 2 randomizing with probability 1/2 between the edges of support Δ are indeed Nash Equilibria. Finally, in an asymmetric setting, the payoffs in the pure strategy equilibrium are not equal, which makes the comparison of payoffs across equilibria more complex. Still, we have pointed out that highest payoff is achieved in the pure strategy equilibrium and the lowest payoff is achieved in the full-support equilibrium.

6 Conclusion

In this paper, we study the role of peer-group benchmarking in the compensation of fund managers. Peer-group benchmarks link the manager's compensation to the return of their fund compared to the returns of similar funds in the industry, which provides alternative incentives to managers and diverts them from the usual optimization between risk and return. Standard benchmark metrics used in practice can take many forms. They can correspond for instance to the return of the best performing fund, the median performing fund, or the average return of funds in the industry. Depending on the number of funds in the group, different benchmarks can thus be more or less competitive. Funds mangers are under much bigger pressure to perform if they need to be the best performing fund among dozens of funds rather than to be above the average return in the industry.

To investigate the effect of the competitiveness of the benchmark on funds investment strategies, we introduce a novel model of tournament between risk-averse fund managers. Managers' portfolio choices are motivated by two rationales. They want to differentiate from other competitors so that they perform well when other firms underperform. This leads to extreme strategies characterized by a high level of risk. The other concern is to limit the risk of the compensation. These motives creates a strategic environment in which managers want to (anti-)coordinate which leads to multiple equilibria. We characterize pure strategy equilibria in which managers solve the coordination issue. We then focus on the unique full-support mixed strategy equilibrium in which managers use all possible investment strategies according to a smooth probability function and use atoms on the two extreme investment strategies.

We use the model to analyze the effects of the competition in the industry and the competitiveness of benchmarks on investment strategies. The model shows that more competition and a more competitive benchmark leads to riskier strategies. These results contribute to the debate on the use of peer-group benchmarks. They can inform better regulation and better practice in the design of compensation contracts.

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8 Appendix

8.1 Appendix

Proof of Proposition 1. Set $u_{iH}(z) = p\frac{1-e^{-\bar{b}_i z}}{b_i}$ and $u_{iL}(x) = (1-p)\frac{1-e^{-\frac{\bar{b}_i \pi_H}{\pi_L} z}}{b_i}$. Given a return $y \in \Delta$ chosen by manager j, manager i chooses a return $x \in \Delta$ to maximize

$$u_{iH}(x-y)H(x-y) + u_{iL}(y-x)H(y-x).$$

If $x \geq y$, then the maximum utility is achieved at $x^* = \frac{1}{\pi_H}$ and the resulting utility is $u_{iH}(\frac{1}{\pi_H} - y)$. Conversely, if $x \leq y$, then the maximum utility is achieved at $x^* = 0$ and the resulting utility is $u_{iL}(y)$. We deduce that $x^* = \frac{1}{\pi_H}$ is the best response if and only if $\varphi(y) \geq 0$ with $\varphi(y) = u_{iH}(\frac{1}{\pi_H} - y) - u_{iL}(y)$. Function φ is strictly decreasing with $\varphi(0) = u_{iH}(\frac{1}{\pi_H}) > 0$ and $\varphi(\frac{1}{\pi_H}) = -u_{iL}(\frac{1}{\pi_H}) < 0$, so there is a unique $\overline{y}_i$ in $\left(0, \frac{1}{\pi_H}\right)$, such that $\varphi(\overline{y}_i) = 0$. Manager i's best response is given by

$$x^* = \begin{cases} \frac{1}{\pi_H}, \text{ if } y \in [0, \overline{y}_i) \\ \in \{0, \frac{1}{\pi_H}\}, \text{ if } y = \overline{y}_i \\ 0, \text{ if } y \in (\overline{y}_i, \frac{1}{\pi_H}]. \end{cases}$$

The desired result follows easily. One can verify that $\lim\limits_{\bar{b}_i \to \infty} \overline{y}_i = \frac{1}{\pi_H}$ (resp. 0) if $p > 1/2$ (resp. $p < 1/2$). When $p = 1/2$, we have $\lim\limits_{\bar{b}_i \to \infty} \overline{y}_i = \frac{\pi_L}{\pi_H} \frac{1}{\pi_H + \pi_L}$. Finally note that

$$\begin{aligned} U_i(x) &= \int_\Delta [u_{iH}(x-y)H(x-y) + u_{iL}(y-x)H(y-x)]dF_j(y) \\ &\leq \int_\Delta \max \ \{u_{iH}(\frac{1}{\pi_H} - y), u_{iL}(y)\} dF_j(y) \\ &\leq \max \ \{p\frac{1-e^{-\frac{\bar{b}_i}{\pi_H}}}{b_i}, (1-p)\frac{1-e^{-\frac{\bar{b}_i}{\pi_L}}}{b_i}\} \text{ as } \int_\Delta dF_j(y) = 1. \ \blacksquare \end{aligned}$$

Proof of Proposition 2. Without loss of generality, we may assume that $k_i W_{i0} = k_j W_{j0} = 1$. For all $x \in \Delta$, we have

$$\begin{aligned} U_i(x) &= p\int_\Delta (x-y)H(x-y)dF_j(y) + (1-p)\frac{\pi_H}{\pi_L}\int_\Delta (y-x)H(y-x)dF_j(y) \\ &= U_i(0) + \frac{p\pi_L + (1-p)\pi_H}{\pi_L}\left[\int_0^x (F_j(z) - \alpha)dz\right], \end{aligned}$$

with $U_i(0) = (1-p)\frac{\pi_H}{\pi_L}\int_\Delta y dF_j(y)$ and $\alpha = \frac{(1-p)\pi_H}{p\pi_L + (1-p)\pi_H}$.

Case 1: Let denote Γ_j^* the distribution that consists in two atoms at $x = 0$ and $x = \frac{1}{\pi_H}$ with

$$\phi_{0j}^* = \alpha \text{ and } \phi_{1j}^* = 1 - \alpha.$$

For all $x \in \Delta$, we have $U_i(x) = U_i(0) = \frac{p(1-p)}{p\pi_L+(1-p)\pi_H}$. Thus Δ_i could be any subset of Δ. Then the indifference condition for player j is given by $U_j(0) = U_j(\frac{1}{\pi_H})$, or equivalently

$$\int_\Delta z dF_i(z) = \frac{1-\alpha}{\pi_H}. \tag{4}$$

It remains to check for profitable deviations by player j. Observe that for all $x \in \Delta$

$$U_j(x) = U_j(0) + \frac{p\pi_L + (1-p)\pi_H}{\pi_L} \left[\int_0^x (F_i(z) - \alpha) dz \right].$$

Then, it is easy to verify that relationship (4) implies $F_i(0^+) \leq \alpha$. As F_i is non-decreasing, function U_j starts (weakly) decreasing and then possibly increases (if $F_i(\frac{1}{\pi_H}^-) > \alpha$). As $U_j(\frac{1}{\pi_H}) = U_j(0)$, we conclude that there is no profitable deviation.

In the remaining of the proof, we assume that no player is using strategy Γ_j^*. In addition, it is easy to verify that if player j only assigns atoms at $x = 0$ or/and $x = \frac{1}{\pi_H}$ other that ϕ_{0j}^* and ϕ_{1j}^*, then the corresponding equilibrium is a pure strategy Nash equilibrium.

Case 2: Assume that $\phi_{0j} > \phi_{0j}^*$, then for all $x \in (0, \frac{1}{\pi_H})$, U_i is increasing: player i may only play at $x = 0$ and $x = \frac{1}{\pi_H}$. If player i only plays at $x = 0$, there is no equilibrium. If player i only plays at $x = \frac{1}{\pi_H}$, then we have a pure Nash equilibrium $(\frac{1}{\pi_H}, 0)$. If player i plays at $x = 0$ and $x = \frac{1}{\pi_H}$, there is no equilibrium as the condition $F_j(0^+) \leq \alpha$ is not met.

Case 3: Assume that $\phi_{0j} < \phi_{0j}^*$. There are several cases. If for all $x \in (0, \frac{1}{\pi_H})$, $F_j(x) < \alpha$, then Δ_i is a subset of $\{0, \frac{1}{\pi_H}\}$, which has been analyzed. If there is $a_i = \inf \{x \in (0, \frac{1}{\pi_H}), F_j(x) \geq \alpha\}$, then again Δ_i is a subset of $\{0, \frac{1}{\pi_H}\}$, which has analyzed.

Case 4: Assume that $\phi_{0j} = \phi_{0j}^*$. Then, as player j does not use Γ_j^*, there exists $x_j < \frac{1}{\pi_H}$ such that $x_j = \inf \{x \in (0, \frac{1}{\pi_H}), F_j(x_j + \varepsilon) \geq \alpha\}$ for all small $\varepsilon > 0$. Thus for all $x \geq x_j + \varepsilon$, U_i is increasing. This implies that Δ_i is a subset of $\{0, \frac{1}{\pi_H}\}$, which has been analyzed.

Utility Level. If player j assigns atoms ϕ_{0j}^* (resp. ϕ_{1j}^*) at $x = 0$ (resp. $x = \frac{1}{\pi_H}$), for all $x \in \Delta_i$, player i's utility is given by

$$U_i(x) = \frac{(1-p)(1-\alpha)}{\pi_L} k_i W_{i0} = \frac{p\alpha}{\pi_H} k_i W_{i0}.$$

Similarly, player j's utility level at $x = 0$ and $x = \frac{1}{\pi_H}$ is given by

$$\begin{aligned} U_j(0) &= k_j W_{j0}(1-p)\frac{\pi_H}{\pi_L}\int_\Delta z dF_i(z) \\ &= \frac{(1-p)(1-\alpha)}{\pi_L} k_j W_{j0} = \frac{p\alpha}{\pi_H} k_j W_{j0}. \end{aligned}$$

Variance. The variance of the returns is given by

$$\begin{aligned} \mathrm{var}(R) &= p\int_\Delta y^2 dF(y) + (1-p)\frac{\pi_H^2}{\pi_L^2}\int_\Delta (\frac{1}{\pi_H} - y)^2 dF(y) - \overline{R}^2 \\ &= \frac{p\pi_L^2 + (1-p)\pi_H^2}{\pi_L^2}\int_\Delta y^2 dF(y) + \frac{(1-p)}{\pi_L^2} - \frac{2(1-p)\pi_H}{\pi_L^2}\int_\Delta y dF(y) - \overline{R}^2. \end{aligned}$$

Since $\int_\Delta y dF(y)$ is constant, $\mathrm{var}(R)$ achieves its minimum (maximum) whenever $\int_\Delta y^2 dF(y)$ does so. By the Cauchy Schwarz inequality we have $\left(\int_\Delta y dF(y)\right)^2 \leq \int_\Delta dF(y) \times \int_\Delta y^2 dF(y)$, so that the minimum variance is achieved for a cdf F such that $\int_\Delta y^2 dF(y) = \left(\int_\Delta y dF(y)\right)^2$, i.e., $dF(y) = \delta_{\frac{1-\alpha}{\pi_H}}(y)$. Then let a_L (resp. a_H) in $[0, \frac{1}{\pi_H}]$ denote the lower (resp. upper) boundaries of the support of cdf F at which atoms ϕ_L and ϕ_H (with possibly zero value) are assigned. Note that it must be the case that $a_L \leq \frac{1-\alpha}{\pi_H} \leq a_H$ so that $\int_\Delta y dF(y) = \frac{1-\alpha}{\pi_H}$. We have

$$\begin{aligned} \int_\Delta y^2 dF(y) &= \phi_L a_L^2 + \phi_H a_H^2 + \int_{a_L^+}^{a_H^-} y^2 dF(y) \\ &\leq \phi_L a_L^2 + \phi_H a_H^2 + a_H^2(1-\phi_L-\phi_H) \\ &= \phi_L a_L^2 + a_H^2(1-\phi_L). \end{aligned}$$

Thus, to find the maximum of $\int_\Delta y^2 dF(y)$ we only need to consider distributions that only assign atoms at a_L and a_H so $\phi_L + \phi_H = 1$ and satisfies $\phi_L a_L + a_H(1-\phi_L) = \frac{1-\alpha}{\pi_H}$. As $a_L \leq \frac{1-\alpha}{\pi_H} \leq a_H$, the maximum is reached when $a_L = 0$, $a_H = \frac{1}{\pi_H}$ and $\phi_L = \alpha$. ■

Proof of Proposition 3.

Step 1: Existence and Uniqueness Assume that Γ_i has full support so that for all $x \in \Delta$

$$U_i(x) = p\int_\Delta \frac{1-e^{-\bar{b}_i(x-y)}}{b_i}H(x-y)dF_j(y) + (1-p)\int_\Delta \frac{1-e^{-\bar{b}_i\frac{\pi_H}{\pi_L}(y-x)}}{b_i}H(y-x)dF_j(y) = \widehat{U}_i. \quad (5)$$

Set $C_i = \int_\Delta e^{\bar{b}_i y} dF_j(y)$ and $D_i = \int_\Delta e^{-\bar{b}_i \frac{\pi_H}{\pi_L} y} dF_j(y)$. Then evaluating relationship (5) at $x = \frac{1}{\pi_H}$ and $x = 0$ respectively leads to

$$b_i \overline{U}_i = p - p e^{-\frac{\bar{b}_i}{\pi_H}} C_i \tag{6}$$

$$b_i \overline{U}_i = 1 - p - (1-p) D_i. \tag{7}$$

Differentiating relationship (5) with respect to x yields for all $x \leq \frac{1}{\pi_H}$

$$p I_i(x) - (1-p) \frac{\pi_H}{\pi_L} J_i(x) = 0, \tag{8}$$

where $I_i(x) = \int_\Delta e^{-\bar{b}_i (x-y)} H(x-y) dF_j(y)$ and $J_i(x) = \int_\Delta e^{-\bar{b}_i \frac{\pi_H}{\pi_L}(y-x)} H(y-x) dF_j(y)$ are defined for all $x \geq 0$.

Then as for $x > \frac{1}{\pi_H}$ we must have $dF_j = 0$ so that $I_i(x) = C_i e^{-\bar{b}_i x}$ and $J_i(x) = 0$ we extend relationship (8) as follows:

$$p I_i(x) - (1-p) \frac{\pi_H}{\pi_L} J_i(x) = p C_i e^{-\bar{b}_i x}.$$

To sum up, for all $x \in \mathbb{R}_+$ we have

$$p I_i(x) - (1-p) \frac{\pi_H}{\pi_L} J_i(x) = p C_i e^{-\bar{b}_i x} 1_{(\frac{1}{\pi_H}, \infty)}. \tag{9}$$

Since F_j has a compact support, the Laplace transform $\widehat{f}_j(s) = \int_\Delta e^{-sy} dF_j(y)$ exists. Then using the property of the Laplace Transform for a convolution, we have $\widehat{I}_i(s) = \frac{\widehat{f}_j(s)}{s + \bar{b}_i}$. Furthermore, we note that

$$dJ_i(x) = \bar{b}_i \frac{\pi_H}{\pi_L} J_i(x) dx - dF_j(x),$$

which implies that $\widehat{J}_i(s) = \frac{\widehat{f}_j(s)}{\bar{b}_i \frac{\pi_H}{\pi_L} - s} - \frac{D_i}{\bar{b}_i \frac{\pi_H}{\pi_L} - s}$. Taking the Laplace transform of relationship (9) yields

$$p \frac{\widehat{f}_j(s)}{s + \bar{b}_i} + (1-p) \frac{\pi_H}{\pi_L} \frac{\widehat{f}_j(s)}{s - \bar{b}_i \frac{\pi_H}{\pi_L}} = p e^{-\frac{\bar{b}_i}{\pi_H}} C_i \frac{e^{-\frac{s}{\pi_H}}}{s + \bar{b}_i} + (1-p) \frac{\pi_H}{\pi_L} \frac{D_i}{s - \bar{b}_i \frac{\pi_H}{\pi_L}}. \tag{10}$$

Then, using relationships (6) and (7) yields

$$\frac{p\pi_L + (1-p)\pi_H}{\pi_L} \frac{s - \lambda_i}{(s + \bar{b}_i)(s - \bar{b}_i \frac{\pi_H}{\pi_L})} \widehat{f}_j(s) = (p - b_i \overline{U}_i) \frac{e^{-\frac{s}{\pi_H}}}{s + \bar{b}_i} + (1 - p - b_i \overline{U}_i) \frac{\pi_H}{\pi_L} \frac{1}{s - \bar{b}_i \frac{\pi_H}{\pi_L}},$$

Rearranging terms, we obtain that

$$\frac{p\pi_L + (1-p)\pi_H}{\pi_L}\widehat{f}_j(s) = (1-p-b_i\overline{U}_i)\frac{\pi_H}{\pi_L} + (p-b_i\overline{U}_i)e^{-\frac{s}{\pi_H}} + (p-b_i\overline{U}_i)(\bar{b}_i\frac{\pi_H}{\pi_L}-\lambda_i)e^{-\frac{\lambda_i}{\pi_H}}\frac{1-e^{\frac{\lambda_i - s}{\pi_H}}}{s-\lambda_i}$$
$$+\frac{(1-p-b_i\overline{U}_i)(\lambda_i+\bar{b}_i)\frac{\pi_H}{\pi_L} - (p-b_i\overline{U}_i)(\bar{b}_i\frac{\pi_H}{\pi_L}-\lambda_i)e^{-\frac{\lambda_i}{\pi_H}}}{s-\lambda_i}.$$

Performing the inversion of the Laplace transform that is injective, there is a unique distribution F_j that is given by

$$dF_j(x) = \phi_0\delta_0(x) + \phi_1\delta_{1/\pi_H}(x) + A_j e^{\lambda_i x}1_{\left[0,\frac{1}{\pi_H}\right]}dx$$
$$+\frac{(1-p-b_i\overline{U}_i)(\lambda_i+\bar{b}_i)\frac{\pi_H}{\pi_L} - (p-b_i\overline{U}_i)(\bar{b}_i\frac{\pi_H}{\pi_L}-\lambda_i)e^{-\frac{\lambda_i}{\pi_H}}}{s-\lambda_i}e^{\lambda_i x}1_{(\frac{1}{\pi_H},\infty)}dx,$$

with

$$\begin{aligned}
\phi_0 &= \frac{\pi_H}{p\pi_L+(1-p)\pi_H}(1-p-b_i\overline{U}_i) \\
\phi_1 &= \frac{\pi_L}{p\pi_L+(1-p)\pi_H}(p-b_i\overline{U}_i) \\
A_j &= \phi_1(\bar{b}_i\frac{\pi_H}{\pi_L}-\lambda_i)e^{-\frac{\lambda_i}{\pi_H}},
\end{aligned}$$

and δ_a denote the Dirac function at point a. In order to have $f \equiv 0$ on $\left(\frac{1}{\pi_H},\infty\right)$, we must have

$$(1-p-b_i\overline{U}_i)(\lambda_i+\bar{b}_i)\frac{\pi_H}{\pi_L} = (p-b_i\overline{U}_i)(\bar{b}_i\frac{\pi_H}{\pi_L}-\lambda_i)e^{-\frac{\lambda_i}{\pi_H}}, \tag{11}$$

or equivalently $A_j = \phi_0(\lambda_i+\bar{b}_i)$. The value of A_j is determined by the normaalization condition $\int_\Delta dF_j(y) = 1$. Finally, observe that distribution Γ_j with cdf F_j has full support. ■

Step 2: Lowest Utility Level.

From relationship (11), we have

$$\overline{U}_i = \frac{p(1-p)(e^{\frac{\lambda_i}{\pi_H}}-1)}{b_i(pe^{\frac{\lambda_i}{\pi_H}}-(1-p))}$$

Let $\overline{U}$ denote the utility level derived by player i in *some* equilibrium. For $x \le \frac{1}{\pi_H}$, we shall have

$$U(x) = p\int_\Delta \frac{1-e^{-\bar{b}_i(x-y)}}{b_i}H(x-y)dF_j(y) + (1-p)\int_\Delta \frac{1-e^{-\bar{b}_i\frac{\pi_H}{\pi_L}(y-x)}}{b_i}H(y-x)dF_j(y) \le \overline{U}.$$

Set $C_i = \int_\Delta e^{\bar{b}_i y} dF_j(y)$ and $D_i = \int_\Delta e^{-\bar{b}_i \frac{\pi_H}{\pi_L} y} dF_j(y)$ observe that

$$\begin{aligned} b_i\overline{U} &\geq p - pe^{-\frac{\bar{b}_i}{\pi_H}} C_i \\ b_i\overline{U} &\geq 1 - p - (1-p)D_i \end{aligned}$$

Then, for $x > \frac{1}{\pi_H}$, as seen before we impose $U(x) = p\frac{1-e^{-\bar{b}_i x}C_i}{b_i}$. Taking the Laplace Transform on both sides of the inequality yields

$$p[\frac{1}{s} - \frac{1}{s+\bar{b}_i}]\widehat{f}_j(s) + (1-p)[\frac{1}{s} - \frac{\widehat{f}_j(s)}{s} - \frac{\widehat{f}_j(s)}{\bar{b}_i\frac{\pi_H}{\pi_L} - s} + \frac{D_i}{\bar{b}_i\frac{\pi_H}{\pi_L} - s}] \leq b_i\overline{U}\frac{1-e^{-\frac{s}{\pi_H}}}{s} + p\frac{e^{-\frac{s}{\pi_H}}}{s} - pC_i\frac{e^{-\frac{s+\bar{b}_i}{\pi_H}}}{s+\bar{b}_i},$$

or equivalently

$$\begin{aligned} \frac{\bar{b}_i}{s}[\frac{p}{s+\bar{b}_i} + \frac{\pi_H}{\pi_L}\frac{1-p}{s-\bar{b}_i\frac{\pi_H}{\pi_L}}]\widehat{f}_j(s) &\leq \frac{1}{s}\Bigg\{[b_i\overline{U} - (1-p)(1-D_i)] - [b_i\overline{U} - p(1-e^{-\frac{\bar{b}_i}{\pi_H}}C_i)]e^{-\frac{s}{\pi_H}} \\ &\quad + p\bar{b}_iC_i\frac{e^{-\frac{s+\bar{b}_i}{\pi_H}}}{s+\bar{b}_i} + (1-p)\bar{b}_i\frac{\pi_H}{\pi_L}\frac{D_i}{s-\bar{b}_i\frac{\pi_H}{\pi_L}}\Bigg\}. \end{aligned}$$

It follows that

$$\begin{aligned} \bar{b}_i\frac{p\pi_L + (1-p)\pi_H}{\pi_L}\frac{s-\lambda_i}{s(s+\bar{b}_i)(s-\bar{b}_i\frac{\pi_H}{\pi_L})}\widehat{f}_j(s) &\leq \frac{1}{s}\Bigg\{[b_i\overline{U} - (1-p)(1-D_i)] - [b_i\overline{U} - p(1-e^{-\frac{\bar{b}_i}{\pi_H}}C_i)]e^{-\frac{s}{\pi_H}} \\ &\quad + p\bar{b}_iC_i\frac{e^{-\frac{s+\bar{b}_i}{\pi_H}}}{s+\bar{b}_i} + (1-p)\bar{b}_i\frac{\pi_H}{\pi_L}\frac{D_i}{s-\bar{b}_i\frac{\pi_H}{\pi_L}}\Bigg\}. \end{aligned}$$

Assume $\lambda_i \geq 0$. Multiplying by s the previous inequality and taking the right limit at $s = \lambda_i$ leads to

$$\begin{aligned} 0 &\leq [b_i\overline{U} - (1-p) + (1-p)D_i] - [b_i\overline{U} - p + pe^{-\frac{\bar{b}_i}{\pi_H}}C_i]e^{-\frac{\lambda_i}{\pi_H}} \\ &\quad + p\bar{b}_ie^{-\frac{\bar{b}_i}{\pi_H}}C_i\frac{e^{-\frac{\lambda_i}{\pi_H}}}{\lambda_i+\bar{b}_i} - (1-p)\bar{b}_i\frac{\pi_H}{\pi_L}\frac{D_i}{\bar{b}_i\frac{\pi_H}{\pi_L} - \lambda_i}, \end{aligned}$$

or equivalently

$$\begin{aligned} 0 &\leq [b_i\overline{U} - (1-p) + (1-p)D_i] - [b_i\overline{U} - p + pe^{-\frac{\bar{b}_i}{\pi_H}}C_i]e^{-\frac{\lambda_i}{\pi_H}} \\ &\quad + \bar{b}_i[\frac{(p-b_i\overline{U})e^{-\frac{\lambda_i}{\pi_H}}}{\lambda_i+\bar{b}_i} - (1-p)\frac{\pi_H}{\pi_L}\frac{1-p-b_i\overline{U}}{\bar{b}_i\frac{\pi_H}{\pi_L} - \lambda_i}] \\ &\quad + \bar{b}_i[\frac{pe^{-\frac{\bar{b}_i}{\pi_H}}C_i - p + b_i\overline{U})e^{-\frac{\lambda_i}{\pi_H}}}{\lambda_i+\bar{b}_i} + \frac{\pi_H}{\pi_L}\frac{1-p-b_i\overline{U} - (1-p)D_i}{\bar{b}_i\frac{\pi_H}{\pi_L} - \lambda_i}], \end{aligned}$$

which leads to

$$\begin{aligned}
\bar{b}_i[\frac{(p-b_i\overline{U})e^{-\frac{\lambda_i}{\pi_H}}}{\lambda_i+\bar{b}_i}-(1-p)\frac{\pi_H}{\pi_L}\frac{1-p-b_i\overline{U}}{\bar{b}_i\frac{\pi_H}{\pi_L}-\lambda_i}] &\geq \frac{\lambda_i}{\bar{b}_i\frac{\pi_H}{\pi_L}-\lambda_i}[b_i\overline{U}-(1-p)+(1-p)D_i] \\
+\frac{\lambda_i}{\lambda_i+\bar{b}_i}[b_i\overline{U}-p+pe^{-\frac{\bar{b}_i}{\pi_H}}C_i]e^{-\frac{\lambda_i}{\pi_H}} &\geq 0
\end{aligned}$$

This implies that

$$\overline{U} \geq \frac{p(1-p)(e^{\frac{\lambda_i}{\pi_H}}-1)}{b_i(pe^{\frac{\lambda_i}{\pi_H}}-(1-p))}.$$

The proof is similar in the case when $\lambda_i < 0$. ■

Proof of Proposition 4.

Atoms. Using the definition of λ_i and the expression of A_j, we have

$$\phi_{0j}^{-1} = \frac{p\pi_L+(1-p)\pi_H}{(2p-1)(1-p)\pi_H}\left(p-1+pe^{\frac{\lambda_i}{\pi_H}}\right).$$

It follows that $\frac{\partial}{\partial \bar{b}_i}\left(\phi_{0j}^{-1}\right) = \frac{p}{(1-p)\pi_H}e^{\frac{\lambda_i}{\pi_H}} > 0$. It is easy to verify that

$$\lim_{\bar{b}_i\to\infty}\phi_{0j} = \begin{cases} \frac{(1-2p)\pi_H}{p\pi_L+(1-p)\pi_H}, \text{ if } p<\frac{1}{2} \\ 0, \text{ if } p>\frac{1}{2}. \end{cases}$$

Similarly we have $\phi_{1j}^{-1} = \frac{p\pi_L+(1-p)\pi_H}{(2p-1)p\pi_L}\left(-(1-p)e^{-\frac{\lambda_i}{\pi_H}}+p\right)$, so that $\frac{\partial}{\partial \bar{b}_i}\left(\phi_{1j}^{-1}\right) = \frac{1-p}{\pi_L}e^{-\frac{\lambda_i}{\pi_H}} > 0$. It is easy to verify that

$$\lim_{\bar{b}_i\to\infty}\phi_{1j} = \begin{cases} 0, \text{ if } p<\frac{1}{2} \\ \frac{(2p-1)\pi_L}{p\pi_L+(1-p)\pi_H}, \text{ if } p>\frac{1}{2}. \end{cases}$$

Pdf. For all $x \in \Delta$, we have $A_j = \frac{p(1-p)(\pi_H+\pi_L)}{p\pi_L+(1-p)\pi_H}\frac{\lambda_i}{p-1+pe^{\frac{\lambda_i}{\pi_H}}}$. It follows that

$$\lim_{\bar{b}_i\to\infty}A_j\frac{e^{\frac{\lambda_i-s}{\pi_H}}-1}{\lambda_i-s} = \begin{cases} \frac{p(\pi_H+\pi_L)}{p\pi_L+(1-p)\pi_H}, \text{ if } p<\frac{1}{2} \\ \frac{(1-p)(\pi_H+\pi_L)}{p\pi_L+(1-p)\pi_H}e^{-\frac{s}{\pi_H}}, \text{ if } p>\frac{1}{2}. \end{cases}$$

We conclude that

$$\lim_{\bar{b}_i\to\infty}\widehat{f}_j(s) = \begin{cases} 1, \text{ if } p<\frac{1}{2} \\ e^{-\frac{s}{\pi_H}}, \text{ if } p>\frac{1}{2}. \end{cases}$$

Utility Level Comparison. From relationship (6) we have $b_i\overline{U}_i = p - pe^{-\frac{\bar{b}_i}{\pi_H}}C_i$ with $C_i = \int_\Delta e^{\bar{b}_i y}dF_j(y)$. As for all $y\in\Delta$, $e^{\bar{b}_i y}\geq 1$, we have $C_i \geq 1$ and $C_i = 1$ iff $dF_j(y) = \delta_0(y)$. Similarly, from relationship (7) we have $b_i\overline{U}_i = 1-p-(1-p)D_i$ with $D_i = \int_\Delta e^{-\bar{b}_i\frac{\pi_H}{\pi_L}y}dF_j(y)$. As for all $y\in\Delta$, $e^{-\bar{b}_i\frac{\pi_H}{\pi_L}y} \geq e^{-\frac{\bar{b}_i}{\pi_L}}$, we have $D_i \geq e^{-\frac{\bar{b}_i}{\pi_L}}$ and $D_i = e^{-\frac{\bar{b}_i}{\pi_L}}$ iff $dF_j(y) = \delta_{1/\pi_H}(y)$. ■

8.2 Appendix 2

The Winner Takes All: Proof of Proposition 5.

Step 1: Derivation of the Equilibrium. We want to show the existence (and uniqueness) of a distribution F such that for all x in Δ

$$p\int_{\Delta}\frac{1-e^{-\bar{b}(x-z)}}{\bar{b}}H(x-z)dF_M(z)+(1-p)\int_{\Delta}\frac{1-e^{-\bar{b}\frac{\pi_H}{\pi_L}(\widehat{z}-x)}}{\bar{b}}H(\widehat{z}-x)dF_m(\widehat{z})=\overline{U}_n,$$

where $\overline{U}_n$ is a constant. Set $C_n=\int_\Delta e^{\bar{b}y}dF_M(y)$, $D_n=\int_\Delta e^{-\bar{b}\frac{\pi_H}{\pi_L}y}dF_m(y)$ and observe that $b\overline{U}_n=p-pe^{-\frac{\bar{b}}{\pi_H}}C_n$ and $b\overline{U}_n=1-p-(1-p)D_n$. Then, making no assumption on distributions F_M and F_m, given the results derived for the two managers' game, we find that

$$p\frac{\widehat{f}_M(s)}{s+\bar{b}}+(1-p)\frac{\pi_H}{\pi_L}\frac{\widehat{f}_m(s)}{s-\bar{b}\frac{\pi_H}{\pi_L}}=pe^{-\frac{\bar{b}}{\pi_H}}C_n\frac{e^{-\frac{s}{\pi_H}}}{s+\bar{b}}+(1-p)\frac{\pi_H}{\pi_L}\frac{D_n}{s-\bar{b}\frac{\pi_H}{\pi_L}}.$$

Multiplying both sides by s and letting s goes to $+\infty$, we obtain that

$$\frac{p\pi_L F_M(0^+)+(1-p)\pi_H F_m(0^+)}{\pi_L}=(1-p)\frac{\pi_H}{\pi_L}D_n,$$

or equivalently

$$\frac{p\pi_L\phi_0^{n-1}+(1-p)\pi_H\left[1-(1-\phi_0)^{n-1}\right]}{\pi_L}=(1-p-b\overline{U}_n)\frac{\pi_H}{\pi_L}. \tag{12}$$

Similarly, multiplying both sides by s and letting $se^{\frac{s}{\pi_H}}$ goes to $-\infty$, we obtain that

$$\frac{p\pi_L\left[1-F_M(1^-)\right]+(1-p)\pi_H\left[1-F_m(1^-)\right]}{\pi_L}=pe^{-\frac{\bar{b}}{\pi_H}}C_n,$$

or equivalently

$$\frac{(1-p)\pi_H\phi_1^{n-1}+p\pi_L\left[1-(1-\phi_1)^{n-1}\right]}{\pi_L}=p-b\overline{U}_n. \tag{13}$$

Eliminating $\overline{U}_n$ leads to $T_n(\phi_0)=\widehat{T}_n(\phi_1)$. Then, we have

$$(1-\alpha)\left[1-\frac{\bar{b}\frac{\pi_H}{\pi_L}-\lambda}{s-\lambda}\right]\widehat{f}_M(s)+\alpha\left[1+\frac{\lambda+\bar{b}}{s-\lambda}\right]\widehat{f}_m(s)=\widehat{Q}(s), \tag{14}$$

where

$$\widehat{Q}(s)=\alpha(1-\frac{b\overline{U}_n}{1-p})+(1-\alpha)(1-\frac{b\overline{U}_n}{p})e^{-\frac{s}{\pi_H}}+\alpha(\lambda+\bar{b})(1-\frac{b\overline{U}_n}{1-p})\frac{1}{s-\lambda}$$
$$-(1-\alpha)(1-\frac{b\overline{U}_n}{p})(\bar{b}\frac{\pi_H}{\pi_L}-\lambda)\frac{e^{-\frac{s}{\pi_H}}}{s-\lambda}.$$

Next observe that $(1-\alpha)(\bar{b}\frac{\pi_H}{\pi_L}-\lambda) = \alpha(\lambda+\bar{b})$; inverting the Laplace Transform equation (14), we obtain for all $x \in \Delta$

$$\int_\Delta (1-\alpha)\delta_0(x-y)dF_M(y)+\int_\Delta \alpha\delta_0(x-y)dF_m(y)+\alpha(\lambda+\bar{b})e^{\lambda x}\int_{0^-}^x e^{-\lambda y}\left[dF_m(y)-dF_M(y)\right] = Q(x),$$

or equivalently

$$\begin{aligned}&(1-\alpha)dF_M(x)+\alpha dF_m(x)+\alpha(\lambda+\bar{b})e^{\lambda x}\int_{0^-}^x e^{-\lambda y}\left[dF_m(y)-dF_M(y)\right]\\&=\left[(1-\alpha)\phi_0^{n-1}+\alpha\left[1-(1-\phi_0)^{n-1}\right]\right]\delta_0(x)+\left[\alpha\phi_1^{n-1}+(1-\alpha)\left[1-(1-\phi_1)^{n-1}\right]\right]\delta_{1/\pi_H}(x)+A_n e^{\lambda x}dx,\end{aligned} \tag{15}$$

with

$$A_n = (\lambda+\bar{b})\left[(1-\alpha)\phi_0^{n-1}+\alpha\left[1-(1-\phi_0)^{n-1}\right]\right],$$

where we used relationships (12) and (13).

Now assume that the (individual) distribution F is of the form

$$dF(x) = \phi_0\delta_0(x)+\phi_1\delta_{1/\pi_H}(x)+f(x)dx,$$

where f is a smooth positive function. This implies that

F_M has an atom of size ϕ_0^{n-1} at $x=0$ and an atom of size $1-(1-\phi_1)^{n-1}$ at $x=\frac{1}{\pi_H}$
F_m has an atom of size $1-(1-\phi_0)^{n-1}$ at $x=0$ and an atom of size ϕ_1^{n-1} at $x=\frac{1}{\pi_H}$,

and F_M and F_m admit a smooth pdf on f_M and f_m respectively with

$$\begin{aligned}f_M(x) &= (n-1)F^{n-2}(x)f(x)\\ f_m(x) &= (n-1)\left[1-F(x)\right]^{n-2}f(x).\end{aligned}$$

This representation of F is (the only one) consistent with relationship (15). It follows that for all $x \in (0,\frac{1}{\pi_H})$, we have

$$(1-\alpha)f_M(x)+\alpha f_m(x)+\alpha(\lambda+\bar{b})e^{\lambda x}\int_0^x e^{-\lambda y}\left[dF_m(y)-dF_M(y)\right] = A_n e^{\lambda x}.$$

Evaluating this relationship at $x=0^+$ leads to

$$(1-\alpha)f_M(0^+)+\alpha f_m(0^+) = (\lambda+\bar{b})\phi_0^{n-1}.$$

Then, differentiating with respect to x leads to

$$(1-\alpha)f'_M(x)+\alpha f'_m(x)+\alpha(\lambda+\overline{b})\left[\lambda e^{\lambda x}\int_0^x e^{-\lambda y}\left[dF_m(y)-dF_M(y)\right]+f_m(x)-f_M(x)\right]=\lambda A_n e^{\lambda x}.$$

Eliminating the term $\int_0^x e^{-\lambda y}\left[dF_m(y)-dF_M(y)\right]$ leads to

$$(1-\alpha)f'_M(x)+\alpha f'_m(x)+\overline{b}\alpha\left[f_m(x)-f_M(x)\right]-\lambda f_M(x)=0.$$

Integrating this equation yields

$$(1-\alpha)f_M(x)+\alpha f_m(x)+\overline{b}\alpha\left[F_m(x)-F_M(x)\right]-\lambda F_M(x)=\overline{A}_n,$$

where $\overline{A}_n$ is a constant to be determined. Evaluating this relationship at $x=0^+$ leads to

$$(1-\alpha)f_M(0^+)+\alpha f_m(0^+)+\overline{b}\alpha\left[F_m(0^+)-F_M(0^+)\right]-\lambda F_M(0^+)=\overline{A}_n,$$

which leads to $\overline{A}_n=\overline{b}\left[(1-\alpha)\phi_0^{n-1}+\alpha(1-(1-\phi_0)^{n-1})\right]$. The desired result follows easily. ■

Step 2: Existence and Uniqueness of the Equilibrium. Given $\phi_0\in(0,1)$, for $[\phi_0,1]$, consider the following ODE

$$J'(z)=R(z,\phi_0), \tag{16}$$

where $R(z,\phi_0)=\frac{(n-1)\left[(1-\alpha)z^{n-1}+\alpha(1-z)^{n-1}\right]}{D_n(z)}$ and function D_n is defined as

$$D_n(z,\phi_0)=T_n(\phi_0)+(\overline{b}\alpha+\lambda)z^{n-1}+\overline{b}\alpha(1-z)^{n-1},$$

and initial condition $J(\phi_0)=0$. We note that $D_n(\phi_0,\phi_0)=(\overline{b}+\lambda)\phi_0^{n-1}>0$ and are looking for a condition on parameter ϕ_0 so that $D_n(z,\phi_0)>0$ for all $z\geq\phi_0$. As $\overline{b}\alpha+\lambda=\overline{b}\frac{\pi_H}{\pi_L}(1-\alpha)>0$, D_n is a convex function that achieves its minimum at $z_n^*=\frac{1}{1+\left[\frac{\overline{b}\alpha+\lambda}{\overline{b}\alpha}\right]^{1/(n-2)}}$ that is independent of ϕ_0. Thus, we need to restrict parameter ϕ_0 to be such that

$$T_n(\phi_0)>-(\overline{b}\alpha+\lambda)(z_n^*)^{n-1}-\overline{b}\alpha(1-z_n^*)^{n-1}.$$

Next, note that $T'_n>0$ with $T_n(0)=-\overline{b}\alpha<0$ and $T_n(1)=\overline{b}(1-\alpha)>0$. As z_n^* is such that $(\overline{b}\alpha+\lambda)(z_n^*)^{n-2}=\overline{b}\alpha(1-z_n^*)^{n-2}$, we find that

$$0>-(\overline{b}\alpha+\lambda)(z_n^*)^{n-1}-\overline{b}\alpha(1-z_n^*)^{n-1}=-\overline{b}\alpha(1-z_n^*)^{n-2}>-\overline{b}\alpha.$$

Thus there exists a unique $\phi_{0,\min} = T_n^{-1}\left[-\bar{b}\alpha(1-z_n^*)^{n-2}\right] \in (0,1)$ such that $D_n(\phi_{0,\min}, z_n^*) = 0$. Next, we verify that $\phi_{0,\min} < z_n^*$, or equivalently $T_n(\phi_{0,\min}) = -\bar{b}\alpha(1-z_n^*)^{n-2} < T_n(z_n^*)$, i.e.,

$$\begin{aligned}
-\bar{b}\alpha(1-z_n^*)^{n-2} &< \bar{b}\left[(1-\alpha)(z_n^*)^{n-1} - \alpha(1-z_n^*)^{n-1}\right] \\
&< (\bar{b}+\lambda)(z_n^*)^{n-1} - \left[(\bar{b}\alpha+\lambda)(z_n^*)^{n-1} + \bar{b}\alpha(1-z_n^*)^{n-1}\right] \\
&= (\bar{b}+\lambda)(z_n^*)^{n-1} - \bar{b}\alpha(1-z_n^*)^{n-2},
\end{aligned}$$

so the condition is satisfied. Thus for all $\phi_{0,\min} < \phi_0 \leq z$, we have $D_n(z,\phi_0) > 0$ so the ODE (16) is well defined; function J is (uniquely) well defined, strictly increasing so it admits an inverse, some function F that is also strictly increasing for any initial condition $F(0) = \phi_0$. For F to be the cfd of an equilibrium distribution, we need to find $(\phi_0, \phi_1) \in (0,1)^2$, such that $T_n(\phi_0) = \widehat{T}_n(\phi_1)$ and $\lim_{x \nearrow \frac{1}{\pi_H}} F(x) = 1 - \phi_1$. Both functions T_n and $\widehat{T}_n$ are strictly increasing so function $\psi_n = \widehat{T}_n^{-1} \circ T_n$ is well defined with $\psi_n' > 0$. As $\phi_0 \geq \phi_{0,\min}$, we must have $\phi_1 \geq \phi_{1,\min}$ with $\phi_{1,\min} = \psi_n(\phi_{0,\min})$. We check that $\phi_{1,\min} > 0$, which is equivalent to is such that $\widehat{T}_n(\phi_{1,\min}) = T_n(\phi_{0,\min}) > T_n(0) = -\bar{b}\alpha$. As $T_n(\phi_{0,\min}) = -\bar{b}\alpha(1-z_n^*)^{n-2}$ and $z_n^* < 1$, the condition is satisfied. Next, we check that $z_n^* < 1 - \phi_{1,\min}$, or equivalently $\widehat{T}_n(1-z_n^*) > -\bar{b}\alpha(1-z_n^*)^{n-2}$, i.e.,

$$\frac{\pi_H + \pi_L}{\pi_L}(\bar{b}\alpha + \lambda) > \frac{\pi_H}{\pi_L}(\bar{b}+\lambda)z_n^*.$$

Using the fact that $\bar{b}\alpha + \lambda = \bar{b}\frac{\pi_H}{\pi_L}(1-\alpha)$ and $\bar{b} + \lambda = \bar{b}(1-\alpha)\frac{\pi_H+\pi_L}{\pi_L}$, the condition is always satisfied as $z_n^* < 1$. Thus we have $0 < \phi_{0,\min} < z_n^* < 1 - \phi_{1,\min}$. Next, we consider the set

$$\{\phi_0 \geq \phi_{0,\min},\, \phi_1 \geq \phi_{1,\min},\, \phi_0 + \phi_1 \leq 1 \text{ and } T_n(\phi_0) = \widehat{T}_n(\phi_1)\}.$$

We have $\phi_0 \leq 1 - \phi_1$ if and only if $\widehat{T}_n(\phi_1) = T_n(\phi_0) \leq T_n(1-\phi_1)$, which is equivalent to $\phi_1 \leq \phi_{1,\max}$ and $\phi_0 \leq \phi_{0,\max}$. Observe $T_n(\phi_{0,\max}) = 0 > -\bar{b}\alpha(1-z_n^*)^{n-2} = T_n(\phi_{0,\min})$, so indeed $\phi_{0,\min} < \phi_{0,\max}$. Similarly, $\widehat{T}_n(\phi_{1,\max}) = 0 > -\bar{b}\alpha(1-z_n^*)^{n-2} = \widehat{T}_n(\phi_{1,\min})$, so that $\phi_{0,\min} < \phi_{0,\max}$. We also note that $\phi_{1,\max} \geq \phi_{0,\max}$ (resp. $\leq \phi_{0,\max}$) if and only if $\alpha \leq \frac{1}{2}$ (resp. $\geq \frac{1}{2}$). Finally, set

$$\Delta_m = \{(\phi_0, \phi_1)\,, \phi_{0,\min} \leq \phi_0 < \phi_{0,\max} \text{ and } \phi_{1,\min} \leq \phi_1 < \phi_{1,\max}\}.$$

In order to have an equilibrium, (ϕ_0, ϕ_1) must be in Δ_m.

Integrating the ODE (16) between ϕ_0 and z leads to $J(z) = \int_{\phi_0}^{z} R(u, \phi_0)du$. To be an equilibrium, we must find a value ϕ_0^* in $(0, 1)$ such that

$$\frac{1}{\pi_H} = \int_{\phi_0^*}^{1-\psi_n(\phi_0^*)} R(u, \phi_0^*)du.$$

Set $\Gamma(z) = \int_{z}^{1-\psi_n(z)} R(u, z)du - \frac{1}{\pi_H}$. We want to show that function Γ has a unique root ϕ_0^* such that $(\phi_0^*, \phi_1^*) \in \Delta_m$, with $\phi_1^* = 1 - \psi_n(\phi_0^*)$. We have

$$\Gamma'(z) = -R(1 - \psi_n(z), z)\psi_n'(z) - R(\psi_n(z), z) + \int_{z}^{1-\psi_n(z)} R_2(u, z)du < 0,$$

as it is easy to verify that $R_2 = \frac{\partial R}{\partial \phi_0} < 0$. Then, recall we have shown that $0 < \phi_{0,\min} < z_n^* < 1 - \phi_{1,\min}$ and $D_n(\phi_{0,\min}, z_n^*) = 0$, we have $\lim\limits_{z \to \phi_{0,\min}} \Gamma(z) = \infty$. If $\alpha \leq \frac{1}{2}$, then, we choose $\overline{z} = \phi_{0,\max} = \frac{1}{1+\left(\frac{1-\alpha}{\alpha}\right)^{1/(n-1)}}$, so we have $(\overline{z}, 1 - \psi_n(\overline{z})) \in \Delta_m^2$ with $1 - \psi_n(\overline{z}) = \overline{z}$. This implies that $\Gamma(\overline{z}) < 0$. Conversely, if $\alpha \geq \frac{1}{2}$, then, we choose $\overline{z}$ such that $\psi_n(\overline{z}) = \phi_{1,\max} = \frac{1}{1+\left(\frac{\alpha}{1-\alpha}\right)^{1/(n-1)}}$, so we have $(\overline{z}, 1 - \psi_n(\overline{z})) \in \Delta_m^2$ with $1 - \psi_n(\overline{z}) = \overline{z}$. Again, this implies that $\Gamma(\overline{z}) < 0$. We deduce that there exists a unique couple $(\phi_0^*, \phi_1^*) \in \Delta_m$, such that $\int_{\phi_0^*}^{1-\phi_1^*} R(u, z)du = \frac{1}{\pi_H}$, which completes the proof. $\blacksquare$

The Winner Takes All: Risk Neutral Managers. Assume that $\alpha < \frac{1}{2}$. We now write $\phi_0 = \phi_0(n)$ and $\phi_1 = \phi_1(n)$. It is easy to verify that when $\alpha < \frac{1}{2}$, $\phi_0(n)$ is increasing in n and $\lim\limits_{n\to\infty} \phi_0(n) = \lim\limits_{n\to\infty} \phi_1(n) = \frac{1}{2}$. Thus for all $n \in \mathbb{N}$, $\phi_0(n) \in (0, \frac{1}{2})$. Then, the expected return can be written $\overline{R} = \frac{p\pi_L+(1-p)\pi_H}{\pi_L\pi_H}[(1-\alpha)\phi_0(n)+\alpha\phi_1(n)]$, which is increasing in n. If $\pi_H = \pi_L$, the average return is equal to $\frac{(1-p)\phi_0(n)+p\phi_1(n)}{\pi_H}$, which converges to $\frac{1}{2\pi_H}$ as n goes to ∞. Finally the variance of the returns is given by $\mathrm{var}[R_1] = \frac{B(\phi_0(n))}{\pi_H^2}$, where $B(x) = -(1-2p)^2x^2 + (1-2p)^2x + p(1-p)$. It is easy to see that B is increasing on $(0, \frac{1}{2})$, which implies that the variance is increasing in n and we have $\lim\limits_{n\to\infty} \mathrm{var}[R_1] = \frac{1}{4\pi_H^2}$. A similar result holds for $\alpha > \frac{1}{2}$. $\blacksquare$

The Winner Takes All: Proof of Proposition 6. Recall that $\phi_0^* > \phi_{0,\min}$ with $\phi_{0,\min}$ implicitly defined by

$$T_n(\phi_{0,\min}) = -\frac{(\bar{b}\alpha + \lambda)}{\left(1 + \left(\frac{\bar{b}\alpha+\lambda}{\bar{b}\alpha}\right)^{\frac{1}{n-2}}\right)^{n-2}}.$$

As n goes to ∞, one can check that

$$\frac{(\bar{b}\alpha+\lambda)}{\left(1+\left(\frac{\bar{b}\alpha+\lambda}{\bar{b}\alpha}\right)^{\frac{1}{n-2}}\right)^{n-2}} = \frac{2\sqrt{\bar{b}\alpha(\bar{b}\alpha+\lambda)}}{2^{n-1}}(1-\frac{1}{8(n-2)}\left[\ln\frac{\bar{b}\alpha+\lambda}{\bar{b}\alpha}\right]^2)+o(1/n).$$

Then, we write $\phi_{0,\min}=\frac{1}{2}-\frac{a_n}{2}$ and we want to show that $\lim\limits_{n\to\infty} a_n=0$. For n large enough, a_n must satisfy the following asymptotic relationship

$$\bar{b}\left[(1-\alpha)(1-a_n)^{n-1}-\alpha(1+a_n)^{n-1}\right] \underset{n\to\infty}{\sim} -2\sqrt{\bar{b}\alpha(\bar{b}\alpha+\lambda)}.$$

Assuming (to be checked later) that $\lim\limits_{n\to\infty} a_n=0$, we take a Taylor expansion of order 1 and solve for a_n to obtain that

$$a_n \underset{n\to\infty}{\sim} \frac{1}{n-1}\left[2\sqrt{\frac{\pi_H}{\pi_L}\alpha(1-\alpha)}+1-2\alpha\right].$$

so indeed $\lim\limits_{n\to\infty} a_n=0$. This expression is valid as long as $2\sqrt{\frac{\pi_H}{\pi_L}\alpha(1-\alpha)}+1-2\alpha\neq 0$. If not, we need to take a Taylor expansion of order 2 and solve for a_n to obtain that

$$|a_n| \underset{n\to\infty}{\sim} \frac{\sqrt{\frac{\pi_H}{\pi_L}\alpha(1-\alpha)}}{\sqrt{2(1-2\alpha)}}\frac{1}{n^{3/2}}\left|\ln\frac{(1-\alpha)\pi_H}{\alpha\pi_L}\right|.$$

Again, we find that $\lim\limits_{n\to\infty} a_n=0$. In addition, recall that

$$\begin{aligned}\phi_{0,\max} &= \frac{1}{1+\left(\frac{1-\alpha}{\alpha}\right)^{1/(n-1)}} \\ &\underset{n\to\infty}{\sim} \frac{1}{2}+\frac{1}{4(n-1)}\ln\frac{1-\alpha}{\alpha}.\end{aligned}$$

Since $\phi_{0,\min}\le\phi_0^*\le\phi_{0,\max}$, and $\lim\limits_{n\to\infty}\phi_{0,\min}=\lim\limits_{n\to\infty}\phi_{0,\max}=\frac{1}{2}$, we can claim that $\lim\limits_{n\to\infty}\phi_0^*=\frac{1}{2}$. A similar analysis for ϕ_1^* leads to the same result: $\lim\limits_{n\to\infty}\phi_1^*=\frac{1}{2}$. ■

8.3 Appendix 3

Order Statistic: Proof of Proposition 7. If manager i chooses $x\ge z$, then $x^*=\frac{1}{\pi_H}$ which leads to utility level $\frac{1}{2}\frac{1-e^{-\bar{b}(\frac{1}{\pi_H}-y)}}{b}$. Conversely, if manager i chooses $x\le y$, then $x^*=0$ which leads to utility level $\frac{1}{2}\frac{1-e^{-\bar{b}z}}{b}$. Finally, if manager i chooses $x\in[y,z]$, her level of utility is maximized at $x^*=\frac{y+z}{2}\in[y,z]$, which leads to utility level $\frac{1-e^{-\bar{b}\frac{z-y}{2}}}{b}$. It follows that $x^*=\frac{1}{\pi_H}$

(resp. $x^* = 0$) dominates $x^* = 0$ (resp. $x^* = \frac{1}{\pi_H}$) iff $z + y \leq \frac{1}{\pi_H}$ (resp. $z + y \geq \frac{1}{\pi_H}$). Then, assume $z + y \geq \frac{1}{\pi_H}$; $x^* = \frac{y+z}{2}$ dominates $x^* = 0$ iff

$$\frac{1 - e^{-\bar{b}\frac{z-y}{2}}}{b} \geq \frac{1}{2}\frac{1 - e^{-\bar{b}z}}{b},$$

which is equivalent to $z \geq L_1(y)$, with $L_1(y) = \frac{2}{\bar{b}} \ln\left[e^{\bar{b}\frac{y}{2}} + \sqrt{e^{\bar{b}y} - 1}\right]$. Similarly, assume $z+y \leq \frac{1}{\pi_H}$; $x^* = \frac{y+z}{2}$ dominates $x^* = \frac{1}{\pi_H}$ iff

$$\frac{1 - e^{-\bar{b}\frac{z-y}{2}}}{b} \geq \frac{1}{2}\frac{1 - e^{-\bar{b}(\frac{1}{\pi_H} - y)}}{b},$$

which is equivalent to $z \geq L_2(y)$, with $L_2(y) = -\frac{2}{\bar{b}} \ln\left[\frac{e^{-\bar{b}\frac{y}{2}} + e^{-\bar{b}(\frac{1}{\pi_H} - \frac{y}{2})}}{2}\right]$. Furthermore, one can check that functions L_1 and L_2 are increasing on Δ with $L_1(0) = 0$ and $L_2(\frac{1}{\pi_H}) = \frac{1}{\pi_H}$ and for all $y \in (0, \frac{1}{\pi_H}]$, $L_1(y) > y$ and for all $y \in [0, \frac{1}{\pi_H})$, $L_2(y) > y$. Finally, note that the curves representing L_1 and L_2 intersect on the line $z + y = \frac{1}{\pi_H}$ at $\overline{y} = \frac{1}{\pi_H} - \frac{\ln[2e^{\frac{\bar{b}}{2\pi_H}} - 1]}{\bar{b}}$. Manager i's best response is given by

$$x^* = \begin{cases} 0, \text{ if } \frac{1}{\pi_H} - z \leq y \leq z \leq L_1(y) \\ \frac{z+y}{2}, \text{ if } z \geq \max\ \{L_1(y), L_2(y)\} \\ \frac{1}{\pi_H}, \text{ if } y \leq z \leq \frac{1}{\pi_H} - y \leq L_2(y). \end{cases}$$

It is then easy to verify that no pure strategy Nash equilibrium may exist. ■

Order Statistic: Proof of Proposition 8.

Step 1: Derivation of the Equilibrium. If the individual distribution F has an atom ϕ_0 (resp. ϕ_1) at $x = 0$ (resp. $x = \frac{1}{\pi_H}$) then distributions F_k and F_{n-k} also have atoms $\phi_{0,k} = L_{n,k}(\phi_0)$ (resp. $\phi_{1,k} = L_{n,n-k}(\phi_1)$) and $\phi_{0,n-k} = L_{n,n-k}(\phi_0)$ (resp. $\phi_{1,n-k} = L_{n,k}(\phi_1)$) respectively at $x = 0$ (resp. $x = \frac{1}{\pi_H}$). Set $C_{n,k} = \int_\Delta e^{\bar{b}y} dF_{n-k}(y)$, $D_{n,k} = \int_\Delta e^{-\bar{b}\frac{\pi_H}{\pi_L}y} dF_k(y)$ and observe that $b\overline{U}_{n,k} = p - pe^{-\frac{\bar{b}}{\pi_H}} C_{n,k}$ and $b\overline{U}_{n,k} = 1 - p - (1-p)D_{n,k}$. Then, making no assumption on distributions F_{n-k} and F_k, given the results derived for the two managers' game, we find that

$$p\frac{\widehat{f}_{n-k}(s)}{s + \bar{b}} + (1-p)\frac{\pi_H}{\pi_L}\frac{\widehat{f}_k(s)}{s - \bar{b}\frac{\pi_H}{\pi_L}} = pe^{-\frac{\bar{b}}{\pi_H}} C_{n,k} \frac{e^{-\frac{s}{\pi_H}}}{s + \bar{b}} + (1-p)\frac{\pi_H}{\pi_L}\frac{D_{n,k}}{s - \bar{b}\frac{\pi_H}{\pi_L}}.$$

Multiplying both sides by s and letting s goes to $+\infty$, we obtain that

$$\frac{p\pi_L F_{n-k}(0^+) + (1-p)\pi_H F_k(0^+)}{\pi_L} = (1-p)\frac{\pi_H}{\pi_L} D_{n,k},$$

or equivalently

$$\frac{p\pi_L \phi_{0,n-k} + (1-p)\pi_H \phi_{0,k}}{\pi_H} = 1 - p - b\overline{U}_{n,k}$$

Similarly, multiplying both sides by s and letting $se^{\frac{s}{\pi_H}}$ goes to $-\infty$, we obtain that

$$\frac{p\pi_L \left[1 - F_{n-k}(1^-)\right] + (1-p)\pi_H \left[1 - F_k(1^-)\right]}{\pi_L} = pe^{-\frac{\bar{b}}{\pi_H}} C_{n,k},$$

or equivalently

$$\frac{p\pi_L \phi_{1,n-k} + (1-p)\pi_H \phi_{1,k}}{\pi_L} = p - b\overline{U}_{n,k}.$$

Eliminating $\overline{U}_{n,k}$ leads to $T_{n,k}(\phi_0) = \widehat{T}_{n,k}(\phi_1)$.

The analysis conducted for the case $k = 1$ remains valid providing that we make the following adjustments. The value of constant A_n is now given by

$$\begin{aligned} A_{n,k} &= \alpha(\lambda + \bar{b})(1 - \frac{b\overline{U}_{n,k}}{1-p}) \\ &= \alpha(\lambda + \bar{b}) \left[(1-\alpha)L_{n,n-k}(\phi_0) + \alpha L_{n,k}(\phi_0)\right]. \end{aligned}$$

Similarly, we have

$$\overline{A}_{n,k} = \bar{b} \left[(1-\alpha)L_{n,n-k}(\phi_0) + \alpha L_{n,k}(\phi_0)\right].$$

Then, functions F_{n-k} and $F_k$satisfy the following first order ODE

$$(1-\alpha)F'_{n-k}(x) + \alpha F'_k(x) + \bar{b}\alpha \left[F_k(x) - F_{n-k}(x)\right] - \lambda F_{n-k}(x) = \overline{A}_{n,k}.$$

Finally, using the fact that $F_k(x) = L_{n,k}\left[F(x)\right]$, $F_{n-k}(x) = L_{n,n-k}\left[F(x)\right]$, $L_{n,k}(y) = (n-1)\binom{n-1}{k-1}\int_0^y t^{k-1}(1-t)^{n-k-1}dt$ and $\overline{A}_{n,k} = T_{n,k}(\phi_0) + \bar{b}\alpha$ yields the desired result. ■

Step 2: Existence and Uniqueness of the Equilibrium. Given $\phi_0 \in (0,1)$, for $[\phi_0, 1]$, consider the following ODE

$$\varphi'_k(z) = R_k(z, \phi_0), \tag{17}$$

where $R_k(z,\phi_0) = \frac{(n-1)\binom{n-2}{k-1}\left[(1-\alpha)z^{n-k-1}(1-z)^{k-1} + \alpha z^{k-1}(1-z)^{n-k-1}\right]}{D_{n,k}(z)}$ and function $D_{n,k}$ is defined as

$$D_{n,k}(z,\phi_0) = T_{n,k}(\phi_0) + \bar{b}\alpha + (\bar{b}\alpha + \lambda)L_{n,n-k}(z) - \bar{b}\alpha L_{n,k}(z),$$

and initial condition $\varphi_k(\phi_0) = 0$. We note that

$$\begin{aligned} D_{n,k}(\phi_0, \phi_0) &= T_{n,k}(\phi_0) + \bar{b}\alpha + (\bar{b}\alpha + \lambda)L_{n,n-k}(\phi_0) - \bar{b}\alpha L_{n,k}(\phi_0) \\ &= (\bar{b} + \lambda)L_{n,n-k}(\phi_0) > 0, \end{aligned}$$

As in the case $k = 1$, we are looking for a condition on parameter ϕ_0 so that $D_{n,k}(z, \phi_0) > 0$ for all $z \geq \phi_0$. Note that

$$\frac{\partial D_{n,k}(z, \phi_0)}{\partial z} = (n-1)\binom{n-1}{k-1} z^{k-1}(1-z)^{n-k-1} \left[(\bar{b}\alpha + \lambda)(\frac{z}{1-z})^{n-2k} - \bar{b}\alpha \right].$$

As $\bar{b}\alpha + \lambda = \bar{b}\frac{\pi_H}{\pi_L}(1-\alpha) > 0$, is easy to verify that function $D_{n,k}$ is a convex function that achieves its minimum at $z = z^*_{n,k} = \frac{1}{1+\left[\frac{\bar{b}\alpha+\lambda}{\bar{b}\alpha}\right]^{1/(n-2k)}}$ that is independent of ϕ_0. The rest of the proof is similar to the case $k = 1$ and is therefore omitted. ■