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Abstract

This paper studies a competitive model of insurance markets in which consumers are privately informed about their risk and risk preferences. We provide a tractable characterization of equilibria, which depend non-trivially on consumers' type distribution, a desirable feature for policy analysis. The use of consumer characteristics for risk classification is modeled as the disclosure of a public informative signal. A novel monotonicity property of signals is shown to be necessary and sufficient for their release to be welfare improving for almost all consumer types. We also study the effect of changes to the risk distribution in the population as the result of demographic changes or policy interventions. When considering the monotone likelihood ratio ordering of distributions, an increase in the risk distribution leads to lower utility for almost all consumer types. In contrast, the effect is ambiguous when considering the first order stochastic dominance ordering.

JEL Classification: G22, D82, I13, I18

Keywords: Risk Classification, Insurance Markets, Risk distribution, Multidimensional private information

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Risk Classification in Insurance Markets with Risk and Preference Heterogeneity*

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June, 2021

Abstract

This paper studies a competitive model of insurance markets in which consumers are privately informed about their risk and risk preferences. We provide a tractable characterization of equilibria, which depend non-trivially on consumers' type distribution, a desirable feature for policy analysis. The use of consumer characteristics for risk classification is modeled as the disclosure of a public informative signal. A novel monotonicity property of signals is shown to be necessary and sufficient for their release to be welfare improving for almost all consumer types. We also study the effect of changes to the risk distribution in the population as the result of demographic changes or policy interventions. When considering the monotone likelihood ratio ordering of distributions, an increase in the risk distribution leads to lower utility for almost all consumer types. In contrast, the effect is ambiguous when considering the first order stochastic dominance ordering.

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1 Introduction

Risk classification is a natural consequence of profit maximization by insurance providers in competitive markets. It consists of using individual information to predict or assess the risk level of potential insurees, which may involve both medical information (e.g, pre-existing conditions or medical history) as well as demographic characteristics (e.g., income, age or gender). As highlighted in Handel et al. (2015) (henceforth HHW), allowing for more precise risk classification has the potential of reducing the information asymmetry between consumers and firms and hence alleviate the problem of adverse selection. But at the same time it implies that consumers with different observables may be offered very different premia for the same coverage. This dispersion in prices and, as a consequence, in consumption may have adverse effects on welfare. Hence, a trade-off arises between reducing adverse selection and price dispersion. Models commonly used in the insurance literature struggle to address this issue because they either assume that agent’s preferences can be summarized by a one-dimensional variable (Rothschild and Stiglitz (1976)), leading to the prediction of equilibrium outcomes that are independent of the type distribution; or exogenous restrictions to either one or two available contracts (Akerlof (1970), HHW). We propose a parsimonious and tractable model with a rich set of contracts and consumer characteristics, allowing us to identify non-trivial effects of changes to the distribution of risk and preferences – which may be the result of demographic changes or policy interventions such as risk classification and insurance mandates – on both prices and welfare.

In our model, consumers are privately informed about both their risk level and risk preferences, the two main dimensions of heterogeneity emphasized in the empirical literature (see, for example, Cohen and Einav (2007)). More specifically, consumers have normally distributed losses and constant absolute risk aversion (following Einav et al. (2013) and Azevedo and Gottlieb (2017) – henceforth AG, among others). Consumers’ private information concerns both the expected value of the loss they face (capturing their risk level), which has a continuous distribution, and their risk aversion, which has a binary distribution. The difference between the two levels of risk aversion is a key parameter of our model, which captures the amount of preference heterogeneity.

We follow Dubey and Geanakoplos (2002), Bisin and Gottardi (2006) and, more recently, AG in assuming that consumers and firms act as price takers and have consistent beliefs regarding the types of consumers trading any contract which may be offered. In equilibrium all contracts that are traded generate zero profits.

We differ from the previous literature on risk classification (HHW, Garcia and Tsur (2021) and Finkelstein et al. (2009)) by allowing for a continuum of different coverages traded in equilibrium. In the presence of a rich set of contracts, our assumption of two-dimensional private information is instrumental to the study of risk classification, as it allows for non-trivial effects of the risk distribution on equilibrium outcomes. In the case of one-dimensional heterogeneity in risk levels considered in most of the literature, following Rothschild and Stiglitz (1976), competitive equilibria lead to full separation, which implies that the price of each contract is determined by the unique risk type choosing it and, as a consequence, only depends on the risk distribution through its support (see also Dubey and Geanakoplos (2002) and AG).¹ A direct, and implausible, implication is that risk classification may become irrelevant. For instance, if the set of possible risk levels for both men and women in the population are the same, the introduction of gender-based pricing has zero effect.

Conversely, in the presence of multiple dimensions of private information, consumers of different types may share the same willingness to pay for coverage at the margin and, as a consequence, equilibria may feature pooling. As each insurance contract is priced based on the average risk level of consumers choosing it, any change in the type distribution has a direct effect on prices since it affects the relative frequency of the different

¹The same outcome is also derived as a free-entry equilibrium in Rothschild and Stiglitz (1976), the unique pure-strategy equilibrium outcome in a game-theoretic version of that model (Farinha Luz (2017)), and as a directed search equilibrium in the presence of search frictions (Gale (1992), Guerrieri et al. (2010)). While a competitive equilibrium always exists, free-entry equilibria and pure strategy equilibria fail to exist for some risk distributions.

types purchasing the same contract. It also has an indirect effect, as price changes affect contract choices and hence which consumer types are pooled together.

Our main results are the following. First, we show that equilibria always exist where a convex set of contracts is traded and exhibit the following pattern of trades: extreme coverage levels are chosen only by one consumer type (that is, exhibit separation), while each intermediate coverage level is selected by multiple consumer types (partial pooling occurs) - see Section 3. The equilibrium is not unique, a common feature when some pooling occurs. Second, we present a method to obtain meaningful comparative static results in spite of the presence of equilibrium multiplicity and obtain clean analytical expressions for the impact of changes in the type distribution on equilibrium prices and allocations, by focusing on the case of small preference heterogeneity (Section 4). Third, we investigate the effect of risk classification by considering the disclosure of a public signal that is informative about consumers' risk, and identify a novel property of signals, referred to as monotonicity, that is necessary and sufficient for its disclosure to be beneficial for almost all types of consumers (Section 5). Finally, we study the effects on equilibrium allocations and welfare of more general changes in the distribution of risk in the market (Section 6).

More specifically, in Section 3 we show that the set of coverages traded in equilibrium can be partitioned into three subsets. Contracts with high or with low levels of coverage feature separation, with each contract in these two regions purchased by a single type, respectively, with high risk and high risk aversion or with low risk and low risk aversion. Contracts with intermediate levels of coverage are purchased instead by two types (an outcome referred to as discrete pooling), which have different levels of risk but have the same marginal willingness to pay for coverage. Finally, at the boundary between each of the two separating regions and the discrete pooling region there is (at most) one coverage level featuring continuous pooling, i.e., a contract that is chosen by a positive mass of types. We show that a continuum of equilibria of this kind exists.

In Section 4, we focus on the case where preference heterogeneity is small in contrast to risk heterogeneity. We characterize how equilibrium prices and allocation vary as a function of the level of preference heterogeneity and show that a well defined (Taylor) approximation of equilibria exists and exhibits important properties: it is independent of the equilibrium selection used and depends on the type distribution in a tractable way. In the remainder of the paper, we focus on the case of small, yet positive and fixed, preference heterogeneity. Our approximation results are then used to obtain novel comparative statics results that are robust to equilibrium selection.

Section 5 studies the price and welfare effects of the disclosure of a public signal that provides partial information to firms regarding consumers' risk level. For example, if the signal corresponds to certain demographic characteristic of consumers, the disclosure of this information to firms allows them to learn something about the consumer's risk if both are correlated in the population. The availability of this signal leads to market segmentation, since firms will treat consumers differently according to their public signal realization. We analyze the welfare impact of the signal release from an interim perspective, by comparing the utility of any consumer's risk-and-preference type in equilibrium without and with public signal, taking for the latter the expectation across signal realizations.

Our main result identifies a property of signals, monotonicity, that is necessary and sufficient for its release to be interim welfare improving, for any type distribution. Monotonicity is expressed in terms of the Kullback-Leibler (KL) divergence measure, or relative entropy. Consider a fixed risk-type A, and compare the distribution of signals generated from A-risk consumers with that coming from consumers with a different risk-type B. Monotonicity requires that the signal distribution from both risk-types become more distinct, in the sense of the KL measure, as risk-type B becomes more distant from A. For binary signals, monotonicity is equivalent to the requirement that the probability of each signal realization, conditional on risk, be monotonic on risk. For richer

signals, monotonicity is shown to be weaker than commonly assumed signal properties.

Intuitively, the disclosure of an informative signal has a direct statistical effect on prices. Keeping the equilibrium allocation fixed, the disclosure of a signal leads to a mean-preserving spread in the price of each coverage, as contracts are priced based on the expected risk level of a pool conditional on all information available to firms. However, in our model with a rich space of contracts the disclosure of a signal has an indirect equilibrium effect by changing the set of types that are pooled in each contract. We show that, if the signal is monotonic, this indirect effect amplifies price reductions and dampens price increases resulting from different signal realizations. As a consequence, signal disclosure leads to an expected price *reduction*. This expected price reduction is shown to generate an expected utility gain for almost all consumer types, a surprising result given the presence of risk aversion in our model.

Section 6 studies the comparative statics properties of equilibria with respect to changes in the type distribution in the population. We focus on changes to the type distribution that increase risk in the population while maintaining the same support (that is, with larger mass on riskier types). Note that such changes would have no effect in the standard one-dimensional model. This is a relevant question for policies such as mandates and risk classification. A common justification for insurance mandates is that they improve the risk distribution in the market by increasing participation, especially among low-risk consumers.² In practice, risk classification is usually based on demographic characteristics such as gender, age or geographic location; and regulatory limits on the use of this information by firms is defended as a way to benefit consumers with characteristics associated with higher risks. Translated to our model, one is interested in comparing the equilibrium utility of a consumer with a particular signal realization with signal disclosure, when prices are based on the corresponding conditional type distribution, and without signal disclosure, with prices based on the prior distribution. This is in contrast with the welfare analysis in Section 5, where we evaluate signal disclosure from an interim expected perspective.

The answer to these questions depends on the ordering used to compare group riskiness. First consider a first order stochastic (FOSD) increase in the risk distribution, i.e., an increase of the mass of consumers in the right tail. We show, through an example, that a FOSD increase in the risk distribution may be beneficial for a positive mass of consumers. The intuition is that the endogenous partial screening of types means that equilibrium prices are affected by changes on the risk distribution within the pool of consumers purchasing each coverage, and not on the overall population. We then show that, considering the stronger ordering given by the monotone likelihood ratio property (MLRP), the relative frequency of riskier types within each pool consuming the same coverage level does increase with an increase in riskiness. As a consequence, a MLRP increase in the risk distribution leads to an increase of prices for almost all coverage levels and, as a consequence, almost all consumer types are worse off.

Finally, we also study changes in the distribution of risk preferences in the population and show that an increase in the share of high-risk-aversion consumers leads to lower equilibrium prices for almost all contracts and higher utility levels for almost all types. Note that both changes in the distribution of types in the population increasing either risk or risk aversion can be interpreted as positive demand shocks (for insurance) and our results show that the effects of such a shock depend critically on its source: a positive demand shock induced by an (MLRP) increase in the risk distribution leads to higher prices, while a positive demand shock induced by an increase in the share of high-risk-aversion consumers leads to lower prices.

²For example, the U.S. code currently states that a health insurance mandate, or requirement, as proposed by the 2010 Affordable Care Act " ...will minimize this adverse selection and broaden the health insurance risk pool to include healthy individuals, which will lower health insurance premiums" (42 U.S.C.A. § 18091(a)(2)(I); see also Parmet (2011)). Our analysis shows that this argument may be incorrect in the presence of screening by firms.

Related literature

Our paper is related to the empirical and theoretical literatures on risk classification and market outcomes with multidimensional private information.

HHW study risk classification in health insurance markets. They consider a model where insurance firms compete for the provision of two exogenously given coverage levels and a mandate is in place, motivated by health care exchanges under the U.S. Affordable Care Act. In this environment, a unique equilibrium is shown to exist. Using data from a single large employer, HHW estimate the joint distribution of demographic characteristics, preferences and risk levels in the population and perform numerical simulations of their model to study different types of risk classification. The paper focuses on risk classification based on age and health status.

The use of age information in pricing does not fully remove the informational asymmetries in this market; and the authors find that all consumers choose the low coverage contract in equilibrium – the same outcome obtained when no individual information is used. Hence, age-based pricing has no effect on coverage levels and only leads to price dispersion, with younger consumers paying lower premia relative to older ones. In contrast, HHW assume that the use of health information in pricing can completely remove the information asymmetry between firms and consumers. As a consequence it leads to an increase in coverage, but also generates large variations in prices, for any given level of coverage. This price dispersion decreases ex-ante welfare and HHW find, based on their estimates for the parameters of the model, that this negative effect dominates the positive one due to higher coverage.

Our analysis differ from theirs in three main ways. First, our model allows for richer screening of types with a continuum of possible coverage levels. As a consequence, more information about consumers is revealed by their coverage choices, even in the absence of risk classification. Second, we restrict attention to signals that are only partially informative about consumers' risk levels. For example, this includes the use of age, but not health status, in pricing. Finally, we focus on interim welfare, and not ex-ante, differentiating consumers who differ in terms of their risk level or their risk aversion, making our Pareto improvement results stronger. Furthermore, the specification of the environment we consider allows us to characterize analytically the effects of risk classification and of changes in the risk-distribution on equilibrium prices and welfare, instead of relying on numerical simulations.

In a related contribution, Finkelstein et al. (2009) study the effects of gender-based pricing in the UK annuities market. Their analysis is based on a binary types model, allows for a large space of contracts, and considers two possible allocations: the equilibrium outcome proposed by Miyazaki (1977); Wilson (1977); Spence (1978), with screening of types based on coverage, and complete pooling with actuarially fair prices. The use of gender in pricing has a pure price effect in the presence of full pooling; but affects both prices and coverage in the presence of screening. Both the type distribution and firms' costs are calibrated based on retirement annuities data and the effect of a gender-based pricing ban is numerically quantified. A ban on gender pricing is shown to hurt women in the population, but this effect is stronger with complete pooling and weaker in the presence of screening. The reason is that the presence of screening leads to endogenous separation of types, and hence less cross-subsidies across genders, even in the absence of gender pricing.

On the empirical side, the presence of multidimensional consumer heterogeneity has been established in the literature on structural estimation of insurance demand. See for instance Cohen and Einav (2007) and Einav et al. (2013). While these papers provide rich estimates of the sources of consumer heterogeneity and drivers of demand, they do not perform model-based counter-factual analysis that fully incorporate supply-side responses to interventions.

Garcia and Tsur (2021) study the problem of designing an informative signal in a competitive insurance market with private risk information, a single exogenous coverage and no mandate. For any signal realization,

they consider the most efficient Akerlof (1970) equilibrium, based on the updated distribution of types. The optimal signal maximizes the ex-ante expected utility of consumers. In order to reduce the price variation consumers are subject to, depending on their risk level, the optimal signal structure is shown to provide some information to firms but also generate large risk heterogeneity within the set of types grouped together through the same signal realization.

A few papers have studied the role of categorical price discrimination in insurance markets focusing on interim welfare notion similar to ours, taking expectations of payoffs across different categories, for any fixed risk type. Crocker and Snow (1986) study how the introduction of a risk-informative signal affects the incentive-constrained Pareto frontier and shows that a signal leads to a strictly larger payoff set, with its introduction leading to constrained Pareto efficient outcomes that are not feasible in the absence of such a signal. Rothschild (2011) considers multiple alternative equilibrium concepts proposed in the literature and shows that firms' access to an informative signal, combined with a government intervention, leads to an interim Pareto improvement. In contrast, we do not allow for government interventions, beyond the release of an exogenously given signal.

The notion of market outcome we consider is related to the broad competitive screening literature which followed the works of Rothschild and Stiglitz (1976); Miyazaki (1977); Riley (1979). We use a Walrasian market equilibrium notion with participants acting as price-takers, following Dubey and Geanakoplos (2002); Bisin and Gottardi (2006); Guerrieri et al. (2010); AG. These papers show that, in the standard setting where private information concerns a one-dimensional parameter, competitive equilibria always exist and the equilibrium allocation features perfect separation, coinciding with Riley (1979)'s outcome. Guerrieri et al. (2010) show that the same outcome also obtains in the presence of search frictions, as a result of a competitive search equilibrium.

The analysis of multidimensional private information in competitive environments has been tackled by a few notable papers. AG established the existence of competitive equilibria in general environments that allow for multidimensional consumer heterogeneity, but no characterization is provided beyond one-dimensional environments. Results in this respect can be found in Guerrieri and Shimer (2018) and Chang (2018),³ using a directed search approach. A common feature of the environments considered in both papers is that the willingness to trade of informed agents depends on a one-dimensional sufficient statistic (given, respectively, by the product and the difference of the two dimensions of the agents' private information in Guerrieri and Shimer (2018) and Chang (2018)). This implies that types can be partitioned based on their willingness to trade, and all types in the same element of the partition are pooled together in equilibrium regardless of the type distribution.⁴ This property is not present in our model: the set of types sharing the same marginal willingness to pay in equilibrium is not exogenously given and varies with the type distribution. This is a general property of insurance models, since consumers' risk aversion only has a small effect on their willingness to pay for additional coverage when their risk exposure is small. Wambach (2000) studies no-entry equilibria, as in Rothschild and Stiglitz (1976), in a two-dimensional model with binary risk and wealth heterogeneity and shows that equilibria may feature pooling, as in our model, as well as positive profits, which does not occur in our model.

2 Model

A continuum of consumers (potential insurees) face income uncertainty due to the possibility of negative shocks. More specifically, suppose that consumers have income W and can suffer a loss \tilde{l} distributed according to the

³See also Williams (2021).

⁴Chang (2018)'s model features pools with a positive mass of types, similar to the continuous pooling in our model, and includes consumers with different willingness to trade. However, her pooling result follows from a monotonicity issue: the average benefit (for the uninformed side) of trading with informed agents is non-monotonic in their willingness to trade. Guerrieri and Shimer (2018), on the other hand, use distributional assumptions that rule out such non-monotonicity and their equilibrium only features zero-mass pools with a continuum of types, which are similar to the discrete pooling in our model.

normal distribution $N(\mu, 1)$. Consumers can purchase insurance contracts which are characterized by a pair $(x, p) \in [0, 1] \times \mathbb{R}_+$, with $x \in (0, 1)$ denoting the insurance coverage (i.e., the fraction of the loss reimbursed) and $p \geq 0$ the premium paid.⁵ Consumer's preferences are described by a constant absolute risk aversion (CARA) utility function with parameter $\rho > 0$. Hence, the expected utility of a consumer choosing contract (x, p) is given by

$$\begin{aligned} v(x, p; \mu, \rho) &\equiv \mathbb{E} \left\{ -\exp \left[-\rho \left(W - (1-x)\tilde{l} - p \right) \right] \right\} \\ &= -\exp \left\{ -\rho \left[W - (1-x)\mu - \frac{\rho}{2}(1-x)^2 - p \right] \right\}, \end{aligned} \quad (1)$$

where the second equality is a consequence of the exponential structure of both the CARA utility and the normal distribution, which gives us a quadratic expression for the consumers' certainty equivalent (see Einav et al. (2013) and AG for applications of the normal-CARA specification).

We assume that consumers are privately informed about both their risk level μ and their level of risk aversion ρ .⁶ The set of possible risk levels in the population is an interval $[\mu_L, \mu_H]$ while, for tractability, the set of possible risk aversion values is given by two values, $\rho_l \equiv \rho_0 - \delta/2$ and $\rho_h \equiv \rho_0 + \delta/2$ for some $\delta > 0$ describing the level of heterogeneity of risk preference in the population. Hence, the type space is $\Theta \equiv [\mu_L, \mu_H] \times \{l, h\}$ and a generic type is denoted by $\theta = (\mu, i)$.

Types are distributed over Θ according to probability distribution described by a pair of strictly positive and twice continuously differentiable functions $\phi = (\phi_l, \phi_h)$ on $[\mu_L, \mu_H]$, i.e., for any measurable set $A \subset [\mu_L, \mu_H]$ and $i \in \{l, h\}$

$$\mathbb{P}[\theta \in A \times \{i\}] = \int_{\mu_L}^{\mu_H} \mathbf{1}_{\{\tilde{\mu} \in A\}} \phi_i(\tilde{\mu}) d\tilde{\mu}.$$

Using expression (1), consumer preferences over contracts (x, p) can be represented by the following quasi-linear utility function:

$$u(x, \theta) - p, \quad (2)$$

where $u(x, \theta) = x\mu - \frac{\rho_i}{2}(1-x)^2$. For now, we work directly with this linear-quadratic utility specification.

The insurees' marginal rate of substitution between coverage and price, which determines their willingness to pay for coverage, is

$$u_x(x, \theta) = \mu + \rho_i(1-x), \quad (3)$$

where u_x stands for the partial derivative notation.

From expression (3), we see that a consumer's high willingness to pay for coverage can be due to a high risk level (high μ) or to a high risk aversion level (high ρ). Importantly, the degree to which risk aversion affects one's willingness to pay depends on the level of coverage. For example, risk aversion has a very small effect on one's marginal willingness to pay for coverage if the starting point is close to full coverage. An important role in the analysis which follows is played by the set of consumers' types who exhibit the same willingness to pay for a given coverage level x . We see from (3) that this set depends on x and hence will be determined in equilibrium together with the allocation of contracts.⁷

⁵The space of possible insurance contracts allows then for reimbursement policies that are linear in losses. This specification is quite rich and appears fairly natural in applications. In the context of principal-agent moral hazard problem where average loss is under control of the consumer, Holmstrom and Milgrom (1987) show that linear contracts are indeed optimal. It is beyond the scope of this paper to show that the linearity contracts is without loss in our adverse selection competitive setting.

⁶The model can also allow for private information in the variance of the loss, in addition to its mean. If $\tilde{l} \sim N(\mu, \sqrt{\sigma})$, it can be shown that the utility of a consumer with contract (x, p) only depends on variance σ and risk aversion ρ through their product $\sigma\rho$. Hence, all the analysis that follows readily extends to this case, provided we redefine the parameter ρ as representing the product of risk aversion and risk variance.

⁷As mentioned in the introduction, this feature distinguishes our model from Guerrieri and Shimer (2018) and Chang (2018),

Firms are risk neutral and their expected profit from selling contract (x, p) to a consumer of type $\theta = (\mu, i)$ is given by the price minus the cost of coverage provision for the risk component of the consumer's type:

$$p - c(x, \theta),$$

where $c(x, \theta) = \mu x$. This means that risk aversion ρ is a private-value component of the private information while risk μ is a common-value component of the private information of an individual as it affects the expected costs of firms providing insurance. The number of firms is irrelevant, as they are price takers and have constant returns to scale in insurance provision.

We consider a Walrasian competitive equilibrium notion, meaning that all consumers and firms take prices for all contracts as given. A price function $p : [0, 1] \rightarrow \mathbb{R}_+$ specifies the cost of each contract.⁸ An allocation is a measurable function $t : \Theta \rightarrow [0, 1]$. We denote the joint distribution over $[0, 1] \times \Theta$ induced by the prior type distribution and an allocation t as \mathbb{P}_t . Firms have beliefs regarding the set of types that would consume each possible coverage level if offered, which are described by $P(\cdot | x) \in \Delta(\Theta)$, for $x \in [0, 1]$; and we denote the expected risk level of consumers if coverage level $x \in [0, 1]$ is chosen, using belief system P , by $\mathbb{E}_P[\tilde{\mu} | x]$.

The first equilibrium restriction is that the allocation represents the optimal coverage level for consumers, given prices $p(\cdot)$, i.e., that $t(\theta)$ solves the following maximization:

$$U(\theta) \equiv \max_{x \in [0, 1]} u(x, \theta) - p(x), \text{ for all } \theta \in \Theta. \quad (4)$$

We assume a continuum of insurance firms with constant returns to scale in coverage. Hence, the profit maximization problem of each firm, given belief system P , has a solution only if no contract generates strictly positive profits, and any contract that is traded in equilibrium makes zero expected profits, i.e.,

$$0 = p(x) - x\mathbb{E}_P[\tilde{\mu} | x] \geq p(\hat{x}) - x\mathbb{E}_P[\tilde{\mu} | \hat{x}], \quad (5)$$

for all $x \in t(\Theta)$ and all $\hat{x} \in [0, 1]$.

A competitive equilibrium is given by a price function, an allocation and a belief system that satisfy consumer and firm optimality, belief consistency and additional restrictions for non-traded contracts. In equilibrium, the belief system should satisfy Bayes' rule, whenever possible. For contracts that are not traded in equilibrium, prices are determined by the highest willingness to pay for such a contract, if offered; and firms beliefs regarding types choosing such contracts should be consistent with this reasoning. This restriction follows the ones proposed by Dubey and Geanakoplos (2002); Bisin and Gottardi (2006); Guerrieri et al. (2010), and is in the tradition of the belief refinement literature (see also AG).

Definition 1. A triple of price, allocation and belief (p, t, P) is an equilibrium if it satisfies:

- (1) firm's optimality: condition (5) holds;
- (2) consumer's optimality: $t(\theta)$ solves (4);
- (3) non-traded contracts' pricing: if $p(x) > 0$, then

$$p(x) = \max \{u(x, \theta) - U(\theta) \mid \theta \in \Theta\}, \quad (6)$$

where $U(\theta)$ is defined in (4);

where the set of types exhibiting the same willingness to pay does not depend on their level of trade. The reason is that in those papers consumers' willingness to trade depends on a one-dimensional sufficient statistic for their type, while this is not true in our environment.

⁸With some abuse of notation, we use p to denote both the price function and the price of a given contract.

(4) belief consistency: $P\left(\cdot \mid t\left(\tilde{\mu}, \tilde{i}\right)\right) = \mathbb{P}_t\left(\cdot \mid t\left(\tilde{\mu}, \tilde{i}\right)\right)$ with probability one; and, for $x \in [0, 1]$, the support of $P(\cdot \mid x)$ is contained in

$$\theta^+(x) \equiv \{\theta \in \Theta \mid U(\theta) = u(x, \theta) - p(x)\}.$$

Conditions (1) and (2) are fairly standard. Condition (3) states that any contract with a non-zero price has its price determined by consumers willing to pay the highest price for that coverage. This implies that for each non-traded coverage level with a strictly positive price in equilibrium, there must be at least one type $\theta \in \Theta$ that is indifferent between choosing this coverage and their equilibrium coverage $t(\theta)$. Condition (4) requires that firms' beliefs be consistent with this reasoning, i.e., they believe that the only consumers potentially purchasing a contract are the ones most willing to pay for it. Conditions (3) and (4) are critical, as they rule out trivial equilibria in which any subset of contracts remains non-traded and their prices are set at an arbitrarily high level. If all types most willing to trade a particular contract have costs strictly below their willingness to pay (per unit), this would generate an opportunity for strictly positive profits by firms.

3 Equilibrium characterization

In this section, we characterize equilibrium allocation and prices, starting from the benchmark case where consumers only differ by their risk level, that is where private information is one-dimensional ($\delta = 0$), and extending it subsequently to the environment where there is also preference heterogeneity ($\delta > 0$).

One-dimensional types

Suppose all consumers have the same risk aversion, i.e., $\delta = 0$. In this case, a unique equilibrium exists, the equilibrium is fully separating, with efficiency at the top, and only depends on the support of the risk distribution (the proof is readily obtained by suitably adapting the arguments in Dubey and Geanakoplos (2002), AG). Let us refer to the map associating to any traded coverage level $x \in [0, 1]$ the risk of the unique type purchasing x in equilibrium as the type assignment function $m_0(x)$.

We refer to the derivative of equilibrium objects with respect to coverage x by using the dot notation, as in $\dot{p}(x)$, to describe the marginal price function. Definition 1 requires that contracts make zero profits and that consumers choose their coverage level optimally:

$$\frac{p(x)}{x} = m_0(x) \text{ and } \dot{p}(x) = u_x(x, m_0(x), i) = m_0(x) + \rho_0(1 - x). \quad (7)$$

These two equations define a simple differential system which has a unique solution satisfying $m_0(1) = \mu_H$ (efficiency at the top), which is given by:

$$m_0(x) = \rho_0(1 - x + \ln x) + \mu_H, \quad (8)$$

for all $x \in [x_L, 1]$, where x_L is lowest traded coverage defined by $m_0(x_L) = \mu_L$. The equilibrium allocation $t_0(\cdot)$ is the inverse of the map $m_0(\cdot)$ and the price function is given by $p_0(x) = xm_0(x)$. These objects will be instrumental for our equilibrium analysis of the case $\delta > 0$.

Two-dimensional types

We proceed now to characterize equilibria when $\delta > 0$, that is, when types are two-dimensional. In this case we will show that all equilibria feature partial pooling. Hence, the allocation $t : \Theta \rightarrow [0, 1]$ has a generalized inverse

represented by a correspondence that gives us the set of types choosing a particular coverage. It may be either empty (non-traded contract) or include multiple types (contract with pooling). We call this correspondence the type assignment, and it can be written as the composition of two mappings $m_i^t : [0, 1] \rightrightarrows [\mu_L, \mu_H]$, for $i \in \{l, h\}$. For each $x \in [0, 1]$, $m_i^t(x)$ represents the risk levels of the types purchasing x and having risk aversion ρ_i , i.e., it is given by $\{\mu \in [\mu_L, \mu_H] \mid t(\mu, i) = x\}$. Alternatively, the set of types choosing coverage x is $[m_l^t(x) \times \{l\}] \cup [m_h^t(x) \times \{h\}]$. Whenever this set is a singleton, we also refer to it as a type assignment function. We say that the correspondence $m_i(\cdot)$ is non-decreasing if, for any $x' < x$ with $x', x \in [0, 1]$ and $m_i(x), m_i(x') \neq \emptyset$, we have that $\sup m_i(x') \leq \inf m_i(x)$. Additionally, we say that $m_l(x) \geq m_h(x)$ if $\sup m_l(x) \geq \sup m_h(x)$ and $\inf m_l(x) \geq \inf m_h(x)$. When the allocation we are referring to is well understood, the dependence on t is omitted. The following lemma provides some important properties of the equilibrium allocation and price functions.

Lemma 1. *If (m_l, m_h) are equilibrium type assignments, then:*

- (i) $m_i(\cdot)$ is non-decreasing;
- (ii) for any x such that $m_l(x), m_h(x) \neq \emptyset$, $m_l(x) \geq m_h(x)$;
- (iii) p is an increasing Lipschitz function.

Lemma 1 (i) and (ii) hold since preferences satisfy the single-crossing property (SCP) on each dimension of the consumer's type (either μ or i), holding the other dimension fixed. Conditional on consumer's level of risk aversion (risk level), an increase in the risk level (risk aversion) increases consumer's willingness to pay for coverage. However, the two-dimensional type space does not satisfy globally the single-crossing property, i.e., one can find types (μ, i) and (μ', i') with indifference curves that cross twice. Lemma 1 (iii) follows directly from the equi-Lipschitz property of $u(\cdot, \mu, i)$.

No gap equilibria We restrict attention to equilibria in which the set of traded contracts is convex and of the form $[x_L, 1]$, and we refer to them as no gap equilibria. In Subsection 3.1 (Proposition 2) we show that such equilibria exist when preference heterogeneity is small. In the benchmark one-dimensional model, with $\delta = 0$, only no-gap equilibria exist.

Our equilibrium characterization result relies on an adverse selection property of the model parameters. Let us define the function $e(\cdot)$ as the expected risk level of the set of types with same marginal willingness to pay q , at a given coverage x :

$$e(q, x) \equiv q - (1 - x) \left\{ \rho_0 + \frac{\delta \phi_h (q - (1 - x) \rho_h) - \phi_l (q - (1 - x) \rho_l)}{2 \phi_h (q - (1 - x) \rho_h) + \phi_l (q - (1 - x) \rho_l)} \right\}, \quad (9)$$

with domain $E \equiv \{(q, x) \in \mathbb{R}_+ \times [0, 1] \mid q \in [\mu_L + (1 - x) \rho_h, \mu_H + (1 - x) \rho_l]\}$. Indeed, for $(q, x) \in E$, the types with marginal willingness to pay q at coverage x are $(q - (1 - x) \rho_i, i)$, for $i = l, h$. The value $e(\cdot)$ represents the weighted average of risk among these two types, with weight

$$\frac{\phi_l (q - (1 - x) \rho_l)}{\phi_h (q - (1 - x) \rho_h) + \phi_l (q - (1 - x) \rho_l)} \in (0, 1)$$

applied to the type with low risk aversion. We can now state the key assumption used in our characterization.

Assumption 1. The function $e(q, x)$ is strictly increasing in q , for all $x \in [0, 1]$.

This is a joint restriction on distribution and other model parameters. It is easy to see, for example, that it holds if types are distributed uniformly. We show, in Appendix A, that this assumption holds for $\delta > 0$ small enough, regardless of the mappings (ϕ_l, ϕ_h) .

Lemma 2. *There exists $\bar{\delta} > 0$ such that Assumption 3 holds if $\delta \in (0, \bar{\delta})$.*

The next result provides a characterization of equilibria under small preference heterogeneity. The set of traded coverages can be divided into three open sub-intervals and two coverage levels in the transition between them. The highest (lowest) levels of coverage traded in equilibrium feature separation, i.e., each of these contracts is purchased by a single type with high (low) risk-aversion and levels of risk on the right (left) tail of the risk distribution. Coverage levels in the middle open interval feature instead partial pooling as they are purchased by two types: one with low risk aversion and higher risk level, and another with high risk aversion and lower risk. We refer to this type of pooling discrete pooling. Finally, the two intermediate coverage levels that determine the boundary between these intervals are purchased by a continuum of types with positive measure, which we refer to as continuous pooling.

Proposition 1. *Suppose that Assumption 1 is satisfied. In any (no gap) equilibrium there are $0 < x_L < x_d \leq x_u < 1$ such that:*

- (i) (separation) *there are two separating intervals $[x_L, x_d)$ and $(x_u, 1]$, with $x_d \leq x_u$; each $x \in (x_L, x_d)$ (resp. $x \in (x_u, 1)$) is chosen by only one type with low (resp. high) risk aversion;*
- (ii) (continuous pooling) *there are non-degenerate intervals of risk types choosing coverage x_d and x_u ;*
- (iii) (discrete pooling) *for each $x \in (x_d, x_u)$ there are exactly one low and one high risk aversion types choosing coverage x .*

Hence, to characterize a no gap equilibrium we need to determine the regions in $[0, 1]$ where separation, continuous and discrete pooling occur, as well as the level of coverage chosen by each type in any of these regions. As specified in Definition 1, firms' beliefs are determined by Bayes' rule for all contracts $x \in [x_L, 1]$. For all contracts $x < x_L$ that are not traded in equilibrium beliefs put all mass on the type most willing to trade x (i.e., type (μ_L, l)) and the price $p(x)$, if positive, is such that it makes this type indifferent between any such contract and coverage x_L .⁹

Separation

For each contract x in a separating region, there exists a unique pair of risk μ and risk aversion level ρ assigned to it. The characterization of equilibrium prices and trades in this region is analogous to the one-dimensional case. That is, the consumers' and firms' optimality conditions are analogous to (7). Once a boundary condition of the derived differential system is determined, we can then solve for the type assignment and price functions in each of these regions. While the initial condition in the top separating region is the same as in the one-dimensional case, the initial condition in the bottom separating region depends on the equilibrium type assignment and prices in the other regions.

Continuous pooling

Lemma 1 and Proposition 1 imply that, in this region, coverage x is chosen by all types with risk aversion ρ_i and risk level in an interval $I_i \subset [\mu_L, \mu_H]$, for each $i \in \{l, h\}$. The zero profit condition (5) implies that the unit price of coverage x is determined by the average risk in the pool of consumers choosing x . While Lemma 1 shows that $p(\cdot)$ is continuous, it is not differentiable at x . Indeed, the marginal utility of coverage for all types choosing x must lie in $[\dot{p}(x^-), \dot{p}(x^+)]$,¹⁰ or otherwise some types would want to deviate from x . Hence, if I_i has

⁹Equilibria with gaps in the set of traded-contract have a similar structure to no gap equilibria characterized in Proposition 1, i.e., the same pattern of pooling and separation of types. The only difference is that a set of non-traded coverage levels may exist in the transition between separation and pooling regions.

¹⁰We use the following convention $f(x^-)$ and $f(x^+)$ for the left and right hand side limit of any function $f(x)$, respectively, whenever it exists.

a positive measure, for either $i = l$ or h , the interval $[\dot{p}(x^-), \dot{p}(x^+)]$ also has positive length. Additionally, if the coverage level x is in the interior of the set of traded contracts, the left- and right-hand side limits of the marginal prices must match exactly the minimum and the maximum willingness to pay for coverage among types pooled at coverage x .¹¹ Formally, this means:

$$\frac{p(x)}{x} = \mathbb{E} \left[\tilde{\mu} | (\tilde{\mu}, \tilde{i}) \in \bigcup_{i \in \{l, h\}} I_i \times \{i\} \right]$$

$$[\dot{p}(x^-), \dot{p}(x^+)] = \{\mu + \rho_i(1-x) | (\mu, i) \in \bigcup_{i \in \{l, h\}} I_i \times \{i\}\}.$$

Alternatively, continuous pooling is necessary to insure the continuity of prices, which must hold from Lemma 1 in the transition between discrete pooling regions, involving types with both high and low risk-aversion, to regions of separation, which only involve types with a single risk-aversion level.

Discrete pooling

For every coverage x in this region, the type assignments $m_l(\cdot)$ and $m_h(\cdot)$ are singletons. This means that contract (x, p) is selected by only two types: a low risk aversion type $(m_l(x), l)$ and a high risk aversion type $(m_h(x), h)$. In this case, the posterior probability of low-risk-aversion type $(m_l(x), l)$, conditional on choice x , is given by $w(x) \equiv \mathbb{P}(\tilde{i} = l | t(\tilde{\mu}, \tilde{i}) = x)$. The equilibrium requirements of zero profits (condition 1) and coverage optimality (condition 2) become:

$$\frac{p(x)}{x} = w(x)m_l(x) + (1-w(x))m_h(x) \text{ and } \dot{p}(x) = m_i(x) + \rho_i(1-x), \text{ for } i = l, h. \quad (10)$$

The calculation of the posterior beliefs at coverage x in the interior of a discrete pooling region is non-trivial, since the equilibrium distribution of coverages is absolutely continuous in a neighborhood of x (the mass of consumers purchasing contract x is zero). For this reason, the posterior $w(x)$ is determined by two components: the prior distribution and the slopes of the type assignment functions at x . The first element is intuitive: a higher mass of consumers with low, versus high, risk aversion – $\phi_l(\cdot)/\phi_h(\cdot)$ – leads to a higher posterior $w(\cdot)$. However, the slope of function $m_i(\cdot)$ is also important, since large slope of $m_l(x)$ means that the mass of consumers with low risk aversion is locally concentrated around x , which is connected with higher posterior $w(x)$. The following lemma formalizes this intuition.

Lemma 3. *In a discrete pooling region, the belief consistency condition 3 is equivalent to*

$$\frac{w(x)}{1-w(x)} = \frac{\phi_l(m_l(x))\dot{m}_l(x)}{\phi_h(m_h(x))\dot{m}_h(x)}. \quad (11)$$

3.1 Equilibrium existence and multiplicity

In this section we tackle the question of existence. If preference heterogeneity is small, a continuum of no gap equilibria exists. Proposition 1 implies that all such equilibria have similar structures, and only differ in the exact coverage levels that determine the transition between different types of pooling or separation.

Proposition 2. *For sufficiently small $\delta > 0$, a continuum of no gap equilibria exist.*

¹¹Notice that, for some $i \in \{l, h\}$, there must be a sequence of types $(\mu_n, i)_n$ and coverage levels $(x_n)_n$ such that x_n is optimal for type (μ_n, i) , $x_n \searrow x$ and $\mu \equiv \lim \mu_n \in I_i$. Hence, $[p(x_n) - p(x)](x_n - x)^{-1} \leq [u(x_n, \mu_n, i) - u(x, \mu_n, i)](x_n - x)^{-1}$. Taking the limit $n \rightarrow \infty$ we have that $u_x(x, \mu, i) \geq \dot{p}(x^+)$, this inequality must hold as an equality. An analogous argument holds for coverages below x .

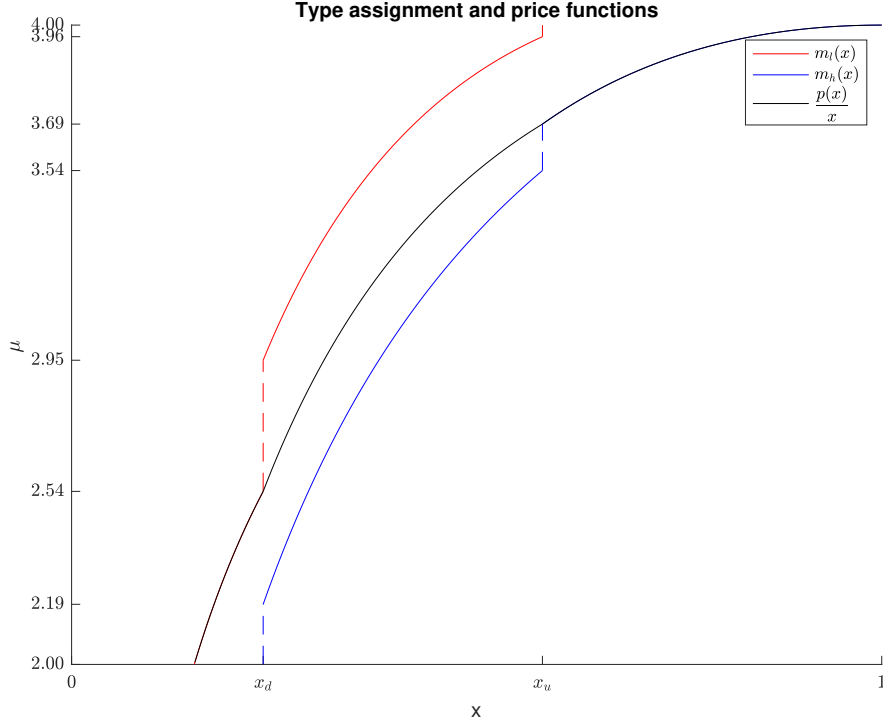


Figure 1: Equilibrium for the case where risk and risk aversion are independently distributed with $\mu \sim U[2, 4]$ and $\mathbb{P}(i = l) = \frac{1}{2}$.

Proposition 2 is proven by construction. The key building blocks of equilibrium construction are the differential equations that govern prices and allocations within discrete pooling and separating regions, which are concatenated according to the structure described in Proposition 1. The boundary condition at $x = 1$ is determined by separation and efficiency at the top. Proceeding from top to bottom, each transition point corresponds to a continuous pooling point, and determines the boundary condition for the subsequent region of traded contracts. As mentioned before, maintaining continuity of prices (a necessary condition, from Lemma 1) in the transition between a separating region, where a single type chooses each coverage, and a discrete pooling region, where types with both risk aversion levels are pooled together, requires the presence of continuous pooling. As a consequence, the equilibrium price function has a kink at these transition points. The multiplicity arises from a degree of freedom in the construction, which is represented by the transition point between the top separating region and the discrete pooling region, which is denoted by x_u .

Figure 1 provides a graphical representation of an equilibrium. The red and blue lines represent the type assignments $m_l(\cdot)$ and $m_h(\cdot)$, respectively. The black line represents the price function, scaled per unit of coverage, which is determined by a convex combination of the two type assignments using the posterior beliefs. The coverage levels above x_u (below x_d) feature separation, with these contracts being purchased by a single type with high (low) risk aversion. In this case the relevant type assignment function is equal to the price per unit of coverage. The middle region between x_d and x_u features discrete pooling, and the type assignment functions $m_i(\cdot)$, $i \in \{l, h\}$, are both singletons. The transition points x_d and x_u feature continuous pooling, which is represented by a discontinuity in the type assignment function. The set of pooled risks is represented by the vertical lines at the discontinuities.

4 Approximation

While the special case of one-dimensional heterogeneity reviewed in Section 3 is quite tractable and allows us to obtain closed form solutions for equilibrium objects, this is not the case when types are two-dimensional, even under strong parametric assumptions on type distributions. In addition, the equilibrium multiplicity established in Proposition 2 makes a meaningful comparative static exercise difficult.¹² Our approach to analyze the properties of equilibria in the two-dimensional model is to look at the one-dimensional environment as a limit case, and study the behavior of equilibria in its neighborhood. For an arbitrary equilibrium selection, we find a second order Taylor approximation for equilibrium prices and show that it exhibits two key properties: its coefficients are independent of the equilibrium selection used, and depend on the type distribution in a tractable and explicit way. We then use this tractable approximation to perform comparative statics exercises with respect to the type distribution, which are robust to equilibrium selection. The advantage of such approach is the possibility to use special properties of the one-dimensional case (uniqueness and distribution independence), while still highlighting novel results of the two-dimensional model, especially the role of the type distribution. In what follows, we make the dependence of equilibrium objects on parameter δ explicit in our notation, e.g., prices are denoted as $p(x; \delta)$, for $x \in [0, 1]$.

4.1 The heuristic approximation

We start by presenting a heuristic derivation by assuming that equilibrium prices are continuously differentiable in the preference heterogeneity parameter, for $\delta > 0$ sufficiently small. This assumption is restrictive as the equilibrium multiplicity result in Proposition 2 implies that there are infinitely many non-differentiable equilibrium selections. In Subsection 4.2 we show that any equilibrium selection is differentiable with respect to δ , at $\delta = 0$. Moreover, we will later show (Corollary 1) that the approximation coefficients obtained here are also valid for equilibrium selections that are non-differentiable for $\delta > 0$. In practical terms, the heuristic approximation consists in taking directly the total derivative of the equilibrium objects with respect to δ and evaluate them at $\delta = 0$.

Another implicit assumption of the heuristic approach is that as δ goes to zero the discrete pooling region overcomes the whole space, i.e., the separating and continuous pooling regions vanish to zero at a smaller rate than δ . Again, in Subsection 4.2 we show that this assumption is without loss of generality. Hence, substituting the first equation of (10) into the second one, after averaging out over both risk aversion levels, we obtain the following ordinary differential equation on prices:

$$\dot{p}(x; \delta) = \frac{p(x, \delta)}{x} + \bar{p}(x; \delta)(1 - x),$$

where the average risk aversion of consumers choosing coverage x is defined as

$$\bar{p}(x; \delta) \equiv w(x; \delta)(\rho_0 - \delta/2) + (1 - w(x; \delta))(\rho_0 + \delta/2).$$

The solution of this ODE with the final condition $p(1; \delta) = \mu_H$ is¹³

$$\frac{p(x; \delta)}{x} = \mu_H - \int_x^1 \left(\frac{1}{z} - 1 \right) \bar{p}(z; \delta) dz.$$

Notice that the equilibrium price at coverage x depends on a cumulative weighted average of the values of risk

¹²In AG, for example, the equilibrium defined and numerically studied may not be unique beyond one-dimensional applications.

¹³Notice that it corresponds to the one-dimensional equilibrium price (8) when $\delta = 0$, i.e., $\bar{p} = \rho_0$.

aversion for all coverage levels above x .

We denote the first- and second-order derivatives of prices with respect to δ as $p_\delta(x; \delta)$ and $p_{\delta\delta}(x; \delta)$, respectively, and use the same convention for other equilibrium objects. Direct differentiation then implies

$$\frac{p_\delta(x; \delta)}{x} = \int_x^1 \left(\frac{1}{z} - 1 \right) \left[w(z; \delta) - \frac{1}{2} + w_\delta(z; \delta) \delta \right] dz.$$

Let us consider the case of independently distributed risk and risk aversion, i.e.,

$$\phi_l(\mu) = \omega_0 \phi(\mu) \quad \text{and} \quad \phi_h(\mu) = (1 - \omega_0) \phi(\mu),$$

for a density function $\phi : [\mu_L, \mu_H] \mapsto \mathbb{R}_{++}$ and some $\omega_0 \in (0, 1)$. Therefore, considering the limit $\delta = 0$, we get an expression for the first-order approximation of prices:

$$\frac{p_\delta(x; 0)}{x} = \int_x^1 \left(\frac{1}{z} - 1 \right) \left(\omega_0 - \frac{1}{2} \right) dz,$$

where, under the assumed independence between risk and risk aversion, $w(x, 0) = \omega_0$ is the prior probability of low risk aversion. Hence, the first-order price approximation depends on the relative distribution of risk aversion. A higher relative share of low-risk-aversion consumers leads to higher prices: consumers with high risk aversion demand higher coverage even when having low risk level, which drives down the average cost/price of such contracts.

Now taking the second-order derivative at $\delta = 0$ we have

$$\frac{p_{\delta\delta}(x; \delta)}{x} = 2 \int_x^1 \left(\frac{1}{z} - 1 \right) w_\delta(z; 0) dz. \tag{12}$$

To compute $w_\delta(z, 0)$ we can differentiate equation (11), which leads to (using the notation $\partial_\delta \equiv \frac{\partial}{\partial \delta}$)

$$\begin{aligned} \frac{w_\delta(x; 0)}{\omega_0(1 - \omega_0)} &= \partial_\delta \left(\log \frac{w(x; \delta)}{1 - w(x; \delta)} \right) \\ &= \frac{\dot{\phi}(m_0(x))}{\phi(m_0(x))} [\partial_\delta (m_l(x; 0) - m_h(x; 0))] + \frac{\partial_\delta (\dot{m}_l(x; 0) - \dot{m}_h(x; 0))}{\dot{m}_0(x)}. \end{aligned}$$

Using then the fact that $m_l(x; \delta) - m_h(x; \delta) = \delta(1 - x)$ (from equation (10)), we get

$$w_\delta(x; \delta) = \omega_0(1 - \omega_0) \left[\frac{\dot{\phi}(m_0(x))}{\phi(m_0(x))} (1 - x) - \frac{1}{\dot{m}_0(x)} \right].$$

Finally, since $\dot{m}_0(x) = \rho_0 \frac{1-x}{x}$, equation (12) can be rewritten as

$$\frac{p_{\delta\delta}(x; \delta)}{x} = 2\rho_0^{-1} \omega_0(1 - \omega_0) \left[\int_{m_0(x)}^{\mu_H} \frac{\dot{\phi}(\mu)}{\phi(\mu)} (1 - t_0(\mu)) d\mu - (1 - x) \right].$$

The second-order approximation term allows us to study how the risk distribution affects equilibrium prices. With a positive yet small amount of preference heterogeneity, each coverage level is consumed by types with similar, yet distinct, risk levels. The relative frequency of the higher-risk-lower-aversion types to the lower-risk-higher-aversion ones is represented by the rate of growth of the risk density, $\frac{\dot{\phi}(\cdot)}{\phi(\cdot)}$. A higher density rate of growth means that, for any given risk level, there are relatively more types in the population with slightly higher risks.

As these two “similar” types are pooled into a single contract, the overall effect on prices is positive. This second-order term is zero in the extreme cases $\omega_0 = 1$ or $\omega_0 = 0$, as the relative share of different risk types is irrelevant in the one-dimensional case.

The results for the independent case discussed here are formally established in Subsection 4.3 (see Corollary 1). First Subsection 4.2 provides the formal derivation of the price approximation terms, relaxing the requirement of continuous differentiability and the assumption of independence of risk and risk preferences. A reader interested purely in the comparative statics exercises obtained from these formulas can skip the next subsections and go directly to Sections 5 and 6, which show how to use these approximation results to evaluate the welfare impact of changes in signal disclosures and in risk distribution.

4.2 Approximation results

In this section, we extend and formalize the results in Subsection 4.1 to the case of non-independent types and arbitrary (potentially non-differentiable) equilibrium selections. Moreover, our approximation results also imply that the separating and continuous pooling regions vanish. In order to study the equilibrium prices, it is necessary to study the limiting behavior of the posterior $w(x)$. In the limit economy without preference heterogeneity, all consumers purchasing the same coverage level share the same risk level. When preference heterogeneity is positive but small, any coverage with discrete pooling features cross-subsidization between types with almost identical level of risk, i.e., the limiting share of low-risk-preference types within a pool is determined by the ratio:

$$\omega_0(x) \equiv \frac{\phi_l(m_0(x))}{\phi_l(m_0(x)) + \phi_h(m_0(x))}, \quad (13)$$

for any $x \in [x_L, 1]$. In the limit $\delta = 0$, all consumers consuming the same coverage level x have a risk level equal to $m_0(x)$. And the limiting share of consumers with low risk aversion,¹⁴ among the ones with risk level $m_0(x)$ is exactly $\omega_0(x)$.

Proposition 2 guarantees the existence of equilibria, for all $\delta > 0$ sufficiently small. We now consider an arbitrary equilibrium selection and define the following limits, if they exist,

$$p_\delta(x) \equiv \lim_{\delta \rightarrow 0} \frac{p(x; \delta) - p(x; 0)}{\delta}$$

and

$$p_{\delta\delta}(x) \equiv 2 \lim_{\delta \rightarrow 0} \frac{\frac{p(x; \delta) - p(x; 0)}{\delta} - p_\delta(x)}{\delta}.$$

Proposition 3. *Consider an arbitrary equilibrium selection and $\delta > 0$ sufficiently small. For all $x \in (x_L, 1)$, the price and posterior exhibit the following limiting behavior:*

(a) *Convergence: the function $p(x; \cdot)$ is continuous at zero,*

$$\lim_{\delta \rightarrow 0} \frac{p(x; \delta)}{x} = m_0(x) \quad \text{and} \quad \lim_{\delta \rightarrow 0} w(x; \delta) = \omega_0(x);$$

(b) *Differentiability: the limit $p_\delta(x)$ exists and is given by*

$$\frac{p_\delta(x)}{x} = \int_x^1 \left(\omega_0(z) - \frac{1}{2} \right) \left(\frac{1}{z} - 1 \right) dz; \quad (14)$$

¹⁴Although in the limit there is just one risk-preference type, the interpretation is that for each risk type there are exactly two risk-preference types with the same risk aversion.

(c) *Second-order differentiability: the limit $p_{\delta\delta}(x)$ exists and is given by*

$$\frac{p_{\delta\delta}(x)}{x} = 2 \int_x^1 w_\delta(z) \left(\frac{1}{z} - 1 \right) dz, \quad (15)$$

where

$$\begin{aligned} w_\delta(x) = & \frac{\dot{\omega}_0(x)}{x\dot{m}_0(x)} p_\delta(x) - \frac{x\omega_0(x)[1-\omega_0(x)]}{\rho_0(1-x)} \\ & + (1-x)[1-\omega_0(x)]\omega_0(x) \left\{ [1-\omega_0(x)] \frac{\dot{\phi}_l(m_0(x))}{\phi_l(m_0(x))} + \omega_0(x) \frac{\dot{\phi}_h(m_0(x))}{\phi_h(m_0(x))} \right\}. \end{aligned} \quad (16)$$

Additionally, all convergence results hold uniformly on any compact subset of $(x_L, 1)$.

Proof. The proof is a combination of the results in Lemmas 17, 18, 19, 22 and 24 in Appendix C. \square

Proposition 3 allows us to use the following approximation of equilibrium prices in order to perform comparative static exercises, for any $x \in (x_L, 1)$:

$$p(x; \delta) = p(x; 0) + \delta p_\delta(x) + \frac{\delta^2}{2} p_{\delta\delta}(x) + o_i(\mu; \delta^2), \quad (17)$$

where $\delta^{-2}|o_i(\mu; \delta)|$ converges to zero uniformly on M , for any compact set $M \subset (\mu_L, \mu_H)$.

The proof of Proposition 3 shows that the approximation of the one-dimensional equilibrium is determined by the approximation on the discrete pooling region of the equilibrium selection, i.e., as δ approaches zero the separating and continuous pooling regions of the equilibrium disappear as $\delta \rightarrow 0$. In the heuristic derivation, this property was taken for granted. The first part of Proposition 3 shows that prices converge pointwise to its one-dimensional counterpart. This occurs as, when the difference of risk-aversion among consumers becomes smaller, the heterogeneity among buyers choosing the same coverage level disappears. Hence, in the limit, the equilibrium allocation and prices fully separate consumers in terms of their risk level. It also determines the limit of the endogenous posterior $w(\cdot)$.

The second part of Proposition 3 is slightly more subtle and shows that the sign of the price first-order approximation does depend on the type distribution through the difference between $\omega_0(x)$ and $\frac{1}{2}$. The reason is that with $\delta > 0$ small, except for very low or very high coverages, all other consumption levels occur in the discrete pooling region where each contract is purchased not by a single risk-type, but by a pair of types: one with low risk and high risk aversion, $(m_h(x), \rho_h)$, and one with high risk and low risk aversion, $(m_l(x), \rho_l)$ (remember that equation (10) implies $m_l(x) > m_h(x)$). Hence, the effect on equilibrium prices of some heterogeneity in risk levels depends on which type is more prevalent. If there are more low risk-aversion consumers, which exhibit a higher risk than the high risk aversion consumers with whom they pool (i.e., $\omega_0(x) > \frac{1}{2}$), prices are increased; if otherwise (i.e., $\omega_0(x) < \frac{1}{2}$), the effect is reversed.

The third part of Proposition 3 is even more subtle and captures how the posterior probability assigned to the low risk averse consumers, exhibiting higher risk ($w(x; \delta)$) differs from its limit $\omega_0(x)$. If $w_\delta(x) > 0$, the introduction of small preference heterogeneity $\delta > 0$ implies that the share of low-risk-aversion-high-risk types consuming each contract is larger, which drives prices up. The way in which the weights vary with the degree of preference heterogeneity δ depends finely on the type distribution. Of special interest to us is the dependence on the risk distribution solely through its rate of growth, computed among consumers with each level of risk aversion. As discussed in Subsection 4.1, this rate of growth represents the frequency of consumers that have higher risk level within their pool, relative to consumers with similar and yet lower risk levels.

For the remainder of the paper, we will use this approximation result to study equilibrium outcomes and interventions of our model, and so we always consider sufficiently small preference heterogeneity, such that an arbitrary equilibrium selection is guaranteed from Proposition 2.

4.3 Independent distribution

Some of our comparative static results will focus on the case of independence between the two dimensions of heterogeneity, so we include an explicit presentation of the approximations coefficients for this case, which coincide with the ones presented and discussed in Subsection 4.1.

Corollary 1. *If risk and risk aversion are independently distributed, then*

$$p_{\delta}(x) = x \left(\omega_0 - \frac{1}{2} \right) (x - 1 + \log x),$$

and

$$\frac{p_{\delta\delta}(x)}{x} = 2\omega_0(1 - \omega_0) \left[\int_x^1 (1 - z) \left(\frac{1}{z} - 1 \right) \frac{\dot{\phi}(m_0(z))}{\phi(m_0(z))} dz - \frac{(1 - x)}{\rho_0} \right].$$

Proof. By direct substitution in Proposition 3. □

5 Signal disclosure

A central policy question in insurance market regulation is the extent to which companies should be allowed to discriminate consumers based on observable characteristics. For example, demographic characteristics are useful for firms when pricing insurance contracts as they are correlated with consumers' risk, even if they are not direct determinants of risk.

We analyze this question by considering the disclosure of an informative signal, or observable characteristic, which can be used in pricing. If firms observe the realization of this signal, their offers can be based on the distribution of types conditional on the signal realization and, as a consequence, equilibrium prices also depend on the signal realization. We assume that the disclosure of the signal realization is an exogenous intervention, and not the result of individual consumers or firms' decisions. If a signal is disclosed, its realization connected with each consumer is observed by the consumer as well as firms. If a signal is not disclosed, firms cannot price based on this information.

Our welfare criterion is interim in the sense that, for any given consumer type, we consider the expected utility in the competitive equilibrium with public disclosure of the signal, taking the expectation across all possible signal realizations. This is the adequate measure of consumer's well-being if consumers do not observe the signal realization prior to the intervention. For example, if the signal is the result of an imprecise health test or, as in HHW, if one evaluates interventions from the point of view of a young consumer's expected lifetime utility, and considers the potential use of individual characteristics that evolve stochastically over time. In many relevant applications, however, such as gender-based pricing, however, the realization of the signals is known prior to the intervention. In those cases, the expected utility describes the aggregate (utilitarian) welfare of all consumers (in the example, all men and women) sharing the same risk and preference characteristics. We then say that a signal is interim Pareto improving if, for almost all types, the expected utility in the equilibrium with signal disclosure is higher than in the equilibrium with no signal. Any Pareto improvement according to this interim criterion implies also an improvement according to an ex-ante criterion. Proposition 4 is our main result and provides necessary and sufficient conditions for a signal to be interim Pareto improving.

We make two main assumptions regarding signals. First, we focus on pure risk signals, i.e., whose realization is independent of risk preferences, conditional on risk. Second, it is assumed that the signal is correlated to consumers' unobservable risk level but contains no additional predictive power on their final losses \tilde{l} relative to their risk type μ , which is known by consumers. In other words, the signal is directly informative for firms, but not for consumers and affects them only through its effect on prices. Since the signal has no direct value to consumers, any positive welfare result (such as Proposition 4) for information disclosure is more surprising.

We denote the (finite) set of possible signal realizations as S . A signal is a function $\pi(\cdot | \cdot) : S \times [\mu_L, \mu_H] \mapsto [0, 1]$, such that $\pi(\cdot | \mu) \in \Delta(S)$ for any $\mu \in [\mu_L, \mu_H]$. That is, for each $\mu \in [\mu_L, \mu_H]$, the mapping $\pi(\cdot | \mu)$ denotes the probability distribution over signal realizations, conditional on risk level μ . We assume that $\pi(\cdot | \cdot)$ is strictly positive, and that, for any $s \in S$, the mapping $\mu \mapsto \pi(s | \mu)$ is continuously differentiable and $\frac{\partial}{\partial \mu} \pi(s | \mu) \neq 0$ for almost all (Lebesgue) $\mu \in [\mu_L, \mu_H]$.¹⁵ The distribution of types conditional on signal s is then given by

$$\phi_i(\mu | s) = \frac{\phi_i(\mu) \pi(s | \mu)}{\Pi(s)}, \quad (18)$$

for $i = l, h$, where $\Pi(s)$ denotes the ex-ante probability of signal realization s :

$$\Pi(s) \equiv \int \pi(s | \mu') [\phi_l(\mu') + \phi_h(\mu')] d\mu'.$$

We use superscript s to refer to equilibrium variables under distribution $(\phi_l(\cdot | s), \phi_h(\cdot | s))$ (e.g., $p^s(x; \delta)$), i.e., when public signal s is received; and superscript 0 when referring to equilibrium variables when no signal is received, i.e., under prior distribution (ϕ_l, ϕ_h) (e.g., $p^0(x; \delta)$). The corresponding equilibrium payoffs of a consumer with type $(\mu, i) \in [\mu_L, \mu_H] \times \{l, h\}$ for $s \in S$ with preference heterogeneity $\delta > 0$, is then

$$V_i^s(\mu; \delta) \equiv v(t_i^s(\mu; \delta), p^s(t_i^s(\mu; \delta); \delta); \mu, \rho_i(\delta)), \quad (19)$$

where $v(\cdot)$ is defined by (1), and $p^s(\cdot)$ and $t^s(\cdot)$ represent equilibrium price and allocation, respectively. Similarly when the superscript is 0.

A signal is interim Pareto improving for a given type distribution (ϕ_l, ϕ_h) and equilibrium selection if, for every $\epsilon > 0$, there exists a $\bar{\delta} > 0$ such that

$$\mathbb{P} \left[(\mu, i) \mid \sum_{s \in S} \pi_i(s | \mu) V_i^s(\mu; \delta) > V_i^0(\mu; \delta) \right] > 1 - \epsilon,$$

for all $\delta < \bar{\delta}$.

The definition of interim Pareto improvement is tailored to our application in three main ways. First, the impact of the signal on each considered type (μ, i) is evaluated based on their expected utility gain, integrating over all signal realizations. Second, it explicitly considers the case of arbitrarily small preference heterogeneity $\delta > 0$, as the tractable signal analysis relies on the price approximation results obtained in Section 4. Finally, our Taylor approximation for equilibrium prices only applies for coverage levels involving discrete pooling, which includes almost all traded coverage levels as $\delta \rightarrow 0$. The ϵ qualifier accounts for the sets of traded contracts not featuring discrete pooling, which have arbitrarily small probability mass.

A signal is interim Pareto improving if it is interim Pareto improving for any prior type distribution with full support and any equilibrium selection. Our main result obtains necessary and sufficient conditions under which a signal is guaranteed to benefit almost all types, regardless of the underlying original type distribution

¹⁵This assumption rules out knife-edge cases where $\pi(s | \cdot)$ is constant in an open interval contained in $[\mu_L, \mu_H]$.

and equilibrium selection.

In Subsection 5.1 we show that the welfare impact of signal disclosure follows directly from its expected effect on prices. Subsection 5.2 uses this result to provide necessary and sufficient conditions for a signal to be interim Pareto improving.

5.1 Welfare and price effects

The disclosure of a signal affects consumers by changing the equilibrium price function they face. The main result in this subsection (Lemma 4) shows that the impact of signal disclosure on the expected utility of each type of consumer is determined by the expected price change generated by the signal, averaging over signal realizations and holding the consumer's coverage level constant. This result is surprising since consumers' preferences, represented in (1), feature risk aversion and the disclosure of a signal introduces a new source of uncertainty as equilibrium prices vary with the signal.

Using the approximation (17), we can approximate the expected price effect of signal disclosure on the coverage originally chosen by type $(\mu, i) \in [\mu_L, \mu_H] \times \{l, h\}$ as

$$\sum_{s \in S} \mu(s | \mu) p^s(t_i^0(\mu; \delta); \delta) - p^0(t_i^0(\mu; \delta); \delta) = \frac{\delta^2}{2} \Delta E[p(t_0(\mu))] + o_i(\delta^2; \mu),$$

where

$$\Delta E[p(x)] \equiv \sum_{s \in S} \pi(s | m_0(x)) [p_{\delta\delta}^s(x) - p_{\delta\delta}^0(x)], \quad (20)$$

we have used the fact that the limit, at $\delta = 0$, and the first-order approximation of prices do not vary with the signal realization $s \in S$. The independence of the first-order derivative $p_{\delta}^s(\cdot)$ on s follows from the fact that the signal contains no information on risk preferences, relative to risk type μ . In other words, the ratio $\omega_0^s(x)$, as defined in (13), does not depend on signal realization s . The following lemma connects the expected price approximation to an approximation of the expected utility gain of consumers in equilibrium.

Lemma 4. *The welfare effect of signal disclosure on a consumer with type $(\mu, i) \in (\mu_L, \mu_H) \times \{l, h\}$ satisfies*

$$\sum_{s \in S} \pi(s | \mu) V_i^s(\mu; \delta) - V_i^0(\mu; \delta) = \frac{\delta^2}{2} \frac{\partial v}{\partial p}(t_0(\mu), p_0(t_0(\mu)), \mu, \rho_0) \Delta E[p(t_0(\mu))] + o_i(\delta^2; \mu), \quad (21)$$

where $\delta^{-2} |o_i(\delta^2; \cdot)|$ converges uniformly to 0 on M , for any compact set $M \subset (\mu_L, \mu_H)$.

Proof. See Appendix C. □

Based on equilibrium utility expression (19), the impact of a signal on a particular type can be decomposed via its effect on prices, holding the coverage originally chosen by that type fixed (price effect), and the change in coverage chosen by the consumer as a response to changes in the equilibrium price function (allocation effect). Intuitively, the proof of Lemma 4 relies on two observations: (i) the price effect of a signal is the dominant one, and (ii) the price effect of a signal on payoffs is determined solely by the expected price change generated by the signal, holding the coverage level constant.

While the signal considered may be very informative about risk levels, its impact on equilibrium prices, and also on allocation, is arbitrarily small for small δ , since the amount of risk heterogeneity within each pool disappears in the limit $\delta = 0$. Hence, observation (i) follows from an envelope-type argument, which implies that the price impact of a shock is of higher order of magnitude than the allocation effect, since consumers are optimally choosing their coverage amount.

Observation (ii) is more subtle. The disclosure of the signal introduces a randomization in the level of consumers' consumption because of the induced randomization in prices. Since equilibrium outcomes in the case of no preference heterogeneity ($\delta = 0$) do not depend on the disclosure of the signal, the impact of signal disclosure on equilibrium prices is *small* for small preference heterogeneity δ , regardless of the informational content of the signal. And since the signal is assumed not to provide additional payoff relevant information to consumers, the realization of the signal is *independent* of their risky losses, conditional on their type. Observation (ii) follows since a consumer with a risky consumption level evaluates small lotteries with realizations independent from their initial consumption according to their expected value.¹⁶

5.2 Welfare improving signals

The main result of this subsection builds on Proposition 3 to show that signals are interim Pareto improving if, and only if, they satisfy an informativeness condition referred to as monotonicity. The information content of a signal regarding consumers' risk profile is determined by how the signal distribution varies with changes in the consumers' risk level. Intuitively, monotonicity means that consumers with more distant risk levels must generate more "distinct" distributions over the possible signal realizations. Hence, defining monotonicity requires a measure of how different any two distributions over signal realizations are. For any two distributions $\pi, \tilde{\pi} \in \Delta(S)$, the Kullback–Leibler (KL) divergence of $\tilde{\pi}$ from π is defined as

$$D_{KL}(\pi || \tilde{\pi}) \equiv \sum_s \pi(s) \log \left(\frac{\pi(s)}{\tilde{\pi}(s)} \right).$$

This measure is also referred to as relative entropy and is non-commutative measure of distribution discrepancy.¹⁷ It is always non-negative and equals to zero if and only if $\pi = \tilde{\pi}$. We say that a signal is monotonic if the gap between the signal distributions, as measured by the KL divergence measure, changes monotonically with the risk level of consumers.

Definition 2. A signal $(\pi(s|\cdot))_{s \in S}$ is monotonic if, for any $\mu \in [\mu_L, \mu_H]$,

$$D_{KL}(\pi(\cdot | \mu) || \pi(\cdot | \tilde{\mu}))$$

is strictly increasing in $\tilde{\mu}$ for $\tilde{\mu} > \mu$.

Consider three ordered risk levels $\mu_1, \mu_2, \mu_3 \in [\mu_L, \mu_H]$ with $\mu_1 < \mu_2 < \mu_3$. As the pair of risk levels (μ_1, μ_3) is more distinct relative to pair (μ_1, μ_2) , monotonicity requires that the pair (μ_1, μ_3) should generate signal distributions that "diverge" more from each other, when compared to the pair (μ_1, μ_2) .

Monotonicity is a generalization of common restrictions on signals present in contract theory, such as monotone likelihood ratio ordering, as we will show in Subsection 5.3.¹⁸ Figure 2 illustrates a non-monotonic binary signal, where one realization of the signal has a higher likelihood for extreme risk levels while the other has higher likelihood for intermediate risk levels. We now state our main result, which relates monotonicity to interim welfare improvements.

¹⁶For any C^1 Bernoulli $v : \mathbb{R} \mapsto \mathbb{R}$, independent and bounded random variables x and y , and $\varepsilon > 0$, $\mathbb{E}[v(x + \varepsilon y)] = \mathbb{E}[v(x)] + \varepsilon \mathbb{E}[v'(x)] \mathbb{E}[y] + o(\varepsilon)$.

¹⁷Information theory (see, for instance, Kraft (1949) and McMillan (1956)) defines this measure as the expected number of extra bits that would be required to code the information if one were to use $\tilde{\pi}$ instead of π . In economics, it is commonly used as a way to model the cognitive cost of processing information in the literature on rational inattention, being referred to as mutual information (Sims (2003)).

¹⁸In an online appendix, we present an alternative and equivalent definition of monotonicity in terms of a statistical property related to the impact of signal realizations on the expected risk of small risk pools.

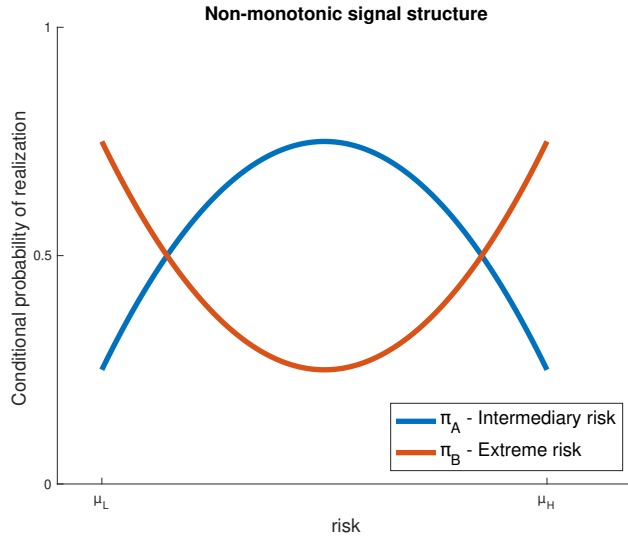


Figure 2: Non-monotonic binary signal.

Proposition 4. *A signal is interim Pareto improving if and only if it is monotonic.*

Proof. From Lemma 4, we know that the expected utility gain of a given type from signal disclosure depends on the expected price change of coverage x , which can be determined by (20) and satisfies, from (15),

$$\Delta E [p(x)] = 2 \int_x^1 \sum_{s \in S} \pi(s | m_0(x)) [w_\delta^s(z) - w_\delta^0(z)] \left(\frac{1}{z} - 1\right) dz, \quad (22)$$

where the expressions of $\omega_\delta^s(\cdot)$ and $\omega_\delta^0(\cdot)$ are given by (16).

Expression (22) can then be further simplified using the following properties: (i) signal realizations have no first order effect on prices, i.e., $p_\delta^s(\cdot) = p_\delta^0(\cdot)$; (ii) the signals convey no information on consumers' risk preference, i.e., the distribution of risk preferences defined in (13) satisfies $\omega^s(\cdot) = \omega^0(\cdot)$; and (iii) Bayes' updating rule (18). As a consequence, we have that

$$\sum_{s \in S} \pi(s | m_0(x)) [w_\delta^s(z) - w_\delta^0(z)] = (1-z)\omega^0(z) [1 - \omega^0(z)] \sum_{s \in S} \pi(s | m_0(x)) \frac{\dot{\pi}(s | m_0(z))}{\pi(s | m_0(z))},$$

where $\dot{\pi}(s | \mu) \equiv \frac{\partial}{\partial \mu} \pi(s | \mu)$.

Direct substitution in (22) and a change of variables in the integration (to $\mu' = m_0(z)$) yield then

$$\Delta E [p(t_0(\mu))] = 2 \int_\mu^{\mu_H} \frac{(1-z)}{\rho_0} \omega^0(t_0(\mu')) [1 - \omega^0(t_0(\mu'))] \left[\frac{\partial}{\partial \tilde{\mu}} D_{KL}(\pi(\cdot | \mu) || \pi(\cdot | \tilde{\mu})) \Big|_{\tilde{\mu}=\mu'} \right] d\mu'.$$

This expression establishes a clear relationship between the sign of the expected price change of a given contract and the monotonicity of the Kullback-Leibler divergence measure. The rest of the proof is in Appendix D. \square

Proposition 4 implies that the disclosure of a monotonic signal leads to an ex-ante welfare gain (i.e., integrating over all possible types), for $\delta > 0$ sufficiently small. It also states that, for any non-monotonic signal, one can find a type distribution such that the disclosure of this signal leads to an expected utility loss to positive mass of types. In fact, one can choose the distribution such that the mass of utility-losing types is large enough so that signal disclosure leads to an ex-ante utility loss.

5.3 Signal-MLRP versus Monotonicity

A common condition on informative signals used in principal-consumer models is that signals satisfy the monotone likelihood ratio property (SMLRP; Mirrlees (1976), Hölmstrom (1979)).

Definition 3. A signal π , with realizations $S = \{s_1, \dots, s_n\}$, satisfies SMLRP if

$$\frac{\pi(s_{k+1} | \mu)}{\pi(s_k | \mu)}$$

is strictly increasing in μ , for any $k \in \{1, \dots, n-1\}$.

Proposition 5. Any signal satisfying MLRP is monotonic.

Proof. See Appendix D. □

In the case of a binary signal, a distribution over signal realizations is described by a single number and both monotonicity and SMLRP are equivalent to the requirement that $\pi(s_2 | \cdot)$ be strictly increasing. Beyond binary signals, SMLRP is stronger requirement in that it makes restrictions on the relative probability of any pair of signal realizations. Monotonicity, on the other hand, uses the coarser information given by KL divergence measure which integrates over all signal realizations.

Remark 1. An alternative restriction on informativeness states that the (absolute) likelihood of a particular signal varies monotonically with the risk level, i.e., we say that a signal π is strongly monotonic (SM) if, for all $s \in S$, $\pi(s | \cdot)$ is strictly increasing or decreasing in μ . This condition means that signals can be classified in two categories: the ones that are more likely with high risks and the others that are more likely with low risks. It is easy to show that (SM) implies monotonicity. While SMLRP and SM are not nested in general, they both coincide in the case of binary signals.

5.4 Independent types

In Subsection 5.1, we showed that the welfare effect of signal disclosure is determined by expected price change, and derived an expression for it in (20). If risk and risk aversion are independently distributed (i.e., $\omega_0(\cdot)$ is constant), this expression can be further simplified after an integration by parts to:

$$\Delta E[p(x)] = -2\omega_0(1 - \omega_0) \int_x^1 D_{KL}(\pi(\cdot | m_0(x)) || \pi(\cdot | m_0(z))) dz.$$

This means that the expected price effect of the signal for coverage x is determined solely by the integral of the divergence measure $D_{KL}(\cdot)$ for all coverage levels above x , which is non-negative regardless of whether monotonicity holds. We are ignoring by assumption the knife-edge cases where $\pi(\cdot | \cdot)$ is constant in risk on an open interval in $[\mu_L, \mu_H]$, so that this integral is strictly positive for $x < 1$. Under the assumption of independence, we can relax the monotonicity requirement for interim welfare improvements to be guaranteed, and we can strengthen the welfare result in Proposition 4 as follows.

Corollary 2. Any signal is interim Pareto improving when the distributions of risk and risk aversion in the population are independent.

Notice that the definition of an interim Pareto improving signal requires that the signal is interim Pareto improving for all possible distributions of types. Corollary 2 shows that, if we impose some restrictions on the set of allowed type distribution, we can find weaker conditions ensuring that a signal is interim Pareto improving. The key restriction to guarantee that any signal is interim Pareto improving is that the mapping $x \rightarrow (1-x)\omega_0(x)[1-\omega_0(x)]$, for any $x \in (x_L, 1)$, is non-increasing. This is guaranteed by independence.

6 Comparative statics

In this section we study the effect of changes in the type distribution on prices and consumers' utility. This question is relevant to the issue of risk classification. Consider a population that is originally segmented into groups A and B , with the population in group B being riskier than in A . Could a ban on risk classification based on this segmentation be beneficial to consumers originally in group B ? This entails the comparison of prices in the market with the type distribution of group B with prices in the presence of a ban, which includes consumers from both groups and hence has a less risky type distribution than group B . As discussed in Section 1, our comparative static analysis is also relevant for the study of insurance mandates. In this section we will be more specific about the comparison of risk distributions and show how interventions leading to a change in the type distribution affect equilibrium prices and utility. We show that quite stringent conditions are needed to ensure that a change in the risk distribution featuring an increase of lower risk types leads to a welfare improvement of consumers in the market. We provide an example of a reduction in the distribution of risks in the sense of first order stochastic dominance that leads to a reduction in welfare for certain types of consumers in the market.

In order to clearly distinguish the source of effects of changes in the distribution between the risk and preference dimensions of consumer heterogeneity, we assume in this section that risk and risk aversion are independently distributed. Hence, the single function ϕ denotes the distribution of risk types in the population, and a single number $\omega \in [0, 1]$ represents the share of low-risk-aversion consumers. We compare prices and consumers' welfare under type distributions (ω^A, ϕ^A) and (ω^B, ϕ^B) , first looking at changes in ϕ and then in ω . We use superscripts to refer to the relevant variables under distributions A and B , e.g., $p^A(\cdot; \delta)$ and \mathbb{P}^A for prices and the probability measure over types under distribution A , respectively.

Definition 4. Type distribution (ω^A, ϕ^A) interim-dominates distribution (ω^B, ϕ^B) if for every $\epsilon > 0$ there exists $\bar{\delta} > 0$ such that

$$\mathbb{P}^k [(\mu, i) \mid V_i^A(\mu; \delta) > V_i^B(\mu; \delta)] > 1 - \epsilon,$$

for any arbitrary equilibrium selection, any $\delta \leq \bar{\delta}$ and $k \in \{A, B\}$.¹⁹

Following the definition of interim Pareto improving signals, definition 4 compares consumer payoffs type-by-type under both distributions, that is, considering the utility of the same type on both cases. The ϵ qualifier accounts for the set of traded coverage levels not featuring discrete pooling, for which our price approximation (17) does not apply but which includes an arbitrarily small mass of consumers under both type distributions for δ small.

6.1 Risk distribution

We first consider changes in the distribution of risk types in the population in the sense of monotone likelihood ratio (MLRP). For any two strictly positive densities ϕ^A and ϕ^B on $[\mu_L, \mu_H]$, we say that ϕ^B MLRP-dominates ϕ^A if the ratio

$$\frac{\phi^B(\mu)}{\phi^A(\mu)}$$

is strictly increasing in μ . This ordering is related to the SMLRP property of signals, in that if a signal satisfies SMLRP, then the risk distributions conditional on any two signal realizations $s' \neq s$ are MLRP ordered.

¹⁹This notion of interim welfare improvement is more meaningful than that of ex-ante improvements when considering changes to the type distribution. A shift in the type distribution may make almost all types strictly better off, but increase the frequency of high-risk types, which have lower utility, enough so that it leads to an ex-ante welfare loss.

In derivative terms, if ϕ^A and ϕ^B are continuously differentiable, MLRP is equivalent to

$$\frac{\dot{\phi}^B(\mu)}{\phi^B(\mu)} > \frac{\dot{\phi}^A(\mu)}{\phi^A(\mu)}, \quad (23)$$

for almost all (Lebesgue) risk levels $\mu \in [\mu_L, \mu_H]$. As shown in Corollary 1, the rate of increase of the density of the risk distribution is related to the second order coefficient of the approximation of equilibrium prices. Building on this, the result which follows establishes that a change in the distribution satisfying MLRP-dominance induces an interim improvement in welfare for almost all consumer types.

Proposition 6. (*Risk distribution effect*) *For any $\omega_0 \in (0, 1)$, if the distribution of risk types ϕ^B MLRP-dominates ϕ^A , then (ω_0, ϕ^A) interim-dominates distribution (ω_0, ϕ^B) .*

Proof. The proof follows a similar argument used in Lemma 4 and Proposition 4 regarding signal disclosure. We show in Appendix C (Lemmas 27 and 26) that consumers' change in utility is determined by the price change induced by the distribution shift:

$$V_i^B(\mu; \delta) - V_i^A(\mu; \delta) = \frac{\delta^2}{2} \frac{\partial v}{\partial p}(t_0(\mu), p_0(t_0(\mu)), \mu, \rho_0) [p_{\delta\delta}^A(t_0(\mu)) - p_{\delta\delta}^B(t_0(\mu))] + o_i(\mu; \delta^2),$$

where $\delta^{-2}|o_i(\mu; \delta)|$ converges to zero uniformly in any compact subset of (μ_L, μ_H) . We have used the fact that $p_{\delta}^A(\cdot) = p_{\delta}^B(\cdot)$, since ω_0 is unchanged on both distributions. Using the expressions for the second derivative of prices derived in Corollary 1 we have

$$\frac{p_{\delta\delta}^B(x) - p_{\delta\delta}^A(x)}{x} = 2\omega_0(1 - \omega_0) \int_x^1 \frac{(1-z)^2}{z} \left[\frac{\dot{\phi}^B(m_0(z))}{\phi^B(m_0(z))} - \frac{\dot{\phi}^A(m_0(z))}{\phi^A(m_0(z))} \right] dz,$$

which is strictly positive for all $x \in (x_L, 1)$ and implies the result. \square

The result established in Proposition 6 is fairly intuitive. It shows that if the distribution of risk types changes towards a distribution with more mass on higher types according to MLRP, then the risk distribution within each pool worsens, which is reflected in higher prices and lower utility for consumers.

We should point out that the ordering of distributions according to the MLRP criterion, although quite demanding, cannot be significantly relaxed, as shown by the following example.

Example 1 (FOSD risk increase reducing prices). Suppose ϕ^A is the uniform distribution on $[\mu_L, \mu_H]$ and ϕ^B is a strictly concave function such that $\dot{\phi}^B(\mu_H) < 0$ and $\phi^B(\mu_H) \geq \phi^A(\mu_H)$.

These properties imply that density ϕ^B crosses ϕ^A only once from below, which ensures that ϕ^B first order stochastically dominates ϕ^A . Figure 3 illustrates an example where ϕ^B is quadratic. Since $\dot{\phi}^B(\mu_H) < 0$, there exists $\mu_0 \in (\mu_L, \mu_H)$ such that, for any $\mu \in [\mu_0, \mu_H]$,

$$\frac{\dot{\phi}^B(\mu)}{\phi^B(\mu)} < 0 = \frac{\dot{\phi}^A(\mu)}{\phi^A(\mu)}.$$

Using Lemma 28, all types with risk level $\mu \in [\mu_0, \mu_H]$ have strictly higher utility and face strictly lower coverage prices in equilibrium with risk distribution ϕ^B than with risk distribution ϕ^A . Hence, this set of consumers benefit from being in a market where there are more high-risk types in the population. The intuition for this result is that each risk pool contains consumers with similar risk levels, for small preference heterogeneity, and hence only the relative frequency of types with similar risk levels matters for equilibrium prices. Since ϕ^B

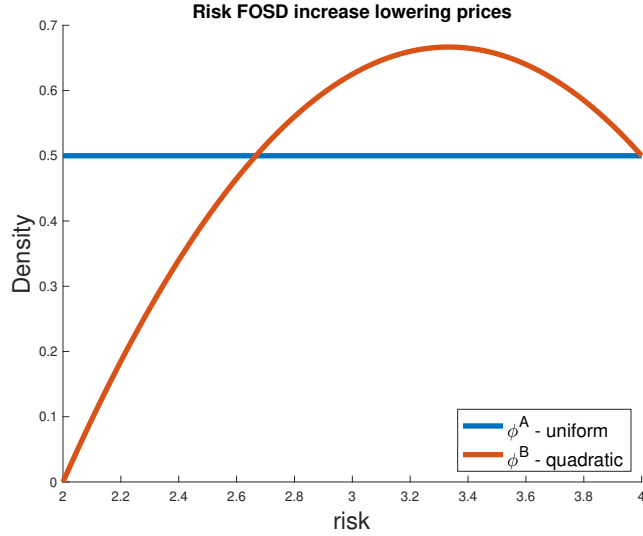


Figure 3: Distribution ϕ^B FOSD dominates ϕ^A and leads to lower prices for high coverage levels.

is decreasing at the top, the relative frequency of the lower risk types *within the pools* that only include similar risk levels above μ_0 is lower than the one under ϕ^A .

The previous example shows that the ordering of distributions according to the first order stochastic dominance criterion is too weak and does not deliver such an unambiguous comparative statics result. It shows that a FOSD increase in the risk distribution may have ambiguous welfare effects, where some consumer types benefit from the increase in the mass of risk types in the population. This result illustrates how the effect of changes in the risk distribution in competitive models of insurance markets can be counter-intuitive and hence any policy analysis in this environment requires careful consideration of such effects.

6.2 Preferences distribution

There are two drivers of consumers demand for coverage in our environment: their risk level, which is the underlying source of adverse selection, and their risk aversion. If consumers have a high willingness to pay for coverage that is unrelated to their risk level, the problem of adverse selection is alleviated since the purchase of higher coverage may not be a strong signal of higher risks. More precisely, in our equilibrium analysis the set of consumers choosing the same coverage level in the discrete pooling region is composed by a *l*-type (with higher risk) and a *h*-type (with lower risk). Hence, an increase in the relative frequency of high-risk-aversion types in the population leads to a lower average risk level in each pool of consumers sharing the same coverage and, as a consequence, to lower prices. The result below shows that this price reduction leads to an interim welfare improvement.

Proposition 7. (*Risk aversion distribution effect*) *An increase in the share of consumers with high-risk aversion (i.e., $\omega^B < \omega^A$) leads to an interim improvement in welfare, i.e., type distribution (ω^B, ϕ) interim-dominates (ω^A, ϕ) .*

Proof. Once again, using Lemma 27, we can relate changes in consumers' utility to price changes:

$$V_i^B(\mu; \delta) - V_i^A(\mu; \delta) = \delta \frac{\partial v}{\partial p}(t_0(\mu), p_0(t_0(\mu)), \mu, \rho_0) [p_\delta^A(t_0(\mu)) - p_\delta^B(t_0(\mu))] + o_i(\mu; \delta),$$

where $\delta^{-1}|o_i(\mu; \delta)|$ converges to zero uniformly on M , for any compact set $M \subset (\mu_L, \mu_H)$. Now, suppose that $\omega^B < \omega^A$, we then have that

$$\frac{p_\delta^A(x) - p_\delta^B(x)}{x} = x(\omega^A - \omega^B)(x - 1 + \log x) > 0.$$

□

7 Conclusion

This paper provides a parsimonious competitive screening model that at the same time illustrates the complex relationship between the distribution of consumer characteristics in the population and the equilibrium pattern of screening, and still allows for tractable analytical results. We also demonstrate how these features of the model are instrumental for the analysis of important policy questions in insurance markets such as risk classification and insurance mandates. We characterize the welfare effect of these interventions and provide conditions under which they are beneficial. The monotonicity condition presented here is novel to the literature, has an intuitive interpretation, and is shown to be central to the welfare impact of risk classification. We provide conditions under which a reduction in the riskiness in the population benefits all consumer types, and show these conditions are fairly tight.

The results obtained here suggest multiple directions for future work. Our analytical characterization of welfare improving signals points to the empirical question of what set of demographic characteristics constitute monotonic signals. On the theoretical side, the characterization of multidimensional models in which preferences do not satisfy the single-crossing property is a notoriously challenging issue. The tractable approach used here can be applied to analyze other competitive settings with rich heterogeneity, such as labour and credit markets. Our analysis assumes public signals, implying that all insurance firms have access to the same information. If some firms have superior information, allowing for detailed risk classification may also hinder competition by limiting the ability of less informed firms to operate in the market. Finally, policy makers in practice may be able to design the information to be used by firms. A simple example is the determination of the age brackets that can be used by firms when pricing consumers in health insurance. Garcia and Tsur (2021) analyze this question in the case of a single available contract, but the economic forces generated by a rich set of contracts would require a different analysis, which would benefit from the results obtained here.

Appendix A - Equilibrium characterization

Throughout the appendix we use the following definitions, with abuse of notation, for $(\mu_l, \mu_h) \in [\mu_L, \mu_H]^2$:

$$\omega(\mu_l, \mu_h) \equiv \frac{\phi_l(\mu_l)}{\phi_l(\mu_l) + \phi_h(\mu_h)}$$

and

$$R(\mu, x, \delta) \equiv \omega(\mu, \mu - (1 - x)\delta). \tag{24}$$

Proof of Lemma 1

Properties (i) and (ii) are equivalent to monotonicity of demand in each dimension, i.e., $t(\cdot, i)$ is non-decreasing for $i \in \{l, h\}$ and $t(\mu, l) \leq t(\mu, h)$. Monotonicity follows from the fact that preferences satisfy the single-crossing property on each dimension, for a given value of the other dimension.

For property (iii): If $p(x), p(\hat{x}) > 0$, using $(m_i(x), i) \in \theta^+(x)$ and $(m_j(\hat{x}), j) \in \theta^+(\hat{x})$, we have

$$u(x, m_i(x), i) - u(\hat{x}, m_i(x), i) \geq p(x) - p(\hat{x}) \geq u(x, m_j(\hat{x}), j) - u(\hat{x}, m_j(\hat{x}), j)$$

which implies that

$$|p(x) - p(\hat{x})| \leq L |x - \hat{x}|$$

where $L = \sup \{|u_x(x, \theta)|; x \in [0, 1] \text{ and } \theta \in \Theta\}$. If $p(\hat{x}) = 0 < p(x)$, the first inequality above still holds and implies the result. If $p(\hat{x}) = p(x) = 0$, the result is trivial.

Proof of Lemma 2

Direct differentiation implies

$$\frac{\partial e(q, x)}{\partial q} = 1 - \delta(1-x) \frac{\phi'_h(q - (1-x)\rho_h) \phi_l(q - (1-x)\rho_l) - \phi'_l(q - (1-x)\rho_l) \phi_h(q - (1-x)\rho_h)}{[\phi_h(q - (1-x)\rho_h) + \phi_l(q - (1-x)\rho_l)]^2}.$$

Defining

$$D \equiv \sup_{\mu, \mu' \in [\mu_L, \mu_H]} \left| \frac{\phi'_h(\mu) \phi_l(\mu') - \phi'_l(\mu') \phi_h(\mu)}{\{\phi_h(\mu) + \phi_l(\mu')\}^2} \right|,$$

we have that if

$$\delta < \frac{1}{D+1},$$

then

$$\frac{\partial e(q, x)}{\partial q} > 1 - (1-x) \frac{D}{D+1} > 0,$$

for all $(q, x) \in E$.

Proof of Proposition 1

Consider a no-gap equilibrium with type assignments (m_l, m_h) , posterior $w(\cdot)$, and interval of traded coverages $X \subset [0, 1]$. From the monotonicity property of the equilibrium in Lemma 1, X can be partitioned into the following sets:

- X_s^i the separating coverages chosen by consumers with risk aversion $i \in \{l, h\}$;
- X_d the discrete pooled coverages chosen by exactly two risk averse types;
- X_c the continuous pooled coverages chosen by intervals of risk types.

For each $x \in X_s^h \cup X_s^l \cup X_d$, the type assignment is either empty or a singleton, and we represent it as a function with image in $[\mu_l, \mu_h] \cup \{\emptyset\}$. The posterior $w(\cdot)$ is extended to the separating regions by using $w(x) = 1$ in X_s^l and $w(x) = 0$ in X_s^h . For convenience, we define the product $0 \cdot \emptyset$ to be zero.

The following lemma shows the basic properties of these sets:

Lemma 5. (a) X_c is a countable set of X ;

(b) If $x \in X_s^h$ and $y \in X \cap [x, 1]$, then $y \in X_s^h$ (i.e., X_s^h is an interval in the higher end of X);

(c) If $x \in X_s^l$ and $y \in X \cap [0, x]$, then $y \in X_s^l$ (i.e., X_s^l is an interval in the lower end of X).

Proof. (a) This is a trivial consequence of the monotonicity of m_i , for $i \in \{l, h\}$.

(b) Let us first show that the set $(X_s^l \cup X_d)' \cap (x, 1]$ has zero Lebesgue measure²⁰, where A' is the set of accumulation points of A . Suppose that this is not the case and let $\underline{y} = \inf \left\{ (X_s^l \cup X_d)' \cap (x, 1] \right\}$. If the interval $[x, \underline{y}]$ is non-degenerated, it must be the union of isolated points in $X_s^l \cup X_d$, points in X_c and points in X_s^h . Since the first two sets are countable, \underline{y} is also the limit of a sequence of points in X_s^h on the left, which is also trivially true in the case $x = \underline{y}$ (i.e., when $[x, \underline{y}]$ is a degenerated interval). We can then find a sequence $(y_n^-)_n$ in X_s^h with $y_n^- \nearrow \underline{y}$ and $(y_n^+)_n$ in $[x, 1] \cap [X_s^l \cup X_d]$ with $y_n^+ \searrow \underline{y}$. Optimality requires that $m_l(y_n^+) \geq m_h(y_n^-) + \delta(1 - y_n^-)$, and Lemma 6 implies that $\lim_{n \rightarrow \infty} w(y_n^+) > 0$, which together with the zero profit condition implies

$$\lim_n \frac{p(y_n^-)}{y_n^-} = \lim_n m_h(y_n^-) < \lim_n [(1 - w(y_n^+)) m_h(y_n^+) + w(y_n^+) m_l(y_n^+)] = \lim_n \frac{p(y_n^+)}{y_n^+},$$

a contradiction with the continuity of $p(\cdot)$. Therefore, $[x, 1] \cap X_s^h$ has full measure on $[x, 1] \cap X$, since $X_c \cup X_s^l \cup X_d$ has also zero measure on this set. Hence, from the equilibrium condition, we have that

$$\frac{p(y)}{y} = m_h(y) \text{ and } w(y) = 0,$$

for almost all $y \in [x, 1] \cap X$. Hence, by the continuity of the function $p(y)/y$ and the monotonicity of m_h , these equalities must hold for all $y \in [x, 1] \cap X$, showing the result.

(c) Let us first show that the set $(X_s^h \cup X_d)' \cap [0, x)$ has zero measure. Suppose that this is not the case and let $\bar{y} = \sup \left\{ (X_s^h \cup X_d)' \cap [0, x) \right\}$. If the interval $[\bar{y}, x]$ is non-degenerated, it must be the union of isolated points in $X_s^h \cup X_d$, points in X_c and points in X_s^l . Since the first two sets are countable, \bar{y} is also the limit of a sequence of points in X_s^l on the right, which is also trivially true in the case $x = \bar{y}$ (i.e., when $[x, \bar{y}]$ is a degenerated interval). We can then find a sequence $(y_n^+)_n$ in X_s^l with $y_n^+ \searrow \bar{y}$, and $(y_n^-)_n$ in $X_s^h \cup X_d$ with $y_n^- \nearrow \bar{y}$. Optimality implies $m_h(y_n^-) \leq m_l(y_n^+) - \delta(1 - y_n^+)$, and Lemma 6 implies $\lim_n w(y_n^-) < 1$, which together with the zero profit condition implies

$$\lim_n \frac{p(y_n^+)}{y_n^+} = \lim_n m_l(y_n^+) > \lim_n [(1 - w(y_n^-)) m_h(y_n^-) + w(y_n^-) m_l(y_n^-)] = \lim_n \frac{p(y_n^-)}{y_n^-},$$

a contradiction with continuity of $p(\cdot)$. Therefore, $[0, x] \cap X_s^l$ has full measure on $[0, x] \cap X$, since $X_c \cup X_s^h \cup X_d$ has zero measure on this set. Hence, from the equilibrium condition, we have that

$$\frac{p(y)}{y} = m_l(y) \text{ and } w(y) = 1,$$

for almost all $y \in [0, x] \cap X$. Hence, from the continuity of the function $p(y)/y$ and monotonicity of m_l , these equalities must hold for all $y \in [0, x] \cap X$, showing the result. \square

The next lemma is the auxiliary result used in the proof of Lemma 5:

Lemma 6. (a) If $(X_s^l \cup X_d) \cap (x, 1]$ has positive measure, then

$$w\left(\inf \left\{ y; y \in (X_s^l \cup X_d)' \cap (x, 1] \right\}\right) > 0;$$

(b) If $X_s^h \cup X_d$ has positive measure, then $\sup \{w(y); y \in X_s^h \cup X_d\} < 1$.

Proof. (a) Define $\underline{y} = \inf \left\{ (X_s^l \cup X_d)' \cap (x, 1] \right\}$. Suppose, by absurd, that $w(\underline{y}) = 0$. We claim that, for every $\epsilon > 0$, $(\underline{y}, \underline{y} + \epsilon) \cap X_d'$ has positive measure. Otherwise, $(\underline{y}, \underline{y} + \epsilon) \cap (X_s^l \cup X_s^h)'$ has full measure for some $\epsilon > 0$.

²⁰In what follows, when we refer to positive or zero measure sets we mean Lebesgue measure.

But this implies that, for a fixed $i \in \{l, h\}$, $X_s^i \cap (\underline{y}, \underline{y} + \epsilon)$ has full measure. Otherwise there exists $y_0 \in (\underline{y}, \underline{y} + \epsilon)$ and sequences $(y_n^-)_n$ and $(y_n^+)_n$ such that $y_n^- \nearrow y_0$, $y_n^+ \searrow y_0$, $y_n^- \in X_s^i$ and $y_n^+ \in X_s^j$, for $i \neq j$. For each element of the sequence we have that

$$\frac{p(y_n^-)}{y_n^-} = m_i(y_n^-),$$

with an analogous equality holding for y_n^+ and j . Defining $m_i^0 = \lim_n m_i(y_n^-)$ and $m_j^0 = \lim_n m_j(y_n^+)$, continuity of prices implies $m_i^0 = m_j^0$ and $\dot{p}(y_0^-) = m_i^0 + (1 - y_0)\rho_i$, $\dot{p}(y_0^+) = m_j^0 + (1 - y_0)\rho_j$. This means the price has a kink at y_0 , which is inconsistent with equilibrium. If $i = h$, the marginal price jumps down at y_0 and type (m_h^0, h) wood choose coverage above y_0 , a contradiction. If $i = l$, monotonicity implies that type $(m_h(y_n^+), l)$ must choose coverage in the interval (y_0, y_n^+) , for each n . Hence, there exists sequence $\{y'_n\}_n$ such that $y'_n \searrow y_0$ and $m_l(y'_n) \searrow m_l^0$. But this implies $\dot{p}(y_0^+) = m_l^0 + (1 - y_0)\rho_l < m_h^0 + (1 - y_0)\rho_h = \dot{p}(y_0^+)$, a contradiction as well. Finally, since $w(\underline{y}) = 0$, we must have that $(\underline{y}, \underline{y} + \epsilon) \subset X_s^h$, which contradicts the definition of \underline{y} .

Since m_i is non-decreasing, it must be differentiable for almost all points in $X_d \cap (\underline{y}, \underline{y} + \epsilon)$, for all $\epsilon > 0$. Notice that

$$\frac{p(x)}{x} = m_h(x) + w(x)(m_l(x) - m_h(x)).$$

Then (10) and (11) imply that (m_l, m_h, w) is the solution to the following ordinary differential equation (ODE) system:

$$\dot{m}_l(x) = \frac{1 - R(m_l(x), x, \delta)}{R(m_l(x), x, \delta) - w(x)} w(x)\delta, \quad (25)$$

$$\dot{w}(x) = \left(\frac{1}{x} - \frac{1}{1-x} \right) (1 - w(x)) + \frac{\rho_l}{x\delta} - \frac{\dot{m}_l(x)}{(1-x)\delta}, \quad (26)$$

$$m_h(x) = m_l(x) - (1-x)\delta,$$

on $X_d \cap (\underline{y}, \underline{y} + \epsilon)$, where $R(\mu, x, \delta)$ is defined in (24). By the same argument above, there are two possible cases:

(i) For every $\epsilon > 0$, $(\underline{y} - \epsilon, \underline{y}) \cap X_d$ has positive measure. Since $w(\underline{y}) = 0$, we must have that $\dot{w}(\underline{y}^-) \leq 0$ and $\dot{w}(\underline{y}^+) \geq 0$ (otherwise, this would imply that $p(x)/x < m_l(x)$, for some x close to \underline{y}). Using these properties and the discrete pooling ODE described above, we get

$$\underline{y} = \frac{\delta + \rho_l}{2\delta + \rho_l},$$

which implies that $\dot{w}(\underline{y}) = 0$, and, taking the second derivative, we get

$$\ddot{w}(\underline{y}) = - \left(1 + \frac{\rho_l}{\delta} \right) \frac{1}{\underline{y}^2} - \frac{1}{(1-\underline{y})^2} < 0,$$

which means that \underline{y} is a local maximum of w . Since $w(\underline{y}) = 0$, this is a contradiction.

(ii) There exists $\epsilon > 0$ such that $(\underline{y} - \epsilon, \underline{y}) \subset X_s^h$. Then, for every $z \in (\underline{y} - \epsilon, \underline{y})$, we have that

$$\frac{p(z)}{z} = m_h(z),$$

which implies that

$$\left. \frac{d}{dy} \left(\frac{p(y)}{y} \right) \right|_{y=\underline{y}^-} = \dot{m}_h(\underline{y}^-).$$

On the other hand,

$$\dot{p}(z) = m_h(z) + (1-z)\rho_h,$$

for almost all $z \in (\underline{y}, \underline{y} + \epsilon)$. Since $\lim_{z \rightarrow \underline{y}^+} w(z) = 0$, we have that m_h is continuous at \underline{y} , which implies that \dot{p} is also continuous at \underline{y} . Moreover,

$$\dot{p}(\underline{y}^+) = [\dot{m}_h(\underline{y}^+) + \dot{w}(\underline{y}^+)(m_l(\underline{y}) - m_h(\underline{y}))] \underline{y} + \frac{p(\underline{y})}{\underline{y}},$$

which implies that

$$\left. \frac{d}{dy} \left(\frac{p(y)}{y} \right) \right|_{y=\underline{y}^+} \geq \dot{m}_h(\underline{y}^+),$$

where the last inequality is strict if and only if $\dot{w}(\underline{y}^+) > 0$ since $m_l(\underline{y}) - m_h(\underline{y}) = (1 - \underline{y})\delta > 0$. By the continuity of \dot{p} at \underline{y} , $\dot{w}(\underline{y}^+) = 0$ and a proof analogous to the case (i) implies $\ddot{w}(\underline{y}) < 0$, a contradiction.

(b) We know that $x \in X_s^h$ implies $w(x) = 0$, and $x \in X_d$ implies

$$w(x) \leq R(m_l(x), x, \delta) \leq \sup_{\mu, \mu' \in [\mu_L, \mu_H]} \omega(\mu, \mu') < 1.$$

□

Lemma 7. *Under Assumption 3, X_d is an interval.*

Proof. If X_d is not an interval, Lemma 5 shows that the only possibility is for a point $x \in X_c$ to exist such that $(x - \epsilon, x + \epsilon) \subset X_c \cup X_d$ for some $\epsilon > 0$. Take sequences $(x_n^-)_n$ and $(x_n^+)_n$ in X_d such that $x_n^- < x < x_n^+$ and $\lim_n x_n^- = x = \lim_n x_n^+$. Define $m_i^0 \equiv \lim_n m_i(x_n^-)$ and $m_i^1 \equiv \lim_n m_i(x_n^+)$, for $i = l, h$. The optimality condition of types in X_d implies that $m_l^0 = m_h^0 + \delta(1 - x)$ and $m_l^1 = m_h^1 + \delta(1 - x)$. We then must have that $m_h^0 < m_h^1$,

$$m_h(x) = [m_h^0, m_h^1]$$

and

$$m_l(x) = [m_h^0 + \delta(1 - x), m_h^1 + \delta(1 - x)].$$

The zero profit condition at coverage x implies that

$$\frac{p(x)}{x} = \frac{\int_{m_h^0}^{m_h^1} e(z + (1 - x)\rho_h, x) [\phi_l(z + \delta(1 - x)) + \phi_h(z)] dz}{\int_{m_h^0}^{m_h^1} [\phi_l(z + \delta(1 - x)) + \phi_h(z)] dz}, \quad (27)$$

where $e(\cdot)$ is defined in (9). But, for each n , the zero profit condition in X_d implies that

$$\frac{p(x_n^-)}{x_n^-} = w(x_n^-) m_l(x_n^-) + (1 - w(x_n^-)) m_h(x_n^-),$$

which is lower than $e(m_h(x_n^-) + (1 - x_n^-)\rho_h, x_n^-)$, since

$$e(m_h(x_n^-) + (1 - x_n^-)\rho_h, x_n^-) = \omega(m_l(x_n^+), m_h(x_n^+)) m_l(x_n^-) + [1 - \omega(m_l(x_n^+), m_h(x_n^+))] m_h(x_n^-)$$

and $w(x_n^-) \leq \omega(m_l(x_n^+), m_h(x_n^+))$. Taking limits, we have that

$$\lim_n \frac{p(x_n^-)}{x_n^-} \leq e(m_h^0 + (1 - x)\rho_h, x),$$

which is strictly lower than $\frac{p(x)}{x}$, from (27), by Assumption 3, and contradicts continuity of $p(\cdot)$. Therefore, the

result follows from Lemma 5. □

The following lemma completes the proof and is also useful in the remaining proofs.

Lemma 8. *There exist $0 \leq x_0 < x_d \leq x_u < 1$ with $p(x_0) = \mu_L x_0$ such that:*

- (i) $X_s^h = (x_u, 1]$, $m_l(x_u) = [\underline{a}_u, \mu_H]$ and $m_h(x_u) = [\bar{a}_u, \bar{b}_u]$;
- (ii) $X_s^l = [x_0, x_d)$, $m_l(x_d) = [\underline{a}_d, \underline{b}_d]$ and $m_h(x_d) = [\mu_L, \bar{b}_d]$;
- (iii) if $x_d < x_u$, then $X_d = (x_d, x_u)$.

Proof. Lemmas 5 and 7 imply the result desired with the weaker inequalities $0 \leq x_0 \leq x_d \leq x_u \leq 1$. We will show the result in three steps.

a) $x_0 < x_d$. If $x_0 = x_d$, the equilibrium restriction for off-path contracts requires that, for $\varepsilon > 0$ small, $\theta^+(x_0 - \varepsilon) = (\mu_L, l)$ and $p(x_0 - \varepsilon) \leq (x - \varepsilon)\mu_L$. But this implies a discontinuity in prices: if $x_d \in X_c$ then $\frac{p(x_d)}{x_d}$ is the average of risks over a positive measure set, which is above μ_L strictly; if $[x_d, x_d + \varepsilon] \in X_d$ for some $\varepsilon > 0$, we must have $\omega(x_d) = 0$ which is ruled out by Lemma 6.

b) $x_d < 1$. If $x_d = 1$, for any $x \in (x_0, x_d = 1)$ optimality implies

$$u_x(1, m_l(x), l) < \dot{p}(1-),$$

but $u_x(1, m_l(x), l) = u_x(1, m_l(x), h)$ and hence type $(m_l(x), h)$ would deviate by lowering his coverage level.

c) $x_u < 1$. If $x_u = 1$, differential equation (26) is satisfied in (x_d, x_u) and, since $w(\cdot) \in [0, 1]$ we must have $w(x) \rightarrow 1$ when $x \rightarrow 1$. But (25) implies that $\dot{m}_l(x) \rightarrow \delta \frac{1 - \omega(\mu_H, \mu_H)}{\omega(\mu_H, \mu_H)} > 0$, and using (26), there exists $\underline{x} < 1$ such that, for $x \in (\underline{x}, 1)$

$$\dot{w}(x) < -\delta \frac{1 - \omega(\mu_H, \mu_H)}{2\omega(\mu_H, \mu_H)} \frac{1}{1 - x},$$

which implies that $w(x) \rightarrow \infty$ as $x \rightarrow 1$, a contradiction. □

Proof of Lemma 3

Suppose that $I = [x - \epsilon, x + \epsilon]$ is an interval of coverage with discrete pooling of an equilibrium (p, m_l, m_h, w) such that m_l and m_h are differentiable at x , with $\epsilon > 0$ sufficiently small. From the consistency condition of equilibrium definition we must have

$$\Pr[x \in I | \theta(x) \in [m_l(x - \epsilon/2), m_l(x + \epsilon/2)] \times \{\rho_l\}] = \frac{\int_{m_l(x - \epsilon/2)}^{m_l(x + \epsilon/2)} \phi_l(z) dz}{\int_{m_l(x - \epsilon/2)}^{m_l(x + \epsilon/2)} \phi_l(z) dz + \int_{m_h(x - \epsilon/2)}^{m_h(x + \epsilon/2)} \phi_h(z) dz}$$

and, taking the limit $\epsilon \rightarrow 0$, the left hand side must converge to $w(x)$. However, dividing the numerator and denominator of the fraction on the right hand side by ϵ and taking the limit, we get

$$w(x) = \frac{\phi_l(m_l(x)) \dot{m}_l(x)}{\phi_l(m_l(x)) \dot{m}_l(x) + \phi_h(m_l(x)) \dot{m}_h(x)},$$

which gives the result.

Appendix B - Existence

Proof of Proposition 2

We can re-write the statement of this proposition as follows:

Proposition 8. *For sufficiently small $\delta > 0$ there exists a continuum of equilibria characterized by (p, m_l, m_h) and $0 < x_d < x_u < 1$ satisfying the following properties:*

- (a) *top separation: $p(x) = m_h(x) = \rho_h(1 - x + \log x) + \mu_H$ and $m_l(x) = \emptyset$, for $x > x_u$;*
- (b) *continuous pooling at x_u : $m_l(x_u) = [\underline{a}_u, \mu_H]$ and $m_h(x_u) = [\bar{a}_u, \bar{b}_u]$ such that:*
 - (b.1) *transversality: $\underline{a}_u = m_l(x_u^-)$, $\bar{a}_u = m_h(x_u^-)$ and $\bar{b}_u = m_h(x_u^+)$;*
 - (b.2) *smooth pasting: $\mathbb{E}[\tilde{\mu}|\tilde{\mu} \in [\underline{a}_u, \mu_H], \rho = \rho_l] + \mathbb{E}[\tilde{\mu}|\tilde{\mu} \in [\bar{a}_u, \bar{b}_u], \rho = \rho_h] = m_h(x_u^+)$;*
- (c) *discrete pooling in the interval (x_d, x_u) : there exists $w : (x_d, x_u) \rightarrow [0, 1]$ such that:*
 - (c.1) *zero profit: $\frac{p(x)}{x} = w(x)m_l(x) + (1 - w(x))m_h(x)$;*
 - (c.2) *optimality: $\dot{p}(x) = m_h(x) + (1 - x)\rho_h = m_l(x) + (1 - x)\rho_l$;*
 - (c.3) *consistency of beliefs: $\frac{w(x)}{1 - w(x)} = \frac{\phi_l(m_l(x))m_l(x)}{\phi_h(m_h(x))m_h(x)}$;*
 - (c.4) *price continuity at x_d and x_u ;*
- (d) *continuous pooling at x_d : $m_l(x_d) = [\underline{a}_d, \underline{b}_d]$ and $m_h(x_d) = [\mu_L, \bar{b}_d]$ such that:*
 - (d.1) *transversality: $\underline{a}_d = m_l(x_d^-)$, $\underline{b}_d = m_l(x_d^+)$ and $\bar{b}_d = m_h(x_d^+)$;*
 - (d.2) *smooth pasting: $\mathbb{E}[\tilde{\mu}|\tilde{\mu} \in [\underline{a}_d, \underline{b}_d], \rho = \rho_l] + \mathbb{E}[\tilde{\mu}|\tilde{\mu} \in [\mu_L, \bar{b}_d], \rho = \rho_h] = m_l(x_d^-)$;*
- (e) *bottom separation: $p(x) = m_l(x) = \rho_l\left(x_d - x + \log\left(\frac{x}{x_d}\right)\right) + \underline{a}_d$ and $m_h(x) = \emptyset$, for $x < x_d$.*

In order to show the existence of a non-gap equilibrium we divide the proof in three parts corresponding to the construction of the pooling regions: (i) top continuous pooling region; (ii) discrete pooling region; (iii) bottom continuous region. Let us denote

$$\bar{b}(x) := (\rho_0 + \delta/2)[1 - x + \log x] + \mu_H \quad (28)$$

the top separating equilibrium part.

Top continuous pooling region

For each $\delta > 0$, a top continuous region is characterized by a vector

$$(\underline{a}, \bar{a}, \underline{b}, \bar{b}, x, w) \in [\mu_L, \mu_H]^4 \times [0, 1]^2$$

that satisfies

$$\underline{b} = \mu_H \quad (29)$$

$$\text{Top separating region: } \bar{b} = \bar{b}(x) \quad (30)$$

$$u_x(x_u, \underline{a}_u, l) = u_x(x_u, \bar{a}_u, h) : \underline{a} = \bar{a} + (1 - x)\delta \quad (31)$$

$$\text{Price continuity: } w\underline{a} + (1 - w)\bar{a} = \bar{b} \Leftrightarrow \bar{a} = \bar{b} - w(1 - x)\delta \quad (32)$$

$$\text{Price continuity: } \frac{\int_{\underline{a}}^{\underline{b}} z\phi_l(z) dz + \int_{\bar{a}}^{\bar{b}} z\phi_h(z) dz}{\int_{\underline{a}}^{\underline{b}} \phi_l(z) dz + \int_{\bar{a}}^{\bar{b}} \phi_h(z) dz} = \bar{b} \quad (33)$$

$$\text{No deviation for } (\mu_H, \bar{b}) : \bar{b} + (1 - x)\delta \geq \mu_H \quad (34)$$

and

$$\text{Weight feasibility: } w \in [0, R(\underline{a}, x, \delta)], \quad (35)$$

where we are dropping the sub-index u for convenience. Notice that $(x, \underline{a}, \bar{a}, \underline{b}, \bar{b}, w)$ defines a top continuous pooling region if and only if $x \in (0, 1)$, $w \in [0, R(\bar{b}(x) + (1-w)(1-x)\delta, x, \delta))$ solves the equation

$$G(x, w) := \int_{\bar{b}(x) + (1-w)(1-x)\delta}^{\mu_H} (z - \bar{b}(x))\phi_l(z)dz + \int_{\bar{b}(x) - w(1-x)\delta}^{\bar{b}(x)} (z - \bar{b}(x))\phi_h(z)dz = 0 \quad (36)$$

and $\bar{b}(x) \geq \mu_H - (1-x)\delta$. In this case, $\underline{a} = \bar{b}(x) + (1-w)(1-x)\delta$, $\bar{a} = \bar{b}(x) - w(1-x)\delta$, $\underline{b} = \mu_H$ and $\bar{b} = \bar{b}(x)$. For simplicity, we will refer only to (x, w) in short instead of the whole vector $(x, \underline{a}, \bar{a}, \underline{b}, \bar{b}, w)$. Define $\underline{x}^\delta \in (0, 1)$ as the interior solution of the equation

$$\bar{b}(x) + (1-x)\delta - \mu_H = 0$$

which represents the coverage level of risk type with high risk aversion in top separating equilibrium part that pools with risk type μ_h with low risk aversion.

Lemma 9. (*Top continuous pooling region*) *The top continuous pooling region is parametrized by a non-degenerated interval $[\underline{x}^\delta, \bar{x}^\delta] \subset [0, 1]$, with $\underline{x}^\delta < \bar{x}^\delta$, satisfying:*

(a) *for each $x \in [\underline{x}_\delta, \bar{x}_\delta]$, there exists $w \in [0, R(\bar{b}(x) + (1-w)(1-x)\delta, x, \delta))$ such that (x, w) defines a top continuous region;*

(b) *if $x = \bar{x}_\delta < 1$, then $w = R(\bar{b}(\bar{x}^\delta) + (1-w)(1-\bar{x}^\delta)\delta, \bar{x}^\delta, \delta)$.*

Proof. We are trying to find a proper solution to equation (36). We start finding a solution for $x = \underline{x}^\delta \in (0, 1)$, and then show that this solution can be extended to the right of \underline{x}^δ . First, for $x \in (0, 1)$ define

$$\underline{w}(x) \equiv \frac{\bar{b}(x) + (1-x)\delta - \mu_H}{(1-x)\delta},$$

$$g(x, w) \equiv \min \{R(\bar{b}(x) + (1-w')(1-x)\delta, x, \delta) - w \mid w' \in [0, w]\}.$$

Notice that $\bar{b}(x) + (1-w)(1-x)\delta \leq \mu_H$ if, and only if, $w \geq \underline{w}(x)$; $\underline{w}(x) > 0$, for $x \in (\underline{x}^\delta, 1)$; and $G(x, \underline{w}(x)) < 0$.

The derivative of G w.r.t. w is

$$G_w(x, w) = (1-w)(1-x)^2\delta^2\phi_l(\bar{b}(x) + (1-w)(1-x)\delta) - w(1-x)^2\delta\phi_h(\bar{b}(x) - w(1-x)\delta),$$

which is positive if and only if

$$w < R(\bar{b}(x) + (1-w)(1-x)\delta, x, \delta). \quad (37)$$

Now considering $x = \underline{x}^\delta$, we have that $\underline{w}(x) = G(\underline{x}^\delta, 0) = 0$,

$$G_w(\underline{x}^\delta, 0) > 0$$

and

$$g(\underline{x}^\delta, 0) = R(\mu_H, \underline{x}^\delta, \delta) > 0.$$

Hence we can find a point $w_0 \in (0, 1)$ such that

$$G(\underline{x}^\delta, w_0) > 0,$$

$$g(\underline{x}^\delta, w_0) > 0.$$

By continuity, we can find $\varepsilon > 0$ sufficiently small such that, for any $x \in (\underline{x}^\delta, \underline{x}^\delta + \varepsilon)$

$$G(x, w_0) > 0 > G(x, \underline{w}(x)),$$

$$g(x, w_0) > 0.$$

From the intermediate value theorem, for each $x \in (\underline{x}^\delta, \underline{x}^\delta + \varepsilon)$, we can find a solution $w^*(x)$ to $G(x, w) = 0$ in $(0, w_0)$. Since $g(x, w_0) > 0$, this solution satisfies $w \in (0, R(\bar{b}(x) + (1-w)(1-x)\delta, x, \delta))$ and hence gives us a top continuous pooling region. Define \bar{x}^δ to be the supremum of all points $x \in (0, 1)$ such that the solution $w^*(\cdot)$ can be extended to $[\underline{x}^\delta, x]$. Our argument above shows that $\underline{x}^\delta < \bar{x}^\delta$. Suppose $\bar{x}^\delta < 1$. If $\underline{w}(\bar{x}^\delta) < w^*(\bar{x}^\delta) < R(\mu_H + w(1-x)\delta, x, \delta)$, the implicit function theorem implies that $w^*(\cdot)$ can be extended beyond \bar{x}^δ , a contradiction. If $\underline{w}(\bar{x}^\delta) < w^*(\bar{x}^\delta)$, we have that $G(\bar{x}^\delta, w^*(\bar{x}^\delta)) < 0$, a contradiction with $w^*(\cdot)$. So we must have $w^*(\bar{x}^\delta) = R(\mu_H + w(1-x)\delta, x, \delta)$. □

Discrete pooling region

For convenience, we repeat the ODE that must be satisfied in a discrete pooling region by type assignment functions and posterior:

$$\begin{aligned} \dot{m}(x) &= \delta \frac{(1 - R(m(x), x, \delta))w(x)}{R(m(x), x, \delta) - w(x)} \\ \dot{w}(x) &= \left(\frac{1}{x} - \frac{1}{1-x} \right) (1 - w(x)) + \delta^{-1} \left(\frac{\rho_0}{x} - \frac{\dot{m}(x)}{1-x} \right) \end{aligned} \quad (38)$$

where we drop the subscript on m_l for convenience.

Lemma 10. *Let $\delta > 0$ be sufficiently small. For each $x_0 \in (0, 1)$, $\mu_0 \in [\mu_L, \mu_H]$ and $w_0 \in [0, R(\mu_0, x_0, \delta)]$, there exists a unique solution (m, w) of (38) with initial condition (x_0, μ_0, w_0) defined in an interval $[x_0, x_1]$ that satisfies:*

- (i) $\dot{m}(x) > 0$;
- (ii) $w(x) \in [0, R(m(x), x, \delta))$, for all $x \in (x_0, x_1]$;
- (iii) $m(x_1) = \mu_H$ or $w(x_1) = 0$.

Proof. Let us consider the following cases:

(a) $w_0 \in [0, R(\mu_0, x_0, \delta))$. By standard theorem for existence and uniqueness of a solution for ODE system (see, for instance, Theorem 3.1 (p. 18) of Hale (1969)), there exists a unique local solution (m, w) of the system (38) with initial condition (x_0, μ_0, w_0) . For this local solution, we have that $\dot{m}(x) > 0$ and $w(x) \in [0, R(m(x), x, \delta))$. We claim that we can extend the solution until we reach x_1 such that $m(x_1) = \mu_H$ or $w(x_1) = 0$, for δ sufficiently small. Otherwise, for arbitrary small δ , we must have $w(x_1) = R(m(x_1), x_1, \delta)$. However, rewriting (38) through a change of variable from x to μ , we can write its solution in terms of μ as:

$$\begin{aligned} \dot{x}(\mu) &= \frac{R(\mu, x(\mu), \delta) - w(\mu)}{\delta w(\mu) (1 - R(\mu, x(\mu), \delta))} \\ \dot{w}(\mu) &= \left(\frac{1}{x(\mu)} - \frac{1}{1-x(\mu)} \right) (1 - w(\mu)) \dot{x}(\mu) + \delta^{-1} \left(\frac{\rho_0 \dot{x}(\mu)}{x(\mu)} - \frac{1}{1-x(\mu)} \right) \end{aligned} \quad (39)$$

with the same initial condition. With some abuse of notation we will consider the same notation for w function

in both ODE systems. We have that $\dot{x}(\mu_1) = 0$ where $\mu_1 = m(x_1)$. Then

$$\dot{w}(\mu_1) = -\frac{\delta^{-1}}{1-x(\mu_1)} \leq -\delta^{-1} < 0.$$

Taking the second derivative of $x(\cdot)$ for $\mu < \mu_1$ and making $\mu \rightarrow \mu_1$ we get

$$\ddot{x}(\mu_1) > 0 \text{ if and only if } \frac{\partial R}{\partial \mu}(\mu_1, x(\mu_1), \delta) - \dot{w}(\mu_1) > 0.$$

Since $\frac{\partial R}{\partial \mu}(\cdot, \cdot, \delta)$ uniformly converges to $\frac{\partial R}{\partial \mu}(\cdot, \cdot, 0)$ and $\dot{w}(\mu_1) \rightarrow -\infty$ when $\delta \rightarrow 0$, $\left| \frac{\partial R}{\partial \mu}(\mu_1, x(\mu_1), \delta) \right| < -\dot{w}(\mu_1)$ for sufficiently small δ , which implies that the function $x(\cdot)$ is locally convex around μ_1 and with zero derivative at μ_1 . This contradicts the fact that $\dot{x}(\mu) > 0$, for all $\mu \in [\mu_0, \mu_1]$.

(b) $w_0 = R(\mu_0, x_0, \delta)$. Rewriting (38) through a change of variable from x to μ , by standard theorem for existence and uniqueness of a solution of ODE system, there exists a unique local solution (x, w) of the system (39) with initial condition (x_0, μ_0, w_0) . By the same argument as in case (i), this local solution can be extended to a maximal interval $[\mu_0, \mu_1]$ with $\dot{x}(\mu) > 0$ and $w(\mu) \in [0, R(\mu, x(\mu), \delta))$, for all $\mu \in (\mu_0, \mu_1]$, and $\mu_1 = \mu_H$ or $w(\mu_1) = 0$. We can change μ back to x and get the solution of the original system. \square

Lemma 11. *For each $\delta \in (0, 1)$, there exists infinitely many different solutions (w, m) of (38) with final condition $(x_1, \underline{a}_1, w_1)$ such that:*

- (i) $x_1 \in (\underline{x}^\delta, \bar{x}^\delta)$ and $w_1 \in (0, R(\underline{a}_1, x_1, \delta))$ define a top continuous pooling region;
- (ii) the maximal interval is $[x_0, x_1]$ such that $x_0 \leq \delta$ or $m(x_0) = \mu_L + (1 - x_0)\delta$.

Proof. (i) By Lemma 9, for each $x_1 \in [\underline{x}^\delta, \bar{x}^\delta]$, there exists $(x_1, \underline{a}_1, \bar{a}_1, \underline{b}_1, \bar{b}_1, w_1)$ which defines a top continuous pooling region. By Lemma 10, the solution (μ, w) of (38) with final condition $(x_1, \underline{a}_1, w_1)$ and maximal interval $[x_0, x_1]$ is such that: (a) $m(x_0) \geq \mu_L$; (b) $\dot{m}(x) > 0$, for $x \in (x_0, x_1]$; (c) $0 \leq w(x_0) \leq R(m(x_0), x_0, \delta)$.

(ii) We start by showing that at least one such solution exists. Suppose not, i.e., $x_0 > \delta$ and $m(x_0) > \mu_L + (1 - x_0)\delta$, for all possible $x_1 \in [\underline{x}^\delta, \bar{x}^\delta]$. There are three possibilities: (a) $w(x_0) = 0$; (b) $w(x_0) = R(m(x_0), x_0, \delta)$; (c) $0 < w(x_0) < R(m(x_0), x_0, \delta)$. Applying Lemma 10, case (c) cannot hold since $[x_0, x_1]$ is the maximal interval. Therefore, let A (resp. B) be the subset of $x_1 \in [\underline{x}^\delta, \bar{x}^\delta]$ such that $m(x_0) > \mu_L + (1 - x_0)\delta$; $\dot{m}(x) > 0$, for $x \in (x_0, x_1]$; and $w(x_0) = 0$ (resp. $w(x_0) = R(m(x_0), x_0, \delta)$).

If $\bar{x}^\delta < 1$, $\underline{x}^\delta \in A$ and $\bar{x}^\delta \in B$, $A \cap B = \emptyset$ and $A \cup B = [\underline{x}^\delta, \bar{x}^\delta]$. We claim that A and B are closed sets, a contradiction.

To show that A is closed, let (x_1^n) be a sequence in A that converges to $x_1 \in [\underline{x}^\delta, \bar{x}^\delta]$. By the definition of A and Lemma 10, we know that there exists a solution (m^n, w^n) of (38) with initial condition $(x_0^n, \mu_0^n, 0)$ in maximal interval $[x_0^n, x_1^n]$ which satisfies: $\dot{m}^n(x) > 0$; $w^n(x) \in [0, R(m^n(x), x, \delta))$, for all $x \in [x_0, x_1]$; and $m^n(x_1^n) = \mu_H$ or $w^n(x_1^n) = 0$. By continuity and regularity of (38) we have that there exists a solution (m, w) of (38) with initial condition $(x_0, \mu_0, 0)$ in maximal interval $[x_0, x_1]$ which satisfies: $\dot{m}(x) > 0$; $w(x) \in [0, R(m(x), x, \delta))$, for all $x \in [x_0, x_1]$; and $m(x_1) = \mu_H$ or $w(x_1) = 0$. Hence, $x_1 \in A$.

To show that B is closed, let (x_1^n) be a sequence in B that converges to $x_1 \in [\underline{x}^\delta, \bar{x}^\delta]$. By the definition of B and Lemma 10, we know that there exists a solution (m^n, w^n) of (38) with initial condition (x_0^n, μ_0^n, w_0^n) in maximal interval $[x_0^n, x_1^n]$ which satisfies: $w_0^n = R(\mu_0^n, x_0^n, \delta)$; $\dot{m}^n(x) > 0$; $w^n(x) \in [0, R(m^n(x), x, \delta))$, for all $x \in (x_0^n, x_1^n]$; and $m^n(x_1^n) = \mu_H$ or $w^n(x_1^n) = 0$. By continuity and regularity of (38) there exists a solution (μ, w) of (38) with initial condition (x_0, μ_0, w_0) in a maximal interval $[x_0, x_1]$ which satisfies: $w_0 = R(\mu_0, x_0, \delta)$; $\dot{\mu}(x) > 0$; $w(x) \in [0, R(m(x), x, \delta))$, for all $x \in (x_0, x_1]$; and $\mu(x_1) = \mu_H$ or $w(x_1) = 0$. Hence, $x_1 \in B$. Therefore, we must have $x_1 \in A \cap B$, which leads to a contradiction by the uniqueness of the ODE solution.

If $\bar{x}^\delta = 1$ and $B = \emptyset$, then take a sequence (x_0^n) in A such that the corresponding sequence of (x_1^n) of the upper bound of the maximal interval which converges to 1 and the solution (w^n, μ^n) of (38). Notice that, by Lemma 13 below, $x_0^n \leq x^\delta < 1$. Using the second equation of (38), we have that

$$(1 - \bar{w}) \int_{a_n}^{x_1^n} \left(\frac{1}{z} - \frac{1}{1-z} \right) dz + \delta^{-1} \rho_0 \int_{a_n}^{x_1^n} \frac{1}{z} dz \geq w^n(x_1^n) - w^n(a_n)$$

or

$$(1 - \bar{w} + \delta^{-1} \rho_0) [\ln x_1^n - \ln a_n] + (1 - \bar{w}) [\ln(1 - x_1^n) - \ln(1 - a_n)] \geq w^n(x_1^n) - w^n(a_n),$$

where $\bar{w} = \sup_{\mu, x} R(\mu, x, \delta)$ and $a_n = \max\{1/2, x_0^n\}$. Notice that the left hand side of the above inequality converge to $-\infty$, which implies that $w^n(x_1^n) \rightarrow -\infty$, when $n \rightarrow \infty$. However, this contradicts Lemma 10 and the definition of x_1^n . Therefore, if $\bar{x}^\delta = 1$, then $B \neq \emptyset$ and the previous argument again applies.

Finally, since A and B are mutually exclusive, closed and do not cover $[\bar{x}^\delta, \underline{x}^\delta]$, their complement is (relatively) open and hence infinite. \square

Lemma 12. *If $\delta > 0$ is sufficiently small, the solution (m, w) of (38) with maximum interval $[x_0, x_1]$ in Lemma 11 satisfies $m(x_0) = \mu_L + (1 - x_0)\delta$.*

Proof. Using the notation of Lemma 11, suppose that $x_0 \leq \delta$ for $\delta > 0$ is small enough. Integrating the second equation of the ODE system, we have that

$$m(x) = \underline{a}_1 + \rho_0 \int_x^{x_1} \left(1 - \frac{1}{z} \right) dz + \delta \int_x^{x_1} \left(2 - \frac{1}{z} \right) (1 - w(z)) dz + \delta \int_x^{x_1} (1 - z) \dot{w}(z) dz,$$

for all $x \in [\delta, x_1]$. Integrating we have

$$\begin{aligned} \bar{m}(x) = & \underline{a}_1 + \rho_0 [x_1 - x + \ln x - \ln x_1] + \delta \int_x^{x_1} \left(2 - \frac{1}{z} \right) (1 - w(z)) dz \\ & + \delta [(1 - x_1)w(x_1) - (1 - x)w(x) - \int_x^{x_1} w(z) dz]. \end{aligned}$$

Since $w(x)$ is uniformly bounded in the interval $[\delta, 1]$, when $\delta \rightarrow 0$ we have that $\mu(x)$ uniformly converges to $\underline{a}_1 + \rho_0 [x_1 - x + \ln x - \ln x_1]$ on the compact interval of $[\delta, x_1]$. Hence, there exist $\delta > 0$ and $\tilde{x} \in [\delta, x_1]$ such that $\mu(\tilde{x}) < \mu_L + (1 - x_0)\delta$, which concludes the proof. \square

Lemma 13. *Suppose that (m, w) is a solution of (38) and $x \in (0, 1)$ is such that $w(x) = 0$. Then, $\dot{w}(x) \geq 0$ if and only if $x \leq x^\delta \equiv \frac{\delta + \rho_0}{2\delta + \rho_0}$.*

Proof. We have that $\dot{m}(x) = 0$ and $\dot{w}(x) = \frac{1}{x} - \frac{1}{1-x} + \frac{\rho_0}{\delta x}$. Then, the result follows. \square

Bottom continuous pooling region

Fix the equilibrium (m, w) of Lemma 12 on the maximal interval $[x_0, x_1]$. A bottom continuous region is characterized by a vector

$$(\underline{a}, \bar{a}, \underline{b}, \bar{b}, x, w) \in [\mu_L, \mu_H]^4 \times [0, 1]^2$$

that satisfies

$$\underline{b} = \mu(x) \tag{40}$$

$$\bar{b} = \underline{b} - (1 - x)\delta \tag{41}$$

$$\bar{a} = \mu_L \tag{42}$$

$$w = w(x) \quad (43)$$

$$\underline{a} = w\underline{b} + (1-w)\bar{b} = \underline{b} - (1-w)(1-x)\delta \quad (44)$$

and

$$\frac{\int_{\underline{a}}^{\underline{b}} z\phi_l(z) dz + \int_{\bar{a}}^{\bar{b}} z\phi_h(z) dz}{\int_{\underline{a}}^{\underline{b}} \phi_l(z) dz + \int_{\bar{a}}^{\bar{b}} \phi_h(z) dz} = \underline{a}, \quad (45)$$

where we are dropping the sub-index d for convenience. Notice that $(x, \underline{a}, \bar{a}, \underline{b}, \bar{b}, w)$ defines a bottom continuous pooling region if and only if $x \in (0, 1)$ solves the equation

$$\int_{m(x)-(1-w(x))(1-x)\delta}^{m(x)} (z - m(x) + (1-w(x))(1-x)\delta)\phi_l(z) dz + \int_{\mu_L}^{\mu(x)-(1-x)\delta} (z - m(x) + (1-w(x))(1-x)\delta)\phi_h(z) dz = 0. \quad (46)$$

It is easy to see that there exists $\bar{x}_0 \in (x_0, 1)$ such that $m(\bar{x}_0) - (1 - \bar{x}_0)\delta = \mu_L$ and, therefore, the left hand side of (46) is non-negative at $x = \bar{x}_0$. Moreover, if δ is sufficiently small, the left hand side of (46) at $x = x_1$ is negative. By the intermediate value theorem, there exists $\tilde{x}_0 \in [\bar{x}_0, x_1)$ for which the equation (46) holds at $x = \tilde{x}_0$.

Global deviation

The last step of the proof is to show that the necessary local deviation conditions of the consumer's problem are sufficient for the global deviation conditions. The following lemma states this property.

Lemma 14. *No consumer type has a profitable deviation, given the price and type assignments constructed.*

Proof. For each coverage $x \in [0, 1]$, let $(m_i(x), i)$ be a type that chooses coverage x . We have to show that $(m_i(x), i)$ does not deviate to any contract $\hat{x} \in [0, 1]$. Let $(m_j(\hat{x}), j)$ be a type that finds \hat{x} optimal. By continuity, without loss of generality, we can assume that $x, \hat{x} \in [x_L, x_d] \cup (x_d, x_u) \cup (x_u, 1]$, i.e., they do not belong to kinks of the price function and are on-the-path equilibrium allocations. Thus, from the FOC of consumer's problem we have that

$$\dot{p}(x) = u_x(x, m_i(x), i) \text{ and } \dot{p}(\hat{x}) = u_x(\hat{x}, m_j(\hat{x}), j).$$

Let us consider the following cases:

(a) $i = j$. From the FOC of the consumer's problem,

$$u(x, m_i(x), i) - p(x) \geq u(\hat{x}, m_i(x), i) - p(\hat{x})$$

if and only if

$$\int_{\hat{x}}^x \int_{m_i(\hat{x})}^{m_i(x)} u_{x\mu}(s, t, i) dt ds \geq 0.$$

Since m_i is a non-decreasing function and $u_{x\mu} > 0$, this is always true.

(b) $i = l$ and $j = h$. Then, $x \in [x_L, x_d] \cup (x_d, x_u)$ and $\hat{x} \in (x_d, x_u) \cup (x_u, 1]$. There are two subcases to consider:

(b.1) $x \leq \hat{x}$. If $\hat{x} \in (x_d, x_u)$, then $u_x(\hat{x}, m_h(\hat{x}), h) = u_x(\hat{x}, m_l(\hat{x}), l)$; if $\hat{x} \in (x_u, 1]$, then $m_l(\hat{x}) = \mu_H$ and

$u_x(\hat{x}, m_h(\hat{x}), h) \geq u_x(\hat{x}, m_l(\hat{x}), l)$.²¹ In both cases we have

$$u_x(x, m_l(x), l) - u_x(\hat{x}, m_h(\hat{x}), h) \leq u_x(x, m_l(x), l) - u_x(\hat{x}, m_l(\hat{x}), l).$$

Since $u_{x\mu} > 0$ and $m_l(x) \leq m_l(\hat{x})$, then

$$\begin{aligned} u_x(x, m_l(x), l) - u_x(\hat{x}, m_l(\hat{x}), l) &\leq u_x(x, m_l(x), l) - u_x(\hat{x}, m_l(x), l) \\ &= \int_{\hat{x}}^x u_x(s, m_l(x), l) ds. \end{aligned}$$

From the FOC of the consumer's problem and combining the above two inequalities, we have the deviation condition

$$u(\hat{x}, m_l(x), l) - p(\hat{x}) \leq u(x, m_l(x), l) - p(x).$$

(b.2) $x > \hat{x}$. Then, $x, \hat{x} \in (x_d, x_u)$ and we can assume $i = j$ without loss of generality. Therefore, the proof is equivalent to case (a).

(c) $i = h$ and $j = l$. The proof is analogous to case (b). \square

Equilibrium existence and multiplicity

Using the construction in the proof above, we showed that for each solution described in Lemma 11 gives us an equilibrium, and there are infinitely many of these.

Appendix C - Equilibrium approximation

In this section, we obtain approximation results for equilibrium objects for $\delta > 0$ sufficiently small and an arbitrary equilibrium selection. Since, as $\delta \rightarrow 0$, $(x_d, x_u) \rightarrow (x_L, 1)$, almost all traded contracts fall in the discrete pooling region eventually. We start by providing results related to the top continuous pooling region, which are then used to obtain approximation results for the discrete pooling region.

Top continuous pooling region

For $\delta > 0$, a top continuous pooling region is defined by coverage level x_u , set of risk levels pooled with low risk aversion, $[\underline{a}_u, \mu_H]$, high risk aversion, $[\bar{a}_u, \bar{b}_u]$, and the posterior at the top of the discrete pooling region, $w(x_u)$. In this section, for brevity, we denote these objects as $(x_u, \underline{a}_u, \bar{a}_u, \bar{b}_u, w)$. They must satisfy equilibrium equations (29)-(35).

We use the notation $(x_u(\delta), \underline{a}_u(\delta), \bar{a}_u(\delta), \bar{b}_u(\delta), w(\delta))$ in order to make the dependence on parameter δ explicit. We will start by characterizing the convergence of $x_u(\cdot)$ to coverage one as $\delta \rightarrow 0$. We proceed by "sandwiching" $x_u(\delta)$: we find a lower bound $\underline{x}_u(\delta)$ such that $|\underline{x}_u(\delta) - 1|$ is $o(\delta)$. We then study the limiting behavior of all other endogenous variables involved in the top continuous pooling region. Finally, we proceed to study the behavior of certain equilibrium objects in the discrete pooling equilibrium region, which includes all types in the limit as $\delta \rightarrow 0$.

For each $\delta > 0$, the function $x \mapsto \rho_h(\delta)(1 - x + \ln x) + \delta(1 - x)$ is strictly concave, zero and strictly decreasing at $x = 1$, and strictly negative for x sufficiently small, we know that it has a unique zero in $(0, 1)$, denoted by $\underline{x}_u(\cdot)$. From (34), we must have $x_u(\delta) \in [\underline{x}_u(\delta), 1)$ for $\delta > 0$. Finally, define $\underline{x}_u(0) \equiv 1$.

In the next lemma we show that the lower bound on $x_u(\delta)$ converges to 1 at rate δ , which means that any selection $x_u(\delta)$ also converges to 1 at rate δ or faster.

²¹Here we are using the properties of the top continuous pooling in this proof. In particular, $\bar{b}(x) \geq \mu_H - (1 - x)\delta$, for all $x \in (x_u, 1]$.

Lemma 15. *The lower bound on the pooling point, $\underline{x}_u(\delta)$, is continuously differentiable and satisfies $\underline{x}_u(0) = 1$ and $\dot{\underline{x}}_u(0) = -\frac{3}{\rho_0}$.*

Proof. For $\delta > 0$, continuous differentiability of $\underline{x}_u(\cdot)$ follows from the implicit function theorem, while continuity at $\delta = 0$ follows from the fact that $1 - x + \ln x < 0$, for any $x \in (0, 1)$.

Notice that, for $\delta > 0$,

$$\dot{\underline{x}}_u(\delta) = -\frac{1}{2} \frac{\frac{\ln \underline{x}_u(\delta)}{\delta} + 3 \left(\frac{1 - \underline{x}_u(\delta)}{\delta} \right)}{\frac{\rho_h(\delta)}{\underline{x}_u(\delta)} \frac{1 - \underline{x}_u(\delta)}{\delta} - 2}, \quad (47)$$

which implies that $\dot{\underline{x}}_u(\delta) < 0$. If there exists a sequence $(\delta_n)_n$ such that $\delta_n \rightarrow 0$ and $\dot{\underline{x}}_u(\delta_n)$ converges, then $\underline{x}_u(\cdot)$ is differentiable at zero and, using (47), satisfies

$$1 = -\frac{1}{\rho_0 \dot{\underline{x}}_u(0) + 2} \implies \dot{\underline{x}}_u(0) = -\frac{3}{\rho_0}.$$

Otherwise, we must have $\lim_{\delta \rightarrow 0} \dot{\underline{x}}_u(\delta) = \lim_{\delta \rightarrow 0} \frac{\underline{x}_u(\delta) - 1}{\delta} = -\infty$, and (47) implies that $\lim_{\delta \rightarrow 0} \underline{x}_u(\delta) = -\frac{1}{\rho_0}$, a contradiction. \square

Corollary 3. *For any arbitrary selection $(x_u(\delta))_{\delta > 0}$, we have that*

$$\lim_{\delta \rightarrow 0} x_u(\delta) = 1 \text{ and } \limsup_{\delta \rightarrow 0} \frac{1 - x_u(\delta)}{\delta} \leq -\dot{\underline{x}}_u(0).$$

Proof. The results follow directly from $1 \leq x_u(\delta) \leq \underline{x}_u(\delta)$ and Lemma 15. \square

This means that, potentially passing to a sub-sequence, we can assume that $\frac{1 - x_u(\delta)}{\delta}$ converges. Define $x_u(0) \equiv \lim_{\delta \rightarrow 0} x_u(\delta) = 1$ and $\dot{x}_u(0) \equiv \lim_{\delta \rightarrow 0} \frac{x_u(\delta) - 1}{\delta}$. The same definition will be extended to equilibrium variables $\bar{b}_u(\delta)$, $\bar{a}_u(\delta)$, $\underline{a}_u(\delta)$, $w(x; \delta)$ and $p(x; \delta)$.

We now characterize the changes in the top continuous pooling parameters around $\delta = 0$.

Lemma 16. *For any arbitrary selection we have that:*

- (i) *Convergence:* $\bar{a}_u(0) = \underline{a}_u(0) = \bar{b}_u(0) = \mu_H$
- (ii) δ -*order convergence:* $\dot{\underline{a}}_u(0) = \dot{\bar{a}}_u(0) = \dot{\bar{b}}_u(0) = 0$;
- (iii) δ^2 -*order convergence:* $\ddot{\bar{b}}_u(0) = -\rho_0 (\dot{x}_u(0))^2$.

Proof. (i) follows directly from taking limits on equations (29)-(35). (ii) follows from subtracting both sides of equations (30), (31) and (32) by μ_H , dividing by δ and taking limits $\delta \rightarrow 0$. To obtain (iii) we rewrite equation (28) as follows:

$$\frac{\bar{b}_u - \mu_H}{\delta^2} = \rho_h(\delta) \left(\frac{x_u(\delta) - 1}{\delta} \right)^2 \frac{\left(\frac{\log x_u(\delta)}{x_u(\delta) - 1} - 1 \right)}{x_u(\delta) - 1}.$$

Taking limits we have that $\ddot{\bar{b}}_u(0) = -\rho_0 [\dot{x}_u(0)]^2$. \square

Discrete pooling approximation

For clarity, our results in this section are grouped according to the equilibrium object they refer to. Let the discrete pooling prices and relative weights be given by $p(x; \delta)$ and $w(x; \delta)$. In the discrete pooling region of any

equilibrium, the price function follows the differential system: from (10) we have

$$\dot{p}(x; \delta) = \frac{p(x; \delta)}{x} + (1-x) \left[\rho_0 + \delta \left(\frac{1}{2} - w(x; \delta) \right) \right], \quad (48)$$

while (10) and (11) imply

$$\dot{w}(x; \delta) = (1-w(x; \delta)) \frac{1-2x}{x(1-x)} + \frac{\rho_0 - \delta/2}{\delta x} - \frac{w(x; \delta) [1 - \tilde{w}(p(x; \delta), w(x; \delta), x; \delta)]}{[\tilde{w}(p(x; \delta), w(x; \delta), x; \delta) - w(x; \delta)] (1-x)}, \quad (49)$$

where we define the following function

$$\tilde{w}(p, w, x; \delta) \equiv \frac{\phi_l \left(\frac{p}{x} + \delta(1-w)(1-x) \right)}{\phi_l \left(\frac{p}{x} + \delta(1-w)(1-x) \right) + \phi_h \left(\frac{p}{x} - \delta w(1-x) \right)}.$$

Notice that the differential equation (48) has following integral form

$$\frac{p(x; \delta)}{x} = \bar{b}_u(\delta) - \int_x^{x_u(\delta)} \left[\rho_0 + \delta \left(\frac{1}{2} - w(z; \delta) \right) \right] \left(\frac{1}{z} - 1 \right) dz. \quad (50)$$

Hence, characterization of the behavior of the posterior is sufficient to study the price function.

The next lemma characterizes the limits of the functions $p(x; \delta)$ and $w(x; \delta)$, and their derivatives when $\delta \rightarrow 0$.

Price convergence

In this section we will use the fact that the one-dimensional prices and type assignment functions satisfy

$$\left(\frac{p_0(x)}{x} \right)' = m_0'(x) = \rho_0 \left(\frac{1}{x} - 1 \right). \quad (51)$$

Lemma 17. *The price function is continuous in δ at zero: for any $x \in (x_L, 1)$, $\lim_{\delta \rightarrow 0} \frac{p(x; \delta)}{x} = m_0(x)$. This convergence is uniform in any compact set $M \subset (x_L, 1)$.*

Proof. The function $p(x; \delta)$ satisfies (50). From the Lemma 16, we know that $\bar{b}_u(\delta) \rightarrow \mu_H$, which, together with (51), imply pointwise convergence. If $z_0 \equiv \inf M$, then for $\delta_0 > 0$ sufficiently small, the family $\left(\frac{p(\cdot; \delta)}{x} \right)_{\delta_0 > \delta > 0}$ is equi-Lipschitz²² in M with constant $L \equiv \left(\frac{1}{z_0} - 1 \right) (\rho_0 + \delta_0)$, which means that the convergence is uniform. \square

Lemma 18. *(First-order price approximation) For any compact $M \subset (x_L, 1)$ and $x \in M$,*

$$\lim_{\delta \rightarrow 0} \frac{\frac{p(x; \delta)}{x} - m_0(x)}{\delta} = \frac{p_\delta(x)}{x} = \int_x^1 \left(\omega_0(z) - \frac{1}{2} \right) \left(\frac{1}{z} - 1 \right) dz,$$

with the convergence being uniform in M .

Proof. From equations (50) and (51) we have

$$\frac{\frac{p(x; \delta)}{x} - m_0(x)}{\delta} = \frac{\bar{b}_u(\delta) - m_0(x_u(\delta))}{\delta} - \int_x^{x_u(\delta)} \left(\frac{1}{2} - w(z; \delta) \right) \left(\frac{1}{z} - 1 \right) dz, \quad (52)$$

²²For any family of function $g_\delta : A \mapsto \mathbb{R}$, with index $\delta \in E$ and $A \subset \mathbb{R}$, we say that this family is equi-Lipschitz in $B \subset A$ if there exists a constant L such that

$$|g_\delta(x) - g_\delta(x')| \leq L|x - x'|,$$

for any $x, x' \in B$ and $\delta \in E$.

which, using $m'_0(1) = 0$ as well as Lemmas 16 and 22, implies pointwise convergence of the object of interest. Uniform convergence comes from the fact that the left-hand side of (52) is equi-Lipschitz with constant $L \equiv \left(\frac{1}{z_0} - 1\right)$, where $z_0 \equiv \inf M$. \square

Lemma 19. (Second-order price approximation) *The second-order approximation of equilibrium prices is given by, for any $x \in (x_L, 1)$,*

$$p_{\delta\delta}(x) = 2 \int_0^1 w_\delta(z) \left(\frac{1}{z} - 1\right) dz,$$

with the convergence being uniform for any compact $M \subset (x_L, 1)$.

Proof. Consider any compact $M \subset (x_L, 1)$ and $x \in M$. Using (52) and Lemma 18 we have

$$\begin{aligned} \frac{1}{\delta} \left[\frac{\frac{p(x;\delta)}{x} - m_0(x)}{\delta} - \frac{p_\delta(x)}{x} \right] &= \frac{\bar{b}_u(\delta) - m_0(x_u(\delta))}{\delta^2} \\ &+ \int_x^{x_u(\delta)} \frac{(w(z;\delta) - \omega_0(z))}{\delta} \left(\frac{1}{z} - 1\right) dz - \frac{1}{\delta} \int_{x_u(\delta)}^1 \left(\omega_0(z) - \frac{1}{2}\right) \left(\frac{1}{z} - 1\right) dz. \end{aligned} \quad (53)$$

Taking the limit $\delta \rightarrow 0$ and using Lemmas 3, 16 and 24 give us pointwise convergence. Now, consider a compact set $M \subset (x_L, 1)$. Using Lemma 24, we can find $\delta_0 > 0$ sufficiently small such that

$$B_M \equiv \sup_{\delta \leq \delta_0} \sup_{z \in M} \left| \frac{\omega_0(z) - w(z;\delta)}{\delta} \right| < \infty.$$

For $\delta > 0$ smaller than δ_0 , the left-hand side of expression (53) is equi-Lipschitz, for $\delta_0 > \delta > 0$ with constant $L \equiv 2 \left(\frac{1}{z_0} - 1\right) B_M$, given $z_0 \equiv \inf M$, and hence the convergence obtained is uniform on M . \square

Weight function convergence

We now obtain some necessary convergence results for the posterior $w(\cdot; \delta)$. For convenience, we refer to the partial derivatives of \tilde{w} as $\tilde{w}_p(p, w, x; \delta) = \frac{\partial}{\partial p} \tilde{w}(p, w, x; \delta)$, with same notation used for derivatives with respect to w , x and δ ; we also denote $\tilde{w}(x; \delta) \equiv \tilde{w}(p(x; \delta), w(x; \delta), x; \delta)$ and the total derivative $\frac{d}{dx} [\tilde{w}(x; \delta)]$ as $d_x \tilde{w}(x; \delta)$. From Lemma 17, it is easy to show that, as $\delta \rightarrow 0$, $\tilde{w}(x; \delta) \rightarrow \omega_0(x)$ and, using direct differentiation,

$$\begin{aligned} (\tilde{w}_p(x; \delta), \tilde{w}_w(x; \delta), \tilde{w}_x(x; \delta)) &\rightarrow \left(\frac{1}{x} \frac{\omega'_0(x)}{m'_0(x)}, 0, -\frac{m_0(x)}{x} \frac{\omega'_0(x)}{m'_0(x)} \right) \\ \tilde{w}_\delta(x; \delta) &\rightarrow (1-x) [1 - \omega_0(x)] \omega_0(x) \left\{ [1 - \omega_0(x)] \frac{\dot{\phi}_l(m_0(x))}{\phi_l(m_0(x))} + \omega_0(x) \frac{\dot{\phi}_h(m_0(x))}{\phi_h(m_0(x))} \right\}. \end{aligned} \quad (54)$$

Lemma 20. *For $\delta > 0$ sufficiently small and interval $I \subseteq (x_d(\delta), x_u(\delta))$ of size $D > 0$, there exists $x \in I$ such that*

$$|\dot{w}(x; \delta)| \leq \frac{1}{D}.$$

Proof. Consider an interval $[a, b]$ with $b - a = D$. Since $w(\cdot; \delta)$ has maximal variation of one in $[a, b]$, the mean value theorem implies the result. \square

Lemma 21. *For $\delta > 0$ sufficiently small and interval $I \subset (x_d(\delta), x_u(\delta))$ of size $D > 0$, there exists $x \in I$ and bound $B(\delta, D, x)$ such that*

$$\frac{|\tilde{w}(x; \delta) - w(x; \delta)|}{\delta} \leq B(\delta, D, x),$$

where $B(\cdot)$ is strictly positive, continuous and satisfies $B(0, D, x) < \infty$.

Proof. From Lemma 20 we can find $x \in I$ such that

$$\left| (1 - w(x; \delta)) \frac{1 - 2x}{x(1-x)} + \frac{1}{\delta} \left[\frac{\rho_0 - \delta/2}{x} - \frac{w(x; \delta) [1 - \tilde{w}(x; \delta)]}{\left[\frac{\tilde{w}(x; \delta) - w(x; \delta)}{\delta} \right] (1-x)} \right] \right| \leq \frac{1}{D}, \quad (55)$$

which implies that, using $\tilde{w}(x; \delta) - w(x; \delta) > 0$ and $\delta > 0$ sufficiently small:

$$\frac{|\tilde{w}(x; \delta) - w(x; \delta)|}{\delta} \leq \frac{xw(x; \delta) [1 - \tilde{w}(x; \delta)]}{(1-x)(\rho_0 - \delta/2) + \delta [1 - w(x; \delta)] (1-2x) - \frac{\delta x(1-x)}{D}}.$$

Now define $B(\delta, D, x)$ as the right-hand side of this last inequality. \square

Lemma 22. (*Level convergence of w*) For any $x \in (x_L, 1)$, $\limsup_{\delta} \left[\sup_{x \in M} \frac{|w(x; \delta) - \tilde{w}(x; \delta)|}{\delta} \right] < \infty$, which implies that $w(x; \delta)$ converges to $\omega_0(x)$. This convergence is uniform for any compact $M \subset (x_L, 1)$.

Proof. Suppose, by way of contradiction, that there exist sequences $(z_n)_n$ in $M \subset (x_L, 1)$ and $(\delta_n)_n$ such that $\frac{|w(z_n; \delta_n) - \tilde{w}(z_n; \delta_n)|}{\delta_n} \rightarrow \infty$. We first show that one can find another convergent sequence $(x_n)_n$ in M such that

$$\frac{|w(x_n; \delta_n) - \tilde{w}(x_n; \delta_n)|}{\delta_n} \rightarrow \infty \text{ and } \dot{w}(x_n) = d_x \tilde{w}(x_n; \delta_n).$$

We then show that the existence of such sequence leads to a contradiction.

Denote $z_0 \equiv \inf M$ and $z_1 \equiv \sup M$. Consider $D \in (0, \frac{1}{2} \min \{z_0 - x_L, 1 - z_1\})$ and define

$$K \equiv 1 + \sup_{x \in [z_0 - D, z_0] \cup [z_1, z_1 + D]} B(0, D, x).$$

Continuity of B implies that, for n sufficiently large,

$$\sup_{x \in [z_0 - D, z_0] \cup [z_1, z_1 + D]} B(\delta_n, D, x) < K.$$

From Lemma 21, there exist sequences $(x_n^-)_n$ in $[z_0 - D, z_0]$ and $(x_n^+)_n$ in $[z_1, z_1 + D]$ such that

$$\max \left\{ \frac{|\tilde{w}(x_n^-; \delta_n) - w(x_n^-; \delta_n)|}{\delta_n}, \frac{|\tilde{w}(x_n^+; \delta_n) - w(x_n^+; \delta_n)|}{\delta_n} \right\} < K,$$

while defining $x_n \equiv \arg \max_{x \in [x_n^-, x_n^+]} |\tilde{w}(x; \delta_n) - w(x; \delta_n)|$, we have that

$$\lim_n \frac{|\tilde{w}(x_n; \delta_n) - w(x_n; \delta_n)|}{\delta_n} \geq \lim_n \frac{|\tilde{w}(z_n; \delta_n) - w(z_n; \delta_n)|}{\delta_n} = \infty.$$

Hence, $x_n \in (z_0, z_1)$ is an interior optimizer and, hence, satisfies the required properties.

We now show that the construction of sequence $(x_n)_n$ leads to a contradiction. The ordinary differential equation:

$$\dot{w}(x; \delta) = (1 - w(x; \delta)) \frac{1 - 2x}{x(1-x)} + \frac{1}{\delta} \left[\frac{\rho_0 - \delta}{x} - \frac{w(x; \delta) [1 - \tilde{w}(x; \delta)]}{\left[\frac{\tilde{w}(x; \delta) - w(x; \delta)}{\delta} \right] (1-x)} \right] \quad (56)$$

implies that $\dot{w}(x_n, \delta_n) \rightarrow \infty$. Now notice that

$$\begin{aligned} \dot{w}(x_n; \delta_n) - d_x \tilde{w}(x_n; \delta_n) &= [1 - \tilde{w}_w(p(x_n; \delta_n), w(x_n; \delta_n), x_n; \delta_n))] \dot{w}(x_n; \delta_n) \\ &\quad - \tilde{w}_p(p(x_n; \delta_n), w(x_n; \delta_n), x_n; \delta_n) \dot{p}(x; \delta_n) - \tilde{w}_x(p(x_n; \delta_n), w(x_n; \delta_n), x_n; \delta_n), \end{aligned}$$

which, using (54), diverges as $n \rightarrow \infty$. This is a contradiction with the construction of $(x_n)_n$. \square

Lemma 23. (*Derivative convergence of posterior*) For any compact $M \subset (x_L, 1)$, $\sup_{x \in M} |\dot{w}(x; \delta) - \dot{\omega}_0(x)|$ converges to zero uniformly on M .

Proof. Suppose, by way of contradiction, the desired convergence result fails. Since both $w(x; \delta)$ and $\omega_0(x)$ are twice continuously differentiable and $|w(x; \delta) - \omega_0(x)|$ converges uniformly to zero, we can find constant $\gamma > 0$ and sequence (x^n, δ^n) such that $x^n \in M$, $\delta^n \rightarrow 0$ and

$$\dot{w}(x^n; \delta^n) - \dot{\omega}_0(x^n) = \gamma, \quad (57)$$

$$\ddot{w}(x^n; \delta^n) - \ddot{\omega}_0(x^n) \leq 0.$$

But, using (49) we have that

$$\begin{aligned} \ddot{w}(x^n; \delta^n) &= -\dot{w}(x; \delta) \frac{1-2x}{x(1-x)} - \frac{d}{dx} \left[\frac{w(x; \delta)(1-\tilde{w}(x; \delta))}{(w(x; \delta) - \tilde{w}(x; \delta))(1-x)} \right] \\ &\quad + (1-w(x; \delta)) \frac{d}{dx} \left[\frac{1-2x}{x(1-x)} \right] + \frac{(\rho_0 - \delta/2)}{\delta} \frac{d}{dx} \left[\frac{1}{x} \right] \\ &= \frac{(\dot{w}(x; \delta) - d_x \tilde{w}(x; \delta)) w(x; \delta) (1-w(x; \delta)) + O(1)}{(\tilde{w}(x; \delta) - w(x; \delta))^2}, \end{aligned} \quad (58)$$

where $O(1)$ represents all the terms that are bounded uniformly in $(x, \delta) \in M \times (0, \bar{\delta})$, for $\bar{\delta} > 0$ sufficiently small. Also notice that (57) implies that \dot{w} is bounded and, hence, using (54) we have that

$$\lim_n d_x \tilde{w}(x; \delta) = \dot{\omega}_0(\lim x^n).$$

This, together with (58) imply that $\lim_n \ddot{w}(x^n; \delta^n) = \infty$, contradicting the definition of $(x^n, \delta^n)_n$. \square

Lemma 24. (*First-order approximation of posterior*) The posterior $w(x; \delta)$ is differentiable in δ at zero, for any $x \in (x_L, 1)$, and its derivative is given by

$$\lim_{\delta \rightarrow 0} \frac{w(x; \delta) - \omega_0(x)}{\delta} = w_\delta(x),$$

with $w_\delta(\cdot)$ described in (16). This convergence holds uniformly in any compact $M \subset (x_L, 1)$.

Proof. Consider a compact $M \subset (x_L, 1)$. From (49), using Lemma 23, we can see that on M

$$\lim_{\delta \rightarrow 0} \frac{1}{\delta} \left[\frac{\rho_0 - \delta/2}{x} - \frac{w(x; \delta) [1 - \tilde{w}(x; \delta)]}{\left[\frac{\tilde{w}(x; \delta) - w(x; \delta)}{\delta} \right] (1-x)} \right] = \dot{\omega}_0(x) - (1 - \omega_0(x)) \frac{1-2x}{x(1-x)}, \quad (59)$$

which, together with Lemma 22, imply that on M

$$\lim_{\delta \rightarrow 0} \frac{\tilde{w}(x; \delta) - w(x; \delta)}{\delta} = \frac{x\omega_0(x) [1 - \omega_0(x)]}{\rho_0(1-x)}. \quad (60)$$

Using the fact that $\omega_0(x) = \tilde{w}(m_0(x), x, w(x; \delta), x; 0)$, continuous differentiability of $\tilde{w}(\cdot)$, Lemma 18 and (54), we have that the following holds uniformly on M

$$\begin{aligned} \lim_{\delta \rightarrow 0} \frac{\tilde{w}(x; \delta) - w(x; \delta)}{\delta} &= \lim_{\delta \rightarrow 0} \frac{\tilde{w}(p(x; \delta), w(x; \delta), x; \delta) - \tilde{w}(m_0(x), x, w(x; \delta), x; \delta)}{\delta} \\ &+ \lim_{\delta \rightarrow 0} \frac{\tilde{w}(m_0(x), x, w(x; \delta), x; \delta) - \omega_0(x)}{\delta} + \lim_{\delta \rightarrow 0} \frac{\omega_0(x) - w(x; \delta)}{\delta} \\ &= \frac{\dot{\omega}_0(x)}{x \dot{m}_0(x)} p_\delta(x) + \lim_{\delta \rightarrow 0} \tilde{w}_\delta(p(x; \delta), w(x; \delta), x; \delta) - w_\delta(x), \end{aligned} \quad (61)$$

and hence (54), (60) and (61) give us the result. \square

Welfare approximation

In this section we obtain expressions for the second order approximation of equilibrium utilities, defined in (19). The equilibrium payoffs are related to quasi-linear payoff

$$U_i(\mu; \delta) = u(t_i(\mu; \delta); \mu, \rho_i(\delta)) - p(t_i(\mu; \delta); \delta),$$

by

$$V^i(\mu; \delta) = \gamma_i(U_i(\mu; \delta), \delta), \quad (62)$$

with function $\gamma_i \in C^2$ is defined by

$$\gamma_i(U, \delta) \equiv -\exp\{-\rho_i(\delta)[W - \mu + U]\}.$$

So we proceed by first characterizing the approximation terms of equilibrium allocation $t_i(\cdot; \delta)$, then quasi-linear utility $U_i(\cdot; \delta)$ and finally utility level $V^i(\mu; \delta)$.

In order to obtain approximation terms for utility levels, we need to use equilibrium behavior of demand functions $t_i(\mu; \delta)$, which satisfy $t_i(m_i(x; \delta); \delta) = x$. This is obtained through a series of lemmas. Notice that the type assignment functions satisfy, in the pooling region:

$$\begin{aligned} m_l(x; \delta) &= \frac{p(x; \delta)}{x} + \delta(1 - w(x; \delta))(1 - x), \\ m_h(x; \delta) &= \frac{p(x; \delta)}{x} - \delta w(x; \delta)(1 - x). \end{aligned}$$

Our previous approximation results imply that $m_i(x; \delta)$ has a quadratic approximation,

$$m_i(x; \delta) = m_0(x) + \delta m_{i,\delta}(x) + \frac{\delta^2}{2} m_{i,\delta\delta}(x) + o(x; \delta^2),$$

where $\lim_{\delta \rightarrow 0} \sup_{x \in M} \frac{o(x; \delta^2)}{\delta^2}$ for any compact $M \subset (x_L, 1)$.

Moreover, the first-order approximation coefficients are

$$\begin{aligned} m_{l,\delta}(x) &= \frac{p_\delta(x)}{x} + (1 - \omega_0(x))(1 - x), \\ m_{h,\delta}(x) &= \frac{p_\delta(x)}{x} - \omega_0(x)(1 - x), \end{aligned} \quad (63)$$

which implies that $m_{i,\delta}(\cdot)$ only depends on the distribution of types through $\omega_0(\cdot)$.

We can now state our limiting results regarding the demand functions.

Lemma 25. *The demand functions $t_i(\mu; \delta)$, for $i \in \{l, h\}$, satisfy, for $\mu \in (\mu_L, \mu_H)$,*

(a) *continuity:*

$$\lim_{\delta \rightarrow 0} t_i(\mu; \delta) = t_0(\mu);$$

(b) *first-order approximation:*

$$t_{i,\delta}(\mu) \equiv \lim_{\delta \rightarrow 0} \frac{t_i(\mu; \delta) - t_0(\mu)}{\delta} = -\frac{m_{i,\delta}(t_0(\mu))}{\dot{m}_0(t_0(\mu))};$$

(c) *second-order approximation:*

$$\begin{aligned} t_{i,\delta\delta}(\mu) &\equiv 2 \lim_{\delta \rightarrow 0} \frac{1}{\delta} \left\{ \frac{t_i(\mu; \delta) - t_0(\mu)}{\delta} - t_{i,\delta}(\mu) \right\} \\ &= -\frac{1}{\dot{m}_0(t_0(\mu))} \left\{ \ddot{m}_0(t_0(\mu)) [t_{i,\delta}(\mu)]^2 + 2\dot{m}_{i,\delta}(t_0(\mu)) t_{i,\delta}(\mu) + m_{i,\delta\delta}(t_0(\mu)) \right\}, \end{aligned}$$

with the convergence being uniform in any compact $C \subset (\mu_L, \mu_H)$. Additionally, $t_{i,\delta}(\mu)$ only depends on the type distribution through $\omega_0(x)$, for any $x \in (x_L, 1)$.

Proof. Consider a compact $C \subset (\mu_L, \mu_H)$, $i \in \{l, h\}$ and $\delta > 0$ sufficiently small. Let $x_1 \equiv t_0(\inf C - \varepsilon)$ and $x_2 \equiv t_0(\sup C + \varepsilon)$, for $\varepsilon < \min\{1 - \sup C, \inf C - x_L\}$. Continuity of $m_i(\cdot; \delta)$ in δ implies that, as $\delta \rightarrow 0$, $m_i(x_1; \delta) \rightarrow \inf C - \varepsilon$ and $m_i(x_2; \delta) \rightarrow \sup C + \varepsilon$. Hence, monotonicity of m_i and t_i imply that $t_i(m_i(x_1; \delta); \delta) < t_i(\mu; \delta) < t_i(m_i(x_2; \delta); \delta)$ and hence $T_i(\delta) \equiv \{t_i(\mu; \delta) \mid \mu \in C\}$ is a compact set contained in $[x_1, x_2] \subset (x_L, 1)$.

Hence, using uniform convergence of $m_i(\cdot; \delta)$ in $[x_1, x_2]$ we have that

$$m_i(t_i(\mu; \delta); \delta) - m_0(t_i(\mu; \delta)) = \delta m_{i,\delta}(t_i(\mu; \delta)) + \frac{\delta^2}{2} m_{i,\delta\delta}(t_i(\mu; \delta)) + o(t_i(\mu; \delta); \delta), \quad (64)$$

where $\sup_{\mu \in C} \frac{o(t_i(\mu; \delta); \delta)}{\delta} \rightarrow 0$.

Finally, using $m_i(t_i(\mu; \delta); \delta) = m_0(t_0(\mu)) = \mu$ we have

$$m_0(t_i(\mu; \delta)) - m_0(t_0(\mu)) = -[m_i(t_i(\mu; \delta); \delta) - m_0(t_i(\mu; \delta))],$$

and, since the right hand side satisfies approximation equation (64), we have a quadratic approximation of the left hand side, which holds uniformly in $\mu \in C$. Since $m_0(\cdot)$ is twice continuously differentiable and has strictly positive derivative in $[x_1, x_2]$, direct differentiation gives us the result. \square

We are now in position to state the main quasi-linear-payoff approximation result. Define $U^0(\mu) \equiv u(t_0(\mu); \mu, \rho_0) - p_0(\mu)$, for $\mu \in [\mu_L, \mu_H]$ and $\dot{p}_\delta(x) \equiv \frac{d}{dx}[p_\delta(x)]$.

Lemma 26. *For any $\mu \in (\mu_L, \mu_H)$ and $i \in \{l, h\}$, the payoff function has the following limiting behavior, for any compact $C \subset (\mu_L, \mu_H)$:*

(i) *continuity:*

$$\lim_{\delta \rightarrow 0} U_i(\mu; \delta) = U^0(\mu);$$

(ii) *first-order approximation:*

$$U_{h,\delta}(\mu) \equiv \lim_{\delta \rightarrow 0} \frac{U_h(\mu; \delta) - U^0(\mu)}{\delta} = -p_\delta(t_0(\mu)) - \frac{1}{4}(1 - t_0(\mu))^2,$$

$$U_{l,\delta}(\mu) \equiv \lim_{\delta \rightarrow 0} \frac{U^l(\mu; \delta) - U^0(\mu)}{\delta} = -p_\delta(t_0(\mu)) + \frac{1}{4}(1 - t_0(\mu))^2;$$

(iii) second-order approximation:

$$U_{h,\delta}(\mu) \equiv 2 \lim_{\delta \rightarrow 0} \frac{\frac{U_h(\mu; \delta) - U^0(\mu)}{\delta} - U_{h,\delta}(\mu)}{\delta} = - \left[\begin{array}{c} p_{\delta\delta}(t_0(\mu)) \\ + [\dot{p}_\delta(t_0(\mu)) - \frac{1}{2}(1 - t_0(\mu))] t_{h,\delta}(\mu) \end{array} \right],$$

$$U_{l,\delta\delta}(\mu) \equiv 2 \lim_{\delta \rightarrow 0} \frac{\frac{U^l(\mu; \delta) - U^0(\mu)}{\delta} - U_{l,\delta}(\mu)}{\delta} = - \left[\begin{array}{c} p_{\delta\delta}(t_0(\mu)) \\ + [\dot{p}_\delta(t_0(\mu)) + \frac{1}{2}(1 - t_0(\mu))] t_{h,\delta}(\mu) \end{array} \right],$$

with the convergence being uniform in C .

Proof. Consider compact $C \subset (\mu_L, \mu_H)$, $\delta > 0$ sufficiently small and $i \in \{l, h\}$ (the argument for $i = l$ is analogous). Uniform continuity in δ and monotonicity in x of $m_i(\cdot; \delta)$ imply that we can find x_1, x_2 such that $t_i(\mu; \delta) \subset [x_1, x_2] \subset (x_L, 1)$ (for details, see the proof of Lemma 25). Notice that the function $H : (x, p, \delta) \mapsto u(x; \mu, \rho_i(\delta)) - p$ is twice continuously differentiable, hence we have that

$$\begin{aligned} U_i(\mu; \delta) - U^0(\mu) &= [H(t_i(\mu; \delta), p(t_i(\mu; \delta); \delta), \delta) - H(t_i(\mu; \delta), p_0(t_i(\mu; \delta)), \delta)] \\ &\quad + [H(t_i(\mu; \delta), p_0(t_i(\mu; \delta)), \delta) - H(t_i(\mu; \delta), p_0(t_0(\mu)), \delta)] \\ &\quad + [H(t_i(\mu; \delta), p_0(t_0(\mu)), \delta) - H(t_0(\mu), p_0(t_0(\mu)), \delta)] \\ &\quad + [H(t_0(\mu), p_0(t_0(\mu)), \delta) - H(t_0(\mu), p_0(t_0(\mu)), 0)], \end{aligned}$$

where twice continuous differentiability of p_0 and Lemmas 18 and 25 imply that each of the four terms in the right hand side have a well-defined quadratic approximation with error term $o(\mu; \delta^2)$ satisfying $\lim_{\delta \rightarrow 0} \sup_{\mu \in C} \frac{o(\mu; \delta^2)}{\delta^2}$, implying that both first- and second-order derivative limits are well defined. The exact formulas obtained come from direct differentiation. \square

Comparing distributions

In Sections 5 and 6, we compare the utility obtained by each type under different type distributions. Consider any two distributions (ϕ_l^k, ϕ_h^k) , for $k \in \{A, B\}$, with the same support. We make the dependence of equilibrium objects by using superscripts, as in $V_i^k(\mu; \delta)$, for $i \in \{l, h\}$. The superscript is omitted whenever the equilibrium object does not depend on k .

Lemma 27. *For any type with risk $(\mu, i) \in (\mu_L, \mu_H) \times \{l, h\}$ and $k \in \{A, B\}$, the following hold:*

$$V^k(\mu, i) \equiv \lim_{\delta \rightarrow 0} V^k(\mu, i; \delta) = \gamma_i(U^0(\mu), 0),$$

$$V_\delta^k(\mu, i) \equiv \lim_{\delta \rightarrow 0} \frac{V^k(\mu, i; \delta) - V^k(\mu, i; 0)}{\delta} = \frac{\partial}{\partial U} \gamma_i(U^0(\mu), 0) U_{i,\delta}^k(\mu) + \frac{\partial}{\partial \delta} \gamma_i(U^0(\mu), 0),$$

and

$$\begin{aligned} V_{\delta\delta}^k(\mu, i) &\equiv 2 \lim_{\delta \rightarrow 0} \frac{1}{\delta} \left[\frac{V^k(\mu, i; \delta) - V^k(\mu, i; 0)}{\delta} - \frac{\partial}{\partial U} \gamma_i(U^0(\mu), 0) U_{i,\delta}^k(\mu) + \frac{\partial}{\partial \delta} \gamma_i(U^0(\mu), 0) \right] = \\ &\quad \left[\begin{array}{c} \frac{\partial^2}{\partial U^2} \gamma_i(U^0(\mu), 0) [U_{i,\delta}^k(\mu)]^2 + \frac{\partial^2}{\partial U \partial \delta} \gamma_i(U^0(\mu), 0) U_{i,\delta}^k(\mu) \\ + \frac{\partial}{\partial U} \gamma_i(U^0(\mu), 0) U_{i,\delta\delta}^k(\mu) \\ + \frac{\partial^2}{\partial \delta \partial U} \gamma_i(U^0(\mu), 0) U_{i,\delta}^k(\mu) + \frac{\partial^2}{\partial \delta^2} \gamma_i(U^0(\mu), 0) \end{array} \right], \quad (65) \end{aligned}$$

with convergence of the limits above being guaranteed uniformly on any compact set $C \subset (\mu_L, \mu_H)$.

Proof. The results follow from definition expression (62), twice differentiability of γ_i and the uniform convergence results in Lemma 26. \square

Proof of Lemma 4

We now consider distributions indexed by $k \in \{0\} \cup S$, with index $k = 0$ represents the prior and $k = s$ represents the conditional type distribution with signal realization $s \in S$. Since the signal considered is a pure risk signal, Lemma 26 and Proposition 3 imply that term $U_\delta(\mu, i)$ and, as a consequence, V_δ^k , do not vary with $k \in \{0, S\}$. So, we omit notation k in their expressions.

From Lemmas 26 and 27 we have that the signal effect is

$$\begin{aligned} \sum_{s \in S} \pi(s | \mu) V_i^s(\mu; \delta) - V_i^0(\mu; \delta) &= \sum_{s \in S} \pi(s | \mu) [V_i^s(\mu; \delta) - V_i^0(\mu; \delta)] \\ &= -\frac{\delta^2}{2} \frac{\partial}{\partial U} \gamma_i(U^0(\mu), 0) \sum_{s \in S} [\pi(s | \mu) p_{\delta\delta}^s(t_0(\mu)) - p_{\delta\delta}^0(t_0(\mu))] + o(\mu; \delta^2), \end{aligned}$$

with $\lim_{\delta \rightarrow 0} \sup_{\mu \in C} \frac{o(\mu; \delta^2)}{\delta^2} = 0$. The equality $\frac{\partial}{\partial U} \gamma_i(U^0(\mu), 0) = \frac{\partial v}{\partial p}(t_0(\mu), p_0(t_0(\mu)), \mu, \rho_0)$ gives us the result.

Appendix D - Signal disclosure and comparative statics

Proof of Proposition 4

(If): Consider an arbitrary $\varepsilon > 0$ and let $M_\varepsilon \subset (\mu_L, \mu_H)$ be compact such that $\int_{M_\varepsilon} \phi_l(\mu') + \phi_h(\mu') d\mu' > 1 - \varepsilon$. From Lemma 4, we can find $\bar{\delta} > 0$ such that, for any $0 < \delta < \bar{\delta}$ and $i = l, h$,

$$\sup_{\mu \in C} \frac{|o_i(\delta^2; \mu)|}{\delta^2} < \inf_{\mu \in M_\varepsilon} \frac{1}{2} \frac{\partial v}{\partial p}(t_0(\mu), p_0(t_0(\mu)), \mu, \rho_0) \Delta E(p(t_0(\mu))),$$

which implies that all types with risk level in M_ε have a strict interim improvement from the signal disclosure.

(Only if): Now suppose that there exist $\mu_1, \mu_2 \in (\mu_L, \mu_H)$ with $\mu_1 < \mu_2$ satisfying $\partial_2 D_{KL}(\pi(\cdot | \mu_1) || \pi(\cdot | \mu_2)) < 0$.

Now consider a sequence of absolutely continuous full-support distributions on $[\mu_L, \mu_H]$ that weakly converge to the Dirac measure $\delta_{\{\mu_2\}}$, with continuously differentiable densities $\{f^n(\cdot)\}_n$. It is easy to show that there exists C^1 functions $\omega_0^n \circ t_0 : [\mu_L, \mu_H] \mapsto (0, 1)$ such that $\omega_0^n \circ t_0(\cdot) [1 - \omega_0^n \circ t_0(\cdot)] = \frac{1}{4\bar{f}^n} f^n(\cdot)$, where $\bar{f}^n \equiv \sup f^n$.²³ Hence, for any density ϕ on $[\mu_L, \mu_H]$, consider type distributions

$$(\phi_l^n(\mu), \phi_h^n(\mu)) = (\omega_0^n \circ t_0(\mu) \phi(\mu), [1 - \omega_0^n \circ t_0(\mu)] \phi(\mu)),$$

for each n . The price effect of disclosing this signal under distribution $(\phi_l^n(\mu), \phi_h^n(\mu))$ on the price of coverage $x = t_0(\mu_1)$ is then given by

$$\Delta E(p^n(x)) = -\frac{1}{2\rho_0 \bar{f}^n} \int_{\mu_1}^{\mu_H} f^n(\mu) (1 - t_0(\mu)) \partial_2 D_{KL}(\pi(\cdot | \mu_1) || \pi(\cdot | \mu)) d\mu,$$

²³A simple calculation shows that ω_0^n must satisfy $\omega_0^n \circ t_0(\mu) = \frac{1}{2} \pm \frac{1}{2} \sqrt{1 - 4f^n(\mu)}$, for all $\mu \in [\mu_L, \mu_H]$. In fact, there are always at least two such solutions.

which converges, as $n \rightarrow \infty$, to

$$\lim_n \Delta E(p^n(x)) \bar{f}^n = -\frac{1}{2\rho_0} (1 - t_0(\mu_2)) \partial_2 D_{KL}(\pi(\cdot | \mu_1) || \pi(\cdot | \mu_2)) > 0.$$

This implies that the expected price effect on coverage $t_0(\mu_1)$ is strictly positive.

Proof of Proposition 5

First notice that

$$\frac{\partial}{\partial \tilde{\mu}} D_{KL}(\pi(\cdot | \mu) || \pi(\cdot | \tilde{\mu})) = \sum_{s \in S} \frac{\dot{\pi}(s | \tilde{\mu})}{\pi(s | \tilde{\mu})} [\pi(s | \tilde{\mu}) - \pi(s | \mu)], \quad (66)$$

where we have used the fact that, for any $\mu \in [\mu_L, \mu_H]$, $\sum_{s \in S} \dot{\pi}(s | \mu) = 0$.

From (66),

$$\frac{\partial}{\partial \tilde{\mu}} D_{KL}(\pi(\cdot | \mu) || \pi(\cdot | \tilde{\mu})) = - \sum_{s \in S} \pi(s | \tilde{\mu}) \frac{\dot{\pi}(s | \tilde{\mu})}{\pi(s | \tilde{\mu})} \left[\frac{\pi(s | \mu)}{\pi(s | \tilde{\mu})} - 1 \right] \quad (67)$$

$$= -\text{cov } \tilde{\mu} \left(\frac{\dot{\pi}(s | \tilde{\mu})}{\pi(s | \tilde{\mu})}, \frac{\pi(s | \mu)}{\pi(s | \tilde{\mu})} \right), \quad (68)$$

where $\text{cov } \tilde{\mu}$ represents the covariance across different signal realizations with respect to the measure described by $\{\pi(s | \tilde{\mu})\}_{s \in S}$.

Condition MLRP implies that, for any $\ell \in \{1, \dots, n-1\}$,

$$\frac{\pi(s_{\ell+1} | \tilde{\mu})}{\pi(s_{\ell} | \tilde{\mu})} > \frac{\pi(s_{\ell+1} | \mu)}{\pi(s_{\ell} | \mu)} \iff \frac{\pi(s_{\ell} | \mu)}{\pi(s_{\ell} | \tilde{\mu})} > \frac{\pi(s_{\ell+1} | \mu)}{\pi(s_{\ell+1} | \tilde{\mu})},$$

and

$$\frac{\dot{\pi}(s_{\ell+1} | \mu)}{\pi(s_{\ell+1} | \mu)} > \frac{\dot{\pi}(s_{\ell} | \mu)}{\pi(s_{\ell} | \mu)},$$

for almost all $\mu \in [\mu_L, \mu_H]$. Summing up, the first term in the covariance (67), evaluated at s_{ℓ} , is increasing in ℓ while the second term is decreasing in ℓ . Hence, the covariance is negative and the expression in (67) is positive.

The following lemma is an auxiliary result and shows that for the welfare comparison it suffices to consider the first- or second-order terms of the price function.

Lemma 28. *Suppose that, for every compact set $M \subseteq (x_L, 1)$, there exists $\delta > 0$ sufficiently small such that*

$$p_{\delta}^A(x) < p_{\delta}^B(x) \text{ or } p_{\delta}^A(x) = p_{\delta}^B(x) \text{ and } p_{\delta\delta}^A(x) < p_{\delta\delta}^B(x),$$

for all $x \in M$. Then, (ω^A, ϕ^A) welfare-dominates (ω^B, ϕ^B) .

Proof. Consider a compact set $M \subset (x_L, 1)$. Proposition 3 implies that

$$\frac{p^A(x; \delta) - p^B(x; \delta)}{\delta} = \frac{p^A(x; \delta) - m_0(x)x}{\delta} - \frac{p^B(x; \delta) - m_0(x)x}{\delta},$$

which converges, uniformly in M , to $p_{\delta}^A(x) - p_{\delta}^B(x)$ and implies the result if $p_{\delta}^A(x) < p_{\delta}^B(x)$. Alternatively, if

$p_\delta^A(x) = p_\delta^B(x)$, for all $x \in M$, we have that

$$\frac{p^A(x; \delta) - p^B(x; \delta)}{\delta^2} = \frac{\frac{p^A(x; \delta) - m_0(x)x}{\delta} - p_\delta^A(x)}{\delta} - \frac{\frac{p^B(x; \delta) - m_0(x)x}{\delta} - p_\delta^B(x)}{\delta},$$

which converges to $\frac{1}{2} [p_{\delta\delta}^A(x) - p_{\delta\delta}^B(x)]$ uniformly over M and implies again the result. The welfare results follow from Lemma 26. \square

Lemma 28 does guarantee uniform dominance with respect to δ .

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Online Appendix (Not for publication)

The statistical content of monotonicity

In this online appendix we show that monotonicity is equivalent to a simple statistical property related to the impact signal realizations have on the expected risk assessment of a small risk pool. Its connection with equilibrium analysis is discussed below.

Consider a set or pool of types with different levels of risk aversion and risk heterogeneity $\varepsilon > 0$, defined by:

$$T(\bar{\mu}, \varepsilon) \equiv \{(\bar{\mu}, \rho_h), (\bar{\mu} + \varepsilon, \rho_l)\},$$

and denote the expected risk level in this set as its cost C . For a signal π , the impact of signal realization $s \in S$ on the cost of pool T is given by

$$\Delta C^s(\bar{\mu}, \varepsilon) \equiv \mathbb{E}[\tilde{\mu} \mid (\tilde{\mu}, \tilde{\rho}) \in T(\bar{\mu}; \varepsilon), \tilde{s} = s] - \mathbb{E}[\tilde{\mu} \mid (\tilde{\mu}, \tilde{\rho}) \in T(\bar{\mu}; \varepsilon)].$$

Our equilibrium characterization shows that the price of a given coverage x is indirectly affected by the type distribution within pools with coverage levels above x . This top-down property of equilibrium prices implies that any changes in the cost of a pool T will indirectly affect the prices of lower coverage contracts consumed by consumers with lower risk level $\underline{\mu} < \bar{\mu}$. We now focus on the *indirect expected* effect that signal disclosure has on consumers with risk level $\underline{\mu}$ *through* its impact on the cost of pool T :

$$\Delta C(\bar{\mu}, \underline{\mu}; \varepsilon) \equiv \sum_{s \in S} \pi(s \mid \underline{\mu}) \Delta C^s(\bar{\mu}, \varepsilon).$$

The result below shows that monotonicity is equivalent to negativity of the indirect cost effect described here.

Proposition 9. *A signal $\pi(\cdot)$ is monotonic if, and only if for any full support continuous type distribution (ϕ_l, ϕ_h) , almost all $\underline{\mu}, \bar{\mu} \in (\mu_L, \mu_H)$ and $\varepsilon > 0$ sufficiently small,*

$$\Delta C(\bar{\mu}, \underline{\mu}, \varepsilon) < 0.$$

Proof. For brevity, define $\pi_0(\cdot) \equiv \frac{1}{\#S}$. We then have that, for $k \in S \cap \{0\}$,

$$\begin{aligned} \Delta C^s(\bar{\mu}, \varepsilon) &\equiv \varepsilon \frac{\pi(s \mid \bar{\mu} + \varepsilon) \phi_l(\bar{\mu} + \varepsilon)}{\pi(s \mid \bar{\mu} + \varepsilon) \phi_l(\bar{\mu} + \varepsilon) + \pi(s \mid \bar{\mu}) \phi_h(\bar{\mu})}, \\ \Delta C(\bar{\mu}, \underline{\mu}, \varepsilon) &= \varepsilon \left[\sum_{s \in S} \pi(s \mid \underline{\mu}) \frac{\pi(s \mid \bar{\mu} + \varepsilon) \phi_l(\bar{\mu} + \varepsilon)}{\pi(s \mid \bar{\mu} + \varepsilon) \phi_l(\bar{\mu} + \varepsilon) + \pi(s \mid \bar{\mu}) \phi_h(\bar{\mu})} - \frac{\phi_l(\bar{\mu} + \varepsilon)}{\phi_l(\bar{\mu} + \varepsilon) + \phi_h(\bar{\mu})} \right] \\ &= \varepsilon \sum_{s \in S} \pi(s \mid \underline{\mu}) \int_0^\varepsilon \frac{\partial}{\partial z} \left[\frac{\phi_l(\bar{\mu} + \varepsilon)}{\phi_l(\bar{\mu} + \varepsilon) + \frac{\pi(s \mid \bar{\mu})}{\pi(s \mid \bar{\mu} + z)} \phi_h(\bar{\mu})} \right] dz. \end{aligned}$$

These imply that

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \frac{\Delta C(\bar{\mu}, \underline{\mu}, \varepsilon)}{\varepsilon^2} &= \sum_{s \in S} \pi(s | \underline{\mu}) \frac{\partial}{\partial z} \left[\frac{\phi_l(\bar{\mu} + \varepsilon)}{\phi_l(\bar{\mu} + \varepsilon) + \frac{\pi(s|\bar{\mu})}{\pi(s|\bar{\mu}+z)} \phi_h(\bar{\mu})} \right] \Big|_{z=\varepsilon=0} \\ &= \sum_{s \in S} \pi(s | \underline{\mu}) \frac{\dot{\pi}(s | \bar{\mu})}{\pi(s | \bar{\mu})} \frac{\phi_l(\bar{\mu}) \phi_h(\bar{\mu})}{[\phi_l(\bar{\mu}) + \phi_h(\bar{\mu})]^2}, \end{aligned}$$

which is strictly negative, for any full support continuous (ϕ_l, ϕ_h) and almost all $\mu_L < \underline{\mu} < \bar{\mu} < \mu_H$ if, and only if $\sum_{s \in S} \pi(s | \underline{\mu}) \frac{\dot{\pi}(s|\bar{\mu})}{\pi(s|\bar{\mu})} < 0$ for almost all $\mu_L < \underline{\mu} < \bar{\mu} < \mu_H$, i.e., monotonicity holds. \square

While Proposition 9 does not use equilibrium objects, these two are connected since, in equilibrium, prices are determined by the average riskiness of risk pools and the top-down property of prices is represented by the indirect cost assessment introduced here.