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Keywords: Blotto budgets, intra-group conflict, threshold public good, collective action, All-pay auction, incomplete information

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1 Introduction

In the episode "The Dragon and the Wolf" of the fantasy drama *Game of Thrones* the bitterly hostile rulers Cersey Lannister and Daenerys Targaryen have a diplomatic meeting and deliberate about possible collaboration. The army of the Dead from the North poses a lethal threat to both dominions. Daenerys whose territory is more immediately threatened by the invasion is interested in a truce. Cersey whose territory is more distant from the point of possible invasion agrees to a truce, but plans to break that agreement, let Daenerys fight alone and defeat Daenerys Targaryen when she has been weakened by the battle with the army from the North. This piece of fiction alludes to a strategic problem of more general nature. It is well-documented for major military conflicts in the last two centuries that the members of military alliances think beyond the time when they might have defeated their enemy. As the former Secretary of State James F. Byrnes (1947, p. 44) describes from his memories of the Yalta conference:

> One statement of Stalin's that interested me was: "It is not so difficult to keep unity in time of war since there is a joint aim to defeat the common enemy, which is clear to everyone. The difficult task will come after the war when diverse interests tend to divide the Allies. It is our duty to see that our relations in peacetime are as strong as they have been in war."

Byrnes (1947, p. 45) reminds us that the "tide of Anglo-Soviet-American friendship had reached a new high" at the Yalta conference, but that the "tide began to ebb" very quickly. The quote illustrates that alliance members think about the continuation game in which they might have to solve their internal quarrels about how to split the rents from winning.[1] This general insight is in line with findings in the experimental laboratory[2] and has been explored in a number of more specific formal

[1] Historians discuss this for the Napoleonic wars (O'Connor 1969), for the Great War (Bunselmeyer, 1975, p. 15) and for the members of the great alliance in the Second World War (Weinberg, 1994, p. 736).

[2] Ke, Konrad and Morath (2013, 2015) find behavior in line with players who anticipate the future conflict subgame inside the winner group and adjust their contributions to their group's efforts inside their alliance.

frameworks.[3] The analysis often assumes that the players mobilize new resources and incur additional fighting costs in the internal conflict that follows if their team wins. However, the strength of a player in the team might be affected, or even determined by the amount of effort contributed to team effort, and might be the residual strength that remains after fighting. Assuming a fixed total amount of resources that a player can use for a sequence of fights, Klumpp, Konrad and Solomon (2019) analyze this in the context of dynamic multi-battle contests. In a conference paper, Klumpp and Konrad (2019) study an inter-group contest followed by intra-group fighting among the winners. Their analysis, like most of the contributions to this literature, assumes that players inside the group have perfect information about each other. In particular, they know their own as well as all other players' precise budgets.

The current analysis studies the most natural perspective of incomplete information about the players' resource budgets in such a framework. Incomplete information gives each of the players an information advantage from being able to make their plans based on precise knowledge of their own budgets. It also moderates and smoothens the conflict, as expending an additional unit of resource for the common good will typically change the win probability only marginally. To illustrate the importance of incomplete information in applications, returning to historical examples, the members of the Grand Alliance might have anticipated what became the Cold War after World War II, but they might have been mutually uncertain about each other's stocks of army equipment, state of weapon technology, the ability to launch a nuclear counterstrike, or the other player's military capacity more generally.

Consider the dilemma of a player with a given stock of fighting resources: using a larger amount of it to support the group's military objectives (joint acquisition or defence) makes them more successful. But it leaves this player in a weaker position once this joint fight was successful. Even though the resources might be given and 'use-it-or-lose-it', as is commonly assumed in Colonel Blotto games, there is an opportunity cost of using a unit of resource for the common good: it has a two-fold positive externality for the other member of the group. It makes success of the group

[3] Katz and Tokatlidu (1996), Wärneryd (1998), Konrad (2004) and Münster (2007) advance the theory about the dilemma of an in-group conflict emerging inside a victorious alliance.

more likely, and it makes it more likely that the other group member can win and acquire the asset for himself in the ensuing fight inside the group. These effects are strongest if the two types of conflicts are fully discriminatory in the way assumed here: in the sense that small differences in the resources expended can make a large difference. And even a small amount of incomplete information about the budget of the co-player becomes highly relevant.

We study this problem for a group of two players who are ex ante symmetric. Two types of symmetric threshold equilibrium are identified. One type is called joint-contribution threshold equilibrium: the players' joint efforts are just sufficient to succeed. The other type is called stand-alone-contribution threshold equilibrium: the group succeeds because one or potentially both players contribute the full amount of resources needed to acquire the asset. This type of equilibrium is characterized by underprovision of group effort for some range of budget combinations, optimal provision of group effort by only one player, or overprovision for another range of budget combinations. All the equilibria described are inefficient ex ante compared to coordinated action that maximizes the sum of payoffs of the group members.

The work is related to the literature on private provision of a discrete public good (see, e.g., Bagnoli and Lipman 1989, 1992) and, in particular, to the contributions studying this problem when the potential contributors have incomplete information about how much they value the public good (Menezes, Monteiro and Temini 2000, Barbieri and Malueg 2008) or about each other's costs (Bliss and Nalebuff 1983, Fudenberg and Tirole 1986). In the Blotto game here the provision occurs with resources that are 'use-it-or-lose-it': The opportunity cost of higher effort emerges indirectly as the resources have an important alternative use if and only if the amounts contributed are sufficient to win the later confrontation with the other group member.

The paper is also related to the guns-and-butter models of conflict and their extensions (see, e.g., Haavelmo 1954, Hirshleifer 1985, Skaperdas 1992, Münster 2007, and the survey by Garfinkel and Skaperdas 2007). These models consider players who have some endowments that can be used for different purposes: 'butter' and 'guns' in the simplest case, where butter becomes the public good for the two players, and with the remainder of their budget they fight about this amount of butter.

The analysis shares with the guns-and-butter literature that a player who devotes more resources to the production of consumable output automatically produces less guns, i.e., is weaker in the distributional conflict between them. Perhaps most close to the analysis here is the guns-and-butter model by Hodler and Yektaş (2012) who consider incomplete information about the resource endowment. There, the distributional conflict is an all-pay-auction without noise, like in the model here, but overall productive output ('butter') is a continuous public good: output is a continuous function of efforts. In our context the group fights against a given threshold: the group's achievement is a discrete public good. This causes a two-fold discontinuity: whether the collective task is achieved, and who wins the internal conflict.

The analysis is also related to the growing literature on conflict between players with given budgets who fight along multiple fronts. Framed in the military context, such games are often described as how Colonel Blotto and his adversary would allocate given amounts of military resources on several battlefields, where each player might allocate idiosyncratic values to winning the various battlefields.[4] A version of this game with incomplete information about competing players' valuations is by Kovenock and Roberson (2011); incomplete information about players' resource endowments is studied by Adamo and Matros (2009). However, these frameworks are constant-sum games in which players are rivals throughout, whereas in our framework the game is not a constant-sum game: players have a common goal ('winning the asset for the group') and adversarial goals ('winning the intra-group contest').

2 The framework

Consider a group with $n = 2$ members. Each of them has an initial budget, and these budgets are denoted by m_i with $i \in \{1, 2\}$. The budget size characterizes the player's type. These budgets are independent random draws from the same cumulative distribution function F with full support on $[0, \bar{m}]$. We assume F to be

[4] The name for this game is attributed to Gross and Wagner (1950). Major more recent contributions are Roberson (2006) and Roberson and Kvasov (2012). For a survey see Roberson and Kovenock (2010).

continuous, differentiable and concave on $[0, \bar{m}]$.[5] This characterizes the type space and distribution of types. Each player knows his own budget and believes that the other player's budget is a random draw from the distribution F.

The players simultaneously choose their contributions $x_1 \in [0, m_1]$ and $x_2 \in [0, m_2]$ that sum up to joint contributions $x_1 + x_2$ and also determine the player's remaining resources $m_i - x_i$. Given the players' beliefs about their co-player's budget, their strategies are functions that map their own budget into contributions: $x_1(m_1)$ and $x_2(m_2)$. Payoff of player 1 is determined as

$$\pi_1 = \begin{cases} 1 & \text{if} \quad x_1 + x_2 \geq b \text{ and } m_1 - x_1 > m_2 - x_2 \\ \frac{1}{2} & \text{if} \quad x_1 + x_2 \geq b \text{ and } m_1 - x_1 = m_2 - x_2 \\ 0 & \qquad\qquad \text{otherwise} \end{cases} \tag{1}$$

and the payoff of player 2 is described analogously, replacing 1 by 2 and vice-versa. The payoff function (1) shows that the budget is 'use-it-or-lose-it'. Players are in a special type of Blotto game. They are members in a fight of their team for a common threshold goal but they also fight inside the own team about the allocation of what the team wins by reaching the goal. The variable $b > 0$ is a given positive constant and observed by the players prior to their effort choices x_1 and x_2. It is the threshold which needs to be matched or topped by the joint efforts of players 1 and 2.

A pair of strategies $x_1^*(m_1)$ and $x_2^*(m_2)$ is a Bayesian Nash equilibrium if, given the players' beliefs about the co-players' budgets, these strategies are mutually optimal replies, i.e., if

$$x_1^*(m_1) = \arg \max_{x_1 \in [0, m_1]} \{ \int_0^{\bar{m}} \pi_1(x_1; x_2^*(m_2)) dF(m_2) \}$$

and analogously for player 2.

The conflict game studied has two discontinuities for each player. The first dis-

[5] In addition to $F(m)$ being a smooth and atomless cumulative distribution function, the concavity assumption deserves to be highlighted. The uniform distribution and many right-skewed distributions comply with this assumption. Prominent examples are the exponential distribution and the Pareto distribution that is empirically particularly relevant as a characterization of incomes (or endowments).

continuity is on the sum of contributions $x_1 + x_2$ that need to match or exceed b in order to generate positive payoff inside the group. The second discontinuity is on the determination of the winner inside the group, conditional on $x_1 + x_2$ exceeding b. As players want to make group victory likely but also to outbid their internal rival exactly in this case of victory, this begs the question of existence of (pure-strategy) equilibrium. Klumpp and Konrad (2019) showed that such equilibria exist with a specific knife-edge property, basically making the discontinuity in whether the group wins interact with the discontinuity for how they split the gain from winning.[6] Incomplete information makes this type of knife-edge equilibrium infeasible, but at the same time smoothens the problem and allows for different types of equilibrium.

We focus on symmetric equilibrium in monotonic threshold strategies, i.e., equilibria in which the mutually optimal replies have the following property: $x_i^*(m_i) = 0$ for all $m_i \in [0, m^*)$ and $x_i^*(m_i) = t$ for all $m_i \in [m^*, \bar{m}]$, for some $m^* \in (0, \bar{m})$ and $t \in (0, m^*]$. The equilibrium concept is Bayesian Nash equilibrium.

2.1 Stand-alone equilibria

First we consider equilibria in which single players might make contributions that are sufficient to 'stand alone', i.e., sufficient to match or exceed b. Hence, in the equilibrium with threshold m^*, the players contribute an amount $x_i^*(m_i) = b$ if and only if $m_i \geq m^*$.

Proposition 1 *Let $b < \bar{m}$ and suppose that F is (weakly) concave on the support $[0, \bar{m}]$. There is a unique symmetric standalone equilibrium with threshold m^* defined by*

$$F(m^*) = F(m^* + b) - F(m^* - b), \tag{2}$$

where m^* satisfies $m^* \in (b, 2b]$ and $m^* < \bar{m}$.

As we prove in the appendix, existence of stand-alone threshold equilibrium mainly requires two conditions. First, types just above the threshold m^* (who con-

[6] For instance, for $m_1 + m_2 > b$ and $\max\{m_1, m_2\} - \min\{m_1, m_2\} < b$ an equilibrium exists for which $m_1 - x_1 = m_2 - x_2$ and $x_1 + x_2 = b$.

tribute an amount b) lose against any other type who contributes. Those types can only win against types with a very low budget: a budget smaller than $m_i - b \to m^* - b$. This requires the threshold m^* to be sufficiently large for standalone contributions to be optimal. Formally, for types $m_i \downarrow m^*$, the probability that their remaining budget is sufficient to beat their internal rival ($\Pr(m_j < m^* - b)$) must be larger than the probability that deviating to a free-riding strategy is successful, which is $\Pr(m_j > m^* \text{ and } m_j - b < m^*)$. A sufficiently high threshold m^* guarantees a sufficiently high win probability even when contributing.

Second, types m_i just below the threshold (who free-ride in the equilibrium) need to face a sufficiently high probability that their internal rival contributes b (otherwise the group does not win) but, net of contributing, has left less than the endowment of type m_i. This requires the threshold m^* to be sufficiently small. Formally, for types $m_i \uparrow m^*$, the probability that their internal rival contributes and has a remaining budget smaller than their own ($\Pr(m_j > m^* \text{ and } m_j - b < m^*)$) must be larger than the probability that deviating to a contribution is successful ($\Pr(m_j < m^* - b)$). These two restrictions on incentives of types just above and just below the threshold characterize a unique symmetric standalone equilibrium.[7]

Figure 1 shows that there are four different sets of player types and that player i's equilibrium expected payoff is non-monotone in m_j for intermediate values budgets m_i. First, there are players i who never win: those with $m_i < m^* - b$ whose resources are not sufficient to beat a player j who contributes. As $m^* \in (b, 2b]$ and, hence, $m^* - b \leq b$, those players do not have sufficient resources to make a stand-alone contribution. Second, there are players who free-ride and get a positive expected payoff: those with $m_i \in (m^* - b, m^*)$. Those players would be able to contribute (at least if m_i is sufficiently close to m^*) but they prefer to contribute zero, hoping that m_j is in some intermediate interval so that they have a chance to beat j in the internal conflict. Third, there are players who contribute (with $m_i \in [m^*, m^* + b)$) but only win if m_j is either below $m_i - b$ or between m^* and m_i. Those players are sufficiently strong to contribute and still win against at least a share of the free-riding players j

[7] Concavity of F ensures that considering incentives to deviate reduces to considering the types around the threshold m^*. For details see the proof in the appendix.

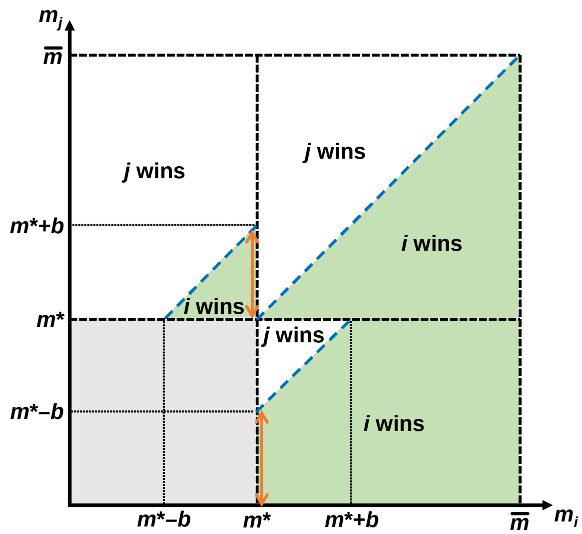

Figure 1: The figure illustrates the different combinations (m_i, m_j) for which either i or j wins in the standalone equilibrium. Focusing on player i, there are four different regions (player types). From left to right, there are (i) players i with $m_i < m^* - b$ who never win since $m_j - b > m_i$ for all j who contribute; (ii) players i with $m_i \in (m^* - b, m^*)$ who win if and only if $m_j \in (m^*, m_i + b)$; (iii) players i with $m_i \in (m^*, m^* + b)$ who win if $m_j < m_i - b$ or if $m_j \in (m^*, m_i)$; (iv) players i with $m_i > m^* + b$ who win if and only if $m_j < m_i$. The expected payoff of a type $\tilde{m}_i$ is obtained by drawing a vertical line at $m_i = \tilde{m}_i$ and calculating the probability that m_j is in the interval where $\tilde{m}_i$ wins. Hence, the arrow just to the left of m^* indicates the expected payoff of types m_i just below m^* and the arrow just to the right of m^* indicates the expected payoff of types m_i just above m^*. In equilibrium, these two payoffs must be identical in order for deviations of types just below and just above m^* to be non-profitable.

(those with $m_j < m_i - b$) but they are beaten by free-riding players j with a budget close to m^*. Finally, there is a group of players with very high resources. Provision of the stand-alone effort to the team is best for them and they never lose against free-riding players j. But still, if such a player fights with another, even stronger team member, then this other member also contributes the stand-alone effort, and still beats this player.

Corollary 1 summarizes the equilibrium strategies for the special case of a uniform distribution, for which a closed form solution for $m^*(b)$ exists.

Corollary 1 *Suppose F is a uniform distribution on $[0,1]$.*
(i) If $b < 1/3$, there is a unique symmetric standalone equilibrium where $x_i^* = b$ if $m_i \geq m^* = 2b$ and $x_i^* = 0$ otherwise.
(ii) If $b \in [1/3, 1)$, there is a unique symmetric standalone equilibrium where $x_i^* = b$ if $m_i \geq m^* = (1+b)/2$ and $x_i^* = 0$ otherwise.

The proof for the more general proposition for weakly concave or concave F in the appendix also proves the corollary, but the uniform distribution case offers some illustration, as probabilities for m_j being in some interval from $[0,1]$ is equal to the length of the interval. Let the required contributions are small (Corollary 1(i)). Players contribute if and only if $m_i \geq 2b$. This means that a player gets zero expected payoff if and only if $m_i < m^* - b = b$ (compare Figure 1). Here, only players with resources smaller than b get zero expected payoff. The players who free ride but still realize a strictly positive expected payoff would all be sufficiently powerful to make a stand-alone contribution. But doing so they reduce their remaining resources so much that they would be defeated by players from the first group. Thus, they prefer to hold back their resources, hoping that the other player has just somewhat more resources, provides the stand-alone effort and can then be beaten.

To illustrate the reason for uniqueness of the threshold $m^* = 2b$, suppose that player 2 chooses a threshold different from $2b$. For a smaller equilibrium candidate threshold $m_2^* \in [b, 2b)$ consider player 1 with $m_1 = m_2^*$. This player's probability to win for $x_1 = b$ is $m_2^* - b$. For $x_1 = 0$ this player's probability to win is b. But $b > m_2^* - b$ for all $m_2^* \in [b, 2b)$. Hence, player 1 with $m_1 = m_2^*$ is strictly better-off contributing 0 than contributing b. This contradicts a symmetric equilibrium with a threshold m^* below $2b$. Suppose next that the candidate threshold is $m_2^* \in (2b, 1-b]$. Consider player 1 with $m_1 = m_2^*$. This player's payoff from $x_1 = b$ is $m_2^* - b$ and from $x_1 = 0$ it is $m_2^* - (m_2^* - b) = b < m_2^* - b$. This rules out that the optimal threshold $m_1^*(m_2^*) = m_2^*$ for player 1. Similar contradictions can be constructed for the remaining parameter values in the corollary.

These stand-alone equilibria can be compared with several generic games of non-cooperative provision of a discrete public good with stand-alone contributions: the various versions of the volunteer's dilemma. A static version of this problem is studied by Diekmann (1985). Its dynamic version is the waiting game, as in Bliss and Nalebuff (1984) and Fudenberg and Tirole (1986). In these games the provision of the public good has a direct cost for the stand-alone contributor, and contributors' individual benefit from provision of the public good exceeds this cost of stand-alone provision. Players are willing to incur the cost of stand-alone provision if the alternative is that the public good is not provided. But if several players might volunteer, players prefer most to free-ride if they anticipate that others make the provision. Players randomize in Diekmann's (1985) static volunteers' game. Zero, one or multiple players might then expend the cost of stand-alone provision in the equilibrium. In the dynamic versions the problem turns into a waiting game, and incomplete information about other players' contribution cost and the choice of timing for own action resolves the coordination problem between the players.

The stand-alone equilibria here start with a very different framework and focus on a different trade-off. A major goal is also the provision of a discrete public good, but contributions to it are made from a given individual budget and do not cause a genuine cost for the contributing player. They are costly only insofar as they reduce the resource endowment of the contributing player in the internal conflict that might follow the public good provision. This opportunity cost matters if the players' contributions are sufficiently high to be successful. As players do not know the resource endowment of the other player in their group, they face strategic uncertainty. This can make none, one or both players contribute independently. Overprovision occurs if both players are very resource-rich. For uniformly distributed budgets with maximum budget $\bar{m} > 3b$ this happens if both have budgets that exceed $2b$. Their independent decisions can also lead to what can be seen as underprovision from the perspective of the group. Think of the case with uniform F, and with $\bar{m} > 3b$. If players have budgets in the range between b and $2b$, each of them has sufficient resources to make the group win, but none of them contributes the standalone effort and the group does not win. This is precisely the range in which the player prefers to free-ride and hope that the other group member is sufficiently rich to make the contribution, but not too rich so that he could still beat the free-riding player.

2.2 Joint contribution equilibria

Now let us turn to a different type of symmetric threshold equilibrium: an equilibrium in which players might contribute zero or half of the necessary joint amount b. Hence, in the equilibrium with contribution threshold $\hat{m}$, the players contribute an amount $\hat{x}_i(m_i) = b/2$ if and only if $m_i \geq \hat{m}$.

Proposition 2 *Let $b < 2\bar{m}$ and suppose that F is (weakly) concave on the support $[0, \bar{m}]$.*
(i) Suppose that $b < 2\bar{m}/3$. If

$$F(\bar{m}) - F(\bar{m} - b/2) - F(b/2) \geq 0, \tag{3}$$

there is a unique symmetric joint contribution equilibrium characterized by $\hat{m} = b/2$. If (3) is violated, no symmetric joint contribution equilibrium exists.
(ii) Suppose that $b \in [2\bar{m}/3, \bar{m})$ and define $\tilde{b}$ as the (unique) solution to

$$F(\bar{m}) - F\left(\bar{m} - \tilde{b}\right) - F\left(\bar{m} - \frac{\tilde{b}}{2}\right) = 0, \tag{4}$$

where $\tilde{b}$ satisfies $\tilde{b} \in [2\bar{m}/3, \bar{m})$. If $b < \tilde{b}$, no symmetric joint contribution equilibrium exists. If $b \geq \tilde{b}$, the set of symmetric joint contribution equilibria is characterized by

$\hat{m} \in [b/2, z]$ where $z \in (b/2, b]$ is the solution to

$$F(\bar{m}) - F(\bar{m} - b) - F(z) = 0. \tag{5}$$

(iii) If $b \in [\bar{m}, 2\bar{m})$, the set of symmetric joint contribution equilibria is characterized by $\hat{m} \in [b/2, \bar{m})$.

The proof of Proposition 2 is in the appendix. To understand the result intuitively it is crucial to note that only incentives to deviate to stand-alone contributions need to be considered. First of all, players who free ride in the candidate equilibrium (with $m_i < \hat{m}$) do not have sufficient resources to beat a player j who contributes in equilibrium, in case they decide to contribute $b/2$ themselves. Second, players who contribute in equilibrium reduce the probability of winning to zero in case they decide to contribute zero.

For large amounts of resource investments b required by the players (Proposition 2(ii) and (iii)), there is a continuum of joint contribution equilibria characterized by contribution thresholds $\hat{m} \in [b/2, z]$ with $z \in (b/2, b]$. This becomes most obvious for the case where $b \geq \bar{m}$ (Proposition 2(iii)). Here, stand-alone contributions are not feasible for types m_i in the support of F. Since no other contribution may constitute a profitable deviation, any threshold $\hat{m} \in [b/2, \bar{m})$ can be supported as part of a symmetric joint contribution equilibrium. Hence, the set of equilibria includes the joint contribution equilibrium with efficient participation (the one with $\hat{m} = b/2$ where players contribute their share whenever they are able to do so) as well as joint contribution equilibria with inefficiently low participation.

For investment thresholds $b \in [2\bar{m}/3, \bar{m})$ as in Proposition 2(ii), a continuum of joint contribution equilibria can exist because incentives to deviate from a candidate equilibrium are, in some range, independent of the threshold $\hat{m}$. To see why, consider the player with the maximum endowment (with $m_i \rightarrow \bar{m}$) and suppose that $\hat{m} \in (\bar{m} - b, \bar{m} - b/2)$. The candidate equilibrium payoff of this player i is $1 - F(\hat{m})$: she wins if and only if j contributes (i wins the internal conflict against all those players j). If i deviates and contributes $x_i = b$, she also wins against some non-contributing players (those with $m_j < m_i - b$) but now wins against the contributing players j

only if m_j is small (that is, if $m_j \in (\hat{m}, m_i - b/2)$). Comparing candidate equilibrium payoff and deviation payoff shows that the incentive to deviate is independent of $\hat{m}$ in this range.[8] If $\hat{m}$ becomes too large, however, then equilibrium participation is low and so are equilibrium payoffs. Thus, condition (5) defines an upper bound on the contribution thresholds $\hat{m}$ that can be supported in equilibrium.

For small amounts b of resource investments required, existence of a joint contribution equilibrium is not guaranteed since deviations to stand-alone contributions are particularly attractive if the required resources are small. For $b < 2\bar{m}/3$, Proposition 2(i) shows that a joint contribution equilibrium may only exist if it involves efficient participation, that is, if $\hat{m} = b/2$ so that all players with $m_i \geq b/2$ contribute in equilibrium. As the two corollaries below show formally, such an equilibrium exists for uniformly distributed budgets but does not exist for strictly concave probability distributions.

Figure 2 illustrates equilibrium payoffs and incentives to deviate to a standalone contribution for the highest possible types ($m_i \to \bar{m}$). Since types $m_i < \hat{m}$ get zero equilibrium expected payoff (the threshold is never reached), we must have $\hat{m} \in [b/2, b]$: If $\hat{m}$ was larger than b, types $m_i \in (b, \hat{m})$ could get a strictly positive payoff (equal to $\Pr(m_j < m_i - b)$) when deviating to a standalone contribution. Types $m_i > \hat{m}$ win in equilibrium whenever $m_j \in (\hat{m}, m_i)$ so that j contributes but has fewer remaining resources than i. The deviation payoff of the highest possible types ($m_i \to \bar{m}$) depends on whether the threshold $\hat{m}$ is smaller or larger than $\bar{m} - b$ and $\bar{m} - b/2$, respectively. Case (i) in the left panel considers the case where b is small. Here, $\hat{m} < \bar{m} - b$ ensures that, when deviating to a standalone contribution, types $m_i \to \bar{m}$ win against all non-contributing types j. But a deviation to a standalone contribution means that i does not win anymore if j contributes and $m_j \in (m_i - b/2, m_i)$. The solid and the dashed arrow in the left panel of Figure 2 show candidate equilibrium payoff and deviation payoff, respectively. In case (i), if

[8] For details see the proof in the appendix. Formally, the left-hand side of equation (4) is the difference between candidate payoff and deviation payoff for the highest possible budget $m_i = \bar{m}$, in case the required resources are $\tilde{b}$. Since the incentive to deviate is stronger the lower b (the left-hand side of (4) is increasing in b), the threshold $\tilde{b}$ defined by (4) is a lower bound for existence of a joint contribution equilibrium in the range of Proposition 2(ii).

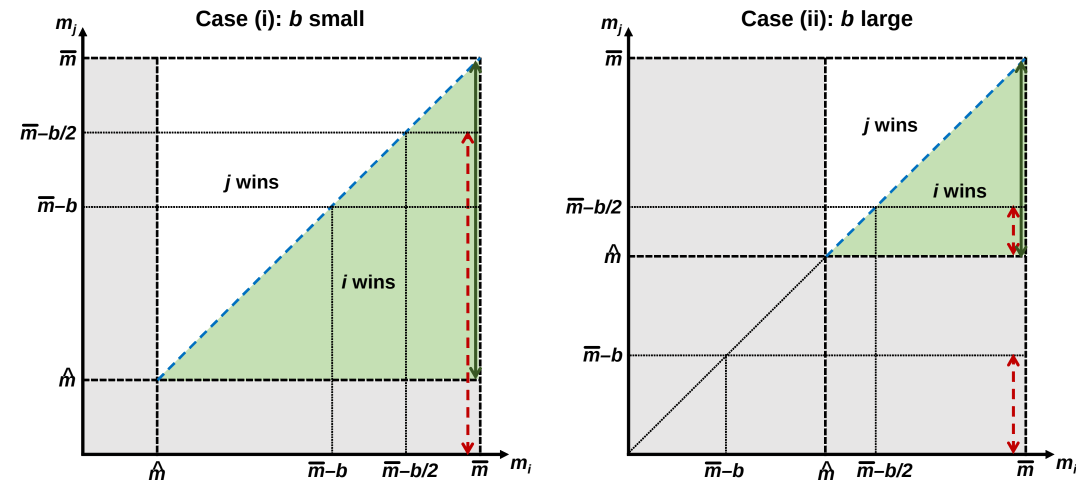

Figure 2: The figure illustrates the different combinations (m_i, m_j) for which either i or j wins in the joint contribution equilibrium, for small vs. large amounts b of resources required. The expected payoff of a type $\tilde{m}_i$ is obtained by drawing a vertical line at $m_i = \tilde{m}_i$ and calculating the probability that m_j is in the interval where $\tilde{m}_i$ wins. Thus, the (solid) arrow at the far right of each of the panels indicates the expected equilibrium payoff of types $m_i \to \bar{m}$. The dashed arrow indicates the expected deviation payoff of types $m_i \to \bar{m}$ when choosing $x_i = b$. Upon deviating, the highest types $m_i \to \bar{m}$ win if $m_j < \bar{m} - b$ or if $m_j \in (\hat{m}, \bar{m} - b/2)$. In case (i) where b is small, high types m_i win against all non-participating types of j when making a contribution b. In case (ii) where b is large, high types m_i lose against some non-participating types of j when making a contribution b.

low values of m_j are particularly likely (as for strictly concave distribution functions F), a symmetric joint contribution equilibrium does not exist.[9]

The right panel of Figure 2 considers a case where b is large and, hence, $\hat{m} > \bar{m} - b$. (Again, necessary condition in equilibrium is $\hat{m} \in [b/2, b]$.) For larger values of b, deviations to standalone contributions are less attractive since, upon deviating to $x_i = b$, i does not only lose against contributing players j with $m_j \in (m_i - b/2, m_i)$, but also loses against non-contributing players j with $m_j \in (m_i - b, \hat{m})$. Whether types $m_i \to \bar{m}$ with $x_i = b$ can still win against contributing players j depends on whether $\hat{m} < \bar{m} - b/2$. As seen in case (ii) of Figure 2 from the solid arrow (candidate equilibrium payoff of types $m_i \to \bar{m}$) and the dashed arrow (deviation payoff of types $m_i \to \bar{m}$ when choosing $x_i = b$), the incentive to deviate is reduced if b is increased (keeping $\hat{m}$ fixed). In this case, a continuum of thresholds $\hat{m}$ can be supported as part of joint contribution equilibria.[10]

The characterization of joint contribution equilibria is simplified when considering the case of uniformly distributed budgets. Corollary 2 illustrates the result of Proposition 2 for this case.

Corollary 2 *Suppose* $F(m) = m$ *on* $[0, 1]$.
(i) If $b < 2/3$*, there is a unique symmetric joint contribution equilibrium characterized by* $\hat{m} = b/2$.
(ii) If $b \in [2/3, 1)$*, the set of symmetric joint contribution equilibria is characterized by* $\hat{m} \in [b/2, b]$.
(iii) If $b \in [1, 2)$*, the set of symmetric joint contribution equilibria is characterized by* $\hat{m} \in [b/2, 1)$.

For the uniform distribution, there is a unique joint contribution equilibrium without efficient participation in case b is small. Once b is large, there is a continuum of equilibria with threshold $\hat{m} \in [b/2, \min\{b, 1\}]$. This set includes the equilibrium

[9] In case (i) of Figure 2, a no-deviation condition is $F(\bar{m}) - F(\bar{m} - b/2) \geq F(\hat{m})$ which, as $\hat{m} \geq b/2$, is violated if F is strictly concave.

[10] In the right panel of Figure 2, if $\hat{m} > \bar{m} - b/2$, then equilibrium payoff for types $m_i \to \bar{m}$ remains $F(\bar{m}) - F(\hat{m})$ whereas the payoff from deviating to $x_i = b$ is $F(\bar{m} - b)$ and, hence, independent of $\hat{m}$. This eplains the upper bound z in Proposition 2(ii) for equilibrium thresholds $\hat{m}$.

where players' participation is efficient as well as equilibria where the threshold is not reached even though the players jointly have sufficient resources. The latter arises because the risk of losing the internal conflict makes the players refrain from increasing their contribution up to b even when they are very resource-rich.

To illustrate this equilibrium further, consider the function Δ that describes the difference between the equilibrium payoff from $x_i = b/2$ and the deviation payoff from $x_i = b$ for $F(m_i) = m_i$ and $m_i = \bar{m} = 1$, which corresponds to (9) in the appendix. As explained above in the context of Figure 2, for the deviation payoff of types $m_i \to \bar{m}$ we need to distinguish whether $\hat{m} > \bar{m} - b$ and $\hat{m} > \bar{m} - b/2$.[11] This yields

$$\Delta(m_i = 1) = \begin{cases} b - \hat{m} & \text{if } 1 > \hat{m} > 1 - \frac{b}{2} \\ \frac{3}{2}b - 1 & \text{if } \hat{m} \in [1 - b, 1 - \frac{b}{2}] \\ \frac{b}{2} - \hat{m} & \text{if } \hat{m} < 1 - b \end{cases}$$

so that the requirement $\Delta\,(m_i = 1) \geq 0$ together with the necessary condition $\hat{m} \in [b/2, b]$ can be mapped into Figure 3. The dark (red) line and the dark (red) area are combinations of b and $\hat{m}(b)$ for which $\Delta(m_i = 1) \geq 0$ holds, i.e., combinations of b and $\hat{m}$ for which a deviation to $x_i = b$ does not pay, not even for player types $m_i = 1$ for which this deviation is most attractive among all player types. Figure 3 illustrates that for $b \in [0, 2/3)$ there is precisely one corresponding value of value of $\hat{m}$ that is feasible and does not invite a profitable deviation. This identifies the unique equilibrium for a given b, for all $b < 2/3$, along the red line.[12] For any larger $b \in (2/3, 2)$ there is a whole set of thresholds $\hat{m}(b) \in [b/2, \min\{b, 1\})$ for which no profitable deviations exist.

The uniform distribution is a special case in that joint contribution equilibria also exist for low values of b. This is no longer true for strictly concave probability distributions. Intuitively, low budgets become more likely if F is concave; the resulting

[11] Referring back to the two cases illustrated in Figure 2, $\Delta(m_i = 1)$ for $\hat{m} < 1 - b$ corresponds to the payoff difference illustrated by the solid and the dashed arrow in case (i) of Figure 2 and $\Delta(m_i = 1)$ for $\hat{m} \in [1 - b, 1 - b/2]$ corresponds to the payoff difference illustrated by the solid and the dashed arrow in case (ii) of Figure 2.

[12] With $\hat{m} \in [b/2, b]$, $\Delta\,(m_i = 1) \geq 0$ requires $\hat{m} \leq b/2$ if $\hat{m} < 1 - b$. If $\hat{m} \geq 1 - b$, $\Delta\,(m_i = 1) \geq 0$ is violated for $b < 2/3$ (compare the respective conditions illustrated in Figure 3).

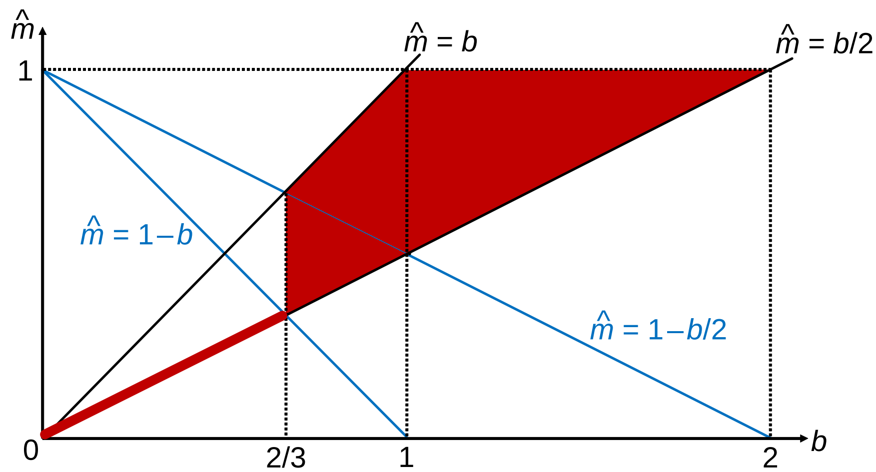

Figure 3: The figure illustrates the set of equilibria characterized in Corollary 2, i.e., for a uniform distribution $F(m_i) = m_i$ on the unit interval $[0, 1]$. The equilibrium combinations of $(b, \hat{m}(b))$ are represented by the highlighted (thick) line for small values of b and the highlighted area for larger values of b. Any equilibrium combination must necessarily be inside the cone generated by the feasibility constraint $\hat{m} \geq b/2$ and the condition $\hat{m} \leq b$ which is a necessary equilibrium condition (see step 3 in the proof of Proposition 2). For $b \in [2/3, 1)$ all $\hat{m}(b) \in [b/2, b]$ are thresholds for which $x_i(m_i) = 0$ if $m_i < \hat{m}$ and $x_i(m_i) = b/2$ if $m_i \geq \hat{m}$ characterize the equilibrium choices. Since $\hat{m}$ cannot be larger than $\bar{m} = 1$ by definition, for $b \in [1, 2)$ the interval for equilibrium thresholds becomes $\hat{m}\,(b) \in [b/2, 1)$. For $b < 2/3$, however, there is one single threshold $\hat{m}(b) = b/2$ for which a symmetric joint contribution equilibrium exists. For small b, if a higher threshold than $\hat{m} = b/2$ is chosen, it is too attractive for resource-rich players to deviate to a standalone contribution which has low cost and still ensures a sufficiently high probability of winning.

low probability that the required resources b are met and the strengthened incentives to deviate to stand-alone contributions cause the non-existence of a symmetric joint contribution equilibrium.

Corollary 3 *Suppose F is strictly concave on $(0,1)$. If $b < \tilde{b}$ where $\tilde{b}$ is defined by (4) and satisfies $\tilde{b} > 2/3$, no symmetric joint contribution equilibrium exists.*

The proof of this corollary is in the appendix. For strictly concave distribution functions F of the budgets, there is either no joint contribution equilibrium (if b is small) or a continuum of equilibria (if b is large), the latter including the one with $\hat{m} = b/2$ as well as joint contribution equilibria with inefficiently low participation.

The joint-contributions results are reminiscent to Bagnoli and Lipman (1989, 1992) who study how several players might efficiently fund a threshold public good. In their framework this is a possible equilibrium outcome if their joint benefits from provision exceed the total cost. A natural non-cooperative equilibrium in their set-up is the one in which the players share the necessary contribution costs evenly.

In the context here the provision of joint effort that is larger or equal to b can be seen as the provision of a threshold public good. If players' resources are sufficient, there is a symmetric equilibrium in which they share the burden. An even split is a symmetric equilibrium that is part of the set of symmetric equilibria, unless players' budgets are likely to be large compared to the amount of team effort needed. Of course, the incentives and conditions are quite different to the standard analysis of non-cooperative provision of a threshold public good. In the provision game the quantity of resources can be freely chosen, but each unit contributed has a deterministic provision cost that enters into the contributor's budget. In the Blotto-alliance the provision occurs with resources that are useless if the provision does not occur: players allocate a "use-it-or-lose-it" budget. The opportunity cost emerges indirectly because the resources have an important alternative use if and only if the amounts contributed are sufficient to reach the group goal. If the total provisions sum up to b or more, then, and only then, the resources become very valuable for a confrontation with the other member of the group that emerges. A player cares about relative resource endowment in this confrontation. A player might increase the contributions, and this might be crucial reaching a given group benefit. But exactly the increased contribution of a player might make that player weaker than the other group member in the fight between them about the benefit. This trade-off differs and is stronger than the free-riding problem in standard voluntary contributions games. There a player might weigh the additional cost of own contributions and the higher likelihood of enjoyment of the public good. The player has private costs of contributing, but they generate a public benefit if they succeed to provide the threshold public good. Here the player who withholds resources not only brings the provision of the public good into question. The player also harms the co-player directly, because the player becomes a stronger rival in the internal conflict once they jointly reach the group goal.

3 Conclusions

Acting as a volunteer in the interest of the group often has major disadvantages: this action dissipates resources that could be used in the power struggle within the group, and in a fully non-cooperative world a player who contributes much to the group effort might generate benefits to the group, but be left with too little resources to succeed in the internal power struggle that decides the allocation of these benefits. So, while a volunteer acquires some desirable goods for the group, protects or rescues the group or accomplishes other tasks that benefit the group, this very act might be unattractive as it weakens this player in the internal struggle. We show that even in the face of such adverse incentives, group members can often coordinate their efforts in a non-cooperative equilibrium to achieve a common goal. This, and a more precise description of the determinants of such non-cooperative but coordinated action in equilibrium is the core result of this work. The analysis focuses on two kinds of equilibria: those in which the group members achieve the common goal by joining forces, and equilibria in which the group achieves the goal because there are members who are willing to achieve the group goal by their own efforts alone. Both types of equilibrium exist for larger and partially overlapping parameter ranges. The joint-contribution equilibrium is particularly relevant if each single player's resources are insufficient to make a stand-alone contribution; the volunteer's (standalone) equilibrium is particularly relevant if range of possible resource endowments includes large endowments.

A Appendix

A.1 Proof of Proposition 1

Before we show equilibrium existence, we show that there is a unique solution $m^* \in (b, 2b]$ to (2). For all $m^* \leq b$, the left-hand side is strictly smaller than the right-hand side (RHS) of equation (2). Moreover, the left-hand side (LHS) strictly increases in m^* whereas the RHS weakly decreases in m^* (given that $F'' \leq 0$). If m^* approaches $\min\{2b, \bar{m}\}$, the LHS is weakly larger than the RHS. To show the latter, suppose first that $2b < \bar{m}$. We need to show that

$$F(2b) \geq F(2b+b) - F(2b-b)$$

which is equivalent to

$$\int_0^{2b} F'(x)\,dx - \int_0^{2b} F'(x+b)\,dx \geq 0.$$

This inequality is true since F is weakly concave; it holds with strict inequality if F is strictly concave on some non-empty interval. If $2b \geq \bar{m}$ and $m^* \to \bar{m}$, the LHS of (2) approaches one while the RHS of (2) approaches $1 - F(\bar{m} - b) < 1$. Thus, there is a unique solution $m^* > b$ to (2) which is weakly smaller than $2b$ if $2b < \bar{m}$ and strictly smaller than $\bar{m}$ (and $2b$) if $2b \geq \bar{m}$.

We first show existence. Consider the candidate standalone equilibrium with threshold $m^* > b$ as given in (2). Suppose first that $m_i \geq m^*$ where i is supposed to choose $x_i^* = b$ and realizes an expected payoff of

$$F(m_i) - F(m^*) + F(\min\{m^*, m_i - b\}),$$

that is, a positive payoff if (i) j contributes but has a budget lower than m_i or (ii) j does not contribute and has a budget lower than $m_i - x_i$. Any $x_i > b$ is strictly dominated by $x_i = b$ and any $x_i \in (0, b)$ leads to a strictly lower payoff than $x_i = 0$ if j follows the candidate strategy. If i deviates to $x_i = 0$, her deviation payoff is

$$F(m_i + b) - F(m^*)$$

since she gets a positive payoff whenever j contributes but has a budget lower than $m_i + b$. The candidate strategy $x_i^* = b$ is a best reply if and only if

$$F(m_i) - F(m^*) + F(\min\{m^*, m_i - b\}) \geq F(m_i + b) - F(m^*).$$

If $m_i \in [m^*, m^* + b)$, this no-deviation condition becomes

$$F(m_i) + F(m_i - b) - F(m_i + b) \geq 0.$$

Since the left-hand side of this inequality is (weakly) increasing in m_i if F is (weakly) concave, a necessary condition for existence of the equilibrium is

$$F(m^*) \geq F(m^* + b) - F(m^* - b), \tag{6}$$

and this inequality holds if m^* is given by (2). Note that (6) requires $m^* > b$: at $m^* = b$, (6) is equivalent to $F(b) \geq F(2b) - F(0)$ which is violated due to $b < \bar{m}$.

If instead $m_i \in [m^* + b, \bar{m})$ (and this interval is non-empty, which, due to $m^* > b$, requires $\bar{m} > 2b$), the no-deviation condition is equivalent to

$$F(m^*) + F(m_i) - F(m_i + b) \geq 0.$$

If F is (weakly) concave, the left-hand side of this inequality is (weakly) increasing in m_i and is, thus, larger than

$$
\begin{aligned}
F(m^*) + F(m^* + b) - F(m^* + b + b) &\geq F(m^*) + F(m^*) - F(m^* + b) \\
&> F(m^*) + F(m^* - b) - F(m^* + b) = 0
\end{aligned}
$$

where the first (weak) inequality uses (weak) concavity of F and the equality uses (2). Altogether, the candidate strategy $x_i^* = b$ is a best reply if and only if $m^* > b$ and (6) holds.

Now suppose that $m_i < m^*$ where i is supposed to choose $x_i^* = 0$ and realize an expected payoff of

$$\max\{F(m_i + b) - F(m^*), 0\},$$

that is, a positive payoff in situations where j contributes (i.e., $m_j \geq m^*$) but $m_j - b < m_i$, provided this interval is non-empty. Consider possible deviations. Any $x_i > b$ is strictly dominated by $x_i = b$; any $x_i \in (0, b)$ does not change the probability that joint contributions are at least b but lowers the probability that i wins against j, as compared to $x_i = 0$. Thus, if $m_i \in [0, b)$, no profitable deviation exists.

From the case of $m_i \geq m^*$ above, it follows that equilibrium existence requires $m^* > b$. If $m_i \in [b, m^*)$ and i deviates to $x_i = b$, i gets a positive payoff if and only if j does not contribute and, in addition, has a budget below $m_i - b$. (If j contributes, too, m_j must be larger than m^* so that $m_j - x_j > m_i - x_i$ in the case of $m_i < m^*$.) Thus, i's deviation payoff is

$$F(\min\{m^*, m_i - b\}) = F(m_i - b),$$

which is strictly positive if m_i is in the (non-empty) interval (b, m^*). The candidate strategy $x_i^* = 0$ is a best reply if and only if

$$\max\{F(m_i + b) - F(m^*), 0\} \geq F(m_i - b)$$

for all $m_i \in (b, m^*)$. First of all, since $F(m_i - b) > 0$ if $m_i > b$, this requires

$m_i + b > m^*$ for all $m_i \in (b, m^*)$, that is, requires

$$m^* \leq 2b,$$

which holds for m^* as given by (2), where $m^* \in (b, 2b]$ if $2b < \bar{m}$ and $m^* \in (b, \bar{m})$ if $2b \geq \bar{m}$ (see above). In this case, the no-deviation condition is equivalent to $F(m^*) \leq F(m_i + b) - F(m_i - b)$ for all $m_i \in (b, m^*)$. Since the right-hand side of this inequality is (weakly) decreasing in m_i if F is (weakly) concave, a necessary condition for existence of the equilibrium is

$$F(m^*) \leq F(m^* + b) - F(m^* - b), \tag{7}$$

and this inequality holds if for a threshold m^* as given by (2).

Uniqueness follows directly from the arguments above. Since necessary conditions for equilibrium existence are (6) and (7), m^* must be given by (2) in any equilibrium. Since there is a unique solution m^* to (2), the equilibrium must be unique in the class of symmetric standalone equilibria.

A.2 Proof of Corollary 1

From Proposition 1 it follows that there is a unique solution $m^* \in (b, \bar{m})$ to (2). For a uniform distribution, condition (2) is equivalent to

$$\frac{m^*}{\bar{m}} = \min\left\{\frac{m^* + b}{\bar{m}}, 1\right\} - \frac{m^* - b}{\bar{m}} \Leftrightarrow 2m^* = \min\{m^* + 2b, \bar{m} + b\}. \tag{8}$$

Suppose $m^* + 2b < \bar{m} + b$. Then, (8) is solved for $m^* = 2b$. In order for $m^* + 2b < \bar{m} + b$ to hold at $m^* = 2b$, we must have $b < \bar{m}/3$. This shows part (i).

Now suppose $m^* + 2b \geq \bar{m} + b$. Then, (8) is solved for $m^* = (\bar{m} + b)/2$. In order for $m^* + 2b \geq \bar{m} + b$ to hold at $m^* = (\bar{m} + b)/2$, we must have $(\bar{m} + b)/2 \geq \bar{m} - b$ or, equivalently, $b \geq \bar{m}/3$. This shows part (ii). Note that $(\bar{m} + b)/2 \leq 2b$ if $b \geq \bar{m}/3$. Uniqueness follows from Proposition 1.

A.3 Proof of Proposition 2

Before we derive the equilibrium set for different values of b, we derive some preliminary results.

Step 1: We show that investigating equilibrium existence reduces to considering deviations to $x_i = b$ (standalone contributions). To see why, consider first types $m_i < \hat{m}$ so that $\hat{x}_i = 0$ in the candidate equilibrium. If $m_i < b/2$, $\hat{x}_i = 0$ is strictly preferred to any $x_i > 0$ if j follows the candidate strategy. If $m_i \geq b/2$, $\hat{x}_i = 0$ is strictly preferred to any $x_i \in (0, b/2)$, and $x_i = b/2$ is strictly preferred to any $x_i \in (b/2, b)$. A deviation to $\check{x}_i = b/2$, however, yields zero expected payoff: the threshold b would only be met if $m_j \geq \hat{m}$, in which case it must hold that $m_j - b/2 \geq \hat{m} - b/2 > m_i - b/2$. Hence, the only choice that may constitute a profitable deviation is $x_i = b$. (All $x_i > b$ are strictly dominated by $x_i = b$.)

Similarly, for types $m_i \geq \hat{m}$, deviations to $x_i < b/2$ cannot be profitable since the threshold b would never be met in this case. Any $x_i \in (b/2, b)$ is strictly worse than the candidate strategy $\hat{x}_i = b/2$ and any $x_i > b$ is strictly dominated by $x_i = b$. Again, the only choice that may constitute a profitable deviation is $x_i = b$.

Step 2: Let $b \geq \bar{m}$. From Step 1 it follows that there are no profitable deviations (since the set of types that can deviate to $x_i = b$ has mass zero). By definition of the joint contribution equilibrium, $\bar{m} > \hat{m} \geq b/2$ and, hence, $b < 2\bar{m}$, which shows part (iii) of Proposition 2.

Step 3: Let $b < \bar{m}$. Suppose that $\hat{m} > b$ and consider types $m_i \in (b, \hat{m})$. Those types' candidate equilibrium payoff is zero (the threshold is never reached since they do not contribute). Deviations to $x_i = b$ yield an expected payoff of at least $\Pr(m_j < m_i - b) = F(m_i - b) > 0$ so that $x_i = 0$ cannot be a best reply for types $m_i \in (b, \hat{m})$. Since $\hat{m} \geq b/2$ by assumption, it follows that, for $b < \bar{m}$, the contribution threshold $\hat{m}$ must satisfy $\hat{m} \in [b/2, b]$ in any symmetric joint contribution equilibrium. With Step 1, $\hat{m} \in [b/2, b]$ implies that the candidate strategy $\hat{x}_i = 0$ is a best reply for types $m_i < \hat{m}$.

Step 4: Let $b < \bar{m}$. For types $m_i \geq \max\{\hat{m}, b\}$, the difference between the candidate equilibrium payoff (under $\hat{x}_i$) and expected payoff when deviating to $x_i = b$, defined as $\Delta(m_i) := \pi_i(m_i; \hat{x}_i) - \pi_i(m_i; x_i = b)$, is (weakly) decreasing in m_i. To show this, note first that the candidate equilibrium payoff is $F(m_i) - F(\hat{m})$ since i wins if and only if $m_j \in [\hat{m}, m_i)$. The deviation payoff under $x_i = b$ depends on whether (i) $m_i - b/2 > \hat{m}$ (so that i can still win against contributors j) and (ii) $m_i - b > \hat{m}$ (so that i wins against all non-contributors j). Formally, if $m_i \in [b, \hat{m} + b/2)$ and this interval is non-empty, i wins with $x_i = b$ if and only if $m_j < m_i - b$. If $m_i \in [\hat{m} + b/2, \hat{m} + b)$ and this interval is non-empty, i wins with $x_i = b$ if $m_j < m_i - b$ or if $m_j \in [\hat{m}, m_i - b/2)$. If $m_i \in [\hat{m} + b, \bar{m}]$ and this interval is non-empty, i wins with $x_i = b$ if $m_j < \hat{m}$ or if $m_j \in [\hat{m}, m_i - b/2)$, that is, if and only if $m_j < m_i - b/2$. Thus, for $m_i \geq b$, the difference $\Delta(m_i)$ between candidate equilibrium payoff and expected payoff when deviating to $x_i = b$ is:

$$\Delta(m_i) = \begin{cases} F(m_i) - F(\hat{m}) - F(m_i - b) & \text{if } m_i \in \left[b, \min\left\{\hat{m} + \frac{b}{2}, \bar{m}\right\}\right) \\ F(m_i) - F(m_i - b) - F\left(m_i - \frac{b}{2}\right) & \text{if } m_i \in \begin{array}{c}\left[\min\left\{\hat{m} + \frac{b}{2}, \bar{m}\right\},\right. \\ \left.\min\left\{\hat{m} + b, \bar{m}\right\}\right)\end{array} \\ F(m_i) - F(\hat{m}) - F\left(m_i - \frac{b}{2}\right) & \text{if } m_i \in \left[\min\left\{\hat{m} + b, \bar{m}\right\}, \bar{m}\right] \end{cases} \tag{9}$$

$\Delta(m_i)$ (weakly) decreases in m_i if F is (weakly) concave. Thus, a necessary and sufficient condition for equilibrium existence is obtained when investigating $\Delta(\bar{m})$ (the incentive to deviate to $x_i = b$ for the highest possible budget).

Step 5: We note that $F(m) \geq m/\bar{m}$ for all $m \in (0, \bar{m})$ if F is weakly concave. With $F(0) = 0$ and $F(\bar{m}) = 1$, concavity forces F to be weakly above the straight line between $(0, 0)$ to $(\bar{m}, 1)$.

To show this, note first that weak concavity of F implies $F'(0) \geq 1/\bar{m}$. If, to the contrary, $F'(0) < 1/\bar{m}$, then $\int_0^{\bar{m}} F'(z)\, dz < \int_0^{\bar{m}} (1/\bar{m})\, dz = 1$, which contradicts $F(\bar{m}) = 1$. Now suppose that $F(m) < m/\bar{m}$ for some $m \in (0, \bar{m})$. If $F'(0) = 1/\bar{m}$, then, by weak concavity of F, $F(\bar{m}) \leq F(m) + (\bar{m} - m) F'(m) < m/\bar{m} + (\bar{m} - m)/\bar{m} = 1$; contraction. If $F'(0) > 1/\bar{m}$ then $F(m) < m/\bar{m}$ for some $m \in (0, \bar{m})$ implies that F must cross the line $y = m/\bar{m}$ from above at some $\breve{m} \in (0, m)$; hence, $F'(\breve{m}) < 1/\bar{m}$. With weak concavity of F, it follows that $F(\bar{m}) < 1$; contradiction. Thus, $F(m) \geq m/\bar{m}$ for all $m \in (0, \bar{m})$. A similar argument shows that $F(m) > m/\bar{m}$ for all $m \in (0, \bar{m})$ if F is strictly concave on some non-empty interval (m', m'').

Since part (iii) of the proposition follows from Step 2 above, it remains to prove parts (i) and (ii). Assume $b < \bar{m}$. By Step 3, let $\hat{m} \in [b/2, b]$. By Steps 1 and 3, we only need to consider behavior of types $m_i \geq \hat{m}$ where, by Step 4, for equilibrium existence it is sufficient to focus on types with budget $m_i \to \bar{m}$.

Part (i): Assume $b < 2\bar{m}/3$ and consider the incentive to deviate from the candidate equilibrium if $m_i \to \bar{m}$. If $b < \bar{m}/2$ then $\bar{m} > 2b \geq \hat{m} + b$ for all $\hat{m}$ under consideration. (In words, this implies that, when deviating to $x_i = b$, types $m_i \to \bar{m}$ win against all non-contributing types of j.) If $b \in [\bar{m}/2, 2\bar{m}/3)$ then $\bar{m} \geq \hat{m} + b$ if $\hat{m} \in [b/2, \bar{m} - b]$ and $\bar{m} \in (\hat{m} + b/2, \hat{m} + b)$ if $\hat{m} \in (\bar{m} - b, b]$. (Here, whether types $m_i \to \bar{m}$ win against non-contributing players j with m_j close to $\hat{m}$ depends on the size of the threshold $\hat{m}$.)

Suppose first that $\hat{m}$ is such that $\bar{m} \geq \hat{m} + b$. Then, with (9), the candidate strategy is a best reply for all $m_i \in [b, \bar{m}]$ if and only if

$$F(\bar{m}) - F(\hat{m}) - F(\bar{m} - b/2) \geq 0. \tag{10}$$

If F is weakly concave, it holds that $F(m) \geq m/\bar{m}$ for all $m \in (0, \bar{m})$ (compare Step 5 above). Thus, the left-hand side of (10) is weakly smaller than

$$1 - \frac{\hat{m}}{\bar{m}} - \frac{\bar{m} - b/2}{\bar{m}} = \frac{1}{\bar{m}} \left(\frac{b}{2} - \hat{m} \right),$$

which is strictly negative if $\hat{m} > b/2$. Thus, (10) is violated for all $\hat{m} > b/2$ and a joint contribution equilibrium can exist only if $\hat{m} = b/2$. Inserting $\hat{m} = b/2$ into (10) yields (3) as necessary and sufficient condition for equilibrium existence. (Existence is ensured under (3) since $\bar{m} \geq \hat{m} + b$ holds at $\hat{m} = b/2$ by assumption of $b < 2\bar{m}/3$

and, hence, (10) is the relevant no-deviation condition.)

Now suppose that $\hat{m}$ is such that $\bar{m} \in (\hat{m}+b/2, \hat{m}+b)$. (This can occur if and only if $b \in [\bar{m}/2, 2\bar{m}/3)$. Here, note that $\hat{m}+b/2 \leq 3b/2 < \bar{m}$ if $b < 2\bar{m}/3$.) With (9), the candidate strategy is a best reply for all $m_i \in [b, \bar{m}]$ if and only if

$$F(\bar{m}) - F(\bar{m}-b) - F\left(\bar{m} - \frac{b}{2}\right) \geq 0. \tag{11}$$

Since weak concavity of F implies $F(m) \geq m/\bar{m}$ for all $m \in (0, \bar{m})$, the left-hand side of (11) is weakly smaller than

$$F(\bar{m}) - \frac{\bar{m}-b}{\bar{m}} - \frac{\bar{m}-\frac{b}{2}}{\bar{m}} = \frac{1}{\bar{m}}\left(\frac{3b}{2} - \bar{m}\right),$$

which is strictly negative due to $b < 2\bar{m}/3$. Thus, (11) is violated and no joint contribution equilibrium can exist with threshold $\hat{m}$ that is such that $\bar{m} \in (\hat{m}+b/2, \hat{m}+b)$. This completes the proof of part (i).

Part (ii): Suppose that $b \in [2\bar{m}/3, \bar{m})$. Here, we have $\hat{m}+b \geq b/2+b \geq \bar{m}$. If $\hat{m}$ is small such that $\bar{m} \geq \hat{m}+b/2$, the candidate strategy is a best reply for all $m_i \in [b, \bar{m}]$ if and only if (11) holds. The left-hand side of (11) is strictly negative if $b \to 0$, strictly increasing in b, and strictly positive if $b \to \bar{m}$. Thus, there exists a unique solution $\tilde{b} \in (0, \bar{m})$ given by (4) such that the candidate strategy is a best reply if and only if $b \geq \tilde{b}$. Since the left-hand side of (11) is weakly smaller than

$$\frac{1}{\bar{m}}\left(\frac{3b}{2} - \bar{m}\right)$$

(see the previous paragraph), it must hold that $\tilde{b} \in [2\bar{m}/3, \bar{m})$. If $b \in \left[2\bar{m}/3, \tilde{b}\right)$, no symmetric joint contribution equilibrium exists.

If $b \in \left[\tilde{b}, \bar{m}\right)$, a joint contribution equilibrium can be supported for any $\hat{m}$ that is sufficiently small such that $\bar{m} \geq \hat{m}+b/2 \Leftrightarrow \hat{m} \leq \bar{m}-b/2$. (This upper bound, $\bar{m}-b/2$, on $\hat{m}$ approaches b if $b \to 2\bar{m}/3$ and approaches $b/2$ if $b \to \bar{m}$. Hence, taking into account that $\hat{m} \in [b/2, b]$, the interval $\hat{m} \in [b/2, \bar{m}-b/2]$ for which a joint contribution equilibrium exists is non-empty.)

Now consider larger thresholds $\hat{m}$ for which $\bar{m} < \hat{m} + b/2 \Leftrightarrow \hat{m} > \bar{m} - b/2$. With (9), the candidate strategy is a best reply for all $m_i \in [b, \bar{m}]$ if and only if

$$F(\bar{m}) - F(\hat{m}) - F(\bar{m} - b) \geq 0. \tag{12}$$

The left-hand side of (12) strictly decreases in $\hat{m}$. If $\hat{m} \downarrow \bar{m} - b/2$, (12) is equivalent to (11) and holds if and only if $b \geq \tilde{b}$; thus, equilibrium existence again requires $b \geq \tilde{b}$. If $\hat{m} \to b$, the left-hand side of (12) is

$$F(\bar{m}) - F(\bar{m} - b) - F(b) \leq 1 - \frac{\bar{m} - b}{\bar{m}} - \frac{b}{\bar{m}} = 0$$

where the weak inequality holds due to $F(m) \geq m/\bar{m}$ for all $m \in (0, \bar{m})$ (due to weak concavity of F). Thus, there is a unique solution $z \in (\bar{m} - b/2, b]$ to (5) such that any $\hat{m} \in [b/2, z]$ can be supported as part of a joint contribution equilibrium. Since (12) is a necessary condition for equilibrium existence, $\hat{m} \in [b/2, z]$ characterizes the full set of symmetric joint contribution equilibria in the case where $b \in \left[\tilde{b}, \bar{m}\right)$.

A.4 Proof of Corollary 2

Part (i): Suppose $b < 2/3$. From Proposition 2(i), there is a unique symmetric joint contribution equilibrium if and only if (3) holds, which is true (it holds with equality) if F is a uniform distribution.

Part (ii): Suppose $b \in [2/3, 1)$. From Proposition 2(ii), equilibrium existence requires $b \geq \tilde{b}$ as given in (4). For a uniform distribution, $\tilde{b} = 2/3$ so that a joint contribution equilibrium exists for all $b \in [2/3, 1)$. Any $\hat{m}$ with $\hat{m} \leq \bar{m} - b/2$ can be supported as equilibrium since (11) holds for all $b \in [2/3, 1)$. For larger $\hat{m}$ (that is, $\hat{m} > \bar{m} - b/2$), necessary and sufficient condition for equilibrium existence is (12), which, for $F(m) = m$, is equivalent to $\hat{m} \leq b$. This shows part (ii).

Part (iii): Follows from Proposition 2(iii).

A.5 Proof of Corollary 3

Suppose $b < 2/3$. From Proposition 2(i), there is a unique symmetric joint contribution equilibrium if and only if (3) holds. Since strict concavity of F implies $F(m) > m/\bar{m} = m$ for all $m \in (0,1)$, the left-hand side of (3) is strictly smaller than

$$1 - (1 - b/2) - b/2 = 0$$

so that (3) is violated.[13]

Suppose $b \geq 2/3$. By Proposition 2(ii), equilibrium existence requires $b \geq \tilde{b}$. Using again $F(m) > m$ in condition (4) shows that $\tilde{b} > 2/3$. Altogether, for small thresholds $b < \tilde{b}$, a symmetric joint contribution equilibrium does not exist. (For larger values of b, the equilibrium is as characterized in Proposition 2.)

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