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Alex Edmans

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Centre for Economic Policy Research
33 Great Sutton Street, London EC1V 0DX, UK
Tel: +44 (0)20 7183 8801
www.cepr.org

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Abstract

Existing theories of debt consider a single contractible performance measure ("output"). In reality, many other performance signals are also available. It may seem that debt is no longer optimal; for example, if the signals are sufficiently positive, the agent should receive a payment even if output is low. This paper shows that debt remains the optimal contract under additional signals -- they only affect the face value of debt, but not the form of the contract. We show how the face value should depend on other signals, providing a theory of performance-sensitive debt.

JEL Classification: D86, G32, G34, J33

Keywords: Informativeness principle, limited liability, performance-sensitive debt

Alex Edmans - aedmans@london.edu
London Business School and CEPR

When Is (Performance-Sensitive) Debt Optimal?*

Pierre Chaigneau
Queen's University

Alex Edmans
LBS, CEPR, and ECGI

Daniel Gottlieb
LSE

July 29, 2021

Abstract

Existing theories of debt consider a single contractible performance measure ("output"). In reality, many other performance signals are also available. It may seem that debt is no longer optimal; for example, if the signals are sufficiently positive, the agent should receive a payment even if output is low. This paper shows that debt remains the optimal contract under additional signals – they only affect the face value of debt, but not the form of the contract. We show how the face value should depend on other signals, providing a theory of performance-sensitive debt.

KEYWORDS: informativeness principle, limited liability, performance-sensitive debt.
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*pierre.chaigneau@queensu.ca, aedmans@london.edu, d.gottlieb@lse.ac.uk.

This paper contains some results from a paper previously circulated under the title "The Informativeness Principle under Limited Liability."

The vast majority of firms issue debt. In some cases, like in most start-up firms, debt is the only external source of financing. A large theoretical literature has therefore aimed to understand when debt contracts are optimal. Most justifications of debt are based on moral hazard. In a costly state verification framework, Townsend (1979) and Gale and Hellwig (1985) show that debt contracts minimize audit costs while inducing truthful reporting of the firm's output. In a model where the entrepreneur can affect both the mean and the dispersion of output, Hébert (2018) shows that debt is optimal because it is the least risky security. Hart and Moore (1998) show that collateralized debt allows for external funding even when the firm's output is not contractible and can be diverted by the entrepreneur. When output is contractible, Innes (1990) demonstrates that debt is the optimal contract if the manager is protected by limited liability and investors' payoff cannot be decreasing in output (the monotonicity constraint). Intuitively, limited liability prevents investors from punishing the manager for low output, so they instead incentivize him by maximizing his rewards for high output. Due to the monotonicity constraint, the manager cannot gain more than one-for-one. He is thus the residual claimant, receiving equity; investors receive debt.

These frameworks assume that output is the only signal of the agent's effort. This assumption seems to be critical in generating debt as the optimal contract. When output q is lower than a threshold q^*, the principal concludes that the agent has shirked and pays him zero – under a debt contract, q^* is the face value of debt and the agent's equity is worthless if firm value q is below it. In reality, principals have access to multiple additional signals of performance, such as sales, profits, market share, credit ratings, or peer performance. If these signals are sufficiently indicative of effort, it may seem optimal to pay the agent a strictly positive amount even if $q < q^*$, and so debt is no longer the optimal contract. Similarly, a negative signal may mean it is optimal to pay the agent less than the residual even if $q > q^*$.

This paper studies whether and how the optimal contract changes if the principal has access to a signal s of effort in addition to output q. The signal could affect the optimal contract in two ways. First, debt might no longer be the optimal contract. Debt is "bang-bang" in that the agent receives the lowest possible amount (zero) below a threshold, and the highest possible amount (the residual) above. It may seem that even an infinitesimally informative signal will perturb the optimal contract so that the agent's payoff optimally lies between the extremes. In contrast, we show that debt remains the optimal contract even under strictly informative signals – and even if the signals are informative everywhere, i.e. provide information about effort regardless of the output level.

Second, the signal could affect the optimal contract by changing the threshold q_s^*. Then, the contract becomes performance-sensitive debt, where the face value depends on the signal and so it is denoted q_s^*. For example, a signal that indicates high effort (such as a high credit rating) could lower q^* and increase the agent's payoff. Indeed, Holmström's (1979) informativeness principle showed that any informative signal has value, i.e. will change the contract. However, we show that a signal may be informative almost everywhere, yet have no value – i.e. affect neither the form of the contract nor the face value of debt. The difference from Holmström (1979) is that there are no binding contracting constraints in his model, and so the principal can always make use of a signal by changing the contract in response. However, when contracting constraints bind, the contract cannot change in response to the signal. If $q < q^*$ and the signal indicates that the manager has shirked (i.e. low q is due to low effort rather than bad luck), the principal cannot use the signal to reduce the payment since the manager is receiving zero anyway: the limited liability constraint binds. Likewise, for $q > q^*$, the principal cannot use the signal to increase the payment since the monotonicity constraint binds.

We derive a new necessary and sufficient condition for a signal to have value under contracting constraints. We show that a signal only has value if it affects the face value of debt. In turn, the face value of debt depends on the likelihood ratio of the event $q \geq q^*$ – in contrast to typical likelihood ratios which concern a single output level. Intuitively, with a binding monotonicity constraint, changing the debt repayment changes the payment for all $q \geq q^*$. Thus, a signal only has value if it affects the likelihood ratio that $q \geq q^*$, i.e. is informative about whether output exceeding the face value is the outcome of effort or luck. This is a much stronger condition than in Holmström (1979): even if a signal is informative almost everywhere, it has no value if it is not informative about this specific event.

Finally, we study how a signal adds value if it is informative – i.e. how debt should be sensitive to performance. The informativeness principle studies *whether* a signal should be incorporated into a contract, but not *how* since, in general, it is impossible to solve for the optimal contract in closed form. We show that there are three channels through which a signal may affect the debt contract. First, it may be individually informative about effort. A signal that individually indicates high effort will optimally increase the agent's payment; under a debt contract, this is achieved by lowering the face value q^*. Second, the signal may indicate that the location of the output distribution has shifted. A signal that indicates that the output distribution has shifted to the right (e.g. good peer performance) should lower the agent's payment for any given output level, which is achieved by increasing q^*. Third, the signal may indicate that output is a more precise measure of effort, either because effort has a greater impact on output, or because output volatility is low. In general, greater precision increases the pay-performance sensitivity of the optimal contract. However, since the slope of the contract is capped at 1 for $q > q^*$, this increase in sensitivity is instead achieved by lowering the face value q^* as doing so raises the "delta" of the agent's equity.

1 The Model

There are two risk neutral parties, a principal (firm), and an agent (manager). The manager exerts an unobservable effort $e \in [0, \bar{e}]$. As is standard, effort can be interpreted as any action that improves output but is costly to the manager, such as working rather than shirking, choosing projects that generate cash flows rather than private benefits, or not extracting rents. The manager's cost of effort $C(\cdot)$ is strictly increasing, strictly convex, twice continuously differentiable in $[0, \bar{e})$, with $C(0) = C'(0) = 0$ and $\lim_{e \nearrow \bar{e}} C'(e) = +\infty$.

Effort affects the probability distribution of output q and a signal s, which are both observable and contractible. Output is continuously distributed with full support on $(\underline{q}, +\infty)$, where $\underline{q}$ is either $-\infty$ or 0. To ensure that an optimal contract exists, we assume that the signal is discrete, $s \in \{s_1, ..., s_S\}$. This formulation allows the signal to have one or multiple dimensions (i.e., signals can be vectors).

The signal is distributed according to the probability mass function $\phi_e^s := \Pr(\tilde{s} = s | \tilde{e} = e)$, which is strictly positive and twice continuously differentiable in e. Output is distributed according to the cumulative distribution function $F(q|e, s)$, which is twice continuously differentiable in q and e and has a strictly positive density $f(q|e, s)$. The joint distribution of output and the signal is $f(q, s|e) = \phi_e^s f(q|e, s)$. We assume that the likelihood ratio of output, $\frac{\frac{\partial f}{\partial e}(q|e,s)}{f(q|e,s)}$, is strictly increasing in output q ("MLRP"). The likelihood ratio associated with the event $(\tilde{q} = q, \tilde{s} = s)$ is:

$$LR_s(q|e) := \frac{\partial \phi_{\hat{e}}^s / \partial e}{\phi_{\hat{e}}^s} + \frac{\frac{\partial f}{\partial e}(q|e, s)}{f(q|e, s)} \tag{1}$$

Consistently with any standard unbounded distributions, we assume that $\lim_{q \nearrow +\infty} \frac{\partial f}{\partial e}(q, s|e) = 0$, which implies that debt with arbitrarily high face value has low effort incentives. Moreover, when the support is unbounded below, we assume that $\lim_{q \nearrow +\infty} LR_s(q|e) = \infty$, and $\lim_{q \searrow -\infty} LR_s(q|e) = -\infty$ for all s. These assumptions simplify expressions by ruling out corner solutions, but are not important for our results.

The firm has full bargaining power and offers the manager a schedule of payments $\{w_s(q)\}$ conditional on each realization of (q, s). As in Innes (1990), both the firm and the manager are protected by limited liability. Because we allow output to be negative, the limited liability constraints can be written as:

$$0 \leq w_s(q) \leq \max\{0, q\}.$$

Limited liability on the manager's side requires payments to be non-negative. Limited liability on the firm's side means that the firm cannot pay more than the entire output. Since payments cannot be negative, limited liability on the firm's side also implies that the firm cannot be forced to make payments when output falls below zero.[1]

We follow Grossman and Hart (1983) and separate the principal's problem into two stages. The first stage determines the optimal contract and the associated cost of implementing each effort. Given this cost, the second stage determines which effort to implement. To induce effort $\hat{e}$, the firm solves the following program:

$$\min_{\{w_s(q)\}} \sum_s \phi_{\hat{e}}^s \int_{\underline{q}}^{\infty} w_s(q)\, f(q|\hat{e}, s) dq \tag{2}$$

subject to

$$\sum_s \phi_{\hat{e}}^s \int_{\underline{q}}^{\infty} w_s(q)\, f(q|\hat{e}, s) dq \geq C(\hat{e}), \tag{3}$$

$$\hat{e} \in \arg\max_e \sum_s \phi_e^s \int_{\underline{q}}^{\infty} w_s(q)\, f(q|e, s) dq - C(e), \tag{4}$$

$$0 \leq w_s(q) \leq \max\{0, q\}, \tag{5}$$

$$q - w_s(q) \text{ non-decreasing in } q. \tag{6}$$

The firm minimizes the expected payment (2) subject to the manager's individual rationality constraint ("IR") (3), incentive compatibility constraint ("IC") (4), limited liability constraints ("LL") (5), and a monotonicity constraint with respect to output (6). The monotonicity constraint is the final ingredient of the Innes (1990) model. It means that a dollar increase in output cannot increase the payment to the manager by more than a dollar (else he would inject his own money into the firm to increase output), or equivalently the payoff to the principal cannot decrease in output (else she would exercise her control rights to "burn" output).

With $C(0) = 0$, the IC (4) and LL (5) imply that the IR (3) is automatically satisfied, and so we ignore it in the analysis that follows. To study a nontrivial incentive problem, we consider $\hat{e} > 0$ (with $\hat{e} = 0$, the optimal contract is simply $w_s(q) = 0$ for all $\{q, s\}$). To ensure that an incentive compatible contract exists, we assume:

$$\sum_s \int_0^{\infty} q \frac{\partial f}{\partial e}(q, s|\hat{e}) dq > C'(\hat{e}). \tag{7}$$

Note that the best contract that can be offered to the agent pays the entire output whenever

[1] When output can be negative, bilateral limited liability requires a third party – e.g., a creditor, supplier, or the government – to bear the loss.

it is positive. The previous condition states that offering this contract is enough to incentivize the agent to choose an effort of at least $\hat{e}$. When this condition fails, there is no contract that induces the agent to choose $\hat{e}$. As in Grossman and Hart (1983), effort levels for which this condition fails can be treated as having "infinite cost."

2 Debt Contracts

2.1 When Is Debt Optimal?

As a preliminary result, Lemma 1 below presents a new condition for the validity of the First-Order Approach (FOA) to the effort choice problem in the above program.[2] Let K_e be defined as:

$$K_e := \sum_s \int_0^\infty q \max\left\{\frac{\partial^2 f}{\partial e^2}(q, s|e), 0\right\} dq.$$

Lemma 1 *Suppose that $K_e < C''(e)\ \forall e \in (0, \bar{e})$. Then the FOA is valid.*

The condition in Lemma 1 relies on the contracting constraints and the associated bounds on the payment $w_s(q)$ to the manager to derive an upper bound on the convexity of the expected payment with respect to effort, K_e. The FOA is then valid if the cost of effort is more convex than this upper bound. We henceforth assume that the condition in Lemma 1 holds. Let

$$\overline{LR}_s(q) := \frac{\partial \phi^s_{\hat{e}}/\partial e}{\phi^s_{\hat{e}}} + \frac{\int_q^\infty \frac{\partial f}{\partial e}(z|\hat{e}, s)dz}{\int_q^\infty f(z|\hat{e}, s)dz} \tag{8}$$

denote the likelihood ratio associated with the event $(\tilde{q} \geq q, \tilde{s} = s)$. The likelihood ratio comprises two terms. The first, $\frac{\partial \phi^s_{\hat{e}}/\partial e}{\phi^s_{\hat{e}}}$, captures how individually informative the signal is about effort. For example, if s is profits, high profits indicate high effort. The second, $\frac{\int_q^\infty \frac{\partial f}{\partial e}(z|\hat{e},s)dz}{\int_q^\infty f(z|\hat{e},s)dz}$, captures the effect of effort on the output density conditional on the signal not just at the output realization q, but over all outputs greater than q. For example, if the signal s is peer firm performance, this likelihood ratio will be lower if peer performance is strong.

In Innes (1990), without an additional signal s, the principal receives debt and the agent receives equity. The manager receives zero if output is less than the face value of debt q^*_s, and the residual $q - q^*_s$ otherwise. The intuition is as follows. Due to MLRP, output is most

[2] Innes (1990) assumes the FOA and gives examples of sufficient conditions for it to hold, such as Rogerson's (1985) condition on the convexity of the cumulative distribution function. However, this condition is not satisfied by many distributions with location and scale parameters, which we use in Section 2.3.

informative about effort in the tails of the distribution of q. The firm cannot incentivize the manager in the left tail by giving negative payments (due to limited liability), so it incentivizes him in the right tail by giving high payments. Under the monotonicity constraint, the maximum possible incentives involve the manager gaining one-for-one from any increase in output, so he receives the residual.

With an additional signal s, is it not clear that the optimal contract remains debt. It may be that, for low outputs, if the signal is sufficiently individually indicative of effort (e.g. $\frac{\partial \phi_{\hat{e}}^{s}/\partial e}{\phi_{\hat{e}}^{s}}$ is high), it becomes optimal to pay the agent a strictly positive amount, rather than zero as under a debt contract. Conversely, it may be that, for high outputs, if the signal is sufficiently individually indicative of shirking, it becomes optimal to pay the agent less than the residual. However, Proposition 1 below shows that the contract actually remains debt.

Separately, it may seem that we can apply the logic in Innes (1990) signal-by-signal to show that the optimal contract remains debt in the presence of an additional signal – specifically, that any feasible initial contract is dominated by debt (i.e. can be replaced by a debt contract that provides the same incentives at lower cost). This turns out not to be the case. Indeed, there might not exist a debt contract that provides the same level of incentives as the initial contract *on a signal-by-signal basis*. The intuition is as follows. As in the standard Innes model, for a given signal, increasing the face value of debt decreases the manager's effort incentives (by lowering the delta of his equity) and reduces the cost of the contract. However, for a signal which is individually bad news about effort ($\frac{\partial \phi_{\hat{e}}^{s}/\partial e}{\phi_{\hat{e}}^{s}}$ is negative), there is a countervailing effect: increasing the face value of debt reduces the manager's expected payment for this signal realization, which encourages him to work harder to avoid this signal. If this second effect dominates, increasing the face value of debt for a signal that is bad news about effort will *increase* incentives. For example, if shirking leads to a low credit rating, and a low credit rating leads to a high face value of debt, the manager will increase effort to avoid the low rating.

Proposition 1 *The optimal contract is* $w_s(q) = \max\{q - q_s^*, 0\}$. *For interior solutions, debt repayments* $\{q_s^*\}$ *are such that* $\overline{LR}_s(q_{s_i}) = \overline{LR}_s(q_{s_j})$, *where* $\overline{LR}_s(q)$ *is strictly increasing in* q.

Proposition 1 shows that, with an additional signal of performance, limited liability and monotonicity continue to bind for any output, so that the optimal contract is still debt. Instead of affecting the form of the optimal contract, which remains debt, the signal realization affects the debt repayment. The intuition is as follows. A negative signal means that it is optimal to pay the manager less, but this reduction can only occur for high output levels where the payment is strictly positive. Conceptually, this decrease could be achieved by lowering the slope of the manager's pay, but it turns out to be optimal instead to lower the debt repayment. Due to MLRP, it is more efficient to provide strong incentives for only high output levels than moderate incentives for a larger range of output levels. Conversely, if output is low, a positive signal only leads to a strictly positive payment if it raises the likelihood ratio (8) above a minimum threshold. Due to MLRP, it is efficient to provide the manager with the minimum possible payment (zero) over a wide range of output levels; thus, a positive signal should lead to a positive payment only at the top end of this range. Overall, the "incentive zone" – the subset of outputs where the manager receives a strictly positive payment – depends on the signal realization. Intuitively, the signal allows the firm to concentrate incentives in states of the world that are stronger positive signals of effort.

Proposition 1 also shows that the optimal debt repayment depends on the likelihood ratio of the event $\tilde{q} \geq q$ conditional on signal s. Note that the relevant likelihood ratio $\overline{LR}_s$ is over a range of outputs $\tilde{q} \geq q$, rather than at a single output level $\tilde{q} = q$. The firm cannot increase the payment at a specific output level in isolation without increasing it at all lower outputs, as this would violate the monotonicity constraint; similarly, it cannot decrease the payment at a specific output level in isolation without decreasing it at all higher outputs.

This optimal contract is consistent with the financing decisions of both mature firms and also young firms since they frequently raise debt and the entrepreneur holds levered equity, as shown by Robb and Robinson (2014) and Hwang, Desai, and Baird (2019). Leary and Roberts (2010) argue that debt issuance behavior is primarily driven by moral hazard, rather than information asymmetry.

2.2 When Is Performance-Sensitive Debt Optimal?

With the debt contract derived in Proposition 1, the principal's only degree of freedom is the face value of debt q_s^*. Thus, the signal realization can only affect the contract via changing the required debt repayment, as with performance-sensitive debt. Part (i) of Proposition 2 gives a necessary and sufficient condition under which the contract is independent of the signal, i.e. $q_s^* = q^* \; \forall s$. Part (ii) gives a sufficient condition for the payment to be independent of the signal, and part (iii) gives a sufficient condition for the debt repayment to optimally be zero.

Proposition 2 *(i) The optimal contract is independent of the signal if and only if* $\overline{LR}_s\left(q_{s_i}^*\right) = \overline{LR}_s\left(q_{s_j}^*\right)$ *for all* s_i, s_j.

(ii) Given output q, *the payment* $w_s(q)$ *is independent of the signal if* $q \leq \min_s \{q_s^*\}$.

(iii) The debt repayment is zero under signal s *if* $\frac{\partial \phi_{\hat{e}}^s / \partial e}{\phi_{\hat{e}}^s}$ *is sufficiently high.*

Part (i) of Proposition 2 asks whether a signal is valuable ex ante – before observing output, would the principal like to make the contract contingent on the signal? It shows that limited liability requires us to refine the informativeness principle. A signal has positive value if and only if it affects the firm's optimal choice of the face value q^*, since this is the only element of the contract that the firm will change according to the signal realization (see Proposition 1). It cannot change the contract for $q < q^*$ because it is already paying zero, nor for $q > q^*$ because it is already paying the residual. The firm optimally sets the same face value q^* if and only if the likelihood ratio that $q \geq q^*$ is the same across signals. With a binding IC, q^* solves the following equation:

$$\sum_s \phi_e^s \int_{q^*}^{\infty} (q - q^*) \frac{\partial f}{\partial e}(q|e,s)dq = C'(e).$$

A signal only has value if it shifts probability mass from below q^* to above q^* (or vice-versa). A signal that redistributes mass within the left tail, or within the right tail, has zero value. A "smoking gun" indicates that a bad event is due to poor performance rather than bad luck, but the bad event will likely lead to the agent being fired and being paid zero anyway.[3] For instance, investors only noticed that Enron was adopting misleading accounting practices when it was already going bankrupt.

Proposition 2 has implications for when debt contracts should be performance-sensitive. In theory, the face value of debt could depend on many signals, but in practice it is often signal-independent. Proposition 2 potentially rationalizes this practice – even if signals are informative about effort, they should not enter the contract if they are only informative in the tails. In addition, Proposition 2 provides conditions under which the repayment *should* depend on additional signals, as in performance-sensitive debt – if and only if the signal is informative about effort conditional on output exceeding the promised repayment. In addition to studying the optimality of performance-sensitive debt, Proposition 2 also allows us to study the conditions under which the entrepreneur's equity claim should depend on performance milestones, as documented empirically by Kaplan and Strömberg (2003) for venture capital contracts.[4]

Part (ii) asks whether a signal is valuable ex post – after observing output, will the payment

[3] The "smoking gun" could be generated by an audit that is only undertaken upon a bad event, in which case the signal realization is zero absent a bad event.

[4] While the original informativeness principle in Holmström (1979) would suggest that contracts should depend on performance milestones, it does not generally deliver debt and equity as optimal contracts. Kaplan and Strömberg (2004) find that the debt and equity contracts used in venture capital are determined primarily by agency problems, not risk-sharing considerations.

to the agent depend on the signal? In other words, while part (i) asks whether the optimal *contract* depends on the signal, part (ii) asks whether the optimal *payment* depends on the signal. If output is sufficiently low, the signal has no value since the agent will be paid zero even under the most favorable signal realization. Thus, even if the signal realization reduced the optimal face value of debt – i.e. changed the optimal contract – it would not change the payment as it remains zero. Part (ii) is relevant if signals are costly, and the principal can observe output before deciding whether to gather the signal.

Part (iii) shows that, if a signal is a sufficiently positive signal of effort, then $q_s^* = 0$. Intuitively, to provide strong incentives, the principal may be willing to completely forgive the debt in rare states that are very positive signals of effort. Indeed, $\frac{\partial \phi_{\hat{e}}^s / \partial e}{\phi_{\hat{e}}^s}$ will be high when effort has a strong effect on the probability of observing signal s, and when the probability $\phi_{\hat{e}}^s$ of observing signal s is low. Note that the debt repayment could not be zero in a model without an additional signal, as the principal would never obtain a return in any state. This also means that the debt repayment may be the same under two different signal realizations, if they are both sufficiently positive that the optimal debt repayment is zero.

We close with two examples that apply Proposition 2 to a real-world setting. First, we consider whether contracts should depend on s, a signal of economic conditions. Economic conditions are informative about effort – for any given level of output, a high s suggests that the output was due to good economic conditions rather than effort, and so it increases the likelihood that the manager has shirked. However, Proposition 2 shows that economic conditions s should only affect the contract if they affect the probability that $q > q^*$ under high versus low effort. This will fail to hold if they affect the level of output but not the probability that output exceeds q^*.[5] For example, consider a start-up which is developing a major new software; the manager's effort affects the probability that the software is adopted by the industry. If the software is adopted, $q > q^*$ (regardless of economic conditions); if it is not adopted, $q < q^*$ (again, regardless of economic conditions). Economic conditions could affect the actual level of q (both if the software is adopted and if it is not), but if they do not affect the probability that $q > q^*$, because they do not affect the likelihood that the software will be adopted, then they should not be included in the contract. In contrast, for an "everyday" software product, where the probability that $q > q^*$ does depend on economic conditions (as well as the manager's effort), then the face value of debt should depend on economic conditions.

As a second example, consider a firm whose production can break down due to a fault, whose probability can depend on managerial effort. If it does, then output is below q^* (regardless of

[5]It will also hold if they affect the probabilities (that $q > q^*$ under high and low effort) by the same proportion.

economic conditions); if it does not, then $q > q^*$ (regardless of economic conditions). As in the previous example, economic conditions could affect the actual level of q (both if production breaks down and if it does not), but if they do not affect the probability that production breaks down, then they should not be included in the contract. In contrast, if demand depends on the state of the economy, rather than a breakdown, then debt should be performance-sensitive. In the first example, what matters is whether the signal is uninformative about the upside (developing new software); in this example what matters is whether the signal is uninformative about the downside (production breaking down).

2.3 How Should Debt Be Performance-Sensitive?

Having derived a condition for performance-sensitive debt to be optimal, we finally study *how* debt should be sensitive to performance if this condition is satisfied, thus providing testable predictions (from a positive perspective), and guidance for contract design (from a normative perspective). To do so, we now parametrize the output distribution. This allows us to model the signal realization as affecting the distribution's parameters, and thus study how the face value of debt varies with these parameters. Specifically, we consider output distributions with a scale parameter σ_s, which can be interpreted as the distribution's volatility, and a location parameter $h_s(e)$ which, for symmetric distributions such as the normal and logistic, is the mean. We assume $h_s'(e) > 0$ for all e (higher effort shifts the distribution rightward). For distributions with location and scale parameters, there exists a function $g(\cdot)$ such that we can rewrite the density as:

$$f(q|e,s) \equiv \frac{1}{\sigma_s} g\left(\frac{q - h_s(e)}{\sigma_s}\right). \tag{9}$$

Without loss of generality, let $h_s(e) = \xi_s + \zeta_s \Upsilon(e)$ and normalize $\Upsilon(\hat{e}) = 0$ and $\Upsilon'(\hat{e}) = 1$, so that $h_s(\hat{e}) = \xi_s$ and $h_s'(\hat{e}) = \zeta_s > 0$. We refer to ξ_s as the equilibrium location parameter and ζ_s as the impact parameter; the latter captures the effect of effort on output.

Proposition 3 shows how the signal realization affects the face value of debt. It holds "all else equal across signals": we are comparing the face value of debt under two different signal realizations s_i and s_j that differ along only one dimension (e.g. the scale parameter σ_s); all other dimensions are constant. Note that we are not undertaking comparative statics (e.g. changing σ_s across all signals) that would change the contracting environment.

Proposition 3 *All else equal across signals:*

(i) If $\frac{\partial \phi_{\hat{e}}^{s_i}/\partial e}{\phi_{\hat{e}}^{s_i}} > \frac{\partial \phi_{\hat{e}}^{s_j}/\partial e}{\phi_{\hat{e}}^{s_j}}$, $q_{s_i}^* \leq q_{s_j}^*$. *Higher individual informativeness decreases the debt repayment.*

(ii) If $\xi_{s_i} < \xi_{s_j}$, $q^*_{s_i} \leq q^*_{s_j}$. A higher equilibrium location parameter increases the debt repayment.

(iii) If $\zeta_{s_i} > \zeta_{s_j}$, $q^*_{s_i} \leq q^*_{s_j}$. A higher impact parameter decreases the debt repayment.

(iv) If $\sigma_{s_i} > \sigma_{s_j}$ and $q^*_s > \max\{q^P_s, \xi_s\}$, $q^*_{s_i} \geq q^*_{s_j}$. A higher scale parameter increases the debt repayment if debt repayments are high across signals.

Part (i) is the "individual informativeness effect". If $\frac{\partial \phi^{s_i}_{\hat{e}}/\partial e}{\phi^{s_i}_{\hat{e}}} > \frac{\partial \phi^{s_j}_{\hat{e}}/\partial e}{\phi^{s_j}_{\hat{e}}}$, then signal realization s_i is individually more indicative of high effort than s_j. Thus, to reward managerial effort, the debt repayment should be lower under s_i than s_j. While it is intuitive that signals that are individually indicative of effort should affect the face value of debt, parts (ii)-(iv) show that debt should be performance-sensitive even if the signal is not individually informative about the manager's "performance". This is because the likelihood ratio (8) depends not only on the individual informativeness of the signal (the first term), but how the signal affects the effort level that the principal infers from observing output (the second term).

Part (ii) is the "location effect". If s_j is associated with a lower equilibrium location parameter ξ_{s_j} than s_i, then it indicates that the output distribution has shifted to the left. Due to MLRP, this shift means that achieving any given output level is more indicative of high effort than low effort. Part (ii) may lead to counterintuitive results, since performance measures that indicate low effort (such as low credit ratings) typically increase the required debt repayment. While a low credit rating is indeed a negative individual signal of performance, it may also shift the output distribution to the left as it restricts the firm's access to financing. Thus, achieving a given output is a more positive signal of effort, and so the universal practice of the debt repayment decreasing in the credit rating may not be optimal.

Parts (i) and (ii) echo the results in the model of Chaigneau, Edmans, and Gottlieb (2020) who study performance-vesting options – the strike price of the option is analogous to the face value of debt, and thus affected by the individual informativeness and location effects in similar ways. However, parts (iii) and (iv) are different. They capture the "precision effect" – how the signal realization affects the precision of output as a measure of effort. In turn, signal precision is increasing in the impact parameter ζ_s and decreasing in the scale parameter σ_s.

Part (iii) shows that, in states in which the impact parameter is high, the principal wishes to provide strong incentives, i.e. a high sensitivity of pay to performance. When options are the optimal contract, as in Chaigneau, Edmans, and Gottlieb (2020), this is achieved by increasing the number of options. However, under a debt contract, the slope is already at its maximum of 1 and thus cannot be increased further. Thus, the agent's payment can only be made more sensitive to performance by lowering the face value of debt q_s, as doing so increases the "delta" of her equity. Thus, the incentive zone is always enlarged when impact is higher.

Part (iv), on the effect of the scale parameter, is generally the opposite of part (iii). The principal wishes to provide weak incentives in states where output volatility is high, which is achieved by increasing the face value of debt to lower the delta of equity. However, the impact and scale parameters do not *always* have opposite effects on the debt repayment, because the scale parameter changes the equilibrium output distribution but the impact parameter does not. A higher volatility parameter not only reduces the slope of the likelihood ratio (similar to a lower impact parameter) but it also spreads out the likelihood ratio, because it spreads out the output distribution. This second effect goes in the same direction as the first for $q_s^* > \max\{q_s^P, \xi_s\}$, explaining the additional condition in part (iv). However, when debt repayments are low across signals, it is possible for this second effect to dominate. In this case – somewhat surprisingly – stronger incentives are provided under more volatile signals – see Example 1. Intuitively, for low debt repayments, the manager is paid unless output is sufficiently bad news about effort. When output is more volatile, the level below which output is sufficiently bad news decreases.

Example 1 *Let s be output volatility, with $s = h$ (l) corresponding to high (low) volatility where $\sigma_h = 1.1$ and $\sigma_l = 1.0$. In all cases, output is normally distributed with a mean of 10, an impact parameter of 1, and volatility is not individually informative about effort. We consider the case with a high marginal cost of effort, $C'(\hat{e}) = 1$, so that debt repayments are low across signals to provide strong incentives. In this case, the second effect from part (iv) of Proposition 3 dominates, and the debt repayment is lower (i.e. stronger incentives are provided) when output volatility is higher. The debt contract is displayed in Figure 1.*

Our rationale for performance-sensitive debt complements existing explanations. Manso, Strulovici, and Tchistyi (2010) model performance-sensitive debt as a mechanism to signal the firm's growth rate in an adverse selection model; there is no moral hazard. Bhanot and Mello (2006) and Koziol and Lawrenz (2010) show that performance-sensitive debt deters risk shifting. While none of these papers model an effort decision, Manso et al. (2010, Section 8) conjecture that performance-sensitive debt "could serve as an additional incentive for the firm's manager to exert effort" and Tchistyi (2009) shows that performance-sensitive debt can deter cash flow diversion. This intuition would suggest that the debt repayment should fall with signals that are individually indicative of effort (part (i) of Proposition 3). However, it does not have implications for the equilibrium location, impact, and scale parameters (parts (ii), (iii) and (iv)).

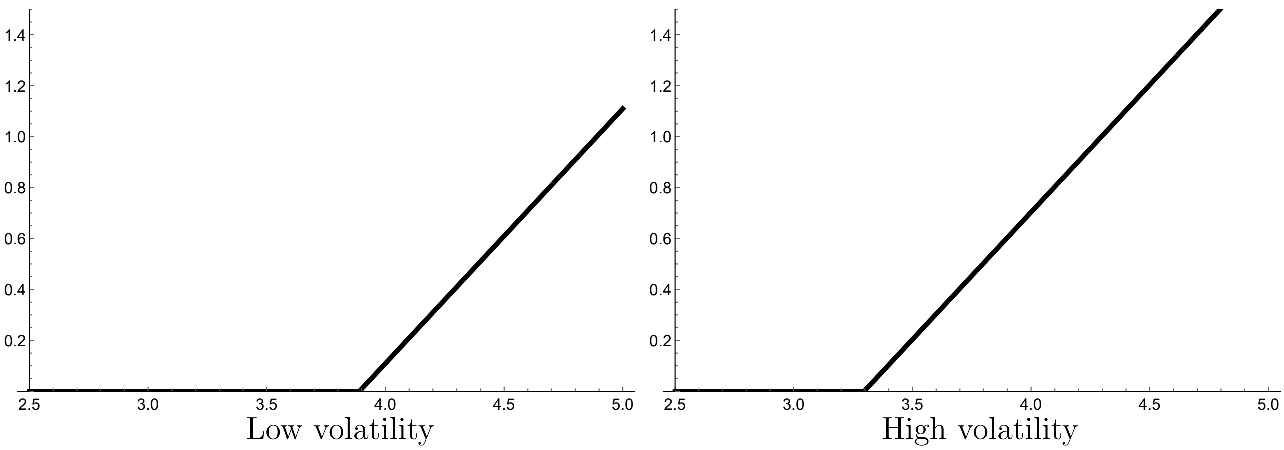

Figure 1: Payoff $w_s(q)$ as a function of q for high and low output volatility.

Innes (1993) derives the optimal contract when profits (which correspond to q in our setting) can be decomposed into output and the output price, i.e. the price is an additional signal that can be used in the contract. He shows that the optimal contract is a price-contingent commodity bond, which has similarities to performance-sensitive debt; however, the only signal that he analyzes is price (i.e. one component of output). We consider a broad set of signals, including signals that are informative about the manager's effort, and signals that affect the output distribution in different ways to the price. Bensoussan, Chevalier-Roignant, and Rivera (2019) model performance-sensitive debt as a solution to debt overhang. Adam and Streitz (2016) test empirically whether performance-sensitive debt is used to reduce hold-up problems, which arise from the information the lender acquires over the course of the lending relationship.[6]

Proposition 3 is an "all else equal" result, which compares two signal realizations that differ only along one parameter, and holds other parameters constant. In reality, signals may differ along multiple parameters, and so more than one out of the individual informativeness, location, and precision effects may be at work. Examples 2 and 3 illustrate such cases.

Example 2 *Let s be economic conditions, with $s = r$ corresponding to a recession and $s = e$ an expansion, with $\phi^r_{\hat{e}} = 0.25$, and $\phi^e_{\hat{e}} = 0.75$. Economic conditions are individually uninformative about effort. The firm's business is procyclical but more volatile in bad times. In a recession, $\xi_r = 10$, $\zeta_r = 1$, $\sigma_r = 1.5$. In an expansion, $\xi_e = 10.5$, $\zeta_e = 1$, $\sigma_e = 1$. In any case, output is*

[6]Asquith, Beatty, and Weber (2005) also conduct an empirical study of performance-sensitive debt. However, in their setting, the debt contract only includes either potential increases in the promised repayment, or decreases in the promised repayment – i.e. debt is performance-sensitive in one direction. In our paper, it is performance-sensitive in both directions.

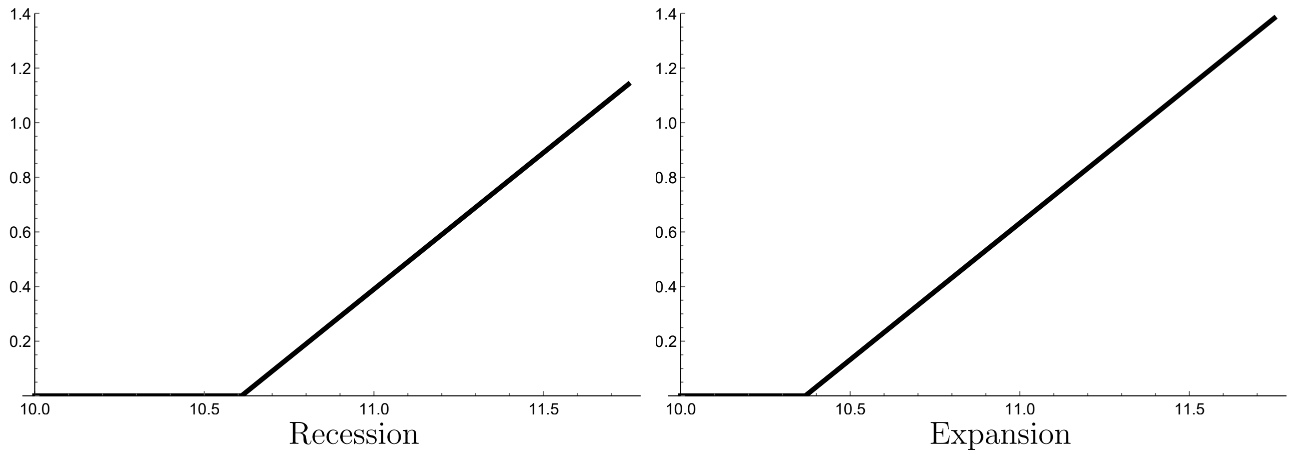

Figure 2: Payoff $w_s(q)$ as a function of q in an economic recession and an expansion.

normally distributed. The marginal cost of effort is $C'(\hat{e}) = 0.5$. The debt contract is displayed in Figure 2. The scale effect dominates the location effect, so that the debt repayment is higher in a recession.

Example 3 *Let s be economic conditions, with $s = r$ corresponding to a recession, and $s = e$ an expansion, with $\phi^r_{\hat{e}} = 0.25$, and $\phi^e_{\hat{e}} = 0.75$. Economic conditions are individually uninformative about effort. The firm's business is not cyclical, but it is more volatile in bad times, and the manager's effort has a stronger impact in bad times. In a recession, $\xi_r = 10$, $\zeta_r = 1.5$, $\sigma_r = 1.1$. In an expansion, $\xi_e = 10$, $\zeta_e = 1$, $\sigma_e = 1$. In any case, output is normally distributed. The marginal cost of effort is $C'(\hat{e}) = 0.5$. The debt contract is displayed in Figure 3. The impact effect dominates the scale effect, so that the debt repayment is lower in a recession.*

Summing up the results of this section, a signal can affect the face value of debt even if it is not individually informative about effort. If the signal indicates that the entire output distribution has improved, such as good industry performance, then all output levels are less indicative of effort and so the face value rises. If the signal indicates that output is a more precise measure of effort, such as a high impact parameter or low industry volatility, then incentives should generally be increased by lowering the face value of debt. However, when the face value of debt is low, higher volatility can instead reduce the face value of debt, as shown in Example 1.

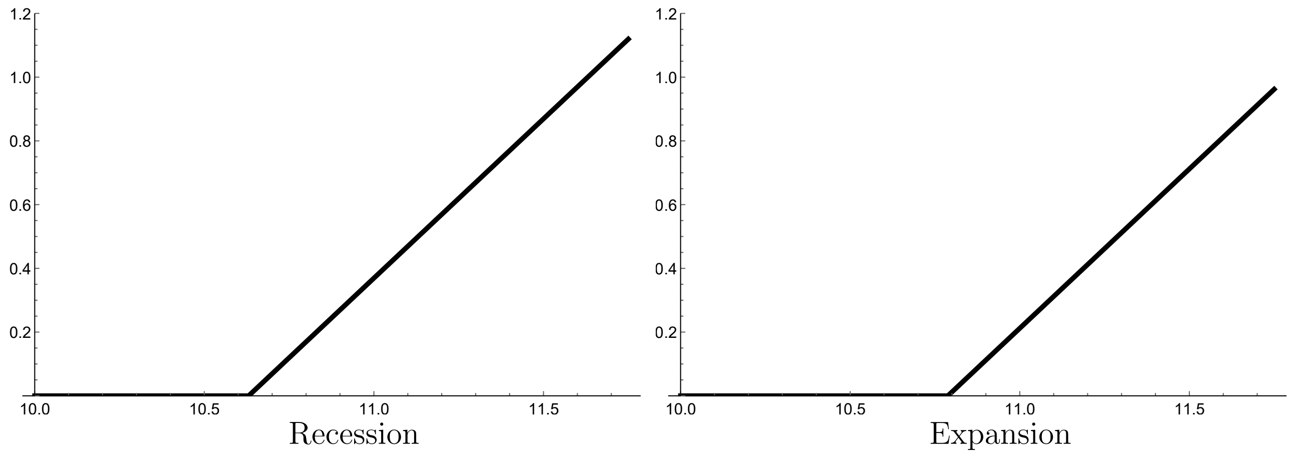

Figure 3: Payoff $w_s(q)$ as a function of q in an economic recession and an expansion.

3 Conclusion

This paper shows that, in the presence of limited liability and monotonicity constraints, the optimal contract remains debt even if the principal has access to additional performance signals. While it may seem intuitive that a good signal should lead to the agent being paid even if output is low, and a bad signal should lead to him not being the residual claimant even if output is high, we show that the signal does not affect the form of the contract, but only the face value of debt. As a result, Holmström's (1979) informativeness principle needs to be refined in the presence of the above constraints – a signal is only valuable if it is informative about whether output exceeds the face value of debt. If this condition is satisfied, then performance-sensitive debt is optimal.

We show how the signal should affect the face value of debt. As is intuitive, signals that individually indicate high effort optimally lower the face value of debt. However, in contrast to the "performance-sensitive debt" terminology, a signal should affect the face value even if it is individually uninformative about performance. Instead, such signals are valuable because they affect the inference that the principal makes about the agent's effort from observing output. If the signal indicates that the distribution of output has shifted to the right, a given output level implies lower effort and the face value rises. If it suggests that output is a more precise measure of effort, it is optimal to provide stronger incentives, which generally involves a lower face value of debt.

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1 Proofs

Proof of Lemma 1: The FOA is valid if the following objective function is concave in e:

$$\sum_s \phi_e^s \int_{\underline{q}}^{\infty} w_s(q) f(q|e,s)dq - C(e).$$

A sufficient condition is:

$$\sum_s \int_{\underline{q}}^{\infty} w_s(q) \frac{\partial^2 f}{\partial e^2}(q,s|e)dq < C''(e) \qquad \forall e. \tag{10}$$

From equation (5), $w_s(q) \in [0, \max\{0,q\}]$ for all q,s, so that $w_s(q)=0$ for $q \le 0$. Then, a sufficient condition for equation (10) is:

$$\sum_s \int_0^{\infty} \max\left\{ q\frac{\partial^2 f}{\partial e^2}(q,s|e), 0\right\} dq = \sum_s \int_0^{\infty} q \max\left\{ \frac{\partial^2 f}{\partial e^2}(q,s|e), 0\right\} dq < C''(e) \quad \forall e. \blacksquare$$

Proof of Proposition 1:

We first prove that the likelihood ratio $\overline{LR}_s(q)$ in equation (8) is increasing in q:

$$\frac{d}{dq}\left\{ \frac{\partial \phi_{\hat{e}}^s/\partial e}{\phi_{\hat{e}}^s} + \frac{\int_q^{\infty} \frac{\partial f}{\partial e}(z|\hat{e},s)dz}{\int_q^{\infty} f(z|\hat{e},s)dz} \right\} = \frac{-\frac{\partial f}{\partial e}(q|\hat{e},s)\int_q^{\infty} f(z|\hat{e},s)dz + f(q|\hat{e},s)\int_q^{\infty}\frac{\partial f}{\partial e}(z|\hat{e},s)dz}{\left(\int_q^{\infty} f(z|\hat{e},s)dz\right)^2}. \tag{11}$$

For $\frac{\partial f}{\partial e}(q|\hat{e},s) \le 0$, we have $-\frac{\partial f}{\partial e}(q|\hat{e},s)\int_q^{\infty} f(z|\hat{e},s)dz \ge 0$. Moreover, $f(q|\hat{e},s)\int_q^{\infty}\frac{\partial f}{\partial e}(z|\hat{e},s)dz > 0$ because of MLRP and $\int_{\underline{q}}^{\infty}\frac{\partial f}{\partial e}(z|\hat{e},s)dz = 0$. In sum, the RHS of equation (11) is positive. For $\frac{\partial f}{\partial e}(q|\hat{e},s) > 0$, the RHS of equation (11) is positive if and only if:

$$\begin{aligned}
& f(q|\hat{e},s)\int_q^{\infty}\frac{\partial f}{\partial e}(z|\hat{e},s)dz \ge \frac{\partial f}{\partial e}(q|\hat{e},s)\int_q^{\infty} f(z|\hat{e},s)dz \\
\Leftrightarrow \quad & \int_q^{\infty} \frac{\frac{\partial f}{\partial e}(z|\hat{e},s)}{\frac{\partial f}{\partial e}(q|\hat{e},s)}dz \ge \int_q^{\infty}\frac{f(z|\hat{e},s)}{f(q|\hat{e},s)}dz \\
\Leftrightarrow \quad & \int_q^{\infty}\left[\frac{\frac{\partial f}{\partial e}(z|\hat{e},s)}{\frac{\partial f}{\partial e}(q|\hat{e},s)} - \frac{f(z|\hat{e},s)}{f(q|\hat{e},s)}\right]dz \ge 0,
\end{aligned}$$

which holds because by MLRP we have $\frac{\frac{\partial f}{\partial e}(z|\hat{e},s)}{f(z|\hat{e},s)} \ge \frac{\frac{\partial f}{\partial e}(q|\hat{e},s)}{f(q|\hat{e},s)}$ for any $q \ge z$.

The rest of the proof is divided into two parts:

Step 1. Conditional on each signal realization, the optimal contract is debt.

Step 1.a. This part of the proof adapts the proof technique from Lemma 1 in Matthews (2001) to a setting with continuous output and an additional signal. Let $(W_s^*)_{s\in\{1,\ldots,S\}}$ (henceforth denoted by (W_s^*) for brevity) be a feasible payment schedule that induces effort $\hat{e}$. For a given signal realization s', consider an alternative payment schedule which is the same as (W_s^*) for any signal other than s', and $W_{s'}^{q_{s'}} = \max\{0, q - q_{s'}\}$ for a given s'. The face value $q_{s'}$ is chosen so that the payment schedules contingent on signal s', $W_{s'}^*$ and $W_{s'}^{q_{s'}}$, have the same expected payment under effort $\hat{e}$:

$$\int_{\underline{q}}^{\infty} W_{s'}^*(q) f(q, s'|\hat{e}) dq = \int_{\underline{q}}^{\infty} W_{s'}^{q_s}(q) f(q, s'|\hat{e}) dq. \tag{12}$$

It is straightforward to show that $W_{s'}^{q_{s'}}$ exists and is unique. We will first show that, for a given s', replacing $W_{s'}^*$ by $W_{s'}^{q_{s'}}$ increases effort.

For a given s', define:

$$W_{s,s'}^{**}(q) := \begin{cases} W_s^*(q) & \text{for } s \neq s' \\ W_s^{q_s}(q) & \text{for } s = s' \end{cases} . \tag{13}$$

In what follows we will compare the original payment schedule (W_s^*) to the payment schedule $\left(W_{s,s'}^{**}\right)$ as defined in equation (13). Let $e_{s'}^D$ be an optimal effort for the agent when the payment schedule is $\left(W_{s,s'}^{**}\right)$ instead of (W_s^*):

$$e_{s'}^D \in \arg\max_{e\in[0,\bar{e}]} \sum_s \int_{\underline{q}}^{\infty} W_{s,s'}^{**} f(q, s|e) dq - C(e).$$

Since the agent chooses $\hat{e}$ when the payment schedule is (W_s^*) and $e_{s'}^D$ when it is $(W_{s,s'}^{**})$, we must have:

$$\sum_s \int_{\underline{q}}^{\infty} W_s^{**}(q) f(q, s|e_{s'}^D) dq - C(e_{s'}^D) \geq \sum_s \int_{\underline{q}}^{\infty} W_{s,s'}^{**}(q) f(q, s|\hat{e}) dq - C(\hat{e}),$$

and

$$\sum_s \int_{\underline{q}}^{\infty} W_s^*(q) f(q, s|\hat{e}) dq - C(\hat{e}) \geq \sum_s \int_{\underline{q}}^{\infty} W_s^*(q) f(q, s|e_{s'}^D) dq - C(e_{s'}^D).$$

Combining these two inequalities, we obtain

$$\sum_s \int_{\underline{q}}^{\infty} \left[W_{s,s'}^{**}(q) - W_s^*(q)\right] \left[f(q, s|e_{s'}^D) - f(q, s|\hat{e})\right] dq \geq 0.$$

Using equation (13), this rewrites simply as:

$$\int_{\underline{q}}^{\infty} \left[W_{s'}^{q_{s'}}(q) - W_{s'}^{*}(q)\right]\left[f(q,s') - f(q,s'|\hat{e})\right] dq \geq 0. \tag{14}$$

Since both contracts have the same expected value under effort $\hat{e}$ by construction, and $W_{s'}^{q_{s'}}$ pays the lowest feasible amount for $q < q_{s'}$ and has the highest possible slope for $q > q_{s'}$, there exists $\bar{q}_{s'} \geq q_{s'}$ such that

$$W_{s'}^{q_{s'}}(q) \left\{ \begin{array}{c} \leq \\ \geq \end{array} \right\} W_{s'}^{*}(q) \text{ for all } q \left\{ \begin{array}{c} \leq \\ \geq \end{array} \right\} \bar{q}_{s'}. \tag{15}$$

We will now show by contradiction that $\hat{e} \leq e_{s'}^{D}$. Suppose that $\hat{e} > e_{s'}^{D}$. Then:

$$\begin{aligned}
0 \; & \leq \textstyle\int_{\underline{q}}^{\infty} \left[W_{s'}^{q_{s'}}(q) - W_{s'}^{*}(q)\right]\left[\frac{f(q,s'|e_{s'}^{D})}{f(q,s'|\hat{e})} - 1\right] f(q,s'|\hat{e})dq \\
& = \textstyle\int_{\underline{q}}^{\infty} \left[W_{s'}^{q_{s'}}(q) - W_{s'}^{*}(q)\right] \frac{f(q,s'|e_{s'}^{D})}{f(q,s'|\hat{e})} f(q,s'|\hat{e})dq - \underbrace{\int_{\underline{q}}^{\infty} \left[W_{s'}^{D}(q) - W_{s'}^{*}(q)\right] f(q,s'|\hat{e})dq}_{=0} \\
& = \textstyle\int_{\underline{q}}^{\bar{q}_{s'}} \left[W_{s'}^{q_{s'}}(q) - W_{s'}^{*}(q)\right] \frac{f(q,s'|e^{D})}{f(q,s'|\hat{e})} f(q,s'|\hat{e})dq + \int_{\bar{q}_{s'}}^{\infty} \left[W_{s'}^{q_{s'}}(q) - W_{s'}^{*}(q)\right] \frac{f(q,s'|e_{s'}^{D})}{f(q,s'|\hat{e})} f(q,s'|\hat{e})dq \\
& < \textstyle\int_{\underline{q}}^{\bar{q}_{s'}} \left[W_{s'}^{q_{s'}}(q) - W_{s'}^{*}(q)\right] \frac{f(\bar{q}_{s'},s'|e_{s'}^{D})}{f(\bar{q}_{s'},s'|\hat{e})} f(q,s'|\hat{e})dq + \int_{\bar{q}_{s'}}^{\infty} \left[W_{s'}^{q_{s'}}(q) - W_{s'}^{*}(q)\right] \frac{f(\bar{q}_{s'},s'|e_{s'}^{D})}{f(\bar{q}_{s'},s'|\hat{e})} f(q,s'|\hat{e})dq \\
& = \textstyle\frac{f(\bar{q}_{s'},s'|e_{s'}^{D})}{f(\bar{q}_{s'},s'|\hat{e})} \int_{\underline{q}}^{\infty} \left[W_{s'}^{q_{s'}}(q) - W_{s'}^{*}(q)\right] f(q,s'|\hat{e})dq = 0,
\end{aligned}$$

where, for every s, the first line divides and multiplies the expression inside the integral in equation (14) by $f(q,s'|\hat{e})$; the second line adds a term that equals zero (due to equation (12)); the third line splits the integral between outputs lower and higher than $\bar{q}_{s'}$; the fourth line uses MLRP supposing that $\hat{e} > e_{s'}^{D}$ and equation (15); the fifth line uses equation (12). These inequalities give us a contradiction ($0 < 0$), showing that $\hat{e} \leq e_{s'}^{D}$.

Step 1.b. For a given initial contract (W_s^*), repeat the same procedure for every $s \in \{s_1, \ldots, s_S\}$ which is such that the payment schedule under this signal realization does not take the form of debt. The resulting contract, which we denote by (W_s^D), is a debt contract, i.e. the payment schedule takes the form of debt for every s. Since the procedure weakly increased the implemented effort for every s, the effort implemented by this debt contract, denoted by e^D, is weakly larger than the effort $\hat{e}$ to be induced (this directly follows from the fact that the LHS of the IC is additive across signals). We now show how to modify this contract to implement the same effort as the initial contract, $\hat{e}$, at a lower cost. Since the resulting contract will still be a debt contract, it satisfies the contracting constraints in equations (5) and (6).

By assumption, the contract (W_s^*) is incentive compatible and the FOA holds, so that:

$$\sum_s \int_{\underline{q}}^{\infty} W_s^*(q) \frac{\partial f}{\partial e}(q, s|\hat{e}) dq \quad = \quad C'(\hat{e}). \tag{16}$$

Let ε be an arbitrarily large constant which satisfies the following two conditions: (i) $\varepsilon > \max\{q_1, \ldots, q_S\}$, and (ii):

$$\sum_s \int_{\varepsilon}^{\infty} (q - \varepsilon) \frac{\partial f}{\partial e}(q, s|\hat{e}) dq \quad < \quad C'(\hat{e}). \tag{17}$$

There exists ε that satisfies condition (17) because of the assumption that $\lim_{q \nearrow +\infty} \frac{\partial f}{\partial e}(q, s|e) = 0$. Consider the subset of $\{s_1, \ldots, s_S\}$ such that:

$$\int_{\varepsilon}^{\infty} (q - \varepsilon) \frac{\partial f}{\partial e}(q, s|\hat{e}) dq < \int_{\underline{q}}^{\infty} W_s^*(q) \frac{\partial f}{\partial e}(q, s|\hat{e}) dq, \tag{18}$$

and denote this subset by $\mathcal{S}$. $\mathcal{S}$ is nonempty (if it were, summing over signals in equation (18) and comparing with equation (17) would yield the contradiction that equation (16) does not hold).

For any $s \in \mathcal{S}$, we claim and establish below that there exists $\hat{q}_s \geq q_s$ which solves:

$$\int_{\hat{q}_s}^{\infty} (q - \hat{q}_s) \frac{\partial f}{\partial e}(q, s|\hat{e}) dq = \int_{\underline{q}}^{\infty} W_s^*(q) \frac{\partial f}{\partial e}(q, s|\hat{e}) dq. \tag{19}$$

For a given $s \in \mathcal{S}$, using the IC with the FOA and the results on effort under the two payment schedules W_s^* and $W_s^{q_s}$ established in Step 1.a. gives the following equation:

$$\int_{q_s}^{\infty} (q - q_s) \frac{\partial f}{\partial e}(q, s|\hat{e}) dq \quad \geq \quad \int_{\underline{q}}^{\infty} W_s^*(q) \frac{\partial f}{\partial e}(q, s|\hat{e}) dq \tag{20}$$

For each signal $s \in \mathcal{S}$, there are two cases. If, for a given s, equation (20) holds as an equality, then set $\hat{q}_s = q_s$, so that equation (19) holds. If, for a given s, equation (20) holds as a strict inequality, then for this s, there is $\hat{q}_s \in (q_s, \varepsilon)$ such that equation (19) holds because of the intermediate value theorem, which for a given s we apply on the interval $[q_s, \varepsilon]$. The theorem applies because of equation (18), equation (20) as a strict inequality, and $\int_z^{\infty} (q - z) \frac{\partial f}{\partial e}(q, s|\hat{e}) dq$ is a continuous function of z.

First, if $\mathcal{S} = \{s_1, \dots, s_S\}$ or if

$$\sum_{\tilde{s} \notin \mathcal{S}} \int_{\varepsilon}^{\infty} (q - \varepsilon) \frac{\partial f}{\partial e}(q, \tilde{s}|\hat{e}) dq + \sum_{\tilde{s} \in \mathcal{S}} \int_{\hat{q}_{\tilde{s}}}^{\infty} (q - \hat{q}_{\tilde{s}}) \frac{\partial f}{\partial e}(q, \tilde{s}|\hat{e}) dq = C'(\hat{e}), \tag{21}$$

where for each $s \in \mathcal{S}$, the face value $\hat{q}_s$ is implicitly defined in equation (19), then for any $s \in \mathcal{S}$ use the payment schedule:

$$W_s^{\hat{q}_s}(q) = \max\{0, q - \hat{q}_s\}, \tag{22}$$

and for any $s \notin \mathcal{S}$ the face value is set at ε.

Second, if $\mathcal{S} \subset \{s_1, \dots, s_S\}$ and the condition in equation (21) does not hold, then let the signals in $\mathcal{S}$ be ordered such that $\mathcal{S} = \{s_1^{\mathcal{S}}, \dots, s_N^{\mathcal{S}}\}$, with $N \geq 1$ (since $\mathcal{S}$ is nonempty). Denote by $\mathcal{S}^c$ the complement of $\mathcal{S}$. For any $s \in \mathcal{S}^c$, set the face value at ε. If

$$\sum_{\tilde{s} \in \mathcal{S}^c \cup \{s_1^{\mathcal{S}}\}} \int_{\varepsilon}^{\infty} (q - \varepsilon) \frac{\partial f}{\partial e}(q, \tilde{s}|\hat{e}) dq + \sum_{\tilde{s} \in \mathcal{S} \setminus \{s_1^{\mathcal{S}}\}} \int_{\hat{q}_{\tilde{s}}}^{\infty} (q - \hat{q}_{\tilde{s}}) \frac{\partial f}{\partial e}(q, \tilde{s}|\hat{e}) dq < C'(\hat{e}), \tag{23}$$

then let $\check{q}_{s_1^{\mathcal{S}}}$ be implicitly defined by:

$$\sum_{\tilde{s} \in \mathcal{S}^c} \int_{\varepsilon}^{\infty} (q - \varepsilon) \frac{\partial f}{\partial e}(q, \tilde{s}|\hat{e}) dq + \sum_{\tilde{s} \in \mathcal{S} \setminus \{s_1^{\mathcal{S}}\}} \int_{\hat{q}_{\tilde{s}}}^{\infty} (q - \hat{q}_{\tilde{s}}) \frac{\partial f}{\partial e}(q, \tilde{s}|\hat{e}) dq + \int_{\check{q}_{s_1^{\mathcal{S}}}}^{\infty} \left(q - \check{q}_{s_1^{\mathcal{S}}}\right) \frac{\partial f}{\partial e}(q, s_1^{\mathcal{S}}|\hat{e}) dq = C'(\hat{e}).$$

$\check{q}_{s_1^{\mathcal{S}}}$ exists and is larger than $\hat{q}_{s_1^{\mathcal{S}}}$ by application of the intermediate value theorem to the interval $[\hat{q}_{s_1^{\mathcal{S}}}, \varepsilon]$, with equations (23) and (24):

$$\sum_{\tilde{s} \in \mathcal{S}^c} \int_{\varepsilon}^{\infty} (q - \varepsilon) \frac{\partial f}{\partial e}(q, \tilde{s}|\hat{e}) dq + \sum_{\tilde{s} \in \mathcal{S}} \int_{\hat{q}_{\tilde{s}}}^{\infty} (q - \hat{q}_{\tilde{s}}) \frac{\partial f}{\partial e}(q, \tilde{s}|\hat{e}) dq > C'(\hat{e}). \tag{24}$$

In turn, we get equation (24) because of equation (16) on the one hand, and on the other hand because for signals in $\mathcal{S}$, the face value $\hat{q}_s$ satisfies equation (19), for signals in $\mathcal{S}^c$ the condition in equation (18) does not hold, and equation (21) does not hold here (see above). If condition (23) holds, then set the face value of signal $s_1^{\mathcal{S}}$ at $\check{q}_{s_1^{\mathcal{S}}}$, and set the face value at $\hat{q}_s$ for other signals in $\mathcal{S}$. If condition (23) does not hold, then set $\hat{q}_{s_1^{\mathcal{S}}} = \varepsilon$, repeat the same steps with signal $s_2^{\mathcal{S}}$ (we omit explicit formulation of these steps for brevity), and continue repeating these steps to additional signals in $\mathcal{S}$ until, for a signal $s_i^{\mathcal{S}}$, with $i \leq N$, condition

$$\sum_{\tilde{s} \in \mathcal{S}^c \cup \{s_1^{\mathcal{S}}, \ldots, s_i^{\mathcal{S}}\}} \int_{\varepsilon}^{\infty} (q - \varepsilon) \frac{\partial f}{\partial e}(q, \tilde{s}|\hat{e}) dq + \sum_{\tilde{s} \in \mathcal{S} \setminus \{s_1^{\mathcal{S}}, \ldots, s_i^{\mathcal{S}}\}} \int_{\hat{q}_{\tilde{s}}}^{\infty} (q - \hat{q}_{\tilde{s}}) \frac{\partial f}{\partial e}(q, \tilde{s}|\hat{e}) dq < C'(\hat{e}) \quad (25)$$

is satisfied, in which case set the face value of signal $s_i^{\mathcal{S}}$ at $\check{q}_{s_i^{\mathcal{S}}}$, which is implicitly defined by:

$$\sum_{\tilde{s} \in \mathcal{S}^c \cup \{s_1^{\mathcal{S}}, \ldots, s_{i-1}^{\mathcal{S}}\}} \int_{\varepsilon}^{\infty} (q - \varepsilon) \frac{\partial f}{\partial e}(q, \tilde{s}|\hat{e}) dq + \sum_{\tilde{s} \in \mathcal{S} \setminus \{s_1^{\mathcal{S}}, \ldots, s_i^{\mathcal{S}}\}} \int_{\hat{q}_{\tilde{s}}}^{\infty} (q - \hat{q}_{\tilde{s}}) \frac{\partial f}{\partial e}(q, \tilde{s}|\hat{e}) dq$$
$$+ \int_{\check{q}_{s_i^{\mathcal{S}}}}^{\infty} \left(q - \check{q}_{s_i^{\mathcal{S}}}\right) \frac{\partial f}{\partial e}(q, s_i^{\mathcal{S}}|\hat{e}) dq = C'(\hat{e}).$$

$\check{q}_{s_i^{\mathcal{S}}}$ exists and is larger than $\hat{q}_{s_i^{\mathcal{S}}}$ because of the same arguments used above. Because of equation (17), condition (25) will be satisfied for a signal $s_i^{\mathcal{S}}$, with $i \leq N$. If $i < N$, for signals $s \in \{s_{i+1}^{\mathcal{S}}, \ldots, s_N^{\mathcal{S}}\}$ in $\mathcal{S}$, set the face value to $\hat{q}_s$ as in equation (19).

In sum, for each given s, the new contract is a debt contract with face value equal to either $\hat{q}_s$ or $\check{q}_s$ or ε, such that $\hat{q}_s \geq q_s$ if $\hat{q}_s$ exists, $\check{q}_s > \hat{q}_s \geq q_s$ if $\check{q}_s$ and $\hat{q}_s$ exist, and $\varepsilon > q_s$. Since by construction the debt contract (W_s^D) with face values q_s has the same cost as the initial contract (W_s^*), and the cost of a debt contract for the principal at a given s is decreasing in the face value at this signal s, the new debt contract achieves the same effort $\hat{e}$ as the initial contract (W_s^*) at a lower cost.

Step 2. Determining the optimal face value of debt.

Since any debt contract satisfies bilateral LL and monotonicity, and since we assumed that the condition for the FOA in Lemma 1 holds, the firm's program becomes:

$$\min_{\{q_s\}_{s=1,\ldots,S}} \sum_s \int_{q_s}^{\infty} (q - q_s) f(q, s|\hat{e}) dq. \quad (26)$$

subject to

$$\sum_s \int_{q_s}^{\infty} (q - q_s) \frac{\partial f}{\partial e}(q, s|\hat{e}) dq = C'(\hat{e}), \quad (27)$$

where $\frac{\partial f}{\partial e}(q,s|\hat{e}) = \frac{\partial \phi^s_{\hat{e}}}{\partial e} f(q|\hat{e},s) + \phi^s_{\hat{e}} \frac{\partial f}{\partial e}(q|\hat{e},s)$. The likelihood ratio can be rewritten as follows:

$$\begin{aligned}
\overline{LR}_s(q) &= \frac{\int_q^\infty \left[\frac{\partial \phi^s_{\hat{e}}}{\partial e} f(z|\hat{e},s) + \phi^s_{\hat{e}} \frac{\partial f}{\partial e}(z|\hat{e},s)\right] dz}{\int_q^\infty \phi^s_{\hat{e}} f(z|\hat{e},s) dz} \\
&= \frac{\int_q^\infty \frac{\partial \phi^s_{\hat{e}}}{\partial e} f(z|\hat{e},s) dz}{\int_q^\infty \phi^s_{\hat{e}} f(z|\hat{e},s) dz} + \frac{\int_q^\infty \phi^s_{\hat{e}} \frac{\partial f}{\partial e}(z|\hat{e},s) dz}{\int_q^\infty \phi^s_{\hat{e}} f(z|\hat{e},s) dz} \\
&= \frac{\partial \phi^s_{\hat{e}}/\partial e}{\phi^s_{\hat{e}}} + \frac{\int_q^\infty \frac{\partial f}{\partial e}(z|\hat{e},s) dz}{\int_q^\infty f(z|\hat{e},s) dz}
\end{aligned}$$

For each fixed κ and signal realization s, construct the threshold $q^*_s(\kappa)$ as follows:

$$q^*_s(\kappa) := \begin{cases} 0 & \text{if } \overline{LR}_s(0) > \kappa \\ \overline{LR}_s^{-1}(\kappa) & \text{if } \overline{LR}_s(0) \leq \kappa \end{cases}. \tag{28}$$

The cutoff κ is implicitly determined by the binding IC:

$$\sum_s \int_{q^*_s(\kappa)}^\infty (q - q^*_s(\kappa)) \frac{\partial f}{\partial e}(q,s|\hat{e}) dq = C'(\hat{e}). \tag{29}$$

The necessary first-order conditions associated with the program in equations (26) and (27) are equation (28) and the binding IC:

$$\sum_s \int_{q^*_s(\kappa)}^\infty (q - q^*_s(\kappa)) \frac{\partial f}{\partial e}(q,s|\hat{e}) dq = C'(\hat{e}). \tag{30}$$

where $\kappa := \frac{1}{\mu}$ and μ is the Lagrange multiplier associated with the IC.

Each κ determines $q^*_s(\kappa)$ according to equation (28). From the Intermediate Value Theorem, there exists κ that solves equation (30): as $\kappa \searrow -\infty$, the LHS of (30) exceeds $C'(\hat{e})$ since then $q^*_s(\kappa) = 0 \; \forall s$ and

$$\sum_s \int_0^\infty q \frac{\partial f}{\partial e}(q,s|\hat{e}) dq \geq C'(\hat{e})$$

by the assumption in equation (7), and it converges to $0 < C'(\hat{e})$ as $\kappa \nearrow +\infty$. Moreover, κ must be unique since our conditions for the validity of the FOA ensure that the agent's program has a unique solution. ■

Proof of Proposition 2. Start with part (i) of the Proposition. From Proposition 1, there are two possible cases in which the optimal contract does not depend on the signal ($q^*_{s_1} = ... =$

$q^*_{s_S} = q^*$): an interior solution $q^* \in (\underline{q}, \overline{q})$ and a boundary solution $q^* \in \{\underline{q}, \overline{q}\}$. Using the conditions from equation (28) for an interior solution establishes:

$$\overline{LR}_{s_i}(q^*) = \overline{LR}_{s_j}(q^*) = \kappa \;\; \forall s_i, s_j. \tag{31}$$

where κ is determined by (29). Using the definition of $\overline{LR}_s(q)$ and rearranging yields the result stated in the proposition.

We now verify that the solution cannot be at the boundary. For a boundary solution we need either $\overline{LR}_s(\underline{q}) > \kappa$ for all s or $\overline{LR}_s(\overline{q}) < \kappa$ for all s. In the first case, the firm always receives zero, which contradicts the optimality of implementing high effort (since the firm can always obtain strictly positive profits by paying zero in all states and implementing low effort). In the second case, the manager always receives zero, violating equation (29) as the IC is not satisfied.

For part (ii) of the Proposition, if $q \leq \min_s \{q^*_s\}$ then $w_s(q) = 0 \; \forall s$, i.e., $w_s(q)$ is independent of s.

For part (iii), given signal realization s, according to the optimal contract in Proposition 1 and to equation (28), the debt repayment is zero if $\overline{LR}_s(q)$ is above κ for any q, where κ is implicitly defined in equation (29). Given that the second term in the likelihood ratio $\overline{LR}_s(q)$ in equation (8) is increasing in q (as established in the proof of Proposition 1) and is bounded from below by 0, a sufficient condition for the payment to be the zero under signal s is that the first term in the likelihood ratio $\overline{LR}_s(q)$ in equation (8) be above κ. ■

Proof of Proposition 3:

For distributions with location and scale parameters, the PDF of output can be written as in equation (9). The likelihood ratio in equation (8) can then be written as:

$$\overline{LR}_s(q) = \frac{\partial \phi^s_{\hat{e}} / \partial e}{\phi^s_{\hat{e}}} - \frac{\zeta_s}{\sigma_s} \frac{\int_q^\infty g'\left(\frac{z-\xi_s}{\sigma_s}\right) dz}{\int_q^\infty g\left(\frac{z-\xi_s}{\sigma_s}\right) dz}.$$

For part (i), suppose that signals s_i and s_j differ only in that $\frac{\partial \phi^{s_i}_{\hat{e}}/\partial e}{\phi^{s_i}_{\hat{e}}} \geq \frac{\partial \phi^{s_j}_{\hat{e}}/\partial e}{\phi^{s_j}_{\hat{e}}}$. Since the likelihood ratio $\overline{LR}_s(q)$ is increasing in q as shown above, and since the face value of debt q^*_s is given by equation (28), with all else equal across signals we have $q^*_{s_i} \leq q^*_{s_j}$.

For part (ii), when $\overline{LR}_{s_i}(q) \geq \overline{LR}_{s_j}(q)$ for any q, since $\overline{LR}_s(q)$ is increasing in q as shown above and the face value q^*_s is given by (28), we have $q^*_{s_i} \leq q^*_{s_j}$. This condition on the likelihood ratios is satisfied for two signals $\{s_i, s_j\}$ such that $\xi_{s_i} \leq \xi_{s_j}$, all else equal across signals.

For part (iii), for single-peaked distributions, there exists $\overline{z}$ such that $g'(z) > 0$ for $z < \overline{z}$ and $g'(z) < 0$ for $z > \overline{z}$, and $\int_{\underline{q}}^{\infty} g'(z)\, dz = 0$. Therefore, all else equal:

$$\frac{\partial \overline{LR}_s(q)}{\partial \zeta_s} = -\frac{1}{\sigma_s} \frac{\int_q^{\infty} g'\left(\frac{z-h_s(\hat{e})}{\sigma_s}\right) dz}{\int_q^{\infty} g\left(\frac{z-h_s(\hat{e})}{\sigma_s}\right) dz} > 0 \tag{32}$$

Consider two signals $\{s_i, s_j\}$ such that $\zeta_{s_i} \geq \zeta_{s_j}$. Then, because of equation (32), we have $\overline{LR}_{s_i}(q) \geq \overline{LR}_{s_j}(q)$ for any q. Since the face value q_s^* is given by (28), with all else equal across signals we have $q_{s_i}^* \leq q_{s_j}^*$.

For part (iv), for a given s, use the change of variables $y = \frac{z-h_s(\hat{e})}{\sigma_s}$ to rewrite the likelihood ratio as:

$$\overline{LR}_s(q) = \frac{\partial \phi_{\hat{e}}^s / \partial e}{\phi_{\hat{e}}^s} - \frac{\zeta_s}{\sigma_s} \frac{\int_{\frac{q-h_s(\hat{e})}{\sigma_s}}^{\infty} g'(y)\, dy}{\int_{\frac{q-h_s(\hat{e})}{\sigma_s}}^{\infty} g(y)\, dy}.$$

Then:

$$\begin{aligned} \frac{\partial \overline{LR}_s(q)}{\partial \sigma_s} = {} & \frac{\zeta_s}{\sigma_s^2} \frac{\int_{\frac{q-h_s(\hat{e})}{\sigma_s}}^{\infty} g'(y)\, dy}{\int_{\frac{q-h_s(\hat{e})}{\sigma_s}}^{\infty} g(y)\, dy} \\ & -\frac{\zeta_s}{\sigma_s} \frac{g'\left(\frac{q-h_s(\hat{e})}{\sigma_s}\right) \frac{q-h_s(\hat{e})}{\sigma_s^2} \int_{\frac{q-h_s(\hat{e})}{\sigma_s}}^{\infty} g(y)\, dy - g\left(\frac{q-h_s(\hat{e})}{\sigma_s}\right) \frac{q-h_s(\hat{e})}{\sigma_s^2} \int_{\frac{q-h_s(\hat{e})}{\sigma_s}}^{\infty} g'(y)\, dy}{\left(\int_{\frac{q-h_s(\hat{e})}{\sigma_s}}^{\infty} g(y)\, dy\right)^2} \end{aligned}$$

The first term on the RHS is negative, for the same reason as in part (iii) above. We now study the sign of the second term on the RHS. Let $\underline{y} \equiv \frac{q-h_s(\hat{e})}{\sigma_s}$. For $q > h_s(\hat{e})$ and $q > q_s^P$ (which implies $g'(y) < 0\ \forall y \geq \underline{y}$), the numerator of the second fraction of the second term on the RHS is positive if and only if:

$$\begin{aligned} & g'\left(\frac{q-h_s(\hat{e})}{\sigma_s}\right) \frac{q-h_s(\hat{e})}{\sigma_s^2} \int_{\frac{q-h_s(\hat{e})}{\sigma_s}}^{\infty} g(y)\, dy - g\left(\frac{q-h_s(\hat{e})}{\sigma_s}\right) \frac{q-h_s(\hat{e})}{\sigma_s^2} \int_{\frac{q-h_s(\hat{e})}{\sigma_s}}^{\infty} g'(y)\, dy > 0 \\ \Leftrightarrow \quad & g'(\underline{y}) \int_{\underline{y}}^{\infty} g(y)\, dy > g(\underline{y}) \int_{\underline{y}}^{\infty} g'(y)\, dy \quad \Leftrightarrow \quad \int_{\underline{y}}^{\infty} \frac{g(y)}{g(\underline{y})} dy < \int_{\underline{y}}^{\infty} \frac{g'(y)}{g'(\underline{y})} dy \\ \Leftrightarrow \quad & \int_{\underline{y}}^{\infty} \left[\frac{g(y)}{g(\underline{y})} - \frac{g'(y)}{g'(\underline{y})}\right] dy < 0. \end{aligned} \tag{33}$$

Since the distribution g is characterized by MLRP, we have $\frac{g'(\underline{y})}{g(\underline{y})} \geq \frac{g'(y)}{g(y)}$ $\forall y \geq \underline{y}$ so that

$\frac{g(y)}{g(\underline{y})} \leq \frac{g'(y)}{g'(\underline{y})}$ $\forall y \geq \underline{y}$. That is, the term in brackets on the same line of equation (33) is negative for all $y \geq \underline{y}$, so that the integral is negative, and the inequality in equation (33) holds. In sum, if $\sigma_{s_i} > \sigma_{s_j}$, all else equal across signals, then for $q > \max\{q_s^P, h_s(\hat{e})\}$, $\overline{LR}_{s_i}(q) < \overline{LR}_{s_j}(q)$. Since the face value q_s^* is given by (28), with all else equal across signals we have $q_{s_i}^* \geq q_{s_j}^*$. ■