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JEL Classification: L13, M37, L65

Keywords: ordered search, product heterogeneity, position externalities, optimal and equilibrium rankings, generalized second price auction, position auction

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Search Direction: Position Externalities and Position Auction Bias*

Simon P. Anderson† and Regis Renault‡

November 10, 2021

Abstract

We formulate a tractable model of pricing under directed search with heterogeneous firm demands. Demand height and width drive bids in a position auction and enable us to bridge insights from the ordered search literature to those in the position auction literature. Equilibrium pricing implies that the marginal consumer's surplus decreases down the search order, so consumers optimally follow the firms' position ordering. A firm suffers from "business stealing" by firms that precede it and "search appeal" from subsequent firms. We find rankings that achieve the maximal joint profit, social welfare, or consumer surplus by constructing firm-specific scores. A generalized second price auction for positions endogenizes equilibrium orders and bids are driven by position externalities that impact incremental profit from switching positions. The joint profit maximization order is upheld when firm heterogeneity concerns mostly demand height. But the consumer welfare order is robust when firms differ mostly over demand width.

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1 Introduction

Internet search is guided by ad positions. These slots are allocated through firms' bids. The extant literature on ordered search has not fully integrated the role of bidding for positions, while the literature on position auctions has not gone deep into firm pricing when search is ordered. The ordered search literature shows that otherwise symmetric firms expecting to be searched earlier charge lower prices so consumers indeed want to start search at those firms. Armstrong (2017) is an invaluable synthesis and buildout of that literature. By contrast, the research on position auctions focuses on how heterogeneous firms bid *per click* to obtain a favorable position in a directed search order. However, it treats prices as exogenous so it does not address some critical conundrums that the existing results on competition with ordered search inevitably raise. How does firm heterogeneity modify pricing to induce consumers to follow the specified search order? Does a firm selling a product which is less appealing to sample drop its price enough to compensate for this disadvantage when it is placed early in the search order? If early firms price low, all other things equal, how can consumers expect them to bid sufficiently high per click to be in such favorable positions? The present paper proposes a comprehensive analysis of these issues while considerably expanding the dimensions of firm heterogeneity that can be accommodated.

Whether a firm will bid more than its rivals to be searched early is impacted by position externalities, which are the effects that a firm has on other firms' profits if it changes its position in the search order. There is an obvious traffic externality, or *business stealing* (Chen and He, 2011), which results from a firm moving ahead in the search order siphoning off customers (who stop there) from following firms. But there is a more subtle reverse impact for firms that are demoted: there is less competition due to the reduced number of firms remaining to be searched. To illustrate, consider the symmetric case, where firms searched later charge higher prices. When a firm moves ahead in the search order, firms jumped over charge higher prices because they are one step closer to the end so it is less attractive for consumers to search on.[1] We call this the *search appeal* externality. These

[1] If all prices are the same as for instance in Athey and Ellison (2011), and if the optimal search rule is

externalities are nuanced when firms have different demand profiles. For example, advancing a popular firm hurts those superseded more than advancing a niche firm. Similarly, the search appeal effect is stronger when consumers get a higher expected surplus with the firm. Our setting provides a precise measure of these externalities and how they are affected by the properties of demand for the various products. It is stripped down to basic component parts that capture the complex manner in which these externalities impact pricing, bidding, and firms' and consumers' welfare. We account for multiple dimensions of firm heterogeneity by isolating factors that affect their pricing, which are expressed in *demand height* and factors that affect their sales, expressed in *demand width*.

The first dimension of demand height we label the product's *quality*: all other things equal, a higher quality product is sold at a higher price in any given slot. A firm's profit is impacted directly by its product's quality but others are not, so this is a private value dimension. The second dimension arises because the pricing of a product is also affected by the distribution of valuations for competing products that are positioned later. In our model, the search appeal of those products is precisely measured by the reservation utility associated with sampling the product as characterized in Weitzman (1979). Increasing firm's search appeal brings down the prices of all firms that are positioned earlier in the search order. We show that the equilibrium price of a firm increases in step with its product's quality and decreases in the cumulative search appeal of all the products positioned later. This pricing behavior yields a higher surplus for the marginal consumer (who is indifferent between buying and searching on) at earlier slots. As a result, it is always optimal for a consumer to follow the directed search order: pricing ensures that this search order is optimal according to the characterization of Weitzman (1979), no matter what the order is.

Demand width is characterized by the product's *market potential* which is the probability that a consumer is interested in buying the product. The product's market potential has a direct impact on the firm's profit by determining the quantity sold to consumers who search up to that firm: a firm positioned in a given slot sells more if its product has a higher market

myopic as in Weitzman (1979), then the number of remaining firms is irrelevant to the consumer's search decision as long as there is at least one left.

potential. Market potential also induces a negative externality on firms farther down in the search order: the higher the market potential of preceding products, the lower the sales of each firm.

Using the three parameters of product heterogeneity (quality, search appeal, and market potential), we can construct product-specific scores to characterize an optimal ordering of firms to maximize either total profit or total welfare or total consumer surplus. Total profit maximization is achieved by prioritizing products with larger demand height parameters and lower demand width. This mitigates the adverse effect of the search appeal and business stealing externalities by ensuring that the prices of early firms, which sell more, are as high as possible while the sales of later firms, which extract more surplus, are as large as possible. By contrast, consumers like low prices early, at slots where they are more likely to buy, and a low probability of purchasing a product positioned late, which extracts a large surplus from them. As a result, they prefer products in top slots to have limited demand height and substantial demand width.

Consider now a position auction with slots going to firms in the order of their bids per click and each firm paying the bid of the next highest bidder. We analyze such a *generalized second price auction* assuming bidders have complete information. Whether such an auction mechanism can achieve joint surplus maximization for the auction participants (joint profit maximization) is a priori ambiguous due to the two position externalities. We show that when product heterogeneity only concerns demand height, if the joint profit maximizing order involves ranking products according to the decreasing order of product qualities, it is always an equilibrium outcome. This is a generalization of results in previous literature that show this is true in pure private value settings, corresponding in our model to the case where only product qualities differ.[2] There are many other equilibrium outcomes if product demands are similar enough. We show however that, if qualities are sufficiently dissimilar, then the equilibrium ordering cannot be too different from that which maximizes total profit.

By contrast, when there is sufficient firm heterogeneity over demand width, maximization

[2] See Varian (2007) and Edelman, Ostrovsky and Schwartz (2007). We also show that the equilibrium we derive satisfies the "no envy" refinement used by these authors.

of total profit is no longer an equilibrium. Indeed, it would require that firms with low market potential bid to be early in the order despite their low value for being in a top slot: the value of the additional clicks they can obtain in such positions is low because these clicks are less likely to be converted into purchases. Furthermore the reverse order, where products with large market potential come first, can always be sustained as an equilibrium outcome. This is typically the order favored by consumers.

Our paper integrates and enriches two streams of literature. An important insight from the analysis of sequential ordered search is that when prices are endogenously chosen by competing sellers there can be equilibria where firms searched earlier are both more attractive to search and earn more profit, so they might be willing to pay for such prominence.[3] A first step forward is made by Armstrong, Vickers, and Zhou (2009),[4] with only one prominent firm and the remaining sellers searched randomly. The setting where all firms are searched in order is explored by Zhou (2011). However, these papers assume i.i.d. consumer tastes for products. We allow for asymmetric match distributions and show that the marginal consumer's surplus is lower at firms searched later, which implies prices increase along the search order when qualities are the same. Asymmetries in product demands are considered in Armstrong, Vickers, and Zhou (2009) and Song (2017). The former allow for heterogeneous product qualities in an extension of their basic setting with only one prominent firm while the latter considers products with asymmetric taste heterogeneity in a duopoly. Our setting allows for multiple dimensions of product heterogeneity and deals with ordered search among any number of competitors. Choi, Dai, and Kim (2018) and Haan, Moraga-Gonzalez, and Petrikaite (2018) study how firms can use posted prices to direct search. In that setting, different consumers follow different search orders based on some prior information about how much they like the various products but firms are symmetric in the aggregate and charge the same price.

[3]Previous literature on sequential search and competition focused on random search and is surveyed in Anderson and Renault (2018). The ordered search models dicussed here build on the setting introduced by Wolinsky (1986) and Anderson and Reenault (1999) with consumer search and horizontal product differentiation.

[4]See Arbatskaya (2007) for an earlier contribution with homogenous products.

The position auctions literature has made valuable progress on the auction side of the slate while suppressing the market competition side.[5] A first group of articles studies the properties of generalized second price auctions with private values and do not account for consumer behavior[6] or surplus. Varian (2007) and Edelman, Ostrovsky, and Schwartz (2007) focus on the existence of an efficient equilibrium, which maximizes the firms' surplus. Gomes and Sweeney (2014) show that such an equilibrium may fail to exist in a sealed bid auction with asymmetric information. In our setting, a joint profit maximizing equilibrium may not exist even though bidders have complete information. Athey and Ellison (2011) use a setting very similar to that of Chen and He (2011) to look at auctions with asymmetric information and then optimal auction design, while assuming that consumers go on searching until a "need" is fulfilled. This yields a business stealing externality similar to ours. However, there is no externality from search appeal because both price and the conversion rate (the probability that a consumer buys conditional on reaching the firm) are exogenous. Furthermore, these two papers establish that it is optimal for consumers to search in the order that arises from the auction because firms with a higher probability of meeting a need bid more. By contrast, in our model it is optimal for consumers to search in the order because of the pricing behavior they expect.

We also contribute to the analysis of auctions with externalities where the bidder willingness to pay is determined in equilibrium and depends on which other bidders are likely to win. That literature has considered auctions for a single object (e.g. Jehiel and Moldovanu, 1996a, 1996b) whereas we allow for any number of slots to be allocated.

Section 2 describes our search and competition environment while optimal ranking scores for maximization of total industry profit, social welfare, and consumer surplus are derived in Section 3. Finally we consider when allocation rules such as auctions used on internet platforms might achieve total profit maximization in Section 4. Section 5 concludes.

[5] Although Chen and He (2011) have endogenous prices, all firms end up charging the monopoly price due to a standard mechanism *à la* Diamond (1971).

[6] An exception is Gomes (2014) who endogenizes consumer click behavior in a two-sided market setting.

2 Market equilibrium

2.1 Competition with ordered search

We first describe a model of oligopolistic competition with ordered consumer search, and find firms' equilibrium prices. There are n firms with Firm i selling product i, with zero production costs. Consumers have unit demand with independent valuations for the n competing products. Let $F_i(v)$ denote the distribution function of a consumer's valuation with product i, $i = 1, ..., n$. We break down $F_i(v)$ into three component parts.

We are thinking of situations in which consumers idiosyncratically either like the product, or they do not, but they have heterogeneous valuations if they like it. For example, a consumer may reject out-of-hand several styles of jacket, but the lowest valuation for a jacket that she will countenance taking home to take up space in the closet is quite high. Nonetheless, there may be several jackets that could interest her if she knows their details.[7]

Let then the probability of rejecting the product outright (regardless of price) be γ_i. Lower γ_i products are more popular, per se. Second, let q_i be the lowest valuation associated to product i, conditional on it being desired. As we elaborate below, we shall assume that q_i is sufficiently large that all consumers who have some appreciation for a product end up buying it in equilibrium when they come across it. The valuation for product i, v_i, therefore has support $\{0\} \cup S_i$, where $\min S_i = q_i$ and it is further assumed that v_i is bounded: let then $B_i \equiv \max_{v_i \in S_i} v_i - q_i < \infty$. The distribution function of v_i is denoted F_i and, to facilitate the exposition, it is assumed to be differentiable on S_i, where f_i denotes the corresponding pdf.[8]

Distribution functions F_i are common knowledge but neither consumers nor firms know the realizations of v_i. Consumers may however learn these realizations through search. Search is sequential, with cost $s > 0$ per search. Searching a firm reveals both its price and the consumer's valuation of the product searched. Searching a firm is necessary for a

[7] This set-up contrasts with Athey and Ellison (2011) in which prices are fixed, and with Chen and He (2011) where each consumer has at most one product that could interest her, regardless of prices.

[8] The analysis goes through without assuming the existence of a p.d.f. on S_i and can accommodate any specification of F_i on S_i including atoms in the distribution.

consumer to be able to buy its product. As is standard in sequential search models the consumer may always purchase from any previously searched firm at no extra search cost. Buying none of the n products nets an outside value utility of zero.[9]

The timing is that firms simultaneously choose prices and consumers choose their search rules based on match values and prices they have found out so far, the distributions at other firms and the prices expected there. We seek a Perfect Bayesian Equilibrium at which search is ordered, meaning that all consumers follow the same search order. Because we have not specified any systematic difference between the n firms (i.e. the match distribution for Firm $i = 1, ..., n$ can be any distribution satisfying the properties described above), there is no loss of generality in assuming that this order is from Firm 1 to Firm n and we then check that this order is indeed optimal for consumers. In our equilibrium, each firm optimally prices so as to sell to all consumers that reach it with a positive draw for its product. This pricing property means that Firm i renders any consumer drawing q_i with it indifferent between buying from i and searching further. Thus a consumer has zero willingness to pay for any product encountered before Product i, and never goes back (as long as prices are strictly positive, which will hold true in equilibrium). We show below that such an equilibrium exists, provided that for all i q_i is large enough and the density f_i is strictly positive at q_i.

To derive the equilibrium, we engage the powerful results of Weitzman (1979) to describe optimal consumer search. He shows that remaining search options can be ordered by simple myopic reservation values such that a consumer searches the option with the highest reservation value next, or else stops searching if she already holds a utility above the highest value (and buys the best option held or buys nothing). These reservation values are summary statistics for options, which set equal the expected costs and benefits of an additional search. They are therefore determined independently of what has already been discovered.

A version of these reservation values is a key ingredient of pricing analysis. For each product we define Δ_i as per standard search analysis as the value that equates the expected

[9] It can be thought of as a continuation value (searching the organic links of a search engine after searching the sponsored links, or purchasing a product off line) for consumers who have searched through all the n firms: equilibrium prices are simply shifted down by the continuation value.

upside gain to the search cost, so

$$\int_{q_i+\Delta_i}^{q_i+B_i} (v - q_i - \Delta_i) dF(v) = s. \tag{1}$$

Integrating by parts, $\int_{q_i+\Delta_i}^{q_i+B_i} [1 - F_i(v)]dv = s$. The LHS strictly increases from 0 to $+\infty$ as Δ_i drops from B_i to $-\infty$. Hence a unique Δ_i always exists. Graphically, the value of Δ_i is determined from the value of the critical valuation $q_i + \Delta_i$ for which the area under the demand curve $(1 - F_i(p))$ equals the search cost s. That is, if the consumer currently held a utility value of $q_i + \Delta_i$ then searching Firm i would be a break-even prospect if it were expected to charge a zero price. (This is illustrated for Firm $i + 1$ and Δ_{i+1} in Figure 1.) With this interpretation, the values $q_i + \Delta_i$, $i = 1, ..., n$, are the reservation values that characterize consumer optimal search behavior if prices are all zero: from the analysis in Weitzman (1979), a consumer should always choose to search next the remaining alternative with the highest reservation utility $q_i + \Delta_i$ or else stop searching if she already holds a higher utility. If prices were equal for all products, consumers would choose to search in the order in which we have indexed the firms only if $q_i + \Delta_i \geq q_{i+1} + \Delta_{i+1}$, $i = 1, ..., n - 1$.

Assume $\Delta_i > 0$. This implies that if a consumer holds valuation q_i with product i and contemplates searching Firm $i+1$, and if price differences were to exactly match base quality differences (i.e., if $p_i - p_{i+1} = q_i - q_{i+1}$), then her expected benefit from searching Firm $i+1$ would be strictly positive.

For our characterization analysis it is useful that we can vary parameters γ_i, q_i, and Δ_i independently from each other. A change in q_i is merely a shift up or down of the support of strictly positive valuations, S_i, so it can be done independently of the value of γ_i (the probability that the product is not desired.) We show in the Appendix that the LHS of (1) can be rewritten appropriately so that it is possible to modify F_i and have Δ_i vary from 0 to $+\infty$ independently of the values specified for q_i and γ_i.

We now move to characterizing the equilibrium pricing and search order.

2.2 Pricing

Consider a Firm $i < n$, and suppose it prices so that even a consumer who draws a match q_i with its product chooses not to search on. Because consumers follow an optimal search order in equilibrium, they compare the utility they currently hold with the highest reservation value among the remaining firms. In an equilibrium where consumers search in order from 1 to n, this highest reservation value should be that for Firm $i+1$. In other words, as per the Weitzman (1979) analysis, the optimal search rule is myopic and only considers the costs and benefits of searching Firm $i+1$ as if it were the only firm remaining. Because consumers expect utility $v_{i+1} - p_{i+1}$ with Firm $i+1$, the reservation utility, r_{i+1}, associated with Firm $i+1$ is the unique solution to

$$\int_{r_{i+1}+p_{i+1}}^{q_{i+1}+B_{i+1}} (v - p_{i+1} - r_{i+1}) dF_{i+1}(v) = s.$$

It is immediate from comparing the above condition to (1) that we can write the reservation valuation as $r_{i+1} = q_{i+1} + \Delta_{i+1} - p_{i+1}$. From this observation, we conclude that the largest price, p_i, that Firm $i = 1, ..., n-1$ can charge such that a consumer with match q_i will decline to search Firm $i+1$ satisfies $q_i - p_i = r_{i+1} = q_{i+1} + \Delta_{i+1} - p_{i+1}$. This equality determines the candidate equilibrium pricing rule as

$$p_i = p_{i+1} + q_i - q_{i+1} - \Delta_{i+1}, \tag{2}$$

which therefore determines a recursive relation. We now need to find an initial condition, which is the price set by Firm n.

So consider Firm n's problem. It knows it is the last to be searched and that all consumers who get to it in equilibrium have zero valuation for all the other products. It is therefore in a monopoly position. As we do for the other firms, we seek an equilibrium price such that all consumers with valuations of at least q_n buy Firm n's product. The largest price n can charge which is consistent with all such consumers buying is $p_n = q_n$. By applying the recursive price relation (2) it follows by induction that equilibrium prices are

$$p_i = q_i - \sum_{j=i+1}^{n} \Delta_j, \quad i = 1, ..., n. \tag{3}$$

We establish below that this pricing sequence induces consumer search in the specified order. We now show (proof in Appendix) that the pricing behavior described above is indeed profit-maximizing as long as q_i is sufficiently large and under the mild condition that $f_i(q_i) > 0$ for all i.

Lemma 1 *Assume $f_i(q_i) > 0$ and $v_i - q_i < B_i < \infty$ for all $v_i \in S$, $i = 1, ..., n$. If consumers search optimally from Firm 1 to Firm n expecting all the firms to price according to (3), then it is optimal for any Firm i to charge price p_i defined by (3) as long as q_i is sufficiently large, $i = 1, ..., n$.*

The intuition for the above result is the following. If all Firms $j > i$ charge p_j defined by (3), then Firm i faces a demand which is completely inelastic up to p_i defined by (3): for all prices up to this level, it serves all consumers with strictly positive valuation for its product and has demand $1 - \gamma_i$. If its price satisfies (3), then the marginal consumer has valuation q_i. Hence i's demand derivative for a price increase (the right derivative) is $f_i(q_i)$. If this is strictly positive, then with a large enough price, i.e. for a large enough q_i, the corresponding price elasticity is above one. A small price deviation is therefore unprofitable. Furthermore, the upper bound on the support of valuations $\bar{v}_i$, implies an upper bound on a potentially profitable price increase and if the equilibrium price is large enough, then the relative price increment is too small to compensate for a resulting drop in demand which is not arbitrarily small (so that large price deviations cannot be profitable either).

Lemma 1 establishes that firms do not wish to deviate from the candidate equilibrium prices. To establish that this is an equilibrium, we merely need to verify that the specified search order is optimal for consumers, that is, $r_i \geq r_{i+1}$ for $i = 1, ..., n-1$. Recall that $r_i = q_i + \Delta_i - p_i$ which, from the pricing expression (3), implies $r_i = \sum_{j \geq i} \Delta_j$. So r_i indeed monotonically decreases as i increases. We summarize with the following Proposition.

Proposition 1 *Under the assumptions of Lemma 1, there exists an equilibrium where consumers search firms in the order of the firm index, $i = 1, ..., n$ and Firm i charges a price given by (3): $p_i = q_i - \sum_{j=i+1}^{n} \Delta_j$ with equilibrium demands $D_i = (1 - \gamma_i) \prod\limits_{j<i} \gamma_j$; $i = 1, ..., n$.*

The pricing sequence in (3) bears the hallmark property that firms that are searched early on extract less surplus from consumers than firms that are searched later. This property is needed for consumers to search in the stipulated order, and generalizes the results in previous studies of products with symmetric match distributions (Armstrong *et al.*, 2009, Zhou, 2011 and Armstrong, 2017) that early firms charge lower prices.[10]

The equilibrium price has two components: a private value measured by q_i and an externality from the remaining firms in the search order measured by $\sum_{j>i} \Delta_j$. The private value is a surplus associated with the consumption of the firm's product, for which it captures any additional dollar through its price. The firm cannot however capture the entirety of this surplus because of the downward pressure on its price resulting from the option consumers have to search on to the firms down the line. The amount by which price is lowered can be interpreted as the total *search appeal* of the remaining products to be checked out by the consumer. Because of the "myopic" search rule used by the consumer, only the search appeal of the next product down, Δ_{i+1}, is directly relevant for Firm i's pricing. However, Firm i must also take into account the pricing behavior of Firm $i+1$, which depends on the search appeal of Firm $i+2$. This is why the total search appeal externality imparted on a firm is the cumulative search appeal of all the remaining firms.

In the benchmark case where firms all have identical product match distributions (so that $\Delta_i = \Delta$ for all i) the search appeal externality is merely $(n-i)\Delta$ for Firm i, which depends on only the *number* of firms following Firm i and prices step up by Δ from one firm to the next. In our setting where products are *ex ante* heterogenous, the externality also depends on the *identity* of the remaining firms. This property is key to the welfare analysis of the optimal ordering of firms for maximal total profit or consumer surplus. It also has important implications for the firms' willingness to pay to be searched earlier rather than later.

The search appeal externality reflects the competition that a firm faces from the following firms. But a firm does not directly compete in price with the preceding firms. This is because

[10] Armstrong (2017) shows this assuming that the c.d.f. of valuations for a product is logconcave. Other articles also find this result in contexts with a very specific form of asymmetry: merged and not merged products in Moraga-Gonzalez and Petrikaite (2012), or products with different degrees of heterogeneity in matches in Song (2013).

consumer search behavior factors in only the expected price at remaining firms: a firm has no way to steal customers from its predecessors by committing to a lower price than expected. However, preceding firms do affect Firm i's profit because they price so that all consumers who have a positive valuation with at least one earlier firm stop searching before reaching Firm i. As a result, only a fraction $\prod_{j<i} \gamma_j$ reach Firm i. This constitutes a *business stealing* externality which was previously analyzed by Chen and He (2011) and Athey and Ellison (2011) in models where prices are exogenous, or effectively so. If firms were *ex ante* symmetric, with $\gamma_i = \gamma$ for all i, market stealing for Firm i would depend only on the number of predecessors and the fraction of consumers reaching Firm i would merely be γ^{i-1}. Again, the identity of firms that are searched prior to Firm i becomes relevant once match distributions differ across products. Our analysis introduces novel insights for the interaction between business stealing and pricing, as shown in Section 3, where we show the tensions for consumers and firms among different search orders when there are position externalities. In Section 4 we endogenize the search order through a position auction.

We conclude this Section with two points about the comparison between equilibrium and socially optimal search behavior. First, at this juncture (i.e., before we endogenize the search order through the position auction), any order of search can constitute an equilibrium as per Proposition 1. Yet the (first-best) optimal order entails searching in decreasing order of the reservation values $q_i + \Delta_i$. To see this, first note that the social optimum entails pricing at marginal cost (here zero). The search problem for social welfare maximization is to achieve the best possible gross valuation net of search costs. The definition of Δ_i implies that the reservation value associated with searching Firm i is $q_i + \Delta_i$ - and hence the optimal order follows this statistic in decreasing order. The consumer should search i if she holds a lower value than $q_i + \Delta_i$.

Second, even for any given search order, *equilibrium search* (constrained by having to follow the stipulated search order) *is too low* compared to the social optimum. This result follows because equilibrium pricing chokes off further search as soon as the consumer gets a positive match value. Instead, a consumer should optimally search Firm i whenever her match is below $q_i + \Delta_i$. Then the benefit from searching Firm i alone is enough to justify incurring search cost s and, whenever the search order is suboptimal, there is also an option value from being able to search on beyond Firm i.[11] In equilibrium, the consumer searches Firm i if and only if her match is zero so that, if $q_{i-1} < q_i + \Delta_i$, then there is not enough search in equilibrium (regardless of whether the search order is optimal) because a consumer holding q_{i-1} at Firm $i-1$ should search but does not due to Firm $i-1$'s price discount.

3 Optimal rankings

The results of the previous Section indicate that ANY order of search can be sustained as an equilibrium to the game in which consumers follow their optimal search protocol and firms set their prices. Prices though differ across these equilibrium search orders when firms are asymmetric, and so the search order matters for various measures of market performance. Typically, the optimal order varies by market performance measure. We here determine the optimal orders, given equilibrium search and pricing, for total industry profit, social welfare, and consumer surplus. For short, call these TIP, W, and CS respectively.

A priori, this is a complicated problem because position order affects all prices and search probabilities: with n active firms there are $n!$ configurations to compare. Nevertheless, the structure of our model delivers a simple and clean characterization for the optimal order under each criterion. The optimal order is described by ordering firms according to a simple summary statistic, which is different for each surplus criterion.

The idea is as follows. For any neighboring pair of firms, A and B, in slots i and $i+1$ respectively, (and for each surplus criterion), we can find a summary statistic Φ_k for Firm k such that the maximand (CS, W, or TIP) evaluated in these two slots is higher if $\Phi_A \geq \Phi_B$. Crucially, the summary statistics are derived solely from parameters of the match distribution of the corresponding product, F_A for Φ_A and F_B for Φ_B. Hence they do not depend on which two slots are flipped (e.g., first and second or fifteenth and sixteenth). The key property of

[11] This option value is zero when the search order is optimal, as reflected in the myopia property of the optimal search rule.

our model is that such a flip affects the welfare objective only through the joint impact in the two consecutive slots: the welfare in all the other slots only depends on the joint externality that the two firms exert, either because they are in front (the business stealing externality) or because they come later in the search order (the search appeal externality that affects prices in those earlier slots). Thus, with $\Phi_A \geq \Phi_B$, A being in front of B (rather than the reverse) yields a higher welfare criterion computed over all the n slots. Clearly, a necessary condition for a maximum is that flipping the order of the two firms in each successive pair does not strictly increase the desired objective function. Because the flipping rule is independent of the positions i and $i+1$ to be flipped, this criterion induces an ordering of firms based on the indices Φ_k as claimed above. Put another way, any alternative order, with at least one pair of consecutive firms violating the pairwise flip condition, cannot be an optimum. Thus the ranking of firms by the size of their summary statistics is a necessary condition for optimality. It is also sufficient because, if there are no ties among firms in the sufficient statistics Φ_k, there is only one such order out of a finite set of possible configurations, and if there are ties, flipping two consecutive firms that are tied leaves the objective unchanged so that the multiple solutions obtained by ranking according to Φ_k are all optimal.

We now derive the specific summary statistics for the different criteria. We also give the intuition for the various orders.

3.1 Total Industry Profit (TIP)

The profit for the firm in position i is

$$\pi_i = (q_i - \kappa_i)\,\lambda_i\,(1-\gamma_i)\,, \quad i = 1, ...n, \tag{4}$$

where we have defined $\kappa_i = \Sigma_{j>i}\Delta_j$ as the sum of all later price steps (where κ_n is taken to be zero), and $\lambda_i = \prod_{j<i}\gamma_j$ for $i > 1$ as the as the probability that a consumer has no interest in any of the previous products (and we let $\lambda_1 = 1$). The term in the first parenthesis in (4) is the equilibrium price (3) and it is multiplied by the probability that the consumer ends up searching Firm i, λ_i, and then buying product i, $1-\gamma_i$.

As explained above, to find the maximum TIP we just need to look at the change in profit from switching Firms A and B between slots i and $i+1$. Thus A precedes B as long as

$$\pi_A^i + \pi_B^{i+1} \geq \pi_B^i + \pi_A^{i+1}, \tag{5}$$

where π_k^i denotes the profit of Firm k when it is in slot i. Writing this out for our model,

$$(1-\gamma_A)(1-\gamma_B)(q_A - q_B) + (1-\gamma_B)\Delta_A - (1-\gamma_A)\Delta_B \geq 0. \tag{6}$$

To derive this, first notice that we can divide through by the total number of consumers who search up to slot i, i.e., λ_i, and then the terms in all prices after $i+1$ (i.e., κ_{i+1}) cancel out. Importantly, the condition is independent of the position in the overall order of the two slots that are switched.

Dividing through (6) by $(1-\gamma_A)(1-\gamma_B)$ delivers the TIP summary statistics such that A should be before B (in any consecutive pair, and hence in the global maximum) as long as

$$\Phi_A^\pi \equiv q_A + \frac{1}{1-\gamma_A}\Delta_A > q_B + \frac{1}{1-\gamma_B}\Delta_B \equiv \Phi_B^\pi.$$

The TIP summary statistic is readily apparent from this inequality, and is given next:

Proposition 2 *An order of firms maximizes Total Industry Profit if and only if it follows the ranking of the summary statistic*

$$\Phi_k^\pi \equiv q_k + \frac{1}{1-\gamma_k}\Delta_k \tag{7}$$

and firms should follow a decreasing order of the Φ_k^π. *Ceteris paribus, higher* q_k, Δ_k, *and* γ_k *should go earlier in the order.*

To understand this result, recall that, in equilibrium, firms that are early in the search order sell more but extract less consumer surplus (they have deeper quality discounts), whereas firms that come later sell less but extract more consumer surplus because their prices are closer to their qualities. TIP maximization is achieved by ensuring that firms that sell more extract as much surplus as possible and firms that extract the most surplus sell as much as possible. The first objective is achieved by having firms with a large quality q_k and a large search appeal Δ_k searched early. Having firms with least popular products (large γ_k) searched first serves the second goal.

One way to see these effects clearly in isolation is by looking at each as the sole source of heterogeneity (so the other parameters are set the same for all firms). A large quality ensures that there is much potential consumer surplus to be extracted by the sellers of such products, which should therefore have the most consumers sampling them. Notice that the quality effect is NOT an externality on the other firms.

High Δ firms cause low prices on all those which precede them. Switching a high-Δ firm with a low-Δ one that was initially earlier, raises the prices for all the firms in between the two slots, and so raises total profits. The idea of stacking up early all the high-Δ firms is to "clear-the-decks" of them to suppress their shadow on all prices that come earlier, which they would otherwise bring down. Put another way, having the firms that are most appealing to search early mitigates the search appeal externality imparted by these firms: they can keep their prices relatively high because consumers are not too eager to search the remaining firms.

Finally, it may seem surprising that firms with less popular products (large γ_k) should be presented first to consumers because these firms are less likely to make a sale. However, early slots have low prices, so the ranking uses up these slots on less likely prospects.[12] Firms that extract the most surplus from their customers have larger sales if the business stealing externality from earlier firms is limited. Both this feature and the search appeal externality already suggest that consumer surplus may run the opposite way from TIP, a property that is confirmed in broad-brush terms.

[12]For example, suppose there were two firms, and $\gamma_A = 0.1$ while $\gamma_B = 0.9$. Then the number of consumers who buy constitute 91 percentage points, regardless of the order of search. Having A first entails 89% buying at the high price, while B first means only 1% do.

3.2 Social Welfare

We next consider the pairwise ranking condition for Welfare (given equilibrium firm pricing). First note that, because pricing ensures that all consumers who are interested in any of the products buy one of them, prices are just a straight transfer between firms and consumers, and so do not enter the calculus.

Conditional on reaching Firm k, the expected social welfare when a consumer searches Firm k may be written as

$$-s + (1 - \gamma_k) q_k + \int_{q_k}^{q_k + B_k} (v - q_k)\, dF_k(v).$$

Substituting in the definition of Δ_k in equation (1) to cancel out s, the above expression may be rewritten

$$(1 - \gamma_k) q_k + [1 - F_k(q_k + \Delta_k)]\Delta_k + \int_{q_k}^{q_k + \Delta_k} (v - q_k)\, dF_k(v). \tag{8}$$

Now define $\beta_k = 1 - F_k(q_k + \Delta_k)$ and $\alpha_k = \int_{q_k}^{q_k + \Delta_k} (v - q_k)\, dF_k(v)$, so that this expected social welfare may be written as

$$(1 - \gamma_k) q_k + \beta_k \Delta_k + \alpha_k. \tag{9}$$

Figure 2 illustrates these components, and how they correspond to (8). The height Δ_k is the quality-price discount the preceding firm $(k - 1)$ must offer to keep on-board all its consumers not drawing a zero match value, the width β_k is the fraction of consumers drawing a match with k of at least $(q_k + \Delta_k)$ so that $\beta_k \Delta_k$ is the value of the discount to these types, and α_k is the remaining surplus triangle. Notice that the expression $(\beta_k \Delta_k + \alpha_k) < 1$ can be treated as one entity in what follows, though we choose to break it into the separate rectangle and "triangle" components. We have also drawn in the Figure the firm surplus, $p_k (1 - \gamma_k)$, so that the consumer surplus (treated next) is also immediately transparent as the difference between social surplus and firm profit.

Consider now the welfare from searching A then B (conditional on having reached A at some position i). Then, comparing it with the converse while using the analogous expression (switching subscripts) for the opposite order yields the condition for the sequence AB (for any consecutive pair) to be preferable to BA as:

$$\begin{aligned} & q_A (1-\gamma_A) + \beta_A \Delta_A + \alpha_A + \gamma_A \left(q_B (1-\gamma_B) + \beta_B \Delta_B + \alpha_B \right) \\ \geq \;\; & q_B (1-\gamma_B) + \beta_B \Delta_B + \alpha_B + \gamma_B \left(q_A (1-\gamma_A) + \beta_A \Delta_A + \alpha_A \right), \end{aligned}$$

and hence

$$\Phi_A^W \equiv q_A + \frac{\beta_A \Delta_A + \alpha_A}{1-\gamma_A} \geq q_B + \frac{\beta_B \Delta_B + \alpha_B}{1-\gamma_B} \equiv \Phi_B^W.$$

The summary statistic is thus the one given next:

Proposition 3 *An order of firms maximizes Social Welfare if and only if it follows the ranking of the summary statistics*

$$\Phi_k^W \equiv q_k + \frac{\beta_k \Delta_k + \alpha_k}{1-\gamma_k} \tag{10}$$

and firms should follow a decreasing order of the Φ_k^W. *Ceteris paribus, higher* q_k, Δ_k, γ_k, *should go earlier in the order.*

As can be seen by comparing (10) with (7) the criterion for determining the search order that maximizes social welfare is remarkably similar to that which we derived to optimize total industry profit: larger values of q_k, Δ_k and γ_k should come first. If all else is the same, the two criteria are perfectly aligned regarding base qualities q_k. This is easily understood: any additional dollar of surplus induced by an increase in q_k is entirely captured by Firm k as can be seen from the equilibrium pricing expression (3).

The fact that both criteria call for having products with large values of Δ_k searched early is somewhat misleading, because the underlying economic reason is quite different. As was explained above, the rationale for having products with large search appeal early in the order when maximizing TIP is to mitigate the search appeal externality imparted on the first firms. By contrast, when considering social welfare, Δ_k is relevant to the extent that it enters into the measure of the total surplus generated in excess of the base quality q_k, for which the expression is $\beta_k \Delta_k + \alpha_k$. Generally, social welfare maximization puts less weight on Δ_k than TIP maximization. There can even be an extreme configuration where Δ_k is essentially irrelevant to social welfare: this happens when there is only a small probability that the match with product k exceeds $q_k + \Delta_k$ so that β_k is nearly zero (and B_k is then large so that a consumer can draw very large matches and (1) is satisfied).[13]

Finally the requirement that less popular products (low values of γ_k) should come first to achieve maximum social welfare may seem even more surprising than the analogous result for TIP maximization. Indeed, if the probability that consumers are not at all interested in the first products they encounter, they will keep searching longer, which seems wasteful. However, as we noted in the previous Section, consumers actually do not search enough in equilibrium. They stop searching as soon as they have a strictly positive valuation with a product whereas the social optimum would have them search as long as their valuation is below $q_k + \Delta_k$. This inefficiency is more severe when consumers draw a strictly positive match early on.

3.3 Consumer Surplus

The consumer surplus case proceeds analogously to the welfare one, except now prices feature explicitly. A second key difference is that the q_k's do not enter because they are priced out.

Letting p_k^i be Firm k's price when in slot i, consumer surplus with product k in slot i is

$$-s + (1 - \gamma_k)(q_k - p_k^i) + \int_{q_k}^{q_k + B_k} (v - q_k) dF_k(v),$$

The pricing rule (3) gives $p_k^i = q_k - \Delta_{i+1} - \kappa_{i+1}$ and $p_k^{i+1} = q_k - \kappa_{i+1}$ where $\kappa_{i+1} = \Sigma_{j>i+1}\Delta_j$ denotes the sum of later price steps. Using the search identity (1) and the pricing equation to substitute out the search cost and the price, consumer surplus can be expressed in a manner similar to the social welfare expression (9), while defining β_k and α_k as before so we have

$$(1 - \gamma_k)\left(\Delta_{i+1} + \kappa_{i+1}\right) + \beta_k \Delta_k + \alpha_k,$$

[13] Also note that α_k could be anywhere between 0 and arbitrarily close to $\Delta_k(1 - \beta_k - \gamma_k)$: the former happens when q_k is drawn with probability $1 - \beta_k - \gamma_k$ and the latter when all the weight of the distribution of matches between q_k and $q_k + \Delta_k$ is concentrated just below $q_k + \Delta_k$.

with Firm k in slot i and

$$(1-\gamma_k)\kappa_{i+1} + \beta_k\Delta_k + \alpha_k,$$

with Firm k in slot $i+1$.

Hence the consumer surplus associated with AB exceeds that of BA (which is found by transposing subscripts again) if

$$(1-\gamma_A)\,\Delta_B + \beta_A\Delta_A + \alpha_A + \gamma_A\,(\beta_B\Delta_B + \alpha_B) \geq (1-\gamma_B)\,\Delta_A + \beta_B\Delta_B + \alpha_B + \gamma_B\,(\beta_A\Delta_A + \alpha_A)$$

where the κ_{i+1} terms all cancel out because they are common to both firms' prices: hence the same calculus applies regardless of which slot i is the base one.

Rearranging yields

$$\Phi_B^{CS} \equiv \frac{1}{1-\gamma_B}\left((\beta_B - 1)\Delta_B + \alpha_B\right) \geq \frac{1}{1-\gamma_A}\left((\beta_A - 1)\Delta_A + \alpha_A\right) \equiv \Phi_A^{CS},$$

It is readily seen that $\alpha_k < \Delta_k(1-\gamma_k-\beta_k)$ so $\beta_k\Delta_k + \alpha_k < \Delta_k$ and all the terms above are negative and the implication for the summary statistic is given next:

Proposition 4 *An order of firms maximizes Consumer Surplus if and only if it follows the ranking of the summary statistics*

$$\Phi_k^{CS} \equiv \frac{1}{1-\gamma_k}\left((\beta_k - 1)\Delta_k + \alpha_k\right) < 0 \tag{11}$$

and firms should follow a decreasing order of the Φ_k^{CS}. The q_k value is irrelevant whereas, ceteris paribus, higher Δ_k and γ_k should go later in the order.

The qualitative implications regarding the ranking of products according to how popular they are (γ_k) and how appealing they are for search (Δ_k) is quite opposite to what we obtained for TIP or social welfare maximization. Indeed the consumer surplus maximization objectives are the reverse of what we found for total profit: consumers retain more surplus for products placed early in the search order because of pricing and also because they expand less search costs to get to them. Hence, the maximization of consumer surplus requires that the likelihood that the early products are bought is as large as possible, which is ensured by having low γ_k products first, and the surplus extracted from consumers at those early slots is as low as possible, which means that products with a large Δ_k should come later.

By explicitly deriving endogenous prices and considering *ex ante* heterogeneous products, we highlight a conflict between the order that is desirable for firms and that which consumers prefer, which has not been identified in previous literature. For instance, in Athey and Ellison (2011), the preferred order for both consumers and firms is that the most popular products are first. This is because for them the order matters only because they assume that search costs are heterogeneous and some consumers stop searching whereas they could have purchased a product that they like (this does not happen in our setting). If we introduce heterogeneous search costs in our search environment, the optimal order for TIP maximization becomes ambiguous but the order for consumer surplus would be unchanged.[14]

4 Auctions

Thus far we have analyzed the market outcome while assuming some set order of search. The analysis in Section 2.2 shows that any search order could be an equilibrium order, where firms price optimally while expecting consumers to follow the search order and consumers search optimally in this order while correctly anticipating firms' prices. Here we endogenize the search order by allowing firms to compete for the best spots. Specifically, we consider a generalized second price auction, in line with previous literature on auction mechanisms used for online search engines. We significantly extend the earlier literature by allowing the firm's position valuations to result from our full-fledged price competition model. Consequently, the auction involves two position externalities, business stealing and search appeal. The auction game we consider is as follows. There are $n \geq 2$ firms competing for n slots. Each firm simultaneously posts a bid per click. Let b^i denote the i^{th} highest bid, where two consecutive bids can be equal. The i^{th} slot, $i = 1, ..., n$, is allocated to the firm with the i^{th} highest bid, and that firm is charged b^{i+1} per click: if two firms have equal bids each is equally likely to

[14] In Chen and He (2011), all consumers have the same search cost and the order of products is irrelevant for TIP or CS maximization in the early slots for which they assume a low search cost. However, it is preferable that these early slots are occupied by the most popular products.

be placed in front. Throughout the analysis below, Firm i, $i = 1, ..., n$ is the firm that is positioned in slot i in a candidate equilibrium. Firm n, which has the lowest bid, b^n, gets the last slot so it is searched last and pays nothing.

Recall from our earlier analysis (see (4)) that the gross profit for Firm i is

$$\pi_i = (q_i - \Sigma_{j>i}\Delta_j)(1-\gamma_i)\prod_{j<i}\gamma_j, \quad i = 1, ...n. \tag{12}$$

We start with some general preliminary analysis that introduces the main concepts and how they can be used to analyze a position auction problem.

4.1 Preliminaries

We write the (generalized second-price) auction position equilibrium problem in terms of firms' profits in different positions, the traffic (incoming clicks) they get in these positions, and the amounts they pay for click traffic at each position. For the moment, we work with a general analysis which encompasses our specific model as well as the others in the extant literature, and should prove useful to other researchers who address such auction equilibria under reasonable restrictions on payoffs. We deliver some conditions that can be readily evaluated from the pay-off structure and which imply that candidate local equilibrium conditions on positions ensure global deviations are not profitable. In the next sub-section we provide a simple setting in which the per click bids are guaranteed to be higher at earlier positions (so that higher bids get better places).

We consider some ordering of firms by their positions, and the (initial) order specifies each firm's type in terms of its characteristics and the externalities that effect it, as expressed through profitability and click traffic in different places (which are in turn determined by pay-off parameters). What happens to the payoff to the firm in the jth position (Firm $j = 1, ..., n$) when it moves to some other position i? In contemplating such a move, we hold fixed the order of the other firms, so that the first $i-1$ firms retain their positions if j moves *up* to reach the ith position ($i < j$), Firms i through $j-1$ are demoted one slot down, and firms below j retain their positions. Conversely, if Firm j contemplates moving *down* to position $i > j$ then Firms 1 through $j-1$ and Firms $i+1$ through n retain their positions, while Firms $j+1$ through i are promoted one slot up.

Let $\pi^{i}(j)$ denote the profit of Firm (type) j when it moves to position i, so $\pi^{j}(j)$ is its profit at the status quo. We hold throughout that profit for any firm type is higher at earlier slots. Let the number of clicks (incoming traffic) to Firm j in position i be denoted $\Gamma_i(j)$. In accord with the earlier pricing equilibrium model, we assume that incoming traffic is the chance that a consumer has not bought from any of the preceding firms, so that $\Gamma_j(j) = \Pi_{\ell<j}\gamma_\ell$, $\Gamma_i(j) = \Pi_{\ell<i}\gamma_\ell$ for $i < j$, and $\Gamma_i(j) = \Pi_{\ell<i+1,\ell\neq j}\gamma_\ell$ for $i > j$. Let $B_i(j)$ denote the *total* amount that Firm j would pay for the clicks it would receive in position i. This total will be defined momentarily in terms of the per click bid of the next firm in the purported search order and incoming traffic.[15]

A *per click position equilibrium* is a set of per click bids such that firms' positions are given in descending order of their bids with each paying the bid per click of the firm immediately below it, and no firm wishing to switch its position. We split the latter conditions into there being no desire to jump down, and none to jump up. We deal with these in turn.

For Firm i (i.e., the incumbent type in position i) to not wish to jump down $k \geq 1$ slots to position $i+k$, requires that

$$B_i(i) - B_{i+k}(i) \leq \pi^{i}(i) - \pi^{i+k}(i), \tag{13}$$

which says that the lost profit from jumping down exceeds the bid cost saving. This gives conditions in terms of $i+1$'s equilibrium per click bid, as we now show.

In order to analyze per click bidding equilibrium conditions, we rewrite the above inequality in terms of per click bids and incremental per click profit values. To this end, write the incoming click traffic to slot i when Firm i is there as $\Gamma_i(i) = \Pi_{\ell<i}\gamma_\ell$; when i has moved down to position $i+k$ its incoming clicks are $\Gamma_{i+k}(i) = \Pi_{\ell<i+k+1}\gamma_\ell/\gamma_i$, which reflects the clicks on those firms i has promoted by dropping behind them. Hence, because i pays the per click bid b^{i+1} of the next Firm $i+1$, its bid payments are $B_i(i) = b^{i+1}\Gamma_i(i)$. Likewise, to get position $i+k$ it drops its per click bid so as to pay the per click bid paid by the erstwhile

[15] These bids are to be determined endogenously in equilibrium from the profit and traffic statistics.

incumbent, which is b^{i+k+1}, so that i will pay $B_{i+k}(i) = b^{i+k+1}\Gamma_{i+k}(i)$ for jumping down k slots. Define

$$IV_{i+k}^{i}(j) = \frac{\pi^{i}(j) - \pi^{i+k}(j)}{\Gamma_{i}(j)},\ j = 1, ..., n;\ i = 1, ..., n-1;\ k = 1, ..., n-i \tag{14}$$

as the *Incremental Value* to Firm j in position i over position $i+k$ ($k \geq 1$) per incoming click to j at position i. This is the differential profit superiority on a per click basis. In these terms, the condition on per click bids for no jumping down (rewriting (13)) is therefore

$$b^{i+1} - \frac{\Gamma_{i+k}(i)}{\Gamma_{i}(i)} b^{i+k+1} \leq IV_{i+k}^{i}(i), \qquad i = 1, ..., n-1;\ k = 1, ..., n-i \tag{15}$$

(where the Γ ratio is just γ_{i+1} for $k = 1$, and is γ^{k} for common γ).

We proceed in an analogous manner for jumps up, with the aim of again getting a condition in terms of the two bids on the LHS of the above equation. To do so requires thinking of Firm $i+k$ jumping up to position i with $k \geq 1$, usurping the incumbent and paying its bid. So, in terms of the total incremental cost-benefit, for Firm $i+k$ to prefer to stay put requires

$$B_{i}(i+k) - B_{i+k}(i+k) \geq \pi^{i}(i+k) - \pi^{i+k}(i+k). \tag{16}$$

In terms of per click bids, this condition is rewritten as

$$b^{i} - \frac{\Gamma_{i+k}(i+k)}{\Gamma_{i}(i+k)} b^{i+k+1} \geq IV_{i+k}^{i}(i+k), \qquad i = 1, ..., n-1;\ k = 1, ..., n-i, \tag{17}$$

where b^{i+k+1} is the per click bid paid by Firm $i+k$ at the status quo, and b^{i} is what it would pay by bidding to take position i away from the incumbent (and the Γ ratio is just γ_{i} for $k = 1$, and is γ^{k} for common γ).

The *per click position equilibrium condition* on the vector of b's is therefore that the two inequalities (15) and (17) hold for all possible jumps that these bids are strictly decreasing (for slots are allocated by highest per click bids: note that we will find conditions to avoid ties).

An *Envy-Free* equilibrium is a more restrictive one in which no firm would wish to move if it were able to pay the per click bid paid by the firm whose position it would usurp.

This concept was introduced as a "no envy" requirement by Edelman et al. (2007) and as "symmetry" by Varian (2007). Notice that any jump down does already entail paying (infinitesimally more than) the usurped firm's bid, so the earlier Nash equilibrium condition for jumps down (15) is the same as the envy-free one. For jumps up though, the envy-free condition is stronger. For example, a firm contemplating jumping up one slot would actually pay the per click bid of the firm above it, which is higher than its own bid in equilibrium, whereas an envy-free equilibrium would entail it not wanting to make the jump if it could pay its *own* bid. Envy-free equilibrium bids are therefore a subset of the Nash equilibrium ones.

For an envy-free equilibrium, the no-jump-up condition is the only one modified. It just involves replacing b^i by b^{i+1} in the earlier condition (17). This yields the necessary condition for no firm to want to jump up if it could do so and pay what the incumbent slot-holder pays as

$$b^{i+1} - \frac{\Gamma_{i+k}(i+k)}{\Gamma_i(i+k)} b^{i+k+1} \geq IV^i_{i+k}(i+k), \quad i = 1, ..., n-1; \; k = 1, ..., n-i. \tag{18}$$

For $k = 1$, (15) and (18) provide conditions on neighboring bids that must hold in equilibrium and which we term *local envy-free equilibrium conditions*. To interpret them, consider the case when γ is common so that the bid sequence must follow the local IV condition

$$b^{i+1} - \gamma b^{i+2} \in \left[IV^i_{i+1}(i+1), IV^i_{i+1}(i)\right]. \tag{19}$$

The LHS of (19) is the extra bid payment (per click incoming to slot i) at slot i over slot $i+1$. The upper bound, $IV^i_{i+1}(i)$, is the extra revenue (per click incoming to slot i) that Firm i gets in slot i over slot $i+1$, and so represents the maximal increment i is willing to pay for slot i over slot $i+1$. Firm i would not want slot i if it had to pay more. The lower bound, $IV^i_{i+1}(i+1)$, is the extra revenue (per click incoming to slot i) that Firm $i+1$ would get in slot i over slot $i+1$, and hence is the maximal increment $i+1$ is willing to pay for slot i over slot $i+1$. If Firm $i+1$ were able to pay what Firm i pays (which is b^{i+1}) then $i+1$ would envy i were b^{i+1} any lower, even if Firm $i+1$ would not necessarily want to make the jump up, for b^i will exceed b^{i+1}: this feature underscores the idea seen in the three-firm example below that other (non envy-free) equilibria can exist.[16] So, while Firm $i+1$ would be prepared to pay up to $IV_{i+1}^{i}(i+1)$ for slot i, it does not want the slot if it had to pay more.

We now provide conditions on IV's (and hence on the underlying payoff structure) which ensure that the local equilibrium conditions – those under which each firm does not want to jump over or behind a neighbor – are also global so larger deviations are unprofitable too.

Lemma 2 *Assume firms are ordered such that* $IV_{i+k+1}^{i+k}(i) \geq IV_{i+k+1}^{i+k}(i+k)$ *for all* $i = 1, ...n-1$ *and* $k = 1, ..., n-i$. *Then if one-step deviations down are not profitable, larger downward deviations are not either.*

Proof. Assume $n \geq 3$. We wish to show that, for all $i = 1, \ldots, n-1$, if (15) holds for $k = 1$, then (15) holds for $k = 1, ...n-i$. This is clearly true for $k = 1$.

Now we just need to show that, for $i = 1, ..., n-2$, if this is true for some $k = 1, ..., n-i-1$, then it is true for $k+1$. If (15) holds for k then Firm i's net profit is larger in slot i than in slot $i+k$ so that

$$\pi^i(i) - \Gamma_i(i)b^{i+1} \geq \pi^{i+k}(i) - \Gamma_{i+k}(i)\, b^{i+k+1}. \tag{20}$$

From the lemma's assumption, Firm $i+k$ does not want to deviate to slot $i+k+1$ so $b^{i+k+1} - \gamma_{i+k+1} b^{i+k+2} \leq IV_{i+k+1}^{i+k}(i+k)$, where we have substituted in $\frac{\Gamma_{i+k+1}(i+k)}{\Gamma_{i+k}(i+k)} = \gamma_{i+k+1}$ (for Firm $i+k$ is jumping over its successor). The Lemma also assumes that $IV_{i+k+1}^{i+k}(i) \geq IV_{i+k+1}^{i+k}(i+k)$, and hence

$$b^{i+k+1} \leq \gamma_{i+k+1} b^{i+k+2} + IV_{i+k+1}^{i+k}(i),$$

which delivers an upper bound for b^{i+k+1}. Substituting this bound into (20) yields

$$\pi^i(i) - \Gamma_i(i)b^{i+1} \geq \pi^{i+k}(i) - \Gamma_{i+k+1}(i)\, b^{i+k+2} - \Gamma_{i+k}(i) IV_{i+k+1}^{i+k}(i),$$

[16] The local IV condition pertaining to Firm $i+1$, without the envy-free stipulation, is $b^i - \gamma b^{i+2} \geq IV_{i+1}^{i}(i+1)$.

with $\Gamma_{i+k+1}(i) = \gamma_{i+k+1}\Gamma_{i+k}(i)$. Applying the definition of incremental values given in (14) we obtain

$$\pi^i(i) - \Gamma_i(i)b^{i+1} \geq \pi^{i+k+1}(i) - \Gamma_{i+k+1}(i)\, b^{i+k+2},$$

so Firm i does not want to deviate to slot $i+k+1$. $\blacksquare$

The result follows from a transitivity argument: "stronger" firms (under the IV condition given) have more incentive to be ahead, so a stronger one will not prefer to jump down a further slot if a weaker one would not.

An analogous construction for jumps up using (18) delivers the next result for envy-free bids. The proof gives a simple two-step argument which the reader might find more transparent than the full version (which is readily furnished along the above lines).

Lemma 3 *Assume firms are ordered such that $IV^i_{i+1}(i+1) \geq IV^i_{i+1}(i+k)$ for all $i = 1, ...n-1$ and $k = 1, ..., n-i$. Then if no firm envies the firm that is in the immediately preceding slot, then no firm envies any firm that is positioned in any slot earlier than its own.*

Proof. Assume $n \geq 3$. Consider some $i = 1, ..., n-2$ where Firm $i+2$ does not envy Firm $i+1$ and Firm $i+1$ does not envy Firm i. If Firm $i+1$ does not envy Firm i we have $b^{i+1} - \frac{\Gamma_{i+1}(i+1)}{\Gamma_i(i+1)}b^{i+2} \geq IV^i_{i+1}(i+1)$. Now if Firm $i+2$ has moved up to slot $i+1$ and is paying b^{i+2}, it would not envy Firm i either because $IV^i_{i+1}(i+2) \leq IV^i_{i+1}(i+1)$. Then, because Firm $i+2$ does not envy Firm $i+1$, it does not envy i either. Applying the same logic recursively implies that, for all $k > 1$, Firm $i+k$ does not envy Firm i. $\blacksquare$

The two lemmas jointly ensure that envy-free holds for any deviation if it holds for one-step deviations. Each lemma requires a key assumption that compares incremental values between firms positioned early and firms positioned late. To better understand the lemmas' implications, suppose for the rest of this sub-section that all γ's are the same (and we look at a tighter restriction on demand parameters in the next sub-section to give the backdrop to the subsequent results).

When the conditions given in the lemmas hold, and when the γ's are identical, then any bid sequence that satisfies no local envy will guarantee no global envy so that a necessary and sufficient condition for no envy is that (19) holds, i.e., $b^{i+1}-\gamma b^{i+2} \in \left[IV_{i+1}^{i}(i+1), IV_{i+1}^{i}(i)\right]$.[17] In order for the interval in (19) to be non-empty, we need $IV_{i+1}^{i}(i) \geq IV_{i+1}^{i}(i+1)$. The conditions used in Lemmas 2 and 3 do not suffice because they do not impose any restrictions on how incremental values compare for two neighboring firms moving in front of one another.

Both lemmas' assumptions are satisfied, along with the local incremental value restriction that guarantees that locally envy free bids can be constructed from (19), if the order satisfies a *Strong Incremental Value condition* that firms' strength rankings (in terms of incremental values) are the same for all pairwise position comparisons:

$$IV_{i+k}^{i}(j) \geq IV_{i+k}^{i}(\ell) \text{ for all } j < \ell. \tag{21}$$

Even if the strong IV condition is satisfied, we still need to verify that the candidate bid sequence is decreasing in the order to have an envy free equilibrium. This is illustrated by the private value setting that we now discuss.

4.2 Private values and decreasing bids

Under private values, firms only differ by their quality q_i so that the search appeal and business stealing externalities are *anonymous*. They do not depend on which firms are in front or behind, but only on j's own quality and on slot i: Firm j's profit in slot i is $\pi^i(j) = (1-\gamma)\gamma^{i-1}(q_j - (n-i)\Delta)$. Then incremental values are

$$IV_{i+k}^{i}(j) = (1-\gamma)\left((1-\gamma^k)(q_j - (n-i-k)\Delta) - k\Delta\right),$$

which increases in q_j, so that a higher quality firm always benefits more from being placed earlier. The strong incremental value condition therefore holds if (and only if) firms are ranked by decreasing quality.[18]

[17] It then suffices for the existence of an envy-free equilibrium that the bids are decreasing in the order, a condition to which we return below.

[18] If some Firm i has deviated down and demoted some other Firm $i+k$ with lower quality, its incremental value for not moving down one more step is higher than that of the firm it has replaced: $IV_{i+k+1}^{i+k}(i) \geq IV_{i+k+1}^{i+k}(i+k)$, which allows Lemma 2 to be applied. Similarly, if some Firm $i+k$ does not envy some Firm i before it (with higher quality), then $IV_i^{i-1}(i+k) \leq IV_i^{i-1}(i)$ so Lemma 2 applies.

The above discussion concerns conditions ensuring that firms neither want to deviate nor envy each other. Equilibrium also requires that the *bid sequence* decreases in i. To gain some intuition for the type of properties that would ensure this, consider the private value setting when there is no search appeal so $\Delta_i = 0$, and each Firm i charges its monopoly price q_i. The incremental values are $IV_{i+k}^{i}(j) = (1-\gamma)\left(1-\gamma^{k}\right)q_j$. We now show firms placed later bid less if bids satisfy (19).

The largest possible value for b^{i+1}, $i = 1, ..., n-1$ (therefore the most stringent condition) is obtained by taking the largest possible incremental bids up to b^{i+1}. This entails $b^j - \gamma b^{j+1} = IV_j^{j-1}(j-1)$, for all $j = i+1, ..., n$, and hence the highest value is $b^{i+1} = \sum_{j=i+1}^{n} \gamma^{j-i-1} IV_j^{j-1}(j-1)$. Since $q_i \geq q_{j-1}$ for $j > i+1$, $b^{i+1} \leq \frac{1-\gamma^{n-i}}{1-\gamma} IV_{i+1}^{i}(i)$. From (19) $b^i - b^{i+1} \geq IV_i^{i-1}(i) - (1-\gamma)b^{i+1} \geq IV_i^{i-1}(i) - (1-\gamma^{n-i})IV_{i+1}^{i}(i) > 0$ (because $IV_i^{i-1}(i) = IV_{i+1}^{i}(i)$). Thus all bids decrease in i.

The argument uses the property that $IV_i^{i-1}(i) \geq IV_j^{j-1}(j-1)$, whenever $j > i$. This property is not related to the assumptions in Lemmas 2 and 3. Nonetheless, for private values with no search appeal, it holds as long as products are ranked by decreasing quality and this validates the lemmas. This property fails if $\Delta > 0$. For instance, if all products have identical quality q, then $IV_i^{i-1}(i) < IV_j^{j-1}(j-1)$ for $j > i$ because earlier firms charge lower prices and the per click benefit from being placed one slot higher is increasing in price. This difficulty arises for all settings we consider below.

The special case above of $\Delta_i = 0$ for all i (i.e., absent search appeal) is effectively the setting of Varian (2007) and Edelman *et al.* (2007). These authors consider firms with different per click valuations which are exogenous and independent of (and decreasing in) slot position. This is the situation when $\Delta_i = 0$. Firm i charges $p_i = q_i$ and its (per click) incremental valuation is $(1-\gamma)q_i$ independent of where it is, while the number of clicks at slot j is just the probability that no preceding firms sells (which is γ^{j-1}). As seen in Section 3, joint profit and social surplus are maximized by ordering firms by quality. Consumers are indifferent to how firms are ordered because any incremental quality accrues to the firm as higher price.

4.3 Ordering of products more generally: 3 firms and beyond

We now characterize the equilibrium outcome of the GSP (Generalized Second Price) auction. The gist of the arguments and intuitions can be captured with three firms. We focus on this simple case for the text discussion and only state the general n firms results and present the full analysis in the Appendix. We start with the symmetric case as a benchmark and then move to the analysis of the firms' ordering when products are heterogeneous. Our analysis of heterogeneous firms distinguishes two key dimensions of demand. The first is *height*, which measures how much interested consumers are willing to pay, and is parameterized by q_i and Δ_i. The second is width, which measures the likelihood that a consumer is interested and is captured by the product's *market potential,* $1-\gamma_i$. We consider heterogeneity in these dimensions in turn. With heterogeneity in height, the quantity sold in a given slot does not depend on which firm is where whereas the price charged depends on where firms are. With heterogeneity in width, the price charged in a given slot does not depend on where firms are but the quantity sold does. Succinctly, with heterogeneity in height, firms differ in how much they *charge* whereas with heterogeneity in width firms differ by how much they *sell*.

Suppose there are three slots to be allocated among three firms. Firm 1's bid b^1 is paid by no firm and can be set as large as needed to deter Firms 2 and 3 from deviating to slot 1. Hence, the characterization of an equilibrium involves specifying per click bids b^2 and b^3 that satisfy 4 equilibrium conditions: Firm 1 should not want to drop to slots 2 or 3, Firm 2 should not wish to drop to slot 3 and Firm 3 should not wish to rise to slot 2. Using (15) and (17), these conditions are respectively

$$b^2 \leq \tilde{b}^2\left(b^3\right) \equiv \gamma_2 b^3 + IV_2^1(1), \tag{22}$$

$$b^2 \leq \bar{b}^2 \equiv IV_3^1(1), \tag{23}$$

$$b^3 \leq \bar{b}^3 \equiv IV_3^2(2), \tag{24}$$

$$b^2 \geq \underline{b}^2 \equiv IV_3^2(3). \tag{25}$$

Consider first the symmetric case with $q_i = q$, $\Delta_i = \Delta$ and $\gamma_i = \gamma$ for all i. This is a special case of the private value setting considered in the preceding subsection and there are

3 relevant incremental values: $IV_2^1(1) = (1-\gamma)((1-\gamma)(q-\Delta)-\Delta)$, for a move from slot 2 to slot 1, $IV_3^2(2) = IV_3^2(3) = (1-\gamma)((1-\gamma)q-\Delta)$ for a move between slots 2 and 3 (which firm moves is irrelevant because of symmetry), and $IV_3^1(1) = (1-\gamma)((1-\gamma^2)q-2\Delta)$ for jumping between slots 3 and 1. Recalling that incremental values are evaluated conditional on the number of clicks at the early slot, their value is given by the increase in the "conditional" probability that a consumer buys if the firm is placed earlier, $((1-\gamma)(1-\gamma^k)$ for a k-slot jump, $k = 1, 2)$, multiplied by the price in the later slot (q in slot 3 and $q - \Delta$ in slot 2) minus the decrease in price associated with being searched earlier ($k\Delta$ for k slots, $k = 1, 2$) times the probability a consumer buys if the firm is in the earlier slot $(1-\gamma)$.[19] Throughout (as per the earlier pricing analysis too), q is taken to be large enough that these incremental values are strictly positive.

Figure 3a describes the set of bid combinations (b^2, b^3) delimited by conditions (22)-(25) in the symmetric case,[20] along with the requirement that $b^2 > b^3$ (which is implied by (24) and (25)). Note that (23) is implied by (22) and (24) which illustrates that Lemma 1 indeed applies in this private value setting (as discussed earlier) so that equilibrium conditions for one-step deviations down suffice for ruling out all deviations down. Notice that the equilibrium set is never empty if $IV_2^1(1) = IV_3^2(3)$. This happens for $\Delta = 0$ so there always exists an equilibrium for Δ small enough. It is straightforward to check that the equilibrium triangle is not empty as long as q is large enough, no matter how large is Δ. Intuitively, if q is large, then the incremental value for jumping from 3 to 1 (which is also the value of b^2 at the top vertex, $\bar{b}^2$) exceeds the incremental value for jumping from 3 to 2 (which is the RHS of (25), $\underline{b}^2$).

We establish equilibrium existence in the n-firm case (see Propositions 5 and 8 below) by engaging the envy-free concept and using Lemmas 2 and 3. There are 6 conditions ensuring no-envy, compared to the 4 equilibrium conditions (22)-(25): although a large enough b^1 deters Firms 2 and 3 from deviating to slot 1, it does not prevent them from envying Firm

[19] Letting the "conditional" probabilities of buying in slots i and $i + k$, and prices in slots i and $i + k$ be denoted respectively Q_i, Q_{i+k}, and P_i, P_{i+k}, the IV is $P_iQ_i - P_{i+k}Q_{i+k} = P_{i+k}\Delta Q + Q_i\Delta P$, with $\Delta Q = Q_i - Q_{i+k}$ and $\Delta P = P_i - P_{i+k}$.

[20] By symmetry, we suppress the firm identity argument of the IV functions in Figures 3a and 3b.

1 which pays only b^2. However, Lemmas 2 and 3 apply as their incremental value conditions hold in this private value setting and hence it suffices to take into account the 4 local one step conditions, which are the downward-jump equilibrium conditions, (22) and (24), along with (in general)

$$b^2 \geq \hat{b}^2\left(b^3\right) \equiv \gamma_1 b^3 + IV_2^1(2), \tag{26}$$

$$b^3 \geq \underline{b}^3 \equiv IV_3^2(3), \tag{27}$$

which respectively ensure Firm 2 does not envy Firm 1 and Firm 3 does not envy Firm 2.

As shown in Figure 3b, the set of envy-free equilibrium bids involve only two lines, and valid bid combinations are on both lines and above the diagonal. This is because incremental values do not depend on firm identity with symmetry and $\gamma_1 = \gamma_2$ so the RHS of (26) and (27) coincide with those of (22) and (24) respectively. Then there is only one envy-free equilibrium candidate, which is $b^2 = \bar{b}^2 = IV_3^1(1)$ (because $\tilde{b}^2(\bar{b}^3) = \bar{b}^2$ as shown in Figure 3a) and $b^3 = \underline{b}^3$. Although this point always exists (as long as incremental values are positive), it may not lie above the diagonal so we could have $b^2 < b^3$ (if $IV_2^1(2)$ is far enough below $IV_3^2(2)$ and γ is small). However, $b^2 > b^3$ is guaranteed for q large enough (as was true too for the equilibrium triangle in Figure 3a to not be empty: in both cases it suffices that $IV_2^1(1) > IV_3^2(2)$). Indeed, the condition for $IV_2^1(1) > IV_3^2(2)$ reduces to $q(1-\gamma)\gamma > \Delta$. This is clearly true for q large enough. And, in concord with the analysis of Section 4.2, it holds true for $\Delta = 0$.

Product heterogeneity, which we examine next, implies that incremental values do not just depend on the two positions involved in a putative move. They also depend on the firm which is contemplating a move, which firms are being jumped over, and which firms follow the lower slot (firms placed before the upper slot are not relevant because incremental values are normalized by the number of clicks in that slot).

4.3.1 Demand Height

Consider first heterogeneity in demand height only, so that $\gamma_1 = \gamma_2 = \gamma_3 = \gamma$. The identity of the firm which is jumped over and those following the lower slot affect the incremental value through the size of the change in the search appeal externality.

Assume that the joint profit maximizing order involves positioning higher quality products earlier: that is $q_i \geq q_j$ if and only if $\Phi_i^\pi \geq \Phi_j^\pi$. Then, in an equilibrium that maximizes joint profit we have $q_1 \geq q_2 \geq q_3$ and $\Phi_1^\pi \geq \Phi_2^\pi \geq \Phi_3^\pi$. We now show that Lemmas 2 and 3 apply under these assumptions. With 3 firms, Lemma 2 merely requires that $IV_3^2(1) \geq IV_3^2(2)$, or $(1-\gamma)\left((1-\gamma)q_1 - \Delta_3\right) \geq (1-\gamma)((1-\gamma)q_2 - \Delta_3)$, and this is clearly the case because $q_1 \geq q_2$. To apply Lemma 3 we need to check that $IV_2^1(2) \geq IV_2^1(3)$ or $(1-\gamma)((1-\gamma)(q_2 - \Delta_3) - \Delta_1) \geq (1-\gamma)((1-\gamma)(q_3 - \Delta_2) - \Delta_1)$. This is equivalent to $q_2 + \Delta_2 \geq q_3 + \Delta_3$. Now

$$q_2 + \Delta_2 \geq q_2 + \Delta_2 - \gamma(q_2 - q_3) \geq (1-\gamma)q_3 + \Delta_3 + \gamma q_3 = q_3 + \Delta_3,$$

where the first inequality follows from $q_2 \geq q_3$ and the second inequality holds because $\Phi_2^\pi \geq \Phi_3^\pi$. From Lemmas 2 and 3 we infer that necessary and sufficient conditions for an envy-free equilibrium are (22), (24), (26) and (27) along with $b^2 \geq b^3$. Figure 4 shows the envy-free region and the diagonal. Because $\Phi_2^\pi \geq \Phi_3^\pi$ we have $IV_3^2(2) \geq IV_3^2(3)$ and hence the ordering of the two vertical lines, which are derived from (24) and (27). Similarly, the positions of the two oblique lines induced by (22) and (26) follow from $\Phi_1^\pi \geq \Phi_2^\pi$. Although the envy-free region is clearly never empty, it is not guaranteed that there exist points in that set that lie above the diagonal (for reasons similar to those already discussed in the symmetric case). However, taking all q's large ensures that the entire envy-free set satisfies the desired bid ordering $b^2 \geq b^3$.

As shown in the Appendix, these arguments generalize to n firms:

Proposition 5 *Assume $\gamma_i = \gamma$ for all i, $q_i \geq q_{i+1}$ and $\Phi_i \geq \Phi_{i+1}$ for $i = 1, ..., n-1$. For q_n sufficiently large, there exists an envy-free equilibrium with Firm i in slot i and bids satisfying*

$$b^i - \gamma b^{i+1} = IV_i^{i-1}(i-1), \; i = 2, ..., n-1, \tag{28}$$

with $b^n = IV_n^{n-1}(n-1)$.

However, any other order can also be an equilibrium outcome if qualities and search appeals are sufficiently similar (by continuity with the symmetric case). Under the assumption in Proposition 5 that joint profit maximization requires placing higher quality products earlier, other equilibrium orders are not joint profit maximizing and hence not envy-free (because any envy-free equilibrium must maximize TIP when γ's are the same.[21]) Nonetheless we now show that, if qualities are heterogeneous enough, then any equilibrium order should approximately rank products in decreasing order of quality. This means that, under the assumptions of Proposition 5, the equilibrium order should be approximately joint profit maximizing. This also shows by means of a counter example, that the coincidence of the decreasing quality order with joint profit maximization is necessary to ensure the existence of a joint profit maximizing equilibrium.

For three firms, if qualities are different enough, then in equilibrium we must necessarily have $q_1 > q_3$. To show this, the necessary equilibrium condition $\underline{b}^2 \leq \bar{b}^2$ implies

$$(1+\gamma)q_1 - q_3 \geq \frac{\Delta_3}{1-\gamma}.$$

This requirement cannot be met if $q_3 > 2q_1$ with $\gamma \in (0, 1)$ and $\Delta_3 \geq 0$. Hence if qualities are heterogeneous enough so that the larger of two qualities is at least twice the smaller one, then the top firm always has a higher quality than the bottom firm in equilibrium.

The graphical intuition for these results is quite simple. If we start from the symmetric case in Figure 3a and increase q_1 or decrease q_3, the vertical distance between both $\tilde{b}^2$ (b^3) and $\bar{b}^2$ (the equilibrium conditions for Firm 1, (22) and (23)), and $\underline{b}^2$ (the equilibrium condition for Firm 3, (25)) widens, which expands the equilibrium set. The reverse happens if q_1 is lowered while q_3 rises. The only constraint on q_2 in order for an equilibrium to exist is that it should not be so small that $\bar{b}^3$ does not lie too far to the left (it should not be to the left of the crossing point between $\tilde{b}^2$ (b^3) and $\underline{b}^2$ whenever these two constraints cross). This is why there is no restriction on how q_2 relates to the other products' qualities, even when qualities

[21] When γ's are the same the LHS of (15) and (18) are the same and so these inequalities directly imply that any local envy free equilibrium maximizes TIP for each consecutive firm pair and so for all firms. The local envy-free condition also implies TIP maximization for per impression bidding regardless of the γ's but not for per click bidding.

are very heterogeneous.

Although this result allows for an order that is quite different from the quality order with only 3 firms, it can be generalized to any number of firms. The following proposition states that, with many firms, the equilibrium order must be approximately that of decreasing product qualities and, under the assumptions of Proposition 5, joint profit is approximately at its maximum: the proposition is proved in the Appendix.

Proposition 6 *Assume $\gamma_i = \gamma$ for all i and that for all $i, j = 1, ..., n$ with $i \neq j$, either $q_i > 2q_j$ or $q_j > 2q_i$. Then, in any equilibrium, the product with the ith highest quality, $i = 1, ..., n$ is placed at slot $i-1$, slot i, or slot $i+1$.*

4.3.2 Demand width

We now look at heterogeneity in demand width alone, so that products only differ in terms of market potential (and have common q and Δ). From Section 3, the preferences of firms and consumers regarding the ranking of products are opposite. Joint profit maximization requires that products with small market potentials (large γ_i) come first, whereas consumers prefer the reverse.[22]

This is the source of product heterogeneity in Chen and He (2011) and Athey and Ellison (2011). A firm's profit depends not only on its position and its own product's popularity, but also on the popularity of the products sold by firms that precede it in the search order. This is the business stealing externality (following Chen and He, 2011).[23]

In the models of Chen and He (2011) and Athey and Ellison (2011), each product meets a consumer's needs with some probability which differs across products and is identical for all consumers. All products are sold at the same fixed price. This is replicated in our framework with no search appeal and identical qualities, where the common price is the

[22]This is the case if the distribution of matches above q is reshuffled in such a way that β_i and α_i are identical across products - as γ_i is decreased, it is possible to shift weight from $v_i = 0$ to $v_i \in [q, q + \Delta)$, while concentrating it close enough to q so α and β do not change.

[23]The externality would introduce more substantial differences with the private value setting under per impression bidding. This is because business stealing from earlier firms affects a firm's profit through the number of clicks it receives, so it does not affect the per click revenue whereas it would affect the per impression revenue.

common quality q and Firm i's product meets a consumer's needs with probability $1-\gamma_i$, the market potential. They characterize an equilibrium of the generalized second price auction where the more popular firms are positioned earlier so $\gamma_i \leq \gamma_{i+1}$ for $i = 1, ..., n-1$. In their settings this is the preferred order for both firms and consumers. From Propositions 2 and 4 this is also the case in the model we consider. Note however that these preferences are weak. Without search appeal and with identical qualities, total profit and consumer surplus are unaffected by the search order. All products are sold at price q and total sales are merely determined by the probability that at least one of the products suits a consumer's preferences, so that total revenue is independent of the firms' order.[24] Consumer surplus is always zero because the expected surplus from buying one of the products is exactly equal to the expected search cost.[25] This preference becomes strict for consumers with $\Delta > 0$, still assuming identical search appeal across firms, but it is reversed for firms: as discussed in Section 3, total industry profit is larger when less popular products are positioned earlier.

Contrary to what happens when demand height is heterogeneous, the joint profit maximizing order cannot be sustained if there is too much heterogeneity in market potential. The equilibrium necessary condition $\bar{b}^2 \geq \underline{b}^2$ implies that

$$((1-\gamma_2\gamma_3)q - 2\Delta)(1-\gamma_1) - ((1-\gamma_2)q - \Delta)(1-\gamma_3) \geq 0. \tag{29}$$

As q tends to infinity, $\frac{((1-\gamma_2\gamma_3)q-2\Delta)(1-\gamma_1)}{(1-\gamma_2)q-\Delta}$ tends to

$$\frac{(1-\gamma_2\gamma_3)(1-\gamma_1)}{(1-\gamma_2)}.$$

Now if $1-\gamma_1 < (1-\gamma_2)^2$, this is bounded above by $(1-\gamma_2\gamma_3)(1-\gamma_2)$ so that, for q large enough, condition (29) requires

$$(1-\gamma_2\gamma_3)(1-\gamma_2) \geq (1-\gamma_3).$$

However, $1-\gamma_2\gamma_3 < 1$, so the above cannot hold if $1-\gamma_2 < 1-\gamma_3$. This illustrates that if market potentials are heterogeneous enough (in the sense that the lower of two

[24]In Athey and Ellison (2011), profit is strictly larger with more popular products first because they assume some heterogeneity in search costs. In Chen and He (2011) what is critical in maximizing joint profit is that the three most popular products are in the top three slots.

[25]With $\Delta_i = 0$ in (11), we have $\alpha_i = 0$ and hence $\Phi_i^{CS} = 0$ for all firms.

market potentials is less than the square of the larger one), then there is no equilibrium with $1-\gamma_1 < 1-\gamma_2 < 1-\gamma_3$ for q large enough. Hence it is not possible to achieve joint profit maximization. This is readily generalized to n firms, which yields the next proposition proved in the Appendix.

Proposition 7 *Assume that q and Δ are common and that for all $i, j = 1, ..., n$ with $i \neq j$, $\min\{1-\gamma_i, 1-\gamma_j\} < (\max\{1-\gamma_i, 1-\gamma_j\})^2$. Then, for q large enough, there is no equilibrium such that $\gamma_i \geq \gamma_{i+1}$, $i = 1, ..., n-1$ (the joint profit maximizing order).*

By contrast, the reverse order - which is the one preferred by consumers - can always be sustained as an equilibrium outcome (for q sufficiently large). It is straightforward to check that Lemma 2 applies. Indeed, with $1-\gamma_1 \geq 1-\gamma_2$, we have $IV_3^2(1) = (1-\gamma_1)((1-\gamma_3)q-\Delta) \geq (1-\gamma_2)((1-\gamma_3)q-\Delta) = IV_3^2(2)$. Similarly, the incremental value condition for Lemma 3 holds because $IV_2^1(2) = (1-\gamma_2)((1-\gamma_1)(q-\Delta)-\Delta) \geq (1-\gamma_3)((1-\gamma_1)(q-\Delta)-\Delta) = IV_2^1(3)$. However, in order for one-step envy-free constraints, (24) and (27) to hold, bids should satisfy $b^3 \in [IV_3^2(3), IV_3^2(2)] = [(1-\gamma_3)((1-\gamma_2)q-\Delta), (1-\gamma_2)((1-\gamma_3)q-\Delta)]$: if $1-\gamma_2 > 1-\gamma_3$, the interval is empty unless $\Delta = 0$, in which case both incremental values are equal to $(1-\gamma_2)(1-\gamma_3)$. In other words, there is no envy-free equilibrium sustaining this order with large firms in front unless $\Delta = 0$. This is because, when looking at two consecutive slots, the larger firm loses more from dropping its price if it is in front: this is also the logic underpinning the optimal joint profit maximizing order, which has smaller firms positioned early.

Our proof of the existence result for n firms (see Proposition 8 below) uses the property that there exist envy-free equilibria with firms with large market potential positioned early for $\Delta = 0$. Although we give below a more direct existence argument for three firms and any $\Delta \geq 0$, we take a short detour to consider the case with no search appeal externality ($\Delta = 0$). It is illustrated by Figure 5a which shows the four local envy-free constraints. Constraints (24) and (27) become a single vertical line because $IV_3^2(2) = IV_3^2(3)$ for $\Delta = 0$. Constraints (22) and (26) are depicted by the two oblique lines, where the steeper one corresponds to

(22) because $\gamma_2 \geq \gamma_1$ (product 2 is less popular). Note that, although the incremental value is the same for Firms 1 and 2 (so the two intercepts coincide), the two constraints differ because there is a difference in incremental cost: it is lower for Firm 1 moving in front of Firm 2 than for Firm 2 moving in front of Firm 1 and this difference increases as the price paid in slot 2, b^3, increases (this is because Firm 1 is more likely than Firm 2 to have to pay for a click if it is in slot 2). These two oblique lines cut the vertical line to determine the segment of envy-free bid combinations. Interestingly, contrary to the envy-free set for heterogeneous demand height, this segment is guaranteed to be above the diagonal because $IV_2^1(1) = (1-\gamma_2)(1-\gamma_1)q \geq (1-\gamma_2)(1-\gamma_3)q = IV_3^2(2)$. We used a similar property to establish that bids are decreasing in the private value setting with $\Delta = 0$ without having to impose any further restriction on q.

We now return to the existence of an equilibrium for any $\Delta \geq 0$ using the graphical approach. Because Lemma 2 applies, we can omit equilibrium condition (23) and we need only represent 3 constraints on the equilibrium set, which results in a triangle. It is depicted in Figure 5b. First notice that $\bar{b}^3 \leq \underline{b}^2$ (because $1-\gamma_2 \geq 1-\gamma_3$). Hence, (24) and (25) ensure that the equilibrium triangle, if it exists, lies above the diagonal, so a sufficient condition for the equilibrium set to be non-empty is that the top vertex is above $\underline{b}^2$. Since $1-\gamma_1 \geq 1-\gamma_3$, a sufficient condition is

$$\gamma_2(1-\gamma_2)(1-\gamma_3)q \geq ((\gamma_2(1-\gamma_2) + (1-\gamma_1)(2-\gamma_2) - (1-\gamma_3))\Delta,$$

which holds for q large enough. The next proposition generalizes the existence result to n firms and is proved in the Appendix.

Proposition 8 *Assume $q_i = q$, $\Delta_i = \Delta$ for all i, and $\gamma_i \leq \gamma_{i+1}$ for $i = 1, ..., n-1$. For q sufficiently large, there exists an equilibrium such that Firm i is in slot i and bids satisfy*

$$b^i - \gamma_i b^{i+1} = IV_i^{i-1}(i-1), \;\; i = 2, ..., n-1, \tag{30}$$

with $b^n = IV_n^{n-1}(n-1)$.

At first glance, the bids in Proposition 8 (different γ's) look like those in Proposition 5 (same γ's). Indeed, both involve each firm bidding up to the highest point at which the firm above does not want to drop down. But there are substantial differences. In Proposition 5 the equilibrium is envy-free for firms to be in the profit-maximizing order,[26] and (weaker) firms are bidding above their incremental values so that they would not want the higher slot even if they got it at their own bid. The equilibrium in Proposition 8 is not envy free so some firms - in particular, the bottom Firm n - would want to be higher if they could get a higher slot at their own equilibrium bids, but are deterred by the sufficiently high bids of stronger firms above.[27] This means that the lowest firm bids below its value of the next slot up in the equilibrium.

Proposition 5 extends the GSP auction results of Varian (2007) and Edelman *et al.* (2007) to allow for search appeal externalities and moreover that these can be firm specific (the Δ_i): there exists an envy-free equilibrium that implements the TIP maximizing order. We go further in Proposition 6 to argue that this order is robust in the sense that any equilibrium must be "close" to this order (under the conditions given). Chen and He (2011) and Athey and Ellison (2011), by suppressing pricing, also do not allow for search appeal externalities (i.e. $\Delta_i = 0$ for them) but they do address heterogeneous business stealing externalities (the γ_i). Then they find that the auction outcome orders firms by decreasing market potential (which both maximizes TIP and consumer surplus).[28] We find in Proposition 7 that this order sustains when we allow for search appeal externalities ($\Delta > 0$). However, it is not TIP maximizing in the presence of search appeal, but instead has firms following the order preferred by consumers. Proposition 8 shows that this is the robust order and TIP maximization is not.

[26] As noted earlier, local envy-free is equivalent to joint profit maximization and so to global profit maximization. Any order that does not maximize joint profits cannot be sustained by local envy-free bids.

[27] To see that Firm n envies Firm $n-1$, Firm n would get profit $\pi^{n-1}(n) - b^n\Gamma_{n-1}$ if it could get the higher slot at its own bid, as opposed to the status quo profit $\pi^n(n)$. It prefers the former if $IV_n^{n-1}(n) > b^n = IV_n^{n-1}(n-1)$, or $q(1-\gamma_{n-1}) - \Delta > q(1-\gamma_n) - \Delta$, which is true because $\gamma_{n-1} < \gamma_n$ in the stipulated order - stronger firms are earlier.

[28] Recall that Athey and Ellison (2011) assume heterogeneous search costs.

5 Conclusions

Ordered search characterizes the lion's share of the modern online economy, which is only growing in importance. Yet research so far has been stymied for lack of a tractable set-up, even in the symmetric case, let alone dealing with the full set of product distinguishers that we do here. One main accomplishment of the paper is to deliver a clean analysis for ordered search under asymmetry, which we effectuate by invoking a positive lowest willingness to pay for interested consumers. We address three forms of asymmetries among firms' products, which are expressed as three key shape parameters which drive bids. Bids are determined from incremental values and incremental costs from moving ahead. These values and costs depend on own demand shape parameters and those of rivals jumped over. Each parameter entails a different effect on other firms' profits (position externality) when a firm bids for a slot in a position auction.

First, higher quality earns higher profit earlier and entails no externalities, so higher-quality firms bid more for earlier spots and so higher qualities get earlier slots, ceteris paribus. This effect encourages maximal total industry profit (TIP).

Second, own search appeal has no effect on a firm's own incremental profit from moving, so it has no direct impact on bids. But a move up entails a positive profit externality on all firms jumped over through increasing their prices. When a firm moves up it raises prices now behind it. If the firms it jumps have high search appeal (which triggers a low price in front), it suffers much in terms of lower price and so is less tempted to jump such firms than jumping firms with low search appeal. This effect sorts to the back the firms with low search appeal, which tends to promote TIP maximization but imparts a negative externality on consumers.

We bring together these first two effects as demand height which impacts pricing. As we show, if firms are the same in terms of demand width (the third parameter) the TIP-maximizing order of firms is an equilibrium order in the auction and is also robust. This is bad for consumers because prices are raised.

Third, demand width (market potential) induces the business stealing effect. Width directly affects firm incremental values (and incremental costs) as well as inflicting a negative externality on the firms jumped over. The direct effect and the externality are stronger for stronger firms. This observation helps understand why the TIP-maximizing outcome is jeopardized when widths differ. Because the stronger-width firms tend to rise to the top, consumers benefit but other firms' profits suffer. As we show, when only widths differ the order of firms that maximizes consumer welfare is an equilibrium to the position auction and is moreover a robust order.

The analysis is facilitated by key properties of the model. The identities of those firms coming before matters for the size of incoming demand, and the identities of those coming after matters to equilibrium prices. However, the order in which predecessors or successors are presented has no bearing on a firm's profit at a particular position in the search order. This key property enables us to determine summary statistics for firms, which are firm specific and independent of position. These statistics enable us to determine optimal rankings of firms that maximize total profits and consumer surplus. Comparing these rankings enables us to determine the tensions between the various parties.

Establishing the existence of an incremental value bidding equilibrium (or indeed, any equilibrium) is challenging in the presence of position externalities. In this regard, it is noteworthy that despite the billions of dollars spent on position auctions, there is little work beyond the classic 2007 papers and Gomes and Sweeney (2014) (which close down the externalities), with the notable exceptions of Chen and He (2011) and Athey and Ellison (2011) who assume prices are exogenous. Athey and Ellison (2011) conclude that the outcome to the position auction is efficient for both firms and for consumers. However, our set-up reveals tensions between orders preferred by consumers and those benefitting firms when prices are endogenous. When demand height heterogeneity is dominant, consumer welfare tends to take the back seat.

Appendix

A1 Results from Section 2

Claim 1 *For any $\gamma_i \in (0,1)$, $q_i > 0$ and $\Delta_i > 0$ there exists a distribution function F_i, whose support has a maximum $q_i + B_i < \infty$, which satisfies (1).*

Proof. It is useful to define, for all $v_i \geq q_i$, $\delta_i = v_i - q_i$: then the support of δ_i should be a subset of $[0, B_i]$. Let $\bar{F}_i$ be the distribution function for δ_i. We can then rewrite (1) using $F_i(v) = \gamma_i + (1-\gamma_i)\bar{F}_i(v - q_i)$ for $v \geq q_i$ as

$$(1-\gamma_i)\int_{\Delta_i}^{B_i}\left(1-\bar{F}_i(\delta)\right)d\delta = s. \tag{31}$$

The left-hand side can be made arbitrarily close to $(1-\gamma_i)(B_i - \Delta_i)$ by moving all the weight of the distribution of δ_i in the neighborhood of B_i (so that $\bar{F}_i$ is nearly 0 on $[0, B_i]$, except in a small neighborhood of B_i) and, equal to 0, by shifting all the weight of δ_i below Δ_i (so $\bar{F}_i(\delta) = 1$ for $\delta > \Delta_i$). Furthermore, $B_i - \Delta_i$ can be made as large as necessary by increasing B_i. Hence, for any $\Delta_i > 0$, $\gamma_i \in (0,1)$, $q_i > 0$ and $s > 0$, it is possible to find some specification of $\bar{F}_i$ such that (31) holds. ■

Lemma 1 *Assume $f_i(q_i) > 0$ and $v_i - q_i < B_i < \infty$ for all $v_i \in S$, $i = 1, ..., n$. If consumers search optimally from Firm 1 to Firm n expecting all the firms to price according to (3), then it is optimal for any Firm i to charge price p_i defined by (3) as long as q_i is sufficiently large, $i = 1, ..., n$.*

Proof. If Firm $i < n$ charges its candidate equilibrium price p_i, it earns per click profit $(1-\gamma_i)p_i$. At this price, it sells to all consumers with strictly positive willingness to pay for its product who have reached it. Hence, it cannot gain additional profit by charging a lower price. Assume therefore that it charges a price that is $\Delta p > 0$ in excess of p_i. Its corresponding profit is then at most

$$(p_i + \Delta p)\left(1 - \gamma_i - (1 - F_{i+1}(q_{i+1} + \Delta_{i+1}))(F_i(q_i + \Delta p) - \gamma_i)\right).$$

This upper bound on deviation profit is obtained as follows. First, if Firm i deviates to $p_i + \Delta p$, then all consumers with valuations less than Δp in excess of q_i search Firm $i+1$

(recall that at price p_i consumers holding match q_i with Firm i are just indifferent between buying product i and searching on). Hence, the probability that a consumer who does not search in equilibrium chooses to search Firm $i+1$ is $F_i(q_i+\Delta p)-\gamma_i$. Among those searching consumers, those with valuations $v_{i+1} > q_{i+1} + \Delta_{i+1}$ with Firm $i+1$, strictly prefer buying product $i+1$ and never return to Firm i (indeed, since the equilibrium price difference is Δ_{i+1}, those consumers would prefer buying product $i+1$ even if Firm i had not increased its price from its equilibrium level and they had chosen to search nonetheless). Hence, Firm i's demand at price $p_i+\Delta p$ is at most $1-\gamma_i-(1-F_{i+1}(q_{i+1}+\Delta_{i+1}))(F_i(q_i+\Delta p)-\gamma_i)$ and the profit gain from the price increase is bounded above by

$$\Delta p(1-\gamma_i)-(p_i+\Delta p)(1-F_{i+1}(q_{i+1}+\Delta_{i+1}))(F_i(q_i+\Delta p)-\gamma_i).$$

First consider a small deviation with Δp close to zero. Because $f_i(q_i)=F'(q_i)$, $\frac{F_i(q_i+\Delta p)-F_i(q_i)}{\Delta p}$ tends to $f_i(q_i)$ as Δp tends to 0. Hence there exists $\bar{\delta}$ such that if $\Delta p < \bar{\delta}$, then $F_i(q_i+\Delta p)-F_i(q_i) > \frac{f_i(q_i)}{2}\Delta p$. Then, because $F_i(q_i)=\gamma_i$, the benefit from deviating is bounded above by

$$\Delta p\left((1-\gamma_i)-(p_i+\Delta p)(1-F_{i+1}(q_{i+1}+\Delta_{i+1}))\frac{f_i(q_i)}{2}\right),$$

which is negative if p_i is large enough, since $f_i(q_i)>0$ and $F_{i+1}(q_{i+1}+\Delta_{i+1})<1$.

Now take a large deviation, $\Delta p > \bar{\delta}$. If follows that $F_i(q_i+\Delta p)\geq F_i(q_i+\bar{\delta})>\gamma_i$. Since a price above q_i+B_i would yield zero profit, an upper bound for the deviation gain is

$$(q_i+B_i-p_i)(1-\gamma_i)-(p_i+\Delta p)(1-F_{i+1}(q_{i+1}+\Delta_{i+1}))(F_i(q_i+\bar{\delta})-\gamma_i).$$

From the pricing expression (3), price p_i is increasing in q_i and q_i-p_i does not depend on q_i. Hence, for q_i large enough, the above upper bound on the profit change from a large price increase is negative, so that such a deviation is not profitable.

The above arguments go through for $i=n$, where $F_{i+1}(q_{i+1}+\Delta_{i+1})$ is replaced by 1 (all the consumers who give up buying product n at $q_n+\Delta p$ select not to buy any product so none of them return to Firm n). $\blacksquare$

A2 Equilibrium ordering of n products (Section 4.3)

Here we provide the general analysis of equilibrium product ordering. We parallel the analysis for 3 products in Section 4.3 text assuming there are $n \geq 2$ products. We examine in turn heterogeneity in demand height and heterogeneity in demand width.

A2.1 Demand heights

We first consider when all products have the same market potential while allowing for heterogeneity in base quality and search appeal. Each Firm $j = 1, ..., n-1$ suffers a negative externality which depends on the search appeal Δ_ℓ of all the firms $\ell = j+1, ..., n-1$ that follow it. The relevant cases for the equilibrium analysis below are when j is positioned weakly earlier than i (which applies for downward deviations by j) and when j is positioned weakly farther down than $i+k$ (which applies for upward deviations by j). We have the following expressions for the incremental value for Firm j for position i over $i+k$. For coming down from an earlier slot:[29]

$$IV_{i+k}^{i}(j) = (1-\gamma)\left((1-\gamma^k)(q_j - \sum_{\ell > i+k} \Delta_\ell) - \sum_{\ell=i+1}^{i+k} \Delta_\ell\right) \quad \text{if } j \leq i, \tag{32}$$

(because if some Firm $j < i$ ends up in slot i, Firm i is promoted to slot $i-1$) and for coming up from a later slot:

$$IV_{i+k}^{i}(j) = (1-\gamma)\left((1-\gamma^k)(q_j - \sum_{\ell \geq i+k; \ell \neq j} \Delta_\ell) - \sum_{\ell=i}^{i+k-1} \Delta_\ell\right), \quad \text{if } j \geq i+k.$$

(because if some Firm $j > i+k$ ends up in slot $i+k$, Firm $i+k$ is demoted down to slot $i+k+1$ and contributes to the search appeal externality in slot $i+k$).

From equations (15) and (17) in the text, equilibrium conditions are

$$b^{i+1} - \gamma^k b^{i+k+1} \leq IV_{i+k}^{i}(i), \quad i = 1, ..., n-1, \ k = 1, ..., n-i,$$

and

$$b^{i} - \gamma^k b^{i+k+1} \geq IV_{i+k}^{i}(i+k), \quad i = 1, ..., n-1, \ k = 1, ..., n-i.$$

[29] For completeness, note that for $i < j < i+k$ (a case that is not used in the analysis), we have $IV_{i+k}^{i}(j) = (1-\gamma)\left((1-\gamma^k)(q_j - \sum_{\ell > i+k} \Delta_\ell) - \sum_{i \leq \ell \leq i+k; \ell \neq j} \Delta_\ell\right)$.

Proposition 5 *Assume $\gamma_i = \gamma$ for all i, $q_i \geq q_{i+1}$ and $\Phi_i \geq \Phi_{i+1}$ for $i = 1, ..., n-1$. For q_n sufficiently large, there exists an envy-free equilibrium with Firm i in slot i and bids satisfying*

$$b^i - \gamma_i b^{i+1} = IV_i^{i-1}(i-1),\ i = 2, ..., n-1,$$

with $b^n = IV_n^{n-1}(n-1)$.

Proof. Incremental values of Firm j can be written out as the sum of a term that depends on q only and a term that depends on Δ only. Define $D_{i+k}^i(j) = IV_{i+k}^i(j) - (1-\gamma)\left(1-\gamma^k\right) q_j$ which is a term that does not depend on q_j and is decreasing in Δ_ℓ for all $\ell > i$.

The argument for bids being decreasing in i builds on the corresponding analysis in the private value case with $\Delta = 0$. In that case we derive a lower bound for the bid difference $b^i - b^{i+1} \geq \gamma^{n-i}(1-\gamma)^2 q_i$, which is strictly positive and linearly increasing in q_i. To deal with the case with $\Delta_j \geq 0$ for all $j > i$, it suffices to add a term $D_j^{j-1}(j-1)$ to the incremental values $IV_j^{j-1}(j-1)$ so the bid difference expression is modified by $\bar{D} = \sum_{j=i}^n \gamma^{j-i} D_j^{j-1}(j-1)$ which does not depend on any q_j, $j = i, ..., n$. Hence, if q_n is large enough so q_i is as well, the bid difference is indeed positive.

Let us now turn to showing that firms do not want to deviate to lower slots. The specification of bids in (28) implies that Firm $i-1$ is indifferent between staying in slot $i-1$ and moving down to slot i, for all $i = 2, ..., n$. Furthermore, because $q_i \geq q_{i+k}$ for all $i = 1, ..., n-1$, $k = 1, .., n-k$, $IV_{i+k+1}^{i+k}(i) \geq IV_{i+k+1}^{i+k}(i+k)$ (notice that the two incremental value expressions depends neither on Δ_i nor on Δ_{i+k} so only the quality differences are relevant) and Lemma 2 applies, so Firm $i-1$ does not want to deviate to any lower slot.

Finally, Lemma 3 can be applied to establish that no firm envies any firm that precedes it so upward deviations are unprofitable. Indeed, $\Phi_{i-1} \geq \Phi_i$ is equivalent to $IV_i^{i-1}(i-1) \geq IV_i^{i-1}(i)$ so that the proposed bids ensure that Firm i would not want to move to slot $i-1$ while paying b^i. In order for Lemma 3 to apply we also need

$$\begin{aligned} IV_{i+1}^i(i+1) &= (1-\gamma)((1-\gamma)(q_{i+1} - \sum_{j>i+1} \Delta_j) - \Delta_i) \\ &\geq IV_{i+1}^i(i+k) = (1-\gamma)((1-\gamma)(q_{i+k} - \Delta_{i+1} - \sum_{j>i+1, j\neq i+k} \Delta_j) - \Delta_i), \end{aligned}$$

for $i = 1, ..., n-2$, $k = 2, ...n-i$. Because $\Phi_{i+1} \geq \Phi_{i+k}$, $q_{i+1}+\Delta_{i+1}-\gamma q_{i+1} \geq q_{i+k}+\Delta_{i+k}-\gamma q_{i+k}$ and using $q_{i+1} \geq q_{i+k}$ we have $q_{i+1} + \Delta_{i+1} \geq q_{i+k} + \Delta_{i+k}$. This is necessary and sufficient to have $IV^i_{i+1}(i+1) \geq IV^i_{i+1}(i+k)$. ■

Proposition 6 *Assume $\gamma_i = \gamma$ for all i and that for all $i, j = 1, ..., n$ with $i \neq j$, either $q_i > 2q_j$ or $q_j > 2q_i$. Then, in any equilibrium, the product with the ith highest quality, $i = 1, ..., n$ is placed at slot $i-1$, slot i, or slot $i+1$.*

Proof. Consider some $i = 2, ..., n-1$ and $r = 1, ..., n-i$. Equilibrium condition (15) evaluated at $k = r+1$ yields

$$b^i - \gamma^{r+1} b^{i+r+1} \leq IV^{i-1}_{i+r}(i-1)$$

and equilibrium condition (17) evaluated at $k = r$ yields

$$b^i - \gamma^r b^{i+r+1} \geq IV^i_{i+r}(i+r).$$

With $\gamma < 1$, $\gamma^{r+1} < \gamma^r$ so we must have

$$IV^{i-1}_{i+r}(i-1) \geq IV^i_{i+r}(i+r)$$

which requires that

$$(1-\gamma^{r+1})(q_{i-1} - \sum_{\ell > i+r} \Delta_\ell) - \sum_{\ell=i}^{i+r} \Delta_\ell \geq (1-\gamma^r)(q_{i+r} - \sum_{\ell > i+r} \Delta_\ell) - \sum_{\ell=i}^{i+r-1} \Delta_\ell$$

Rearranging, we must have

$$(1-\gamma^{r+1})q_{i-1} - (1-\gamma^r)q_{i+r} \geq (\gamma^r - \gamma^{r+1}) \sum_{\ell > i+r} \Delta_\ell + \Delta_{i+r}$$

The RHS is positive so the LHS should be positive as well, and we must have

$$\frac{1-\gamma^{r+1}}{1-\gamma^r} q_{i-1} \geq q_{i+r}.$$

Now,

$$\frac{1-\gamma^{r+1}}{1-\gamma^r} = 1 + \gamma^r \frac{1-\gamma}{1-\gamma^r},$$

which is decreasing in r. Hence it is bounded above by $1+\gamma$ and this is at most 2 because $\gamma < 1$. So we must have

$$q_{i-1} > \frac{1}{2} q_{i+r}.$$

Under the assumptions in the Proposition this requires that $q_{i-1} > q_{i+r}$. This must hold for all $i = 2, ..., n-1$ and $r = 1, ..., n-i$. So the result holds. ■

A2.2 Demand widths

We assume common $\Delta \geq 0$ and common quality q high enough for all i. Products merely differ as to how popular they are with consumers, as measured by the market potential $1-\gamma_i$ for product i. Because of the business stealing externality, the exact formulation for incremental values depends on where Firm j is positioned in equilibrium with respect to the range of slots i to $i+k$ impacted by the move that is being contemplated by Firm j. The incremental value for some Firm j being placed in some slot $i = 1, ..., n-1$ rather than in slot $i+k$, $k = 1, ..., n-i$ is[30]

$$IV_{i+k}^{i}(j) = (1-\gamma_j)\left((1-\Pi_{\ell=i+1}^{i+k}\gamma_\ell)(q-(n-i-k)\Delta) - k\Delta\right), \text{ if } j \leq i, \tag{33}$$

for a firm that would have initially moved down to slot i, while

$$IV_{i+k}^{i}(j) = (1-\gamma_j)\left((1-\Pi_{\ell=i}^{i+k-1}\gamma_\ell)(q-(n-i-k)\Delta) - k\Delta\right), \text{ if } j \geq i+k, \tag{34}$$

for a firm that would have initially moved up to slot $i+k$.

Equilibrium conditions (15) and (17) in the text are

$$b^{i+1} - \prod_{\ell=i+1}^{i+k} \gamma_\ell b^{i+k+1} \leq IV_{i+k}^{i}(i),\ i = 1, ..., n-1,\ k = 1, ..., n-i, \text{with } b^{n+1} = 0 \tag{35}$$

and

$$b^{i} - \prod_{\ell=i}^{i+k-1} \gamma_\ell b^{i+k+1} \geq IV_{i+k}^{i}(i+k),\ i = 1, ..., n-1,\ k = 1, ..., n-i, \text{with } b^{n+1} = 0. \tag{36}$$

[30]Again, the case $i < j < i+k$ is not relevant to the analysis: we would then have $IV_{i+k}^{i}(j) = (1-\gamma_j)\left((1-\Pi_{i\leq\ell\leq i+k,\ell\neq j}\gamma_\ell)(q-(n-i-k)\Delta) - k\Delta\right)$.

Proposition 7 *Assume that q and Δ are common and that for all $i, j = 1, ..., n$ with $i \neq j$, $\min\{1-\gamma_i, 1-\gamma_j\} < (\max\{1-\gamma_i, 1-\gamma_j\})^2$. Then, for q large enough, there is no equilibrium such that $\gamma_i \geq \gamma_{i+1}$, $i = 1, ..., n-1$ (the joint profit maximizing order).*

Proof. Consider some $i = 2, ..., n-1$. Equilibrium conditions (15) and (17) imply

$$IV_{i+1}^{i-1}(i-1) \geq b^i - \gamma_i \gamma_{i+1} b^{i+2} \geq b^i - \gamma_i b^{i+2} \geq IV_{i+1}^{i}(i+1),$$

where the second inequality holds because $\gamma_{i+1} < 1$. Hence we must have

$$\begin{aligned} & \left((1-\gamma_i\gamma_{i+1})(q-(n-i-2)\Delta)-2\Delta\right)(1-\gamma_{i-1}) \\ \geq \; & \left((1-\gamma_i)(q-(n-i-1)\Delta)-\Delta\right)(1-\gamma_{i+1}), \end{aligned}$$

or

$$\frac{(1-\gamma_i\gamma_{i+1})(q-(n-i-2)\Delta)-2\Delta}{(1-\gamma_i)(q-(n-i-1)\Delta)-\Delta}(1-\gamma_{i-1}) \geq (1-\gamma_{i+1}).$$

As q tends to infinity the LHS tends to $\frac{(1-\gamma_i\gamma_{i+1})}{(1-\gamma_i)}(1-\gamma_{i-1})$. Now if $1-\gamma_{i-1} \leq 1-\gamma_i$ then the assumption in the proposition implies that $1-\gamma_{i-1} < (1-\gamma_i)^2$, so for q large enough equilibrium would require that

$$(1-\gamma_i\gamma_{i+1})(1-\gamma_i) > (1-\gamma_{i+1}).$$

Because $1-\gamma_i\gamma_{i+1} < 1$ the above inequality cannot hold if $1-\gamma_i \leq 1-\gamma_{i+1}$. ■

Proposition 8 *Assume $q_i = q$, $\Delta_i = \Delta$ for all i, and $\gamma_i \leq \gamma_{i+1}$ for $i = 1, ..., n-1$. For q sufficiently large, there exists an equilibrium such that Firm i is in slot i and bids satisfy*

$$b^i - \gamma_i b^{i+1} = IV_i^{i-1}(i-1), \; i = 2, ..., n-1,$$

with $b^n = IV_n^{n-1}(n-1)$.

Proof. Incremental values can be written out as the sum of a term that depends on q only and a term that depends on Δ only. Define $D_{i+k}^i(j) = IV_{i+k}^i(j) - (1-\gamma_j)\left(1-\Pi_{\ell=i}^{i+k-1}\gamma_\ell\right)q$ for $j \geq i+k$ and $D_{i+k}^i(j) = IV_{i+k}^i(j) - (1-\gamma_j)\left(1-\Pi_{\ell=i+1}^{i+k}\gamma_\ell\right)q$ for $j \leq i$, which is a term that does not depend on q and is decreasing in Δ.

We first show that bids that satisfy (30) are strictly decreasing in position i. To ease notation, let $d_i = D_i^{i-1}(i=1)$. For $i = n-1$, $b^{i+1} = b^n = IV_n^{n-1}(n-1)$ and, for $i = 2, ..., n-2$,

$b^{i+1} = IV^i_{i+1}(i) + \sum_{j=i+1}^{n-1}\prod_{\ell=i+1}^{j}\gamma_\ell IV^j_{j+1}(j) = (1-\gamma_i)(1-\gamma_{i+1})q + \sum_{j=i+1}^{n-1}(\prod_{\ell=i+1}^{j}\gamma_\ell - \prod_{\ell=i+1}^{j+1}\gamma_\ell)(1-\gamma_j)q + d_{i+1} + \sum_{j=i+1}^{n-1}\prod_{\ell=i+1}^{j}\gamma_\ell d_{j+1}$. Now, because $(1-\gamma_j) \leq (1-\gamma_i)$ for all $j > i$ we have $b^{i+1} \leq (1-\prod_{\ell=i+1}^{n}\gamma_\ell)(1-\gamma_i)q + \bar{D}$ where $\bar{D} = \sum_{j=i+1}^{n-1}\prod_{\ell=i+1}^{j}\gamma_\ell d_{j+1}$ (canceling out terms in the sum). Thus we have

$$b^i - b^{i+1} \geq IV^{i-1}_i(i-1) - (1-\gamma_{i-1})b^{i+1} \geq \prod_{\ell=i+1}^{n}\gamma_\ell(1-\gamma_{i-1})(1-\gamma_i)q + d_i + (1-\gamma_{i-1})\bar{D}.$$

The above lower bound on bid difference is strictly positive for q sufficiently large and it can be made arbitrarily large by increasing q.

We now turn to showing that firms do not want to deviate to lower slots. The specification of bids in (30) implies that Firm $i-1$ is indifferent between staying in slot $i-1$ and moving down to slot i, for all $i = 2, ..., n$. Furthermore, because $1-\gamma_j \geq 1-\gamma_{j+k}$ for all $j = 1, ..., n-1$, $k = 1, .., n-j$, $IV^{j+k}_{j+k+1}(j) \geq IV^{j+k}_{j+k+1}(j+k)$ and Lemma 2 applies, so Firm $i-1$ does not want to deviate to any lower slot.

To deal with upward jumps consider a deviation upwards by some Firm $i+k$, $i = 1, ..., n-1$, $k = 1, ..., n-i$ to slot i. Assume first $\Delta = 0$. Then we have $IV^{i-1}_i(i-1) = IV^{i-1}_i(i)$ and the bids defined by (30) ensure that Firm i does not envy Firm $i-1$, for $i = 2, ..., n$. Besides, for all $k \geq 1$ we have $1-\gamma_{i+k} \leq 1-\gamma_i$. As a result, $IV^i_{i+1}(i) > IV^i_{i+1}(i+k)$. Hence, for $\Delta = 0$, Firm $i+k$ does not envy Firm i as a result of Lemma 3. That is to say, Firm $i+k$ does not want to move to slot i even if it could do so while paying b^{i+1}. Hence we have

$$b^{i+1} - \Pi_{\ell=i}^{i+k-1}\gamma_\ell b^{i+k+1} \geq IV^i_{i+k}(i+k),$$

or

$$(1-\gamma_{i-1})(1-\gamma_i)q + \sum_{j=i+1}^{i+k}\Pi_{\ell=i}^{j-1}\gamma_\ell\,(1-\gamma_{j-1})\,(1-\gamma_j)q \geq (1-\gamma_{i+k})\left(1-\Pi_{\ell=i}^{i+k-1}\gamma_\ell\right)q. \quad (37)$$

Now consider some $\Delta > 0$. If Firm $i+k$ deviates to slot i, it pays b^i and this is not profitable if $b^i - \Pi_{\ell=i}^{i+k-1}\gamma_\ell b^{i+i+1} \geq IV_{i+k}^i(i+k)$ or, equivalently,

$$\begin{aligned} b^i - b^{i+1} &\geq (1-\gamma_{i-1})(1-\gamma_i)q + \sum_{j=i+1}^{i+k} \Pi_{\ell=i}^{j-1}\gamma_\ell (1-\gamma_{j-1})(1-\gamma_j)q \\ &\quad + D_i^{i-1}(i-1) + \sum_{j=i+1}^{i+k} \Pi_{\ell=i}^{j-1}\gamma_\ell D_j^{j-1}(j-1) \\ &\geq (1-\gamma_{i+k})\left(1-\Pi_{\ell=i}^{i+k-1}\gamma_\ell\right) q + D_{i+k}^i(i+k). \end{aligned}$$

The above equilibrium condition differs from the envy-free condition (37) because of the bid difference $b^i - b^{i+1}$ which is positive and can be made arbitrarily large by choosing q large and because of terms on both sides that account for Δ possibly being different from zero and that do not depend on q. Hence, for q large enough, Firm $i+k$ does not want to deviate to slot i, which completes the proof. ■

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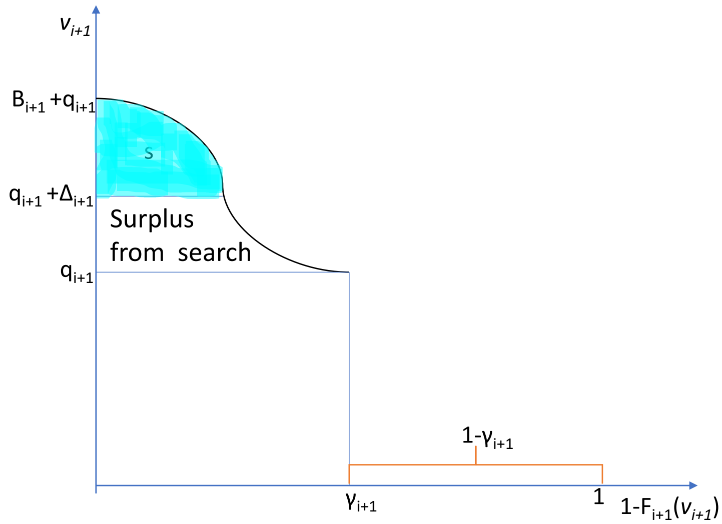

FIGURE 1: If price difference equals quality difference, $p_i = p_{i+1} + (q_i - q_{i+1})$, consumer holding q_i at Firm i will buy at Firm $i + 1$ if $v_{i+1} \geq q_{i+1}$

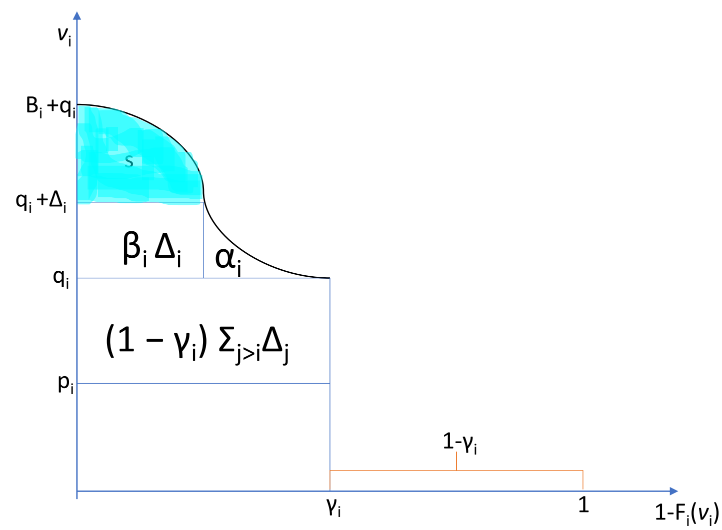

FIGURE 2: Consumer surplus

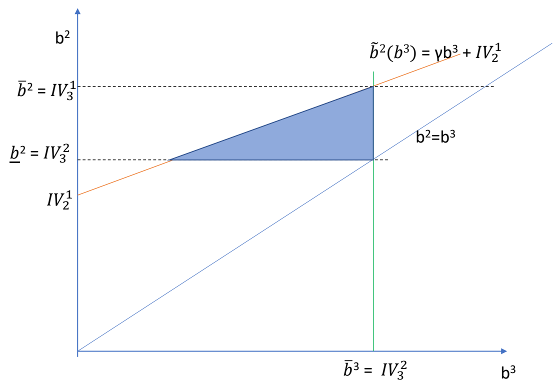

FIGURE 3a: Equilibrium bid set (shaded) under symmetry.

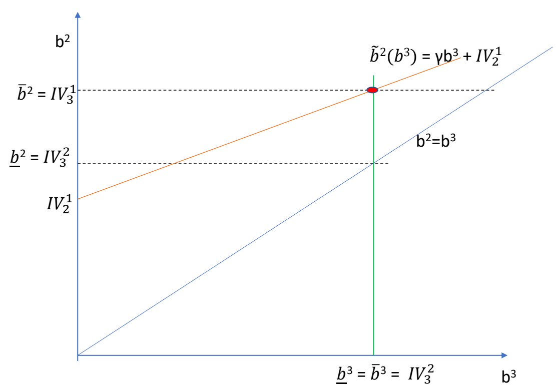

FIGURE 3b: Envy-free equilibrium bids under symmetry, (b^2,b^3) = (IV_3^1, IV_3^2).

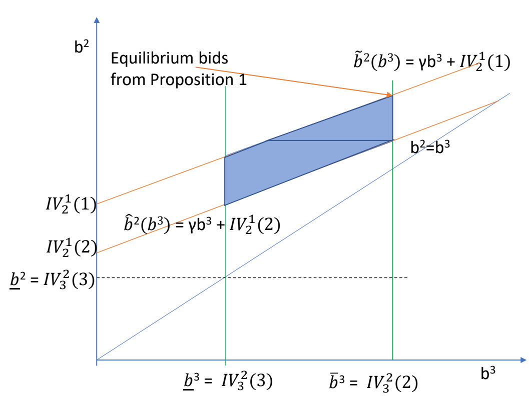

FIGURE 4: Envy-free equilibrium bids (shaded) with
demand height heterogeneity only

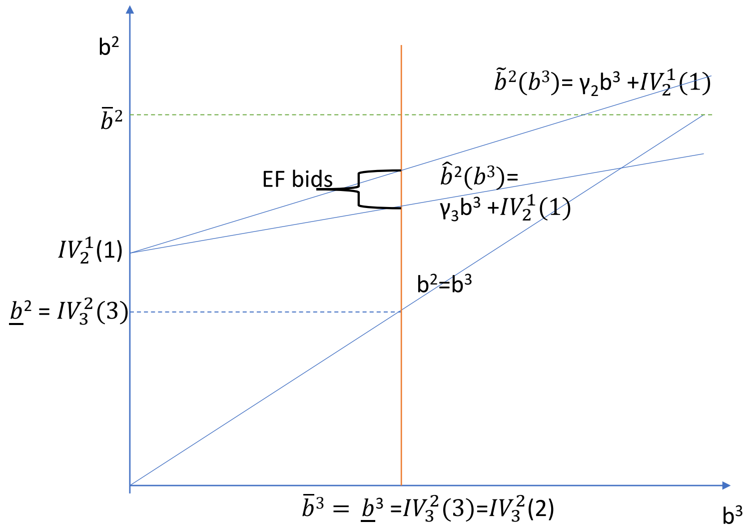

FIGURE 5a: Envy-free equilibrium bid interval for heterogeneous widths, Δ=0, with firms ordered by decreasing market potential

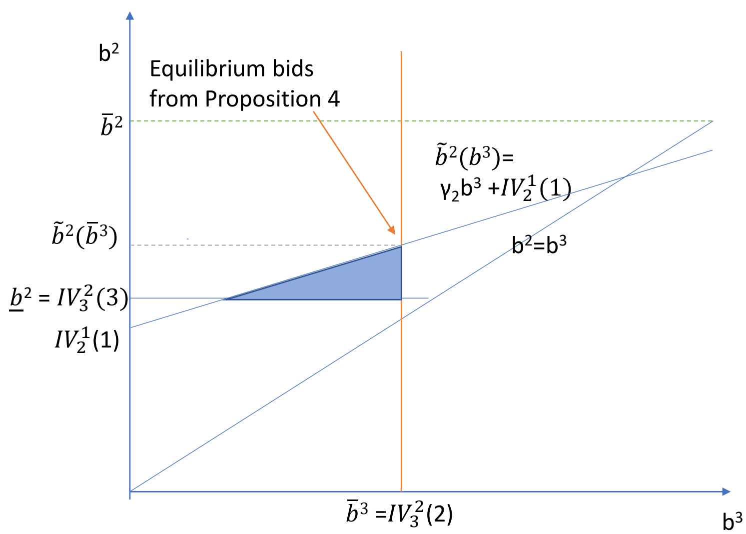

FIGURE 5b: Equilibrium bid set for heterogeneous widths with firms ordered by decreasing market potential

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