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JEL Classification: G02, E03, E44, G11, G41

Keywords: behavioral finance, money market, aggregate growth, Stochastic discount factor

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Do the Effects of Individual Behavioral Biases Cancel Out?*

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June 27, 2021

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A major criticism of behavioral economics is that it has not shown that the idiosyncratic biases of individual investors lead to aggregate effects. We construct a model of a general-equilibrium production economy with a large number of firms and investors. Investors' beliefs about stock returns are determined endogenously based on their psychological distances from firms; consequently, investors are optimistic about some stocks and pessimistic about others. We consider two examples: one where portfolio errors cancel out and the other in which the behavioral biases cancel out when aggregated across investors. We show asset prices and macroeconomic aggregates are still distorted.

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1 Introduction and Motivation

The rational-expectations paradigm is the traditional approach to explaining phenomena in financial markets and the macroeconomy. It assumes that investors are risk-averse utility optimizers with unbiased Bayesian forecasts. However, the rational paradigm has been criticized because these assumptions are descriptively false and its predictions fail to explain the data (Gennaioli and Shleifer, 2018). An alternative behavioral paradigm has been developed, which relaxes the assumptions of rational expectations and consequently is much more successful in explaining the observed behavior of individual investors. But the behavioral paradigm has been criticized on the grounds that it has not shown that the psychological biases exhibited by individual investors lead to aggregate effects. For instance, while Fama (2012) agrees that behavioral finance is very good at describing individual behavior, he argues that the "jumps that [behaviorists] make from there to markets aren't validated by the data." Similarly, Scholes (2009) says that the trouble with behavioral economics is that "it really hasn't shown in aggregate how it affects prices." Hirshleifer (2001, p. 1540) summarizes this perspective stating that "Economists often argue that errors are independent across individuals, and therefore cancel out in equilibrium." Motivated by this observation, our paper aims to provide a model of behavioral finance to demonstrate that even if individual "errors" in decisions or beliefs cancel out in equilibrium, their impact on asset prices and the macroeconomy will not.

There are two building blocks of behavioral finance—psychological biases of investors and limits to arbitrage.[1] The model we develop relies on the first—psychological biases in decision making. We do not impose exogenous belief configurations on investors. Instead we endogenize beliefs, following the lead of Hansen and Sargent (2007), who argue that doing so ensures deviations from the rational benchmark are reasonable. However, the framework of Hansen and Sargent (2007) can only generate endogenous pessimism.[2] We therefore develop a novel model of belief formation where both optimism and pessimism can arise endogenously.

Both psychology and neuroscience find that human behavior is the result of a complex interaction between cognitive and emotional responses to stimuli (Kahneman, 2003, Camerer, Loewenstein, and Prelec, 2004, Pessoa, 2008); for instance, studies in neuroscience identify distinct brain modules that specialize in different activities, with the amygdala associated with

[1] Excellent surveys of these two building blocks are provided by Shleifer (2000), Barberis and Thaler (2003), Hirshleifer (2001, 2015), Shefrin (2007, 2010), and Statman (2010, 2011).

[2] Similarly, Bhamra, Uppal, and Walden (2021) only generate pessimism, while Brunnermeier and Parker (2005) and Bracha and Brown (2012) can generate only endogenous optimism.

emotions and the prefrontal cortex with analytical thinking (Reisberg, 2001). Motivated by these findings, we develop a framework in which we postulate an emotional process for belief formation together with a rational process that, for a given set of beliefs, chooses actions to maximize expected utility.

Specifically, an investor's decisions are determined in two stages. In the first stage, each investor chooses her beliefs about the expected returns for each firm based on her psychological distance from that firm. Trope and Liberman (2010) explain that psychological distance is a subjective experience. Its reference point is the self, here and now, and the different ways in which an object might be removed from that point—in time, space, social distance, and likelihood—constitute different distance dimensions. This endogenous selection of beliefs leads to biases relative to the rational-expectations benchmark. The bias in beliefs can be positive, which signifies optimism, or negative, signifying pessimism. Importantly, an investor's degree of bias can vary across firms. In the second stage, the investor chooses her optimal consumption and portfolio, given her beliefs. Effectively, the investor engages in a simultaneous-move intrapersonal game. The simultaneous choice of beliefs and actions represents the interaction of two processes: a psychological process governing belief selection and a rational process governing action choices.

In order to study the effect of behavioral biases on macroeconomic quantities, we combine our model of belief formation with the Cox, Ingersoll, and Ross (1985) framework, where both asset prices and economic growth are endogenous. As in Cox, Ingersoll, and Ross (1985), we consider a setting with a finite number of heterogeneous firms whose physical capital is subject to exogenous shocks. But, in contrast with Cox, Ingersoll, and Ross, we have heterogeneous investors with Epstein and Zin (1989) and Weil (1990) preferences coupled with beliefs that deviate from rational expectations. We then specify the behavioral biases so that they "cancel out" when aggregated across all investors. We show that, even in this case, the individual biases in beliefs affect both the interest rate and the market price of risk; that is, both the mean and volatility of the stochastic discount factor. The belief biases have an effect also on individual consumption, which does not cancel out in aggregate, and hence influences macroeconomic quantities such as aggregate investment and growth.

We consider two distinct approaches to ensuring biases "cancel out." First, we ensure that the portfolio errors resulting from biased beliefs cancel out when aggregated across investors. To implement this approach, we consider the case where each investor is more optimistic about a small subset of firms, which are psychologically close to her, and pessimistic about the rest, which are psychologically distant. Hence, investors tilt their portfolios towards the firms about which they are optimistic. We specify that investors are optimistic and pessimistic about different firms, so there is heterogeneity in portfolio errors that cancels out when added across investors. We also restrict the mean level of bias to be identical across investors. Even though portfolio errors cancel out, each investor has the same biased demand for the bond, which distorts the interest rate. In the second approach, we specify the biased beliefs themselves to cancel out when aggregated across investors and firms. However, these biases still distort the interest rate, because as the wealth distribution fluctuates over time, the relative importance of an individual's biases changes. Thus, fluctuations in relative wealth lead to a dislocation in the money market, which changes both the mean and volatility of the stochastic discount factor.[3]

Hirshleifer (2001, page 1534) states that "Over time I believe that the purely rational paradigm will be subsumed by a broader psychological paradigm that includes full rationality as a significant special case. ...The central task of asset pricing is to examine how expected returns are related to risk and to investor misvaluation." Our framework provides a way to connect investor misvaluations of expected returns to psychological distance. Importantly, our model needs only two free parameters to characterize fully the deviation of beliefs from rational expectations for all investors, with rational expectations a special case of our general framework.

The single key driver of the results in our model, that investors hold biased portfolios, is one for which economists have gathered a great deal of empirical evidence. Guiso, Haliassos, and Jappelli (2002), Haliassos (2002), Campbell (2006), Calvet, Campbell, and Sodini (2007), and Guiso and Sodini (2013) highlight underdiversification in the portfolios of individual investors. Polkovnichenko (2005), using data from the Survey of Consumer Finances, finds that for investors that invest in individual stocks directly, the median number of stocks held was two from 1983 until 2001, when it increased to three. Barber and Odean (2000) and Goetzman and Kumar (2008) report similar findings based on data for individual investors at a U.S. brokerage firm. This evidence can be interpreted as suggesting that investors are optimistic about a small number of stocks and pessimistic about the rest.

Huberman (2001) introduces the idea that investors are optimistic about assets with which they are "familiar," that is, assets that are psychologically close, and provides evidence of this

[3]Our focus on the money market complements the analysis of the effects of speculation on asset prices in exchange economies in Atmaz and Basak (2018) and Martin and Papadimitriou (2021), where the risk-free interest rate is exogenous.

in a multitude of contexts. Grinblatt and Keloharju (2001) find that not just temporal distance but also other aspects of an asset, such as the language in which the annual reports are published and the cultural background of the chief executive, may affect investors' portfolio holdings. Massa and Simonov (2006) also find that investors bias their portfolios toward stocks that are geographically and professionally close to the investor. Keloharju, Knüpfer, and Linnainmaa (2012) find that people tend to invest in firms they know through their product-market experiences, and that this bias is linked to preferences as opposed to information. The most striking example of biased investing is the holding of "own-company stock," that is, stock of the company where the person is employed (Haliassos, 2002).[4] Cohen (2009) shows that this biased perspective of own-company stock may be driven by loyalty considerations. French and Poterba (1990) and Cooper and Kaplanis (1994) document that investors bias their portfolios toward "home equity" rather than diversifying internationally.

A paper close to our work is Heyerdahl-Larsen and Walden (2021) that studies in a production economy the effects of disagreement amongst investors on both financial and real quantities. We also study a production economy with disagreement, but with Epstein and Zin (1989) preferences and under the restriction that either portfolio differences or belief differences cancel out. Moreover, our model allows for multiple trading dates, which makes it possible to study the impact of changes in the wealth distribution—a key driver of our results. Our paper is related also to other theoretical models where investors choose not to invest in all available assets. For example, Merton (1987) develops a static mean-variance model where each investor is aware only of a subset of the available securities. Cao, Han, Hirshleifer, and Zhang (2011) develop a model to explain how the portfolio holdings in Merton (1987) can arise endogenously when investors are averse to changes from the status quo. Garleânu, Panageas, and Yu (2014) use distance-dependent participation costs to generate differences in portfolio holdings across investors and study the implications for crashes and contagion in financial markets.

The rest of this paper is organized as follows. Section 2 describes the general model. In Section 3, we consider the setting where the portfolio errors of investors cancel out, but the implications of their errors do not cancel out. In Section 4, we consider the setting where the biased beliefs of investors cancel out, but the implications of their biases do not. We conclude in Section 5. Proofs for all our results are presented in the appendix.

[4] Mitchell and Utkus (2004) report that five million Americans have over sixty percent of their retirement savings invested in company stock and that about eleven million participants in 401(k) plans invest more than twenty percent of their retirement savings in their employer's stock.

2 The General Model

In this section, we describe a model of a production economy where investors have biased beliefs and recursive utility. In the next two sections, we then make particular parametric assumptions in order to characterize the equilibrium in this economy in closed form, which makes it easy to understand the economic intuition underlying the key results.

2.1 Firms and Wealth Dynamics of Investors

There are N firms indexed by $n \in \{1, \ldots, N\}$. Firms are modeled as in Cox, Ingersoll, and Ross (1985). The value of the capital stock in each of the N firms at date t is denoted by $K_{n,t}$ and the output flow by

$$Y_{n,t} = \alpha_n K_{n,t}, \tag{1}$$

for some constant technology level $\alpha_n > 0$. The level of a firm's capital stock can be increased by investing at the rate $I_{n,t}$. We thus have the following capital accumulation equation for an individual firm:

$$dK_{n,t} = I_{n,t}\, dt + K_{n,t}\, \sigma_n dZ_{n,t},$$

where $dZ_{n,t}$ is the increment in a standard Brownian motion, such that $E_t^{\mathbb{P}}[dZ_{n,t} dZ_{m,t}] = 0$ for $n \neq m$. Firm-level heterogeneity creates benefits from diversifying investments across firms. A firm's output flow is divided between its investment flow and dividend flow:

$$Y_{n,t} = I_{n,t} + D_{n,t}. \tag{2}$$

Using (1) and (2), we can therefore write the capital accumulation equation as

$$dK_{n,t} = \big(\alpha_n K_{n,t} - D_{n,t}\big) dt + K_{n,t}\, \sigma_n dZ_{n,t}. \tag{3}$$

In the Cox, Ingersoll, and Ross (1985) model, the return on a firm's physical capital, α_n, equals the return on its stock. Similarly, the volatility of the return on a firm's capital, σ_n, equals the volatility of the return on its stock. In addition to investing in these N risky firms, or equivalently, in the stocks of these firms, investors put $B_{h,t}$ in the risk-free asset at date t, which has an interest rate i_t.

Given that the investor's wealth, $W_{h,t}$, is held in either the risk-free asset or invested in a risky firm, we have that:

$$W_{h,t} = B_{h,t} + \sum_{n=1}^{N} K_{hn,t}.$$

Denoting the proportion of an investor's wealth invested in firm n by ω_{hn}, we have that the amount of investor h's wealth invested in firm n is $K_{hn,t} = \omega_{hn} W_{h,t}$ and the amount invested in the risk-free asset is $B_{h,t} = \big(1 - \sum_{h=1}^{N} \omega_{hn}\big) W_{h,t}$.

The dividends distributed by firm n are consumed by investor h:

$$C_{hn,t} = D_{hn,t} = \frac{K_{hn,t}}{K_{n,t}} D_{n,t},$$

where $C_{hn,t}$ is the consumption rate of investor h from the dividend flow of firm n. The resulting dynamic budget constraint for investor h is given by

$$\frac{dW_{h,t}}{W_{h,t}} = \Big(1 - \sum_{n=1}^{N} \omega_{hn,t}\Big)\, i_t dt + \sum_{n=1}^{N} \omega_{hn,t} \Big(\alpha_n dt + \sigma_n dZ_{n,t}\Big) - \frac{C_{h,t}}{W_{h,t}} dt,$$

where $C_{h,t} = \sum_{n=1}^{N} C_{hn,t}$ is the consumption rate of investor h.

2.2 Psychological Distance and Psychological Bias

Motivated by the empirical evidence in Bhamra, Uppal, and Walden (2021), we develop a framework in which an investor's biases about stocks are determined by the psychological distance between the investor and the stock. In contrast to Bhamra, Uppal, and Walden (2021), where investors can only be pessimistic about stocks (with the level of pessimism depending on the psychological distance between a firm and the investor), our model allows an investor to be optimistic about some firms and pessimistic about others.[5] As we show below, the level of optimism an investor has about a particular firm decreases with her psychological distance from that firm. Once psychological distance is beyond some threshold, optimism turns to pessimism.

We denote the psychological distance between an investor h and firm n by d_{hn}, which could represent geographical distance, temporal distance, and/or social distance. We now explain how the *psychological bias* of an investor h to firm n, denoted by b_{hn}, is related to psychological distance.

We assume that each investor has a trust region of length $2S$ centered on the investor, where S is a random variable that is identically and independently distributed across investors.

[5] There is a large literature in psychology that documents that individual are optimistically biased when making certain decisions. For instance, when assessing the likelihood of getting Covid or being involved in a car accident, individuals tend to overestimate the likelihood of favorable outcomes (Weinstein, 1980, Slovic, Fischhoff, and Lichtenstein, 1980, Slovic, 1987). These psychological biases are present also when entrepreneurs decide to launch new businesses (Baker and Wurgler, 2013) and chief executives pursue mergers (Malmendier and Tate, 2008). Similarly, Shiller (2015) describes how "wishful thinking on the part of investors that blinds us to the truth of our situation" leads to irrational exuberance in financial markets.

Investors know the mean of S, denoted by $E[S]$, but do not know S. When $d_{hn} < S$, the firm is within the investor's trust region and when $d_{hn} > S$, the firm is outside the trust region. The fraction of the trust region which is penetrated by the firm is $\frac{S-d_{hn}}{2S}$. We now define the level of trust investor h has in expected return measurements for firm n as being directly proportional to the mean of the positive part of this fraction.

Definition 2.1. *The* level of trust *an investor h has in measurements about expected returns for firm n is a random variable given by*

$$T_{hn} = \overline{T} \max \left(\frac{S - d_{hn}}{S}, 0 \right),$$

where $\overline{T} > 1$ is the maximum level of trust.

From the above definition, we can see that when $d_{hn} = 0$, then $T_{hn} = \overline{T}$. Also, when $d_{hn} > S$, we have $T_{hn} = 0$. Thus, the level of trust investor h has about expected returns for firm n is a random variable taking values between 0 and $\overline{T}$. Trust will have a mean value, which we use to define *psychological bias* as shown below.

Definition 2.2. *The* psychological bias *investor h has about firm n is given by*

$$b_{hn} = E[T_{hn}] - 1. \tag{4}$$

In order to connect explicitly the psychological bias of an investor about a firm to the psychological distance between them, we need to make an assumption about the probability distribution for S, the size of the trust region. We compute the expected level of trust, and hence, the psychological bias by updating the prior distribution for the trust region. A key element of this is choosing the prior. We choose the least-informative prior according to the Principle of Maximum Entropy (Jaynes, 1957, 1986). Choosing the least-informative prior is motivated by the desire for cautiousness. We update the prior using Bayes' Law, as in Shepard (1987). Using the resulting posterior, we compute expected trust to obtain the expression for psychological bias shown in the proposition below.

Proposition 2.1. *The* psychological bias *of investor h about firm n, in terms of the psychological distance between investor h and firm n, is*

$$b_{hn} = e^{-\kappa(d_{hn} - \overline{d})} - 1 \in (-1, \overline{T} - 1], \tag{5}$$

where the decay factor *κ is given by $\kappa = \frac{2}{E[S]}$ and $\overline{d} = \frac{1}{\kappa} \ln \overline{T}$.*

Observe that b_{hn} decays exponentially, where the *decay factor* κ is inversely proportional to the mean value of the size of the investor's trust region, $E[S]$. At $d_{hn} = 0$, an investor's psychological bias about a firm is at its maximum positive value, $e^{\kappa\bar{d}} - 1 = \overline{T} - 1 > 0$, indicating optimism. As d_{hn} increases, b_{hn} falls, reducing the level of optimism until the rational-expectations threshold, $d_{hn} = \bar{d}$, is reached so that investor h has rational beliefs, in which case $b_{hn} = 0$. Beyond the rational-expectations threshold, $b_{hn} < 0$, indicating a pessimistic bias. The rational-expectations threshold psychological distance at which optimism turns to pessimism is increasing in the mean size of the trust region and the maximum level of trust.

2.3 Biased Beliefs, Martingales, and Relative Entropy

While under the rational-expectations measure $\mathbb{P}$ the expected rate of return on firm n's stock is α_n, investor h believes it to be $\alpha_n + \nu_{hn,t}$. We can summarize how investor h's beliefs about expected returns on the N stocks deviate from $\mathbb{P}$ by the vector $\boldsymbol{\nu}_{h,t} = (\nu_{h1,t}, \ldots, \nu_{hN,t})^\top$. We denote investor h's subjective beliefs by the probability measure $\mathbb{P}^h$. Before describing how the belief $\mathbb{P}^h$ is formed, we describe how one can use an exponential martingale to distort the rational beliefs $\mathbb{P}$ to obtain $\mathbb{P}^h$ and to measure the distortion of $\mathbb{P}^h$ relative to $\mathbb{P}$.

For expositional ease, we start by considering the special case where the expected return on a single firm n is distorted, i.e. $\nu_{hn,t} \neq 0$, but $\nu_{hn',t} = 0$ for all $n' \neq n$.[6] We denote the resulting probability measure by $\mathbb{P}^{hn}$.[7] We denote the exponential martingale which distorts $\mathbb{P}$ into $\mathbb{P}^{hn}$ via M_{hn}, where

$$M_{hn,t} = \exp\left(-\frac{1}{2}\int_0^t \left(\frac{\nu_{hn,u}}{\sigma_n}\right)^2 du + \int_0^t \frac{\nu_{hn,u}}{\sigma_n} dZ_{n,u}\right).$$

We start by using the exponential martingale M_{hn} to define the expected returns on firm n over the interval $[t, t+dt)$ under the probability measure $\mathbb{P}^{hn}$, $E_t^{\mathbb{P}^{hn}}[dR_{n,t}]$, where

$$E_t^{\mathbb{P}^{hn}}[dR_{n,t}] = E_t\left[\frac{M_{hn,t+dt}}{M_{hn,t}} dR_{n,t}\right]. \tag{6}$$

To understand how the martingale distorts the rational-expectations probability measure $\mathbb{P}$, observe that $M_{hn,t+dt} = M_{hn,t} + dM_{hn,t}$ and so $M_{hn,t+dt}/M_{hn,t} = 1 + dM_{hn,t}/M_{hn,t}$. Therefore, (6) implies that

$$E_t^{\mathbb{P}^{hn}}[dR_{n,t}] = E_t[dR_{n,t}] + E_t\left[\frac{dM_{hn,t}}{M_{hn,t}} dR_{n,t}\right].$$

[6] We provide a heuristic explanation for why a distortion in a probability must be carried out via an exponential martingale in Appendix C.

[7] The probability measure $\mathbb{P}^h$ is obtained by combining the probability measures $\mathbb{P}^{h1}, \ldots, \mathbb{P}^{hN}$. Formally, the probability measure $\mathbb{P}^h$ is the product measure $\mathbb{P}^{h1} \times \ldots \times \mathbb{P}^{hN}$.

The above equation is useful, because we can see that the expected return under $\mathbb{P}$ is distorted by the amount $E_t\left[\frac{dM_{hn,t}}{M_{hn,t}}dR_{n,t}\right]$ to give the expected return under $\mathbb{P}^{hn}$. This is known as Girsanov's theorem.

We can measure the size of the distortion in the probabilities by using the concept of relative entropy.

Definition 2.3. *The conditional relative entropy per unit time from the personal belief $\mathbb{P}^{hn}$ to the objective belief $\mathbb{P}$ is*

$$D^{KL}[\mathbb{P}|\mathbb{P}^{hn}] = -\frac{1}{dt}E_t^{\mathbb{P}}\left[\ln\frac{M_{hn,t+dt}}{M_{hn,t}}\right],$$

which, using Ito's Lemma, can be rewritten as

$$= \frac{1}{2}\frac{1}{dt}E_t^{\mathbb{P}}\left[\left(\frac{dM_{hn,t}}{M_{hn,t}}\right)^2\right]$$

$$= \frac{1}{2}\left(\frac{\nu_{hn,t}}{\sigma_n}\right)^2.$$

The size of the distortion in beliefs represented by $\mathbb{P}^{hn}$ relative to $\mathbb{P}$ is given by $\frac{1}{2}\left(\frac{\nu_{hn,t}}{\sigma_n}\right)^2$. When beliefs about expected returns for all N stocks can change, we obtain the relative entropy from $\mathbb{P}^h$ to $\mathbb{P}$,

$$D^{KL}[\mathbb{P}|\mathbb{P}^{h}] = \frac{1}{2}\sum_{n=1}^{N}\left(\frac{\nu_{hn,t}}{\sigma_n}\right)^2.$$

The relative entropy, $D^{KL}[\mathbb{P}|\mathbb{P}^h]$, will play a key role in the way we generate biased beliefs endogenously.

2.4 Epstein-Zin Preferences with Biased Beliefs

An investor h's time-t utility level, $U_{h,t}$, is defined by an intertemporal aggregation of date-t consumption flow, $C_{h,t}$, and the time-t certainty-equivalent of date $t+dt$ utility:[8]

$$U_{h,t} = \mathcal{A}(C_{h,t}, \mu_{h,t}^{\mathbb{P}^h}[U_{h,t+dt}]),$$

[8]The only difference with Epstein and Zin (1989) is that we work in continuous time, whereas they work in discrete time. The continuous-time version of recursive preferences is known as stochastic differential utility (SDU), and is derived formally in Duffie and Epstein (1992). Schroder and Skiadas (1999) provide a proof of existence and uniqueness for the finite-horizon case.

where $\mathcal{A}(\cdot,\cdot)$ is the time aggregator, defined by

$$\mathcal{A}(x,y) = \left[(1-e^{-\delta dt})x^{1-\frac{1}{\psi}} + e^{-\delta dt}y^{1-\frac{1}{\psi}}\right]^{\frac{1}{1-\frac{1}{\psi}}},$$

in which $\delta > 0$ is the rate of time preference, $\psi > 0$ is the elasticity of intertemporal substitution. The expression $\mu_{h,t}^{\mathbb{P}^h}[U_{h,t+dt}]$ is the certainty-equivalent operator of utility at time $t+dt$ under biased beliefs and is defined below.

Definition 2.4. *The date-t (biased) certainty equivalent of date-$t+dt$ utility is given by*

$$\mu_{h,t}^{\mathbb{P}^h}[U_{h,t+dt}] = \widehat{\mu}_{h,t}^{\mathbb{P}^h}[U_{h,t+dt}] + U_{h,t}L_{h,t}dt, \tag{7}$$

where the certainty equivalent $\widehat{\mu}_{h,t}^{\mathbb{P}^h}[U_{h,t+dt}]$ is defined by

$$u_{\gamma_h}\left(\widehat{\mu}_{h,t}^{\mathbb{P}^h}[U_{h,t+dt}]\right) = E_t^{\mathbb{P}^h}[u_{\gamma_h}(U_{h,t+dt})], \tag{8}$$

γ_h is investor h's relative risk aversion, and

$$L_{h,t}[\mathbb{P}|\mathbb{P}^h] = -\frac{1}{\gamma_h}\sum_{n=1}^{N}\frac{1+b_{hn}}{b_{hn}}D^{KL}[\mathbb{P}|\mathbb{P}^{hn}] = -\frac{1}{\gamma_h}\sum_{n=1}^{N}\frac{1}{1-e^{\kappa(d_{hn}-\bar{d})}}D^{KL}[\mathbb{P}|\mathbb{P}^{hn}]. \tag{9}$$

The definition of the certainty equivalent in (7) consists of two new elements. First, in (8) we see the standard definition of a certainty equivalent but where the expectation is taken under the investor's personal beliefs $\mathbb{P}^h$. Second, (7) includes a penalty for deviating from the rational-expectations belief $\mathbb{P}$, which is given in (9).

The following proposition makes the intuition underlying Definition 2.4 more transparent.

Proposition 2.2. *The date-t biased certainty equivalent of date-$t+dt$ utility is equivalent to*

$$\mu_{h,t}^{\mathbb{P}^h}[U_{h,t+dt}] = E_t^{\mathbb{P}^h}[U_{h,t+dt}] - \frac{1}{2}\gamma_h U_{h,t}Var_t\left[\frac{dU_{h,t}}{U_{h,t}}\right] + U_{h,t}L_{h,t}[\mathbb{P}|\mathbb{P}^h]. \tag{10}$$

We can now see that the investor's beliefs enter naturally into the certainty equivalent via the term $E_t^{\mathbb{P}^h}[U_{h,t+dt}]$. There is also, as is standard, a penalty for risk, given by $\frac{1}{2}\gamma_h U_{h,t}Var_t\left[\frac{dU_{h,t}}{U_{h,t}}\right]$. The third term, which is a novel, reflects the penalty for deviating from the rational belief $\mathbb{P}$. It is important to note that the penalty $L_{h,t}[\mathbb{P}|\mathbb{P}^h]$ contains a weighted-sum of Kullback-Leibler divergences, where the weight $\frac{1}{1-e^{\kappa(d_{hn}-\bar{d})}}$ is positive if and only if $d_{hn} < \bar{d}$. The change in sign of the weights occurs in just the right way, making it possible to obtain endogenous optimism and pessimism.

Investor h aims to maximize her time-t utility by optimally choosing her beliefs, portfolio weights, and consumption according to the following Bellman equation

$$U_{h,t} = \sup_{C_{h,t}} \mathcal{A}\Big(C_{h,t}, \sup_{\boldsymbol{\omega}_{h,t}} \inf_{\{\mathbb{P}^{hn}\}_{n\in A_P}} \sup_{\{\mathbb{P}^{hn}\}_{n\in A_O}} \mu^{\mathbb{P}^h}_{h,t}[U_{h,t+dt}]\Big), \tag{11}$$

where $U_{h,t}$ is now optimized time t utility, A_O is the set of firms about which the investor is optimistic, i.e. $d_{hn} < \overline{d}$, and A_P is the set of firms about which the investor is pessimistic, i.e. $d_{hn} > \overline{d}$. Recall that the investor's subjective beliefs are represented by the probability measure $\mathbb{P}^h$, which be written as the product measure $\mathbb{P}^h = \mathbb{P}^{h1} \times \cdots \times \mathbb{P}^{hN}$, as in Section 2.3.

Proposition 2.3. *The investor's consumption-portfolio choice problem in* (11) *can be rewritten as the following Hamilton-Jacobi-Bellman equation:*

$$0 = \sup_{C_{h,t}} \delta_h\, u_{\psi_h}\left(\frac{C_{h,t}}{U_{h,t}}\right) + \sup_{\boldsymbol{\omega}_{h,t}} \inf_{\{\mathbb{P}^{hn}\}_{n\in A_P}} \sup_{\{\mathbb{P}^{hn}\}_{n\in A_O}} \frac{1}{U_{h,t}} \mu^{\mathbb{P}^h}_{h,t}\left[\frac{dU_{h,t}}{dt}\right], \tag{12}$$

where $u_{\psi_h}(x) = \frac{x^{1-\frac{1}{\psi_h}}}{1-\frac{1}{\psi_h}}$.

In general, for the economy described above, the choices of individual investors and the resulting equilibrium cannot be characterized in closed form. The main challenge in obtaining a closed-form solution arises because investors demand to hedge against a stochastic investment opportunity set, which in our model is a consequence of the interest rate being stochastic. There are two conditions under which investors would not have a demand for intertemporal hedging, and hence, it would be possible to solve the model in closed form. One, if the interest rate were constant, and two, if investors were to have unitary risk aversion.[9]

We provide two examples below, in Sections 3 and 4, that correspond to these two conditions. In each example, we characterize the equilibrium in closed form and show that, even when the biases are specified in such a way that they cancel out, they still impact asset prices. We provide two complementary definitions of what it means for biases to "cancel out." Our first definition, given in Section 3, is framed in terms of portfolio errors relative to the case where all investors are rational. The second definition, given in Section 4, is defined directly in terms of the biases themselves. The first example shows that even if the portfolio biases of investors cancel out, there is still an impact on asset prices and macroeconomic variables. The second example shows that even if the biases themselves cancel out, they still distort asset prices.

[9]If the interest rate were constant, then the investment opportunity set would be nonstochastic and so there would be nothing to hedge. And, if investors were to have unitary risk aversion, they would be myopic and have no desire to hedge against future changes investment opportunities—their intertemporal hedging demand would be zero.

3 Example 1: Economy where Portfolio Biases Cancel Out

In this section we show that, even if behavioral biases lead to "errors" in the portfolios of individual investors that cancel out in the aggregate, the biases still impact asset prices and macroeconomic variables.

Our focus is on the impact of behavioral biases, not heterogeneous preferences, so we assume the N investors have the same time discount rate δ, the same elasticity of intertemporal substitution ψ, and identical relative risk aversion γ.

In order to characterize the equilibrium in closed form, we make the following assumption.

Assumption 3.1. *In the economy considered in this example, we assume expected returns and return volatilities are the same across risky assets; that is, $\alpha_n = \alpha$ and $\sigma_n = \sigma$. We also assume that the mean bias is the same across all investors; that is, $\mu_{b,h} = \mu_b$, where*

$$\mu_{b,h} = \frac{1}{N}\sum_{n=1}^{N} b_{hn}.$$

The assumption that the moments of asset returns are identical across assets implies that the optimal (unbiased) portfolio is one that invests $1/N$ in each of the risky assets. This allows us to define precisely the "canceling out" of portfolio errors when these errors are aggregated across investors.

Definition 3.1. *Suppose investor h's optimal portfolio weight in equilibrium for firm n is*

$$\omega_{hn} = \frac{1}{N} + \epsilon_{hn},$$

where $\frac{1}{N}$ is the unbiased portfolio weight and ϵ_{hn} is the 'error' of investor h's portfolio when investing in firm n. The portfolio errors ϵ_{hn} "cancel out across investors" if

$$\forall n,\ \frac{1}{H}\sum_{h=1}^{H} \epsilon_{hn} = 0. \tag{13}$$

It is important to understand that the above definition is not just market clearing in disguise, because it pertains to the portfolio errors for a single stock n.

3.1 Portfolio and Consumption Choices of Individual Investors

We assume for now, and confirm in Proposition 3.4 below, that Assumption 3.1 leads to an equilibrium where the interest rate is constant. In this case, the Hamilton-Jacobi-Bellman equation can be decomposed into two parts: *a single-period linear-quadratic* optimization problem for the joint determination of beliefs and portfolios and an intertemporal consumption choice problem.

In the following proposition, we see that our model provides a tractable formulation of both endogenous optimism and pessimism—existing work that endogenizes beliefs, such as Brunnermeier and Parker (2005), Hansen and Sargent (2007), and Bhamra, Uppal, and Walden (2021), cannot generate both optimism and pessimism.

Proposition 3.1. *The investor's optimization problem consists of two parts, a linear-quadratic joint beliefs-portfolio optimization problem*

$$\max_{\boldsymbol{\omega}_{h,t}} \min_{\{\mathbb{P}^{hn}\}_{n\in A_P}} \max_{\{\mathbb{P}^{hn}\}_{n\in A_O}} LQ_h(\boldsymbol{\omega}_h, \boldsymbol{\nu}_h), \tag{14}$$

and an intertemporal consumption choice problem

$$0 = \sup_{C_h} \left(\delta\, u_\psi \left(\frac{C_h}{U_h} \right) - \frac{C_h}{W_h} + \max_{\boldsymbol{\omega}_{h,t}} \min_{\{\mathbb{P}^{hn}\}_{n\in A_P}} \max_{\{\mathbb{P}^{hn}\}_{n\in A_O}} LQ_h(\boldsymbol{\omega}_h, \boldsymbol{\nu}_h) \right), \tag{15}$$

where

$$LQ_h(\boldsymbol{\omega}_h, \boldsymbol{\nu}_h) = E_t^{\mathbb{P}^h}[dR_{h,t}] - \frac{1}{2}\gamma Var_t[dR_{h,t}] + L_{h,t}[\mathbb{P}|\mathbb{P}^h], \tag{16}$$

where A_O is the set of firms about which the investor is optimistic ($d_{hn} < \overline{d}$), A_P the set about which the investor is pessimistic ($d_{hn} \geq \overline{d}$), and the penalty $L_{h,t}[\mathbb{P}|\mathbb{P}^h]$ is defined in (9).

The expression in (16) is the standard definition of mean-variance utility, but with two changes. One, the expectation is taken under the investor's personal beliefs $\mathbb{P}^h$. Two, there is a penalty for deviating from the rational-expectations belief $\mathbb{P}$, which is given in (9). In the above optimization problem, optimal portfolios and beliefs about expected returns are determined simultaneously via an intrapersonal game. Belief selection is the psychological aspect of the game, whereby beliefs are determined for a given portfolio. Portfolio choice is the rational aspect of the game, in which an optimal portfolio is selected based on beliefs.

To understand this intrapersonal game between the psychological and rational selves of an individual, we first focus on the psychological aspect, i.e.

$$\min_{\{\mathbb{P}^{hn}\}_{n\in A_P}} \max_{\{\mathbb{P}^{hn}\}_{n\in A_O}} \sum_{n=1}^{N} (E_t^{\mathbb{P}^{hn}} - E_t^{\mathbb{P}}) \left[\frac{dR_{h,t}}{dt} \right] - \frac{1}{\gamma} \sum_{n=1}^{N} \frac{1}{1 - e^{\kappa(d_{hn} - \overline{d})}} D^{KL}[\mathbb{P}|\mathbb{P}^{hn}],$$

which we can separate into the selection of the optimistic beliefs

$$\max_{\{\mathbb{P}^{hn}\}_{n\in A_O}} \sum_{n\in A_O} (E_t^{\mathbb{P}^{hn}} - E_t^{\mathbb{P}}) \left[\frac{dR_{h,t}}{dt} \right] - \frac{1}{\gamma} \sum_{n\in A_O} \frac{1}{1 - e^{\kappa(d_{hn} - \overline{d})}} D^{KL}[\mathbb{P}|\mathbb{P}^{hn}],$$

and the pessimistic beliefs

$$\min_{\{\mathbb{P}^{hn}\}_{n\in A_P}} \quad \sum_{n\in A_P} (E_t^{\mathbb{P}^{hn}} - E_t^{\mathbb{P}}) \left[\frac{dR_{h,t}}{dt}\right] - \frac{1}{\gamma} \sum_{n\in A_P} \frac{1}{1-e^{\kappa(d_{hn}-\overline{d})}} D^{KL}[\mathbb{P}|\mathbb{P}^{hn}].$$

Observe that the separation of beliefs into optimistic and pessimistic depends on psychological distance: investor h has optimistic beliefs about firm h if $d_{hn} < \overline{d}$ and pessimistic beliefs if $d_{hn} > \overline{d}$. The case of $d_{hn} = \overline{d}$ corresponds to rational beliefs about firm n.

The optimistic beliefs are selected by trading off higher perceived expected portfolio returns, achieved by making the term $E_t^{\mathbb{P}^h}[dR_{h,t}/dt] - E_t^{\mathbb{P}}[dR_{h,t}/dt] = \sum_{n=1}^N \nu_{hn,t}\omega_{hn,t}$ more positive, against a penalty for deviating from the rational-expectations belief. The penalty is a weighted sum of the relative entropies of the probability measures $\mathbb{P}^{h1}, \ldots, \mathbb{P}^{hN}$, which together form $\mathbb{P}^h$. The weights, $\frac{1}{1-e^{\kappa(d_{hn}-\overline{d})}}$, are positive because $d_{hn} < \overline{d}$ and become smaller as the psychological distance d_{hn} decreases. Therefore, an optimistic deviation in beliefs from $\mathbb{P}$ is penalized less severely when the psychological distance between the investor and firm is closer to zero—this generates increasingly optimistic beliefs for $d_{hn} < \overline{d}$ as d_{hn} gets closer to zero. Beliefs are most optimistic when $d_{hn} = 0$.

The pessimistic beliefs are selected by trading off higher perceived expected portfolios returns, acheived by making the term $E_t^{\mathbb{P}^h}[dR_{h,t}/dt] - E_t^{\mathbb{P}}[dR_{h,t}/dt] = \sum_{n=1}^N \nu_{hn,t}\omega_{hn,t}$ more negative, against a penalty for deviating from the rational-expectations belief. In contrast to the case of optimistic beliefs, the weights in the penalty function are now negative (because $d_{hn} > \overline{d}$) and become smaller in magnitude as the psychological distance d_{hn} increases. Therefore a pessimistic deviation in beliefs from $\mathbb{P}$ is penalized less severely when the psychological distance between the investor and firm is larger.

The solution to the optimal belief problem for a given portfolio $\boldsymbol{\omega}_{h,t}$ is

$$\nu_{hn,t} = \gamma_h \frac{b_{hn}}{1+b_{hn}} \omega_{hn,t}\sigma^2 = \gamma_h(1-e^{\kappa(d_{hn}-\overline{d})})\omega_{hn,t}\sigma^2. \tag{17}$$

We see that for a positive portfolio weight ($\omega_{hn,t} > 0$), the distorted expected return on a firm is increased when the investor has a positive bias toward a firm ($b_{hn} > 0$, because $d_{hn} < \overline{d}$) and decreased when an investor has a negative bias towards a firm, ($b_{hn} < 0$, because $d_{hn} > \overline{d}$).

Substituting (17) into (16) and simplifying gives

$$\begin{aligned}
&\max_{\boldsymbol{\omega}_{h,t}} \quad E_t^{\mathbb{P}^h}[dR_{h,t}] - \frac{1}{2}\gamma Var_t[dR_{h,t}] - \frac{1}{2}\gamma\sigma^2 \sum_{n=1}^N \frac{b_{hn}}{1+b_{hn}} \omega_{hn,t}^2 \\
&= \max_{\boldsymbol{\omega}_{h,t}} \quad i + \sum_{n=1}^N (\alpha - i)\omega_{hn,t} - \frac{1}{2}\gamma\sigma^2 \sum_{n=1}^N \frac{1}{1+b_{hn}} \omega_{hn,t}^2.
\end{aligned} \tag{18}$$

We can now see that a positive bias reduces the penalty for risk, while a negative bias increases it. Thus, the investor chooses the following beliefs and portfolio.

Proposition 3.2. *The investor's deviation in beliefs is*

$$\nu_{hn} = (\alpha - i)b_{hn} = (\alpha - i)\left(e^{-\kappa(d_{hn}-\bar{d})} - 1\right), \tag{19}$$

implying that investor h's beliefs about the expected risk premium for firm n are

$$\alpha - i + \nu_{hn} = (\alpha - i)(1 + b_{hn}) = (\alpha - i)e^{-\kappa(d_{hn}-\bar{d})},$$

and the investor's portfolio choice is

$$\omega_{hn} = \frac{1}{\gamma}\frac{\alpha - i + \nu_{hn}}{\sigma^2} = \frac{1}{\gamma}\frac{\alpha - i}{\sigma^2}(1 + b_{hn}) = \frac{1}{\gamma}\frac{\alpha - i}{\sigma^2}e^{-\kappa(d_{hn}-\bar{d})}. \tag{20}$$

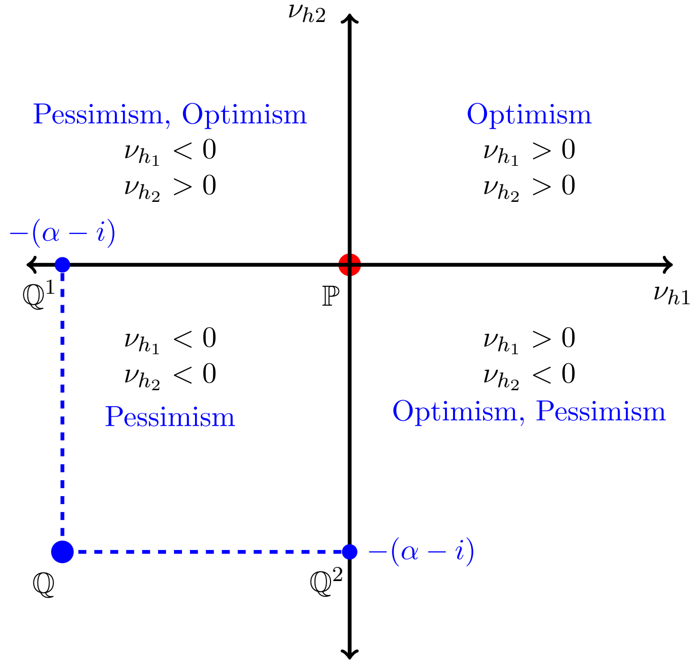
From (19), we see that an investor h has optimistic beliefs about firm n's expected risk premium when $d_{hn} < \bar{d}$, rational beliefs when $d_{hn} = \bar{d}$, and pessimistic beliefs when $d_{hn} > \bar{d}$. The expression for the portfolio weights in (20) is the standard Merton portfolio weight but using investor h's personal subjective expectation of stock n's return. We see from this expression that if an investor's psychological distance from a firm is less than the threshold, $\bar{d}$, she has a positive bias towards a firm, and so she overweights it in her portfolio; and if her psychological distance from a firm exceeds the threshold $\bar{d}$, she has a negative bias, so she underweights it.

In Figure 1, we show geometrically how biases impact beliefs. To fix ideas, we set the number of firms to two, $N = 2$. In this case the rational-expectations belief $\mathbb{P}$ is summarized by the vector $\boldsymbol{\nu}_h = (\nu_{h1}, \nu_{h2})^\top = (0, 0)^\top$, placing it at the origin in (ν_{h1}, ν_{h2})-space. For reference, we also show the location of beliefs represented by the risk-neutral measure, $\mathbb{Q}$, summarized by the vector $\boldsymbol{\nu}_h = (-(\alpha - i), -(\alpha - i))^\top$, which lies in the third quadrant and its constituent measures $\mathbb{Q}^1$ and $\mathbb{Q}^2$, represented by $\boldsymbol{\nu}_h = (-(\alpha - i), 0)^\top$ and $\boldsymbol{\nu}_h = (0, -(\alpha - i))^\top$, respectively.

Figure 1 also shows how optimism and pessimism can be visualized in the space of beliefs. The belief $\mathbb{P}^h$ is summarized by the vector $\boldsymbol{\nu}_h = ((\alpha - i)b_{h1}, (\alpha - i)b_{h2})^\top$, which lies in the first quadrant if the investor has positive bias with respect to both firms ($b_{h1} > 0$ and $b_{h2} > 0$), the second quadrant if $b_{h1} < 0$ and $b_{h2} > 0$, the third quadrant if $b_{h1} < 0$ and $b_{h2} < 0$, and the fourth quadrant if $b_{h1} > 0$ and $b_{h2} < 0$. That is, optimistic beliefs lie in the first quadrant and pessimistic beliefs in the third quadrant. Beliefs which are pessimistic about firm 1, but optimistic about firm 2 lie in the second quadrant and beliefs which are optimistic about firm 1, but pessimistic about firm 2 lie in the fourth quadrant.

Figure 1: The beliefs of an individual investor

This figure shows optimism and pessimism in the space of beliefs. The rational-expectations belief $\mathbb{P}$ is summarized by the vector $\boldsymbol{\nu}_h = (\nu_{h1}, \nu_{h2})^\top = (0,0)^\top$. The location of beliefs represented by the risk-neutral measure, $\mathbb{Q}$, summarized by the vector $\boldsymbol{\nu}_h = (-(\alpha - i), -(\alpha - i))^\top$, lies in the third quadrant. The belief $\mathbb{P}^h$ is summarized by the vector $\boldsymbol{\nu}_h = ((\alpha - i)b_{h1}, (\alpha - i)b_{h2})^\top$, which lies in the first quadrant if the investor has positive bias with respect to both firms ($b_{h1} > 0$ and $b_{h2} > 0$), the second quadrant if $b_{h1} < 0$ and $b_{h2} > 0$, the third quadrant if $b_{h1} < 0$ and $b_{h2} < 0$, and the fourth quadrant if $b_{h1} > 0$ and $b_{h2} < 0$.



When the investment opportunity set is constant, the individual's optimal consumption is given by the following proposition.

Proposition 3.3. *The optimal consumption-wealth ratio is identical across investors and given by*

$$c = \frac{C_{h,t}}{W_{h,t}} = \psi\delta + (1-\psi)\left(i + \frac{1}{2\gamma}\left(\frac{\alpha - i}{\sigma_{1/N}}\right)^2 (1+\mu_b)\right), \tag{21}$$

where $\sigma_{1/N} = \sigma/\sqrt{N}$ is the volatility of the equally weighted portfolio.

Optimism and pessimism impact the consumption to wealth ratio in opposing directions. When the income effect dominates (i.e., $\psi > 1$), optimism about a stock, that is, greater return expectations leads to increased savings and decreased consumption. The opposite is true when the substitution effect dominates (i.e., $\psi < 1$).

3.2 Equilibrium Risk-free Interest Rate

By imposing market clearing in the risk-free bond market, we obtain the equilibrium risk-free interest rate.

Proposition 3.4. *If Assumption 3.1 holds, then the equilibrium risk-free interest rate is given by the constant*

$$i = i_{NB} + \gamma \frac{\mu_b}{1+\mu_b}\,\sigma^2_{1/N}, \tag{22}$$

where i_{NB} is the equilibrium risk-free interest rate when there are no biases, i.e.

$$i_{NB} = \alpha - \gamma\,\sigma^2_{1/N}.$$

We can see that investors' biases impact the equilibrium interest rate, even when their portfolio errors cancel out. To see the intuition, suppose for the sake of simplicity that we have two investors and two firms, whose biases are symmetric in the following sense: Investor 1 is unbiased with respect to Firm 1, but biased with respect to Firm 2, with a level of bias equal to b; Investor 2 is unbiased with respect to Firm 2, but biased with respect to Firm 1, with a level of bias equal to b. The mean bias of each investor is the same and equal to $\mu_b = b/2$, thereby satisfying Assumption 3.1. If the mean bias, $\mu_b = b/2$, is positive, then there is greater demand for risky assets relative to the case where all investors are fully rational. Hence, there is reduced demand for the bond, which increases the equilibrium risk-free interest rate. If the mean bias is negative, the opposite is true and the risk-free interest rate is lower.[10]

Rearranging Equation (22), we see that both the aggregate stock market equity premium and the expected return on each stock in excess of the risk-free return, are given by:

$$\alpha - i = \gamma\,\sigma^2_{1/N} - \gamma \frac{\mu_b}{1+\mu_b}\,\sigma^2_{1/N}. \tag{23}$$

From the right-hand side of the above expression, we see in the case where all investors are fully rational ($\mu_b = 0$), the expected excess return is given by $\gamma\,\sigma^2_{1/N}$. When $\mu_b > 0$, optimism increases overall demand for risky assets, thereby depressing the equilibrium excess return, while pessimism ($\mu_b < 0$) has the opposite effect.[11]

[10] A simple back-of-the-envelope calculation based on (22) allows us to see that even for small values of μ_b, the distortion to the equilibrium interest rate is significant. For example, with $\mu_b = \pm 0.05$, $\gamma = 5$, and $\sigma_{1/N} = 20$ percent per annum, the distortion is equal to approximately 1 percent per annum.

[11] Substituting the expression for the equilibrium interest rate in (22) into the expression for the optimal portfolio weights in (20) leads to the optimal portfolio in equilibrium, $\omega_{hn} = \frac{1}{N} + \epsilon_{hn}$, where $\epsilon_{hn} = \frac{1}{N}\frac{b_{hn}-\mu_b}{1+\mu_b}$. Aggregating the expression for ϵ_{hn} across investors, as in (13), confirms that the portfolio errors , ϵ_{hn}, "cancel out across investors" if $\frac{1}{H}\sum_{h=1}^{H} b_{hn} = \mu_b$.

3.3 Aggregate Investment and Growth

Above, we have examined the effect of the belief biases on the risk-free interest rate and the equity risk premium. We now study how belief biases impact aggregate investment and growth.

Proposition 3.5. *The aggregate growth rate of the economy is the aggregate investment-capital ratio,*

$$g = \frac{I_t^{agg}}{K_t^{agg}} = \psi(\alpha - \delta) + (1 - \psi)\frac{1}{2}\left(\frac{\gamma}{1 + \mu_b}\right)\sigma_{1/N}^2. \tag{24}$$

An increase in an individual investor's optimism, that is μ_b becoming more positive, decreases $\frac{\gamma}{1+\mu_b}\sigma_{1/N}^2$. When the substitution effect dominates ($\psi > 1$), a more optimistic investor will consume less and invest more, causing the aggregate investment-capital ratio to rise. Naturally, an increase in the aggregate investment-capital ratio boosts output growth.

3.4 Equilibrium Price of the Aggregate Stock Market

We denote by p_t^{agg} the price-dividend ratio of the aggregate capital stock, or equivalently, the aggregate wealth-consumption ratio:

$$p_t^{\text{agg}} = \frac{K_t^{\text{agg}}}{C_t^{\text{agg}}} = \frac{W_t^{\text{agg}}}{C_t^{\text{agg}}}.$$

The following proposition gives that the aggregate price-dividend ratio, first in terms of the endogenous expected growth rate of aggregate output, g, and then in terms of exogenous variables.

Proposition 3.6. *The aggregate price-dividend ratio is*

$$p_t^{agg} = \frac{1}{i + \frac{\gamma}{1+\mu_b}\sigma_{1/N}^2 - g} \tag{25}$$

$$= \frac{1}{\psi\delta + (1 - \psi)\left(\alpha - \frac{1}{2}\frac{\gamma}{1+\mu_b}\sigma_{1/N}^2\right)}. \tag{26}$$

where i is the risk-free interest rate given in (22), g is the endogenous expected growth rate of aggregate output in (24), α and $\sigma_{1/N}$ are the expected return and volatility of the equally-weighted portfolio return, γ is the risk aversion of investors, and μ_b is their mean level of bias.

Observe that (25) is the well-known Gordon-growth formula for a perpetuity with constant growth rate g, but where the discount rate and the growth rate are linked explicitly to the behavioral economics of time and risk. The discount rate is seen to be the sum of the constant risk-free rate and the risk premium, $\frac{\gamma}{1+\mu_b}\sigma^2_{1/N}$.

In (26) where the aggregate price-dividend ratio is expressed purely in terms of exogenous variables, the term $\alpha - \frac{1}{2}\frac{\gamma}{1+\mu_b}\sigma^2_{1/N}$ can be interpreted as the expected return on the aggregate stock market *adjusted for risk and belief biases.* Thus, the denominator is a weighted sum of the rate of time preference and the adjusted expected return, with the weights depending on the elasticity of intertemporal substitution, ψ. A decrease in the mean level of bias, μ_b, reduces the equilibrium expected return adjusted for risk and biases, $\alpha - \frac{1}{2}\frac{\gamma}{1+\mu_b}\sigma^2_{1/N}$. The effect of this on the aggregate price-dividend ratio will depend on whether ψ is greater or less than unity, which determines whether the substitution or income effect dominates.

3.5 Stochastic Discount Factors

In this section, we see that each investor has her own stochastic discount factor, which implies that each investor prices assets using her own individual factor model. The following proposition gives the stochastic discount factor for investor h.

Proposition 3.7. *Investor h's stochastic discount factor is given by*

$$\frac{d\Lambda_{h,t}}{\Lambda_{h,t}} = -i dt - \gamma\frac{1}{N}\sigma\left[\sum_{n=1}^{N} dZ^h_{n,t} + \frac{1}{1+\mu_b}\sum_{n=1}^{N}(b_{hn}-\mu_b)dZ^h_{n,t}\right],$$

where Z^h_n, $n \in \{1,\ldots,N\}$ are standard Brownian motions under $\mathbb{P}^h$ such that $E^{\mathbb{P}^h}_t[dZ^h_{n,t}dZ^h_{n',t}] = 0$ when $n \neq n'$. The volatility of investor h's stochastic discount factor is given by

$$\gamma\,\sigma_{1/N}\sqrt{1+\left(\frac{\sigma_{b,h}}{1+\mu_b}\right)^2},$$

where $\sigma_{b,h}$ is the standard deviation of her biases

$$\sigma_{b,h} = \sqrt{\frac{1}{N}\sum_{n=1}^{N}(b_{hn}-\mu_b)^2}.$$

Individuals agree on the equilibrium risk-free rate. However, heterogeneity in their biases for different firms drives individuals to perceive firm-level risk differently. The key determinant of investor-specific risk prices for a given firm is an investor's level of bias with respect to a firm, b_{hn} *relative* to the mean bias, μ_b, as we can see from the term $b_{hn}-\mu_b$ inside the expression for investor's h risk price for firm n: $\gamma\frac{1}{N}\sigma\left(1+\frac{b_{hn}-\mu_b}{1+\mu_b}\right)$. When an investor is relatively more optimistic about a firm than the mean investor, i.e. $b_{hn} - \mu_b$, her perceived risk price is distorted upwards relative to the case of no bias. The volatility of investor h 's stochastic discount factor is given by γ multiplied by the volatility of her portfolio, which is given by $\sigma_{1/N}\sqrt{1+\left(\frac{\sigma_{b,h}}{1+\mu_b}\right)^2}$.

The following proposition gives the equilibrium stochastic discount factor.

Proposition 3.8. *The equilibrium stochastic discount factor is given by* Λ*, where*

$$\frac{d\Lambda_t}{\Lambda_t} = -idt - \gamma\frac{1}{1+\mu_b}\frac{\sigma}{N}\sum_{n=1}^{N} dZ_{n,t}. \tag{27}$$

The volatility of the equilibrium stochastic discount factor is given by

$$\gamma\frac{1}{1+\mu_b}\sigma_{1/N}.$$

This proposition shows that the equilibrium price of risk is distorted: relative to the case where all investors are fully rational, the price of risk for firm n is $\gamma\frac{1}{1+\mu_b}\frac{\sigma}{N}$ instead of just $\gamma\frac{\sigma}{N}$. When the mean bias μ_b is positive, i.e. there is optimism, the risk price for each firm is reduced. For the case of pessimism, the opposite is the case.

4 Example 2: Economy where Biases Themselves Cancel Out

In the previous section, we showed that even if the portfolio errors resulting from behavioral biases canceled out when aggregated across investors, the biases would still impact asset prices. One might object to this on the grounds that even though *portfolio errors* cancel out, the mean bias across firms for each individual investor, $\mu_{b,h} = \frac{1}{N}\sum_{n=1}^{N} b_{hn}$, is not zero. In this section, we address this potential objection by showing that even if the behavioral biases themselves cancel out in the aggregate, they still impact asset prices.

A formal definition of what exactly we mean by behavioral biases "canceling out" across both firms *and* investors is given below.

Definition 4.1. *The psychological biases cancel out across investors if*

$$\forall n \in \{1\ldots,N\}, \quad \sum_{h=1}^{H} b_{hn} = 0.$$

The psychological biases cancel out across firms if

$$\forall h \in \{1\ldots,H\}, \quad \sum_{n=1}^{N} b_{hn} = 0.$$

In order to characterize the equilibrium in closed form for the version of the economy considered in this section, we make the following assumption, which ensures Definition 4.1 is satisfied.

Assumption 4.1. *In the economy considered in this section, asset returns are distinct across the N firms, but $H = N$ and $\gamma_h = 1$ for all investors h, with investor-firm psychological biases given by*

$$b_{hn} = \begin{cases} b, & n = h \\ -\frac{b}{N-1}, & n \neq h \end{cases},$$

where $b > 0$, which via (5) is equivalent to the following investor-firm psychological distances

$$d_{hn} = \begin{cases} \overline{d} - \frac{1}{\kappa}\ln(1+b) < \overline{d}, & n = h \\ \overline{d} - \frac{1}{\kappa}\ln\left(1 - \frac{b}{N-1}\right) > \overline{d}, & n \neq h \end{cases}.$$

The above definition implies that investor h is optimistic about firm h and pessimistic about all other firms, because $d_{hh} < \overline{d}$ and $d_{hn} > \overline{d}$ for $n \neq h$.

We can define the psychological bias of all investors towards firm n as the wealth-weighted average of each investor's biases towards firm n,

$$\hat{b}_{n,t} = \sum_{h=1}^{H} w_{h,t} b_{hn,t},$$

where $w_{h,t} = \frac{W_{h,t}}{\sum_{i=1}^{H} W_{i,t}}$ is the fraction of aggregate wealth held by investor h. Under Assumption 4.1, $\hat{b}_{n,t}$ reduces to

$$\hat{b}_{n,t} = b\left(w_{n,t} - \frac{1}{N-1}\sum_{i\neq n} w_{i,t}\right) = b\left(w_{n,t} - \frac{1}{N-1}(1-w_{n,t})\right) = b\frac{N}{N-1}\left(w_{n,t} - \frac{1}{N}\right). \tag{28}$$

If we define the psychological distance of firm n from the representative investor by $d_{n,t} = -\frac{1}{\kappa}\ln\sum_{h=1}^{H} w_{h,t}e^{-\kappa d_{hn}}$, to ensure that

$$\hat{b}_{n,t} = e^{-\kappa(d_{n,t}-\overline{d})} - 1, \tag{29}$$

we see that the psychological distance of firm n from the representative investor varies with the wealth distribution. We shall see below that the impact of fundamental shocks on asset prices will be amplified by their effect on the wealth distribution. Importantly this amplification occurs even when biases cancel out.

The aggregate psychological bias in the economy, defined as the following arithmetic mean,

$$\hat{b}_t = \frac{1}{N}\sum_{n=1}^{N} \hat{b}_{n,t}, \tag{30}$$

is constant and equal to zero under Assumption 4.1 because in (28) the sum of the wealth shares equals 1 and $\frac{1}{N} \times N = 1$. Thus, behavioral biases canceling out implies that the aggregate psychological bias in the economy is zero. Correspondingly, by defining the psychological distance of the representative investor from the representative firm by d_t, where

$$e^{-\kappa(d_t - \overline{d})} = 1 + \hat{b}_t,$$

we see that the psychological distance of the representative investor from the representative firm reduces to the rational expectations threshold under Assumption 4.1; i.e., $d_t = \overline{d}$.

4.1 Equilibrium Risk-Free Interest Rate

Proposition 4.1. *In the absence of any investor-level biases, i.e., $b_{hn} = 0$ for all h and n, the equilibrium interest rate i_{NB} is given by the constant*

$$i_{NB} = \overline{\alpha} - \frac{1}{N}\overline{\sigma}^2, \tag{31}$$

where $\overline{\alpha}$ is the risk-weighted mean of firm-level expected returns and $\overline{\sigma}^2$ is the harmonic mean of the firms' return variances

$$\overline{\alpha} = \frac{\sum_{n=1}^{N} \frac{\alpha_n}{\sigma_n^2}}{\sum_{n=1}^{N} \frac{1}{\sigma_n^2}},$$

$$\overline{\sigma}^2 = \left(\frac{1}{N} \sum_{n=1}^{N} \frac{1}{\sigma_n^2} \right)^{-1}.$$

On the other hand, in the presence of biases that "cancel out," the equilibrium interest rate is given by

$$i_t = \hat{\alpha}_t - \frac{\hat{\sigma}_t^2}{N}, \tag{32}$$

where $\hat{\alpha}_t$ is the distorted risk-weighted mean of firm-level expected returns and $\hat{\sigma}_t$ is the distorted harmonic mean of the firms' return variances:

$$\hat{\alpha}_t = \frac{\sum_{n=1}^{N} \frac{\alpha_n}{\sigma_n^2}(1 + \hat{b}_{n,t})}{\sum_{n=1}^{N} \frac{1}{\sigma_n^2}(1 + \hat{b}_{n,t})} = \frac{\sum_{n=1}^{N} \frac{\alpha_n}{\sigma_n^2} e^{-\kappa d_{n,t}}}{\sum_{n=1}^{N} \frac{1}{\sigma_n^2} e^{-\kappa d_{n,t}}}, \tag{33}$$

$$\hat{\sigma}_t^2 = \left(\frac{1}{N} \sum_{n=1}^{N} \frac{1}{\sigma_n^2}(1 + \hat{b}_{n,t}) \right)^{-1} = \left(\frac{1}{N} \sum_{n=1}^{N} \frac{1}{\sigma_n^2} e^{-\kappa(d_{n,t} - \overline{d})} \right)^{-1}. \tag{34}$$

The equilibrium interest rate in the absence of any investor-level biases, i_{NB}, serves as a benchmark. In the single-firm setting we know that the interest rate is the difference between the expected return and the return variance (recall that we have assumed relative risk aversion $\gamma = 1$). When there are multiple firms, we obtain in (31) a similar expression, but where the expected returns are risk-weighted and the return variance is the harmonic mean of individual firms variances, which is multiplied by $1/N$ to reflect the impact of diversification. The expression for the interest rate in (31) satisfies the usual comparative static: when expected returns are higher or risk is lower, the risk-free bond becomes less attractive, leading to an increase in the interest rate.

In the presence of behavioral biases that "cancel out," the expression for the interest rate in (32) still has the familiar form of the difference between an average expected return and an average return variance. However, both the averages are distorted by behavioral biases, as can be seen in (33) and (34). That is, even though the behavioral biases in (30) cancel out when summed across firms, they do not cancel out once they are weighted by firm-level expected returns or variances.

The difference between the interest rate in the presence of behavioral biases that "cancel out", i_t, and the interest rate in the absence of any investor-level biases, i_{NB}, is, as we show in the proof of Proposition 4.1, given by

$$i_{\epsilon,t} = i_t - i_{NB} = b\frac{1}{N-1}\overline{\sigma}^2 \frac{\sum_{h=1}^{H}\left(w_{h,t} - \frac{1}{N}\right)\frac{\alpha_h - i_{NB}}{\sigma_h^2}}{1 + b\frac{1}{N-1}\overline{\sigma}^2 \sum_{h=1}^{H}\left(w_{h,t} - \frac{1}{N}\right)\frac{1}{\sigma_h^2}}, \tag{35}$$

where $w_{h,t}$ is the wealth share of investor h. We see from (35) that the dislocation of the money market is driven by shocks to the portfolios of investors. The distortion vanishes momentarily only when the fraction of aggregate wealth held by each investor is the same, but returns immediately as new shocks arrive. This makes it clear that the distortion in the interest rate is driven by wealth inequality, which is a consequence of portfolio heterogeneity driven by behavioral biases.

4.2 Stochastic Discount Factors

The following proposition gives the equilibrium stochastic discount factor.

Proposition 4.2. *If Assumption 4.1 holds, then the equilibrium stochastic discount factor is*

$$\begin{aligned}\frac{d\Lambda_t}{\Lambda_t} &= -i_t dt - \sum_{n=1}^{N}\left(\Theta_{NB,n} - \frac{1}{\sigma_n} i_{\epsilon,t}\right) dZ_{n,t} \\ &= -i_t dt - \sum_{n=1}^{N}\frac{\alpha_n - i_t}{\sigma_n} dZ_{n,t},\end{aligned}$$

where $\Theta_{NB,n}$ is the risk price when there are no biases ($b = 0$):

$$\Theta_{NB,n} = \frac{\alpha_n - i_{NB}}{\sigma_n} = \frac{\alpha_n - \overline{\alpha} + \frac{1}{N}\overline{\sigma}^2}{\sigma_n}. \tag{36}$$

The price of risk for shocks to firm n is given by $\Theta_{NB,n} - \frac{1}{\sigma_n} i_{\epsilon,t}$, where $\Theta_{NB,n}$ is the risk price when there are no biases, i.e. $b = 0$. We therefore see that distortions in the money market lead to distortions in the price of risk, impacting the entire financial market. In particular, greater demand for precautionary savings implies $i_{\epsilon,t} < 0$, leading to increases in the price of risk for each firm.

The following proposition gives investor-specific stochastic discount factors.

Proposition 4.3.

$$\frac{d\Lambda_{h,t}}{\Lambda_{h,t}} = -i_t dt - (1+b)\left(\Theta_{NB,h} - \frac{1}{\sigma_h} i_{\epsilon,t}\right) dZ^h_{h,t} - \left(1 - \frac{b}{N-1}\right) \sum_{n \neq h} \left(\Theta_{NB,n} - \frac{1}{\sigma_n} i_{\epsilon,t}\right) dZ^h_{n,t}.$$

Here, we can see each investor prices risk according to her own biases. An investor who is optimistic about a specific firm will have a higher price of risk for that firm than an investor who is pessimistic about the firm. The higher risk price will lead to a greater expected excess return.

5 Conclusion

In this paper, we develop a model where investors' beliefs about expected stock returns are determined endogenously based on their psychological distances from firms. Consequently, investors are optimistic about some stocks and pessimistic about others, which results in biased portfolios that are underdiversified. We then embed these beliefs in a general-equilibrium production economy with a large number of heterogeneous firms and investors. We characterize in closed form the equilibrium in two examples of this economy. In the first example, the portfolio "errors" resulting from the behavioral biases of individual investors cancel out when aggregated across all investors. In the second example, the behavioral biases themselves cancel out when aggregated across all investors. For both examples, the behavioral biases lead to a dislocation in the money market, which distorts both aggregate financial variables and macroeconomic quantities. In particular, both the drift and the volatility of the stochastic discount factor are affected. This implies that behavioral biases of individual investors will have an affect on the prices of all assets even though the biases cancel out in aggregate.

The two specific examples we have constructed show that the effects of idiosyncratic behavioral biases at the individual level do not wash out in aggregate even under very restrictive assumptions that lead to a canceling out of the portfolio errors caused by psychological biases or a canceling out of the biases themselves. This implies that under less restrictive assumptions, the effects of psychological biases will be evident in asset prices and macroeconomic variables. Other behavioral biases, such as overconfidence about some assets relative to others, will also have a similar impact. Thus, our work indicates that the effects of behavioral biases on financial markets and the macroeconomy are pervasive and economists should focus on investigating their impact instead of suggesting that such biases wash out in aggregate.

A Proofs for Propositions

In this appendix, we provide the derivations for all the results in the main text.

Proof of Proposition 2.1

We shall derive (5) from (4). We see that

$$b_{hn} = \overline{T} \int_0^{\infty} p_1(s) \max\left(\frac{s - d_{hn}}{s}, 0\right) ds - 1,$$

where $p_1(s)$ is the probability density for the trust region.

The Principle of Maximum Entropy (Jaynes, 1957, 1986) states that if nothing is known about a distribution except that it belongs to a certain class, then the distribution with the largest entropy should be chosen as the least informative. Thus, the following static optimization problem determines the prior distribution of S, which defines the size of the trust region.

$$\max_{p(s)} - \int_0^{\infty} p(s) \ln p(s) ds$$

subject to

$$\int_0^{\infty} p(s) ds = 1, \tag{A1}$$

and

$$\int_0^{\infty} s p(s) ds = \mu, \tag{A2}$$

where $\mu = E[S]$ is the known mean of S.

To solve the above constrained maximization problem, we form the Lagrangian

$$\mathcal{L} = -\int_0^{\infty} p(s) \ln p(s) ds - \kappa_0 \left(1 - \int_0^{\infty} p(s) ds\right) - \kappa \left(\mu - \int_0^{\infty} s p(s) ds\right).$$

We have the first order condition

$$0 = -\ln p(s) - 1 + \kappa_0 + \kappa s,$$

and so

$$p(s) = e^{-(1-\kappa_0)} e^{-\kappa s}, \; s \geq 0.$$

From (A1), we see that

$$p(s) = \frac{1}{\kappa} e^{-\kappa s}, \; s \geq 0.$$

We now use the constraint (A2) to determine κ, starting with

$$\mu = \kappa \int_0^\infty s e^{-\kappa s} ds.$$

It follows that

$$\frac{1}{\kappa} = \mu.$$

Starting from the prior density $p(s)$, the posterior density for S is given by Bayes' Law via

$$p_1(s) \propto s p(s),$$

where s is the likelihood of an object being in its trust region. Therefore,

$$p_1(s) = \kappa s e^{-\kappa s},\ s \geq 0.$$

Hence,

$$\begin{aligned} b_{hn} &= \overline{T}\kappa \int_0^\infty e^{-\kappa s} \max\left(s - d_{hn}, 0\right) ds - 1 \\ &= \overline{T}\kappa \int_{d_{hn}}^\infty e^{-\kappa s} \left(s - d_{hn}\right) ds - 1 \\ &= \overline{T}\frac{e^{-\kappa d_{hn}}}{\kappa} - 1. \end{aligned}$$

We define

$$\overline{d} = \frac{1}{\kappa} \ln \overline{T},$$

Therefore

$$b_{hn} = e^{-\kappa(\overline{d} - d_{hn})} - 1,$$

where $\overline{d}$ is the threshold psychological distance at which investor h has zero bias for firm n. $\square$

One can show that if biases are additive, then psychological distances are subadditive, which has important consequences for how psychological distances can be reduced.

Proposition A.1. *Consider an investor h with biases $b_{h1}, \ldots, b_{hN}$ generated by the psychological distances, $d_{h1}, \ldots, d_{hN}$ via (5). The mean bias of investor h is given by*

$$\mu_{b,h} = \frac{1}{N} \sum_{n=1}^{N} b_{hn}.$$

If the psychological distance of investor h from the representative firm for the group of firms $1, \ldots, N$ is given by d_h, where

$$\mu_{b,h} = e^{-\kappa(d_h - \overline{d})} - 1,$$

then

$$d_h = -\frac{1}{\kappa} \ln \frac{1}{N} \sum_{n=1}^{N} e^{-\kappa d_{hn}}, \tag{A3}$$

and

$$d_h \leq \frac{1}{N} \sum_{n=1}^{N} d_{hn}, \tag{A4}$$

with equality if and only if $d_{h1} = \ldots = d_{hN}$.

Proof of Proposition A.1

The result follows by applying Jensen's inequality to (A3). □

The expression (A3) is useful, because it tells us straight away via Jensen's inequality that psychological distance is subadditive, i.e., (A4) holds. Volatility is also subadditive and it is precisely this property which makes it possible to diversify risk by holding more risky assets. By analogy with volatility, we can see that the psychological distance of an investor from a group of firms with a mean psychological distance d is actually smaller than d. Hence, the overall level of bias in an economy can be reduced by spreading out the firms, even while their keeping their mean psychological distances from investors fixed.

We can make the consequences of subadditivity more concrete by considering an investor and 2 firms, with psychological distances from the investor of $\overline{d} - y$ and $\overline{d} + y$. The mean psychological distance is clearly $\overline{d}$. However, the psychological distance of the investor from the representative firm is less than or equal to $\overline{d}$, as shown below:

$$\begin{aligned} d_h &= -\frac{1}{\kappa} ln \frac{1}{2}(e^{-\kappa(\overline{d}-y)} + e^{-\kappa(\overline{d}+y)}) \\ &= \overline{d} - \frac{1}{\kappa} ln \frac{1}{2}(e^{\kappa y} + e^{-\kappa y}) \\ &= \overline{d} - \frac{1}{\kappa} ln \cosh(\kappa y) \leq \overline{d}, \quad \text{with equality if and only if } y = 0. \end{aligned}$$

In this example, we can see clearly that as the two firms move farther apart, i.e., y increases, $ln \cosh(\kappa y)$ increases, and so the psychological distance of the investor from the representative firm decreases, reaching an exterior minimum when $y = \overline{d}$.

Proof of Propostion 2.2

Equation (8) implies that

$$\widehat{\mu}^{\mathbb{P}^h}_{h,t}[U_{h,t+dt}] = \left(E^{\mathbb{P}^h}_t [u_{\gamma_h} (U_{h,t+dt})] \right)^{\frac{1}{1-\gamma_h}},$$

Therefore

$$\widehat{\mu}_{h,t}^{\mathbb{P}^h}[U_{h,t+dt}] = E_t^{\mathbb{P}^h}\left[U_{h,t+dt}^{1-\gamma_h}\right]^{\frac{1}{1-\gamma}} = E_t^{\mathbb{P}^h}\left[U_{h,t}^{1-\gamma_h} + d(U_{h,t}^{1-\gamma_h})\right]^{\frac{1}{1-\gamma_h}}.$$

Applying Ito's Lemma, we obtain

$$\begin{aligned} d(U_{h,t}^{1-\gamma_h}) &= (1-\gamma_h)U_{h,t}^{-\gamma_h}dU_{h,t} - \frac{1}{2}(1-\gamma_h)\gamma_h U_{h,t}^{-\gamma_h-1}(dU_{h,t})^2 \\ &= (1-\gamma_h)U_{h,t}^{1-\gamma_h}\left[\frac{dU_{h,t}}{U_{h,t}} - \frac{1}{2}\gamma_h\left(\frac{dU_{h,t}}{U_{h,t}}\right)^2\right]. \end{aligned}$$

Therefore,

$$\begin{aligned} \widehat{\mu}_{h,t}^{\mathbb{P}^h}[U_{h,t+dt}] = E_t^{\mathbb{P}^h}\left[U_{h,t+dt}^{1-\gamma_h}\right]^{\frac{1}{1-\gamma_h}} &= U_{h,t}\left(E_t^{\mathbb{P}^h}\left[1+(1-\gamma_h)\left[\frac{dU_{h,t}}{U_{h,t}} - \frac{1}{2}\gamma_h\left(\frac{dU_{h,t}}{U_{h,t}}\right)^2\right]\right]\right)^{\frac{1}{1-\gamma_h}} \\ &= U_{h,t}\left(1+(1-\gamma_h)\left[E_t^{\mathbb{P}^h}\left[\frac{dU_{h,t}}{U_{h,t}}\right] - \frac{1}{2}\gamma_h E_t^{\mathbb{P}^h}\left[\left(\frac{dU_{h,t}}{U_{h,t}}\right)^2\right]\right]\right)^{\frac{1}{1-\gamma_h}}. \end{aligned}$$

Hence,

$$\widehat{\mu}_{h,t}^{\mathbb{P}^h}[U_{h,t+dt}] = U_{h,t}\left(1+E_t^{\mathbb{P}^h}\left[\frac{dU_{h,t}}{U_{h,t}}\right] - \frac{1}{2}\gamma E_t^{\mathbb{P}^h}\left[\left(\frac{dU_{h,t}}{U_{h,t}}\right)^2\right]\right) + o(dt).$$

Therefore, in the continuous time limit, we obtain

$$\frac{\widehat{\mu}_{h,t}^{\mathbb{P}^h}[dU_{h,t+dt}]}{dt} = \frac{\widehat{\mu}_{h,t}^{\mathbb{P}^h}[U_{h,t+dt}] - U_{h,t}}{dt} = U_{h,t}\left(E_t^{\mathbb{P}^h}\left[\frac{dU_{h,t}}{U_{h,t}}\right] - \frac{1}{2}\gamma_h E_t^{\mathbb{P}^h}\left[\left(\frac{dU_{h,t}}{U_{h,t}}\right)^2\right]\right).$$

The result therefore follows from (7). □

Proof of Proposition 2.3

Writing out (11) explicitly gives

$$\frac{U_{h,t}^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}} = (1-e^{-\delta dt})\sup_{C_{h,t}}\frac{C_{h,t}^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}} + e^{-\delta dt}\sup_{\boldsymbol{\omega}_{h,t}}\inf_{\{\mathbb{P}^{hn}\}_{n\in A_P}}\sup_{\{\mathbb{P}^{hn}\}_{n\in A_O}}\frac{\left(\mu_{h,t}^{\mathbb{P}^h}[U_{h,t+dt}]\right)^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}}.$$

Now

$$\begin{aligned} \left(\widehat{\mu}_{h,t}^{\mathbb{P}^h}[U_{h,t+dt}]\right)^{1-\frac{1}{\psi}} &= \left(U_{h,t} + \mu_{h,t}^{\mathbb{P}^h}[dU_{h,t}]\right)^{1-\frac{1}{\psi}} \\ &= U_{h,t}^{1-\frac{1}{\psi}}\left(1+\mu_{h,t}^{\mathbb{P}^h}\left[\frac{dU_{h,t}}{U_{h,t}}\right]\right)^{1-\frac{1}{\psi}} \\ &= U_{h,t}^{1-\frac{1}{\psi}}\left(1+\left(1-\frac{1}{\psi}\right)\mu_{h,t}^{\mathbb{P}^h}\left[\frac{dU_{h,t}}{U_{h,t}}\right]\right) + o(dt). \end{aligned}$$

Hence,

$$\frac{U_{h,t}^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}} = \delta \sup_{C_{h,t}} \frac{C_{h,t}^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}} dt + \sup_{\boldsymbol{\omega}_{h,t}} \inf_{\{\mathbb{P}^{hn}\}_{n\in A_P}} \sup_{\{\mathbb{P}^{hn}\}_{n\in A_O}} \frac{U_{h,t}^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}} \left(1 + \left(1-\frac{1}{\psi}\right) \mu_{h,t}^{\mathbb{P}^h}\left[\frac{dU_{h,t}}{U_{h,t}}\right]\right)$$
$$- \delta \frac{U_{h,t}^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}} dt + o(dt),$$

from which we obtain (12). $\square$

Proof of Proposition 3.1

We assume the investment opportunity set is constant. The only state variable therefore is wealth, so we start with the Ansatz $U_{h,t} = \kappa_h W_{h,t}$. Therefore, $dU_{h,t}/U_{h,t} = dW_{h,t}/W_{h,t}$. Consequently, (10) implies that

$$\mu_{h,t}^{\mathbb{P}^h}\left[\frac{dU_{h,t}}{U_{h,t}}\right] = E_t^{\mathbb{P}^h}\left[\frac{dW_{h,t}}{W_{h,t}}\right] - \frac{1}{2}\gamma Var_t\left[\frac{dW_{h,t}}{W_{h,t}}\right] - \frac{1}{\gamma}\sum_{n=1}^{N}\frac{1+b_{hn}}{b_{hn}} D^{KL}[\mathbb{P}|\mathbb{P}^{hn}]. \tag{A5}$$

Therefore, (12) reduces to (15). $\square$

Proof of Proposition 3.2

The unique solution to (18) is

$$\omega_{hn} = \frac{1}{\gamma}\frac{\alpha - i}{\sigma^2}(1 + b_{hn}),$$

and substituting the above expression into (17) gives (19). $\square$

Proof of Proposition 3.3

We assume the investment opportunity set is constant. Therefore, (A5) holds and the beliefs-portfolio choice problem $\sup_{\boldsymbol{\omega}_{h,t}} \inf_{\{\mathbb{P}^{hn}\}_{n\in A_P}} \sup_{\{\mathbb{P}^{hn}\}_{n\in A_O}} \mu_{h,t}^{\mathbb{P}^h}[dU_{h,t}/dt]$ reduces to (16). We substitute in the optimal beliefs and portfolios, given in (19) and (20), respectively, to obtain

$$\sup_{\boldsymbol{\omega}_{h,t}} \inf_{\{\mathbb{P}^{hn}\}_{n\in A_P}} \sup_{\{\mathbb{P}^{hn}\}_{n\in A_O}} \mu_{h,t}^{\mathbb{P}^h}\left[\frac{dU_{h,t}}{dt}\right] = U_{h,t}\left[i + \frac{1}{2\gamma}\left(\frac{\alpha - i}{\sigma^2/N} + \frac{\alpha - i}{\sigma^2/N}\frac{1}{N}\sum_{n=1}^{N} b_{hn}\right) - \frac{C_{h,t}}{W_{h,t}}\right].$$

We now observe that

$$\frac{\alpha - i}{\sigma^2/N} + \frac{\alpha - i}{\sigma^2/N}\frac{1}{N}\sum_{n=1}^{N} b_{hn} = \frac{\alpha - i}{\sigma^2/N}(1 + \mu_{b,h}),$$

where $\mu_{b,h} = \mu_b$ is constant across investors. Therefore,

$$\sup_{\boldsymbol{\omega}_{h,t}} \inf_{\{\mathbb{P}^{hn}\}_{n\in A_P}} \sup_{\{\mathbb{P}^{hn}\}_{n\in A_O}} \mu^{\mathbb{P}^h}_{h,t} \frac{1}{U_{h,t}} \left[\frac{dU_{h,t}}{dt}\right] = i + \frac{1}{2\gamma}\left(\frac{\alpha - i}{\sigma_{1/N}}\right)^2 (1+\mu_b) - \frac{C_{h,t}}{W_{h,t}},$$

where $\sigma_{1/N} = \sigma/\sqrt{N}$.

The optimal consumption problem is

$$\sup_{C_{h,t}} \delta \frac{\left(\frac{C_{h,t}}{U_{h,t}}\right)^{1-\frac{1}{\psi}} - 1}{1-\frac{1}{\psi}} - \frac{C_{h,t}}{W_{h,t}}.$$

The FOC is

$$\delta \left(\frac{C_{h,t}}{U_{h,t}}\right)^{-\frac{1}{\psi}} \frac{1}{U_{h,t}} = \frac{1}{W_{h,t}}.$$

Therefore,

$$\frac{C_{h,t}}{W_{h,t}} = \delta^{\psi} \kappa_h^{1-\psi},$$

and so

$$\sup_{C_{h,t}} \delta \frac{\left(\frac{C_{h,t}}{U_{h,t}}\right)^{1-\frac{1}{\psi}} - 1}{1-\frac{1}{\psi}} - \frac{C_{h,t}}{W_{h,t}} = \frac{\delta^{\psi_h}\kappa_h^{1-\psi} - \psi\delta}{1-\psi_h}.$$

Thus, the HJB equation simplifies to

$$0 = \frac{\delta^{\psi_h}\kappa^{1-\psi} - \psi\delta}{1-\psi} - \left[i + \frac{1}{2\gamma}\left(\frac{\alpha - i}{\sigma_{1/N}}\right)^2 (1+\mu_b)\right],$$

and so the consumption-wealth ratio is identical across investors and given by

$$c = \frac{C_{h,t}}{W_{h,t}} = \delta^{\psi}\kappa_h^{1-\psi} = \psi\delta + (1-\psi)\left(i + \frac{1}{2\gamma}\left(\frac{\alpha - i}{\sigma_{1/N}}\right)^2 (1+\mu_b)\right),$$

and the optimized value of the investor's utility function per unit wealth is

$$\frac{U_{h,t}}{W_{h,t}} = \left(\frac{\psi\delta + (1-\psi)\Big(i + \frac{1}{2\gamma}\left(\frac{\alpha-i}{\sigma_{1/N}}\right)^2 (1+\mu_b)\Big)}{\delta^{\psi}}\right)^{\frac{1}{1-\psi}}.$$

$\square$

Proof of Proposition 3.4

From (20), we see that the optimal portfolio policy is given by

$$\omega_{hn} = \frac{1}{\gamma}\frac{\alpha - i}{\sigma^2}(1 + b_{hn}).$$

Therefore, π_h, the fraction of wealth held in risky assets by investor h is

$$\pi_h = \sum_{n=1}^{N} \omega_{hn} = \frac{1}{\gamma}\frac{\alpha - i}{\sigma^2/N}(1 + \mu_{b,h}).$$

If $\mu_{b,h} = \mu_b$, then π_h is identical across investors and the proportion of wealth each investor desires to hold in the bond is the same and given by

$$\frac{B_{h,t}}{W_{h,t}} = \left(1 - \frac{1}{\gamma}\frac{\alpha - i}{\sigma^2/N}(1 + \mu_b)\right).$$

Market clearing in the bond market implies $\sum_{h=1}^{H} B_{h,t} = 0$, and so

$$0 = \left(1 - \frac{1}{\gamma}\frac{\alpha - i}{\sigma^2/N}(1 + \mu_b)\right)\sum_{h=1}^{H} W_{h,t}.$$

We know that $\sum_{h=1}^{H} W_{h,t} > 0$, and so

$$i = \alpha - \frac{\gamma}{1 + \mu_b}\frac{\sigma^2}{N},$$

which implies (22) holds. □

Proof of Proposition 3.5

We want to derive an expression for trend output growth, g, defined as

$$g\,dt = E_t\left[\frac{dY_t^{\text{agg}}}{Y_t^{\text{agg}}}\right].$$

Equation (1) implies

$$Y_t^{\text{agg}} = \sum_{n=1}^{N} Y_{n,t} = \alpha\sum_{n=1}^{N} K_{n,t},$$

and Equation (3) implies

$$dE_t\left[\sum_{n=1}^{N} K_{n,t}\right] = E_t\left[d\sum_{n=1}^{N} K_{n,t}\right] = \alpha\sum_{n=1}^{N} K_{n,t} - \sum_{n=1}^{N} D_{n,t}dt.$$

In equilibrium $\sum_{n=1}^{N} K_{n,t} = W_t^{\text{agg}}$ and $\sum_{n=1}^{N} D_{n,t} = C_t^{\text{agg}}$. Therefore,

$$E_t\left[\frac{dW_t^{\text{agg}}}{W_t^{\text{agg}}}\right] = \left(\alpha - \frac{C_t^{\text{agg}}}{W_t^{\text{agg}}}\right)dt.$$

We also know that

$$\frac{dW_t^{\text{agg}}}{W_t^{\text{agg}}} = \frac{dY_t^{\text{agg}}}{Y_t^{\text{agg}}}$$

and so

$$g\,dt = E_t\left[\frac{dY_t^{\text{agg}}}{Y_t^{\text{agg}}}\right] = \left(\alpha - \frac{C_t^{\text{agg}}}{W_t^{\text{agg}}}\right)dt.$$

The aggregate investment flow must be equal to aggregate output flow less the aggregate consumption flow:

$$I_t^{\text{agg}} = \alpha K_t^{\text{agg}} - C_t^{\text{agg}}.$$

Therefore

$$\frac{I_t^{\text{agg}}}{K_t^{\text{agg}}} = \alpha - \frac{C_t^{\text{agg}}}{K_t^{\text{agg}}} = \alpha - \frac{C_t^{\text{agg}}}{W_t^{\text{agg}}} = g.$$

Now we observe that

$$\frac{C_t^{\text{agg}}}{W_t^{\text{agg}}} = \frac{\sum_{h=1}^{H} C_{h,t}}{\sum_{h=1}^{H} W_{h,t}} = \frac{c\sum_{h=1}^{H} W_{h,t}}{\sum_{h=1}^{H} W_{h,t}},$$

because the consumption-wealth ratio $c = C_{h,t}/W_{h,t}$ is identical across investors. Therefore,

$$g = \frac{I_t^{\text{agg}}}{K_t^{\text{agg}}} = \alpha - c. \tag{A6}$$

We substitute the equilibrium interest rate given in (22) into the expression in (21) for the consumption-wealth ratio to obtain the general-equilibrium consumption-wealth ratio:

$$\frac{C_{h,t}}{W_{h,t}} = c,$$

where

$$c = \psi\delta + (1-\psi)\left(\alpha - \frac{1}{2}\frac{\gamma}{1+\mu_b}\sigma_{1/N}^2\right). \tag{A7}$$

Substituting the above expression into $\alpha - c$ gives the desired result. □

Proof of Proposition 3.6

The aggregate price-dividend ratio is equal to the aggregate wealth-consumption ratio. Therefore,

$$p_t^{\text{agg}} = \frac{1}{c}.$$

In equilibrium the consumption-wealth ratio c is given by (A7), and so

$$p_t^{\text{agg}} = \frac{1}{\psi\delta + (1-\psi)\left(\alpha - \frac{1}{2}\frac{\gamma}{1+\mu_b}\sigma_{1/N}^2\right)}.$$

Using (A6), we obtain

$$p_t^{\text{agg}} = \frac{1}{\alpha - g} = \frac{1}{i + \alpha - i - g}.$$

The expected risk premium $\alpha - i$ is given in (23), which can be rewritten as

$$\alpha - i = \gamma \frac{1}{1 + \mu_b} \sigma_{1/N}^2.$$

Therefore, we obtain the Gordon growth formula:

$$p_t^{\text{agg}} = \frac{1}{i + \gamma \frac{1}{1+\mu_b} \sigma_{1/N}^2 - g}.$$

□

Proof of Proposition 3.7

Each investor has her own stochastic discount factor. We denote the date-t stochastic discount factor of investor h via $\Lambda_{h,t}$. Observe that

$$\frac{d\Lambda_{h,t}}{\Lambda_{h,t}} = -idt - \gamma \left(\frac{dW_{h,t}}{W_{h,t}} - E_t^{\mathbb{P}^h} \left[\frac{dW_{h,t}}{W_{h,t}} \right] \right),$$

provided the interest rate is constant. We know that

$$\frac{dW_{h,t}}{W_{h,t}} - E_t^{\mathbb{P}^h} \left[\frac{dW_{h,t}}{W_{h,t}} \right] = \sum_{n=1}^{N} \omega_{hn,t} \sigma dZ_{n,t}^h,$$

where Z_n^h, $n \in \{1, \dots, N\}$ are standard Brownian motions under $\mathbb{P}^h$ such that $E_t^{\mathbb{P}^h}[dZ_{n,t}^h dZ_{n',t}^h] = 0$ when $n \neq n'$.

In equilibrium $\omega_{hn,t}$ is given by the expression in Footnote 11 . Hence, we obtain

$$\frac{dW_{h,t}}{W_{h,t}} - E_t^{\mathbb{P}^h} \left[\frac{dW_{h,t}}{W_{h,t}} \right] = \frac{1}{N} \sigma \left[\sum_{n=1}^{N} dZ_{n,t}^h + \frac{1}{1 + \mu_b} \sum_{n=1}^{N} (b_{hn} - \mu_b) dZ_{n,t}^h \right].$$

Therefore,

$$\frac{d\Lambda_{h,t}}{\Lambda_{h,t}} = -idt - \gamma \frac{1}{N} \sigma \left[\sum_{n=1}^{N} dZ_{n,t}^h + \frac{1}{1 + \mu_b} \sum_{n=1}^{N} (b_{hn} - \mu_b) dZ_{n,t}^h \right].$$

The volatility of investor h's stochastic discount factor is given by

$$\gamma \frac{1}{N} \sigma \sqrt{N + 2 \frac{1}{1 + \mu_b} \sum_{n=1}^{N} (b_{hn} - \mu_b) + \frac{1}{(1 + \mu_b)^2} \sum_{n=1}^{N} (b_{hn} - \mu_b)^2}.$$

By assumption, $\sum_{n=1}^{N}(b_{hn} - \mu_b) = 0$, and so the stochastic discount factor volatility reduces to

$$\gamma \frac{\sigma}{\sqrt{N}} \sqrt{1 + \frac{1}{(1+\mu_b)^2} \frac{1}{N} \sum_{n=1}^{N} (b_{hn} - \mu_b)^2} = \gamma \frac{\sigma}{\sqrt{N}} \sqrt{1 + \left(\frac{\sigma_b}{1+\mu_b}\right)^2},$$

where σ_b is the standard deviation of the biases

$$\sigma_b = \sqrt{\frac{1}{N} \sum_{n=1}^{N} (b_{hn} - \mu_b)^2}.$$

□

Proof of Proposition 3.8

The equilibrium stochastic discount factor is given Λ, where

$$\frac{d\Lambda_t}{\Lambda_t} = -i dt - \sum_{n=1}^{N} \Theta_{n,t} dZ_{n,t},$$

where $\Theta_{n,t}$, $n \in \{1, \dots, N\}$ are to be determined. Markets are dynamically complete, so shocks to the stochastic discount factor over the time interval $[t, t+dt)$ can depend only on the Brownian increments, $dZ_{1,t}, \dots, dZ_{N,t}$. If Λ is a stochastic discount factor, then

$$\begin{aligned} \alpha - i = \gamma \frac{1}{1+\mu_b} \frac{\sigma^2}{N} &= -\frac{1}{dt} E_t \left[\sigma dZ_{n,t} \left(-\sum_{n=1}^{N} \Theta_{n,t} dZ_{n,t} \right) \right] \\ &= \sigma \Theta_{n,t}, \end{aligned}$$

and so

$$\Theta_{n,t} = \gamma \frac{1}{1+\mu_b} \frac{\sigma}{N}.$$

Hence, we obtain (27) and the volatility of the equilibrium stochastic discount factor. □

Proof of Proposition 4.1

With $\gamma = 1$, the optimal portfolio policy of investor h is

$$\omega_{hn} = \frac{\alpha_n - i_t}{\sigma_n^2}(1 + b_{hn}).$$

Consequently, the proportion of wealth invested in risky assets by investor h is

$$\pi_h = \sum_{n=1}^{N} \omega_{hn} = \sum_{n=1}^{N} \frac{\alpha_n - i_t}{\sigma_n^2}(1 + b_{hn}). \tag{A8}$$

The market-clearing condition for the bond, which is in zero net supply, is

$$\sum_{h=1}^{H}(1-\pi_{h,t})W_{h,t}=0,$$

which is equivalent to

$$\sum_{h=1}^{H}\pi_{h,t}w_{h,t}=1, \tag{A9}$$

where $w_{h,t}=W_{h,t}/\sum_{h=1}^{H}W_{h,t}$. Substituting (A8) into (A9) gives

$$\sum_{h=1}^{H}\sum_{n=1}^{N}\frac{\alpha_n-i_t}{\sigma_n^2}(1+b_{hn})w_{h,t}=1.$$

Therefore,

$$\sum_{n=1}^{N}\frac{\alpha_n-i_t}{\sigma_n^2}\left(1+\sum_{h=1}^{H}w_{h,t}b_{hn}\right)=1.$$

Define

$$\hat{b}_{n,t}=\sum_{h=1}^{H}w_{h,t}b_{hn}.$$

Thus,

$$\sum_{n=1}^{N}\frac{\alpha_n-i_t}{\sigma_n^2}\left(1+\hat{b}_{n,t}\right)=1.$$

We now make i_t the subject of the above equation, giving

$$i_t=\frac{\sum_{n=1}^{N}\frac{\alpha_n}{\sigma_n^2}(1+\hat{b}_{n,t})-1}{\sum_{n=1}^{N}\frac{1}{\sigma_n^2}(1+\hat{b}_{n,t})}. \tag{A10}$$

Therefore,

$$i_t=\frac{\sum_{n=1}^{N}\frac{\alpha_n}{\sigma_n^2}(1+\hat{b}_{n,t})}{\sum_{n=1}^{N}\frac{1}{\sigma_n^2}(1+\hat{b}_{n,t})}-\frac{1}{N}\left(\frac{1}{N}\sum_{n=1}^{N}\frac{1}{\sigma_n^2}(1+\hat{b}_{n,t})\right)^{-1},$$

which implies (32), where $\hat{\alpha}_t$ and $\hat{\sigma}_t$ are given in terms of $\hat{b}_{n,t}$, $n\in\{1,\ldots,N\}$.

We now observe that (29) implies

$$\frac{1+\hat{b}_{n,t}}{\sigma_n^2}=\frac{e^{-\kappa(d_{n,t}-\bar{d})}}{\sigma_n^2}.$$

Therefore, we obtain the final expression in (34). Furthermore,

$$\frac{1+\hat{b}_{n,t}}{\sigma_n^2}=e^{-\kappa\bar{d}}\frac{e^{-\kappa d_{n,t}}}{\sigma_n^2},$$

and

$$\sum_{n=1}^{N} \frac{1+\hat{b}_{n,t}}{\sigma_n^2} = e^{-\kappa \bar{d}} \frac{e^{-\kappa d_{n,t}}}{\sigma_n^2}.$$

Therefore, we obtain the final expression in (33). So far we have only assumed that all investors have relative risk aversion of one. If Assumption 4.1 holds, then $\hat{b}_{n,t}$ reduces to the expression given in (28) and $\frac{1}{N}\sum_{n=1}^{N} \hat{b}_{n,t} = 0$.

Furthermore, if we set all individual level biases to be zero, $b_{hn} = 0$ for all h and n, then (32) reduces to (31).

We can now rewrite (A10) as

$$i_t = \frac{i_{NB} + \left(\frac{1}{N}\sum_{n=1}^{N} \frac{\alpha_n}{\sigma_n^2}\hat{b}_{n,t}\right)\overline{\sigma}^2}{1 + \overline{\sigma}^2 \frac{1}{N}\sum_{n=1}^{N} \frac{1}{\sigma_n^2}\hat{b}_{n,t}}. \tag{A11}$$

The above expression can be rewritten as

$$i_t = i_{NB} + \frac{\frac{1}{N}\sum_{n=1}^{N} \frac{\alpha_n - i_{NB}}{\sigma_n^2}\hat{b}_{n,t}}{1 + \left(\frac{1}{N}\sum_{n=1}^{N} \frac{1}{\sigma_n^2}\hat{b}_{n,t}\right)\overline{\sigma}^2}\overline{\sigma}^2. \tag{A12}$$

Under Assumption 4.1, $\hat{b}_{n,t}$, reduces to the final expression in (28), and so (A12) implies (35). □

Proof of Proposition 4.2

Markets are dynamically complete, so the equilibrium stochastic discount factor must be of the form

$$\frac{d\Lambda_t}{\Lambda_t} = -i_t dt - \sum_{n=1}^{N} \Theta_{n,t} dZ_{n,t},$$

where $\Theta_{n,t}$, $n \in \{1, \ldots, N\}$ are to be determined. The equilibrium stochastic discount factor must satisfy

$$\alpha_n - i_t = -\frac{1}{dt} E_t\left[\sigma_n dZ_{n,t} \frac{d\Lambda_t}{\Lambda_t}\right],$$

and so

$$\alpha_n - i_t = \sigma_n \Theta_{n,t}.$$

When there are no biases,

$$\alpha_n - i_{NB} = \sigma_n \Theta_{NB,n},$$

giving (36). With biases,

$$\alpha_n - i_{NB} - i_{\epsilon,t} = \sigma_n \Theta_{n,t},$$

giving

$$\Theta_{n,t} = \Theta_{NB,n} - \frac{1}{\sigma_n} i_{\epsilon,t}.$$

□

Proof of Proposition 4.3

Investors agree on the interest rate, so investor h's stochastic discount factor must be of the form

$$\frac{d\Lambda_{h,t}}{\Lambda_{h,t}} = -i_t dt - \sum_{n=1}^{N} \Theta_{hn,t} dZ_{n,t}^h,$$

where $\Theta_{hn,t}$, $n \in \{1, \ldots, N\}$ are to be determined. The SDF of investor h must satisfy

$$(\alpha_h - i_t)(1+b) = -\frac{1}{dt} E_t \left[\sigma_h dZ_{h,t} \frac{d\Lambda_{h,t}}{\Lambda_{h,t}} \right],$$

and

$$(\alpha_n - i_t)\left(1 - \frac{b}{N-1}\right) = -\frac{1}{dt} E_t \left[\sigma_n dZ_{n,t} \frac{d\Lambda_{h,t}}{\Lambda_{h,t}} \right], \; n \neq h.$$

Therefore,

$$(\alpha_h - i_t)(1+b) = \sigma_h \Theta_{hh,t},$$

and

$$(\alpha_n - i_t)\left(1 - \frac{b}{N-1}\right) = \sigma_n \Theta_{hn,t}, \; n \neq h.$$

Hence, when there are no biases,

$$\Theta_{hn,t} = \Theta_{NB,n} = \frac{\alpha_n - i_t}{\sigma_n} = \frac{\alpha_n - \overline{\alpha} + \frac{1}{N}\overline{\sigma}^2}{\sigma_n}, \; \forall h \in \{1, \ldots, H\}, \; \forall n \in \{1, \ldots, N\},$$

and with biases

$$\Theta_{hh,t} = (1+b)\left(\Theta_{NB,h} - \frac{1}{\sigma_h} i_{\epsilon,t}\right),$$

$$\Theta_{hn,t} = -\left(1 - \frac{b}{N-1}\right)\left(\Theta_{NB,n} - \frac{1}{\sigma_n} i_{\epsilon,t}\right), \; h \neq n.$$

□

B Hamilton-Jacobi-Bellman Equation for Example 2

In this section, we derive the optimal consumption policies for each investor (for the case where $N = H = 2$) in terms of solutions to a coupled system of ordinary differential equations.[12]

We define

$$x_t = \frac{W_{1,t}}{W_{1,t} + W_{2,t}}.$$

Therefore,

$$x_t = \frac{1}{1 + e^{\Delta_t}} \quad \text{where} \quad \Delta_t = \ln W_{2,t} - \ln W_{1,t}.$$

From Ito's Lemma, we obtain

$$\begin{aligned} d\Delta_t = &\left[(\boldsymbol{\omega}_{2,t} - \boldsymbol{\omega}_{1,t})^\top (\boldsymbol{\alpha} + \boldsymbol{\nu}_{h,t} - \mathbf{1} i_t) - \frac{1}{2}\boldsymbol{\omega}_{1,t}^\top V \boldsymbol{\omega}_{1,t} + \frac{1}{2}\boldsymbol{\omega}_{2,t}^\top V \boldsymbol{\omega}_{2,t} + \frac{C_{1,t}}{W_{1,t}} - \frac{C_{2,t}}{W_{2,t}} \right] dt \\ &+ (\boldsymbol{\omega}_{2,t} - \boldsymbol{\omega}_{1,t})^\top \Sigma d\boldsymbol{Z}_t^h, \end{aligned}$$

where $\Sigma = \text{diag}(\sigma_1, \sigma_2), V = \Sigma^\top \Sigma = \text{diag}(\sigma_1^2, \sigma_2^2), \boldsymbol{\omega}_{h,t} = (\omega_{h1,t}, \omega_{h2,t})^\top, \mathbf{1} = (1,1)^\top, \boldsymbol{Z}_t^h = (Z_{1,t}^h, Z_{2,t}^h)^\top$, and where $Z_{1,t}^h$ and $Z_{2,t}^h$ are standard Brownian motions under $\mathbb{P}^h$ such that $E_t^{\mathbb{P}^h}[dZ_{1,t}^h dZ_{2,t}^h] = 0$. We also have

$$dx_t = -x_t(1 - x_t)(d\Delta_t - \frac{1}{2}(1 - 2x_t)(d\Delta_t)^2).$$

Therefore,

$$\begin{aligned} dx_t = x_t(1 - x_t) &\left[(\boldsymbol{\omega}_{1,t} - \boldsymbol{\omega}_{2,t})^\top (\boldsymbol{\alpha} + \boldsymbol{\nu}_{h,t} - \mathbf{1} i_t) - \frac{1}{2}\boldsymbol{\omega}_{1,t}^\top V \boldsymbol{\omega}_{1,t} + \frac{1}{2}\boldsymbol{\omega}_{2,t}^\top V \boldsymbol{\omega}_{2,t} - \frac{C_{1,t}}{W_{1,t}} + \frac{C_{2,t}}{W_{2,t}} \right. \\ &\left. - \frac{1}{2}(2x_t - 1)(\boldsymbol{\omega}_{1,t} - \boldsymbol{\omega}_{2,t})^\top V (\boldsymbol{\omega}_{1,t} - \boldsymbol{\omega}_{2,t}) \right] dt + x_t(1 - x_t)(\boldsymbol{\omega}_{1,t} - \boldsymbol{\omega}_{2,t})^\top \Sigma d\boldsymbol{Z}_t^h \\ = x_t(1 - x_t) &\left[(\boldsymbol{\omega}_{1,t} - \boldsymbol{\omega}_{2,t})^\top (\boldsymbol{\alpha} + \boldsymbol{\nu}_{h,t} - \mathbf{1} i_t - V(x_t \boldsymbol{\omega}_{1,t} + (1 - x_t)\boldsymbol{\omega}_{2,t}) - \frac{C_{1,t}}{W_{1,t}} + \frac{C_{2,t}}{W_{2,t}} \right] dt \\ &+ x_t(1 - x_t)(\boldsymbol{\omega}_{1,t} - \boldsymbol{\omega}_{2,t})^\top \Sigma d\boldsymbol{Z}_t^h. \end{aligned}$$

The HJB equation for investor h is

$$\begin{aligned} 0 = &\sup_{C_{h,t}} \delta u_\psi \left(\frac{C_{h,t}}{U_{h,t}} \right) + \sup_{\boldsymbol{\omega}_{h,t}} \inf_{\{\nu_{hn,t}\}_{n \in A_P}} \sup_{\{\nu_{hn,t}\}_{n \in A_O}} \frac{W_{h,t} U_{h,W_{h,t}}}{U_{h,t}} \frac{1}{dt} E_t^{\mathbb{P}^h} \left[\frac{dW_{h,t}}{W_{h,t}} \right] + \frac{U_{h,x_t}}{U_{h,t}} E_t^{\mathbb{P}^h} \left[\frac{dx_t}{dt} \right] \\ &+ \frac{1}{2} \frac{W_{h,t}^2 U_{h,W_{h,t} W_{h,t}}}{U_{h,t}} \frac{1}{dt} E_t^{\mathbb{P}^h} \left[\left(\frac{dW_{h,t}}{W_{h,t}} \right)^2 \right] + \frac{W_{h,t} U_{W_{h,t} x_t}}{U_{h,t}} \frac{1}{dt} E_t^{\mathbb{P}^h} \left[\frac{dW_{h,t}}{W_{h,t}} dx_t \right] + \frac{1}{2} \frac{U_{h,x_t x_t}}{U_{h,t}} E_t^{\mathbb{P}^h} \left[\frac{(dx_t)^2}{dt} \right] \\ &- \frac{1}{2}\gamma_h \left(\left(\frac{W_{h,t} U_{h,W_{h,t}}}{U_{h,t}} \right)^2 \frac{1}{dt} E_t^{\mathbb{P}^h} \left[\left(\frac{dW_{h,t}}{W_{h,t}} \right)^2 \right] + 2 \frac{W_{h,t} U_{h,W_{h,t}}}{U_{h,t}} \frac{U_{h,x_t}}{U_{h,t}} \frac{1}{dt} E_t^{\mathbb{P}^h} \left[\frac{dW_{h,t}}{W_{h,t}} dx_t \right] + \frac{1}{2} \left(\frac{U_{h,x_t}}{U_{h,t}} \right)^2 E_t^{\mathbb{P}^h} \left[\frac{(dx_t)^2}{dt} \right] \right) \end{aligned}$$

[12] Their numerical solution is available upon request.

$$- \frac{1}{\gamma_h} \sum_{n=1}^{2} \frac{1+b_{hn}}{b_{hn}} D^{KL}[\mathbb{P}|\mathbb{P}^{hn}].$$

With the Ansatz

$$U_{h,t} = g_h(x_t) W_{h,t},$$

the above HJB reduces to

$$\begin{aligned}
0 = &\sup_{C_{h,t}} \delta u_\psi \left(\frac{C_{h,t}/W_{h,t}}{g_h(x_t)} \right) \\
&+ \sup_{\boldsymbol{\omega}_{h,t}} \inf_{\{\nu_{hn,t}\}_{n \in A_P}} \sup_{\{\nu_{hn,t}\}_{n \in A_O}} \frac{1}{dt} E_t^{\mathbb{P}^h} \left[\frac{dW_{h,t}}{W_{h,t}} \right] - \frac{1}{2} \gamma_h \frac{1}{dt} E_t^{\mathbb{P}^h} \left[\left(\frac{dW_{h,t}}{W_{h,t}} \right)^2 \right] + (1-\gamma_h) \frac{g_h'(x_t)}{g_h(x_t)} \frac{1}{dt} E_t^{\mathbb{P}^h} \left[\frac{dW_{h,t}}{W_{h,t}} dx_t \right] \\
&- \frac{1}{\gamma_h} \sum_{n=1}^{2} \frac{1+b_{hn}}{b_{hn}} D^{KL}[\mathbb{P}|\mathbb{P}^{hn}] + \frac{g_h'(x_t)}{g_h(x_t)} E_t^{\mathbb{P}^h} \left[\frac{dx_t}{dt} \right] + \frac{1}{2} \left[\frac{g_h''(x_t)}{g_h(x_t)} - \gamma_h \left(\frac{g_h'(x_t)}{g_h(x_t)} \right)^2 \right] E_t^{\mathbb{P}^h} \left[\frac{(dx_t)^2}{dt} \right].
\end{aligned} \quad \text{(B13)}$$

The controlled state variable is the investor's wealth W_h, while the evolution of the exogenous state variable x is taken as given. The investor's joint belief-portfolio choice problem is therefore

$$\begin{aligned}
&\sup_{\boldsymbol{\omega}_{h,t}} \inf_{\{\nu_{hn,t}\}_{n \in A_P}} \sup_{\{\nu_{hn,t}\}_{n \in A_O}} \frac{1}{dt} E_t^{\mathbb{P}^h} \left[\frac{dW_{h,t}}{W_{h,t}} \right] - \frac{1}{2} \gamma_h \frac{1}{dt} E_t^{\mathbb{P}^h} \left[\left(\frac{dW_{h,t}}{W_{h,t}} \right)^2 \right] \\
&+ (1-\gamma_h) \frac{g_h'(x_t)}{g_h(x_t)} \frac{1}{dt} E_t^{\mathbb{P}^h} \left[\frac{dW_{h,t}}{W_{h,t}} dx_t \right] - \frac{1}{\gamma_h} \sum_{n=1}^{2} \frac{1+b_{hn}}{b_{hn}} D^{KL}[\mathbb{P}|\mathbb{P}^{hn}].
\end{aligned}$$

When $\gamma_h = 1$, this reduces to the following linear-quadratic problem

$$\sup_{\boldsymbol{\omega}_{h,t}} \inf_{\{\nu_{hn,t}\}_{n \in A_P}} \sup_{\{\nu_{hn,t}\}_{n \in A_O}} \frac{1}{dt} E_t^{\mathbb{P}^h} \left[\frac{dW_{h,t}}{W_{h,t}} \right] - \frac{1}{2} \frac{1}{dt} E_t^{\mathbb{P}^h} \left[\left(\frac{dW_{h,t}}{W_{h,t}} \right)^2 \right] - \sum_{n=1}^{2} \frac{1+b_{hn}}{b_{hn}} D^{KL}[\mathbb{P}|\mathbb{P}^{hn}],$$

which is equivalent to (14) with $\gamma_h = 1$. Using the optimal beliefs and portfolios given in Proposition 3.2, we see that

$$\begin{aligned}
&\sup_{\boldsymbol{\omega}_{h,t}} \inf_{\{\nu_{hn,t}\}_{n \in A_P}} \sup_{\{\nu_{hn,t}\}_{n \in A_O}} \frac{1}{dt} E_t^{\mathbb{P}^h} \left[\frac{dW_{h,t}}{W_{h,t}} \right] - \frac{1}{2} \frac{1}{dt} E_t^{\mathbb{P}^h} \left[\left(\frac{dW_{h,t}}{W_{h,t}} \right)^2 \right] - \sum_{n=1}^{2} \frac{1+b_{hn}}{b_{hn}} D^{KL}[\mathbb{P}|\mathbb{P}^{hn}] \\
&= i_t + \frac{1}{2} \sum_{n=1}^{2} \left(\frac{\alpha_n - i_t}{\sigma_n} \right)^2 (1+b_{hn})^2 - \frac{1}{2} \sum_{n=1}^{2} \left(\frac{\alpha_n - i_t}{\sigma_n} \right)^2 b_{hn}^2 - \frac{C_{h,t}}{W_{h,t}} \\
&= i_t + \frac{1}{2} \sum_{n=1}^{2} \left(\frac{\alpha_n - i_t}{\sigma_n} \right)^2 (1+b_{hn}) - \frac{C_{h,t}}{W_{h,t}}.
\end{aligned}$$

The HJB (B13) then becomes

$$0 = \sup_{C_{h,t}} \delta u_\psi \left(\frac{C_{h,t}/W_{h,t}}{g_h(x_t)} \right) - \frac{C_{h,t}}{W_{h,t}} + i_t + \frac{1}{2} \sum_{n=1}^{2} \left(\frac{\alpha_n - i_t}{\sigma_n} \right)^2 (1+b_{hn})$$

$$+\frac{g_h'(x_t)}{g_h(x_t)}E_t^{\mathbb{P}^h}\left[\frac{dx_t}{dt}\right]+\frac{1}{2}\left[\frac{g_h''(x_t)}{g_h(x_t)}-\gamma_h\left(\frac{g_h'(x_t)}{g_h(x_t)}\right)^2\right]E_t^{\mathbb{P}^h}\left[\frac{(dx_t)^2}{dt}\right]. \tag{B14}$$

The FOC for consumption is

$$\delta\left(\frac{C_{h,t}/W_{h,t}}{g_h(x_t)}\right)^{-\psi}\frac{1/W_{h,t}}{g_h(x_t)}=\frac{1}{W_{h,t}},$$

which implies that the optimal consumption policy for investor h is given by

$$C_{h,t}=\delta^{\psi}g_h(x_t)^{1-\psi}W_{h,t}.$$

Substituting the above optimal policy into (B14) gives the coupled system of ordinary differential equations:

$$0=\frac{\delta^{\psi}g_h(x_t)^{1-\psi}-\psi\delta}{\psi-1}+i_t+\frac{1}{2}\sum_{n=1}^{2}\left(\frac{\alpha_n-i_t}{\sigma_n}\right)^2(1+b_{hn})$$
$$+\frac{g_h'(x_t)}{g_h(x_t)}E_t^{\mathbb{P}^h}\left[\frac{dx_t}{dt}\right]+\frac{1}{2}\left[\frac{g_h''(x_t)}{g_h(x_t)}-\left(\frac{g_h'(x_t)}{g_h(x_t)}\right)^2\right]E_t^{\mathbb{P}^h}\left[\frac{(dx_t)^2}{dt}\right],\ h\in\{1,2\}. \tag{B15}$$

By setting $N=H=2$ in the expression shown in (A11), we obtain

$$i_t=\frac{\overline{\alpha}-\overline{\sigma}^2+\overline{\sigma}^2\left[x_t\sum_{n=1}^2\frac{\alpha_nb_{1n}}{\sigma_n^2}+(1-x_t)\sum_{n=1}^2\frac{\alpha_nb_{2n}}{\sigma_n^2}\right]}{1+\overline{\sigma}^2\left[x_t\sum_{n=1}^2\frac{b_{1n}}{\sigma_n^2}+(1-x_t)\sum_{n=1}^2\frac{b_{2n}}{\sigma_n^2}\right]}$$
$$=\frac{i_{NB}+\overline{\sigma}^2\left[x_t\sum_{n=1}^2\frac{\alpha_nb_{1n}}{\sigma_n^2}+(1-x_t)\sum_{n=1}^2\frac{\alpha_nb_{2n}}{\sigma_n^2}\right]}{1+\overline{\sigma}^2\left[x_t\sum_{n=1}^2\frac{b_{1n}}{\sigma_n^2}+(1-x_t)\sum_{n=1}^2\frac{b_{2n}}{\sigma_n^2}\right]},$$

which reduces to

$$i_t=\frac{i_{NB}+b\overline{\sigma}^2\sum_{h=1}^2 w_{h,t}\left(\frac{\alpha_h}{\sigma_h^2}-\frac{\overline{\alpha}}{\overline{\sigma}^2}\right)}{1+b\overline{\sigma}^2\sum_{h=1}^2 w_{h,t}\left(\frac{1}{\sigma_h^2}-\frac{1}{\overline{\sigma}^2}\right)},$$

when $b_{11}=-b_{12}=b_{22}=-b_{21}=b$, with $x_t=w_{1,t}$.

We can now specify the boundary conditions for (B15). We have

$$i(0)=i_t|_{x_t=0}=\frac{i_{NB}+\overline{\sigma}^2\left[\sum_{n=1}^2\frac{\alpha_nb_{2n}}{\sigma_n^2}\right]}{1+\overline{\sigma}^2\left[\sum_{n=1}^2\frac{b_{2n}}{\sigma_n^2}\right]},$$

and

$$i(1)=i_t|_{x_t=1}=\frac{i_{NB}+\overline{\sigma}^2\left[\sum_{n=1}^2\frac{\alpha_nb_{1n}}{\sigma_n^2}\right]}{1+\overline{\sigma}^2\left[\sum_{n=1}^2\frac{b_{1n}}{\sigma_n^2}\right]}.$$

Now, setting $x_t=0$ and $x_t=1$ in (B15), we obtain the following boundary conditions

$$g_h(0)=\left[\frac{\psi\delta+(1-\psi)\left(i(0)+\frac{1}{2}\sum_{n=1}^2\frac{\alpha_n-i(0)}{\sigma_n^2}(1+b_{hn})\right)}{\delta^{\psi}}\right]^{\frac{1}{1-\psi}},\ h\in\{1,2\},$$

$$g_h(1)=\left[\frac{\psi\delta+(1-\psi)\left(i(1)+\frac{1}{2}\sum_{n=1}^2\frac{\alpha_n-i(1)}{\sigma_n^2}(1+b_{hn})\right)}{\delta^{\psi}}\right]^{\frac{1}{1-\psi}},\ h\in\{1,2\}.$$

C Martingales and Beliefs

To see the mechanics of how the rational-expectations probability measure can be distorted into the subjective probability measure, we focus on the special case where only the expected return on firm n is distorted, i.e. $\nu_{hn,t} \neq 0$, but $\nu_{hn',t} = 0$ for all $n' \neq n$. We denote the resulting probability measure by $\mathbb{P}^{hn}$.

We define the normalized return

$$d\epsilon_{n,t} = \frac{dR_{n,t} - \alpha_n dt}{\sigma_n \sqrt{dt}}.$$

We observe that $d\epsilon_{n,t}$ is normally distributed, such that under $\mathbb{P}$, $d\epsilon_{n,t} \sim N[0,1]$, with probability density function

$$p(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}.$$

Under $\mathbb{P}^{hn}$ only the mean of the normalized return is changed, so that $d\epsilon_{n,t} \sim N\left[\frac{\nu_{hn,t}}{\sigma_n}\sqrt{dt}, 1\right]$, with probability density function $p_{hn}(x) = p\left(x - \frac{\nu_{hn,t}}{\sigma_n}\sqrt{dt}\right)$. The ratio of the $\mathbb{P}^{hn}$ probability density function to the $\mathbb{P}$ probability density function, $p_{hn}(x)/p(x)$, simplifies to give

$$\frac{p_{hn}(x)}{p(x)} = \exp\left(-\frac{1}{2}\left(\frac{\nu_{hn,t}}{\sigma_n}\right)^2 dt + \frac{\nu_{hn,u}}{\sigma_n}\sqrt{dt}\, x\right).$$

It is then straightforward to express an expectation under $\mathbb{P}^{hn}$ as an expectation under $\mathbb{P}$:

$$E_t^{\mathbb{P}^{hn}}\left[d\epsilon_{n,t}\right] = \int_{-\infty}^{\infty} p_{hn}(x) x dx = \int_{-\infty}^{\infty} p(x)\frac{p_{hn}(x)}{p(x)} x dx = E_t\left[\frac{p_{hn}(d\epsilon_{n,t})}{p(d\epsilon_{n,t})} d\epsilon_{n,t}\right], \tag{C1}$$

where the random variable $\frac{p_{hn}(d\epsilon_{n,t})}{p(d\epsilon_{n,t})}$ distorts rational-expectations probabilities into subjective probabilities.

We can use the ratio of probability densities to define a stochastic process, M_{hn}, via

$$\frac{M_{hn,t+dt}}{M_{hn,t}} = \frac{p_{hn}(dZ_{n,t}/\sqrt{dt})}{p(dZ_{n,t}/\sqrt{dt}))} = \exp\left(-\frac{1}{2}\left(\frac{\nu_{hn,t}}{\sigma_n}\right)^2 dt + \frac{\nu_{hn,t}}{\sigma_n} dZ_{n,t}\right),$$

which is equivalent to

$$M_{hn,t} = \exp\left(-\frac{1}{2}\int_0^t \left(\frac{\nu_{hn,u}}{\sigma_n}\right)^2 du + \int_0^t \frac{\nu_{hn,u}}{\sigma_n} dZ_{n,u}\right),$$

an exponential martingale under $\mathbb{P}$.

It follows from (C1) that the exponential martingale, M_{hn}, distorts the rational-expectations probability measure, resulting in the subjective probability measure $\mathbb{P}^{hn}$,

$$E_t^{\mathbb{P}^{hn}}\left[d\epsilon_{n,t}\right] = E_t\left[\frac{M_{hn,t+dt}}{M_{hn,t}} d\epsilon_{n,t}\right],$$

which is equivalent to

$$E_t^{\mathbb{P}^{hn}}\left[dR_{n,t}\right] = E_t\left[\frac{M_{hn,t+dt}}{M_{hn,t}} dR_{n,t}\right] = E_t[dR_{n,t}] + E_t\left[\frac{dM_{hn,t}}{M_{hn,t}} dR_{n,t}\right],$$

which tell us that a distortion of probabilities (i.e., a change in the probability measure), changes the expected return (i.e., the drift). This is Girsanov's theorem (Girsanov, 1958).

References

Atmaz, A., and S. Basak, 2018, "Belief dispersion in the stock market," *Journal of Finance*, 73, 1225–1279.

Baker, M., and J. Wurgler, 2013, "Behavioral Corporate Finance: An Updated Survey," in Espen Eckbo (ed.), *Handbook of the Economics of Finance*, vol. 2, . pp. 357–424, Elsevier.

Barber, B. M., and T. Odean, 2000, "Trading Is Hazardous to Your Wealth: The Common Stock Investment Performance of Individual Investors," *Journal of Finance*, 55, 773–806.

Barberis, N., and R. H. Thaler, 2003, "A Survey of Behavioral Finance," in George M. Constantinides, Milton Harris, and René M. Stulz (ed.), *Handbook of the Economics of Finance*, Elsevier, Amsterdam.

Bhamra, H. S., R. Uppal, and J. Walden, 2021, "Psychological Distance and Deviations from Rational Expectations," Working paper.

Bracha, A., and D. J. Brown, 2012, "Affective Decision Making: A Theory of Optimism Bias," *Games and Economic Behavior*, 75, 67–80.

Brunnermeier, M. K., and J. A. Parker, 2005, "Optimal Expectations," *American Economic Review*, 95, 1092–1118.

Calvet, L. E., J. Y. Campbell, and P. Sodini, 2007, "Down or Out: Assessing the Welfare Costs of Household Investment Mistakes," *Journal of Political Economy*, 115, 707–747.

Camerer, C. F., G. Loewenstein, and D. Prelec, 2004, "Neuroeconomics: Why Economics Needs Brains," *Scandinavian Journal of Economics*, 106, 555–579.

Campbell, J. Y., 2006, "Household Finance," *Journal of Finance*, 61, 1553–1604.

Cao, H. H., B. Han, D. Hirshleifer, and H. H. Zhang, 2011, "Fear of the Unknown: Familiarity and Economic Decisions," *Review of Finance*, 15, 173–206.

Cohen, L., 2009, "Loyalty-Based Portfolio Choice," *Review of Financial Studies*, 22, 1213–1245.

Cooper, I., and E. Kaplanis, 1994, "Home Bias in Equity Portfolios, Inflation Hedging and International Capital Market Equilibrium," *Review of Financial Studies*, 7, 45–60.

Cox, J. C., J. Ingersoll, Jonathan E., and S. A. Ross, 1985, "An Intertemporal General Equilibrium Model of Asset Prices," *Econometrica*, 53, 363–84.

Duffie, J. D., and L. G. Epstein, 1992, "Stochastic Differential Utility," *Econometrica*, 60, 353–394.

Epstein, L. G., and S. Zin, 1989, "Substitution, Risk Aversion and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework," *Econometrica*, 57, 937–969.

Fama, E. F., 2012, "Unapologetic After All These Years: Eugene Fama Defends Investor Rationality and Market Efficiency," Interview with Mark Harrison: http://annual.cfainstitute.org/2012/05/14/eugene-fama-defends-investor-rationality-and-market-efficiency/.

French, K., and J. Poterba, 1990, "Japanese and U.S. Cross-Border Common Stock Investments," *Journal of the Japanese and International Economies*, 4, 476–493.

Garleânu, N., S. Panageas, and J. Yu, 2014, "Financial Entanglement: A Theory of Incomplete Integration, Leverage, Crashes, and Contagion," forthcoming in *American Economic Review.*

Gennaioli, N., and A. Shleifer, 2018, *A Crisis of Beliefs: Investor Psychology and Financial Fragility*, Princeton University Press.

Girsanov, I. V., 1958, "On Transforming a Certain Class of Stochastic Processes by Absolutely Continuous Substitution of Measures," *Theory Probab. Appl.*, 5, 285–301.

Goetzman, W. N., and A. Kumar, 2008, "Equity Portfolio Diversification," *Review of Finance*, 12, 433–463.

Grinblatt, M., and M. Keloharju, 2001, "How Distance, Language, and Culture Influence Stockholdings and Trades," *Journal of Finance*, 56, 1053–1073.

Guiso, L., M. Haliassos, and T. Jappelli, 2002, *Household Portfolios*, MIT Press, Cambridge, MA.

Guiso, L., and P. Sodini, 2013, "Household Finance: An Emerging Field," in George M. Constantinides, Milton Harris, and René M. Stulz (ed.), *Handbook of the Economics of Finance*, Elsevier, Amsterdam.

Haliassos, M., 2002, "Stockholding: Recent Lessons from Theory and Computations," Working Papers in Economics, University of Cyprus.

Hansen, L. P., and T. J. Sargent, 2007, *Robustness*, Princeton University Press.

Heyerdahl-Larsen, C., and J. Walden, 2021, "Distortions and Efficiency in Production Economies with Heterogeneous Beliefs," forthcoming in *Review of Financial Studies.*

Hirshleifer, D., 2001, "Investor Psychology and Asset Pricing," *Journal of Finance*, 56, 1533–1597.

Hirshleifer, D., 2015, "Behavioral Finance," *Annual Review of Financial Economics*, 7, 133–159.

Huberman, G., 2001, "Familiarity Breeds Investment," *Review of Financial Studies*, 14, 659–680.

Jaynes, E. T., 1957, "Information Theory and Statistical Mechanics," *The Physical Review*, 106, 620–630.

Jaynes, E. T., 1986, "Monkeys, Kangaroos, and N," in J. Justice (ed.), *Maximum Entropy and Bayesian Methods in Applied Statistics: Proceedings of the Fourth Maximum Entropy Workshop University of Calgary*, pp. 26–58. Cambridge University Press.

Kahneman, D., 2003, "Maps of Bounded Rationality: Psychology for Behavioral Economics," *American Economic Review*, 93, 1449–1475.

Keloharju, M., S. Knüpfer, and J. Linnainmaa, 2012, "Do Investors Buy What they Know? Product Market Choices and Investment Decisions," *Review of Financial Studies*, 25, 2921–2958.

Malmendier, U., and G. Tate, 2008, "Who Makes Acquisitions? CEO Overconfidence and the Market's Reaction," *Journal of Financial Economics*, 89, 20–43.

Martin, I., and D. Papadimitriou, 2021, "Sentiment and Speculation in a Market with Heterogeneous Beliefs," Working paper, London School of Economics.

Massa, M., and A. Simonov, 2006, "Hedging, Familiarity and Portfolio Choice," *Review of Financial Studies*, 19, 633–685.

Merton, R. C., 1987, "A Simple Model of Capital Market Equilibrium with Incomplete Information," *Journal of Finance*, 42, 483–510.

Mitchell, S. O., and S. Utkus, 2004, "The Role of Company Stock in Defined Contribution Plans," in S. Olivia Mitchell, and Kent Smetters (ed.), *The Pension Challenge: Risk Transfers and Retirement Income Security*, Oxford University Press.

Pessoa, L., 2008, "On the Relationship Between Emotion and Cognition," *Nature Reviews Neuroscience*, 9, 148–158.

Polkovnichenko, V., 2005, "Household Portfolio Diversification: A Case for Rank Dependent Preferences," *Review of Financial Studies*, 18, 1467–1502.

Reisberg, D., 2001, *Cognition: Exploring the Science of the Mind, 2nd*, WW Norton, New York.

Scholes, 2009, "*Efficiency and Beyond*," Interview of Myron Scholes reported in The Economist, 16 July.

Schroder, M., and C. Skiadas, 1999, "Optimal Consumption and Portfolio Selection with Stochastic Differential Utility," *Journal of Economic Theory*, 89, 68–126.

Shefrin, H., 2007, *Beyond Greed and Fear: Understanding Behavioral Finance and the Psychology of Investing*, Oxford University Press.

Shefrin, H., 2010, "Behavioralizing Finance," *Foundations and Trends in Finance*, 4, 1–184.

Shepard, R. N., 1987, "Toward a Universal Law of Generalization for Psychological Science," *Science*, 237, 1317–1323.

Shiller, R. J., 2015, *Irrational Exuberance: Revised and Expanded Third Edition*, Princeton university press.

Shleifer, A., 2000, *Inefficient Markets: An Introduction to Behavioral Finance*, Oxford University Press, Oxford.

Slovic, P., 1987, "Perception of Risk," *Science*, 236, 280–285.

Slovic, P., B. Fischhoff, and S. Lichtenstein, 1980, "Facts and Fears: Understanding Perceived Risk," in *Societal risk assessment* . pp. 181–216, Springer.

Statman, M., 2010, "What is Behavioral Finance," in A. Wood (ed.), *Behavioral Finance and Investment Management*, Research Foundation of CFA Institute.

Statman, M., 2011, *What Investors Really Want*, McGraw-Hill.

Trope, Y., and N. Liberman, 2010, "Construal-Level Theory of Psychological Distance," *Psychological Review*, 117, 440–463.

Weil, P., 1990, "Nonexpected Utility in Macroeconomics," *Quarterly Journal of Economics*, 105, 29–42.

Weinstein, N. D., 1980, "Unrealistic Optimism about Future Life Events," *Journal of Personality and Social Psychology*, 39, 806.