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Abstract

We study the propagation of monetary shocks in a sticky-price general-equilibrium economy where firms set prices subject to strategic complementarities with the decision of other firms. In the dynamic equilibrium the firm's price-setting decisions depend on aggregates, which in turn depend on firms' decisions. We cast this fixed-point problem as a Mean Field Game (MFG) and establish several analytic results. We study existence and uniqueness of the equilibrium and analytically characterize the impulse response function (IRF) of output following an aggregate "MIT" shock. We prove that strategic complementarities make the IRF larger at each horizon, in a convex fashion. We establish that complementarities may give rise to a non-monotone IRF, with a hump-shaped profile. As the complementarity becomes large enough the IRF diverges and at a critical point there is no equilibrium. Finally, we show that the amplification effect of the strategic interactions is similar across models. For instance, the Calvo model and the Golosov-Lucas model display a comparable amplification, in spite of the fact that the non-neutrality in Calvo is much larger.

JEL Classification: E3, E5

Keywords: Monetary Economics, sticky prices, strategic complementarities, dynamic equilibria, mean field games, singular stochastic control

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Price Setting with Strategic Complementarities as a Mean Field Game*

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Abstract

We study the propagation of monetary shocks in a sticky-price general-equilibrium economy where firms' pricing strategy feature a complementarity with the decisions of other firms. In the dynamic equilibrium the firm's price-setting decisions depend on aggregates, which in turn depend on firms' decisions. We cast this fixed-point problem as a Mean Field Game (MFG) and establish several analytic results. We study existence and uniqueness of the equilibrium and analytically characterize the impulse response function (IRF) of output following an aggregate "MIT" shock. We prove that strategic complementarities make the IRF larger at each horizon, in a convex fashion. We establish that complementarities may give rise to a non-monotone IRF, with a hump-shaped profile. As the complementarity becomes large enough the IRF diverges and at a critical point there is no equilibrium. Finally, we show that the amplification effect of the strategic interactions is similar across models. For instance, the Calvo model and the Golosov-Lucas model display a comparable amplification, in spite of the fact that the non-neutrality in Calvo is much larger.

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1 Introduction

The seminal contributions of Bils and Klenow (2004) and Golosov and Lucas (2007) renewed interest in state-dependent sticky price models, and triggered substantive progress both in the empirical front, uncovering patterns about price setting behavior, as well as in the theoretical characterization of the forces that determine the aggregate monetary non-neutrality.[1] In spite of this progress, the need for tractability led most models to either abstract from the interactions between firms' decisions in price setting – as in Golosov and Lucas (2007) – or to explore their effects numerically – as in Klenow and Willis (2016) and Mongey (2021) –, or to abstract from idiosyncratic shocks – as in Caplin and Leahy (1997) and Wang and Werning (2020). In this paper we give a detailed analytic characterization of the effect of a monetary shock in a relatively rich state-dependent model, featuring both idiosyncratic shocks and strategic complementarities/substitutabilities.

The issue is relevant because absent strategic complementarities the current quantitative macro models seem unable to produce the persistent non-neutrality of nominal shocks that is seen in the aggregate data. Strategic complementarities in pricing decision are a key source of "real rigidities" in a variety of sticky-price models, as in e.g. the classic state-dependent model of Caplin and Leahy (1997).[2] More recently, Nakamura and Steinsson (2010) and Klenow and Willis (2016) explored the role of strategic complementarities in state-dependent models with idiosyncratic shocks, a feature that allows the theory to connect with a wealth of micro data on price-setting. Moreover, several empirical studies suggest the presence of non-negligible complementaries, e.g. Cooper and Haltiwanger (1996); Amiti, Itskhoki, and Konings (2014, 2019); Beck and Lein (2020).

A rigorous treatment of strategic complementarities in a general equilibrium model is

[1] Among others, see the contributions of Klenow and Malin (2010), Nakamura and Steinsson (2010), Caballero and Engel (1999, 2007), Midrigan (2011), Alvarez and Lippi (2014), Alvarez, Le Bihan, and Lippi (2016), Alvarez, Beraja, Gonzalez-Rozada, and Neumeyer (2019).

[2] The idea that strategic complementarities may contribute to amplify the aggregate stickiness has a long tradition in macroeconomics, and was formalized by Cooper and John (1988) in a static setup, and surveyed by Leahy (2011).

involved, as emphasized by Caplin and Leahy (1997): decisions depend on aggregate variables, which in turn depend on individual decisions. An analytic characterization of this fixed point problem is difficult, especially so in a model with lumpy behavior, where the optimal decisions are non-linear and time-varying (Ss rules). A recent analysis by Wang and Werning (2020) presents analytic results for a dynamic oligopoly model. In this insightful paper, with rich strategic behaviour, tractability is obtained by assuming that the timing of the firm's price adjustments is exogenous and that the state-space has a finite dimension related to the finite number of firms in the market. Our approach shares with Caplin and Leahy (1997) and Wang and Werning (2020) a quest for analytic results on the propagation of aggregate shocks with strategic complementarity. An important difference with respect to these papers is that we consider a problem with idiosyncratic shocks at the firm level. This feature allows us to relate to the micro-data on price changes, which have been shown to encode powerful information about shock propagation.[3] Instead, due to the absence of idiosyncratic shocks, in Caplin and Leahy (1997) and Wang and Werning (2020) all price changes are either increases or decreases at a point in time.[4]

We present a set of analytical results that characterize the firm's optimal policy and the general equilibrium in a dynamic environment featuring strategic complementarities (or substitutabilities). The key breakthrough is obtained by casting the problem using the mathematical structure of Mean Field Games (MFG), as laid out by Lasry and Lions (2007). The problem takes the form of a system of two coupled partial differential equations: one Bellman equation describing individual decisions, and one Kolmogorov equation describing aggregation. The usefulness of employing the MFG framework to study the dynamic behavior of high-dimensional cross-sections is highlighted by Achdou, Han, Lasry, Lions, and Moll (2022); Ahn, Kaplan, Moll, Winberry, and Wolf (2018) where numerical methods are discussed. Rel-

[3]In Alvarez, Le Bihan, and Lippi (2016) and Alvarez, Lippi, and Oskolkov (2021) it is argued that the kurtosis of the distribution of price changes and the frequency of price changes are a sufficient statistic for the cumulative impulse response of monetary shocks. Alvarez et al. (2021) and Gautier, Marx, and Vertier (2021) explore whether cross sectional evidence is consistent with such theoretical prediction.

[4]Of course, not including idiosyncratic shocks both simplifies the analysis and allows the authors to address other issues, as we further discuss below.

ative to the MFG literature, and its applications to economics, this paper innovates in two dimensions. First, we focus on an analytic characterization of the dynamics that ensue following a perturbation of the stationary equilibrium, i.e. an MIT shock.[5] The presence of strategic complementarities can create, even in simple static models, lack of equilibrium or multiplicity, which makes analytical, as opposed to purely numerical methods, necessary.[6] Second, we consider an impulse control problem, instead of drift control, i.e. we deal with the case of lumpy adjustments which is the relevant one for price-setting. The case of lumpy adjustments, appearing in several economic contexts, motivates our interest in this problem and is mathematically more delicate since it requires to solve a problem with time-varying boundaries. A notable example of a rigorous early analysis of a MFG with impulse control is Bertucci (2017).

We consider an economy with random menu costs of the Calvo-plus type considered in Nakamura and Steinsson (2010). We concentrate on an economy where, following Klenow and Willis's (2016) terminology, we can capture both micro and macro complementarities (or substitutability) in the decision problem of the firm. These originate from the fact that the firm's flow profit in each period depends on its own markup and the markup (or price) of the average firm, with a positive (negative) cross derivative. This model spans price-setting models in between the pure Ss model of Golosov and Lucas (2007) to the pure time-dependent model of Calvo (1983). The dynamic equilibrium of this model for an economy *without strategic interactions* was solved analytically in Alvarez and Lippi (2021). The MFG framework allows us to study analytically the effect of such interactions on the firm's optimal Ss rules after the shock as well as its effect on the aggregate dynamics.

[5] By an MIT shock we mean to solve for the equilibrium triggered by a small unexpected arbitrary perturbation of the stationary distribution – see Boppart, Krusell, and Mitman (2018) for numerical techniques to solve for the similar type of perturbation, and for the same interpretation of the resulting equilibrium as an impulse response.

[6] Note that the well-known "monotonicity condition" for uniqueness, developed by Lasry-Lions and used in almost all papers in these area, corresponds to the case of strategic substitutability.

Main results. The results allow us to make progress on substantive economic questions. First, we establish conditions for the existence and uniqueness of the perturbed equilibrium and analytically characterize the impulse response function (IRF) of output. We show that the presence of the strategic complementarity makes the output IRF of a monetary shock larger at each horizon. Not only the effect is larger at each horizon for higher strategic complementarities, but it is also convex in the degree of strategic interactions. Indeed, there is a critical value of the strength of the strategic complementarity at which the IRF becomes arbitrarily large, and then the equilibrium ceases to exist. For strategic complementarities larger than that critical value the equilibrium may not exists, or it may not be well behaved (e.g. not necessarily continuous as function of the parameters). On the other hand, as substitutability becomes arbitrarily large, the IRF converges to zero.

Second, we show that the presence of large enough strategic complementarities makes the IRF hump-shaped as a function of time elapsed since the shock, in models where it is otherwise monotone decreasing. This is a novel result that illustrates the substantive economic consequences of strategic interactions.

Third, while we concentrate on the effect of a single shock and trace its impulse response, we also characterize the unconditional variance of output if monetary shocks are i.i.d, an experiment similar to the one in the classic article by Caplin and Leahy (1997). We show that in this case the unconditional variance of output is an increasing function of the strength of strategic complementarity. We also note that, while most of our analysis focuses on a small monetary shock, our results are derived to characterize the effect on aggregate output and prices after *any* small perturbation of the initial distribution. For instance, we can use our analysis to study the impulse response to a shock to the average markup or to the idiosyncratic volatility, or in general to any permanent perturbation which affects the economy's steady-state distribution.

Fourth, we show that for the models in the Calvo-plus class the strategic complementarities amplify the Cumulative Impulse Response (CIR) by a measure that is roughly the same for all models within this class. For instance, the Calvo model and the Golosov-Lucas model display a comparable amplification, in spite of the fact that the level of the CIR in these models differs by a factor of 6.

Related Literature. Our model shares with the classic article by Caplin and Leahy (1997) that we feature fixed cost of adjustment and that the firm's objective function (a quadratic form) depends on both its own markup as well as on the average markup. One difference is that their framework does not feature idiosyncratic shocks, while ours does.[7] Caplin and Leahy (1997) study an equilibrium where the aggregate nominal shocks follow a driftless brownian motion, while we mostly focus on an impulse response after a once and for all shock, which makes it easier to connect to e.g. the VAR evidence.

As mentioned, the work by Nakamura and Steinsson (2010) and Klenow and Willis (2016) is closely related to ours. The DSGE models in both papers consider an input-output structure, which makes the (sticky) price of other industries part of the cost of each industry (i.e. "macro strategic complementarities"). Both papers, as well as ours, consider a frictionless labor market, idiosyncratic shocks at the firm level, and menu cost paid by firms to adjust prices. Nakamura and Steinsson (2010) allows, as we do, for a random menu cost. Klenow and Willis (2016) allow, as we do, for a non-constant demand elasticity at the firm level, which yields what they call "micro-strategic complementarities". We show that, up to second order, the two types of complementarities are additive, so we capture both of them in a single parameter. Both papers use numerical techniques to characterize the effect of monetary shocks in aggregate output and prices while we provide analytic results.

Our analysis also relates to Wang and Werning (2020), who analyze the propagation of shocks in a sticky-price economy in the presence of strategic complementarities. They present an insightful analytic solution for a case where firms follow a time dependent rule a la Calvo. Some features of the underlying environment are similar: several forces creating

[7] An interesting feature of Caplin and Leahy (1991, 1997) is to produce a state-dependent reaction to monetary shocks, perhaps the only model where a clear notion of "overheating" due to monetary policy appears.

complementarities (variable demand elasticity, decreasing returns, non-zero Frisch elasticity) are fully summarized by a single parameter. Other modeling aspects are different: first they consider a dynamic oligopoly without idiosyncratic shocks, while we focus on oligopolistically competitive markets with idiosyncratic shocks, a feature that allows us to connect to the distribution of price changes in the data, as mentioned above. Second, the timing of adjustment is exogenous in their paper, while the firms in our setup optimally choose both the timing as well as the size of the price adjustments. The simplification of the exogenous-timing and no-idiosyncratic shocks allows them to connect with the standard Phillips-curve and to analyze the importance of strategic complementarities in the standard New Keynesian setup. Third, the strategic complementarities are global in our set up while they are local in theirs. This allows their work to relate to the concentration within an industry, a feature that our formulation cannot address.

The paper is organized as follows. The next section lays out the general equilibrium environment of the problem and the origins of strategic interactions. Section 3 sets up the dynamic equilibrium as a MFG. Section 4 studies a linearized version of the MFG and derives key results for the equilibrium analysis. Section 5 characterizes the dynamic equilibrium and discusses the economic implications of strategic interactions. Section 6 concludes and discusses future work.

2 General Equilibrium setup and Complementarities

This section presents an economy where households maximize the present value of lifetime utility and firms maximize profits subject to costly price adjustments. We show that non-negligible complementarities between the price setting strategies of firms can arise through two channels. First, consumers' preferences yield a demand system with a non-constant price elasticity, a phenomenon that the literature dubbed *micro-complementarities* as in Kimball (1995). Second, we consider a production structure that generates pricing complementarities through sticky intermediate goods, as in Klenow and Willis (2016); Nakamura and Steinsson (2010), referred to as *macro-complementarities.* We will show that the effects of both channels on the firm's pricing strategy can be approximated by a single parameter and that at a symmetric equilibrium the firm's problem can be approximated by a quadratic return function that depends on the own price and the aggregate price, as in the classic work of Caplin and Leahy (1997).

Households: We consider a continuum of households with time discount ρ and lifetime utility

$$\int_0^\infty e^{-\rho\, t} \left(U(\mathcal{C}(t)) - a\, L(t) + \log \frac{M(t)}{P(t)} \right) dt$$

where U denotes a CRRA utility function over the consumption composite $\mathcal{C}$, the labor supply is L, M is the money stock, P is the consumption deflator, and $a > 0$ is a parameter. The linearity of the labor supply and the log specification for real balances are convenient simplifications also used in Golosov and Lucas (2007) and Woodford (2009). We follow Kimball (1995) in modeling the consumption composite $\mathcal{C}$ using an implicit aggregator over a continuum of varieties k as follows

$$1 = \left(\int_0^1 \Upsilon \left(\frac{c_k(t)}{\mathcal{C}(t)}\, A_k(t) \right) dk \right)$$

where A_k denotes a preference shock for variety k, and $\Upsilon(1) = 1$, $\Upsilon' > 0$ and $\Upsilon'' < 0$. The Kimball aggregator defines $\mathcal{C}$ implicitly, featuring an elasticity of substitution that varies with the relative demand $c_k/\mathcal{C}$. The standard CES demand is obtained as a special case when Υ is a power function.

The representative household chooses c_k, money demand and labor supply to maximize lifetime utility subject to the budget constraint

$$M(0)+\int_0^{\infty}\mathcal{B}(t)\left[\tilde{\Pi}(t)+\tau(t)+(1+\tau_L)W(t)L(t)-R(t)M(t)-\int_0^1\tilde{p}_k(t)c_k(t)dk\right]dt=0$$

where $R(t)$ is the nominal interest rates, $\mathcal{B}(t)=\exp\left(-\int_0^t R(s)ds\right)$ the price of a nominal bond, $W(t)$ the nominal wage, $\tau(t)$ a lump sum nominal transfers, τ_L a constant labor subsidy, $\tilde{\Pi}(t)$ the aggregate (net) nominal profits of firms, and $\tilde{p}_k$ the price of each variety.

Firms. There is a continuum of firms indexed by $k\in[0,1]$, that use a labor input L_k and an intermediate-good input I_k to produce the final consumption good using the CRS technology (omit time index)

$$c_k+q_k=\left(\frac{L_k}{Z_k}\right)^{\alpha}I_k^{1-\alpha}$$

Note that final goods are used by consumers, c_k, and that they are also an input in the production of the intermediate good Q through the production function (the same Kimball aggregator) $1=\int_0^1\Upsilon\left(\frac{q_k}{Q}A_k\right)dk$. The total demand of intermediate goods is $Q=\int_0^1 I_k\,dk$. Labor productivity is $1/Z$ and we assume that $Z_k(t)=\exp\left(\sigma\mathcal{W}_k(t)\right)$ where $\mathcal{W}_k$ are standard BM's, independent across k, so that the log of Z_k follows a driftless Brownian motion with variance σ^2. Note that the aggregates Q and $\mathcal{C}$ have the same unit price, P, since they are produced with identical inputs and the same function Υ. Finally the labor supply by households, L is used to produce each of the k goods and to provide adjustment cost services ℓ, so $L=\int_0^1 L_k dk+\ell$.

The demand for final goods. The consumers' first order conditions yield the demand system, whose exact form depends on the function Υ. Given a total expenditure E (equal to aggregate consumption in equilibrium) the demand for variety k, evaluated at a symmetric equilibrium, is

$$c_k = \frac{1}{\Upsilon^{-1}(1)}\,\frac{E}{PA_k} D\left(\frac{p}{P}\right) \quad \text{where} \quad D\left(\frac{p}{P}\right) \equiv (\Upsilon')^{-1}\left(\frac{p}{P}\,\Upsilon'\left(\Upsilon^{-1}(1)\right)\right) \quad \text{where} \qquad p \equiv \tilde{p}/A\ .$$

The firm's profit function. Let the nominal wage W be the numeraire, and $\tilde{p}_k = pA_k$ be the firm's price. Notice that the firm's marginal (and average) cost is $(Z_k W)^{\alpha} P^{1-\alpha}$ where P is the price of intermediate inputs. We can write the firm's (nominal) profit as $c_k \cdot (pA_k - (Z_k W)^{\alpha} P^{1-\alpha})$. If we assume that $Z_k^{\alpha} = A_k$, i.e. that preference shocks are proportional to marginal cost shocks, then we have that each firm maximizes $\Pi(\hat{p}, P) = c_k A_k\, W\left(\frac{p}{W} - \left(\frac{P}{W}\right)^{1-\alpha}\right)$ so the profits of the individual firm do not depend on Z_k since $c_k A_k = \frac{E}{\Upsilon^{-1}(1)\,P} D\left(\frac{p}{P}\right)$. The notation emphasizes that the firm's decision depends on both the own price, p, and the aggregate price P, and that prices are homogenous in W.

Let us write the profit in terms of the demand $D(p/P)$ and the cost function $\chi = \chi(P)$ giving the marginal cost. We have $\Pi(p, P)/W = D(p/P)\,(p - \chi(P))$. The first order condition for optimality gives:

$$p^*(P) = \frac{\eta(p/P)}{\eta(p/P) - 1}\,\chi(P) \text{ where } \ \eta(p/P) \equiv -\frac{p}{D(p/P)}\frac{\partial D(p/P)}{\partial p}$$

so η is the elasticity of the demand D with respect to the own price p.

We have the following result:

Proposition 1. *Consider a value for P such that $p^*(\bar{P}) = \bar{P}$. Assume that D is decreasing and that $\Pi(p, P)$ is strictly concave at $(p^*(\bar{P}), \bar{P}) = (\bar{P}, \bar{P})$. We have*

$$\frac{\bar{P}}{p^*(\bar{P})}\frac{\partial p^*(\bar{P})}{\partial P} = \frac{1}{1 + \frac{\eta'(1)}{\eta(1)(\eta(1)-1)}}\left[\underbrace{\frac{\eta'(1)}{\eta(1)(\eta(1)-1)}}_{\text{micro elasticity}} + \underbrace{\frac{P}{\chi(P)}\frac{\partial \chi(P)}{\partial P}}_{\text{macro elasticity}}\right] \qquad (1)$$

where $\eta(1) > 1$ and $1 + \frac{\eta'(1)}{\eta(1)(\eta(1)-1)} > 0$. Moreover:

$$\frac{\Pi(p,P)}{\Pi(\bar{P},\bar{P})} = 1 - \frac{1}{2} B \left(\frac{p-\bar{P}}{\bar{P}} + \theta \frac{P-\bar{P}}{\bar{P}} \right)^2 + \iota(P) + o\left(\left\| \frac{p-\bar{P}}{\bar{P}}, \frac{P-\bar{P}}{\bar{P}} \right\|^2 \right) \tag{2}$$

where $\iota(\cdot)$ is a function that does not depend on p, and where:

$$B \equiv -\frac{\Pi_{11}(\bar{P},\bar{P})}{\Pi(\bar{P},\bar{P})} \bar{P}^2 = [\, \eta'(1) + \eta(1)(\eta(1)-1) \,] > 0 \quad \text{and} \quad \theta \equiv \frac{\Pi_{12}(\bar{P},\bar{P})}{\Pi_{11}(\bar{P},\bar{P})} = -\frac{\bar{P}}{p^*(\bar{P})} \frac{\partial p^*(\bar{P})}{\partial P} \Big|_{p^*=\bar{P}} .$$

A few remarks are in order. First, equation (2) shows that the profit maximization problem of the firm can be approximated as the minimization of the quadratic period return $B(x - \theta X)^2$ where $x = \frac{p-\bar{P}}{\bar{P}}$ and $X = \frac{P-\bar{P}}{\bar{P}}$ denote the percent deviation from the symmetric equilibrium of the own and the aggregate price, respectively.

Second, as announced above, the extent of strategic interactions between the own price and the other firms' prices is captured by a single parameter, θ. Notice that static profits are maximized by setting $x = -\theta X$. The parameters θ measures the presence of strategic interactions. The firm's strategy exhibits strategic complementarity if $\theta < 0$, and it exhibits strategic substitutability if $\theta > 0$. Clearly, if $\theta \neq -1$ the only static equilibrium is $X = 0$.

Third, in the absence of macro complementarity, e.g. if $\frac{\partial \chi}{\partial P} = 0$, we have $\theta = -\frac{\eta'}{\eta(\eta-1)+\eta'}$ so that $\theta < 0$ occurs if $\eta'(p/P) > 0$. This condition has a clear economic explanation: if $\eta' > 0$ a higher P lowers the demand elasticity, which induces the firm to raise its markup. Thus $\eta' > 0$ implies that the own price and the aggregate price are strategic complements. Note moreover that if $\frac{\partial \chi}{\partial P} = 0$ the strength of strategic complementarities is bounded, since $-\theta < 1$. Instead, as $\frac{\partial \chi}{\partial P} > 0$, the size of strategic complementarities can be $-\theta > 1$, a case that will be of interest in the discussion of the existence of the solution.

Finally we note that for small shocks we don't need to consider any other equilibrium effects, beyond the path of $X(t)$, in the objective function of the firm. In particular, in the set up described above, one can show that the path of nominal wages and nominal interest rates are only functions of the path of money supply. Moreover, while there are other general equilibrium effects, such as changes on real interest rates, etc, they are higher than second order. More rigorously, Proposition 7 in Alvarez and Lippi (2014) can be adapted to show the validity of the second order approximation to the set up of this paper.

Impulse response of Output to a monetary shock. Note that an increase in the common component of cost for all firms reduces the average deviation of markups from its optimal value, i.e. it lowers X. One of the most interesting objects of the solution of the MFG interpreted as a price setting problem is the path of $X(t)$ after a small displacement from the steady state, i.e. we use an initial condition $m_0(x)$ with $m_0(x) = \tilde{m}(x + \delta)$, where $\tilde{m}$ is the stationary density. The value of $X(t)$ is inversely proportional to the deviation from steady-state output t periods after the monetary shock δ, or the impulse response function for output. We will actually consider a general perturbation $m_0(x) = \tilde{m}(x) + \delta\nu(x)$.

3 General Equilibrium model of Price Setting as a Mean Field Game

We describe the problem of a firm whose value function u depends on the state x and time t. The one dimensional state x represents a deviation from an ideal price, which when uncontrolled follows a Brownian motion with variance per unit of time σ^2 and no drift. We let X denote the cross sectional average of x. The firm seeks to minimize the discounted value of the sum of flow cost F and fixed cost of adjustment ψ, where $\rho \geq 0$ is the discount rate. Additionally, with a Poisson probability rate $\zeta > 0$ the firm firm can change its price without paying any cost. The flow cost of the firm discussed in Proposition 1 is

$$F(x, X) = B\,(x + \theta X)^2 \quad \text{with } B > 0 \,.$$

We consider the problem of a firm that takes as given a path for $\{X(t)\}$ for $t \in [0, T)$, and a terminal value function $u_T(x)$. We study the cases when T is finite, and also the limit as $T \to \infty$. The optimal decision rule of the firm at each time t consists on dividing the state space in a region where control is not exercised, the inaction region, and a complementary region where control is exercised and the state is reset by an impulse. Three time paths describe the decision rule: $\underline{x}(t), \bar{x}(t)$ and $x^*(t)$ for $t \in [0, T)$. At a given time t the optimal rule is represented by the interval $[\underline{x}(t), \bar{x}(t)]$ so that if $x(t)$ is in this interval the firm does *not* exercise control, i.e. inaction is optimal, but if $x(t) \notin (\underline{x}(t), \bar{x}(t))$ the firm exercises control, and immediately changes its price from $x(t^-)$ to $x(t^+) = x^*(t)$. The firm will also reset its price so that $x(t^+) = x^*(t)$, if t is a time where a free adjustment opportunity occurs. We refer to $\underline{x}(t)$ and $\bar{x}(t)$ as the boundaries of the range of inaction, to $x^*(t)$ as the optimal return point. The value function of the firm $u(x, t)$ solves the Hamilton-Jacobi-Bellman (HJB) equation (3), with appropriate boundary conditions, given in (6) - (8). Because of the time dependence of $X(t)$ the value function u must depend on time.

Likewise, given time paths for the decision rules, $\underline{x}, \bar{x}$ and x^*, and the initial condition for the cross sectional distribution $m_0(x)$, one can derive the evolution of the cross sectional distribution $m(x, t)$, which satisfies the Kolmogorov forward equation (KFE) in (4), with appropriate boundary conditions given in (9) - (11). Because of the time varying decision rules (as well as the initial condition), the cross sectional density must depend on time. The evolution of u and m solve a fixed point problem, requiring that the average value $X(t) = \int_{\underline{x}}^{\bar{x}} x m(x, t) dx$. We summarize these conditions below.

A Mean Field Game (MFG), given initial and terminal conditions m_0, u_T, is given by functions u, m, mapping $\mathbb{R} \times [0, T]$ to $\mathbb{R}$, and functions $\underline{x}, \bar{x}, x^*, X$ mapping $[0, T]$ to $\mathbb{R}$. The equilibrium of the MFG is given by the solution of the coupled system of partial differential equations: the HJB for the firm's value function u, and the KFE for the cross sectional density m. For all $t \in [0,T]$ and for all $x \in [\underline{x}(t), \bar{x}(t)]$ these equations are

$$0 = u_t\left(x,t\right) - \rho u(x,t) + \frac{\sigma^2}{2} u_{xx}(x,t) + F(x, X(t)) + \zeta \left[u(x^*(t),t) - u(x,t)\right] \tag{3}$$

$$0 = -m_t\left(x,t\right) + \frac{\sigma^2}{2} m_{xx}(x,t) - \zeta m(x,t) \quad \text{ and } x \neq x^*(t) \tag{4}$$

where, for all $t \in [0,T]$

$$X(t) = \int_{\underline{x}(t)}^{\bar{x}(t)} x\, m(x,t) dx \quad \text{and} \quad x^*(t) = \arg\min_x u(x,t) \tag{5}$$

Additionally we have the boundary and terminal conditions for u are:

$$u_x\left(\bar{x}(t),t\right) = u_x\left(\underline{x}(t),t\right) = u_x\left(x^*(t),t\right) = 0 \text{ for all } t \in [0,T] \tag{6}$$

$$u\left(\bar{x}(t),t\right) = u\left(\underline{x}(t),t\right) = u\left(x^*(t),t\right) + \psi \text{ for all } t \in [0,T] \tag{7}$$

$$u(x,T) = u_T(x) \text{ for all } x \tag{8}$$

The boundary and initial conditions for m are

$$0 = m\left(\bar{x}(t),t\right) = m\left(\underline{x}(t),t\right) \text{ for all } t \in [0,T] \tag{9}$$

$$1 = \int_{\underline{x}(t)}^{\bar{x}(t)} m(x,t) dx \text{ for all } t \in [0,T] \tag{10}$$

$$m(x,0) = m_0(x) \text{ for all } x \tag{11}$$

We now comment on the assumptions used above. First, the boundary conditions for the HJB in equation (6) are typically referred to as "smooth pasting and "optimal return point", and the ones in equation (7) are referred to as "value matching". They follow from optimality and are a consequence of our assumption that for each t the value function $u(\cdot,t)$ is once differentiable for all x, and twice differentiable in the range of inaction. In particular, for any x outside the range of inaction, the value function must satisfy $u(x,t) = u(x^*(t),t) + \psi$. See

Appendix H for the variational inequalities of the general case without smoothness.

Second, we will assume throughout that the inaction region is connected, i.e. given by a single interval, namely $[\underline{x}(t), \bar{x}(t)]$. In principle, the inaction region could be a union of such intervals.[8]

Third, under the assumption that the range of inaction is given by a single interval, then there is zero density outside of the inaction, so $m(x,t) = 0$ for all $x \notin [\underline{x}(t), \bar{x}(t)]$. Then, assuming continuity of $m(\cdot, t)$ for all x we obtain the boundary condition in equation (9). This is the condition to be expected at the boundaries of the range of inaction, since no mass can accumulate at these "exit" points. Likewise, the Kolmogorov forward equation should not hold at $x = x^*(t)$ since this is an "entry" point, i.e. a point where the flux of density that exits from $x = \underline{x}(t)$ and $\bar{x}(t)$ is entering. The integral condition in equation (10) states that for every t, the function $m(\cdot, t)$ is a density and hence integrates to one, i.e mass is preserved. Finally we require that $m(x,t) \geq 0$ for all x, t. See Bertucci (2020) for a rigorous derivation of the boundary conditions in a related problem.

Fourth, recall that in the static pricing game of Section 3 the condition $\theta < 0$ corresponds to the case of strategic complementarities, and $\theta > 0$ to the case of strategic substitutability. We are particularly interested in $\theta < 0$ but we will cover both cases. The standard case treated in the MFG literature considers $\theta > 0$, which corresponds to "monotonicity" condition that is at the center of the argument for uniqueness.[9]

No mass points. We have written the evolution of the cross sectional distribution under the assumption that it has no mass point for all $t \geq 0$. This will follow if the initial distri-

[8] For the stationary problem, one can show that this is not the case, but in the MFG the argument is more involved. This is a moot point when we analyze the perturbation, since we explore variations of the problem nearby the stationary solution.

[9] In term of the notions used for MFGs, letting m_i be an arbitrary measure and $X_i \equiv \int x dm_i$, the definition of monotonicity applied to the period return $F(x,X) = B(x+\theta X)^2$ is that for any two $m_1 \neq m_2$ must satisfy the following inequality

$$0 < \int \left(B(x+\theta X_1)^2 - B(x+\theta X_2)^2\right)(dm_1(x) - dm_2(x)) = 2B\theta(X_1 - X_2)^2$$

Hence, the monotonicity condition in MFGs corresponds to $\theta > 0$, or strategic substitutability.

bution m_0 has no mass points, and if the equilibrium decision rules are such the distribution $m(\cdot, t)$ will not have mass points for all $t \geq 0$. These conditions will be satisfied given the perturbation method we will use.

Steady State: Initial and Terminal Conditions. We describe the stationary version of the MFG. Let $\bar{x}_{ss}$, $\underline{x}_{ss}$ and x^*_{ss} be three time-invariant thresholds, and let $\tilde{u}$ and $\tilde{m}$ be two time-invariant functions with domain in $[\underline{x}_{ss}, \bar{x}_{ss}]$ solving:

$$0 = -\rho \tilde{u}(x) + \frac{\sigma^2}{2} \tilde{u}_{xx}(x) + F(x, X_{ss}) + \zeta \left(\tilde{u}(x^*_{ss}) - \tilde{u}(x) \right) \text{ for all } x \in [\underline{x}_{ss}, \bar{x}_{ss}] \tag{12}$$

$$0 = \frac{\sigma^2}{2} \tilde{m}_{xx}(x) - \zeta \tilde{m}(x) \text{ for all } x \in [\underline{x}_{ss}, \bar{x}_{ss}], x \neq x^*_{ss} \tag{13}$$

where $X_{ss} = \int_{\underline{x}_{ss}}^{\bar{x}_{ss}} x \, \tilde{m}(x) dx$, with boundary conditions: $\tilde{u}_x \left(\bar{x}_{ss} \right) = \tilde{u}_x \left(\underline{x}_{ss} \right) = \tilde{u}_x \left(x^*_{ss} \right) = 0$, $\tilde{u} \left(\bar{x}_{ss} \right) = \tilde{u} \left(\underline{x}_{ss} \right) = \tilde{u} \left(x^*_{ss} \right) + \psi$, and $0 = \tilde{m}(\underline{x}_{ss}) = \tilde{m}(\bar{x}_{ss})$.

When $\zeta > 0$ we have the symmetric stationary distribution $\tilde{m}$ given by

$$\tilde{m}(x) = \frac{\ell}{2} \, \frac{e^{\ell(2\bar{x}_{ss} - x)} - e^{\ell x}}{\left(1 - e^{\ell \bar{x}_{ss}}\right)^2} \quad \text{for } x \in [0, \bar{x}_{ss}] \tag{14}$$

where $\tilde{m}(x) = \tilde{m}(-x)$ for $x \in [-\bar{x}_{ss}, 0]$, and $\ell \equiv \sqrt{\frac{2\zeta}{\sigma^2}}$.

In our model where $F(x, X) = B(x + \theta X)^2$ we have that $X_{ss} = x^*_{ss} = 0$ and $\bar{x}_{ss} = -\underline{x}_{ss}$. Note that the steady state is independent of the value of θ. In this case the solution for $\tilde{u}$ can be obtained, up to an implicit equation in $(\rho + \zeta)/\sigma^2$, a feature that we explore in Lemma 7.

Next we state a proposition on the uniqueness of the stationary state.

Proposition 2. If $\theta \neq -1$, then $X_{ss} = 0$ is the only stationary state and it is independent of θ. If $\theta = -1$ then any X_{ss} is a steady state.

Notice that this result is reminiscent of the trivial result for the static game described above. Nevertheless the result in Proposition 2 is non trivial given that the firm problem has genuine dynamics and features adjustment costs.

4 Equilibrium of the MFG for a small perturbation

In this section we develop results to analyze the dynamic response to a monetary shock in the presence of strategic interactions. As standard in the analysis of dynamic equilibrium models we analyze the effect of a shock by solving an equilibrium starting with an initial condition different from the steady state, what is sometimes referred to as an "MIT shock". In our case the state is given by an infinite dimensional object, i.e. a cross sectional distribution. Moreover, to preserve analytic clarity and tractability, we analyze the equilibrium that follows a *small* perturbation of the economy around the steady state.

The section is organized in three main parts. In Section 4.1 we linearize the HJB equation for the firm's problem and solve it analytically. In Section 4.2 we linearize the KFE for the dynamics of the cross sectional distribution and solve it analytically. In Section 4.3 we derive the fixed point implied by the HJB and the KFE equations and provide a characterization of the resulting kernel that will be central in the analysis of the equilibrium.

Terminal and Initial conditions for MFG. We use the stationary solution to define the initial density m_0 and the terminal value function u_T. For the initial condition we consider a perturbation ν of the stationary density $\tilde{m}$, where we use the parameter δ to index the size of the perturbation, so:

$$m_0(x) = \tilde{m}(x) + \nu(x)\delta, \text{ where} \int_{\underline{x}_{ss}}^{\bar{x}_{ss}} \nu(x)dx = 0, \quad \text{for all } x \in [\underline{x}_{ss}, \bar{x}_{ss}]. \tag{15}$$

In particular, we are interested in an initial condition that corresponds to the effect of an unanticipated aggregate nominal shock δ, where δ is small. The interpretation of this initial condition is that, after the monetary shock δ, the nominal cost jumps immediately by this amount, and hence the value of the state x for each firms jumps from x to $x - \delta$, so that the density before any decision is taken is $m_0(x) = \tilde{m}(x + \delta)$.

For the terminal condition we set:

$$u_T(x) = \tilde{u}(x) \text{ for all } x \in [\underline{x}_{ss}, \bar{x}_{ss}] \text{ and } u_T(x) = \tilde{u}(x^*_{ss}) + \psi \text{ for all } x \notin [\underline{x}_{ss}, \bar{x}_{ss}]$$

so that at time $t = T$ the continuation corresponds to the steady state value function. The interpretation of the terminal condition $u_T(x) = \tilde{u}(t)$ is that the problem of the firm can be regarded as an infinite horizon problem. In this case T measures the horizon over which the strategic interactions apply.

Cases for T and ρ. We will consider the following combinations for T and ρ: (i) $\rho > 0$ and $T < \infty$, (ii) $\rho > 0$ and $T \to \infty$, and (iii) $\rho = 0$ and $T < \infty$, in which case we mean the limit as $\rho \downarrow 0$ and $T < \infty$.

Normalization. To simplify the exposition we normalize the parameters of the problem so that at steady state $\bar{x}_{ss} = 1$. In particular, given $\{\sigma^2, B, \rho, \zeta\}$ we set the fixed cost ψ so that $\bar{x}_{ss} = 1$. This amounts to measure the shock δ in units of standard deviation of steady state price changes, i.e. in units of $\sqrt{Var(\Delta p)}$. Moreover we also define

$$k \equiv \frac{\sigma^2}{2} \quad , \quad \eta \equiv \sqrt{\frac{\rho + \zeta}{k}} \quad , \quad \ell \equiv \sqrt{\frac{\zeta}{k}}$$

For future reference, the average number of price changes in steady state is given by

$$N = \zeta \left(\frac{\cosh(\ell)}{\cosh(\ell) - 1} \right) \text{ for } \zeta > 0 \text{ and } N = \sigma^2 = 2k \text{ for } \zeta = 0.$$

The benchmark initial condition. In general $m_0 : [-1, 1] \to \mathbb{R}$ given by equation (15) for some $\nu(x)$. In most of the analysis we focus on $\nu(x) = \tilde{m}_x(x)$. Direct computation on equation (14) gives

$$\tilde{m}_x(x) = \begin{cases} -\frac{\ell^2}{2}\,\frac{e^{\ell(2-x)}+e^{\ell x}}{\left(1-e^{\ell}\right)^2} & \text{for } \ell > 0 \text{ and } x \in (0,1] \\ -1 & \text{for } \ell = 0 \text{ and } x \in (0,1] \end{cases} \tag{16}$$

where for $x \in [-1, 0)$ we use that $\tilde{m}_x$ is antisymmetric i.e. $\tilde{m}_x(x) = -\tilde{m}_x(-x)$.

Equilibrium for symmetric initial conditions. Next we show that if the initial distribution m_0 is symmetric, i.e. if $m_0(x) = m_0(-x)$, then the equilibrium cross-section average has no dynamics $X(t) = X_{ss} = 0$, i.e. a flat impulse response. This result is important because it will allow us to ignore the symmetric component of the initial perturbation $\nu(x)$, and to focus on the antisymmetric part in Proposition 5. We have:

Proposition 3. Let $m_0(x)$ be a symmetric distribution with support on $[-1, 1]$, i.e. $m_0(x) = m_0(-x)$ and $\int_{-1}^{1} m_0(x)dx = 1$. Then there exists an equilibrium with $X(t) = X_{ss} = 0$, $\bar{x}(t) = \bar{x}_{ss} = 1$, $\underline{x}(t) = \underline{x}_{ss}$, and $x^*(t) = x^*_{ss} = 0$ for all $t \in [0, T]$ and where $m(x, t)$ is symmetric in x for all $t \in [0, T]$. This equilibrium is unique in the class of symmetric m.

A few comments are in order. First, while $X(t) = X_{ss} = 0$, the distribution $m(\cdot, t)$ evolves through time. Second, the proposition establishes uniqueness of the equilibrium only among those in which m is symmetric. We return to uniqueness when we consider a perturbation. Third, a symmetric displacement can be generated e.g. by shocking once and for all the variance of the fundamental shocks σ^2, or the market power of firms e.g. B. Fourth, we can relax the condition that the support is the same, at the cost of a slightly more involved proof.

4.1 Linearization and Solution of the HJB equation

This section derives a linearization of the HJB. To do this we linearize the MFG defined above. We consider an equilibrium with $\{\bar{x}(t,\delta), \underline{x}(t,\delta), x^*(t,\delta), X(t,\delta), u(x,t,\delta), m(x,t,\delta)\}$, where δ indexes the perturbation of the initial condition for a given ν. We differentiate all the equilibrium objects with respect to δ and evaluate them at $\delta = 0$. For all $t \in [0, T]$ we denote these derivatives as follows:

$$
\begin{aligned}
v(x,t) &\equiv \frac{\partial}{\partial \delta} u(x,t,\delta)|_{\delta=0} \text{ for all } x \in [-1,1] \\
n(x,t) &\equiv \frac{\partial}{\partial \delta} m(x,t,\delta)|_{\delta=0} \text{ for all } x \in [-1,1], x \neq 0 \\
\bar{z}(t) &\equiv \frac{\partial}{\partial \delta} \bar{x}(t,\delta)|_{\delta=0} \,,\ \underline{z}(t) \equiv \frac{\partial}{\partial \delta} \underline{x}(t,\delta)|_{\delta=0} \,,\ z^*(t) \equiv \frac{\partial}{\partial \delta} x^*(t,\delta)|_{\delta=0} \text{ and} \\
Z(t) &\equiv \frac{\partial}{\partial \delta} X(t,\delta)|_{\delta=0}
\end{aligned}
$$

In this subsection we study the evolution of the (derivative) of the value function, $v(x,t)$, as function of the path of the average price gap $\{Z(t)\}$. To do so we first obtain the pde and boundary conditions that $v(\cdot, t)$ satisfies. We then look for an explicit solution of $v(\cdot, t)$, which we use to compute the thresholds $\{\underline{z}(t), z^*(t), \bar{z}(t)\}$ as a function of the path of $\{Z(t)\}$.

Linearization of the HJB and its boundary conditions. We differentiate the HJB equation (3) for $u(x,t,\delta)$ with respect to δ at each (x,t) and use the boundary conditions to obtain

$$
0 = -(\rho + \zeta) v(x,t) + v_t(x,t) + k v_{xx}(x,t) + 2B\theta x Z(t) \text{ in } x \in [-1,1], t \in (0,T) \tag{17}
$$

Furthermore, differentiating the two value matching boundary conditions for $u(\bar{x}(t,\delta), t, \delta) = \psi + u(x^*(t,\delta), t, \delta)$ and $u(\underline{x}(t,\delta), t, \delta) = \psi + u(x^*(t,\delta), t, \delta)$ with respect to δ for each t and evaluating them at $\delta = 0$ we get for all $t \in (0, T)$:

$$
v(-1,t) + \tilde{u}_x(-1)\underline{z}(t) = v(0,t) + \tilde{u}_x(0) z^*(t) \ ,\ v(1,t) + \tilde{u}_x(1)\bar{z}(t) = v(0,t) + \tilde{u}_x(0) z^*(t) \tag{18}
$$

where we use the steady state value function $\tilde{u}(x)$.

We also use the boundary condition at $t = T$, which imposes we go to steady state, or more generally to a function independent of δ, gives:

$$0 = v(x,T) \text{ all } x \in [-1,1] \tag{19}$$

Solution of the HJB equation. We prove two intermediate results before characterizing the optimal thresholds.

Lemma 1. The function $v(x,t)$ is antisymmetric in x for each t, i.e. $v(x,t) = -v(-x,t)$ for all $x \in [-1,1]$ and $t \in [0,T]$, and hence it satisfies the boundary condition:

$$0 = v(-1,t) = v(1,t) = v(0,t) \text{ all } t \in (0,T) \tag{20}$$

We can solve the p.d.e. for v given by equation (17) for all t, x, which is the heat equation with source $2B\theta x Z(t)$, with a zero space boundary at $t = T$, and with the boundary conditions implied by value matching. We summarize this in the following lemma.

Lemma 2. Given the source $Z(t)$ for all $t \in [0,T]$, then the unique solution of the heat equation (17) with the two Dirichlet boundary conditions and the condition at $x = 0$ in equation (20) for all $t \in [0,T]$, and with the terminal space condition $v(x,T) = 0$ for all $x \in [0,1]$ is:

$$v(x,t) = -4B\theta \int_t^T \sum_{j=1}^{\infty} e^{(\eta^2+(j\pi)^2)\,k(t-\tau)}\, Z(\tau) \frac{(-1)^j}{j\pi} \sin(j\pi x) d\tau \tag{21}$$

Given this lemma, the next proposition summarizes the nature of the optimal decision rules for a firm facing a path of future values for the cross sectional average price gap or markup:

Proposition 4. Taking as given a path $Z(t)$ for $t \in [0,T]$ the solution to the firm's problem implies the following path for its optimal thresholds $\{\underline{z}(t), z^*(t), \bar{z}(t)\}$:

$$\bar{z}(t) = \bar{T}(Z)(t) \equiv \theta \bar{A} \int_t^T \bar{H}(\tau - t) Z(\tau) d\tau \text{ for all } t \in [0, T) \tag{22}$$

$$z^*(t) = T^*(Z)(t) \equiv \theta A^* \int_t^T H^*(\tau - t) Z(\tau) d\tau \text{ for all } t \in [0, T) \tag{23}$$

where $\underline{z}(t) = \bar{z}(t)$ and where $\bar{H}$ and H^* are defined as:

$$\bar{H}(s) \equiv \sum_{j=1}^{\infty} e^{-(\eta^2 + (j\pi)^2)\, k\, s} \geq 0 \ , \ H^*(s) \equiv \sum_{j=1}^{\infty} e^{-(\eta^2 + (j\pi)^2)\, k\, s} (-1)^j \leq 0 \text{ for all } s > 0 \tag{24}$$

$$\bar{A} \equiv \frac{4B}{\tilde{u}_{xx}(1)} = k \, \frac{2\eta^2}{[1 - \eta \coth(\eta)]} < 0 \, , \text{ and } A^* \equiv \frac{4B}{\tilde{u}_{xx}(0)} = k \, \frac{2\eta^2}{[1 - \eta \operatorname{csch}(\eta)]} > 0 \tag{25}$$

The ratio $A^*/|\bar{A}|$ is strictly increasing in η, with $\frac{\eta^2}{[1-\eta \operatorname{csch}(\eta)]} \to 6, |\frac{\eta^2}{[1-\eta \coth(\eta)]}| \to 3$ as $\eta \to 0$.

A few comments are in order. First, the current value of the thresholds $z^*(t)$ and $\bar{z}(t)$, depends on future values of the average price gap $Z(\tau)$ with $\tau \in (t, T)$. In this sense, this mapping is forward-looking.

Second, the result that $\bar{z}(t) = \underline{z}(t)$ means that the width of the inaction region, but not its position, is constant through time. The economics of this result is that the width of the inaction region reflects the option value of waiting, that is mainly affected by σ^2, the curvature of the payoff function and the fixed costs. Since none of these objects is affected by the monetary shock, the width of the inaction region stays constant. While the width is constant, its position and the location of the optimal return point within it change through time.

Third, θ only appears multiplicatively in the expressions for z^* and $\bar{z}$, since neither $\bar{A}, A^*$ nor $\bar{H}, H^*$ depend on it. Thus, in the special case without strategic interactions, $\theta = 0$, the thresholds are kept at the steady state values, i.e. $z^* = \bar{z} = 0$.

Fourth, given the sign of the expressions above, if there is strategic complementarity ($\theta < 0$) a firm facing higher values of $Z(\tau)$ for $\tau \geq t$, sets a higher value of the optimal return $z^*(t)$, and a larger value of both the upper and lower thresholds of the inaction band,

$\bar{z}(t), \underline{z}(t)$. If $\theta > 0$ the result is the opposite. The strength of the result depends on θ as well as on $\eta = \sqrt{2(\rho + \zeta)/\sigma^2}$. Also, as expected, values of $Z(\tau)$ closer to t receive higher weight on the firm's decision for its optimal return point and width of the inaction band. The parameter η also enters into the expressions for $\bar{A}$ and A^*, which reflect how the curvature of the value function changes as η changes. The reason that $\tilde{u}_{xx}$ appears in the expressions is because we are perturbing the economy around the steady state. Equation (25) shows that the curvature of the steady state value function $\tilde{u}_{xx}$, characterized in Lemma 7, affects the speed of convergence.

4.2 Linearization and Solution of the KF Equation

In this subsection we study the evolution of $n(x,t)$ as function of the path of thresholds $\{\underline{z}(t), z^*(t), \bar{z}(t)\}$. To do so we first obtain the pde and boundary conditions that $n(\cdot,t)$ satisfies. We then look for an explicit solution of $n(\cdot,t)$, which we use to compute $Z(t)$ as a function of the path of thresholds $\{\underline{z}(t), z^*(t), \bar{z}(t)\}$.

Linearization of the KFE and its boundary conditions. We differentiate the KFE for $m(x,t,\delta)$ given in equation (4) with respect to δ at each (x,t) to obtain:

$$0 = -n_t(x,t) + k n_{xx}(x,t) - \zeta n(x,t) \text{ in } x \in [-1,1], t \in (0,T), x \neq 0 \tag{26}$$

Differentiating the boundary condition $m(\bar{x}(t,\delta),t,\delta) = 0$ in equation (9) with respect to δ for each t we get $0 = n(1,t) + \tilde{m}_x(1)\bar{z}(t)$. Likewise, differentiating the boundary condition $m(\underline{x}(t,\delta),t,\delta) = 0$ with respect to δ we get $0 = n(-1,t) + \tilde{m}_x(-1)\underline{z}(t)$. Then the boundary conditions are

$$n(1,t) = -\tilde{m}_x(1)\bar{z}(t) \quad \text{and} \quad n(-1,t) = -n(1,t) \text{ all } t \in (0,T) \tag{27}$$

where we used that $\bar{z}(t) = \underline{z}(t)$ from Proposition 4 and where the expression for $\tilde{m}_x(1)$ is given in equation (16). The reason why $\tilde{m}_x$ appears is because we are perturbing the economy around the steady state.

Differentiating the mass preservation equation (10) with respect to δ we obtain: $0 = \int_{-1}^{1} n(x,t)dx$ for all $t \in (0,T)$. Differentiating this equation with respect to time and using the KFE in equation (26) we have:

$$0 = n_x(1,t) - n_x(0^+,t) + n_x(0^-,t) - n_x(-1,t) \text{ all } t \in (0,T) \tag{28}$$

The initial condition for n comes from differentiating $m_0(x)$ with respect to δ, this gives

$$n(x,0) = \nu(x) \quad \text{for} \quad x \in (-1,1) \tag{29}$$

which in the benchmark case of the small monetary shock is $n(x,0) = \tilde{m}_x(x)$, whose expression is given by equation (16). Given n we can compute $Z(t)$ as:

$$Z(t) = \int_{-1}^{1} x\, n(x,t)dx \text{ all } t \in (0,T) \ . \tag{30}$$

Equilibrium of the perturbed Mean Field Game. The equilibrium of the MFG with initial condition given by the perturbation ν is described by functions $\{Z, \bar{z}, z^*, n\}$ that solve equations (22), (23), (26), (27), (28), (29) and (30).

Irrelevance of the symmetric component of the perturbation ν. Any perturbation ν can be written as the sum of a symmetric component and an antisymmetric component. Given the linearity of the system, the equilibrium for a given ν is obtained as the sum of the equilibrium that corresponds to each of the components. Next we argue that the equilibrium when ν is symmetric has the feature that $Z(t) = 0$ for all t. Because of this we will focus below on ν that are antisymmetric. We summarize this result next:

Proposition 5. Let $\nu(x)$ be symmetric around $x = 0$. Then there is an equilibrium for this initial condition with $Z(t) = 0$ for all $t \in [0, T]$. This equilibrium is unique in the class of symmetric $n(x, t)$.

Intuitively, a symmetric displacement of the steady state distribution has no effect on the mean of the distribution, Z. Give the symmetric law of motion for x, the mean remains at the steady state value. The proof of this proposition follows directly from Proposition 3 where we showed a related result for an equilibrium with an arbitrary symmetric initial condition, not just a perturbation. The perturbation can be obtained using $n(x, t) = (m(x, t) - \tilde{m}(x))/\delta$, including $\nu(x) = (m_0(x, t) - \tilde{m}(x))/\delta$.

Solution of the KFE equation for an antisymmetric ν. We will look for a solution of n that satisfies the p.d.e. given in equation (26), its boundary condition in equation (27), mass preservation as given by equation (28), and the initial condition for $n(\cdot, 0)$.

First, we define the right and left limits of $n(\cdot, t)$ as $a(t)$ and $b(t)$ respectively:

$$n(0^+, t) = b(t) \text{ all } t \geq 0 \quad \text{and} \quad n(0^-, t) = a(t) \text{ all } t \geq 0$$

Given the conditions for boundary conditions and the initial conditions it is natural to look for antisymmetric solutions. Indeed the next lemma shows that this has to be the case.

Lemma 3. If n satisfies the p.d.e. equation (26), the boundary conditions equation (27), the mass preservation equation (28), the initial conditions is antisymmetric, i.e. $\nu(x) = \nu(-x)$, and $a(t) + b(t)$ is continuous as a function of time on $(0, T]$, then $n(x, t)$ is antisymmetric in x for all t, and thus $a(t) = -b(t)$ for all $t \in [0, T]$.

Note that once that n is antisymmetric mass preservation holds automatically. Next we use the antisymmetric nature of n to find an expression for $b(t) - a(t)$ in terms of the threshold $z^*(t)$.

Lemma 4. Assume that $m(x^*(t, \delta), t, \delta)$ is continuous, and right and left differentiable at

$\delta = 0$. Then $z^*(t) = \frac{a(t)-b(t)}{2\,\tilde{m}_x(0^+)}$.

The antisymmetric nature of n, which implies that $a(t) = -b(t)$, and Lemma 4 have the important implication that:

$$b(t) = n(0^+, t) = -\tilde{m}_x(0^+)\, z^*(t) = -n(0^-, t) = -a(t) \text{ for all } t \geq 0$$

Next we are going to give a pde that $n(x,t)$ has to satisfy. The key simplification is that due to the antisymmetric nature of $n(x,t)$ it suffices to define it for $x \in (0,1]$, for every t. Moreover, being antisymmetric, the mass preservation is satisfied. Finally, the characterization in Lemma 4 gives us a boundary condition at $x = 0$ for all t. Hence the system given by equation (26), (27), (28) and (29) becomes the following system:

$$n_t(x,t) = k n_{xx}(x,t) - \zeta n(x,t) \qquad \text{for } x \in [0,1] \quad \text{and } t > 0 \tag{31}$$

$$n(1,t) = -\tilde{m}_x(1)\bar{z}(t) \text{ and } n(0,t) = -\tilde{m}_x(0^+)\, z^*(t) \quad \text{for all } t > 0 \tag{32}$$

$$n(x,0) = \nu(x) \quad \text{for } x \in [0,1] \tag{33}$$

The above system is well understood. It corresponds to a one dimensional heat equation with a bounded spatial domain, an initial spatial condition, and a specification of time varying values on the boundaries of the domain (see Chapter 6 in Cannon (1984)). The initial condition is given by ν and the time varying boundaries are given by z^* and $\bar{z}$. This equation has a unique solution that can be written in terms of these three functions. The solution is a linear functional of z^*, $\bar{z}$ and ν, it is algebraic intensive and explicit expressions are given in Lemma 8 in Appendix A. We use this explicit solution to write the impulse response of the mean $Z(t)$ for given path of the thresholds $\{\bar{z}(t), z^*(t)\}$, using the expression for $Z(t)$ in equation (30). We have:

Proposition 6. Taking as given the paths of $\{z^*(t), \bar{z}(t)\}$, and an initial condition given by an antisymmetric perturbation $\nu(x)$, the solution of the KFE gives the following path for the average value $\{Z(t)\}$:

$$Z(t) = T_Z(z^*, \bar{z})(t) \equiv Z_0^\nu(t) + 4k \int_0^t G^*(t-\tau) z^*(\tau) d\tau + 4k \int_0^t \bar{G}(t-\tau)\bar{z}(\tau) d\tau \tag{34}$$

for all $t \in [0, T]$ and where $\bar{G}$, G^* and Z_0^ν, are defined as

$$\bar{G}(s) \equiv -\tilde{m}_x(1) \sum_{j=1}^{\infty} e^{-(\ell^2 + (j\pi)^2)\, k\, s} > 0 \quad \text{and} \quad G^*(s) \equiv -\tilde{m}_x(0^+) \sum_{j=1}^{\infty} (-1)^{j+1} e^{-(\ell^2+(j\pi)^2) k\, s} > 0$$

for all $s \geq 0$, $\tilde{m}_x(1)$ and $\tilde{m}_x(0^+)$ are given in equation (16), and

$$Z_0^\nu(t) \equiv -4 \sum_{j=1}^{\infty} (-1)^j \frac{e^{-(\ell + (j\pi)^2)\, k\, t}}{j\pi} \int_0^1 \sin(j\pi x) \nu(x) dx\,.$$

This proposition gives the evolution of the average price gap or markup, $Z(t)$, as a function of the path of decisions up to time t, summarized by the boundaries of the inaction region and the optimal return point, i.e. $\{z^*(\tau), \bar{z}(\tau)\}$ for $0 \leq \tau \leq t$. The current value of the average markup $Z(t)$, depends on past values of the thresholds $z^*(\tau)$ and $\bar{z}(\tau)$ with $\tau \in (0, t)$. In this sense, the mapping is backward-looking.

A few comments are in order. First, the expression for $Z(t)$ is made of two parts: the first one, $Z_0^\nu(t)$, gives the dynamics of the average price gap due to the displacement ν of the initial distribution when the thresholds are constant, i.e. $\bar{z} = z^* = 0$. It corresponds to the impulse response of the average price gap in an economy where there are no strategic interactions, i.e. $\theta = 0$, and it is studied in detail in Alvarez and Lippi (2021). The other part, given by the two integrals, describes the effect on $Z(t)$ caused by past changes of the thresholds.

Second the mapping is monotone, in that larger values of past thresholds, lead to larger values of the average markup $Z(t)$, i.e. $G^*(s) > 0$ and $\bar{G}(s) > 0$ for all $s > 0$. Finally, notice that the values of the pairs $(z^*(\tau), \bar{z}(\tau))$ for τ close to t have a higher weight than those further away in time. Given our normalization, the mapping T_Z depends only on $k \equiv \sigma^2/2$ and ℓ.

Third, for the benchmark case of the initial condition for a monetary shock where $\nu = \tilde{m}_x$, as in equation (16), we have

$$Z_0^\nu(t) = 2\sum_{j=1}^{\infty} \frac{\ell^2}{\ell^2 + (j\pi)^2} \left(\frac{(-1)^j \left(1 + e^{2\ell}\right) - 2e^\ell}{\left(1 - e^\ell\right)^2} \right) e^{-(\ell^2 + (j\pi)^2)kt} \,. \tag{35}$$

For any value of ℓ the function $Z_0^\nu(0) = -1$ and $Z_0^\nu(t)$ is increasing in t and converges to zero as $t \to \infty$. For the special case when $\zeta = 0$, corresponding to the pure Ss problem, the expressions for the derivatives of $\tilde{m}$ simplify to $\tilde{m}_x(1) = \tilde{m}_x(0^+) = -1$ and we have

$$Z_0^\nu(t) = 4\sum_{j=1}^{\infty} \frac{[(-1)^j - 1]}{(j\pi)^2} e^{-(j\pi)^2\, k\, t} \quad .$$

4.3 Deriving the fixed point

In this section we put together the solution for the HBJ and KFE derived in Proposition 4 and in Proposition 6 respectively to arrive to a single linear equation that $\{Z(t)\}$ must solve. We denote the fixed point by $Z = \mathcal{T}(Z)$. The mapping $\mathcal{T}$ is the composition of T_Z with $\bar{T}$ and T^* described above, i.e. $\mathcal{T}(Z) = T_Z\left(T^*(Z), \bar{T}(Z)\right)$. Direct computation gives:

Proposition 7. Let ν be an arbitrary perturbation. The equilibrium of a MFG must solve $Z = \mathcal{T}(Z)$ given by:

$$Z(t) = \mathcal{T}(Z)(t) \equiv Z_0^\nu(t) + \theta \int_0^T K(t,s) Z(s) ds \text{ all } t \in [0,T] \tag{36}$$

where Z_0^ν is given by

$$Z_0^\nu(t) \equiv -2\sum_{j=1}^{\infty} (-1)^j \frac{e^{-(\ell + (j\pi)^2)\, k\, t}}{j\pi} \int_{-1}^{1} \sin(j\pi x)\nu(x) dx \,, \tag{37}$$

and where the kernel K is:

$$K(t,s) = \tag{38}$$
$$4 \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \left[\bar{A}_\ell - A_\ell^* \,(-1)^{j+i}\right] \frac{\left[e^{[(j\pi)^2+(i\pi)^2+\eta^2+\ell^2]k(t\wedge s)} - 1\right] e^{-(j\pi)^2 kt - \ell^2 kt - (i\pi)^2 ks - \eta^2 ks}}{(j\pi)^2 + (i\pi)^2 + \eta^2 + \ell^2}$$

with $\bar{A}_\ell \equiv -\tilde{m}_x(1)\,\bar{A}$ and $A_\ell^* \equiv -\tilde{m}_x(0^+)\,A^*$, where $\tilde{m}_x$ is given in equation (16) and $\bar{A}$ and A^* in equation (25).

Equation (36) is a non-homogeneous Fredholm integral equation of the second kind, where the parameter is given by θ. The path $\{Z_0^\nu\}$ is the solution of the MFG when there are no strategic interactions, i.e. when $\theta = 0$, and the perturbation is given by ν. In our benchmark case of a monetary shock $\nu = \tilde{m}_x$, and then Z_0^ν is given by equation (35). The kernel K, given in equation (38), is independent of θ as well as of the initial perturbation ν. This means that the effect of strategic interactions on the equilibrium path Z depends on θ only as a scalar multiplying the kernel K.[10]

We define three objects, related to the kernel, that will be used below. The first is a notion of inner product between vectors, which we apply to functions of time. For any two functions V, W, we define the inner product $\langle \cdot, \cdot \rangle$ using weights given by time discount as follows:

$$\langle V, W \rangle \equiv \frac{\rho}{1 - e^{-\rho T}} \int_0^T V(t) W(t) e^{-\rho t}\, dt \tag{39}$$

The second is a linear operator, $\mathcal{K}$, akin to a matrix multiplication:

$$(\mathcal{K})\,(V)(t) \equiv \int_0^T K(t,s) V(s) ds \text{ for all } t \in [0, T] \tag{40}$$

for any function $V : [0, T] \to \mathbb{R}$. The third is a bound on the kernel K. This comes in two types that are used for different analysis of the fixed point. One is a Lipschitz bound and

[10]Notice that since the kernel K arises from the composition of a backward and a forward operator, then $K(t, s)$ is different from zero for all t, s.

the other is a form of L_2 bound.

$$\text{Lip}_K \equiv \sup_{t\in[0,T]} \int_0^T |K(t,s)|ds \text{ and } ||K||_2^2 \equiv \frac{\rho^2}{(1-e^{-\rho T})^2} \int_0^T \int_0^T K^2(t,s)\, e^{-\rho(t+s)}\, dt\, ds \tag{41}$$

The next lemma gathers important properties of the kernel K that will be used to characterize the equilibrium. The lemma considers the case where $\zeta = 0$, which corresponds to the pure Ss model, as well as the case where $\ell^2 = \zeta/k > 0$, which typically regularizes the kernel.[11]

Lemma 5. Consider the Kernel in equation (38) and the inner product in equation (39).

1. K is symmetric if $\rho = 0$, i.e. $K(t,s) = K(s,t)$ for all (t,s). For $\rho \geq 0$, the operator $\mathcal{K}$ is self-adjoint, i.e. for any V, W we have $\langle \mathcal{K}V, W\rangle = \langle V, \mathcal{K}W\rangle$:

$$\int_0^T \int_0^T K(t,s)V(s)W(t)e^{-\rho t}\, ds\, dt = \int_0^T \int_0^T K(t,s)W(s)V(t)e^{-\rho t}\, ds\, dt\,,$$

2. All elements of K are negative, i.e. $K(t,s) < 0$ for all $(t,s) \in (0,T)^2$

3. K is negative semidefinite, $\langle \mathcal{K}V, V\rangle \leq 0$, i.e. $\int_0^T \int_0^T K(t,s)V(t)V(s)e^{-\rho t} dt ds \leq 0$,

4. If $\zeta/k = \ell^2 = 0$, then $\text{Lip}_K < \frac{\eta^2}{18}\left(\frac{1}{1-\eta\,\text{csch}(\eta)} - \frac{4}{1-\eta\coth(\eta)}\right)$. Moreover, for small ρ we have $\text{Lip}_K < 1 - \frac{7}{180}\eta^2 + o(\eta^2)$,

5. Let $K(t,s;\eta,\ell)$ be the kernel as a function of η, ℓ. Then $|K(t,s;\eta,\ell)| \leq |\tilde{m}_x(0^+)|\, |K(t,s;\eta,0)|$ for all $t,s \in [0,T]$.

6. If $\ell^2 = 0$, and $\rho \geq 0$, then $||K||_2^2 < c_0 \frac{\rho^2 T}{(1-e^{-\rho T})^2}\left(\frac{\eta^2}{[1-\eta\,\text{csch}(\eta)]} - \frac{\eta^2}{[1-\eta\coth(\eta)]}\right)$ for a constant $c_0 > 0$ independent of any other parameters.

7. If $\ell \geq 0$ and $\rho > 0$, then $||K||_2^2 < \rho\left[\frac{1-e^{-2\rho T}+6\rho}{(1-e^{-\rho T})^2}\right] c_1$ for a constant $c_1 > 0$ independent of ρ and T.

[11] We obtain a quite complete characterization even though the kernel $K(t,s)$ diverges to $-\infty$ for $t = s$ (see Appendix H.3 for the explicit calculation).

A few remarks are in order. The lemma establishes that the operator $\mathcal{K}$ is self adjoint (point 1). This property is key to the existence of an orthonormal basis for K and represent the impulse response using standard eigenvalue-eigenfunction projection methods. The negative-definiteness of K (point 2), implies that the eigenvalues are all negative. Second, the fact that K is negative for all t, s implies the monotonicity of the equilibrium for $\theta < 0$. Third, the lemma establishes bounds that allow us to study existence, uniqueness, and a characterization of the solution. The Lipschitz bound (points 4 and 5) is used to find values of θ for which the right hand side of equation (36) is a contraction in the case where T is unbounded. Likewise, the bound for the norm $||K||_2$ (points 6 and 7) is used to establish the compactness of the operator $\mathcal{K}$, which together with the self-adjointness of K, allows us to establish conditions for existence, uniqueness, and a characterization of the solution for the case where T is finite.

5 Equilibrium Characterization for the Monetary Shock

In this section we characterize the dynamic equilibrium. As initial condition we consider a perturbation ν to the stationary density, focusing on the monetary shock described in equation (16). We cover both the pure Ss model ($\zeta/k \equiv \ell^2 = 0$) as in Golosov and Lucas (2007)-Klenow and Willis (2016), as well as the Calvo-plus model ($\zeta/k \equiv \ell^2 > 0$) as in Nakamura and Steinsson (2008) and Alvarez, Le Bihan, and Lippi (2016). In these models output is negatively proportional to price gaps, so that letting $Y_\theta(t)$ the impulse response of output to a small monetary shock we have $Y_\theta(t) = -Z(t)$ where we index the impulse response by the parameter θ. Note that $Y_0(t) \equiv -Z_0^\nu(t)$ where $\nu(x) = -1$ for $x \in (0, 1]$ and $\nu(-x) = -\nu(x)$, which gives the interpretation of a monetary shock, as in equation (16). The impulse response function solves $Y_\theta = \mathcal{T} Y_\theta$ as follows:

$$Y_\theta(t) = (\mathcal{T} Y_\theta)(t) \equiv Y_0(t) + \theta \int_0^T K(t, s) Y_\theta(s) ds \text{ all } t \in [0, T] \tag{42}$$

Characterization. We study the existence and uniqueness of Y_θ, solving the integral equation (42), for different cases. In Section 5.1 we restrict $|\theta|$ to be bounded and allow T to be infinite provided that $\rho > 0$. In Section 5.2 we restrict $T < \infty$ and consider θ arbitrary and $\rho \geq 0$: the finite T allows us to use projection methods to solve for the equilibrium impulse response $Y_\theta(t)$ and obtain an explicit expression for it. Each of these cases provides different insights into the nature of the solution. A key result shows that the equilibrium exists, it is unique, and it is well posed, provided that the strength of strategic complementarity is smaller than some critical value (a bound on $|\theta|$). We also give a characterization of the impulse response as a function of θ, showing that the size of the response to a monetary shock at any given time t is bigger, the larger the strength of strategic complementarity (smaller θ). Moreover we show that larger strategic complementarity increase the variance of output due to monetary shocks. Finally we show that for sufficiently strong strategic complementarity the impulse response is hump shaped; we provide an expression for the impulse response, based on the eigenvalues and eigenfunction of K, that provides a straightforward method for numerical analysis. Our first simple result shows that all IRF start at the same point.

Proposition 8. Let Y_θ be the solution of equation (43). Then its value at $t = 0$ is the same as $Y_\theta(0) = Y_0(0) = 1$.

5.1 Equilibrium with bounded strategic interactions

In this section we analyze the case where the strength of the strategic interactions θ is bounded. For future reference we define the series

$$S_\theta(t) = \sum_{r=0}^{\infty} \theta^r \left(\mathcal{K}\right)^r \left(Y_0\right)(t) \text{ for all } t \in [0, T] \tag{43}$$

where $\mathcal{K}^r$ is the r^{th} iteration of $\mathcal{K}$ defined in equation (40), i.e.:

$$\left(\mathcal{K}\right)^{r+1} \left(V\right)(t) \equiv \int_0^T K(t,s) \left(\mathcal{K}\right)^r \left(V\right)(s) ds$$

The next proposition gives a characterization of the equilibrium for the case of strategic complementarity ($\theta < 0$) and for initial perturbations such that $Y_0(t) > 0$.

Proposition 9. Assume that $T < \infty$ if $\rho = 0$, but otherwise these parameters take arbitrary values. Let ν be any perturbation such that $Y_0(t) > 0$, and $||Y_0||_\infty < \infty$ and $Y_0(t)$ is continuous. Let $\theta \in (\underline{\theta}, 0]$, where $\underline{\theta}$ is such that the series S_θ in equation (43) converges. The unique solution of equation (42) has the following properties:

1. For each $t \in (0, T)$ the fixed point is positive, i.e. $Y_\theta(t) > 0$,
2. For each $t \in (0, T)$, the fixed point $Y_\theta(t)$ is (strictly) monotone decreasing in θ,
3. For each $t \in (0, T)$, the fixed point $Y_\theta(t)$ is (strictly) convex in θ.

The proof of this proposition is straightforward, using that $K \leq 0$ (Lemma 5), and thus for $\theta < 0$ we have that $\theta\mathcal{K}$ is monotone, it has a Lipschitz bound, and preserves the sign of Y_0. The positivity, and the monotonicity and convexity on θ whenever $\theta < 0$, follow since each term of the series for S_θ satisfies these properties. A few comments are in order. First, if $\nu = \tilde{m}_x(x)$, then Y_0 satisfies the hypothesis for Y_0 for the proposition, as can be seen in equation (35). Second, and most importantly, this proposition shows that *as the strategic complementarity gets larger (more negative θ), then the aggregate response to the shock t is larger at each horizon, i.e. $Y_\theta(t)$ is decreasing in θ*. This proposition shows that $Y_\theta(t)$ is a convex function of θ at each t. The monotonicity and convexity properties yield the following important corollary:

Corollary 1. The assumptions of Proposition 9 imply that there is a $0 > \underline{\theta} > -\infty$ such that $S_\theta(t) = +\infty$.

Thus, for sufficiently strong strategic complementarity the series S_θ does not converge. This, in itself, does not imply that there is no equilibrium. We return to this question in the next section, where we show that indeed for values of θ sufficiently large (in absolute value) the model is not well posed: it may fail to have an equilibrium or, even when it has one, the equilibrium may not change continuously as a function of the parameters.

The next proposition establishes a bound for $|\theta|$, in terms of the fundamental model parameters, that ensures existence and uniqueness. In particular, we use Lemma 5 to verify the conditions for the Banach contraction fixed point theorem. This establishes existence and uniqueness of the solution of equation (42) for a range of θ including both positive (strategic substitution) and negative values (strategic complementarity). Additionally, the proposition allows for any arbitrary initial perturbation ν.

Proposition 10. Assume that $T < \infty$ if $\rho = 0$, but otherwise these parameters take arbitrary values. Consider any perturbation ν. A sufficient condition for the existence and uniqueness of the equilibrium IRF, i.e. of the uniqueness and existence of a solution to equation (42) in $L^1([0,T])$ is that $|\theta| \operatorname{Lip}_K < 1$. In this case, $Y_\theta(t) = S_\theta(t)$ as in equation (43). A sufficient condition $|\theta| \operatorname{Lip}_K < 1$ is :

$$|\theta| \frac{\ell^2}{2} \frac{e^{2\ell}}{(1-e^{\ell})^2} \frac{\eta^2}{18} \left(\frac{1}{1-\eta \operatorname{csch}(\eta)} - \frac{4}{1-\eta \coth(\eta)} \right) < 1$$

For the special case of $\ell^2 \equiv \zeta/k = 0$ this gives $|\theta| \frac{\eta^2}{18} \left(\frac{1}{1-\eta \operatorname{csch}(\eta)} - \frac{4}{1-\eta \coth(\eta)} \right) < 1$.

The proof of this proposition is an immediate application of the contraction theorem. The modulus of the contraction is given by the $\theta \operatorname{Lip}_K$ bound that was characterized in part 4 of Lemma 5 for the $\zeta = 0$ case, and extended to the case of $\ell^2 = \zeta/k > 0$ in part 5. For the pure Ss case, i.e. when $\zeta = 0$, we can use the approximation for small ρ in 4 of Lemma 5 to obtain an expression for small η: $|\theta| \left(1 - \frac{7}{180}\eta^2\right) < 1$. Thus for practical purposes in the pure Ss case we can take the sufficient conditions for a contraction to be $|\theta| \leq 1$.[12]

While Proposition 9 was shown only for an interval of strictly negative values of θ, the

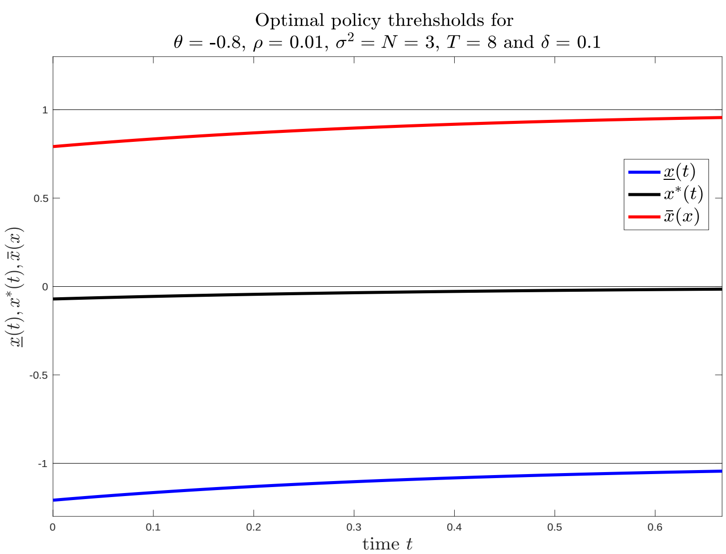
[12]For this case $2k = \sigma^2 = N\,Var(\Delta p) = N$, where N is the expected number of price changes per unit of time in steady state, and where we use the normalization $\bar{x}_{ss} = 1$ and the definition of k. Thus when $\eta^2 = \rho/k$ we can write the bound as $\frac{1}{|\theta|} > 1 - \frac{7}{90}\frac{\rho}{N}$.

same properties hold in a neighbourhood of $\theta = 0$. In particular

$$\frac{\partial}{\partial\theta} Y_\theta(t)|_{\theta=0} = (\mathcal{K})\,(Y_0)(t) < 0 \text{ and } \frac{\partial^2}{\partial\theta^2} Y_\theta(t)|_{\theta=0} = 2\,(\mathcal{K})^2\,(Y_0)(t) > 0$$

and thus the monotonicity and convexity hold also in an interval of positive values, so that the result extends (locally) to the case of strategic substitutability. This is shown by direct computation since by Proposition 10 the series in equation (43) converge uniformly. Indeed, numerically, we find all the properties in Proposition 9 hold for all positive values of θ.

Figure 1: Equilibrium path of thresholds

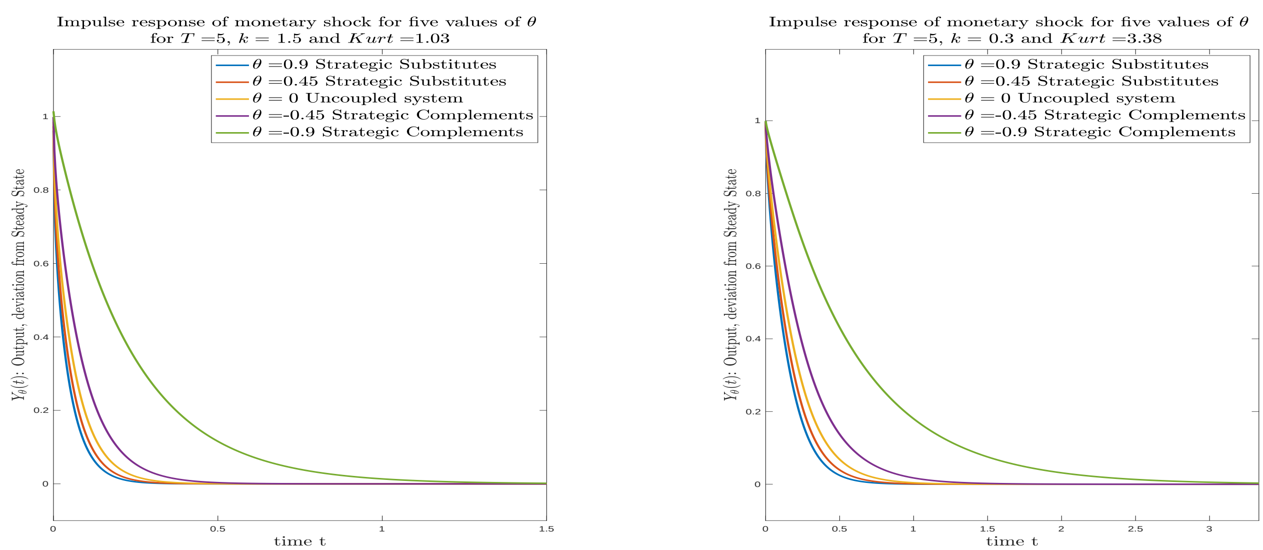


In Figure 1 we display the time path of the equilibrium thresholds $\bar{x}(t), x^*(t)$ and $\underline{x}(t)$ based on the linear approximation. The figure consider the case of $\delta = 0.05$ and $\theta = -0.8$. The black thin lines are the steady state values of the thresholds, and the color solid lines are the linear approximation to the equilibrium thresholds. The thresholds start just at the edge of the initial displaced distribution, m_0, and then evolve according to the equilibrium. As shown above, the paths for both boundaries of the range of inaction $\bar{x}(t)$ and $\underline{x}(t)$, as well as the path for the optimal return $x^*(t)$, deviate from their steady state values with the same sign, determined by θ. The fact that strategic complementarities lowers the thresholds is what makes the impulse response larger, since fewer firms increase prices and, when they do so, they return to a lower value of the price gap.

In the left panel of Figure 2 we display the IRF Y_θ for five values of θ and for $\ell^2 = \zeta/k = 0.01$, so it is essentially the pure Ss model. The figure illustrates Proposition 9: at each t it can be seen that $Y_\theta(t)$ decreases in θ, in a convex fashion. Also, since all IRFs start at the same value, i.e. $Y_\theta(0) = 1$, then for larger strategic complementarity the IRF has to be more protracted. The right panel displays the IRF for $\ell^2 = \zeta/k = 3$, i.e. for a version of the Calvo-plus model. Note that the time scale is different across the panels. As in the pure Ss case, the IRF are decreasing and convex in θ for each t. But comparing the two IRFs for the same θ across the two figures, it can be seen that the Calvo-plus model has a larger IRF than the one for the pure Ss model.

Figure 2: Impulse response of Monetary Shock

Golosov-Lucas: $\zeta/k = \ell^2 = 0.01$ Calvo-plus: $\zeta/k = \ell^2 = 5$



5.2 Equilibrium characterization with a finite T

In this section we focus on a finite horizon $T < \infty$ and analyze how the equilibria vary as a function of θ. A main result is to provide an expression for the IRF Y_θ in terms of the projections onto an orthonormal base, and the associated eigenvalues, implied by the kernel K. We begin by introducing a norm for linear operators, which is the analogue of the square of the trace of matrix:

$$||\mathcal{K}||^2_{HS} \equiv \sum_{i,j} |\langle \mathcal{K} f_i, f_j \rangle|^2 = \sum_{i,j} \left(\frac{\rho}{1 - e^{-\rho T}} \int_0^T \int_0^T K(t,s)\, f_i(s)\, f_j(t)\, e^{-\rho t}\, ds\, dt \right)^2 \tag{44}$$

where $\{f_j\}$ is any orthonormal base for the linear separable Hilbert space $\mathcal{H}$ of functions $V : [0,T] \to \mathbb{R}$ with $\langle V, V \rangle < \infty$. The next proposition, which uses the results of Lemma 5, gives the necessary preliminary results.

Proposition 11. Assume that $T < \infty$. The HS norm is bounded by $||\mathcal{K}||^2_{HS} \leq T^2 ||K||^2_2$. In this case the operator $\mathcal{K}$ is self-adjoint and compact, and thus it has countably many eigenvalues and eigenfunctions that we denote by $\{\mu_j, \phi_j\}_{j=1}^{\infty}$. The eigenvalues μ_j are real, negative, and ordered as $|\mu_1| > |\mu_2| > |\mu_3| \ldots$, and they converge to zero $|\mu_j| \to 0$ as $j \to \infty$. There are at most finitely many eigenfunctions associated with each non-zero eigenvalue. The eigenfunctions $\{\phi_j\}_{j=1}^{\infty}$ form an orthonormal base for $\mathcal{H}$.

The proposition is an instance of the spectral theorem for compact self-adjoint operators, a basic result in functional analysis, see section 5 of Chapter II in Conway (2007). That the operator is self-adjoint was shown in part 1 of Lemma 5. That the operator is compact follows from finite Hilbert-Schmidt norm, which as stated in equation (44) it is equal to the L^2 norm of the kernel found in part 7 of Lemma 5. That the eigenvalues are negative follows directly from part 3 of Lemma 5.

Our first result determines the values of θ for which the solution exists and is unique, and provides a partial characterization through an explicit solution written in terms of the eigenvalues and eigenfunctions of $\mathcal{K}$.

PROPOSITION 12. Assume that $T < \infty$. Then

1. For all $\theta < 1/\mu_1$, then there exists a unique equilibrium solving equation (42) given by

$$Y_\theta(t) = \sum_{j=1}^{\infty} \frac{\langle Y_0, \phi_j \rangle}{1 - \theta \mu_j} \phi_j(t) \quad \text{for all } t \in (0, T) \tag{45}$$

2. If $\theta \to +\infty$, then $Y_\theta(t) \to 0$ for all $t \in (0, T)$.

3. If $\theta = 1/\mu_1$, and ν is such that $Y_0 \geq 0$, then there is no solution to equation (42), i.e. there is no equilibrium.

4. Assume that ν is such that $Y_0 \geq 0$. Then, there is no equilibrium for that value of θ, and there is pole at $\theta = 1/\mu_1$, i.e. for all $t \in (0, T)$:

$$\lim_{\theta \downarrow 1/\mu_1} Y_\theta(t) = +\infty \text{ and } \lim_{\theta \uparrow 1/\mu_1} Y_\theta(t) = -\infty\,. \tag{46}$$

5. If $\theta < 1/\mu_1$, and ν is such that $Y_0 \geq 0$, then the equilibrium is not well posed, i.e. it may not exist, and when it exist $Y_\theta(t)$ may not be continuous on θ.

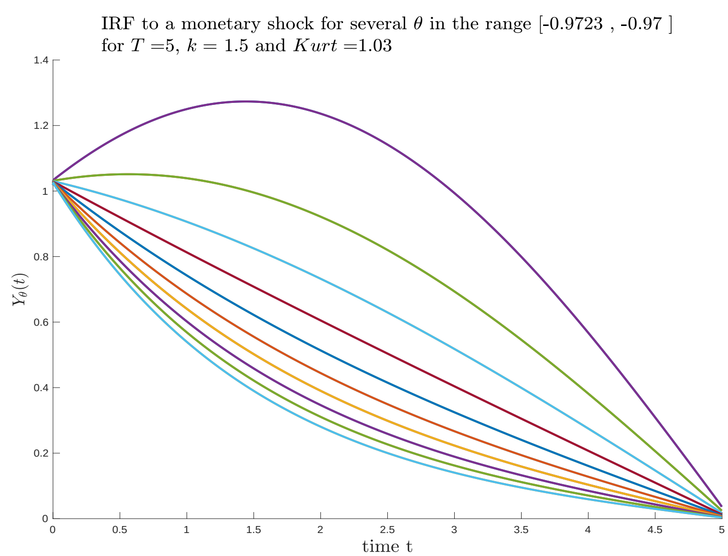
6. If ν is such that $Y_0 \geq 0$, $Y_0(\cdot)$ is continuous, and $Y_0(0) > 0$, then there are countably many values of $\theta < 1/\mu_1$ for which the equilibrium does not exist, and where $Y_\theta(t)$ has a pole at that value.

A few comments are in order. This proposition shows that an equilibrium exists and is unique for $\theta > 1/\mu_1$. Hence, it gives a generalization of our results in Proposition 10, in that it covers all positive θ, i.e. all values of strategic substitutability, and also gives the limit of the region where the equilibrium exists, i.e. $\theta \in (1/\mu_1, \infty)$, as opposed to a sufficient condition. Second, note that it also complements Proposition 9, showing that for θ very large the IRF converges to the flexible price case. Third, note that as the strategic complementarity gets closer to the critical value, i.e. $\theta \downarrow 1/\mu_1$, the IRF $Y_\theta(t)$ gets arbitrarily large. Since $Y_\theta(0) = 1$ for all θ this has the important corollary that the IRF becomes humped shaped.

Corollary 2. Assume that $T < \infty$. If the strategic complementarity is strong enough, i.e. if θ approaches $1/\mu_1$ from above, the IRF $Y_\theta(t)$ has an increasing segment.

Figure 3 illustrates the effect of large values of the strategic complementarity, i.e. θ close to $1/\mu_1$, in producing hump shaped impulse responses.

Figure 3: Impulse response for the Monetary Shock



Note: $\zeta/k = \ell = 0.01$, Golosov-Lucas case.

5.3 Output variance due to monetary shocks

Starting with the seminal analysis of Caplin and Leahy (1997) several well known papers have used the output variance induced by monetary shocks as a summary measure of monetary nonneutrality, as in e.g. Nakamura and Steinsson (2010); Midrigan (2011).

The linear expression for the impulse response given in equation (45) can be used to define a stochastic process for the deviation of output outside of the steady state. In particular, assume that the monetary shock $\{d\epsilon(\tau)\}$ where $\epsilon(\tau)$ is a continuous time process with independent changes and $E[d\epsilon] = 0$ and $E[d\epsilon] = \sigma_\delta^2\, dt$ for some parameter $\sigma_\delta > 0$. Our preferred example is a composite Poisson process for $\{\epsilon(\tau)\}$, where with probability $\varrho > 0$ per unit of time $\epsilon(\tau)$ has a jump of size $\pm\delta$, each jump with probability 1/2. In this case $\sigma_\delta^2 = \varrho\delta^2$. The process for $\{\epsilon(\tau)\}$ generates the stationary stochastic process $\{y\}$ as follows:

$$y(t) = \int_{-T}^{t} Y_\theta(t-\tau) d\epsilon(\tau) \text{ for all } t \geq 0 \tag{47}$$

using the impulse response $Y_\theta(t)$. The unconditional variance of this process is given by:

$$Var_\theta(y) = \sigma_\delta^2 \int_0^T Y_\theta^2(s)\, ds \tag{48}$$

Proposition 13. Assume that $\rho = 0$, $T < \infty$ and that $\theta > 1/\mu_1$. Assume the monetary shocks are $i.i.d.$ and bounded. Then the unconditional variance of output $Var_\theta(y)$ decreases with θ, i.e. $Var_\theta(y) = \sum_{j=1}^{\infty} \frac{\langle \phi_j, Y_0 \rangle^2}{(1-\theta\mu_j)^2}$ and $0 > \frac{1}{Var_\theta(y)} \frac{\partial Var_\theta(y)}{\partial \theta} = 2\sum_{j=1}^{\infty} \omega_j(\theta) \frac{\mu_j}{1-\theta\mu_j} > 2\frac{\mu_1}{1-\theta\mu_1}$ where the $\omega_j(\theta) \equiv \frac{\langle \phi_j, Y_0 \rangle^2}{(1-\theta\mu_j)^2 Var_\theta(y)}$ are weights.

This proposition shows that the strength of strategic complementarities increases the unconditional variance of output –recall that $\theta < 0$ for strategic complementarities, and $\theta > 0$ for substitutability. This proposition complements the result in Proposition 9 that at each t the impulse response increases with the strength of strategic complementarity. We note that variance is also one of the measures used by Nakamura and Steinsson (2010). Note that in the expression for $Var_\theta(y)$ the parameter θ only enters in the factors $1/(1-\theta\mu_j)^2$, since Y_0, ϕ_j, μ_j do not depend on it. The functions Y_0, ϕ_j, μ_j depend on the particular price setting model, i.e. Golosov-Lucas, Calvo, or any variant of Calvo-plus.

5.4 The case of the "pure" Calvo model: $\bar{x}(t) = -\underline{x}(t) \to \infty$

In this simple time dependent model a firm can *only* change prices at exogenously randomly distributed times, independently of their state. In particular in each period a firm can change its price with probability $\zeta > 0$ per unit of time. The simple case of a time dependent model with a constant hazard rate is the most common case analyzed in the literature, due to its tractability, introduced by Calvo. The analysis we use here can draws on Alvarez, Borovicka, and Shimer (2021) Appendix C.3, where a simple closed form expression for the impulse response in the presence of strategic interactions is obtained. We can recast the problem as a Mean Field Game, where the firm's problem becomes

$$\rho u(x,t) = B(x + \theta X(t))^2 + u_t(x,t) + \frac{\sigma^2}{2} u_{xx}(x,t) + \zeta \left(u(x^*(t),t) - u(x,t) \right) \text{ for all } x, \text{ and } t \in [0,T]$$

and final boundary condition $u(x,T) = \tilde{u}(x)$, where $\tilde{u}$ is the stationary solution which corresponds to the problem with $\theta = 0$. Compared to our benchmark model, in this case the barriers are exogenously set at $\bar{x}(t) = +\infty$ and $\underline{x}(t) = -\infty$. The corresponding KFE for the measure $m(x,t)$ is:

$$0 = \frac{\sigma^2}{2} m_{xx}(x,t) - \zeta m(x,t) - m_t(x,t) \text{ for all } x \neq x^*(t), \text{ and } t \in [0,T]$$

with $1 = \int_{-\infty}^{\infty} m(x,t)dx$ for all $t \in [0,T]$ and initial condition $m(x,0) = \tilde{m}(x+\delta)$, where $\tilde{m}$ is the stationary density of the problem with $\theta = 0$, which is a Laplace distribution.

Adapting the arguments in Alvarez, Borovicka, and Shimer (2021), we obtain a simple closed form expression for $Y_\theta(t)$ in the pure Calvo model:

Proposition 14. Consider the Calvo model: $\bar{x}(t) = -\underline{x}(t) \to \infty$. Let μ be the negative root of the quadratic equation: $(\mu - \rho - \zeta)(\zeta + \mu) - \theta(\rho + \zeta)\zeta = 0$. For $T \to \infty$ we get $\lim_{T\to\infty} Y_\theta(t) = e^{\mu t}$ and $\lim_{\rho \downarrow 0} \lim_{T\to\infty} Y_\theta(t) = e^{-\zeta\sqrt{1+\theta}\, t}$ for all $t \geq 0$.

It is remarkable that, as is the case in the Calvo model of Wang and Werning (2020), the impulse response of this involved problem is a simple exponential function (for the case with $T \to \infty$ and $\rho \downarrow 0$). Some features seen above for the state dependent problem also appear here: the impulse response tends to vanish as strategic substitutability gets large ($\theta \to \infty$). On the contrary, large strategic complementarity $\theta \to -1$ yield a very persistent impulse response. Finally, in this simple case the impulse response is monotone, i.e. it can not display a hump shaped pattern.

5.5 Strategic Complementarity and Selection Effects

In this section we return to the analysis of the Calvo-plus , i.e. the model where we let $\ell > 0$, the pure Ss model, the model with $\ell = 0$, and the pure Calvo model described above. We are interested in the relationship between strategic interactions, as measured by θ and the selection effect in the price setting behaviour, measured by ζ. We focus on the cumulative impulse function CIR_θ as a summary measure of the effect of a monetary shock. The main result of this section is that the effect of strategic interactions (θ) is approximately multiplicative separable with the effect of selection in price setting (ζ).

Cumulative impulse response. We define the cumulative impulse response function as $CIR_\theta \equiv \int_0^T Y_\theta(t)dt$. The cumulative IRF is useful as it summarizes the IRF with a single number.

Recall that absent strategic interactions, i.e. when $\theta = 0$, Alvarez, Le Bihan, and Lippi (2016) showed that the scaled cumulative response function $CIR_0/N \equiv \int_0^\infty Y(t)dt/N$ depends only on $\ell^2 = \frac{\zeta \bar{x}_{ss}^2}{\sigma^2/2}$.[13] Motivated by these facts, we analyze (and display) the impulse response for different values of $\ell = \sqrt{k/\sigma^2}$ where for each ζ we adjust σ^2 so that we keep constant the steady state number of price changes N (we keep the normalization $\bar{x}_{ss} = 1$).

[13] Indeed, in that paper it is shown that $CIR_0 = Kurt(\ell)/(6N)$, where $Kurt(\ell)$ is the kurtosis of the price changes using the stationary distribution $\tilde{m}$, and statistic that depends only on ℓ.

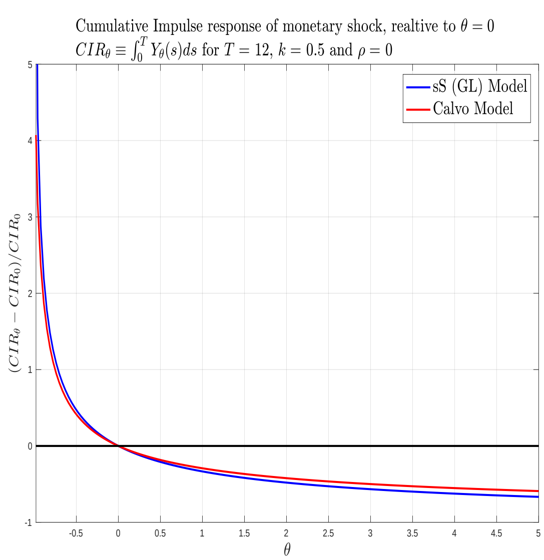
CIR_θ for the "pure" Ss Model, i.e. $\zeta = 0$. The next proposition shows the effect on the cumulative response function CIR_θ of a small change of the coupling parameter θ. The approximation is obtained by differentiating $Y_\theta(t) = Y_0(t) + \theta \int_0^T K(t,s)Y_\theta(s)ds$ with respect to θ and evaluating it at $\theta = 0$ obtaining $\frac{\partial}{\partial\theta}Y_\theta(t)|_{\theta=0} = \int_0^T K(t,s)Y_0(s)ds$.

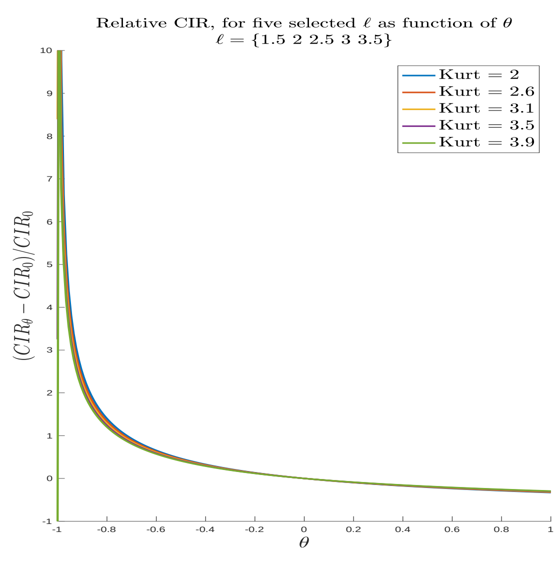
Proposition 15. *Assume that $\zeta = 0$. Consider the CIR_θ for the undiscounted case in a long horizon. Then*

$$\lim_{\rho \downarrow 0} \lim_{T\to\infty} \frac{1}{CIR_\theta}\frac{dCIR_\theta}{d\theta}|_{\theta=0} = 192 \sum_{m=1,3,5,\ldots} \left(\frac{1}{m\pi}\right)^5 [\operatorname{csch}(m\pi) - \coth(m\pi)] \approx -0.578 \quad (49)$$

The left panel of Figure 4 plots $(CIR_\theta - CIR_0)/CIR_0$ for a range of θ that includes both strategic substitutes ($\theta > 0$) and complements ($\theta < 0$). It can be seen that the relative slope around θ is close to 0.6. Also we can see that as θ becomes more negative, and gets closer to the reciprocal of the dominant eigenvalue, then CIR_θ diverges as predicted by Proposition 12.

Figure 4: Impulse response of Monetary Shock





CIR_θ for the "pure" Calvo Model. Using the characterization of Proposition 14 we compute the CIR_θ for the pure Calvo model obtaining:

$$\lim_{\rho \downarrow 0} \lim_{T \to \infty} CIR_\theta^{Calvo} = \frac{1}{\zeta \sqrt{1+\theta}}\,, \text{ and } \lim_{\rho \downarrow 0} \lim_{T \to \infty} \frac{1}{CIR_\theta^{Calvo}} \frac{dCIR_\theta^{Calvo}}{d\theta} |_{\theta=0} = -\frac{1}{2} \tag{50}$$

Note that in the Calvo model the proportional effect of θ on the cumulative impulse response CIR_θ at $\theta \approx 0$ is slightly smaller but overall very close to the value obtained for the pure Ss model. In the Calvo model this elasticity is -0.5, as shown in equation (50), where in the baseline Ss model the elasticity is about -0.578 –see equation (49) in Proposition 15. It is intuitive that the elasticity will be higher in the baseline Ss model, since the firm can also decide when prices are changed. Recall that while the elasticities are similar, the level of the CIR_0 are very different between the baseline Ss model and the Calvo model.[14]

The left panel of Figure 4 compares the CIR for the baseline Ss model and for the Calvo model, over a large range of values of θ. In both cases the CIR_θ is decreasing and convex in θ, diverges towards $+\infty$ at a critical (negative) value of θ, and converges to zero as $\theta \to \infty$. What is remarkable is that the effect of θ in both models is very similar (not just at $\theta \approx 0$), as both curves are very close over the whole domain. The right panel analyzes five Calvoplus models where $0 < \ell < \infty$. For each of these models we study the CIR_θ relative to CIR_0. Overall, the figure shows that across several models, from the pure Ss to the Calvo model, the effect of strategic interactions is approximately multiplicative across a large range of values of the strategic complementarities. This means that in spite of the large *level* differences of the CIR in these models, as in e.g. Calvo being approximately 6 times larger that the Ss model when $\theta \approx 0$, the introduction of strategic interactions affects these models in a quantitatively similar way.

[14] As mentioned, $CIR_0^{Calvo} = 6 \times CIR_0^{Ss}$, provided that both models have the same steady state frequency of price changes –as can be seen in Alvarez, Le Bihan, and Lippi (2016).

6 Conclusions

We studied the propagation of monetary shocks in a sticky-price general-equilibrium economy where firms set prices subject to strategic complementarities with the decision of other firms. In the dynamic equilibrium the firm's price-setting decisions depend on aggregates, which in turn depend on firms's decisions. We cast this fixed-point problem as a perturbation of a Mean Field Game (MFG) and established several analytic results on equilibrium existence and on the analytic characterization of an impulse response.

We think the framework develop in this paper is useful to study the dynamics of equilibrium in related problems. For instance, we are applying it the closely related topic of time dependent price setting Alvarez, Borovicka, and Shimer (2021), and to the case of technology adoption, in particular for the introduction of digital currencies, in Alvarez, Argente, Lippi, Mendez-Chacon, and Van Patten (2022).

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A Proofs

Proof. (of Proposition 1.) Define the markup $m(p/P) \equiv \frac{\eta(p/P)}{\eta(p/P)-1}$. Let us totally differentiate the first order condition $p^*(P) = m(p^*(P)/P)\,\chi(P)$ with respect to P to obtain:

$$\frac{\partial p^*}{\partial P} = m'(p^*/P)\left[\frac{\partial p^*}{\partial P}\frac{1}{P} - \frac{p^*}{P}\frac{1}{P}\right]\chi(P) + m(p^*/P)\frac{\partial \chi(P)}{\partial P}$$

Completing elasticities we have:

$$\frac{P}{p^*}\frac{\partial p^*}{\partial P} = m'(p^*/P)\left[\frac{P}{p^*}\frac{\partial p^*}{\partial P} - \frac{P}{p^*}\frac{p^*}{P}\right]\frac{\chi(P)}{P} + \frac{m(p^*/P)}{p^*}P\frac{\partial \chi(P)}{\partial P}$$

solving for $\frac{P}{p^*}\frac{\partial p^*}{\partial P}$ and rearranging terms

$$\frac{P}{p^*}\frac{\partial p^*}{\partial P} = -\frac{m'(p^*/P)\frac{\chi(P)}{P}}{1 - m'(p^*/P)\frac{\chi(P)}{P}} + \frac{\frac{m(p^*/P)}{p^*}P\frac{\partial \chi(P)}{\partial P}}{1 - m'(p^*/P)\frac{\chi(P)}{P}}$$

Completing elasticities

$$\frac{P}{p^*}\frac{\partial p^*}{\partial P} = -\frac{m(p^*/P)\frac{\chi(P)}{P}}{1 - m'(p^*/P)\frac{\chi(P)}{P}}\left(\frac{m'(p^*/P)}{m(p^*/P)}\right) + \frac{m(p^*/P)\frac{\chi(P)}{p^*}}{1 - m'(p^*/P)\frac{\chi(P)}{P}}\left(\frac{P}{\chi(P)}\frac{\partial \chi(P)}{\partial P}\right)$$

Evaluating this expression at $p^* = P$ gives

$$\frac{P}{p^*}\frac{\partial p^*}{\partial P}\Big|_{p^*=P} = -\frac{m(1)\frac{\chi(P)}{p^*}}{1 - m'(1)\frac{\chi(P)}{p^*}}\left(\frac{m'(1)}{m(1)}\right) + \frac{m(1)\frac{\chi(P)}{p^*}}{1 - m'(1)\frac{\chi(P)}{p^*}}\left(\frac{P}{\chi(P)}\frac{\partial \chi(P)}{\partial P}\right)$$

and using that $\chi(P)/p^* = 1/m(1)$:

$$\frac{P}{p^*}\frac{\partial p^*}{\partial P}\Big|_{p^*=P} = \left[\frac{1}{1 - \frac{m'(1)}{m(1)}}\right]\left[-\frac{m'(1)}{m(1)} + \frac{P}{\chi(P)}\frac{\partial \chi(P)}{\partial P}\right]$$

To get the expression in equation (1) note that $m(x) \equiv \frac{\eta(x)}{\eta(x)-1}$ so

$$m'(x) = \frac{\eta'(x)(\eta(x)-1) - \eta(x)\eta'(x)}{(\eta(x)-1)^2} = -\frac{\eta'(x)}{(\eta(x)-1)^2}$$

and hence:

$$\frac{m'(1)}{m(1)} = -\frac{\eta'(1)}{(\eta(1)-1)^2}\frac{(\eta(1)-1)}{\eta(1)} = -\frac{\eta'(1)}{\eta(1)(\eta(1)-1)}$$

That $\eta(1) > 1$ is implied by the first order optimality condition.

Next we show that $1 + \frac{\eta'(1)}{\eta(1)(\eta(1)-1)} > 0$. Recall the second order condition for a maximum

$$\Pi_{11}(p^*, P) = D''(p^*/P)(p^* - \chi(P))/P^2 + 2D'(p^*/P)/P < 0$$

Note that $D' < 0$ and that $U/p^* = 1/m$ and rewrite the second order condition as

$$\frac{D''(p^*/P)}{D'(p^*/P)P}\left(1 - \frac{1}{m}\right) + 2 > 0 \tag{51}$$

Next, let us differentiate the elasticity $\eta(x) \equiv -\frac{\partial D(x)}{\partial x}\frac{x}{D(x)}$ and evaluate it at $x = 1$. We get

$$\eta'(1) = -\frac{D''(1)}{D(1)} + \left(\frac{D'(1)}{D(1)}\right)^2 - \frac{D'(1)}{D(1)} = -\frac{D''(1)}{D(1)} + \eta^2 + \eta$$

where the second equality uses the elasticity definition. We can thus write the second order condition equation (51) as

$$\frac{D''(1)}{D(1)}\frac{D(1)}{D'(1)P}\frac{1}{\eta} + 2 > 0$$

or, using the expression for D''/D

$$\left(\eta' - \eta^2 - \eta\right)\frac{1}{\eta^2} + 2 = \frac{\eta' + \eta(\eta - 1)}{\eta^2} > 0$$

which establishes that $1 + \frac{\eta'}{\eta(\eta-1)} > 0$, where all η are evaluated at $p/P = 1$.

Finally, the expression for $B \equiv -\frac{\Pi_{11}(\bar{P},\bar{P})}{\Pi(\bar{P},\bar{P})}\bar{P}^2$, is obtained by direct computation evaluating the objects at $p = \bar{P}$. We get

$$\frac{\Pi_{11}}{\Pi} = \frac{D''\left(1 - \frac{1}{m}\right)\frac{p^*}{P^2} + 2\frac{D'}{P}}{D\,P\left(1 - \frac{1}{m}\right)} = \frac{1}{P^2}\left(\frac{D''}{D} + 2\frac{D'}{D}\eta\right) = -\frac{1}{P^2}\left(\eta' + \eta(\eta - 1)\right) \ .$$

Proof. (of Proposition 2) Here we argue that, if $\theta \neq -1$, then the stationary solution displayed above is unique. On the other hand, if $\theta = -1$, then any number X_{ss} corresponds to a steady state.

As a preliminary comment, we note that the same property holds in the simple case of a static game with costless adjustment case, i.e. in the case where $\rho \to \infty$ and where $\psi = 0$. In this case the firm best response is $x^*(X) = \arg\min_x B(x + \theta X)^2$ where X now represent the common choice of all the other firms. Thus, trivially, $x^*(X) = -\theta X$. Then "solving" for the fixed point, $x^*(X) = X$ we get $X = -\theta X$, obtaining the desired result that if $\theta \neq -1$, only $X = 0$ is a Nash equilibrium, but if $\theta = -1$ any X is a Nash equilibrium.

Now we argue that the result for the static game also holds for the stationary state. For this, define $w \equiv x + \theta X_{ss}$. Consider the value function $\hat{u}$ corresponding to the control problem:

$$\hat{u}(w) = \min_{\{\tau_i, \Delta w_i\}} \mathbb{E}\left[\int_0^\infty e^{-\rho t} B w^2(t) + \sum_{i=1}^\infty \psi 1_{\{\tau_i \neq t_i\}} e^{-\rho \tau_i} \,|\, w(0) = w\right]$$

where $dw = \sigma dW$ for $t \in [\tau_i, \tau_i)$ and $w(\tau_i^+) = w(\tau_i^-) + \Delta w_i$ and where t_i are the realizations of the exogenously given times at which the fixed cost is zero, which are exponentially distributed with parameter ζ.

We start making two claims about this problem, and then a third claim about the stationary distribution. First, the value function $\hat{u}$ is symmetric around zero, i.e. $\hat{u}(w) = \hat{u}(-w)$ for all w. This follows because the flow cost Bw^2 is symmetric around zero, and because a standard BM W has, for any collection of times, increments that are normally distributed, and hence symmetric around zero. Second, if the solution of the value function is C^2 then it must satisfy (primes denote derivatives):

$$(\rho + \zeta)\hat{u}(w) = Bw^2 + \hat{u}_{ww}(w)\tfrac{\sigma^2}{2} + \zeta u(w^*) \text{ for all } w \in [-\underline{w}, \bar{w}]$$

with boundary conditions:

$$\hat{u}(\bar{w}) = \hat{u}(\underline{w}) = \hat{u}(w^*) + \psi \text{ and } 0 = \hat{u}_w(\bar{w}) = \hat{u}_w(\underline{w}) = \hat{u}_w(w^*)$$

Thus, since $\hat{u}$ is symmetric, it must be the case that $\bar{w} = -\underline{w}$ and $w^* = 0$.

Third, and finally, using the symmetry of the thresholds $\{\underline{w}, w^*, \bar{w}\}$, we can find the stationary density $\hat{m}(w)$ which is the unique solution of

$$0 = \hat{m}_{ww}(w)\tfrac{\sigma^2}{2} - \zeta\hat{m}(w) \text{ for all } w \in [\underline{w}, w^*) \cup (w^*, \bar{w}]$$

with boundary conditions:

$$0 = \hat{m}(\bar{w}) = \hat{m}(\underline{w}), \ \lim_{w \uparrow w^*} \hat{m}(w) = \lim_{w \downarrow w^*} \hat{m}(w), \ \text{ and } 1 = \int_{\underline{w}}^{\bar{w}} \hat{m}(w) dw\,.$$

Importantly, the density $\hat{m}$ must be symmetric, centered at $w^* = 0$.[15] Hence, $\int_{\underline{w}}^{\bar{w}} w\,\hat{m}(w)dw = 0$. Thus, a stationary equilibrium solution of the original problem requires:

$$x_{ss}^* = w^* - \theta X_{ss},\ \underline{x}_{ss} = \underline{w} - \theta X_{ss},\ \bar{x}_{ss} = \bar{w} - \theta X_{ss},$$
$$X_{ss} = \int_{\underline{w}}^{\bar{w}} \hat{m}(w)\left(w - \theta X_{ss}\right) dw = \int_{\underline{w}}^{\bar{w}} \hat{m}(w) w dw - \theta X_{ss} \int_{\underline{w}}^{\bar{w}} \hat{m}(w) dw$$

and thus we can construct a stationary state if and only if:

$$X_{ss} = -\theta X_{ss}$$

Hence, just as in the static case with no adjustment cost, if $\theta \neq -1$, then $X_{ss} = 0$ is the only stationary state, and if $\theta = -1$ one can construct a stationary state for any X_{ss}. □

Proof. (of Proposition 3). The proof proceed in five parts.

1. Optimal decision rules. First we argue that if $X(t) = 0$, then it is optimal for the firm to set $\bar{x}(t) = \bar{x}_{ss} = 1, \underline{x}(t) = \underline{x}_{ss}$ and $x^*(t) = x^*_{ss} = 0$. This is immediate since

[15] This can be shown since for $[\underline{w}, 0]$ and $[0, \bar{w}]$, the density is a linear combination of the same two exponentials. Using the boundary conditions at $\underline{w}$ and $\bar{w}$ we express each the density in each segment as function of one constant of integration. Finally by continuity at $w = 0$ we find that the distribution must be symmetric.

given $X(t) = 0$ the period flow cost for the firm is $F(x, X) = B(x + \theta X_{ss})^2 = Bx^2$, which is identical to the one for the stationary problem whose HJB is in equation (12). Hence the optimal policy must be the same as the one for the stationary problem.

2. Symmetry of solution of KFE. Now we turn to show that m is symmetric. Let m be a solution to

$$
\begin{aligned}
m_t(x,t) &= m_{xx}(x,t) - \zeta m(x,t) \text{ for } (x,t) \in (-1,0) \cup (0,1) \times [0,T] \\
m(-1,t) &= m(1,t) = 0 \text{ for } t \in \times(0,T] \\
m(x,0) &= m_0(x) \text{ with } m_0(x) = m_0(-x) \text{ for all } x \in [0,1] \text{ and} \\
\int_{-1}^{1} m(x,t)dx &= 1,
\end{aligned}
$$

and where $m(x,t)$ is continuous at $x = 0$. Then $m(\cdot,t)$ is also symmetric on x

Define $M(x,t) = m(x,t) - m(-x,t)$. Then:

$$
\begin{aligned}
M_t(x,t) &= M_{xx}(x,t) - \zeta M(x,t) \text{ for } (x,t) \in (-1,0) \cup (0,1) \times [0,T] \\
M(-1,t) &= M(1,t) = 0 \text{ for } t \in \times(0,T] \\
M(x,0) &= 0 \text{ for all } x \in [0,1] \text{ and } M(0,t) = 0 \text{ for all } t \in [0,T] \\
\int_{-1}^{1} M(x,t)dx &= 0,
\end{aligned}
$$

Differentiating M we get $M_x(x,t) = m(x,t) + m(-x,t)$. Let $0 < \epsilon < 1$, so

$$
M_x(\epsilon,t) = m_x(\epsilon,t) + m_x(-\epsilon,t) \text{ and } M_x(-\epsilon,t) = m_x(-\epsilon,t) + m_x(\epsilon,t)
$$

Taking $\epsilon \downarrow 0$:

$$
M_x(0^+,t) = m_x(0^+,t) + m_x(0^-,t) \text{ and } M_x(0^-,t) = m_x(0^-,t) + m_x(0^+,t)
$$

Thus, $M(\cdot,t)$ is once differentiable at $x = 0$ for all t.

Next we show that for any smooth function with $\phi(-1,t) = \phi(1,t) = 0$ all $t \in [0,T]$ and with $\phi(x,T) = 0$ for $x \in [-1,1]$, then

$$
0 = \int_0^T \int_{-1}^1 M(x,t) \left[-\phi_t(x,t) - k\phi_{xx}(x,t) + \zeta\phi(x,t) \right] dxdt \tag{52}
$$

To see why equation (52) must hold, we proceed in three steps. Fix $x \in [-1,1]$, and integrating by parts

$$
\int_0^T M(x,t)\phi_t(x,t)dt = \int_0^T M_t(x,t)\phi(x,t)dt - M(x,t)\phi(x,t)|_0^T
$$

using the boundary conditions $\phi(x,T) = M(0,x) = 0$ for all $x \in [-1,1]$ we have:

$$\int_{-1}^{1}\int_{0}^{T} M(x,t)\phi_t(x,t)dtdx = \int_{-1}^{1}\int_{0}^{T} M_t(x,t)\phi(x,t)dt$$

Fix any $t \in (0,T)$, using that M_x is continuous in x:

$$\int_{-1}^{1} M(x,t)\phi_{xx}(x,t)dx = -\int_{-1}^{1} M_x(x,t)\phi_x(x,t)dx + M(x,t)\phi_x(x,t)|_{-1}^{1}$$

and using that $M(-1,t) = M(1,t)$ we have

$$\int_{-1}^{1} M(x,t)\phi_{xx}(x,t)dx = -\int_{-1}^{1} M_x(x,t)\phi_x(x,t)dx$$

Integrating by parts again:

$$\int_{-1}^{1} M\phi_{xx}dx = \int_{-1}^{0} M_{xx}\phi dx + \int_{0}^{1} M_{xx}\phi_x - M_x\phi|_{-1}^{0} - M_x\phi|_{0}^{1}$$

using that $M_x(x,t)$ and $\phi(x,t)$ are continuous in $x = 0$, and that $\phi(-1,t) = \phi(1,t) = 0$, then

$$\int_{-1}^{1} M(x,t)\phi_{xx}(x,t)dx = \int_{-1}^{1} M_{xx}(x,t)\phi(x,t)dx$$

Third, integrating with respect to t the last expression, and adding to the first we get:

$$\begin{aligned}&\int_{0}^{T}\int_{-1}^{1} M(x,t)\left[-\phi_t(x,t) - k\phi_{xx}(x,t) + \zeta\phi(x,t)\right]dxdt\\ =&\int_{0}^{T}\int_{-1}^{1} \phi(x,t)\left[M_t(x,t) - kM_{xx}(x,t) + \zeta M(x,t)\right]dxdt\end{aligned}$$

But since $M_t(x,t) = kM_{xx}(x,t) + \zeta M(x,t)$ for all $(x,t) \in (-1,0)\cup(0,1)\times[0,T]$ the second integral is zero.

Now we show that if equation (52) holds for such ϕ function, it must be that $M(x,t) = 0$ for all $(x,t) \in (-1,0)\cup(0,1)\times[0,T]$. Let ϕ be the solution of the KBE equation $\phi_t(x,t) + k\phi_{xx}(x,t) - \zeta\phi(x,t) = -M(x,t)$ for $(x,t) \in [-1,1]\times[0,T]$ with final boundary $\phi(x,T) = 0$ for $x \in [-1,1]$. The solution of this p.d.e. is smooth, in particular C^2 in (x,t) and hence satisfies the hypothesis. We have:

$$0 = \int_{0}^{T}\int_{-1}^{1} (M(x,t))^2\, dxdt$$

Finally, since $M(x,t) = 0$ is the solution, thus $m(x,t) = m(-x,t)$, i.e. m is symmetric.

3. Existence of solution to KFE. Finally, we turn to the existence of a solution to the p.d.e. with the relevant boundary conditions. We will only sketch the argument, which is based on finding a fixed point for a path $A : [0,T] \to \mathbb{R}_+$. Given the symmetry of m it suffices to find m in half of its domain solving:

$$\begin{aligned} m_t(x,t) &= k m_{xx}(x,t) - \zeta m(x,t) \text{ for all } (x,t) \in [0,1] \times [0,T] \\ m(0,t) &= A(t), \text{ and } m(1,t) = 0 \text{ for all } t \in [0,T] \text{ and} \\ m(x,0) &= m_0(x) \text{ for all } x \in [0,1] \end{aligned}$$

Note that the solution depend on A. This solution can be found, given A and given the Fourier coefficients of the function m_0 denoted by $\langle m_0, \varphi_j \rangle$ and the ones for $(1-x)$ denoted by $\langle 1-x, \varphi_j \rangle$

$$\begin{aligned} & m(x,t) = A(t)(1-x) \\ & + \sum_{j=1}^{\infty} \left([\langle m_0, \varphi_j \rangle - A(0)\langle 1-x, \varphi \rangle] e^{-\lambda_j t} - \langle 1-x, \varphi_j \rangle \int_0^t e^{\lambda_j(\tau - t)} [A'(\tau) + \zeta A(\tau)] \, d\tau \right) \varphi_j(x) \end{aligned}$$

where $\varphi_j(x) = \sin(j\pi x)$ and where $\lambda_j = (j\pi)^2 k + \zeta$. Using

$$\int_0^t e^{\lambda_j(\tau - t)} A'(\tau) d\tau = -\int_0^t \lambda_j e^{\lambda_j(\tau - t)} A(\tau) d\tau + A(t) - e^{-\lambda_j t} A(0)$$

Then

$$\begin{aligned} m(x,t) &= A(t)(1-x) + \sum_{j=1}^{\infty} [\langle m_0, \varphi_j \rangle - A(0)\langle 1-x, \varphi \rangle] e^{-\lambda_j t} \varphi_j(x) \\ & - \sum_{j=1}^{\infty} \left(\langle 1-x, \varphi_j \rangle \left(A(t) - A(0)e^{-\lambda_j t} + \int_0^t e^{\lambda_j(\tau - t)} A(\tau) [-\lambda_j + \zeta] \, d\tau \right) \right) \varphi_j(x) \end{aligned}$$

Simplifying

$$m(x,t) = \sum_{j=1}^{\infty} \left(\langle m_0, \varphi_j \rangle + \langle 1-x, \varphi_j k (j\pi)^2 \int_0^t e^{\lambda_j \tau} A(\tau) d\tau \right) e^{-\lambda_j t} \varphi_j(x)$$

The fixed point is obtained by requiring:

$$\frac{1}{2} = \int_0^1 m(x,t) dx \text{ for all } t \in [0,T]$$

or

$$\frac{1}{2} = \sum_{j=1}^{\infty} e^{-\lambda_j t} \frac{1 - \cos(j\pi)}{j\pi} \left(\langle m_0, \varphi_j \rangle + \langle 1-x, \varphi_j \rangle k (j\pi)^2 \int_0^t e^{\lambda_j \tau} A(\tau) d\tau \right)$$

where we use that

$$\int_0^1 \varphi_j(x) dx = \frac{1 - \cos(j\pi)}{j\pi}$$

We can rewrite this equation as:

$$
\begin{aligned}
\frac{1}{2} &= \sum_{j=1}^{\infty} e^{-\lambda_j t} \left(\frac{1-\cos(j\pi)}{j\pi}\right) \langle m_0, \varphi_j \rangle \\
&+ \sum_{j=1}^{\infty} e^{-\lambda_j t} \left(\frac{1-\cos(j\pi)}{j\pi}\right) \langle 1-x, \varphi_j \rangle k (j\pi)^2 \int_0^t e^{\lambda_j \tau} A(\tau) d\tau
\end{aligned}
$$

or

$$
\begin{aligned}
\frac{1}{2} &= \sum_{j=1}^{\infty} e^{-\lambda_j t} \left(\frac{1-\cos(j\pi)}{j\pi}\right) \langle m_0, \varphi_j \rangle \\
&+ \int_0^t \left[\sum_{j=1}^{\infty} e^{-\lambda_j t} \left(\frac{1-\cos(j\pi)}{j\pi}\right) \langle 1-x, \varphi_j \rangle k (j\pi)^2 e^{\lambda_j \tau} \right] A(\tau) d\tau
\end{aligned}
$$

Use that

$$
\langle 1-x, \varphi_j \rangle = \frac{\int_0^1 (1-x)\varphi_j(x)dx}{\int_0^1 \varphi_j(x)^2 dx} = \frac{2}{\pi j}
$$

$$
1 - \cos(j\pi) = 2 \text{ if } j \text{ is odd and } 0 \text{ otherwise}
$$

Thus we have

$$
\frac{1}{2} = \sum_{j=1,3,\ldots}^{\infty} e^{-\lambda_j t} \frac{2}{j\pi} \langle m_0, \varphi_j \rangle + 4k \int_0^t \left[\sum_{j=1,3,\ldots}^{\infty} e^{\lambda_j(\tau - t)} \right] A(\tau) d\tau
$$

Note that if $||A||_\infty < \infty$ for all $t \in [0, T]$ then:

$$
\begin{aligned}
&\left| 4k \int_0^t \left[\sum_{j=1,3,\ldots}^{\infty} e^{\lambda_j(\tau - t)} \right] A(\tau) d\tau \right| \leq 4k ||A||_\infty \sum_{j=1,3,\ldots}^{\infty} \int_0^t e^{\lambda_j(\tau - t)} d\tau \\
&\leq ||A||_\infty \sum_{j=1,3,\ldots}^{\infty} \frac{4}{(j\pi)^2} = ||A||_\infty \frac{1}{6}
\end{aligned}
$$

This is a first order Volterra integral equation with a difference kernel, for which we can obtain a solution to A.

4. Path of $X(t)$**.** Having established that given $\bar{x}(t) = \bar{x}_{ss}, \underline{x}(t) = \underline{x}_{ss}$ and $x^*(t) = x^*_{ss}$, there $m(x,t)$ exists and it is symmetric in x for each t, then $X(t) = \int_{-1}^1 m(x,t)dx = 0 = X_{ss}$.

5. Uniqueness. That the solution is unique on the class of symmetric m, follows from two observations. First, that if m is symmetric, then $X(t) = 0$. Second, that the solution to the KFE in step 3 is unique.

□

Proof. (of Lemma 1). First we show that v is antisymmetric. For that we use that the source

$2B\theta x Z(t)$ is antisymmetric as a function of x. To see this, define $w : [0,1]\times[0,T]$ as $w(x,t) = v(x,t)+v(-x,t)$, which is identically zero and solves $0 = w_t(x,t)+kw_{xx}(x,t)-\rho w(x,t)$ with boundary conditions $w(1,t) = v(1,t) + v(-1,t) = 2v(0,t)$ and $w(0,t) = 2v(0,t)$ all t and $w(x,T) = 0$ all x.

We can use the maximum principle that shows that the maximum and minimum of w has to occur at the given boundaries, i.e. at either $x \in \{0,1\}$ and any $t \in [0,T)$ or at any $x \in [0,1]$ and $t = T$. To see this, notice that since $w(x,T) = 0$ for all $x \in [0,1]$, then if a minimum will be interior, i.e. if it will occur at $0 < \tilde{x} < 1$ and $0 \le \tilde{t} < T$, then $w(\tilde{x},\tilde{t}) < 0$. Hence, $w_t(\tilde{x},\tilde{t}) = -kw_{xx}(\tilde{x},\tilde{t}) + \rho w(\tilde{x},\tilde{t}) < 0$ since $w_{xx}(\tilde{x},\tilde{t}) \le 0$ because $(\tilde{x},\tilde{t})$ is an interior minimum and $k > 0$, and since $w(\tilde{x},\tilde{t}) < 0$. Hence $w(\tilde{x},t') < w(\tilde{x},\tilde{t})$ for t' close to $\tilde{t}$, a contradiction with $(\tilde{x},\tilde{t})$ being an interior minimum. A similar argument shows that there can't be an interior maximum.

Now we show that the maximum and minimum has to occur at $t = T$. For this we use that $w(x,t) = v(x,t) + v(-x,t)$ implies $w_x(0,t) = v_x(0,t) - v_x(0,t) = 0$ for all $t < T$. Thus, suppose that the minimum occurs at $(x,t) = (0,t_1)$ where $t_1 < T$. Then $w(0,t_1) = 2v(0,t_1)$ and $w_t(0,t_1) = 2v_t(0,t_1)$, so $2\rho v(0,t_1) = kw_{xx}(0,t_1)+2v_t(0,t_1)$. Since $(0,t_1)$ is a minimum, we have $v_t(0,t_1) \ge 0$ and since the minimum occurs at $t_1 < T$, then $v(0,t_1) < 0$, so $w_{xx}(0,t_1) < 0$. But since $w_x(0,t_1) = 0$, then we obtain a contradiction with $(0,t_1)$ being a minimum. A similar argument shows that the maximum cannot occur at $(x,t) = (0,t_2)$ where $t_2 < T$. Thus the minimum and maximum occur at $t = T$, where $w(x,T) = 0$.

So we have shown that $w(x,t) = 0$ for all (x,t), and hence $v(x,t) = -v(-x,t)$ all (x,t). Since v is antisymmetric we have $v(0,t) = -v(-0,t)$ and hence $v(0,t) = 0$.

Second, using smooth pasting at the boundaries ($\tilde{u}_x(-1) = \tilde{u}_x(1) = 0$) and optimality at $x^* = 0$ ($\tilde{u}_x(0) = 0$) in equation (18), we can write the boundary conditions as

$$v(-1,t) = v(0,t) = v(1,t) = 0 \quad \text{all } t \in (0,T)$$

which gives the desired result. □

Lemma 6. Let f be the solution of the heat equation

$$0 = f_t(x,t) + kf_{xx}(x,t) - \rho f(x,t) + s(x,t) \text{ for all } x \in [-1,1] \text{ and } t \in [0,T) \tag{53}$$

and boundaries

$$f(1,t) = \bar{\phi}(t) \text{ and } f(-1,t) = \underline{\phi}(t) \text{ for all } t \in (0,T) \tag{54}$$

and

$$f(x,T) = \Phi(x) \text{ for all } x \in [-1,1] \tag{55}$$

for functions $\bar{\phi}, \underline{\phi}, \Phi$ and s. Assume that $\rho \ge 0$ and $k > 0$. The solution is unique.

Proof. (of Lemma 6). As a contradiction, assume that there are two solutions f^1 and f^2. Let $F(x,t) \equiv f^2(x,t) - f^1(x,t)$. Note that the p.d.e. in equation (53) is linear, so that F must satisfy

$$0 = F_t(x,t) + kF_{xx}(x,t) - \rho F(x,t) \text{ for all } x \in [-1,1] \text{ and } t \in (0,T) \tag{56}$$

with boundaries:

$$F(1,t)=0 \text{ and } F(-1,t)=0 \text{ for all } t\in(0,T) \text{ and} \tag{57}$$
$$F(x,T)=0 \text{ for all } x\in[-1,1] \tag{58}$$

We can either use the Maximum principle or a conservation of energy type of argument. We pursue the second.

Define $I(t)\equiv\int_{-1}^{1}(F(x,t))^2dx\geq 0$ for $t\in[0,T]$. Then use the boundary condition $I(T)=0$ to write $0=I(T)=I(0)+\int_0^T I'(t)dt$. Next compute:

$$\begin{aligned} I'(t) &= \int_{-1}^{1}\tfrac{d}{dt}(F(x,t))^2dx = 2\int_{-1}^{1}F(x,t)F_t(x,t)dx = 2\int_{-1}^{1}F(x,t)[\rho F(x,t)-kF_{xx}(x,t)]dx \\ &= 2\rho\int_{-1}^{1}F(x,t)^2dx + 2k\left(\int_{-1}^{1}F_x(x,t)^2dx - F(x,t)F_x(x,t)\Big|_{-1}^{1}\right) \end{aligned}$$

where we have substituted the p.d.e. and integrated by parts. Using the boundary conditions in equation (57) we have:

$$I'(t)=2\rho\int_{-1}^{1}F(x,t)^2dx+2k\int_{-1}^{1}F_x(x,t)^2dx\geq 0$$

Thus $I(T)=0$ only if I is zero for almost all t, and hence $F(x,t)=0$ for almost all x, which in turns implies that $f^1=f^2$. □

Proof. (of Lemma 2) Uniqueness follows from the argument given in Lemma 6.

That equation (21) satisfies the zero boundary condition at $t=T$ follows immediately since at $t=T$ equation (21) becomes an integral with zero length. That the Dirichlet boundary condition holds at $x=1$ and $x=-1$ follows since $\sin(xj\pi)=0$ for all integers j. Note also that the $v(0,t)=0$ since $\sin(0)=0$. It only remains to show that equation (21) satisfies the heat equation with source $CxZ(t)$, where $C\equiv 2B\theta$. Direct computation gives

$$\begin{aligned} v_t(x,t) &= CZ(t)\,2\sum_{j=1}^{\infty}\frac{(-1)^j}{j\pi}\sin(j\pi x) \\ &\quad - 2C\int_t^T\sum_{j=1}^{\infty}e^{(\rho+k(j\pi)^2)(t-\tau)}(\rho+k(j\pi)^2)Z(\tau)\frac{(-1)^j}{j\pi}\sin(j\pi x)d\tau \\ v_{xx}(x,t) &= 2C\int_t^T\sum_{j=1}^{\infty}e^{(\rho+k(j\pi)^2)(t-\tau)}Z(\tau)\frac{(-1)^j}{j\pi}(j\pi)^2\sin(j\pi x)d\tau \end{aligned}$$

and notice that the Fourier series for x in the interval $[0,1]$ is $x=-2\sum_{j=1}^{\infty}\frac{(-1)^j}{j\pi}\sin(j\pi x)$, since $\int_0^1 x\sin(j\pi x)dx/\int_0^1\sin^2(j\pi x)dx=-2\frac{(-1)^j}{j\pi}$. Replacing these expressions in the equation for $v_t(x,t)$ we can verify that $0=v_t(x,t)+kv_{xx}(x,t)-\rho v(x,t)+CxZ(t)$ for all $x\in(-1,1)$ and $t\in[0,T)$. □

For use in Proposition 4 we compute the expressions for the second derivative of $\tilde{u}$ when we use the normalization $\bar{x}_{ss} = 1$, i.e. the choice of ψ so that is attained.

Lemma 7. Fix the parameters σ, B, ζ and ρ and let ψ be such that $\bar{x}_{ss} = 1$. For such case the second derivatives of $\tilde{u}$ evaluated at the thresholds are given by:

$$0 < \tilde{u}_{xx}(0) = \frac{2B}{\rho+\zeta}\left[1 - \eta \,\mathrm{csch}(\eta)\right] , \text{ and } 0 > \tilde{u}_{xx}(1) = \frac{2B}{\rho+\zeta}\left[1 - \eta \coth(\eta)\right] \tag{59}$$

where $\eta \equiv \sqrt{(\rho+\zeta)/k}$. Moreover $|\tilde{u}_{xx}(0)| < |\tilde{u}_{xx}(1)|$.

Proof. (of Lemma 7). The solution for $\tilde{u}$ is of the form of a sum of the particular solution $a_0 + a_2 x^2$ and the two homogenous solutions, which given the symmetry can be written as $A\cosh(\eta x)$, so that $\tilde{u}(x) = a_0 + a_2 x^2 + A\cosh(\eta x)$. From the o.d.e. of $\tilde{u}$ we obtain that $\eta = \sqrt{(\rho+\zeta)/k}$. To determine the coefficients a_0, a_2 note the particular solution must satisfy:

$$(\rho+\zeta)(a_0 + a_2 x^2) = Bx^2 + k2a_2 + \zeta(a_0 + a_2(x^*)) = Bx^2 + k2a_2 + \zeta a_0$$

where we use that $x^* = 0$, and hence $a_2 = B/(\rho+\zeta)$ and $a_0 = 2kB/(\rho(\rho+\zeta))$. It remains to find the value of A. For this we use smooth pasting at $\bar{x} = 1$. We have:

$$\tilde{u}_x(\bar{x}) = 0 = \frac{2B}{\rho+\zeta}\bar{x} + A\eta\sinh(\eta\bar{x})$$

and using $\bar{x} = 1$ we get

$$A = -\frac{2B}{(\rho+\zeta)\eta\sinh(\eta)}$$

Since $\tilde{u}_{xx}(x) = \frac{2B}{\rho+\zeta} + A\eta^2\cosh(\eta x)$ then the second derivatives are:

$$\begin{aligned}
\tilde{u}_{xx}(0) &= \frac{2B}{\rho+\zeta} + A\eta^2 = \frac{2B}{\rho+\zeta} - \frac{2B\eta^2}{(\rho+\zeta)\eta\sinh(\eta)} = \frac{2B}{\rho+\zeta}\left[1 - \eta\,\mathrm{csch}(\eta)\right] \\
\tilde{u}_{xx}(1) &= \frac{2B}{\rho+\zeta} + A\eta^2\cosh(\eta) = \frac{2B}{\rho+\zeta} - \frac{2B\eta^2\cosh(\eta)}{(\rho+\zeta)\eta\sinh(\eta)} = \frac{2B}{\rho+\zeta}\left[1 - \eta\coth(\eta)\right]
\end{aligned}$$

The inequality is equivalent to:

$$1 - \frac{\eta}{\sinh(\eta)} < -1 + \frac{\eta\cosh(\eta)}{\sinh(\eta)} \text{ or } 2 < \eta\frac{1+\cosh(\eta)}{\sinh(\eta)} \text{ or } 2\sinh(\eta) < \eta(1+\cosh(\eta))$$

□

Proof. (of Proposition 4). Consider the smooth pasting and optimal return conditions from the original problem, i.e.

$$0 = u_x(\underline{x}(t,\delta), t, \delta)\,, \quad 0 = u_x(\bar{x}(t,\delta), t, \delta), \quad \text{and} \quad 0 = u_x(x^*(t,\delta), t, \delta)$$

Differentiate them w.r.t. δ to find $\bar{z}, \underline{z}$ and z^*:

$$\bar{z}(t) = -\frac{v_x(1,t)}{\tilde{u}_{xx}(1)} \text{ for all } t \in [0,T)$$
$$\underline{z}(t) = -\frac{v_x(-1,t)}{\tilde{u}_{xx}(-1)} = \bar{z}(t) \text{ for all } t \in [0,T)$$
$$z^*(t) = -\frac{v_x(0,t)}{\tilde{u}_{xx}(0)} \text{ for all } t \in [0,T)\,.$$

Differentiating equation (21) obtained in Lemma 2 we obtain:

$$v_x(1,t) = -2C\int_t^T \sum_{j=1}^{\infty} e^{-(\rho+k(j\pi)^2)(\tau-t)} Z(\tau) d\tau$$
$$v_x(0,t) = -2C\int_t^T \sum_{j=1}^{\infty} e^{-(\rho+k(j\pi)^2)(\tau-t)} Z(\tau)(-1)^j d\tau$$

The equality of $\bar{z} = \underline{z}$ follows from the antisymmetry of v established in Lemma 1 and from $\bar{z}(t) = -\frac{v_x(1,t)}{\tilde{u}_{xx}(1)}$ and $\underline{z}(t) = -\frac{v_x(-1,t)}{\tilde{u}_{xx}(-1)}$ since $\tilde{u}$ is symmetric, and hence $\tilde{u}_{xx}(-1) = \tilde{u}_{xx}(1)$.

The expressions for $\bar{A}$ and A^* in equation (25) follow from Lemma 7.

That $\bar{H}(s) > 0$ is immediate using that k and s are positive. That $H^*(s) < 0$ follows from grouping each pair of consecutive terms as in

$$H^*(s) = -\sum_{j=1,3,5,\ldots} e^{-(\eta^2+(j\pi)^2)\,k\,s}\left[1 - e^{-(\eta^2+((j+1)^2-j^2)\pi^2)\,k\,s}\right] < 0$$

where the inequality follows because k and s are strictly positive. □

Proof. (of Lemma 3.) The proof strategy is to define $N(x,t) = n(x,t) + n(-x,t)$ defined in $(x,t) \in [0,1] \times [0,T]$ satisfying:

$$N_t(x,t) = kN_{xx}(x,t) - \zeta N(x,t) \text{ for } (x,t) \in [0,1]\times[0,T]$$
$$N(x,0) = \nu(x) + \nu(-x) = 0 \text{ for all } x \in [0,1]$$
$$N(1,t) = n(1,t) + n(-1,t) = 0 \text{ for all } t \in [0,T]$$
$$N(0,t) = b(t) + a(t) \equiv C(t) \text{ for all } t \in [0,T]$$
$$\int_0^1 N(x,t)dx = \int_{-1}^0 n(x,t)dx + \int_0^1 n(x,t)dx = 0 \text{ for all } t \in [0,T]$$

for some function $A(t)$. We will show that $A(t) = 0$ for all t and that $N(x,t) = 0$ for all $(x,t) \in [0,1] \times [0,T]$.

The proof proceed by contradiction. Suppose that $\max_{\{(x,t)\in[0,1]\times[0,T]\}} N(x,t) > 0$ and $\min_{\{(x,t)\in[0,1]\times[0,T]\}} N(x,t) < 0$. The two extremes has to have different sign since $\int_0^1 N(x,t)dx = 0$ and $N(1,t) = 0$ for all t. We argue that the maximum and the minimum of $N(x,t)$ on the set $[0,1]\times[0,T]$ has to occur on $\{(x,t) : t = 0\} \cup \{(x,t) : x = 0\} \cup \{(x,t) : x = 1\}$. This is based on the strong maximum/minimum principle for the case for $\zeta \geq 0$, see Evans (2010)

Theorem 12, Section 7.1.c. But since $N(1,t) = 0$ for all t, and $N(x,0) = 0$ for all x, then the maximum and the minimum are attained at $x = 0$ for two values $0 \le \underline{t} < \bar{t} \le T$. Assume, without loss of generality, that $C(\bar{t}) > 0 > C(\underline{t})$. Since $C(t)$ is non-zero, there must be some $0 < t_0 < T$ for which $C(t)$ does not change and it attains a strictly either positive or negative value. Assume, without loss of generality, that it attains a positive value. Then by redefining the p.d.e. considered above in the range $t \in [0, t_0]$ we have that $C(t) \ge 0$ and $C(t_1) > 0$ for some $t' \in [0, t_0]$. But in this case, using the comparison principle, $N(x,t)$ will be positive everywhere in this domain, which is a contradiction.

□

Proof. (of Lemma 4) In this lemma we use that $m(x,t,\delta)$ is continuous around $x = x^*(t,\delta)$ for all t and δ. Under the assumption that $m(x,t,\delta)$ is right and left differentiable at $x = x^*(t,\delta)$, we have

$$m(x,t,\delta) = \begin{cases} m(0,t,0) + m_x(0^-,t,0)\frac{\partial}{\partial\delta}x^*(0,0)\delta + \frac{\partial}{\partial\delta}m(0^-,t,0)\delta + o(\delta) & \text{if } x < x^*(t,\delta) \\ m(0,t,0) + m_x(0^+,t,0)\frac{\partial}{\partial\delta}x^*(0,0)\delta + \frac{\partial}{\partial\delta}m(0^+,t,0)\delta + o(\delta) & \text{if } x > x^*(t,\delta) \end{cases}$$

We can write these expressions in the notation developed above:

$$m(x,t,\delta) = \begin{cases} \tilde{m}(0) + \tilde{m}_x(0^-)z^*(t)\delta + n(0^-,t)\delta + o(\delta) & \text{if } x < x^*(t,\delta) \\ \tilde{m}(0) + \tilde{m}_x(0^+)z^*(t)\delta + n(0^+,t)\delta + o(\delta) & \text{if } x > x^*(t,\delta) \end{cases}$$

Using the continuity of m, we equate both expansions to obtain:

$$\tilde{m}(0) + \tilde{m}_x(0^-)z^*(t)\delta + n(0^-,t)\delta + o(\delta) = \tilde{m}(0) + \tilde{m}_x(0^+)z^*(t)\delta + n(0^+,t)\delta + o(\delta)$$

using that $\tilde{m}_x(0^-) = -\tilde{m}_x(0^+) > 0$, and the notation $a(t) = n(0^-,t)$ and $b(t) = n(0^+,t)$ we have: $-\tilde{m}_x(0^+)z^*(t) + a(t) + o(\delta)/\delta = z^*(t)\tilde{m}_x(0^+) + b(t) + o(\delta)/\delta$ or taking $\delta \to 0$:

$$z^*(t) = \frac{b(t) - a(t)}{-2\,\tilde{m}_x(0^+)}$$

□

Lemma 8. The solution of the heat equation given by equation (31),(32) and (33) is

$$n(x,t) = r(x,t) + \sum_{j=1}^{\infty} c_j(t)\varphi_j(x) \text{ all } x \in [0,1] \text{ and } t > 0 \text{ where}$$
$$r(x,t) = w^*(t) + x\,[\bar{w}(t) - w^*(t)] \text{ all } x \in [0,1],\ t > 0$$

where $w^*(t) = -\tilde{m}_x(0^+)\, z^*(t)$ and $\bar{w}(t) = -\tilde{m}_x(1)\bar{z}(t)$ and for all $j = 1, 2, \ldots$ we have:

$$
\begin{aligned}
&\varphi_j(x) = \sin(j\pi x) \text{ for all } x \in [0,1]\,,\ \langle \varphi_j, h\rangle \equiv \int_0^1 h(x)\varphi_j(x)dx \\
&c_j(t) = c_j(0)e^{-\lambda_j t} + \int_0^t q_j(\tau)e^{\lambda_j(\tau - t)}d\tau \text{ all } t > 0\,, \text{ where } \lambda_j = (\ell^2 + (j\pi)^2)k \quad , \\
&q_j(t) = \frac{\langle \varphi_j\,, -r_t(\cdot,t) - \zeta r(\cdot,t)\rangle}{\langle \varphi_j, \varphi_j\rangle} = 2\left[\frac{\cos(j\pi) - 1}{j\pi}\right] w^{\star\prime}(t) + 2\frac{(-1)^j}{j\pi}\left[\bar{w}'(t) - w^{\star\prime}(t)\right] \\
&\qquad + 2\zeta\left[\frac{\cos(j\pi) - 1}{j\pi}\right] w^\star(t) + 2\zeta\frac{(-1)^j}{j\pi}\left[\bar{w}(t) - w^\star(t)\right] \text{ all } t > 0 \\
&c_j(0) = \frac{\langle \varphi_j\,, \nu - r(\cdot,0)\rangle}{\langle \varphi_j, \varphi_j\rangle} = \frac{\langle \varphi_j\,, \nu\rangle}{\langle \varphi_j, \varphi_j\rangle} + 2\left[\frac{\cos(j\pi) - 1}{j\pi}\right] w^\star(0) + 2\frac{(-1)^j}{j\pi}[w(0) - w^\star(0)]
\end{aligned}
$$

where for the benchmark case of $\nu = \tilde{m}_x$ we get:

$$
\frac{\langle \varphi_j\,, \nu\rangle}{\langle \varphi_j, \varphi_j\rangle} = \frac{\langle \varphi_j\,, \tilde{m}_x\rangle}{\langle \varphi_j, \varphi_j\rangle} = \begin{cases} -\frac{\ell^2 j\pi}{\ell^2 + (j\pi)^2}\left(\frac{1 + e^{\ell}(-1)^{j+1}}{\left(1 - e^{\ell}\right)^2} + \frac{1 + e^{-\ell}(-1)^{j+1}}{\left(1 - e^{-\ell}\right)^2}\right) & \text{if } \zeta > 0 \\ -2\frac{1 + (-1)^{j+1}}{j\pi} & \text{if } \zeta = 0 \end{cases} \tag{60}
$$

Proof. (of Lemma 8) This follows from the explicit solution of the heat equation in $\{(x,t) : x \in [0,1], t \in [0,T]\}$ and using $n(x,t) = n(-x,t)$ to extend it to the negative values of x. We use the general solution of the heat equation using Fourier series with two moving boundaries at $x = 0$ and $x = 1$, a given initial condition, and no source. We reproduce this general solution in Proposition 16. In terms of the notation in Proposition 16 we set $w(x,t) = n(x,t)$, no source, i.e. $s(x,t) = 0$, initial conditions given by $f(x) = \nu(x)$, lower and upper space boundaries $A(t) = -\tilde{m}_x(0^+)\, z^*(t)$, $B(t) = -\tilde{m}_x(1)\bar{z}(t)$ and killing rate $\iota = \zeta$. $\square$

Proof. (of Proposition 6.)

We replace the expression from Lemma 8 for n into the integral for Z obtaining:

$$
\begin{aligned}
Z(t) = 2\int_0^1 xn(x,t)dx &= w^*(t)\frac{2}{2} + [\bar{w}(t) - w^*(t)]\frac{2}{3} + 2\sum_{j=1}^{\infty} c_j(t)\int_0^1 x\sin(j\pi x)dx \\
&= w^*(t) + [\bar{w}(t) - w^*(t)]\frac{2}{3} - 2\sum_{j=1}^{\infty} c_j(t)\frac{(-1)^j}{j\pi}
\end{aligned}
$$

Note that using the expression in Lemma 8 we can write

$$
\begin{aligned}
c_j(t) = &\left(\frac{\langle \varphi_j \, , \, \nu \rangle}{\langle \varphi_j, \varphi_j \rangle} + 2 \left[\frac{\cos(j\pi) - 1}{j\pi} \right] w^*(0) \right) e^{-\lambda_j t} + 2 \frac{(-1)^j}{j\pi} [\bar{w}(0) - w^*(0)] e^{-\lambda_j t} \\
&+ 2 \left[\frac{\cos(j\pi) - 1}{j\pi} \right] \int_0^t w^{*\prime}(\tau) e^{\lambda_j (\tau - t)} d\tau + 2 \frac{(-1)^j}{j\pi} \int_0^t [\bar{w}'(\tau) - w^{*\prime}(\tau)] e^{\lambda_j (\tau - t)} d\tau \\
&+ \zeta 2 \left[\frac{\cos(j\pi) - 1}{j\pi} \right] \int_0^t w^*(\tau) e^{\lambda_j (\tau - t)} d\tau + \zeta 2 \frac{(-1)^j}{j\pi} \int_0^t [\bar{w}(\tau) - w^*(\tau)] e^{\lambda_j (\tau - t)} d\tau
\end{aligned}
$$

Integration by parts, and using the expression for $\lambda_j = \zeta + (j\pi)^2 k$ and cancelling the terms with ζ we get:

$$
\begin{aligned}
c_j(t) = &\left(\frac{\langle \varphi_j \, , \, \nu \rangle}{\langle \varphi_j, \varphi_j \rangle} + 2 \left[\frac{\cos(j\pi) - 1}{j\pi} \right] w^*(0) \right) e^{-\lambda_j t} + 2 \frac{(-1)^j}{j\pi} [\bar{w}(0) - w^*(0)] e^{-\lambda_j t} \\
&- 2 \left[\frac{\cos(j\pi) - 1}{j\pi} \right] (j\pi)^2 k \int_0^t w^*(\tau) e^{\lambda_j (\tau - t)} d\tau - 2 \frac{(-1)^j}{j\pi} (j\pi)^2 k \int_0^t [\bar{w}(\tau) - w^*(\tau)] e^{\lambda_j (\tau - t)} d\tau \\
&+ 2 \left[\frac{\cos(j\pi) - 1}{j\pi} \right] \left[w^*(t) - w^*(0) e^{-\lambda_j t} \right] + 2 \frac{(-1)^j}{j\pi} \left[\bar{w}(t) - w^*(t) - (\bar{w}(0) - w^*(0)) \, e^{-\lambda_j t} \right]
\end{aligned}
$$

Cancelling the terms evaluated at $t = 0$, and simplifying:

$$
\begin{aligned}
c_j(t) = &\frac{\langle \varphi_j \, , \, \nu \rangle}{\langle \varphi_j, \varphi_j \rangle} e^{-\lambda_j t} \\
&- 2 \left[\cos(j\pi) - 1 \right] j\pi k \int_0^t w^*(\tau) e^{\lambda_j (\tau - t)} d\tau - 2(-1)^j j\pi k \int_0^t [\bar{w}(\tau) - w^*(\tau)] e^{\lambda_j (\tau - t)} d\tau \\
&+ 2 \left[\frac{\cos(j\pi) - 1}{j\pi} \right] w^*(t) + 2 \frac{(-1)^j}{j\pi} \left[\bar{w}(t) - w^*(t) \right]
\end{aligned}
$$

Multiplying the expression for $c_j(t)$ by $2 \frac{(-1)^j}{j\pi}$

$$
\begin{aligned}
2 \frac{(-1)^j}{j\pi} c_j(t) = &\, 2 \frac{(-1)^j}{j\pi} \frac{\langle \varphi_j \, , \, \nu \rangle}{\langle \varphi_j, \varphi_j \rangle} e^{-\lambda_j t} \\
&- 4(-1)^j \left[\cos(j\pi) - 1 \right] k \int_0^t w^*(\tau) e^{\lambda_j (\tau - t)} d\tau - 4k \int_0^t [\bar{w}(\tau) - w^*(\tau)] e^{\lambda_j (\tau - t)} d\tau \\
&+ 4(-1)^j \left[\frac{\cos(j\pi) - 1}{(j\pi)^2} \right] w^*(t) + 4 \frac{1}{(j\pi)^2} \left[\bar{w}(t) - w^*(t) \right]
\end{aligned}
$$

using that $\cos(j\pi) = (-1)^j$:

$$
\begin{aligned}
2\frac{(-1)^j}{j\pi} c_j(t) = {} & 2\frac{(-1)^j}{j\pi}\frac{\langle \varphi_j\,,\,\nu\rangle}{\langle \varphi_j,\varphi_j\rangle} e^{-\lambda_j t} \\
& + 4\left[(-1)^j - 1\right] k \int_0^t w^*(\tau) e^{\lambda_j(\tau-t)} d\tau - 4k \int_0^t [\bar{w}(\tau) - w^*(\tau)] e^{\lambda_j(\tau-t)} d\tau \\
& - 4\left[\frac{(-1)^j - 1}{(j\pi)^2}\right] w^*(t) + 4\frac{1}{(j\pi)^2}\left[\bar{w}(t) - w^*(t)\right]
\end{aligned}
$$

Replacing the $2\frac{(-1)^j}{j\pi} c_j(t)$ back into $Z(t)$ we

$$
\begin{aligned}
Z(t) = {} & w^*(t) + [\bar{w}(t) - w^*(t)]\frac{2}{3} - \sum_{j=1}^{\infty} 4\left[(-1)^j - 1\right] k \int_0^t w^*(\tau) e^{\lambda_j(\tau-t)} d\tau \\
& + \sum_{j=1}^{\infty} 4k \int_0^t [\bar{w}(\tau) - w^*(\tau)] e^{\lambda_j(\tau-t)} d\tau \quad + \sum_{j=1}^{\infty} 4\left[\frac{(-1)^j - 1}{(j\pi)^2}\right] w^*(t) \\
& - \sum_{j=1}^{\infty} 4\frac{1}{(j\pi)^2}\left[\bar{w}(t) - w^*(t)\right] \ - 2\sum_{j=1}^{\infty} \frac{(-1)^j}{j\pi}\frac{\langle \varphi_j\,,\,\nu\rangle}{\langle \varphi_j,\varphi_j\rangle} e^{-\lambda_j t}
\end{aligned}
$$

collecting terms and simplifying:

$$
\begin{aligned}
Z(t) = {} & w^*(t)\left[\frac{1}{3} + 4\sum_{j=1}^{\infty}\frac{(-1)^j}{(j\pi)^2}\right] + \bar{w}(t)\left[\frac{2}{3} - 4\sum_{j=1}^{\infty}\frac{1}{(j\pi)^2}\right] \quad - \sum_{j=1}^{\infty} 4\left[(-1)^j - 1\right] k \int_0^t w^*(\tau) e^{\lambda_j(\tau-t)} d\tau \\
& + \sum_{j=1}^{\infty} 4k \int_0^t [\bar{w}(\tau) - w^*(\tau)] e^{\lambda_j(\tau-t)} d\tau \quad - 2\sum_{j=1}^{\infty} \frac{(-1)^j}{j\pi}\frac{\langle \varphi_j\,,\,\nu\rangle}{\langle \varphi_j,\varphi_j\rangle} e^{-\lambda_j t}
\end{aligned}
$$

Using that

$$
\sum_{j=1}^{\infty}\frac{(-1)^j}{(j\pi)^2} = -\frac{1}{12} \text{ and } \sum_{j=1}^{\infty}\frac{1}{(j\pi)^2} = \frac{1}{6}
$$

we get

$$
\begin{aligned}
Z(t) = {} & -\sum_{j=1}^{\infty} 4k\left[(-1)^j - 1\right] \int_0^t w^*(\tau) e^{\lambda_j(\tau-t)} d\tau \quad + \sum_{j=1}^{\infty} 4k \int_0^t [\bar{w}(\tau) - w^*(\tau)] e^{\lambda_j(\tau-t)} d\tau \\
& - 2\sum_{j=1}^{\infty} \frac{(-1)^j}{j\pi}\frac{\langle \varphi_j\,,\,\nu\rangle}{\langle \varphi_j,\varphi_j\rangle} e^{-\lambda_j t}
\end{aligned}
$$

collecting the terms inside the integrals:

$$
Z(t) = \sum_{j=1}^{\infty} 4k(-1)^{j+1} \int_0^t w^*(\tau) e^{\lambda_j(\tau-t)} d\tau \quad + \sum_{j=1}^{\infty} 4k \int_0^t \bar{w}(\tau) e^{\lambda_j(\tau-t)} d\tau \quad - 2\sum_{j=1}^{\infty} \frac{(-1)^j}{j\pi}\frac{\langle \varphi_j\,,\,\nu\rangle}{\langle \varphi_j,\varphi_j\rangle} e^{-\lambda_j t}
$$

Using the definition of $w^*(t) = -\tilde{m}_x(0^+)z^*(t)$ and $\bar{w}(t) = -\tilde{m}_x(1)\bar{z}(t)$ and exchanging the integral with the sum and replacing $\lambda_j = (\ell^2 + (j\pi)^2)k$ we get:

$$\begin{aligned} Z(t) = 4k \int_0^t & \left(-\tilde{m}_x(0^+) \sum_{j=1}^{\infty} (-1)^{j+1} e^{(\ell^2+(j\pi)^2)k(\tau-t)} \right) z^*(\tau) d\tau \\ + 4k \int_0^t & \left(-\tilde{m}_x(1) \sum_{j=1}^{\infty} e^{(\ell^2+(j\pi)^2)k(\tau-t)} \right) \bar{z}(\tau) d\tau \quad - 2 \sum_{j=1}^{\infty} \frac{(-1)^j}{j\pi} \frac{\langle \varphi_j \, , \, \nu \rangle}{\langle \varphi_j, \varphi_j \rangle} e^{-(\ell^2+(j\pi)^2)kt} \end{aligned}$$

Finally computing the projections for ν:

$$\begin{aligned} Z(t) = 4k \int_0^t & \left(-\tilde{m}_x(0^+) \sum_{j=1}^{\infty} (-1)^{j+1} e^{(\ell^2+(j\pi)^2)k(\tau-t)} \right) z^*(\tau) d\tau \\ & + 4k \int_0^t \left(-\tilde{m}_x(1) \sum_{j=1}^{\infty} e^{(\ell^2+(j\pi)^2)k(\tau-t)} \right) \bar{z}(\tau) d\tau \\ & - 4 \sum_{j=1}^{\infty} (-1)^j \frac{e^{-(\ell^2+(j\pi)^2)kt}}{j\pi} \int_0^1 \sin(j\pi x) \nu(x) dx \end{aligned}$$

which gives the expression for T_Z given the definitions of $\bar{G}, G^*$ and Z_0^η.

That $\bar{G}(s) > 0$ is immediate. That $G^*(s) \geq 0$ follows by noticing that we can write:

$$G^*(s) = \sum_{j=1,3,5,\ldots} e^{-(\ell^2+(j\pi)^2)k\,s} \left[1 - e^{-((j+1)^2-j^2)\pi^2 k\,s} \right]$$

and each term $\left[1 - e^{-((j+1)^2-j^2)\pi^2 k\,s} \right] > 0$ since k and s are positive. $\square$

Proof. (of Proposition 7)

First we note that we can decompose ν into its symmetric and antisymmetric part. By linearity, the solution is the sum of the solutions for each part. But, due to Proposition 5 the solution for the symmetric part is zero, so we can assume without loss of generality that ν is antisymmetric. Given Z, we replace $z^* = T^*(Z)$, given by equation (23), and $\bar{z} = \bar{T}(Z)$, given by equation (22), into $T_Z(z^*, \bar{z})$, given by equation (34), to get $\mathcal{T}(Z) = T_Z\left(T^*(Z), \bar{T}(Z)\right)$. Note that, except for the term with Z_0^ν, each term is a double integral. Changing the order of integration and using that $\bar{G}, \bar{H}$ and G^*, H^* satisfy:

$$-\tilde{m}_x(1)\bar{H}(s) = e^{-\rho\,s}\,\bar{G}(s) \geq 0 \text{ and } -\tilde{m}_x(0^+)H^*(s) = e^{-\rho\,s}\,G^*(s) \leq 0 \text{ for all } s > 0 \tag{61}$$

we obtain:

$$Z(t) = Z_0^\nu(t) + \theta \int_0^T K(t,s) Z(s) ds$$

where

$$K(t,s) = 4\,k \int_0^{\min\{t,s\}} e^{-\rho\,(s-\tau)} \left[\bar{A}_\ell \, \frac{\bar{G}(s-\tau)}{\tilde{m}_x(1)} \frac{\bar{G}(t-\tau)}{\tilde{m}_x(1)} - A_\ell^* \, \frac{G^*(s-\tau)}{\tilde{m}_x(0^+)} \frac{G^*(t-\tau)}{\tilde{m}_x(0^+)} \right] d\tau \quad (62)$$

Performing the integration of the exponentials we obtain the desired expression.

The expression for Z_0^ν uses that, since the sin is antisymmetric, for any function we have:

$$\int_0^1 \sin(j\pi x)\nu(x)dx = \tfrac{1}{2} \int_{-1}^1 \sin(j\pi x)\nu(x)dx\,.$$

□

Proof. (of Lemma 5.)

The symmetry of K when $\rho = 0$ in 1 follows directly from its definition in equation (62). That $K \leq 0$ as in 2 uses the expression equation (62) and that $G^* \geq 0$, $A^* > 0$, $\bar{G} \geq 0$, and $\bar{A} < 0$.

For part 1 with $\rho > 0$ and 3 we use the expression for the kernel K derived in the proof of Proposition 7 (see equation (62)). Using that expression we can write K:

$$\begin{aligned} K(t,s) &= -\left(\int_0^{\min\{t,s\}} e^{-\rho(s-\tau)} G_1(s-\tau) G_1(t-\tau) d\tau + \int_0^{\min\{t,s\}} e^{-\rho(s-\tau)} G_2(s-\tau) G_2(t-\tau) d\tau \right) \\ &= -\left(\int_0^T e^{-\rho(s-\tau)} G_1(s-\tau) G_1(t-\tau) d\tau + \int_0^T e^{-\rho(s-\tau)} G_2(s-\tau) G_1(t-\tau) d\tau \right) \end{aligned}$$

where $G_1(s) = 4k|\bar{A}_\ell|\bar{G}(s) > 0$ for $s \geq 0$ and $G_1(s) = 0$ otherwise. Likewise $G_2(s) = 4k|A^*_\ell|G^*(s) > 0$ for $s \geq 0$ and $G_2(s) = 0$ otherwise.

Part 1 establishes that K is self adjoint. For this we compute

$$\begin{aligned} \tilde{K}_{ab} &\equiv \int_0^T \int_0^T K(t,s) V_a(s)\, ds\, V_b(t) e^{-\rho t}\, dt \\ &= -\sum_{j=1}^{2} \int_0^T \int_0^T \int_0^T \left[e^{-\rho(s-\tau)} G_j(s-\tau) G_j(t-\tau) d\tau \right] V_a(s)\, ds\, V_b(t) e^{-\rho t} dt \\ &= -\sum_{j=1}^{2} \int_0^T e^{\rho\tau} \left[\int_0^T e^{-\rho s} G_j(s-\tau) V_a(s)\, ds \int_0^T G_j(t-\tau)\, V_b(t) e^{-\rho t} dt \right] d\tau \end{aligned}$$

Likewise we compute

$$
\begin{aligned}
\tilde{K}_{ba} &\equiv \int_0^T \int_0^T K(t,s) V_b(s)\, ds\, V_a(t) e^{-\rho t}\, dt \\
&= -\sum_{j=1}^{2} \int_0^T \int_0^T \int_0^T \left[e^{-\rho(s-\tau)} G_j(s-\tau) G_j(t-\tau) d\tau \right] V_b(s)\, ds\, V_a(t) e^{-\rho t} dt \\
&= -\sum_{j=1}^{2} \int_0^T e^{\rho\tau} \left[\int_0^T e^{-\rho s} G_j(s-\tau) V_b(s)\, ds \int_0^T G_j(t-\tau)\, V_a(t) e^{-\rho t} dt \right] d\tau
\end{aligned}
$$

Clearly $\tilde{K}_{ab} = \tilde{K}_{ab}$, which establishes the desired result.

Part 3 establishes that K is negative definite. We will show that

$$
Q_i \equiv -\int_0^T \int_0^T \left(\int_0^T e^{-\rho(s-\tau)} G_i(s-\tau) G_i(t-\tau) d\tau \right) V(s) V(t)\, e^{-\rho t}\, ds\, dt < 0
$$

To see why this has to hold, we write:

$$
\begin{aligned}
Q_i &= -\int_0^T \int_0^T \int_0^T e^{-\rho(s-\tau)} G_i(s-\tau) G_i(t-\tau) V(s) V(t) e^{-\rho t}\, d\tau\, ds\, dt \\
&= -\int_0^T e^{\rho\tau} \int_0^T \int_0^T e^{-\rho s} G_i(s-\tau) V(s) G_i(t-\tau) e^{-\rho t} V(t)\, ds\, dt\, d\tau \\
&= -\int_0^T e^{\rho\tau} \left(\int_0^T G_i(s-\tau) e^{-\rho s} V(s)\, ds \right) \left(\int_0^T G_i(t-\tau) e^{-\rho t} V(t)\, dt \right) d\tau \\
&= -\int_0^T e^{\rho\tau} \left(\int_0^T G_i(s-\tau) V(s) e^{-\rho s}\, ds \right)^2 d\tau \leq 0
\end{aligned}
$$

with strictly inequality if $V \neq 0$.

Part 4 of the proof establishes the bounds for the integral $\int_0^T |K(t,s)| ds$.

As a preliminary step we write $\int_0^T |K(t,s)| ds \leq \int_0^\infty |K(t,s)| ds$ as:

$$
\int_0^\infty |K(t,s)| ds = 4 \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \left[\bar{A} - A^* (-1)^{j+i} \right] \kappa_{i,j} \text{ where}
$$

$$
\kappa_{i,j}(t) \equiv \int_0^\infty \frac{\left[e^{[(j\pi)^2 + (i\pi)^2 + \eta^2] k (t \wedge s)} - 1 \right] e^{-(j\pi)^2 kt - (i\pi)^2 ks - \eta^2 ks}}{(j\pi)^2 + (i\pi)^2 + \eta^2} ds
$$

Direct computation gives

$$\begin{aligned}
\kappa_{i,j}(t) &= \int_0^t \frac{e^{[(j\pi)^2+(i\pi)^2+\eta^2]k(t\wedge s)}e^{-(j\pi)^2kt-(i\pi)^2ks-\eta^2ks}}{(j\pi)^2+(i\pi)^2+\eta^2}ds + \int_t^\infty \frac{e^{[(j\pi)^2+(i\pi)^2+\eta^2ks]k(t\wedge s)}e^{-(j\pi)^2kt-(i\pi)^2ks-\eta^2ks}}{(j\pi)^2+(i\pi)^2+\eta^2}ds \\
&- \int_0^\infty \frac{e^{-(j\pi)^2kt-(i\pi)^2ks-\eta^2ks}}{(j\pi)^2+(i\pi)^2+\eta^2}ds \\
&= \int_0^t \frac{e^{[(j\pi)^2+(i\pi)^2+\eta^2]ks}e^{-(j\pi)^2kt-(i\pi)^2ks-\eta^2ks}}{(j\pi)^2+(i\pi)^2+\eta^2}ds + \int_t^\infty \frac{e^{[(j\pi)^2+(i\pi)^2+\eta^2]kt}e^{-(j\pi)^2kt-(i\pi)^2ks-\eta^2ks}}{(j\pi)^2+(i\pi)^2+\eta^2}ds \\
&- e^{-(j\pi)^2kt}\int_0^\infty \frac{e^{-(i\pi)^2ks-\eta^2ks}}{(j\pi)^2+(i\pi)^2+\eta^2}ds \\
&= e^{-(j\pi)^2kt}\int_0^t \frac{e^{(j\pi)^2ks}}{(j\pi)^2+(i\pi)^2+\eta^2}ds + e^{(i\pi)^2kt+\eta^2kt}\int_t^\infty \frac{e^{-(i\pi)^2ks-\eta^2ks}}{(j\pi)^2+(i\pi)^2+\eta^2}ds \\
&- \frac{1}{(j\pi)^2+(i\pi)^2+\eta^2}\frac{e^{-(j\pi)^2kt}}{k(i\pi)^2+k\eta^2} \\
&= \frac{\left(1-e^{-(j\pi)^2kt}\right)}{(j\pi)^2+(i\pi)^2+\eta^2}\frac{1}{(j\pi)^2k} + \frac{1}{(j\pi)^2+(i\pi)^2+\eta^2}\frac{1}{(i\pi)^2k+\eta^2k} - \frac{1}{(j\pi)^2+(i\pi)^2+\eta^2}\frac{e^{-(j\pi)^2kt}}{k(i\pi)^2+\eta^2k} \\
&= \frac{\left(1-e^{-(j\pi)^2kt}\right)}{(j\pi)^2+(i\pi)^2+\eta^2}\left(\frac{1}{(j\pi)^2k}+\frac{1}{(i\pi)^2k+\eta^2k}\right) \\
&= \frac{\left(1-e^{-(j\pi)^2kt}\right)}{(j\pi)^2+(i\pi)^2+\eta^2}\frac{1}{k}\left(\frac{(j\pi)^2+(i\pi)^2+\eta^2}{((j\pi)^2)\left((i\pi)^2+\eta^2\right)}\right) \\
&= \frac{1-e^{-(j\pi)^2kt}}{k\left((j\pi)^2\right)\left((i\pi)^2+\eta^2\right)}
\end{aligned}$$

Thus we get:

$$\kappa_{i,j}(t) = \frac{1-e^{-(j\pi)^2kt}}{k\,(j\pi)^2\,\left((i\pi)^2+\eta^2\right)}$$

We expand κ_{ij} around $\eta=0$ to obtain:

$$\begin{aligned}
\kappa_{i,j}(t) &= \frac{1-e^{-(j\pi)^2kt}}{k\left((j\pi)^2\right)\left((i\pi)^2+\eta^2\right)} = \frac{1-e^{-(j\pi)^2kt}}{k\left((j\pi)^2\right)\left((i\pi)^2\right)}\frac{(i\pi)^2}{\left((i\pi)^2+\eta^2\right)} \\
&= \frac{1-e^{-(j\pi)^2kt}}{k\,(j\pi)^2\,(i\pi)^2}\left(1-\frac{\eta^2}{(i\pi)^2}+o(\eta^2)\right)
\end{aligned}$$

Thus

$$
\begin{aligned}
\int_0^T |K(t,s)|ds &\le -\int_0^\infty K(t,s)ds \\
&= 4\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\left[-\bar{A}+A^*\left(-1\right)^{j+i}\right]\left(\frac{1-e^{-(j\pi)^2kt}}{k}\right)\frac{1}{(j\pi)^2(i\pi)^2}\left(1-\frac{\eta^2}{(i\pi)^2}\right)+o(\eta^2) \\
&\le 4\frac{-\bar{A}}{k}\left[\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\frac{1}{(i\pi)^2}\frac{\left(1-e^{-(j\pi)^2kt}\right)}{(j\pi)^2}\right]\left(1-\frac{\eta^2}{(i\pi)^2}\right) \\
&+ 4\frac{A^*}{k}\left[\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}(-1)^{j+i}\frac{1}{(i\pi)^2}\frac{\left(1-e^{-(j\pi)^2kt}\right)}{(j\pi)^2}\right]\left(1-\frac{\eta^2}{(i\pi)^2}\right)+o(\eta^2) \\
&< 4\frac{-\bar{A}}{k}\left[\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\frac{1}{(i\pi)^2}\frac{1}{(j\pi)^2}\right]\left(1-\frac{\eta^2}{(i\pi)^2}\right) \\
&+ 4\frac{A^*}{k}\left[\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}(-1)^{j+i}\frac{1}{(i\pi)^2}\frac{1}{(j\pi)^2}\right]\left(1-\frac{\eta^2}{(i\pi)^2}\right)+o(\eta^2)
\end{aligned}
$$

were we use that $1-e^{-(j\pi^2)kt}<1$ and that :

$$
\begin{aligned}
\frac{-\bar{A}}{k} &= -\frac{2\eta^2}{1-\eta\coth(\eta)} = 6+\frac{2}{5}\eta^2+o(\eta^2) \\
\frac{A^*}{k} &= \frac{2\eta^2}{1-\eta\,\mathrm{csch}(\eta)} = 12+\frac{7}{5}\eta^2+o(\eta^2)
\end{aligned}
$$

to write:

$$
\begin{aligned}
\int_0^T |K(t,s)|ds &\leq -\int_0^\infty K(t,s)ds \\
&< 4\frac{-\bar{A}}{k}\left[\sum_{j=1}^\infty\sum_{i=1}^\infty \frac{1}{(i\pi)^2}\frac{1}{(j\pi)^2}\right]\left(1-\frac{\eta^2}{(i\pi)^2}\right) \\
&+ 4\frac{A^*}{k}\left[\sum_{j=1}^\infty\sum_{i=1}^\infty (-1)^{j+i}\frac{1}{(i\pi)^2}\frac{1}{(j\pi)^2}\right]\left(1-\frac{\eta^2}{(i\pi)^2}\right) + o(\eta^2) \\
&= 4\left(6+\frac{2}{5}\eta^2\right)\left[\sum_{j=1}^\infty\sum_{i=1}^\infty \frac{1}{(i\pi)^2}\frac{1}{(j\pi)^2}\right]\left(1-\frac{\eta^2}{(i\pi)^2}\right) \\
&+ 4\left(12+\frac{7}{5}\eta^2\right)\left[\sum_{j=1}^\infty\sum_{i=1}^\infty (-1)^{j+i}\frac{1}{(i\pi)^2}\frac{1}{(j\pi)^2}\right]\left(1-\frac{\eta^2}{(i\pi)^2}\right) + o(\eta^2) \\
&= 4\times 6\left[\sum_{j=1}^\infty\sum_{i=1}^\infty \frac{1}{(i\pi)^2}\frac{1}{(j\pi)^2}\right] + 4\times 12\left[\sum_{j=1}^\infty\sum_{i=1}^\infty (-1)^{j+i}\frac{1}{(i\pi)^2}\frac{1}{(j\pi)^2}\right] \\
&+ 4\left[\sum_{j=1}^\infty\sum_{i=1}^\infty \frac{1}{(i\pi)^2}\frac{1}{(j\pi)^2}\right]\left(\frac{2}{5}-\frac{6}{(i\pi)^2}\right)\eta^2 \\
&+ 4\left[\sum_{j=1}^\infty\sum_{i=1}^\infty (-1)^{j+i}\frac{1}{(i\pi)^2}\frac{1}{(j\pi)^2}\right]\left(\frac{7}{5}-\frac{12}{(i\pi)^2}\right)\eta^2 + o(\eta^2)
\end{aligned}
$$

Using the values for the following series into the previous expression

$$
\sum_{j=1}^\infty \frac{1}{(j\pi)^2} = \frac{1}{6}\ ,\ \sum_{j=1}^\infty \frac{(-1)^{j+1}}{(j\pi)^2} = \frac{1}{12}\ ,\ \sum_{j=1}^\infty \frac{1}{(j\pi)^4} = \frac{1}{90} = \frac{1}{6}\frac{1}{15} \text{ and } \sum_{j=1}^\infty \frac{(-1)^{j+1}}{(j\pi)^4} = \frac{7}{720} = \frac{1}{12}\frac{7}{60}
$$

we obtain:

$$
\begin{aligned}
\int_0^T |K(t,s)|ds &\leq -\int_0^\infty K(t,s)ds \\
&< 4\times 6\frac{1}{6^2} + 4\times 12\frac{1}{6^2}\frac{1}{4} + 4\left(\frac{1}{6^2}\frac{2}{5}-\frac{1}{6}\frac{6}{90}\right)\eta^2 + 4\left(\frac{7}{5}\frac{1}{6^2}\frac{1}{4}-\frac{1}{6}\frac{12}{12}\frac{7}{60}\right)\eta^2 + o(\eta^2) \\
&= 1-\frac{7}{180}\eta^2 + o(\eta^2)
\end{aligned}
$$

which is the expression for the case of small ρ.

To obtain the bound in 4 for any η and $t \geq 0$ we note we note that

$$
\kappa_{i,j}(t) = \frac{1-e^{-(j\pi)^2kt}}{k\left((j\pi)^2\right)\left((i\pi)^2+\eta^2\right)} < \hat{\kappa}_{i,j} \equiv \frac{1}{k\,(j\pi)^2\,(i\pi)^2}
$$

hence

$$\int_0^\infty |K(t,s)|dt = 4\sum_{j=1}^\infty\sum_{i=1}^\infty \left[\bar{A}\,-A^*\,(-1)^{j+i}\right]\kappa_{i,j}(t) \le 4\sum_{j=1}^\infty\sum_{i=1}^\infty \left[\bar{A}\,-A^*\,(-1)^{j+i}\right]\hat{\kappa}_{i,j}$$

Again, following the same steps as above we get:

$$\begin{aligned}\int_0^T |K(t,s)|ds \le -\int_0^\infty K(t,s)ds \le 4\frac{-\bar{A}}{k}\left[\sum_{j=1}^\infty\sum_{i=1}^\infty \frac{1}{(i\pi)^2}\frac{1}{(j\pi)^2}\right] \\ +\,4\frac{A^*}{k}\left[\sum_{j=1}^\infty\sum_{i=1}^\infty (-1)^{j+i}\frac{1}{(i\pi)^2}\frac{1}{(j\pi)^2}\right]\end{aligned}$$

Using the series obtained above we have:

$$\int_0^T |K(t,s)|ds < \frac{4}{6^2}\left(\frac{-\bar{A}}{k}+\frac{A^*}{k}\frac{1}{4}\right)$$

Using the expressions for $-\bar{A}/k$ and A^*/k we have:

$$\int_0^T |K(t,s)|ds < \frac{\eta^2}{18}\left(\frac{1}{1-\eta\operatorname{csch}(\eta)}-\frac{4}{1-\eta\coth(\eta)}\right)$$

We establish part 5, a bound for the kernel when $\ell > 0$ in terms of the kernel for $\ell = 0$. The bound uses the expression derived in the proof of Proposition 6, which shows in equation (62)

$$\begin{aligned}K(t,s;\ell,\eta) = & 4\,k\int_0^{\min\{t,s\}} e^{-\rho\,(s-\tau)}\Big[\\ & \bar{A}_\ell\,\frac{\bar{G}(s-\tau;\ell)}{-\tilde{m}_x(1)}\frac{\bar{G}(t-\tau;\ell)}{-\tilde{m}_x(1)} - A^*_\ell\,\frac{G^*(s-\tau;\ell)}{-\tilde{m}_x(0^+)}\frac{G^*(t-\tau;\ell)}{-\tilde{m}_x(0^+)}\Big]d\tau\end{aligned}$$

where direct computation gives

$$0<\frac{\bar{G}(s;\ell)}{-\tilde{m}_x(1)} = \sum_{j=1}^\infty e^{-(\ell^2+(j\pi)^2)\,k\,s} \le \frac{\bar{G}(s;0)}{1} = \sum_{j=1}^\infty e^{-(j\pi)^2\,k\,s} \quad \text{and}$$

$$0<\frac{G^*(s;\ell)}{-\tilde{m}_x(0^+)} = \sum_{j=1}^\infty (-1)^{j+1}e^{-(\ell^2+(j\pi)^2)k\,s} \le \frac{G^*(s;0)}{1} = \sum_{j=1}^\infty (-1)^{j+1}e^{(j\pi)^2 k\,s}$$

where we use that for $\ell = 0$ we have $\tilde{m}_x(x) = -1$ all $x > 0$. Finally, using

$$\bar{A}_\ell = -\tilde{m}_x(1)\bar{A} < 0 \text{ and } A^*_\ell = -\tilde{m}_x(0^+)A^* > 0$$

Thus fix a t, s and τ

$$
\begin{aligned}
&|\bar{A}_\ell| \frac{\bar{G}(s-\tau;\ell)}{-\tilde{m}_x(1)} \frac{\bar{G}(t-\tau;\ell)}{-\tilde{m}_x(1)} + |A^*_\ell| \frac{G^*(s-\tau;\ell)}{-\tilde{m}_x(0^+)} \frac{G^*(t-\tau;\ell)}{-\tilde{m}_x(0^+)} \\
&\leq |\bar{A}_\ell| \, \bar{G}(s-\tau;0)\bar{G}(t-\tau;0) + |A^*_\ell| \, G^*(s-\tau;0)G^*(t-\tau;0) \\
&= |\tilde{m}_x(1)||\bar{A}| \, \bar{G}(s-\tau;0)\bar{G}(t-\tau;0) + |\tilde{m}_x(0^+)||A^*| \, G^*(s-\tau;0)G^*(t-\tau;0) \\
&\leq |\tilde{m}_x(0^+)| \left[|\bar{A}| \, \bar{G}(s-\tau;0)\bar{G}(t-\tau;0) + |A^*| \, G^*(s-\tau;0)G^*(t-\tau;0)\right]
\end{aligned}
$$

where we use that $|\tilde{m}_x(0^+)| > |\tilde{m}_x(1)|$. Integrating with respect to τ we obtain the desired bound.

Now we establish the bound in 6. We do so by proving a stronger bound, i.e. we find a bound for

$$
\frac{\rho^2}{(1-e^{-\rho T})^2} \int_0^T \int_0^T K(t,s)^2 e^{-\rho(t+s)} ds dt \leq \frac{\rho^2}{(1-e^{-\rho T})^2} \int_0^T \int_0^T K(t,s)^2 ds dt
$$

which covers the case where $\rho = 0$. The proof for the bound on the integral of K^2 consists on a long computation of the double integral.

Note that

$$
|K(t,s)| \leq 4 \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} | \left[\bar{A} - A^* (-1)^{j+i}\right] | \left| \frac{\left[e^{[(j\pi)^2+(i\pi)^2+\eta^2]k(t\wedge s)} - 1\right] e^{-(j\pi)^2 kt-(i\pi)^2 ks-\eta^2 ks}}{(j\pi)^2+(i\pi)^2+\eta^2} \right|
$$

Thus using a change on variables we have:

$$
\int_0^T \int_0^T K^2(t,s) dt ds \leq \left[|\bar{A}| + |A^*|\right] \frac{4}{k^2\pi^6} \int_0^Q \int_0^Q \tilde{K}^2(t,s) dt ds \tag{63}
$$

where

$$
\tilde{K}(t,s) \equiv \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} \frac{\left[e^{[j^2+i^2+d](t\wedge s)} - 1\right] e^{-j^2 t - i^2 s - ds}}{j^2+i^2+d} \text{ with } d \equiv \frac{\eta^2}{\pi^2} \text{ and } Q \equiv Tk\pi^2 \tag{64}
$$

We define

$$
f(\tau) \equiv \left(e^{(j^2+i^2+d)\tau} - 1\right)\left(e^{(l^2+m^2+d)\tau} - 1\right) \tag{65}
$$

and then write:

$$
\tilde{K}^2(t,s) = \sum_j \sum_i \sum_l \sum_m \frac{f(t\wedge s) e^{-(j^2+l^2)t-(i^2+d+m^2+d)s}}{(j^2+i^2+d)(m^2+l^2+d)}
$$

Fix j, i, m, l, and consider the double integral in s and t:

$$\int_0^Q \int_0^Q f(t \wedge s) e^{-(j^2+l^2)t-(i^2+d+m^2+d)s} ds dt = \mathcal{A} + \mathcal{B} \tag{66}$$
$$\equiv \int_0^Q \int_0^t f(t \wedge s) e^{-(j^2+l^2)t-(i^2+d+m^2+d)s} ds dt + \int_0^Q \int_t^Q f(t \wedge s) e^{-(j^2+l^2)t-(i^2+d+m^2+d)s} ds dt$$

where $\mathcal{A}$ and $\mathcal{B}$ were implicitly defined. Solving the integral for A by parts we have:

$$\begin{aligned}
\mathcal{A} &= \int_0^Q \left(\int_0^t f(s) e^{-(i^2+d+m^2+d)s} ds \right) e^{-(j^2+l^2)t} dt \\
&= \left(\int_0^{t'} f(s) e^{-(i^2+d+m^2+d)s} ds \right) \left(\frac{e^{-(j^2+l^2)t'}}{-(l^2+j^2)} \right) \Big|_0^Q - \int_0^Q f(t) e^{-(i^2+d+m^2+d)t} \frac{e^{-(j^2+l^2)t}}{-(l^2+j^2)} dt \\
&= -\frac{e^{-(j^2+l^2)Q}}{(l^2+j^2)} \int_0^Q f(s) e^{-(i^2+d+m^2+d)s} ds + \frac{1}{(l^2+j^2)} \int_0^Q f(t) e^{-(j^2+i^2+d+l^2+m^2+d)t} dt
\end{aligned}$$

We also have:

$$\begin{aligned}
\mathcal{B} &= \int_0^Q f(t) e^{-(j^2+l^2)t} \left(\int_t^Q e^{-(i^2+d+m^2+d)s} ds \right) dt \\
&= \int_0^Q f(t) e^{-(j^2+l^2)t} \left(\frac{e^{-(i^2+d+m^2+d)Q} - e^{-(i^2+d+m^2+d)t}}{-(i^2+d+m^2+d)} \right) dt \\
&= \frac{1}{(i^2+d+m^2+d)} \int_0^Q f(t) e^{-(j^2+i^2+d+l^2+m^2+d)t} dt \\
&\quad - \frac{1}{(i^2+d+m^2+d)} \int_0^Q f(t) e^{-(j^2+l^2)t} e^{-(i^2+d+m^2+d)Q} dt
\end{aligned}$$

Since $f(s) \geq 0$ we can write:

$$\begin{aligned}
\mathcal{A} &= -\frac{e^{-(j^2+l^2)Q}}{(l^2+j^2)} \int_0^Q f(s) e^{-(i^2+d+m^2+d)s} ds + \frac{1}{(l^2+j^2)} \int_0^Q f(t) e^{-(j^2+i^2+d+l^2+m^2+d)t} dt \\
&\leq \frac{1}{(l^2+j^2)} \int_0^Q f(t) e^{-(j^2+i^2+d+l^2+m^2+d)t} dt \qquad (67)
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{B} &= \frac{1}{(i^2+d+m^2+d)} \int_0^Q f(t) e^{-(j^2+i^2+d+l^2+m^2+d)t} dt \\
&\quad - \frac{1}{(i^2+d+m^2+d)} \int_0^Q f(t) e^{-(j^2+l^2)t} e^{-(i^2+d+m^2+d)Q} dt \\
&\leq \frac{1}{(i^2+d+m^2+d)} \int_0^Q f(t) e^{-(j^2+i^2+d+l^2+m^2+d)t} dt \qquad (68)
\end{aligned}$$

Thus

$$\mathcal{A}+\mathcal{B} \le \mathcal{C}(j,i,l,m) \equiv \left(\frac{1}{(l^2+j^2)}+\frac{1}{(i^2+d+m^2+d)}\right)\int_0^Q f(t)e^{-(j^2+i^2+d+l^2+m^2+d)t}dt \tag{69}$$

Thus we want to compute the upper bound:

$$\int_0^Q\int_0^Q \tilde{K}^2(t,s)dsdt \le \sum_j\sum_i\sum_l\sum_m \frac{\mathcal{C}(j,i,l,m)}{(j^2+i^2+d)(l^2+m^2+d)} \tag{70}$$

The next step is to compute the integral $\int_0^Q f(t)e^{-(j^2+i^2+d+l^2+m^2+d)t}dt$. We have

$$\begin{aligned}
&f(t)e^{-(j^2+i^2+d+l^2+m^2+d)t} \\
&\equiv \left(e^{(j^2+i^2+d)t}-1\right)\left(e^{(l^2+m^2+d)t}-1\right)e^{-(j^2+i^2+d+l^2+m^2+d)t} \\
&= \left[e^{(j^2+i^2+d+l^2+m^2+d)t}+1-e^{(j^2+i^2+d)t}-e^{(l^2+m^2+d)t}\right]e^{-(j^2+i^2+d+l^2+m^2+d)t} \\
&= 1+e^{-(j^2+i^2+d+l^2+m^2+d)t}-e^{-(l^2+m^2+d)t}-e^{-(j^2+i^2+d)t}
\end{aligned} \tag{71}$$

Now we compute the time integral:

$$\begin{aligned}
&\int_0^Q\left(1+e^{-(j^2+i^2+d+l^2+m^2+d)t}-e^{-(l^2+m^2+d)t}-e^{-(j^2+i^2+d)t}\right)dt \\
&= Q+\frac{1-e^{-(j^2+i^2+d+l^2+m^2+d)Q}}{(j^2+i^2+d+l^2+m^2+d)}-\frac{1-e^{-(l^2+m^2+d)Q}}{(l^2+m^2+d)}-\frac{1-e^{-(j^2+i^2+d)Q}}{(j^2+i^2+d)} \\
&\le Q+\frac{1}{(j^2+i^2+d+l^2+m^2+d)}+\frac{1}{(l^2+m^2+d)}+\frac{1}{(j^2+i^2+d)}
\end{aligned}$$

Hence

$$\begin{aligned}
\mathcal{C}(j,i,l,m) \le &\left(\frac{1}{(l^2+j^2)}+\frac{1}{(i^2+d+m^2+d)}\right) \\
&\times\left(Q+\frac{1}{(j^2+i^2+d+l^2+m^2+d)}+\frac{1}{(l^2+m^2+d)}+\frac{1}{(j^2+i^2+d)}\right)
\end{aligned}$$

and thus we have:

$$\begin{aligned}
&\int_0^Q\int_0^Q \tilde{K}^2(t,s)dsdt \\
&\le \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{1}{(j^2+i^2+d)(l^2+m^2+d)}\right)\left(\frac{1}{(l^2+j^2)}+\frac{1}{(i^2+d+m^2+d)}\right) \\
&\times\left(Q+\frac{1}{(j^2+i^2+d+l^2+m^2+d)}+\frac{1}{(l^2+m^2+d)}+\frac{1}{(j^2+i^2+d)}\right)
\end{aligned} \tag{72}$$

We have

$$\int_0^Q \int_0^Q \tilde{K}^2(t,s)dsdt \le 4\,Q\,\mathcal{D}$$
$$\mathcal{D} \equiv \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{1}{(j^2+i^2+d)(l^2+m^2+d)}\right)\left(\frac{1}{(l^2+j^2)}+\frac{1}{(i^2+d+m^2+d)}\right)$$

In turn, it suffices to show that

$$\mathcal{E} \equiv \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(j^2+i^2+d)(l^2+m^2+d)}\frac{1}{(l^2+j^2)} < \infty$$

To find a bound for this series we use the following integral:

$$\mathcal{F} \equiv \int_1^{\infty}\int_1^{\infty}\int_1^{\infty}\int_1^{\infty}\frac{1}{(x_1^2+x_2^2+d)}\frac{1}{(y_1^2+y_2^2+d)}\frac{1}{(x_1^2+y_1^2)}dx_1dx_2dy_1dy_2$$

Thus using $\int_1^{\infty} 1/(z^2+a^2)dz = \tan^{-1}(a)/a$ we have:

$$\begin{aligned}
\mathcal{F} &= \int_1^{\infty} dx_1 \int_1^{\infty} dy_1 \frac{1}{(x_1^2+y_1^2)}\int_1^{\infty}\frac{1}{(x_1^2+x_2^2+d)}dx_2\int_1^{\infty}\frac{1}{y_1^2+y_2^2}dy_2 \\
&= \int_1^{\infty} dx_1 \int_1^{\infty} dy_1 \frac{1}{(x_1^2+y_1^2)}\int_1^{\infty}\frac{1}{(x_1^2+x_2^2+d)}dx_2\frac{\tan^{-1}(y_1)}{y_1^2} \\
&\le \int_1^{\infty} dx_1 \int_1^{\infty} dy_1 \frac{1}{(x_1^2+y_1^2)}\int_1^{\infty}\frac{1}{(x_1^2+x_2^2)}dx_2\frac{\tan^{-1}(y_1)}{y_1^2} \\
&= \int_1^{\infty} dx_1 \int_1^{\infty} dy_1 \frac{1}{(x_1^2+y_1^2)}\frac{\tan^{-1}(x_1)}{x_1}\frac{\tan^{-1}(y_1)}{y_1}
\end{aligned}$$

Using that $\tan^{-1}(z) \le \pi/2$ for $z \ge 1$ we have

$$\mathcal{F} \le \frac{\pi^2}{4}\int_1^{\infty}\int_1^{\infty}\frac{1}{(x_1^2+y_1^2)}\frac{1}{x_1}\frac{1}{y_1}dx_1dy_1$$

Using that $\int_1^{\infty}\frac{1}{(z^2+a^2)}\frac{1}{z}dz = \log(a^2+2)/(2a)$ we have

$$\begin{aligned}
\mathcal{F} &\le \frac{\pi^2}{4}\int_1^{\infty}\frac{\log(y_1^2+2)}{2y_1}\frac{1}{y_1}dy_1 = \frac{\pi^2}{8}\int_1^{\infty}\frac{\log(y_1^2+2)}{y_1^2}dy_1 \\
&\le \frac{\pi^2}{8}\int_1^{\infty}\frac{\log(y_1^2)}{y_1^2}dy_1 = \frac{\pi^2}{4}\int_1^{\infty}\frac{\log(y_1)}{y_1^2}dy_1 = \frac{\pi^2}{4} < \infty
\end{aligned}$$

since $\int_1^{\infty}\frac{\log(z)}{z^2}dz = 1$.

Combining all the inequalities obtained above we have:

$$\int_0^T \int_0^T K^2(t,s)dsdt \leq 4\,Q\frac{\pi^2}{4}\frac{4}{k^2\pi^6}\left(A^* - \bar{A}\right) = Tk\pi^2\frac{4}{k^2\pi^4}\left(A^* - \bar{A}\right)$$
$$= T\frac{4}{k\pi^2}\left(A^* - \bar{A}\right)$$

Since

$$\bar{A} = k\tfrac{2\eta^2}{[1-\eta\coth(\eta)]} < 0 \;\text{ and }\; A^* = k\,\tfrac{2\eta^2}{[1-\eta\,\mathrm{csch}(\eta)]} > 0$$

We have

$$\int_0^T \int_0^T K^2(t,s)dsdt \leq \frac{8}{\pi^2}\,T\,\left(\tfrac{\eta^2}{[1-\eta\,\mathrm{csch}(\eta)]} - \tfrac{\eta^2}{[1-\eta\coth(\eta)]}\right)$$

and

$$\frac{\rho^2}{(1-e^{-\rho T})^2}\int_0^T \int_0^T K^2(t,s)dsdt \leq c_0\frac{\rho^2 T}{(1-e^{-\rho T})^2}\left(\tfrac{\eta^2}{[1-\eta\,\mathrm{csch}(\eta)]} - \tfrac{\eta^2}{[1-\eta\coth(\eta)]}\right)$$

The proof for the bound of $||K||_2^2$ for the general case, i.e. the proof of part 7, follows the same steps as the proof of part 6, except that: i) we use the bound for the case of $\ell^2 > 0$ between the two Kernels, and ii) the discount $e^{-\rho(t+s)}$ in the definition of $||K||_2^2$ is introduced in the relevant expressions. Given the similarity of the calculations, we only present the steps that given rise to different expressions.

Using the same change of variables as above we can write:

$$\int_0^T \int_0^T K^2(t,s)e^{-\rho(t+s)}dtds \leq \left[|\bar{A}| + |A^*|\right]\frac{4|\tilde{m}_x(0^+)|}{k^2\pi^6}\int_0^Q \int_0^Q \tilde{K}^2(t,s)e^{-r(t+s)}dtds \tag{73}$$

where $r \equiv \rho/\pi^2$, and where we use the same definitions of $\tilde{K}$ and f as in equation (64) and equation (65) respectively. We proceed as above and define $\mathcal{A}+\mathcal{B}$ incorporating the term with discount, so that we get:

$$\int_0^Q \int_0^Q f(t \wedge s)e^{-(j^2+r+l^2)t-(i^2+r+d+m^2+d)s}dsdt = \mathcal{A}+\mathcal{B} \tag{74}$$

Following exactly the same steps we arrive to the following inequality:

$$\mathcal{A}+\mathcal{B} \leq \mathcal{C}(j,i,l,m) \tag{75}$$
$$\equiv \left(\frac{1}{(l^2+j^2)} + \frac{1}{(i^2+d+m^2+d)}\right)\int_0^Q f(t)e^{-(2r+j^2+i^2+d+l^2+m^2+d)t}dt$$

We thus get:

$$\int_0^Q \int_0^Q \tilde{K}^2(t,s)e^{-r(t+s)}dsdt \leq \sum_j \sum_i \sum_l \sum_m \frac{\mathcal{C}(j,i,l,m)}{(j^2+i^2+d)(l^2+m^2+d)} \tag{76}$$

The next step is to compute the integral $\int_0^Q f(t)e^{-(2r+j^2+i^2+d+l^2+m^2+d)t}dt$. We have

$$\begin{aligned} & f(t)e^{-(2r+j^2+i^2+d+l^2+m^2+d)t} \\ & = e^{-2rt} + e^{-(2r+j^2+i^2+d+l^2+m^2+d)t} - e^{-(2r+l^2+m^2+d)t} - e^{-(2r+j^2+i^2+d)t} \end{aligned} \tag{77}$$

Following the same steps as in the previous case:

$$\begin{aligned} & \int_0^Q \left(e^{-2rt} + e^{-(2r+j^2+i^2+d+l^2+m^2+d)t} - e^{-(2r+l^2+m^2+d)t} - e^{-(2r+j^2+i^2+d)t}\right) dt \\ & \leq \frac{1-e^{-2rQ}}{2r} + \frac{1}{(j^2+i^2+d+l^2+m^2+d)} + \frac{1}{(l^2+m^2+d)} + \frac{1}{(j^2+i^2+d)} \end{aligned}$$

Following the same steps we obtain:

$$\begin{aligned} & \int_0^Q \int_0^Q \tilde{K}^2(t,s)e^{-r(t+s)}dsdt \\ & \leq \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty} \left(\frac{1}{(j^2+i^2+d)(l^2+m^2+d)}\right)\left(\frac{1}{(l^2+j^2)} + \frac{1}{(i^2+d+m^2+d)}\right) \\ & \times \left(\frac{1-e^{-2rQ}}{2r} + \frac{1}{(j^2+i^2+d+l^2+m^2+d)} + \frac{1}{(l^2+m^2+d)} + \frac{1}{(j^2+i^2+d)}\right) \end{aligned} \tag{78}$$

and thus we have

$$\begin{aligned} & \int_0^Q \int_0^Q \tilde{K}^2(t,s)e^{-r(t+s)}dsdt \leq \left[\frac{1-e^{-2rQ}}{2r} + 3\right] \mathcal{D} \\ & \mathcal{D} \equiv \sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty} \left(\frac{1}{(j^2+i^2+d)(l^2+m^2+d)}\right)\left(\frac{1}{(l^2+j^2)} + \frac{1}{(i^2+d+m^2+d)}\right) \end{aligned}$$

In the previous case we have shown that the series $\mathcal{D}$ converges to a finite limit. Going back to the original variables for the integration, we obtain the desired bound. In particular we get:

$$\begin{aligned} \frac{\rho^2}{(1-e^{-\rho T})^2} \int_0^T \int_0^T K^2(t,s)e^{-\rho(t+s)}dsdt & \leq \frac{\rho^2}{(1-e^{-\rho T})^2}\left[\frac{1-e^{-2\rho T}}{2\rho} + 3\right] \mathcal{D} \\ & = \rho \left[\frac{1-e^{-2\rho T}+6\rho}{(1-e^{-\rho T})^2}\right] \frac{\mathcal{D}}{2} \end{aligned}$$

□

Proof. (of Proposition 8)The proof of Proposition 8 is immediate, since using the definition of K in equation (38), it is straightforward to compute $K(0,s) = 0$ for all $s \in [0,T]$ hence $Y_\theta(0) = Y_0(0) + \theta \int_0^T K(0,s) Y_\theta(s) ds = Y_0(0)$. Finally that $Y_0(0) = -Z_0^\nu(0) = 1$ follows from evaluation of the series equation (35) for any $\ell \geq 0$. $\square$

Proof. (of Proposition 9)

That the series in equation (43), whenever it converges, is the solution of equation (42) follows from replacing the series into the integral equation.

That $Y_\theta(0) = 1$ follows from the fact that $Y_0(0) = 1$ and that $K(0,s) = 0$ for all $s \in (0,T)$.

To establish that $Y_\theta(t) > 0$ and $\theta < 0$, so we have $\theta K(t,s) > 0$ for all $(t,s) \in (0,T)^2$ and hence $(\theta \mathcal{K})^r (Y_0) > 0$ for $t \in (0,T)$. Note that, for each t, the sequence $S_n(\theta,t) \equiv \sum_{r=0}^n \theta^r (\mathcal{K})^r (Y_0)(t)$ is monotone increasing in n, and, by assumption converges. Hence, $Y_\theta(t) > 0$. Moreover if $\theta' < \theta < 0$ we have $S_n(\theta',t) > S_n(\theta,t)$. Thus, the limit preserves this inequality.

To establish that $Y_\theta(t)$ is convex, we differentiate twice the series with respect to θ, obtaining:

$$\frac{\partial^2}{\partial \theta^2} Y_\theta(t) = \sum_{r=2}^{\infty} r(r-1)\theta^{r-2} \left(\mathcal{K}\right)^r (Y_0)(t)$$

for $t \in (0,T)$. If r is even we have $\theta^{r-2} > 0$ and $\left(\mathcal{K}\right)^r (Y_0)(t) > 0$. If r is odd we have $\theta^{r-2} < 0$ and $\left(\mathcal{K}\right)^r (Y_0)(t) < 0$, hence all the terms in the sum are strictly positive, and thus $\frac{\partial^2}{\partial \theta^2} Y_\theta(t) > 0$.

$\square$

Proof. (of Proposition 11.)

We show here a bound for the HS operator norm in terms of the L^2 norm of the kernel. We use that

$$||K||_2^2 \equiv \frac{\rho^2}{(1-e^{-\rho T})^2} \int_0^T \int_0^T K^2(t,s)\, e^{-\rho(s+t)}\, ds\, dt \tag{79}$$

$$= \sum_{i,j} \left(\frac{\rho^2}{(1-e^{-\rho T})^2} \int_0^T \int_0^T K(t,s)\, f_i(s)\, f_j(t)\, e^{-\rho(s+t)}\, ds\, dt \right)^2 \tag{80}$$

This equality follows from projecting $K(t,s)$ first as a function of s into $\{f_i(s)\}$. In particular, fix a t:

$$K(t,s) = \sum_{i=1}^{\infty} \langle K(t,\cdot), f_i \rangle f_i(s) = \frac{\rho}{1-e^{-\rho T}} \sum_{i=1}^{\infty} \int_0^T K(t,s') f_i(s') e^{-\rho s'} ds'\, f_i(s)$$

And then project this expression as a function of t into the base $\{f_j(t)\}$

$$K(t,s) = \frac{\rho^2}{(1-e^{-\rho T})^2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \int_0^T \int_0^T K(t',s') f_j(t') f_i(s') e^{-\rho s'} e^{-\rho t'} ds' dt'\, f_i(s) f_j(t)$$

To simplify we can write this expression as:

$$K(t,s) = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \kappa_{i,j}\, f_i(s) f_j(t)$$

Now we can write:

$$(K(t,s))^2 = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \kappa_{i,j}\kappa_{m,n}\, f_i(s) f_j(t) f_m(s) f_n(t)$$

Then integrating with respect to $\rho^2 e^{-\rho(t+s)}/(1-e^{-\rho T})^2$ then:

$$\begin{aligned}
&\frac{\rho^2}{(1-e^{-\rho T})^2}\int_0^T\int_0^T (K(t,s))^2\, e^{-\rho(t+s)} dt ds \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \kappa_{i,j}\kappa_{m,n} \frac{\rho}{1-e^{-\rho T}}\int_0^T f_i(s) f_m(s) e^{-\rho s} ds \frac{\rho}{1-e^{-\rho T}}\int_0^T f_j(t) f_n(t) e^{-\rho t} dt \\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \kappa_{i,j}\kappa_{m,n}\delta_{i,m}\delta_{j,n} = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} (\kappa_{i,j})^2
\end{aligned}$$

where we use $\{f_i\}$ are orthonormal, and $\delta_{\cdot,\cdot}$ is the Kroneker symbol, and thus we obtain equation (80).

Let K_ρ be defined as $K_\rho(t,s) = K(t,s)e^{\rho s}$. Then

$$\begin{aligned}
||K_\rho||_2^2 &= \sum_{i,j}\left(\frac{\rho^2}{(1-e^{-\rho T})^2}\int_0^T\int_0^T K_\rho(t,s)\, f_i(s)\, f_j(t)\, e^{-\rho(s+t)}\, ds\, dt\right)^2 \\
&= \sum_{i,j}\left(\frac{\rho^2}{(1-e^{-\rho T})^2}\int_0^T\int_0^T K(t,s)\, f_i(s)\, f_j(t)\, e^{-\rho t}\, ds\, dt\right)^2
\end{aligned}$$

and using Cauchy-Schwarz

$$||K_\rho||_2^2 \le ||K||_2^2\, ||e^{\rho s}||_2^2 = ||K||_2^2 \frac{(\rho T)^2}{(1-e^{-\rho T})^2}$$

so

$$||\mathcal{K}||_{HS}^2 \le \frac{(1-e^{-\rho T})^2}{\rho^2}\frac{(\rho T)^2}{(1-e^{-\rho T})^2}||K||_2^2 = T^2 ||K||_2^2$$

Thus, using this inequality and the results in Lemma 5 we obtain the bound on $||\mathcal{K}||_{HS}$, and thus operator is compact. The rest of the proof is directly from the spectral theorem. $\square$

Proof. (of Proposition 15) We set $T=\infty$. For this value we want to compute

$$\tfrac{d}{d\theta}CIR_\theta|_{\theta=0} = \int_0^\infty \tfrac{d}{d\theta}Y_\theta(t)|_{\theta=0}dt = \int_0^\infty \int_0^\infty K(t,s)Y_0(t)\,ds\,dt$$

which can be written as

$$Q \equiv \int_0^\infty \int_0^\infty K(t,s)Y_0(s)ds\,dt = \sum_{m=1}^{\infty} Q_m \text{ where } Q_m = 4\int_0^\infty \int_0^\infty K(t,s)\frac{1-\cos(m\pi)}{(m\pi)^2}ds\,dt$$

where we have replaced the expression for Y_0

Replacing the expression for K we get that for each m

$$Q_m = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} 16\,(1-\cos(m\pi))\left(\bar{A} - A^*(-1)^{i+j}\right)\tilde{\omega}_{i,j,m}$$

where $\tilde{\omega}_{i,j,m}$ is defined as

$$\begin{aligned}
\tilde{\omega}_{i,j,m} &= \frac{1}{k^2\pi^8}\frac{1}{(i^2+j^2+r^2)m^2}\,\omega_{i,j,m} \text{ and} \\
\omega_{i,j,m} &= \int_0^\infty \int_0^\infty \left(e^{(j^2+i^2+r^2)s\wedge t} - 1\right) e^{-j^2t-i^2s-r^2s-m^2s}ds\,dt
\end{aligned}$$

were we have used a change on variables for t, and where we use $r \equiv \eta^2/\pi^2$.

Now we compute $\omega_{i,j,m}$ letting $\rho \downarrow 0$, or equivalently $r \to 0$. For this note that we can write the inner integral in $\omega_{i,j,m}$ as follows:

$$\begin{aligned}
&\int_0^t e^{-j^2t}e^{-(m^2-j^2)s}ds + \int_t^\infty e^{i^2t}e^{-(i^2+m^2)s}ds - \int_0^\infty e^{-j^2t}e^{-(i^2+m^2)s}ds \\
&= e^{j^2t}\frac{\left[1-e^{-(m^2-j^2)t}\right]}{(m^2-j^2)} + \frac{e^{i^2t}e^{-(i^2+m^2)t}}{(i^2+m^2)} - \frac{e^{-j^2t}}{(i^2+m^2)} \\
&= \frac{e^{-j^2t}-e^{m^2t}}{(m^2-j^2))} + \frac{e^{-m^2t}-e^{-j^2t}}{(i^2+m^2)}
\end{aligned}$$

Then, integrating the resulting expression with respect to t between 0 and ∞ we get:

$$\begin{aligned}
\omega_{i,j,m} &= \frac{1}{(m^2-j^2)}\left[\frac{1}{j^2}-\frac{1}{m^2}\right] + \frac{1}{(i^2+m^2)}\left[\frac{1}{m^2}-\frac{1}{j^2}\right] = \frac{1}{m^2j^2} + \frac{1}{(i^2+m^2)}\frac{(j^2-m^2)}{m^2j^2} \\
&= \frac{1}{m^2j^2}\left(\frac{i^2+j^2}{i^2+m^2}\right)
\end{aligned}$$

Now we replace this expression into $\tilde{\omega}_{i,j,m}$

$$\begin{aligned}\omega_{i,j,m} &= \frac{1}{k^2\pi^8}\frac{1}{m^2}\frac{1}{(j^2+i^2)}\omega_{i,j,m} = \frac{1}{k^2\pi^8}\frac{1}{m^2}\frac{1}{(j^2+i^2)}\frac{1}{m^2j^2}\left(\frac{i^2+j^2}{i^2+m^2}\right)\\ &= \frac{1}{k^2\pi^8}\frac{1}{m^2}\frac{1}{m^2j^2}\left(\frac{1}{i^2+m^2}\right) = \frac{1}{k^2}\frac{1}{(m\pi)^4}\frac{1}{(j\pi)^2}\frac{1}{(i^2\pi^2+m^2\pi^2)}\end{aligned}$$

Finally we want to compute the infinite sums of the expression for $\omega_{i,j,m}$ over i, j, m. For this we will use that when m is odd:

$$\begin{aligned}\sum_{i=1}^{\infty}\frac{1}{i^2\pi^2+m^2\pi^2} &= \frac{m\pi\coth(m\pi)-1}{2m^2\pi^2}\\ \sum_{i=1}^{\infty}\frac{(-1)^i}{i^2\pi^2+m^2\pi^2} &= \frac{m\pi\,\mathrm{csch}(m\pi)-1}{2m^2\pi^2}\\ \sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{i^2\pi^2+m^2\pi^2} &= \frac{1-m\pi\,\mathrm{csch}(m\pi)}{2m^2\pi^2}\end{aligned}$$

and we will also use that

$$\sum_{j=1}^{\infty}\tfrac{1}{(j\pi)^2} = \frac{1}{6} \text{ and } \sum_{j=0}^{\infty}\frac{1}{\pi^2(j+1)^2} = \frac{1}{8}\,.$$

We write $Q = \mathcal{Q}_I - \mathcal{Q}_{II}$:

$$\begin{aligned}\mathcal{Q}_I &= \sum_{m=1,3,5,\dots} 2\times 16\,\bar{A}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\tilde{\omega}_{i,j,m} = \sum_{m=1,3,5,\dots} 32\frac{\bar{A}}{k}\frac{1}{k}\frac{1}{(m\pi)^4}\sum_{j=1}^{\infty}\frac{1}{(j\pi)^2}\sum_{i=1}^{\infty}\frac{1}{(i^2\pi^2+m^2\pi^2)}\\ &= \sum_{m=1,3,5,\dots}\frac{32}{6}\frac{\bar{A}}{k}\frac{1}{k}\frac{1}{(m\pi)^4}\sum_{i=1}^{\infty}\frac{1}{(i^2\pi^2+m^2\pi^2)}\\ &= \sum_{m=1,3,5,\dots}\frac{32}{12}\frac{\bar{A}}{k}\frac{1}{k}\frac{1}{(m\pi)^6}\left(m\pi\coth(m\pi)-1\right)\end{aligned}$$

Now we write the second term of Q:

$$\mathcal{Q}_{II} = \frac{32}{k}\frac{A^*}{k}\sum_{1,3,5,\ldots}\frac{1}{(m\pi)^4}\sum_{j=1}^{\infty}\frac{1}{j^2\pi^2}\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{\pi^2 i^2+\pi^2 m^2} = \frac{32}{k}\frac{A^*}{k}\sum_{m=1,3,5,\ldots}\frac{1}{(m\pi)^4}\left(\mathcal{O}+\mathcal{E}\right) \text{ where}$$

$$\begin{aligned}
\mathcal{O} &= \sum_{j=1,3,5,\ldots}\frac{1}{(\pi j)^2}\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{(i^2\pi^2+m^2\pi^2))} = \sum_{j=0}^{\infty}\frac{1}{\pi^2(j+1)^2}\frac{(1-m\pi\operatorname{csch}(m\pi))}{2m^2\pi^2}\\
&= \frac{1}{8}\frac{(1-m\pi\operatorname{csch}(m\pi))}{2m^2\pi^2} \text{ and}\\
\mathcal{E} &= \sum_{j=2,4,6,\ldots}\frac{1}{(\pi j)^2}\sum_{i=1}^{\infty}\frac{(-1)^{i}}{(i^2\pi^2+m^2\pi^2))} = \left[\frac{1}{6}-\frac{1}{8}\right]\sum_{i=1}^{\infty}\frac{(-1)^{i}}{(i^2\pi^2+m^2\pi^2))}\\
&= \frac{1}{8}\frac{1}{3}\frac{(m\pi\operatorname{csch}(m\pi)-1)}{2m^2\pi^2}
\end{aligned}$$

Thus

$$\begin{aligned}
\mathcal{Q}_{II} &= \frac{32}{k}\frac{A^*}{k}\sum_{m=1,3,5,\ldots}\frac{1}{(m\pi)^4}\left(\mathcal{O}+\mathcal{E}\right) = \frac{32}{k}\frac{A^*}{k}\frac{1}{8}\left(\frac{1}{3}-1\right)\sum_{m=1,3,5,\ldots}\frac{1}{(m\pi)^4}\frac{(m\pi\operatorname{csch}(m\pi)-1)}{2m^2\pi^2}\\
&= \frac{32}{k}\frac{A^*}{k}\frac{1}{8}\frac{1}{3}\sum_{m=1,3,5,\ldots}\frac{1-m\pi\operatorname{csch}(m\pi)}{(m\pi)^6}
\end{aligned}$$

Recall that as $\rho \to 0$ then $\bar{A}/k \to -6$ and $A^*/k \to 12$, and thus

$$\begin{aligned}
Q = \mathcal{Q}_I - \mathcal{Q}_{II} &= \sum_{m=1,3,5,\ldots}\frac{32}{12}\frac{\bar{A}}{k}\frac{1}{k}\frac{1}{(m\pi)^6}\left(m\pi\coth(m\pi)-1\right) - \frac{32}{k}\frac{A^*}{k}\frac{1}{8}\frac{1}{3}\sum_{m=1,3,5,\ldots}\frac{1-m\pi\operatorname{csch}(m\pi)}{(m\pi)^6}\\
&= \sum_{m=1,3,5,\ldots}\frac{32}{12}6\frac{1}{k}\frac{1}{(m\pi)^6}\left(1-m\pi\coth(m\pi)\right) - \frac{32}{k}12\frac{1}{8}\frac{1}{3}\sum_{m=1,3,5,\ldots}\frac{1-m\pi\operatorname{csch}(m\pi)}{(m\pi)^6}\\
&= \frac{16}{k}\sum_{1,3,5,\ldots}\left(\frac{1-m\pi\coth(m\pi)}{(m\pi)^6}-\frac{1-m\pi\operatorname{csch}(m\pi)}{(m\pi)^6}\right)\\
&= \frac{16}{k}\sum_{m=1,3,5,\ldots}\frac{\operatorname{csch}(m\pi)-\coth(m\pi)}{(m\pi)^5}
\end{aligned}$$

Finally we have:

$$\begin{aligned}
CIR_0 &= \int_0^{\infty} Y_0(t)dt = \sum_{1,3,5,\ldots}8\int_0^{\infty}\frac{e^{-\pi^2 m^2 kt}}{(m\pi)^2}dt\\
&= \frac{8}{k}\sum_{1,3,5,\ldots}\frac{1}{(m\pi)^4} = \frac{8}{k}\frac{1}{96} = \frac{1}{12\,k}
\end{aligned}$$

Thus

$$\frac{1}{CIR_\theta}\frac{dCIR_\theta}{d\theta}|_{\theta=0} = \frac{Q}{CIR_0} = 16 \times 12 \sum_{m=1,3,5,\ldots} \frac{\text{csch}(m\pi) - \coth(m\pi)}{(m\pi)^5}$$

and using $16 \times 12 = 192$ we get our final result.

□

Proof. (of Proposition 14.)

Since firms can only change prices at times independent to their state x, writing the control problem of the firm we obtain that the solution for $x^*(t)$ is:

$$\begin{aligned} x^*(t) &= \arg\min_x \int_t^\infty e^{-(\rho+\zeta)s}\mathbb{E}\left[\left(x + \sigma W(s) + \theta X(t+s) 1_{\{t+s\leq T\}}\right)^2 \mid W(t) = 0\right] ds \\ &= -\theta(\zeta+\rho)\int_0^{T-t} e^{-(\zeta+\rho)\tau} X(t+\tau) d\tau \\ &= -\theta(\zeta+\rho)\int_t^{T} e^{-(\zeta+\rho)(s-t)} X(s) ds \text{ for all } t \geq 0 \end{aligned}$$

and thus we get the o.d.e.:

$$\frac{d}{dt}x^*(t) \equiv \dot{x}^*(t) = \theta(\zeta+\rho)X(t) + (\zeta+\rho)x^*(t) \text{ for all } t \geq 0$$

In this simple case we can solve for the dynamics of the cross-sectional average evolves $X(t)$ directly, without solving for the entire density. At time t a fraction $\zeta e^{-\zeta\tau} d\tau$ of firms have prices that have change at time $t - \tau$. At this times, they set the price to be $x^*(t-\tau)$. We also use that before the initial period, i.e. $t \leq 0$, the optimal reset price $x^*(t) = -0$, so boundary condition right after the shock is $X(0) = -1$, using the normalization $\delta = 1$. We thus have

$$X(t) = \zeta \int_0^t e^{-\zeta\tau} x^*(t-\tau) d\tau - e^{-\zeta t} \text{ for all } t \geq 0$$

which implies

$$\frac{d}{dt}X(t) \equiv \dot{X}(t) = \zeta\left(x^*(t) - X(t)\right) \text{ for all } t \geq 0$$

We can write a simple constant coefficient o.d.e. for the vector $(X(t), x^*(t))$ as

$$\begin{pmatrix} \dot{x}^*(t) \\ \dot{X}(t) \end{pmatrix} = \begin{pmatrix} \rho+\zeta & \theta(\rho+\zeta) \\ \zeta & -\zeta \end{pmatrix} \begin{pmatrix} x^*(t) \\ X(t) \end{pmatrix}$$

Letting μ be the eigenvalues of the matrix, we have $(\mu - \rho - \zeta)(\zeta + \mu) - \theta(\rho+\zeta)\zeta = 0$. For instance if $\rho = 0$ we get $(\mu+\zeta)(\mu-\zeta) = \theta\zeta^2$, with solution $\mu = \pm\zeta a$, so that $(a+1)(a-1) = \theta$ or $a^2 - 1 = \theta$, so $\mu = \pm\sqrt{1+\theta}$.

□

Online Appendix:

Price Setting with Strategic Complementarities as a Mean Field Game

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G Solution to the Heat Equation

In this appendix we collect well known results for the solution of one dimensional heat equation with given initial (or terminal) conditions, defined in a strip, with time varying boundaries, and allowing for source.

Consider the heat equation in the domain $(x,t) \in [0,1] \times \mathbb{R}_+$, with a source s, and with time boundaries given by the time varying functions A, B. In particular to solve for $w : [0,1] \times \mathbb{R}_+ \to \mathbb{R}$ given parameter $k > 0$, $\iota \geq 0$, source $s : [0,1] \times \mathbb{R}_+ \to \mathbb{R}$, space boundary at time zero $f : [0,1] \times \mathbb{R}$, and value at the boundaries given by $a, b : \mathbb{R}_+ \to \mathbb{R}$ satisfying:

$$
\begin{aligned}
0 &= -w_t(x,t) - \iota w(x,t) + k w_{xx}(x,t) + s(x,t) \text{ all } x \in [0,1] \text{ and } t > 0 \\
w(x,0) &= f(x) \text{ all } x \in [0,1] \\
w(0,t) &= A(t) \text{ all } t > 0 \text{ and} \\
w(1,t) &= B(t) \text{ all } t > 0
\end{aligned}
$$

Proposition 16. The solution for the KFE equation for w is given by:

$$
\begin{aligned}
w(x,t) &= r(x,t) + \sum_{j=1}^{\infty} a_j(t) \varphi_j(x) \text{ all } x \in [0,1] \text{ and } t > 0 \text{ where} \\
r(x,t) &= A(t) + x[B(t) - A(t)] \text{ all } x \in [0,1],\ t > 0
\end{aligned}
$$

and where for all $j = 1, 2, \ldots$ we have:

$$
\begin{aligned}
\varphi_j(x) &= \sin(j\pi x) \text{ for all } x \in [0,1]\,,\ \langle \varphi_j, h \rangle \equiv \int_0^1 h(x)\varphi_j(x)dx \\
a_j(t) &= a_j(0)e^{-\lambda_j t} + \int_0^t q_j(\tau) e^{\lambda_j(\tau - t)} d\tau \text{ all } t > 0\,, \\
q_j(t) &= \frac{\langle \varphi_j\,,\ s(\cdot,t) - r_t(\cdot,t) - \iota r(\cdot,t) \rangle}{\langle \varphi_j, \varphi_j \rangle} \text{ all } t > 0 \\
\lambda_j &= \iota + (j\pi)^2 k \ \text{ and } a_j(0) = \frac{\langle \varphi_j\,,\ f - r(\cdot,0) \rangle}{\langle \varphi_j, \varphi_j \rangle}\,.
\end{aligned}
$$

The proof can be done by verifying that the equation hold at the boundaries, that for $t > 0$ the p.d.e. holds in the interior since

$$a_j'(t) = -\lambda_j a_j(t) + q_j(t) \text{ for all } t > 0 \text{ and } j = 1, 2, \dots$$

and since $\{\varphi_j(x)\}$ form an orthogonal bases for functions on $\{h : [0,1] \to \mathbb{R}\}$, and finally that the boundary holds at $t = 0$ for all x.

Consider now the KBE equation, which only changes the sign of the time derivative, the range of time, and the time at which the space boundary condition holds, so $w : [0,1] \times [0,T] \to \mathbb{R}$, where:

$$\begin{aligned}
0 &= w_t(x,t) - \iota w(x,t) + k w_{xx}(x,t) + s(x,t) \text{ all } x \in [0,1] \text{ and } t > 0 \\
w(x,T) &= f(x) \text{ all } x \in [0,1], \\
w(0,t) &= A(t) \text{ all } t \in [0,T], \text{ and} \\
w(1,t) &= B(t) \text{ all } t \in [0,T]
\end{aligned}$$

Proposition 17. The solution for the KBE for w is given by:

$$\begin{aligned}
w(x,t) &= r(x,t) + \sum_{j=1}^{\infty} a_j(t)\varphi_j(x) \text{ all } x \in [0,1] \text{ and } t \in [0,T] \text{ where} \\
r(x,t) &= A(t) + x[B(t) - A(t)] \text{ all } x \in [0,1],\ t \in [0,T]
\end{aligned}$$

and where for all $j = 1, 2, \dots$ we have:

$$\begin{aligned}
\varphi_j(x) &= \sin(j\pi x) \text{ for all } x \in [0,1]\,,\ \langle \varphi_j, h\rangle \equiv \int_0^1 h(x)\varphi_j(x)dx \\
a_j(t) &= a_j(T)e^{-\lambda_j(T-t)} + \int_t^T q_j(\tau)e^{\lambda_j(t-\tau)}d\tau \text{ all } t \in [0,T)\,, \\
q_j(t) &= \frac{\langle \varphi_j\,,\, s(\cdot,t) + r_t(\cdot,t) - \iota r(\cdot,t)\rangle}{\langle \varphi_j, \varphi_j\rangle} \text{ all } t \in [0,T) \\
\lambda_j &= \iota + (j\pi)^2 k \text{ and } a_j(T) = \frac{\langle \varphi_j\,,\, f - r(\cdot,T)\rangle}{\langle \varphi_j, \varphi_j\rangle}\,.
\end{aligned}$$

As in the previous case the proof can be done by verifying that the equation hold at the boundaries, that for $t \in [0,T]$ the p.d.e. holds in the interior since

$$-a_j'(t) = -\lambda_j a_j(t) + q_j(t) \text{ for all } t \in [0,T) \text{ and } j = 1, 2, \dots$$

Note that $q_j(t)$ and $a_j(t)$ are also defined differently than for the KFE.

H Additional material

H.1 Variational Inequality

In general, we should write the problem of the firm as solving the following variational inequalities:

$$\rho u(x,t) = \tag{81}$$
$$\min\left\{u_t\left(x,t\right) + \frac{\sigma^2}{2}u_{xx}(x,t) + F(x,X(t)) + \zeta\left(\min_{x'} u(x',t) - u(x,t)\right),\ \rho\left(\psi + \min_{x'} u(x',t)\right)\right\}$$

which must hold for all $t \in [0,T]$ and for all x. We can define $x^*(t) = \arg\min_x u(x,t)$. Note that this formulation does not assume that $u(\cdot,t)$ is once differentiable, nor that range of inaction is given by a single interval. If the value function is well behaved, we can write equation (81) as the classical formulation which we described above, i.e. as the p.d.e equation (3) and the boundary conditions equation (6)-equation (7).

H.2 Equations for the $\zeta = 0$ case.

If $\zeta = 0$, the stationary distribution $\tilde{m}$ given by a triangular tent-map:

$$\tilde{m}(x) = \begin{cases} \frac{2}{\bar{x}_{ss}-\underline{x}_{ss}} - (x - x^*_{ss})\frac{2}{(\bar{x}_{ss}-\underline{x}_{ss})(\bar{x}_{ss}-x^*_{ss})} & \text{for } x \in [x^*_{ss}, \bar{x}_{ss}] \\ \frac{2}{\bar{x}_{ss}-\underline{x}_{ss}} + (x - \underline{x}_{ss})\frac{2}{(\bar{x}_{ss}-\underline{x}_{ss})(x^*_{ss}-\underline{x}_{ss})} & \text{for } x \in [\underline{x}_{ss}, x^*_{ss}] \end{cases} \tag{82}$$

H.3 Kernel evaluation on the diagonal.

Consider the case where $\zeta = 0$. The limit of $K(s,t)$ for $0 < t = s < \infty$ gives

$$\begin{aligned} |K(t,t)| &= \left| 4\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\left[\bar{A} - A^*\left(-1\right)^{j+i}\right]\frac{1-e^{-(j\pi)^2kt-(i\pi)^2kt-\eta^2kt}}{(j\pi)^2+(i\pi)^2+\eta^2}\right| \\ &\geq 4|\bar{A}|\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\frac{1-e^{-(j\pi)^2kt-(i\pi)^2kt-\eta^2kt}}{(j\pi)^2+(i\pi)^2+\eta^2} \\ &= -4|\bar{A}|\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\frac{e^{-(j\pi)^2kt-(i\pi)^2kt-\eta^2kt}}{(j\pi)^2+(i\pi)^2+\eta^2} + 4|\bar{A}|\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\frac{1}{(j\pi)^2+(i\pi)^2+\eta^2} \end{aligned}$$

The first term of the last equality converges for $t > 0$, and j integer since

$$\frac{e^{-(j\pi)^2kt-(i\pi)^2kt-\eta^2kt}}{(j\pi)^2+(i\pi)^2+\eta^2} < \frac{e^{-\pi^2ktj}}{(i\pi)^2}$$

and so

$$\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\frac{e^{-(j\pi)^2kt-(i\pi)^2kt-\eta^2kt}}{(j\pi)^2+(i\pi)^2+\eta^2} < \sum_{j=1}^{\infty}e^{-\pi^2ktj}\sum_{i=1}^{\infty}\frac{1}{(i\pi)^2} = \frac{1}{1-e^{-\pi^2kt}}\frac{1}{6}$$

The second term of the last equality diverges since the $\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\frac{1}{(j\pi)^2+(i\pi)^2+\eta^2}$ diverges to $+\infty$.

I The demand system with Kimball's aggregator

Basic Set up. Let $\mathcal{C}(c_1, c_2, \ldots, c_K)$ be a homogeneous of degree one aggregator defined implicitly by:

$$1 = \sum_{s=1}^{K} \Upsilon\left(\frac{c_s}{\mathcal{C}}\right)\frac{1}{K} \tag{83}$$

Note that the derivative of $\mathcal{C}$ with respect to c_k is given by:

$$\frac{\partial \mathcal{C}}{\partial c_k}(c_1, c_2, \ldots, c_K) = \frac{\Upsilon'\left(\frac{c_k}{\mathcal{C}}\right)\frac{1}{K}}{\sum_{s=1}^{K}\Upsilon'\left(\frac{c_s}{\mathcal{C}}\right)\frac{c_s}{\mathcal{C}}\frac{1}{K}}$$

Household maximization problem. We are interested in solving:

$$U(p_1, p_2, \ldots, p_K, E) \equiv \max_{\{c_1, c_2, \ldots, c_K\}} \mathcal{C}(c_1, c_2, \ldots, c_K) + \lambda\left(E - \sum_{s=1}^{K} p_s c_s \frac{1}{K}\right) \tag{84}$$

The first order conditions can be written as:

$$\lambda p_k \frac{1}{K} = \frac{\partial \mathcal{C}}{\partial c_k} = \frac{\Upsilon'\left(\frac{c_k}{\mathcal{C}}\right)\frac{1}{K}}{\sum_{s=1}^{K}\Upsilon'\left(\frac{c_s}{\mathcal{C}}\right)\frac{c_s}{\mathcal{C}}\frac{1}{K}} \text{ for } k = 1, 2, \ldots, K$$

or

$$\lambda p_k = \frac{\Upsilon'\left(\frac{c_k}{\mathcal{C}}\right)}{\sum_{s=1}^{K}\Upsilon'\left(\frac{c_s}{\mathcal{C}}\right)\frac{c_s}{\mathcal{C}}\frac{1}{K}} \text{ for } k = 1, 2, \ldots, K \tag{85}$$

We can write the expenditure as:

$$E = \sum_{s=1}^{K} p_s c_s \frac{1}{K} = \frac{1}{\lambda}\sum_{s=1}^{K}\frac{\partial \mathcal{C}}{\partial c_s} c_s \tag{86}$$

Define the (relative) demand function for good k

$$q_k \equiv \frac{c_k}{\mathcal{C}(c_1, c_2, \ldots, c_K)}$$

Thus we can write the solution to the maximization problem above as $K + 2$ variables

$\{q_1, q_2, \ldots, q_K, \mathcal{C}, \lambda\}$ solving the following $K+2$ equations:

$$\lambda p_k = \frac{\Upsilon'(q_k)}{\sum_{s=1}^{K} \Upsilon'(q_s)\, q_s \frac{1}{K}} \text{ for } k = 1, 2, \ldots, K$$
$$1 = \sum_{s=1}^{K} \Upsilon(q_s) \frac{1}{K} \quad , \quad \mathcal{C} = \frac{E}{\sum_{s=1}^{K} p_s q_s \frac{1}{K}}$$

We note that this corresponds to the following continuum case as $K \to \infty$:

$$\lambda p_k = \frac{\Upsilon'(q_k)}{\int_0^1 \Upsilon'(q_s)\, q_s ds} \text{ for } k \in [0,1] \quad , \quad 1 = \int_0^1 \Upsilon(q_s)\, ds \quad , \quad \mathcal{C} = \frac{E}{\int_0^1 p_s q_s ds}$$

Symmetric limit case. Returning to the finite case, we are interested in the case where $k=1$ has a price $p_1 = p$ and $p_2 = p_3 = \cdots = p_K = P$ for the rest of the goods. In this case we will let $q_1 = q$ and $q_k = \bar{q}$ for $k = 2, 3, \ldots, K$, and we can write the system as

$$1 = \Upsilon(q) \frac{1}{K} + \Upsilon(\bar{q}) \frac{K-1}{K} \quad , \quad \mathcal{C} = \frac{E}{pq\frac{1}{K} + P\bar{q}\frac{K-1}{K}}$$
$$\lambda p = \frac{\Upsilon'(q)}{\Upsilon'(q)\, q\frac{1}{K} + \Upsilon'(\bar{q})\, \bar{q}\frac{K-1}{K}} \text{ and } \quad , \quad \lambda P = \frac{\Upsilon'(\bar{q})}{\Upsilon'(q)\, q\frac{1}{K} + \Upsilon'(\bar{q})\, \bar{q}\frac{K-1}{K}}$$

And if we let $K \to \infty$ we obtain the simple recursive system:

$$1 = \Upsilon(\bar{q}) \quad , \quad \lambda P \bar{q} = 1 \quad , \quad \mathcal{C} = \frac{E}{P\bar{q}} \text{ and } \quad , \quad \lambda p = \frac{\Upsilon'(q)}{\Upsilon'(\bar{q})\, \bar{q}}$$

Which we can solve as:

$$1 = \Upsilon(\bar{q}) \implies \bar{q} = \Upsilon^{-1}(1) \, , \, \lambda = 1/(P\bar{q}) \text{ and } \mathcal{C} = \frac{E}{P\Upsilon^{-1}(1)}$$
$$\Upsilon'(q) = \frac{p}{P}\Upsilon'(\bar{q}) \implies q = (\Upsilon')^{-1}\left(\frac{p}{P}\Upsilon'\left(\Upsilon^{-1}(1)\right)\right)$$

Preference shocks. Finally, we introduce preference shocks A_s in each good to have: $1 = \sum_{s=1}^{K} \Upsilon\left(\frac{c_s}{\mathcal{C}}, A_s\right) \frac{1}{K}$. In particular we assume the following multiplicative form:

$$1 = \sum_{s=1}^{K} \Upsilon(q_s A_s) \frac{1}{K}$$

This implies:

$$\lambda \frac{p_k}{A_k} = \frac{\Upsilon'(q_k A_k)}{\sum_{s=1}^{K} \Upsilon'(q_s A_s)\, A_s q_s \frac{1}{K}} \text{ for } k = 1, 2, \ldots, K \quad , \quad \mathcal{C} = \frac{E}{\sum_{s=1}^{K} p_s q_s \frac{1}{K}} \tag{87}$$

One case of interest is to consider $A's$ such that $p_s = PA_s$ so the shocks happens to be proportional to the prices. In this case we can write $Q_s \equiv q_s A_s$ and get:

$$\lambda P = \frac{\Upsilon'(Q_k)}{\sum_{s=1}^{K} \Upsilon'(Q_s) Q_s \frac{1}{K}} \text{ for } k = 1, 2, \ldots, K$$
$$1 = \sum_{s=1}^{K} \Upsilon(Q_s) \frac{1}{K} \text{ and } \quad , \quad \mathcal{C} = \frac{E}{P \sum_{s=1}^{K} Q_s \frac{1}{K}}$$

which clearly has a solution with $Q_k = Q$.

Let us consider the case where $A_1 = A$ is arbitrary and $p_k = PA_k$ for $k = 2, \ldots, K$, as before. We have

$$1 = \Upsilon(qA) \frac{1}{K} + \Upsilon(Q) \frac{K-1}{K} \quad , \quad \mathcal{C} = \frac{E}{pq\frac{1}{K} + PQ\frac{K-1}{K}}$$
$$\lambda \frac{p}{A} = \frac{\Upsilon'(qA)}{\Upsilon'(qA)\, qA\frac{1}{K} + \Upsilon'(Q)\, Q\frac{K-1}{K}} \text{ and } \quad , \quad \lambda P = \frac{\Upsilon'(Q)}{\Upsilon'(qA)\, qA\frac{1}{K} + \Upsilon'(Q)\, Q\frac{K-1}{K}}$$

whose limit as $K \to \infty$ is:

$$1 = \Upsilon(Q) \quad , \quad \mathcal{C} = \frac{E}{PQ} \quad , \quad \lambda \frac{p}{A} = \frac{\Upsilon'(qA)}{\Upsilon'(Q)\, Q} \text{ and } \quad , \quad \lambda P Q = 1$$

The demand function can be written as:

$$\Upsilon'(qA) = \frac{p}{PA} \Upsilon'\left(\Upsilon^{-1}(1)\right) \implies \qquad q = \frac{1}{A} (\Upsilon')^{-1} \left(\frac{p}{PA} \Upsilon'\left(\Upsilon^{-1}(1)\right)\right)$$

and letting $c_1 = c = q\,\mathcal{C}$

$$c = \frac{\mathcal{C}}{A} (\Upsilon')^{-1} \left(\frac{p}{PA} \Upsilon'\left(\Upsilon^{-1}(1)\right)\right) = \frac{1}{\Upsilon^{-1}(1)} \frac{E}{PA} (\Upsilon')^{-1} \left(\frac{p}{PA} \Upsilon'\left(\Upsilon^{-1}(1)\right)\right) \tag{88}$$

The firm's real profit function. Let the nominal wage W be the numeraire, Z be the firm's real marginal cost and $p = \hat{p}A$ be the firm's price. We can write the firm's profit function as

$$\Pi(\hat{p}, P, Z) = c \cdot \left(\frac{p}{W} - Z\right) = \frac{E}{\Upsilon^{-1}(1) P} (\Upsilon')^{-1} \left(\frac{\hat{p}}{P} \Upsilon'\left(\Upsilon^{-1}(1)\right)\right) \left(\frac{\hat{p}}{W} - \frac{Z}{A}\right)$$

where the second equality uses equation (88). If we assume that $Z = A$, i.e. that preference shocks are proportional to marginal cost shocks, then we have that each firm solves

$$\max_{\hat{p}} \Pi(\hat{p}, P) = \frac{E}{\Upsilon^{-1}(1) P W} \left[\max_{\hat{p}} \ D\left(\frac{\hat{p}}{P}\right)(\hat{p} - W) \right] \tag{89}$$

$$\text{where} \ \ \hat{p} \equiv p/A \ , \ \text{ and } \ D\left(\frac{\hat{p}}{P}\right) \equiv (\Upsilon')^{-1}\left(\frac{\hat{p}}{P} \, \Upsilon'\left(\Upsilon^{-1}(1)\right)\right) \tag{90}$$

so the profit of the individual firm does not depend on Z.

The first order condition for profit maximization gives

$$\Pi_1(\hat{p}, P) = D'\left(\frac{\hat{p}}{P}\right)\frac{\hat{p} - W}{P} + D\left(\frac{\hat{p}}{P}\right) = 0 \tag{91}$$

evaluated at a symmetric equilibrium where $\hat{p} = P$ we have that the optimal price $\hat{p}^*$ solves

$$D'(1)\,\frac{\hat{p}^* - W}{\hat{p}^*} + D(1) = 0$$

or that the profit maximizing markup, $\mu = \hat{p}^*/W$, satisfies $\frac{\mu - 1}{\mu} = \frac{\hat{p}^* - W}{\hat{p}^*} = \frac{D(1)}{-D'(1)}$.

Comparative statics for the optimal pricing. We want to characterize how the optimal price u varies as a function of the aggregate price P around an optimum.

Recall the first order condition

$$\Pi_1(\hat{p}, P) = D'\left(\frac{\hat{p}}{P}\right)\frac{\hat{p} - W}{P} + D\left(\frac{\hat{p}}{P}\right) = 0 \tag{92}$$

We first notice that the aggregate expenditure E/P enters the profit function multiplicatively in equation (89). This implies that changes in aggregate expenditure will have no first order effect on the price setting choice of the firm (around the optimal choice), or that $\left.\frac{\partial \hat{p}}{\partial E}\right|_{\hat{p}^*} = 0$.

From now on we omit the argument of the function $D(\cdot)$ and simply write D. From the first order condition $\Pi_1(\hat{p}, P) = 0$ we have that

$$\frac{\partial \hat{p}}{\partial P}\frac{P}{\hat{p}} = -\frac{\Pi_{12}}{\Pi_{11}}\frac{P}{\hat{p}} \tag{93}$$

Compute the cross partial derivative

$$\Pi_{12} = -D''\left(\frac{\hat{p} - W}{P}\frac{\hat{p}}{P^2}\right) - D'\left(\frac{\hat{p} - W}{P^2} + \frac{\hat{p}}{P^2}\right) = -D''\left(\frac{D}{-D'}\frac{\hat{p}}{P^2}\right) - D'\left(\frac{D}{-D'P} + \frac{\hat{p}}{P^2}\right) \tag{94}$$

where the second equality uses equation (92).

Let us define the own price elasticity

$$\eta \equiv -\frac{\partial D}{\partial \hat{p}} \frac{\hat{p}}{D} = -\frac{D'}{D} \frac{\hat{p}}{P} \tag{95}$$

Using this definition we compute

$$\frac{\partial \eta}{\partial \hat{p}} = \frac{1}{DP} \left(-D'' \frac{\hat{p}}{P} - D' + \frac{(D')^2}{D} \frac{\hat{p}}{P} \right) \tag{96}$$

Using the definition in equation (95) we rewrite equation (94) as

$$\frac{P}{\hat{p}} \Pi_{12} = \frac{1}{\eta \ P} \left(-D'' \frac{\hat{p}}{P} - D' + \frac{(D')^2}{D} \frac{\hat{p}}{P} \right) = D \ \frac{1}{\eta} \ \frac{\partial \eta}{\partial \hat{p}} \tag{97}$$

where the second equality uses the expression in equation (96).

We have

$$\frac{\partial \hat{p}}{\partial P} \frac{P}{\hat{p}} = \frac{D}{-\Pi_{11}} \ \frac{1}{\eta} \ \frac{\partial \eta}{\partial \hat{p}}$$

Note for instance that if the demand system is CES, so that the function D is a power function, the elasticity eta has a zero elasticity w.r.t. P at the symmetric equilibrium where $\hat{p} = P$, or $\frac{\partial \eta}{\partial \hat{p}} = 0$ as can be readily verified from equation (96). This implies that the optimal firm price u is unresponsive to the aggregate price P at the symmetric equilibrium.