DISCUSSION PAPER SERIES

DP17040

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ORGANIZATIONAL ECONOMICS

CEPR

ISSN 0265-8003

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Discussion Paper DP17040
Published 16 February 2022
Submitted 12 February 2022

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www.cepr.org

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JEL Classification: N/A

Keywords: N/A

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On market prices in double auctions

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February 11, 2022

Abstract

We address some open issues regarding the characterization of double auctions. Our model is a two-sided commodity market with either finitely or infinitely many traders. We first unify existing formulations for both finite and infinite markets and generalize the characterization of market clearing in the presence of ties. Second, we define a mechanism that achieves market clearing in any, finite or infinite, market instance and show that it coincides with the k-double auction by Rustichini et al. (1994) in the former case. In particular, it clarifies the consequences of ties in submissions and makes common regularity assumptions obsolete. Finally, we show that the resulting generalized mechanism implements Walrasian competitive equilibrium.

1 Motivation

Double auctions (DAs) are ubiquitous, in particular for running commodity markets. Potential buyers and sellers submit bids and asks to a central clearinghouse which then establishes who deals with whom and at what prices. An important class of DAs are running 'call markets' where bids and asks are collected, and deals are executed at a single market-clearing price. This price is commonly characterized as equilibrating (revealed) demand and supply.[1] For markets with finitely many traders, the best-known mechanism of this kind is the k-DA (Rustichini et al., 1994), which provides an explicit formula for calculating market-clearing prices. Market clearing, i.e., equating supply and demand–by folk wisdom–implements Walrasian competitive equilibrium (Walrasian CE) in DAs.[2] A Walrasian CE consists of an allocation and a price structure, together with "the property that at the price structure, no trader can, with the value of his initial bundle, buy a bundle that he prefers to his part of the allocation."(Aumann, 1964) In the absence of ties in the reported values, the folk wisdom is unambiguous, because demand-supply equilibration is

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[1]See, for example, Friedman and Rust (1993) for details regarding DAs in theory and practice.

[2]Paying tribute to Walras' foundational works (Walras, 1874, 1883), we use the terminology Walrasian CE, which the literature also refers to as equilibrium, Walrasian equilibrium or competitive equilibrium.

unambiguous. In the presence of ties, however, markets will leave some excess (in either supply or demand), and the connection of market clearing to Walrasian CE is incomplete.

We make three contributions in this article: (i) we unify existing formulations for both finite and infinite markets; (ii) we generalize the characterization of market-clearing in the presence of ties, and (iii) we show that the resulting generalized mechanism implements Walrasian CE and therefore also a core allocation. We next motivate these three contributions and discuss their connection to existing work on the topic.

(i) Prior theory has primarily focused on large k-DAs (Rustichini et al., 1994; Cripps and Swinkels, 2006; Azevedo and Budish, 2019), for which infinite markets are a useful approximation (Aumann, 1964). The explicit formulae that calculate market-clearing prices in these markets, however, do not apply in infinite markets. Infinitely large markets are therefore treated using different models with additional regularity assumptions on reported value distributions (like continuity and strict monotonicity). Such distributional assumptions allow to define a market clearing price by equating supply and demand, (Reny and Perry, 2006) or, when considering finite markets via equilibrating expected demand and expected supply (Tatur, 2005). To unify the existing characterizations, our first contribution is to define a generalized DA mechanism that coincides with the k-DA for finite markets, but equally applies to infinite markets.

(ii) Our second contribution is related to the existence of ties in finite markets, that is, to situations where demand and/or supply amass around potential market-clearing prices. In the presence of such ties, the k-DA may require selecting which traders are involved in trade via a lottery, and market-clearing needs some redefinition to account for that. This issue is acknowledged in principle, but has been sidestepped in earlier work via continuity assumptions on type and action spaces that render ties probability zero events.[3] While technically inconvenient, ties may actually arise quite naturally in DA contexts from strategic or boundedly rational behavior.[4] Our contribution is to provide a general mechanism that works for any market instance, with or without ties.

(iii) Finally, we tighten the link between the concepts of market-clearing and Walrasian CE, that is, the outcomes that are generally considered as desirable (Arrow and Debreu, 1954). Folk wisdom is that the k-DA, and mechanisms that equate supply and demand more generally, result in Walrasian CE. Indeed, market-clearing price and Walrasian CE price are often used interchangeably in the context of DAs –although this has, to our knowledge, not been formalized.[5] Especially in the case of ties where the connection to market-clearing is ambiguous it is not clear whether k-DAs lead to CE. Our contribution is to make this connection explicit by proving that our generalized mechanism does indeed implement Walrasian CE. Moreover, it is well-known that every competitive equilibrium is also a core allocation and conversely, the core approaches the set of Walrasian CE as the number of traders grow (Debreu and Scarf, 1963; Aumann, 1964). It follows that our generalized mechanism implements a core allocation and is therefore a

[3] Indeed, Rustichini et al. (1994) points out that excess might remain, but then addresses other issues in their analysis, and leaves open the connection with the balance of demand and supply, and with the concept of market-clearing.

[4] See the literatures on equilibria in discontinuous games with discrete action spaces (Jackson et al., 2002), and on models of imitation, herding, social influence, etc. (Devenow and Welch, 1996; Shiller, 2000).

[5] "The intersection of these curves define the price and quantity at which neoclassical economic theory predicts trading will occur, the competitive equilibrium (CE) solution."(Rust et al., 2018)

core-selecting auction (Ausubel and Milgrom, 2002; Day and Milgrom, 2008).

Our model is a two-sided commodity market with either finitely or infinitely many traders. We formalize demand and supply schedules in such markets for strict and weak preferences of traders, which permits rigorous definitions of sets of market-clearing prices in general. For strict preferences these schedules are functions, for weak preferences they are correspondences. Weak preferences arise naturally in markets where traders have prices that make them properly indifferent between trading and not trading. We show how weaker notions of market clearing–present in the literature–may result in existence problems or ill-defined market outcomes. We revisit folk results related to the analytical properties of demand and supply, and the topological properties of the set of market-clearing prices. In Theorem 1, we show that the identified set of market-clearing prices can be used to fully characterize the finite k-DA, and that even with ties the mechanism results in a market-clearing price. With this characterization, we then generalize the mechanism to markets with an infinite number of traders, yielding market-clearing prices together with randomly rationed allocations given any market instance, including the presence of ties. Finally, in Theorem 2, we formalize the folk result that market-clearing prices coincide with Walrasian CE prices, that is, prices that allow to balance trade, are individually rational, and stable.

Our generalized DA mechanism provides a unified reference for implementation in practice. Many real-world markets are implemented as DAs, and sometimes–with reference to implementing Walrasian CE–motivated by maximizing volume or gains of trade, even though this is not sufficient to ensure Walrasian CE.

2 The market

Consider a two-sided market with traders interested in either buying or selling an indivisible good. Assume that traders have sub-additive preferences. This allows us to assume, without loss of generality, that each buyer wants to buy one unit and each seller wants to sell one unit. Let $\mathcal{B}, \mathcal{S} \subset \mathbb{R}$ be sets of buyers ($b \in \mathcal{B}$) and sellers ($s \in \mathcal{S}$). We consider both the *finite* and *infinite* case; for finite markets, with m buyers $\mathcal{B} = \{1, 2, ..., m\}$, and n sellers $\mathcal{S} = \{1, 2, ..., n\}$; and for infinite markets with $\mathcal{B} \subset \mathbb{R}$ and $\mathcal{S} \subset \mathbb{R}$ being two closed intervals. Let μ_B and μ_S be the *distributions* of buyers and sellers on $\mathcal{B}$ and $\mathcal{S}$.[6]

Every trader i reports a *value* $t_i \in T = [\underline{t}, \overline{t}]$, where T is the *space* of possible values. For buyers values specify *bids* representing the maximum willingness to pay. For sellers values specify *asks* representing the minimum willingness to sell. In a non-strategic setting, these values can be thought of as true reservation prices, often referred to as *types*. In a strategic setting, they represent reported values or *actions*.

Given distributions of values for both buyers and sellers, a DA chooses a *market outcome*, defined by an *allocation* identifying subsets of traders, $\mathcal{B}^* \subset \mathcal{B}$ and $\mathcal{S}^* \subset \mathcal{S}$, who will be involved in trade at the *market price* P that each active buyer pays and each active seller receives.

Throughout the paper we will come back to the following example of a market.

[6] Distributions are counting measures in finite markets, and Lebesgue-measures in infinite markets.

Example (Market)**.** There are two buyers $\mathcal{B} = \{b_1, b_2\}$ and one seller $\mathcal{S} = \{s_1\}$ with reported values $t_b^1 \geq t_b^2 > t_s^1$. We consider two scenarios, *without ties*, where $t_b^1 > t_b^2$, and *with ties*, where $t_b^1 = t_b^2$.

3 Mechanisms

3.1 Finite k-double auction

For finite markets, Rustichini et al. (1994) introduced the k-DA.

Finite k-DA (Rustichini et al., 1994)

Denote by t the set of all $m+n$ values by the m buyers and n sellers. Denote by $t^{(l)}$ the l'th smallest element in t.

Market price. Given some $k \in [0,1]$, the market price is set at

$$\Pi^* = kt^{(m)} + (1-k)t^{(m+1)}.$$

Allocation. The allocation is carried out to maximize trade by assigning priority to sellers starting with smallest and to buyers starting with largest. If $t^{(m)} \neq t^{(m+1)}$, market excess is equal to zero. Otherwise, either excess supply or excess demand exists, in which case a fair lottery selects active traders (from those bidding or asking exactly at the market price) on the long side of the market.

We illustrate how the k-DA works in our example:

Example (Finite k-DA)**.** In the market without ties, $t^{(2)} = t_b^2$ and $t^{(3)} = t_b^1$. The market price is $\Pi^* = kt_b^2 + (1-k)t_b^1$, and the allocation is $\mathcal{S}^* = \{s_1\}$ and $\mathcal{B}^* = \{b_1\}$. In the market with ties, $t^{(2)} = t_b^2 = t^{(3)} = t_b^1$. The market price is equal to $\Pi^* = t_b^2 = t_b^1$. The allocation is $\mathcal{S}^* = \{s_1\}$ and $\mathcal{B}^* = \{b_1\}$ or $\mathcal{B}^* = \{b_2\}$ (the latter chosen by a fair coin toss).

3.2 Definitions and observations

Because Rustichini et al. (1994)'s k-DA computes the interval of market prices $[t^{(m)}, t^{(m+1)}]$ via an explicit formula on a finite set of values, the same approach does not directly extend to infinite markets. In this section, we therefore extend the k-DA to the infinite case by defining a non-empty interval $[\underline{\Pi^*}, \overline{\Pi^*}]$, corresponding to the k-DA's $[t^{(m)}, t^{(m+1)}]$, with allocations specified accordingly. These prices are market-clearing prices that equilibrate demand and supply. We show that the generalized mechanism indeed properly extends the finite k-DA, proving that $[\underline{\Pi^*}, \overline{\Pi^*}] = [t^{(m)}, t^{(m+1)}]$ for finite markets.

Throughout the section, we formalize demand and supply, and market-clearing prices for general market instances. We shall state some results as *observations*. These are results that are generally known in the literature (Mas-Colell et al., 1995), but have not been stated formally in a single unified framework.[7]

[7] For completeness, we include their proofs in the Appendix.

We start with a formal definition of demand and supply. Consider the set of all traders whose reported value is (strictly) above or below a price P. For a relation $\mathcal{R} \in \{\geq, >, =, <, \leq\}$, we introduce the shorthand notation $\mathcal{B}_{\mathcal{R}}(P) = \{b \in \mathcal{B} : t_b \mathcal{R} P\}$ and $\mathcal{S}_{\mathcal{R}}(P) = \{s \in \mathcal{S} : t_s \mathcal{R} P\}$.

Definition (Demand and supply functions)**.** The *demand* and *supply functions* at price P are defined as $D^f(P) = \mu_B(\mathcal{B}_{\geq}(P))$ and $S^f(P) = \mu_S(\mathcal{S}_{\leq}(P))$, that is, by the mass of all traders who weakly prefer trading over not trading at price P.

Observation 1 (Analytic properties of demand and supply functions)**.** *The demand function is non-increasing, left-continuous and has right limits. The supply function is non-decreasing, right-continuous and has left limit. It holds that $D^f(P+) = \mu_B(\mathcal{B}_{>}(P))$ and $S^f(P-) = \mu_S(\mathcal{S}_{<}(P))$.*

$\mathcal{B}_{>}(P)$ and $\mathcal{S}_{<}(P)$ are the sets of traders who strictly prefer trading over not trading at price P.

Definition (Demand and supply correspondences)**.** The *demand* and *supply correspondences* are the set-valued functions $D^c(P) = [D^f(P+), D^f(P)]$ and $S^c(P) = [S^f(P-), S^f(P)]$.

Demand and supply correspondences account for both, traders with strict trading preference at the given price and for traders who are indifferent at that price. The resulting demand and supply correspondences can therefore by visualized as demand and supply functions with additional vertical lines instead of jump discontinuities. See Figure 1 for an illustration in our example.

Observation 2 (Connection between demand and supply functions and correspondences)**.** $D^f(P) = D^c(P) \Leftrightarrow \mu_B(\mathcal{B}_{=}(P)) = 0 \Leftrightarrow$ *the demand function is continuous at P. Similarly,* $S^f(P) = S^c(P) \Leftrightarrow \mu_S(\mathcal{B}_{=}(\mathcal{S})) = 0 \Leftrightarrow$ *the supply function is continuous at P.*

Example (Demand and supply)**.** In our example, demand and supply functions are

$$D^f(P) = \begin{cases} 2 & \text{if } P \leq t_b^2 \\ 1 & \text{if } t_b^2 < P \leq t_b^1 \\ 0 & \text{if } P > t_b^1 \end{cases} \text{ and } S^f(P) = \begin{cases} 0 & \text{if } P < t_s^1 \\ 1 & \text{if } P \geq t_s^1. \end{cases}$$

For the supply correspondence, it follows from Observation 2 that $S^c(P) = S^f(P)$ for $P \neq t_s^1$ and $S^c(t_s^1) = [0, 1]$. The demand correspondence differs according to whether ties exist or not. Without ties, it holds that $D^c(t_b^1) = [1, 2]$, $D^c(t_b^2) = [0, 1]$, and $D^c(P) = D^f(P)$ otherwise. With ties, it holds that $D^c(t_b^1) = D^c(t_b^2) = [0, 2]$, and again $D^c(P) = D^f(P)$ otherwise. Figure 1 illustrates both cases.

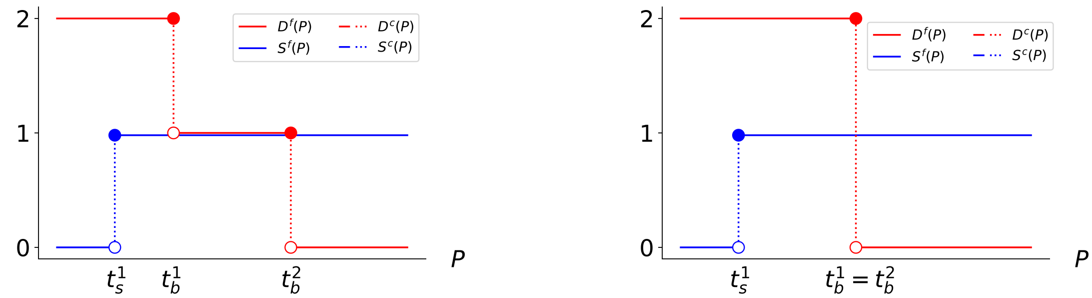
The common definition of DA market-clearing prices equates demand and supply functions.

Definition (Strong market-clearing price)**.** Price P is a *strong market-clearing price* if $D^f(P) = S^f(P)$. Denote the set of all strong market-clearing prices by $\mathcal{P}^{SMC}$.

The analytical properties of the demand and supply functions yield that:

Observation 3 (Topology of $\mathcal{P}^{SMC}$)**.** *The set $\mathcal{P}^{SMC}$ is a convex subset of T.*

Rustichini et al. (1994) show that any price in $(t^{(m)}, t^{(m+1)})$ is a strong market-clearing price, hence:



(a) Market without ties

(b) Market with ties

Figure 1: Demand and supply in the examples without and with ties.

Observation 4 (In finite DAs: $(t^{(m)}, t^{(m+1)}) \subset \mathcal{P}^{SMC}$)**.** *If $t^{(m)} \neq t^{(m+1)}$, then for $k \in (0,1)$ the k-DA results in a strong market-clearing price.*

However, if $k \in \{0,1\}$, the k-DA might not result in a strong market-clearing price.[8] Moreover, if ties exist, the set of strong market-clearing prices might be empty.[9] This can be illustrated in our example:

Example (Strong market-clearing prices)**.** In the market without ties, it follows from Figure 1a that the set of strong market-clearing prices $\mathcal{P}^{SMC}$ is equal to the half-open interval $\mathcal{P}^{MC} = (t_b^2, t_b^1]$. Only for $k = 0$, the k-DA does not choose a strong market-clearing price, in which case the demand function is equal to 2, but the supply function is equal to 1. In the market with ties, it follows from Figure 1b that there exists no strong market-clearing price at all, that is, the set $\mathcal{P}^{MC}$ is empty. The k-DA does not choose a strong market-clearing price at all, regardless of k.

The example illustrates that the k-DA cannot be defined generally via strong market-clearing. Instead, the following holds:

Observation 5 ($\mathcal{P}^{SMC}$ maximizes trading volume)**.** *A strong market-clearing price maximizes the* trading volume*, that is* $Q(P) = \min(D^f(P), S^f(P))$.

Maximizing gains of trade and maximizing trade volume are the two appealing properties from a social welfare perspective, but they are not sufficient to characterize the k-DA, as there exist market prices that maximize gains of trade and trade volume but are not realized for any $k \in [0,1]$ (see example 'Trading volume and excess' below). The following must also hold for general market instances:

Observation 6 ($\mathcal{P}^{SMC}$ minimizes trading excess)**.** *A strong market-clearing price minimizes the* trading excess $Ex(P) = |D^f(P) - S^f(P)|$.

Adding excess minimization to the characterization of the k-DA tightens the concept, but is still insufficient, because for different values of k the excess of the market price in the k-DA may

[8] These two mechanisms, called the *seller's* and *buyer's DA*, are often studied separately from the case $k \in (0,1)$ (Satterthwaite and Williams, 1989; Williams, 1991).

[9] Rustichini et al. (1994); Williams (1991) acknowledge that if $t^{(m)} = t^{(m+1)}$, market excess might exist and the k-DA may require rationing.

differ, or there may exist prices with the same trading volume and excess as outcomes of the k-DA, but are unrealized.

Example (Trading volume and excess)**.** In the market without ties, any price in $(t_b^1, t_b^2]$ maximizes market volume at 1 and minimizes excess at 0. But at t_b^1, the trading excess is equal to 1 and not minimized. Therefore, for $k = 0$, the k-DA does not minimize trading excess. In the market with ties, the k-DA chooses $t_b^1 = t_b^2$ as the market price, which maximizes trading volume at 1 and minimizes trading excess at 1. But every other price in $[t_s^1, t_b^1 = t_b^2)$ has the same trading volume and excess.

Strong market-clearing prices equate demand and supply functions, which does not work in general. A weaker notion of balancing demand and supply based on intersections of demand and supply correspondences always yields market-clearing prices. That is, all traders who strictly prefer to trade are included, but only some of the traders who weakly prefer to trade at a price, requiring some lottery for reported values at the market price.

Definition (Market-clearing price)**.** A price P is a *market-clearing price*, if $D^c(P) \cap S^c(P) \neq \emptyset$. Denote the set of all quasi-market-clearing prices by $\mathcal{P}^{MC}$.

This can also be expressed in terms of demand and supply functions.[10]

Observation 7 ($\mathcal{P}^{SMC}$ via D^f and S^f)**.** *P is a market-clearing price if and only if $D^f(P) \geq S^f(P)$ and $D^f(P+) \leq S^f(P)$ (*type I*) or $S^f(P) \geq D^f(P)$ and $S^f(P-) \leq D^f(P)$ (*type II*).*

Type I (II) refers to the existence of possible demand (supply) excess at P. Market-clearing prices are an extension of strong market-clearing prices:

Observation 8 ($\mathcal{P}^{SMC} \subset \mathcal{P}^{MC}$)**.** *Every strong market-clearing price is a market-clearing price (of type I and II).*

Observation 7 can be used to determine useful bounds on the set $\mathcal{P}^{MC}$ in terms of demand and supply functions.

Definition (Lower and upper bounds)**.** A price P is a *lower bound*, if for all $P' < P$ it holds that $D^f(P') > S^f(P')$ and an *upper bound*, if for all $P' > P$ it holds that $S^f(P') > D^f(P')$.

Observation 9 (Bounds on $\mathcal{P}^{MC}$)**.** *Consider a price P. If P is a lower bound, it holds that $P \leq \inf \mathcal{P}^{MC}$ and if P is an upper bound property, it holds that $P \geq \sup \mathcal{P}^{MC}$. Therefore, if additionally $P \in \mathcal{P}^{MC}$ holds, then $P = \min \mathcal{P}^{MC}$ or $P = \max \mathcal{P}^{MC}$ respectively.*

While it was shown (see example 'Strong market-clearing prices' above) that strong market-clearing prices might not exist, the set of market-clearing prices has the right topology as a candidate for an alternative description of the market prices in the k-DA. Moreover, the connection between $\mathcal{P}^{SMC}$ and $\mathcal{P}^{MC}$ from Observation 8 can be strengthened by showing that $\mathcal{P}^{MC}$ can be viewed as the minimal extension of $\mathcal{P}^{SMC}$ to guarantee existence:

Observation 10 (Topology of $\mathcal{P}^{MC}$)**.** *The set of market-clearing prices is non-empty, convex and closed. The set $\mathcal{P}^{MC} \setminus \mathcal{P}^{SMC}$ has Lebesgue-measure zero. More precisely, if $\mathcal{P}^{SMC} \neq \emptyset$, then $\mathcal{P}^{MC} = \overline{\mathcal{P}^{SMC}}$, and if $\mathcal{P}^{SMC} = \emptyset$, then $\mathcal{P}^{MC}$ is a singleton.*

[10] In the absence of a strong market-clearing price, Tatur (2005) mentions a similar construction, when demand and supply functions are given by probability distributions with jump-discontinuities.

Example (Market-clearing prices)**.** In the market without ties, the set of market-clearing prices $\mathcal{P}^{MC}$ is equal to the closed interval $[t_b^1, t_b^2]$, see Figure 1a. Therefore, in line with Observation 3, $\mathcal{P}^{MC} = \overline{\mathcal{P}^{SMC}}$ holds, because it was shown above that $\mathcal{P}^{SMC} = (t_b^1, t_b^2]$. In the market with ties, there exists a unique market clearing price, that is $\mathcal{P}^{MC} = \{t_b^1 = t_b^2\}$, see Figure 1b. Recall that it was shown above that $\mathcal{P}^{SMC} = \emptyset$. Therefore, again in line with Observation 3, it holds that $\mathcal{P}^{MC}$ is a singleton.

3.3 The generalized k-double auction

We are now able to show that the finite k-DA can be fully characterized by the set of market-clearing prices.[11] Note that this result will immediately imply that the the identified mechanism applies to both finite and infinite markets.

Theorem 1 (Characterization of the finite k-DA via $\mathcal{P}^{MC}$)**.** *In finite markets with m buyers and n sellers $\mathcal{P}^{MC} = [t^{(m)}, t^{(m+1)}]$.*

Proof. Consider two cases separately: (i) $t^{(m)} \neq t^{(m+1)}$, and (ii) $t^{(m)} = t^{(m+1)}$. For (i), it follows from Observation 4 that $P \in (t^{(m)}, t^{(m+1)})$ is a strong market-clearing price, and therefore by Observation 8 $P \in \mathcal{P}^{SMC}$. Next, consider $t^{(m)}$. $S^f(t^{(m)}) = S^f(P)$ for P in $(t^{(m)}, t^{(m+1)})$. If there is a bid equal to $t^{(m)}$, then it holds that $D^f(t^{(m)}) > D^f(P)$. If not, then $D^f(t^{(m)}) = D^f(P)$. This shows that $D^f(t^{(m)}) \geq D^f(P)$. It therefore holds that $D^f(t^{(m)}) \geq D^f(P) = S^f(P) = S^f(t^{(m)})$. To show that $t^{(m)} \in \mathcal{P}^{MC}$, it is by Observation 2 sufficient to show that $D^f(t^{(m)}+) \leq S^f(t^{(m)})$. It holds that $D^f(t^{(m)}+) = D^f(P)$, as there are no values in $(t^{(m)}, t^{(m+1)})$. This finally yields $D^f(t^{(m)}+) = D^f(P) = S^f(P) = S^f(t^{(m)})$. A similar argument shows that $t^{(m+1)} \in \mathcal{P}^{MC}$. Finally, we show that $t^{(m+1)}$ is an upper bound and $t^{(m)}$ is a lower bound. Observation 9 then implies that $t^{(m)} = \min \mathcal{P}^{MC}$ and $t^{(m+1)} = \max \mathcal{P}^{MC}$, which finishes the proof for (i). Consider $P > a^{(m+1)}$. It holds that $D^f(P) < S^f(P)$, as demand increases or supply increases due to the reported values at $t^{(m+1)}$. Therefore $t^{(m+1)}$ is an upper bound. Similar arguments yield that $t^{(m)}$ is a lower bound. For (ii), write $t = t^{(m)} = t^{(m+1)}$ for ease of notation. We will show $t \in \mathcal{P}^{MC}$ and that this price is both a lower and upper bound. Observation 9 then implies that $\mathcal{P}^{MC} = \{t\}$. For a relation $\mathcal{R} \in \{\geq, >, =, <, \leq\}$, denote by $t_{\mathcal{R}}$ the number of values (strictly) above, below or equal to t. Denote by $t_{\mathcal{R},B}$ and $t_{\mathcal{R},S}$ the restriction to either bids or asks. It holds that $t_< \leq m-1$, $t_= \geq 2$, $t_> \leq n-1$, and $t_< + t_= + t_> = m+n$. Note that $D^f(t) = t_{\geq,B} = t_{=,B} + t_{>,B} \geq 1$. That is because there are at most $m-1$ values strictly below t and there is a total of m buyers, which proves that at least one bid is greater or equal to t. Because the total number of sellers is n, it holds that $S^f(t) = t_{\leq,S} = n - t_{>,S} = n - t_> + t_{>,B}$. Take $P > t$. It holds that $D^f(P) \leq t_{>,B}$, because $t_{=,B}$ bids at t are lost and $S^f(P) \geq n - t_> + t_{>,B}$, because supply is non-decreasing. This implies that $S^f(P) - D^f(P) \geq n - t_>$. It follows from $t_> \leq n-1$ that $S^f(P) - D^f(P) \geq 1$, which implies that $S^f(P) > D^f(P)$. Therefore t has the upper bound property. Take $P < t$. It holds that $S^f(P) \leq n - t_> + t_{>,B} - t_= + t_{=,B}$, because $t_= - t_{=,B}$ asks at t are lost and $D^f(P) \geq t_{=,B} + t_{>,B}$, because demand is non-increasing. This

[11] This result is already present without a formal proof in the literature. Rustichini et al. (1994) justifies it with Observation 4, and Cripps and Swinkels (2006) states that it can be seen after "a little time with the appropriate figure" of demand and supply schedules. Both do not provide a rigorous definition of market-clearing prices.

implies that $D^f(P) - S^f(P) \geq t_= + t_> - n$. But it follows from $t_< + t_= + t_> = m + n$ and $t_< \leq m - 1$ that $t_= + t_> = m + n - t_< \geq n + 1$, which implies that $D^f(P) - S^f(P) \geq 1 > 0$. Therefore, $D^f(P) > S^f(P)$, which shows that t has the lower bound property. $\square$

As the concept of market-clearing prices solely depends on demand and supply correspondences (or functions by Observation 2), we can thus define the (generalized) k-DA for both finite and infinite markets:

(Generalized) k-DA

Market price. For $k \in [0, 1]$ set the market price as

$$\Pi^* = k \cdot \min \mathcal{P}^{MC} + (1 - k) \cdot \max \mathcal{P}^{MC}.$$

Allocation. The following allocations are carried out:

$$\begin{aligned} &\mathcal{S}^* = \mathcal{S}_{\leq}(\Pi^*) \text{ and } \mathcal{B}^* = \mathcal{B}_{>}(\Pi^*) \cup \tilde{\mathcal{B}} && \text{if } \Pi^* \text{ is of type I} \\ &\mathcal{B}^* = \mathcal{B}_{\geq}(\Pi^*) \text{ and } \mathcal{S}^* = \mathcal{S}_{<}(\Pi^*) \cup \tilde{\mathcal{S}} && \text{if } \Pi^* \text{ is of type II} \end{aligned}$$

where $\tilde{\mathcal{B}} \subset \mathcal{B}_=(\Pi^*)$ (respectively $\tilde{\mathcal{S}} \subset \mathcal{S}_=(\Pi^*)$) are uniformly random compact sets selecting players in case there is market excess, with $\mu_B(\tilde{B}) = \mu_S(\mathcal{S}^*) - \mu_B(\mathcal{B}_>(\Pi^*))$ (respectively $\mu_S(\tilde{S}) = \mu_B(\mathcal{B}^*) - \mu_S(\mathcal{S}_<(\Pi^*))$).[a]

[a] A uniform random compact set satisfies that for all $b \in \mathcal{B}_=(\Pi^*)$ it holds that $\mathbb{P}[b \in \tilde{\mathcal{B}}] \equiv$ *const* (respectively for all $s \in \mathcal{S}_=(\Pi^*)$ it holds that $\mathbb{P}[s \in \tilde{\mathcal{S}}] \equiv$ *const*).

Note that if Π^* is a strong market-clearing price, the allocation simplifies: Set $\mathcal{B}^* = \mathcal{B}_{\geq}(\Pi^*)$ and $\mathcal{S}^* = \mathcal{S}_{\leq}(\Pi^*)$, that is, no random rationing is needed.

4 Walrasian competitive equilibrium

It is folk wisdom that a DA results in a Walrasian CE, and the terms market-clearing price and (competitive) equilibrium price are used interchangeably, albeit having in principle different definitions (recall the discussion in Section 1). Here, we formalize Walrasian CE for DAs, and prove that the folk wisdom indeed holds for the (generalized) k-DA. In finite markets, Arrow and Debreu (1954) define CE in pure exchange economies via a set of explicit conditions, that have been generalized to infinite markets in Aumann (1964). Many of the technical intricacies of their models can be neglected for our purposes, as we deal with a single commodity and unit demand. For clarity and easier interpretation of results, we give here a definition of Walrasian CE tailored specifically to our model. In plain English, in a Walrasian CE, every trader should weakly prefer their allocated bundle over every other affordable bundle. This can be formalized in two conditions: Every trader involved in trade must weakly prefer trading over not trading (*individual rationality*). Moreover, given their report every trader who strictly prefers trading over not trading at a given price must be involved in trade (*stability*). Finally, to facilitate trade, the number of active traders on both market sides must be equal, as items cannot be created or destroyed (*trade-balance*). This leads to the following definition of Walrasian CE:

Definition (Walrasian CE)**.** A market outcome $(P, \mathcal{B}^*, \mathcal{S}^*)$ is a *Walrasian CE*, if

- it *balances trade*, that is $\mu_B(\mathcal{B}^*) = \mu_S(\mathcal{S}^*)$,
- it is *individually rational*, that is $\mathcal{B}^* \subset \mathcal{B}_{\geq}(P)$ and $\mathcal{S}^* \subset \mathcal{S}_{\leq}(P)$, and
- it is *stable*, that is $\mathcal{B}_{>}(P) \subset \mathcal{B}^*$ and $\mathcal{S}_{<}(P) \subset \mathcal{S}^*$.

Theorem 2 ($\mathcal{P}^{MC} = \mathcal{P}^{EQ}$)**.** *A price is a market-clearing price if and only if it is a Walrasian CE price. Thus, the generalized k-DA results in a Walrasian CE.*

Proof. $\mathcal{P}^{MC} \subset \mathcal{P}^{EQ}$: First assume that P is of type I, that is $D^f(P) \geq S^f(P)$ and $D^f(P+) \leq S^f(P)$. Set $\mathcal{S}^* = \mathcal{S}_{\leq}(P)$. Consider the set $\mathcal{B}_{>}(P)$. It follows from Observation 1 that $D^f(P+) = \mu_B(\mathcal{B}_{>}(P))$. Let $x = S^f(P) - \mu_B(\mathcal{B}_{>}(P)) \geq 0$ and let $\tilde{\mathcal{B}}$ be a subset of $\mathcal{B}_{=}(P)$ with μ_B-measure equal to x. Such a set exists because $D^f(P) = \mu_B(\mathcal{B}_{\geq}(P)) = \mu_B(\mathcal{B}_{>}(P)) + \mu_B(\mathcal{B}_{=}(P)) \geq S^f(P)$ and $D^f(P+) = \mu_B(\mathcal{B}_{>}(P)) \leq S^f(P)$. Set $\mathcal{B}^* = \mathcal{B}_{>}P \cup \tilde{\mathcal{B}}$. We claim that $(P, \mathcal{S}^*, \mathcal{B}^*)$ is a Walrasian CE. It balances trade by construction, because $\mu_B(\mathcal{B}^*) = \mu_B(\mathcal{B}_{>}(P)) + \mu_B(\tilde{(}\mathcal{B})) = \mu_B(\mathcal{B}_{>}(P)) + S^f(P) - \mu_B(\mathcal{B}_{>}(P))$. Individual rationality follows, because $\mathcal{B}^* \subset \mathcal{B}_{\geq}(P)$ and $\mathcal{S}^* = \mathcal{S}_{\leq}(P)$. It is also stable, because $\mathcal{B}_{>}(P) \subset \mathcal{B}^*$ and $\mathcal{S}_{<}(P) \subset \mathcal{S}^*$, as $\mathcal{B}^* = \mathcal{B}_{>}P \cup \tilde{\mathcal{B}}$ and $\mathcal{S}^* = \mathcal{S}_{\leq}(P)$. If there exists a market-clearing price P of type II, that is $S^f(P) \geq D^f(P)$ and $S^f(P-) \leq D^f(P)$, one can construct the Walrasian CE analogously.

$\mathcal{P}^{EQ} \subset \mathcal{P}^{MC}$: We show that $P \notin \mathcal{P}^{MC} \Rightarrow P \notin \mathcal{P}^{EQ}$. One of two cases must hold: (i) $D^f(P) > S^f(P)$ and $D^f(P-) > S^f(P)$ or (ii) $S^f(P) > D^f(P)$ and $S^f(P-) > D^f(P)$. For (i) there exists a price $P' > P$ with $D^f(P') > S^f(P)$. Assume that there exist sets $\mathcal{B}^*$ and $\mathcal{S}^*$, such that $(P, \mathcal{B}^*, \mathcal{S}^*)$ is a Walrasian CE. Individual rationality implies that it must hold that $\mathcal{B}^* \subset \mathcal{B}_{\geq}(P)$ and $\mathcal{S}^* \subset \mathcal{S}_{\leq}(P)$. Next, stability implies that $\mathcal{B}_{>}(P) \subset \mathcal{B}^*$ and $\mathcal{S}_{<}(P) \subset \mathcal{S}^*$ holds. Those two inclusions imply that $\mu_B(\mathcal{B}_{\geq}(P)) \geq \mu_B(\mathcal{B}^*) \geq \mu_B(\mathcal{B}_{>}(P))$ and $\mu_S(\mathcal{S}_{\leq}(P)) \geq \mu_S(\mathcal{S}^*) \geq \mu_S(\mathcal{S}_{<}(P))$. Note that for a price $P' > P$ it holds that $\mu_B(\mathcal{B}_{>}(P)) \geq \mu_B(\mathcal{B}_{\geq}(P'))$. It therefore holds that

$$\mu_B(\mathcal{B}^*) \geq \mu_B(\mathcal{B}_{>}(P)) \geq \mu_B(\mathcal{B}_{\geq}(P')) = D^f(P') > S^f(P) \geq \mu_S(\mathcal{S}^*). \tag{1}$$

This proves that $(P, \mathcal{B}^*, \mathcal{S}^*)$ is not a Walrasian CE, because it does not balance trade. For (ii), the proof is analogous. □

5 Conclusion

DAs are ubiquitous both in theory and practice. By revisiting and clarifying necessary conditions for market clearing prices in both infinite and finite markets we were able to provide a mechanism for any market instance. Showing its connection to Walrasian CE highlights that simplified, folk definitions such as maximizing trade are insufficient to guarantee equilibrium outcomes. In this vein we hope that our work can, on the one hand, allow researchers to avoid unnecessary smoothness assumptions that limit the scope of analysis and, on the other hand, allow practitioners to better understand what is required–and what is insufficient–to build markets that yield equilibrium.

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A Additional Proofs

Proof of Observation 1: Analytic properties of demand and supply. Let μ_S^t be the pushforward measures of μ_S with respect to t_S, that is $\mu_S^t(\cdot) = \mu_S(t_S^{-1}(\cdot))$. μ_S^t is σ-additive and finite on T. Denote by $\underline{T}(P)$ the interval $[\underline{t}, P]$. It holds that $S^f(P) = \mu_S^t(T_S(P))$. $\underline{T}(P_2) \subset \underline{T}(P_1)$, if $P_1 > P_2$. The σ-subadditivity of μ_S^t yields $S^f(P_1, a) = \mu_S^t(\underline{T}(P_1)) \geq \mu_S^t(\underline{T}(P_2)) = S^f(P_1, a)$, which proves that $S^f(\cdot)$ is non-decreasing. Every monotonic function has limits from the right and the left for every point in its domain, see ... Next, consider a strictly decreasing sequence of prices $P_n \downarrow P$. $\underline{T}(P_n)$ is a decreasing sequence of sets, that is $\underline{T}(P_{n+1}) \subset \underline{T}(P_n)$. It holds that $\lim_{n\to\infty} \underline{T}(P_n) = \bigcap_{n=1}^{\infty} \underline{T}(P_n) = \underline{T}(P)$. As a finite measure on $\mathbb{R}$, μ_S^t is continuous from above, see e.g. Folland (1999). That is if $\{A_i\}_i \subset T$ is a sequence of sets with $A_1 \supset A_2 \supset A_3 \supset ...$, then $\mu_S^t\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{i\to\infty} \mu_S^t(A_i)$. This yields

$$\lim_{P_n \downarrow P} S^f(P_n) = \lim_{n\to\infty} \mu_S^t(\underline{T}(P_n)) = \mu_S^t\left(\bigcap_{n=1}^{\infty} \underline{T}(P_n)\right) = \mu_S^t(\underline{T}(P)) = S^f(P), \tag{2}$$

which proves the right-continuity of $S^f(\cdot)$. To show that $S^f(P-) = \mu_S(\mathcal{S}_<(P)) = \mu_S^t([\underline{t}, P))$, note μ_S^t is a σ-additive Borel-measure on $\mathbb{R}$ and therefore regular, see Bogachev (2007). That is, for all Borel-sets A $\mu_S^t(A) = \sup\{\mu_S^t(F) | F \subset A, F \text{ compact Borel set}\}$. It is therefore sufficient to approximate the interval $[\underline{t}, P)$ by compact sets $[\underline{t}, P']$ with $P' < P$. It finally holds that $\sup\{\mu_S^t(F) | F \subset A, F \text{ compact Borel set}\} = \lim_{P'\uparrow P} \mu_S^t(\underline{t}, P']) = \lim_{P'\uparrow P} S^f(P') = S^f(P-)$, which implies that $S^f(P-) = \mu_S(\mathcal{S}_<(P))$. The proof that demand is non-increasing, left-continuous, and has right limits, as well as $D^f(P+) = \mu_B(\mathcal{B}_>(P))$ is analogous. $\square$

Proof of Observation 2: Connection between demand and supply functions and correspondences. Recall that $D^c(P) = [D^f(P+), D^f(P)]$, $D^f(P) = \mu_B(\mathcal{B}_\geq(P))$ and by Observation 1 $D^f(P+) = \mu_B(\mathcal{B}_>(P))$. The σ-additivity of μ_B implies that $D^f(P) = \mu_B(\mathcal{B}_\geq(P)) = \mu_B(\mathcal{B}_>(P)) + \mu_B(\mathcal{B}_=(P)) = D^f(P+) + \mu_B(\mathcal{B}_=(P))$. Because $D^f(P) = \tilde{D^f}(P)$ if and only if $D^f(P+) = D^f(P)$, this is equivalent to $\mu_B(\mathcal{B}_=(P)) = 0$. Furthermore, $D^f(P+) = D^f(P)$ is right-continuity. As $D^f(\cdot)$ is left-continuous by Observation 1, $D^f(P) = \tilde{D^f}(P)$ is equivalent to continuity at P. The proof for supply is analogous. $\square$

Proof of Observation 3: Topology of $\mathcal{P}^{SMC}$. As the empty set is convex by convention, assume that $\mathcal{P}^{SMC} \neq \emptyset$. Consider $P_1, P_2 \in \mathcal{P}^{SMC}$ with $P_1 \leq P_2$. The monotonicity of $D^f(\cdot)$ and $S^f(\cdot)$, see Observation 1, implies that $D^f(P_1) \geq D^f(P_2) = S^f(P_2) \geq S^f(P_1)$, which proves that $D^f(\cdot)$ and $S^f(\cdot)$ are constant on $[P_1, P_2]$. Therefore for any price $P \in [P_1, P_2]$ it holds that $P \in \mathcal{P}^{SMC}$. For $P < \underline{t}$ $D^f(P) = \mu_B(\mathcal{B}) > 0 = S^f(P)$ and for $P > \bar{t}$ $D^f(P) = 0 < \mu_S(\mathcal{S}^*) = S^f(P)$ holds, which implies that $\mathcal{P}^{SMC} \subset T$. $\square$

Proof of Observation 4. We show that $P \in (t^{(m)}, t^{(m+1)}) \Rightarrow P \in \mathcal{P}^{SMC}$. Suppose that $D^f(P) = k \geq 0$. It holds that $D^f(P) = D^f(t^{(m+1)}$ and $S^f(P) = S^f(t^{(m)}$. The set $\{t^{(m+1)}, ..., t^{(m+n)}\}$ has cardinality n and the number of bids in it is $D^f(t^{(m+1)})$ and therefore k. Therefore the number of asks in it is $n - k$. As there is a total number of n asks, the number of asks in the set $\{t^{(1)}, ..., t^{(m)}\}$ is k. As this number is equal to $S^f(t^{(m)}$, it holds that $S^f(P) = k = D^f(P)$. $\square$

Proof of Observation 5: $\mathcal{P}^{SMC}$ maximizes trading volume. For $P \in \mathcal{P}^{SMC}$, it holds by definition that $Q = D^f(P) = S^f(P)$. For $P' \geq P$ it holds that $D^f(P') \leq D^f(P)$, because $D^f(\cdot)$ is non-increasing and for $P' \leq P$ it holds that $S^f(P') \leq S^f(P)$, because $S^f(\cdot)$ is non-decreasing. Therefore for any $P' \neq P$ it holds that $Q(P') \leq Q(P)$. $\square$

Proof of Observation 6: $\mathcal{P}^{SMC}$ minimizes trading excess. For $P \in \mathcal{P}^{SMC}$ it holds by definition that $D^f(P) = S^f(P)$ and therefore also $Ex(P) = 0$. $\square$

Proof of Observation 7: Market-clearing prices via demand and supply functions. If $D^f(P) \geq S^f(P)$, then $D^c(P) \cap S^c(P) \neq \emptyset \Leftrightarrow S^f(P-) \geq D^f(P)$. If $D^f(P) \leq S^f(P)$, then $D^c(P) \cap S^c(P) \neq \emptyset \Leftrightarrow D^f(P+) \geq S^f(P)$. $\square$

Proof of Observation 8: $\mathcal{P}^{SMC} \subset \mathcal{P}^{MC}$. For $P \in \mathcal{P}^{SMC}$ it holds by definition that $D^f(P) = S^f(P)$. Because $D^f(\cdot)$ is non-increasing it follows that $D^f(P+) \geq D^f(P) = S^f(P)$, which proves that P is a market-clearing price of type I. Because $S^f(\cdot)$ is non-decreasing it follows that $S^f(P-) \leq S^f(P) = D^f(P)$, which proves that P is a market-clearing price of type II. $\square$

Proof of Observation 9. Consider that P is a lower bound. It suffices to prove that for a price $P' < P$ it holds that $P' \notin \mathcal{P}^{MC}$. If $P \in \mathcal{P}^{MC}$ additionally, it then follows directly that $P = \min \mathcal{P}^{MC}$. Because $D^f(P') > S^f(P')$, it is sufficient to prove that $D^f(P'-) > S^f(P')$. For P'' in (P', P) it holds that $D^f(P'') > S^f(P'')$. The monotonicity of $D^f(\cdot)$ and $S^f(\cdot)$

therefore yields $D^f(P'-) \geq D^f(P'') > S^f(P'') \geq S^f(P')$. If P is an upper bound, the proof is analogous. $\square$

Proof of Observation 10: Topology of $\mathcal{P}^{MC}$. Consider the set $\hat{\mathcal{P}} = \{P : D^f(P) \geq S^f(P)\}$. The monotonicity of $D^f(\cdot)$ and $S^f(\cdot)$ yields that $\hat{\mathcal{P}}$ is convex. For $P < \underline{t}$ it holds that $D^f(P) = \mu_B(\mathcal{B}) > 0 = S^f(P)$, proving that $\hat{\mathcal{P}}$ is non-empty. For $P > \bar{t}$ it holds that $D^f(P) = 0 < \mu_S(\mathcal{S}^*) = S^f(P)$, proving that $\hat{\mathcal{P}}$ is bounded from above. For such a set the supremum exists and is unqiue, see ... . Denote it by $P^* = \sup \hat{\mathcal{P}}$. We show that $P^* \in \mathcal{P}^{MC}$. Two cases need to be considered separately: (i) $P^* \in \hat{\mathcal{P}}$ and (ii) $P^* \notin \hat{\mathcal{P}}$. For (i), $D^f(P^*) \geq S^f(P^*)$ and for all $P' > P^*$ $D^f(P') < S^f(P')$. The right-continuity of $S^f(\cdot)$ implies $D^f(P'+) \leq S^f(P')$, which proves that P is a market-clearing price of type I. For (ii), $D^f(P^*) < S^f(P^*)$ holds. The monotonicity of $D^f(\cdot)$ and $S^f(\cdot)$ implies that for all $P' < P^*$ $D^f(P') \geq S^f(P')$ holds. Left-continuity of $D^f(\cdot)$ yields that $D^f(P') \geq S^f(P'-)$, which proves that P^* is a market-clearing price of type II.

Next, assume that $\mathcal{P}^{SMC} = \emptyset$. To prove that P^* is the unique market-clearing price, it suffices to prove by Observation 9 that P^* is both a lower and upper bound. P^* is either of type I, that is $D^f(P^*) > S^f(P^*)$ and $D^f(P^*+) \leq S^f(P^*)$ or of type II, that is $S^f(P^*) > D^f(P^*)$ and $S^f(P^*-) \leq D^f(P^*)$. It follows from monotonicity of $D^f(\cdot)$ and $S^f(\cdot)$ and the emptyness of $\mathcal{P}^{MC}$ that for all $P' < P$ it holds that $D^f(P') > S^f(P')$ and for all $P' > P$ it holds that $D^f(P') < S^f(P')$. Therefore P^* is indeed both a lower and upper bound.

Finally, assume that the interval $\mathcal{P}^{SMC} \neq \emptyset$. To show that $\mathcal{P}^{MC} = \overline{\mathcal{P}^{SMC}}$, by Observation 9 it suffices to prove that $\underline{P} \in \mathcal{P}^{MC}$ is a lower bound and $\overline{P} \in \mathcal{P}^{MC}$ is an upper bound. $D^f(\underline{P}) \geq S^f(\underline{P})$ by monotonicity of $D^f(\cdot)$ and $S^f(\cdot)$. By definition, for every P with $\overline{P} > P > \underline{P}$ it holds that $D^f(P) = S^f(P)$. It follows from the left continuity of $S^f(\cdot)$ that $D^f(\underline{P}+) = S^f(\underline{P})$, which proves that $\underline{P} \in \mathcal{P}^{MC}$. For every $P' < \underline{P}$ we have that $D^f(P') > S^f(P')$, which yields the lower bound property. Similar arguments yield that $\overline{P} \in \mathcal{P}^{MC}$ with the upper bound property. $\square$