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Centre for Economic Policy Research
33 Great Sutton Street, London EC1V 0DX, UK
Tel: +44 (0)20 7183 8801
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Keywords: real options, mean-reverting growth, Irreversible investment, real-estate construction, urban growth

Rutger-Jan Lange - rutger-jan@outlook.com
Erasmus University Rotterdam

Coen N Teulings - c.n.teulings@outlook.com
Utrecht University and CEPR

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Rutger-Jan Lange

Coen N. Teulings*

September 16, 2021

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1 Introduction

The street map of Manhattan bears witness to the persistence of urban structures. The rectangular grid between Houston Street and 155th Street was laid out in 1811. Two centuries later, the areas to the north and south of Houston street still look worlds apart. Many of today's buildings were constructed more than a century ago. Similarly, the grid of Amsterdam's famous canal zone was laid out in the 17th century. Most houses still lining the canals today were the first ever erected there, on land previously devoted to agriculture. In Paris, Haussmann's famous boulevards were constructed in the second half of the 19th century, with many present-day buildings built around the same time. Today's investment decisions are therefore likely to give rise to urban structures that remain in place for a century or more. Given the cost of retroactively changing the density of construction (i.e. floorspace per unit of land, see Glaeser and Gyourko, 2005), it is natural to assume that (the density of) construction on a given plot of land is irreversible, in both the up- and downward directions.

Beyond urban structures, urban growth rates, too, are highly persistent. The growth rate of the population of urban regions mean reverts slowly, at a rate of only ∼15% per annum for the United States (Campbell et al., 2009, table 3). A city that is currently growing faster than nationwide average can therefore be expected to continue doing so for ∼6 years. The rental cash flow at a fixed location within the city tends to be positively related to the city's population (e.g. Albouy et al., 2018; Combes et al., 2019; Davis et al., 2021). Persistence in population growth thus implies persistence in rental price growth. This has indeed been found empirically, with rental price growth being a strong predictor of price-to-rent ratios (Sinai and Souleles, 2005, table 3).

The main contribution of this paper is to show that the combination of two forms of persistence, in urban structures and urban growth rates, has far-reaching implications for the optimal size and timing of investment in new construction. One would expect investors to engage in new construction when growth is high and the city is booming, while postponing investment when growth is low. Taking into account both forms of persistence, however, we reach the opposite conclusion. A rational investor postpones investment not because a city's prospects are gloomy, but because they are bright. When growth rates are high, she optimally retains the option to invest: it may (or may not) be optimal to erect a larger structure at a later point in time, suppressing construction today. The investor rationally postpones investment to await greater clarity on this point of uncertainty. This is not, we argue, an esoteric mathematical finding. For empirically relevant parameter values, investors optimally postpone investment after a positive growth shock.

This finding contrasts with that of classic real-options models, which take cash flows to be a geometric random walk.[1] In that case, there is only one state variable (the cash flow), which facilitates a closed-form solution for the optimal investment strategy. The option value of vacant land in the proximity of a city is then independent of the city's current growth rate, which, in the classic model, is not predictive of future growth rates. To account for the persistence in growth rates found empirically, we deviate from the classic model by taking the *growth rate* of the potential cash flow to be an Ornstein-Uhlenbeck process.

[1]See e.g. Titman (1985), Geltner (1989), Smith (1984), Quigg (1993), Williams (1993), Grenadier (1996), Merton (1998), Foo Sing (2001), Plazzi et al. (2008) and Peng (2016) for applications in real estate.

This deviation implies that both the *level* and the *growth rate* of the cash flow are state variables of the problem. We might expect a trade-off between both state variables at the moment of investment, in the sense that the level and the growth rate could be substitutes when considering the investment decision: at least one should be high. In solving the model, however, we find them to be complements rather than substitutes. When growth is high, the investor also requires higher cash flows in order to invest. In other words, the critical level of the cash flow typically increases with the growth rate, entailing a positive relation between growth rates and cash flows at the (optimal) moment of investment.

If initially puzzling, this positive relation can be explained by the fact that the investment is variable a priori but fixed a posteriori. The irreversibility of the investment decision is crucial in that if the investor were able to adjust the size of the building over time, she would start with a small, less capital-intensive structure and modify it later. When growth is high, the investor may be inclined to erect a large structure. However, current rental prices may not cover the capital cost of such a large structure. In this case, it is optimal for developers to wait until a high growth rate has pushed rental prices up before committing to a structure of fixed size. Waiting also generates valuable information, allowing the developer to better tailor the investment to dynamic market conditions—and such new information is, of course, only valuable before the (permanent) structure has been realised.

The above arguments in favour of waiting are contingent on the investor's flexibility in determining the scale of the project. When the size of the building is a priori fixed, due to e.g. regulation or in the case of a Leontief technology, the value of waiting is substantially reduced. In this case, we arrive at the standard finding that the level and the growth rate of the cash flow are substitutes. Our counterintuitive finding that they are complements rather than substitutes applies to investment decisions for which (i) the growth of cash flows is, to some extent, predictable and (ii) the scale of the project is variable, but only at the moment of construction, after which it is fixed. This mechanism can be expected to play a role in all investment decisions concerning urban development and land use more generally.

We start our analysis at the level of an investor who owns a single plot of land facing an exogenous dynamic process for the rental cash flow, of which the growth rate follows an Ornstein-Uhlenbeck process. Using standard production functions (either Cobb-Douglas or Stone-Geary) turning land and investment into rentable floorspace, we prove the existence of a critical growth rate above which investment is never optimal, regardless of the current cash flow level generated by each unit of floorspace. For the Cobb-Douglas case, an investor should invest as soon as the growth rate drops below this critical threshold. For the Stone-Geary case, which is empirically the most relevant, the investment decision depends on both the level of the cash flow and its growth rate. For the majority of investment decisions—80% at our benchmark parameters—these are positively related at the moment of investment, implying that they act as complements. In these cases, a positive growth shock raises the minimum cash flow required in order to invest, making investment less rather than more likely.

Next, we show that this analysis can be naturally extended to the level of a city, where the city's population size and the rental cash flow at a particular location are equilibrium outcomes determined by supply and demand. The demand for floorspace is a function of the city's productivity, the growth rate of which again follows an Ornstein-Uhlenbeck process. Received wisdom suggests that larger cities tend to have denser structures, i.e. higher floorspace densities. However, classic real-options models incorporating irreversibility cannot replicate this empirical fact. In our model, the floorspace density of new construction is determined by the city's growth rate at the moment of construction. This goes against the standard monocentric model of cities, where the density of new construction depends only on the distance to the city centre and the size of the city. By accounting for the city's growth rate, our model is better able to explain high-density construction on the city's perimeter, as has occurred in e.g. New York City, Paris, and Amsterdam.

This article contributes to current policy debates on why "superstar" cities (so dubbed by Gyourko et al., 2013) attract relatively low levels of investment in construction even as rental rates and housing prices soar. The discouraging answer may be that this behaviour is a rational and efficient response in the context of persistent city growth and irreversible investment decisions. In short, the option to build in superstar cities is so valuable that it stifles current investment (even, as recent evidence suggests, after all regulatory hurdles have been cleared (Murray, 2020)).

Persistent growth in equity prices[2] and dividends[3] has been explored by authors in finance, but has received little attention in the real-options literature. Persistent dividend growth could help explain high price-to-earnings ratios of growth firms, as market expectations of continued growth boost equity prices (e.g. La Porta, 1996; Chan et al., 2003; Chen, 2017). That housing can be viewed as an asset that generates "dividend" (i.e. rent) is well known (e.g. Sinai and Souleles, 2005; Fairchild et al., 2015). To the best of our knowledge, however, the problem of determining the net present value of a cash-flow stream involving persistent growth has not been solved. Our closed-form solution may thus be applicable more widely, including to the valuation of growth stocks.

Consideration of persistent growth rates may also contribute to our understanding of the excess volatility in asset prices relative to stable underlying cash flows. This is in line with Bansal and Yaron (2004), who argue that growth rate changes can lead to asset price fluctuations, and Giglio and Kelly (2017), who show that growth expectations may lead to excess volatility, especially when these expectations subsequently fail to be realised.

It is beyond the scope of this introduction to provide a comprehensive overview of the many previous applications of real-options theory to investment decisions; here we mention just a few of the most relevant. In their seminal text, Dixit and Pindyck (1994) discuss the option value when the level (rather than the growth) of the cash flow is mean-reverting. Several authors have assumed investment to be irreversible or the size of investment to be endogenous (e.g. Bertola and Caballero, 1994; Abel and Eberly, 1996). In the context of cities, Arnott and Lewis (1979) and Capozza and Helsley (1989) apply a model with permanent growth rate differentials between cities to show that land close to fast-growing cities (i) commands a higher option value and (ii) will be developed with a higher density of construction. The model that most closely resembles ours is that of Capozza and Li (1994), who allow for stochastic non-persistent growth around a deterministic trend that differs between cities.[4] The inclusion of persistent but uncertain growth rates goes beyond classic real-options models, e.g. Titman (1985). In particular, persistent growth

[2]See e.g. Keim and Stambaugh (1986), Lee (1992), McQueen and Roley (1993), Kandel and Stambaugh (1996), Stambaugh (1999), Patelis (1997) and Huang and Liu (2007).

[3]See e.g. Scheinkman and Xiong (2003), Dumas et al. (2009) and Andrei and Hasler (2014).

[4]See their discussion on the comparative statics of growth rate g on pp. 896-897. They suggest using a model with stochastic rather than deterministic growth rates, as we do in this paper. They conjecture some of the results we document, but not the upward-sloping relation between level and growth of cash flow along the optimal investment boundary.

rates imply that land on the outskirts of superstar cities should be more valuable than would be the case if the valuation were based on standard assumptions (e.g. no persistence in growth) as in Davis et al. (2014) and Combes et al. (2017).

Our analysis is demanding from a technical point of view, since (i) the problem has two state variables: the level of the cash flow and its growth; (ii) the option is of the American type, meaning that the decision maker can exercise the option at any time; and (iii) the stochastic process is parabolic rather than elliptic, implying that classic tools such as smooth pasting do not apply in both spatial directions. Nevertheless, we are able to provide an analytic expression, involving the generalised Hermite polynomial and Kummer's (confluent hypergeometric) function, which determines the critical growth rate above which investment is never optimal.

From a technical perspective, our work is related to the growing body of literature on optimal stopping in multidimensional models, e.g. Rogers (2002), Andersen and Broadie (2004), Bally and Printems (2005) and Strulovici and Szydlowski (2015). The latter argue that there is a need "for a better understanding of the properties of optimal policies and value functions with a multidimensional state space" and "for constructing explicit solutions" (Strulovici and Szydlowski, 2015, p. 1042). To solve our most general model, we apply a recent, robust method for constructing solutions, known as the Poisson optional stopping times (POST) method (Lange et al., 2020), which finds the value function as an increasing sequence of lower bounds; this property persists after discretisation when using standard finite-difference stencils. The theoretical properties of the algorithm (monotone and geometric convergence) imply that the discretised problem can be solved reliably.

In Section 2, we set out the model for a single plot of land, while Section 3 states the optimality conditions for a single investor. Section 4 presents the solution to the model, involving three value functions: (i) for an existing structure with a fixed floorspace, and for the option value of a vacant plot of land assuming (ii) a Cobb-Douglas production function and (iii) a Stone-Geary production function, which includes Leontief as a special case. For the first two cases, we present closed-form solutions, while for the third we use numerical methods. Section 5 generalises the model for a single plot of land to a model for cities. Section 6 positions our findings in the wider debate on agglomeration externalities and possible market failure in cities, suggesting that rational investor behaviour may play a larger role in the postponement of construction than previously thought. Proofs are contained in the Appendix.

2 The Model

We consider an investor who owns a vacant plot of land on which she may erect a building. Both the timing and the size of investment are chosen optimally. She has access to a supply of capital at a real cost of capital for real estate $\rho > 0$, which is assumed to be constant over time.[5] The building, when erected, yields a cash flow of rents Y_t per unit of floorspace. For the sake of simplicity we assume the building can be constructed instantly (i.e. we ignore construction time) and starts producing revenues immediately. As of the moment of investment, the floorspace $F \geq 0$ of the building is fixed and can no longer be adjusted to changes in the supply of and demand for floorspace.

[5]While this assumption is somewhat unrealistic, Campbell and Shiller (1988) and Campbell et al. (2009) found that changes in the interest rate do not substantially affect asset prices and housing prices, respectively. Further, our solution method in Section 4.1 could be extended to allow for time variation in the interest rate.

The exponential growth rate of the cash-flow process $\{Y_t\}$ at time t is given by $\mu + X_t$, where $\mu \geq 0$ is the long-run growth rate—assumed for the sake of simplicity to be non-negative—and $\{X_t\}$ represents the mean-reverting "excess" growth rate, which follows an Ornstein-Uhlenbeck process with mean-reversion parameter $\theta > 0$ and infinitesimal variance of innovations in the growth rate $\sigma^2\theta^2 > 0$, i.e.

$$\mathrm{d}\ln Y_t = (\mu + X_t)\,\mathrm{d}t, \tag{2.1}$$

$$\mathrm{d}X_t = \theta\left(-X_t\,\mathrm{d}t + \sigma\,\mathrm{d}W_t\right). \tag{2.2}$$

Here $\mathrm{d}W_t$ is the increment of a standard Wiener process. Process (2.1)–(2.2) implies that there may be prolonged periods during which the cash flow $\{Y_t\}$ grows at rates either lower or higher than μ, especially when the rate of mean reversion θ is low. Lemma 1 in Appendix A demonstrates that the distribution of $(X_t, \ln Y_t)$ conditional on $(X_0, \ln Y_0)$ is bivariate normal. For future reference, Lemma 1 also implies that

(i) the steady-state distribution of X_t is $\mathrm{N}\left(0, \theta\sigma^2/2\right)$,

(ii) the infinitesimal variance of $\ln Y_t$ is zero, i.e. $\lim_{t\to 0} t^{-1}\mathrm{Var}(\ln Y_t|X_0, Y_0) = 0$,

(iii) the long-run variance of $\ln Y_t$ per unit of time is constant, i.e. $\lim_{t\to\infty} t^{-1}\mathrm{Var}(\ln Y_t|X_0, Y_0) = \sigma^2$,

(iv) The growth rate X_t and the level Y_t are positively correlated for all $t \geq 0$. A first-order Taylor expansion in time yields

$$\mathrm{Cor}(X_t, \ln Y_t|X_0, Y_0) = \frac{\sqrt{3}}{2}\left(1 - \frac{\theta}{8}t\right) + O(t^2), \tag{2.3}$$

which equals $\sqrt{3}/2 \approx 0.866$ for $t = 0$.

The new model (2.1)-(2.2) may be contrasted with the widely used classic model of real estate investment, which takes $\ln Y_t$ to be a Brownian motion with drift. Conveniently, as Part 4 of Lemma 1 shows, this classic model can be recovered as a limiting case of model (2.1) by taking $\theta \to \infty$:

$$\mathrm{d}\,\ln\,Y_t \;=\; \mu\,\mathrm{d}t + \sigma\,\mathrm{d}W_t, \tag{2.4}$$

in which X_t no longer features as a state variable. In the classic model, the variance of $\ln Y_t$ conditional on $\ln Y_0$ increases linearly with time. As such, the variance of $\ln Y_t$ per unit of time is constant at σ^2. In the new model (2.1)-(2.2), the variance per unit of time is constant at σ^2 only in the long run; recall point (iii) above. While the long-run variance increases linearly with time in both models, the term structure of risk in the short to medium term is different. In the new model, risk in the very short term is zero (recall point (ii) above), while in the intermediate term it is substantial, as it is driven by the stochastic drift $(\mu + X_t)\mathrm{d}t$, which may turn negative for extended periods of time. Furthermore, the positive correlation between X and Y (recall point (iv) above) implies that locations featuring the highest cash flows (high Y) tend to see the highest growth (high X). It follows that any target level Y exceeding Y_0 is more likely to be achieved at high rather than low growth rates X. Replacing the classic model (2.4) with the more general model (2.1)–(2.2) will have profound implications for the optimal investment decision.

Following Quigg (1993), we assume the investor is faced with a Stone-Geary production function transforming the plot of land and the investment K into rentable floorspace $F(K)$:

$$F(K) = (K - \phi)^{\alpha}, \qquad K \geq \phi, \tag{2.5}$$

where $\alpha \in [0, 1)$ and $\phi \geq 0$ are parameters. The total cash flow generated by the building equals $Y_t F(K)$, where Y_t is varying over time while $F(K)$ is fixed as of the moment of construction. The production function (2.5) has two important special cases:

(i) for $\phi = 0, \alpha \in (0, 1)$, we obtain a Cobb-Douglas production function with elasticity of substitution between land and construction equal to unity;

(ii) for $\phi > 0, \alpha = 0$, we obtain a Leontief production function with elasticity of substitution equal to zero, where the project requires a fixed investment ϕ yielding a fixed floorspace equal to unity.

For intermediate cases, i.e. $\alpha \in (0, 1)$ and $\phi > 0$, the elasticity of substitution is between zero and one, implying that the share of the cost of construction in the value of the new building decreases as the price of vacant land increases. The parameter ϕ can be interpreted either as (i) the fixed cost of construction that must be paid irrespective of the floorspace $F(K)$ created, or as (ii) the present value of the revenues from vacant land arising from e.g. agricultural use (this interpretation requires the present value of agricultural use to be constant over time).

For the interpretation of our results, we state benchmark parameter values of the model. All (fixed) model parameters are written in Greek font as follows:

Definition 1 (Benchmark Parameters)**.** *The benchmark parameters are $\rho = 0.06$, $\mu = 0.01$, $\sigma = 0.04$ and $\theta = 0.15$ measured on an annual time scale and $\alpha = 0.70$, where $\phi = 0$ for the Cobb-Douglas case and ϕ is normalised to unity for the Stone-Geary case.*

Drawing on Jordà et al. (2019, Table III), we set the real cost of capital for real estate at $\rho = 0.06$. For the unconditional growth rate of real rents, we take the average growth in land productivity found in Davis et al. (2014, p. 732) of $\mu = 0.01$. While the persistence of rent growth in fixed locations is difficult to measure empirically, Section 5 demonstrates that rent growth is driven by population growth. Our benchmark mean-reversion parameter $\theta = 0.15$ implies an annual autocorrelation of $1 - 0.15 = 0.85$, which is consistent with the observed autocorrelation of population growth in US metropolitan areas (Campbell et al., 2009, Table 3). Our benchmark value $\sigma = 0.04$ implies that the steady-state distribution of X is normal with mean zero and standard deviation $\sigma\sqrt{\theta/2} \approx 1.1\%$.

The parameter $\alpha = 0.70$ falls in the range $0.60 - 0.80$ reported in the literature (e.g. Davis et al., 2014). An elasticity of substitution of less than one has been reported in the production function of housing services between land and construction (Combes et al., 2017) and in consumption between land use and other consumption (Teulings et al., 2018).[6] The parameter ϕ is a scaling parameter that determines the levels of Y and X for which construction becomes profitable. Holding constant the excess growth rate X, increasing the fixed cost ϕ implies that the cash flow Y for which investment becomes optimal increases by some constant. Thus, our normalisation $\phi = 1$ does not entail a loss of generality (see Davis et al., 2021, for a similar argument).

[6] Using a Cobb-Douglas utility function for housing services and other consumption, the latter also implies an elasticity of substitution in the production of housing services of less than one.

3 Optimality Conditions for a Single Investor

The investor chooses the size and timing of construction to maximise the expected profits. Both depend on two state variables, the potential cash flow per unit of floorspace Y and its excess growth rate X. The value of the vacant plot of land *before* investment is denoted by $V(X,Y)$, while the value of a unit of floorspace *after* investment is $B(X,Y)$. The value $B(X,Y)$ equals the discounted integral of expected rents, as produced by one unit of floorspace. As is standard, the infinitesimal generator corresponding to process (2.1)–(2.2) is

$$\mathrm{L} := (\mu + X)\, Y \,\frac{\mathrm{d}}{\mathrm{d}Y} \;-\; \theta\, X\, \frac{\mathrm{d}}{\mathrm{d}X} \;+\; \frac{1}{2}\theta^2\sigma^2\, \frac{\mathrm{d}^2}{\mathrm{d}X^2}. \tag{3.1}$$

Using Bellman's dynamic-programming principle[7], we then have

$$\text{before investment:} \qquad \rho\, V(X,Y) \;=\; \mathrm{L}\, V(X,Y), \tag{3.2}$$

$$\text{after investment:} \qquad \rho\, B(X,Y) \;=\; \mathrm{L}\, B(X,Y) + Y. \tag{3.3}$$

Intuitively, equation (3.2) indicates that the return on the vacant land, $\rho\, V(X,Y)$, is driven only by the expected change in the state variables, as measured by $\mathrm{L}V(X,Y)$. After investment, the return on each unit of floorspace, $\rho B(X,Y)$, additionally contains the rent Y per unit of floorspace. Both differential equations are subject to the boundary condition $0 = V(X,0) = B(X,0)$ for all $X \in \mathbb{R}$: if the current cash flow Y equals zero, so will future cash flows (under the assumption of exponential growth).

When deciding to start construction, the investor chooses the investment K to maximise the total ("net") project value, i.e. the value after investment minus the cost of construction. The optimal investment $K^*(X,Y)$ maximises the value of the building minus construction costs:

$$K^*(X,Y) := \arg\max_K \left\{(K-\phi)^\alpha\; B(X,Y) \;-\; K\right\} = \left[\alpha B(X,Y)\right]^{1/(1-\alpha)} + \phi. \tag{3.4}$$

Standard manipulations and the definition of production function (2.5) result in

$$K^*(X,Y) - \phi = \alpha F[K^*(X,Y)]B(X,Y).$$

The term $F[K^*(X,Y)]B(X,Y)$ on the right-hand side, i.e. the amount of floorspace multiplied by its price per unit, denotes the value of the constructed building. By setting $\phi = 0$, we obtain a standard result for the Cobb-Douglas production function, i.e. investment $K^*\,(X,Y)$ is a fixed share α of the value of the (realised) building.

We define $B^*(X,Y)$ as the value of the building minus its construction cost, i.e. the value of the building net of investments, where the level of investment K is chosen optimally given the value of the state variables (X,Y):

$$B^*(X,Y) := \max_K \left\{(K-\phi)^\alpha\; B(X,Y) \;-\; K\right\} = \frac{1-\alpha}{\alpha}\left[\alpha B(X,Y)\right]^{1/(1-\alpha)} - \phi. \tag{3.5}$$

Hence $B^*(X,Y)$ is the *total net value* of an optimally sized building conditional on current state variables (X,Y). Conversely, recall that $B(X,Y)$ is a normalised value, i.e. the value

[7] Alternatively, equations (3.2) and (3.3) can be derived using contingent-claims analysis, in which case L is interpreted as the infinitesimal generator under the risk-neutral measure (see e.g. Dixit and Pindyck, 1994, p. 120).

of an existing unit of floorspace.

For investment to be optimal for some combination of the state variables $(X, Y) = (X^*, Y^*)$, the classic value-matching and smooth-pasting conditions[8] must hold:

$$V(X^*, Y^*) = B^*(X^*, Y^*), \tag{3.6}$$

$$\left.\frac{\mathrm{d}V(X,Y)}{\mathrm{d}X}\right|_{(X^*,Y^*)} = \left.\frac{\mathrm{d}B^*(X,Y)}{\mathrm{d}X}\right|_{(X^*,Y^*)}, \tag{3.7}$$

respectively. The smooth-pasting condition is imposed in the X direction but not the Y direction, as the cash-flow process $\{Y_t\}$ is deterministic on time scales of order $\mathrm{d}t$; in this case, the principle of smooth pasting is not guaranteed.[9]

4 The Solution

This section derives the value of empty land, i.e. $V(X, Y)$, and the value of an existing unit of floorspace, i.e. $B(X, Y)$. To arrive at the optimal investment policy, we start by computing the latter. While the derivation is more complicated than in the classic model (2.4), we are able to provide a closed-form solution for finite θ, which involves an integral that can be computed numerically. Next we derive the main results of this paper: the value of empty land and the optimal investment policy. Section 4.2 solves the Cobb-Douglas case ($\phi = 0$), for which we find that a rational investor should invest if and only if the growth rate X is *below* some critical threshold X^*. In other words, when the demand for floorspace is booming, developers should optimally postpone investment. Section 4.3 solves the Stone-Geary case ($\phi = 1$), using recently developed numerical methods. In this case, the optimal moment of investment depends on both X and Y, but the counterintuitive Cobb-Douglas finding persists: developers should not invest when the growth rate exceeds X^*. Only when fixing the size of the investment (e.g. in the case of a Leontief production function) can we do away with this result.

4.1 Value of Existing Structures

Proposition 1 gives a closed-form expression for the net present value (NPV) of a cash-flow stream $\{Y_t\}$ featuring persistent growth as in model (2.1)–(2.2).

Proposition 1 (Value of an Existing Unit of Floorspace)**.** *Define*

$$\rho_0 := \rho - \mu - \sigma^2/2. \tag{4.1}$$

Assume $\rho_0 > 0$ *and* $0 < \theta < \infty$.

1. *The present value of a unit of floorspace, i.e.* $B(X, Y)$*, equals*

$$B(X, Y) = Y\, b(X), \qquad \forall (X, Y) \in \mathbb{R} \times \mathbb{R}_{\geq 0}, \tag{4.2}$$

[8] See e.g. Bergemann and Välimäki (2000), Moscarini and Smith (2001), DeMarzo and Sannikov (2006) and DeMarzo et al. (2012).

[9] This is a subtle but often neglected point in option-valuation models with multiple state variables. Smooth-pasting conditions must be imposed in the direction of state variables that are continuous but non-differentiable with respect to time, such as (geometric) Brownian motions, but not in the direction of state variables that are both continuous and differentiable, such as our Y_t, in which case a value-matching condition is sufficient.

where the function $b(X)$ is defined for all $X \in \mathbb{R}$ as

$$b(X) := \int_0^\infty \exp\left[-\rho_0\, t + \left(X - \sigma^2\right)\frac{1-\mathrm{e}^{-\theta t}}{\theta} + \frac{\sigma^2}{2}\frac{1-\mathrm{e}^{-2\theta t}}{2\theta}\right]\mathrm{d}t. \qquad (4.3)$$

2. *The left- and right-hand tails of $b\,(X)$ satisfy*

$$1 = \lim_{X\to-\infty} b\,(X)\;(\rho - \mu - X)\,, \qquad (4.4)$$

$$\frac{1}{\theta}\Gamma\left(\frac{\rho_0}{\theta}\right) = \lim_{X\to\infty} b(X)\left(\frac{X}{\theta}\right)^{\rho_0/\theta}\exp\left(-\frac{X-\sigma^2}{\theta} - \frac{\sigma^2}{4\theta}\right), \qquad (4.5)$$

where $\Gamma(\cdot)$ is the Gamma function.

3. *Function $b\,(X)$ satisfies the differential equation*

$$1 + (\mu - \rho + X)\,b(X) - \theta\,X\,b'(X) + \frac{1}{2}\theta^2\sigma^2 b''(X) = 0. \qquad (4.6)$$

4. *Define*

$$b_k(X) := \theta^k b^{(k)}(X)/b(X), \quad \forall X \in \mathbb{R}, \forall k \in \mathbb{N}^+, \qquad (4.7)$$

where $b^{(k)}(X)$ denotes the k-th derivative of $b\,(X)$. The functions $b_1(X)$ and $b_2(X)$ are sigmoid functions, i.e. they are (i) strictly increasing in X, (ii) bounded between zero and one, while (iii) achieving these bounds in the limit $X \to -\infty$ and $X \to \infty$, respectively. Furthermore

$$b_1(X) > b_2(X) > 0.$$

5. *Function $\ln B(X,Y)$ satisfies the stochastic differential equation*

$$\mathrm{d}\ln B(X,Y) = \left\{\mu + [1 - b_1\,(X)]\,X + \frac{\sigma^2}{2}\left[b_2\,(X) - b_1\,(X)^2\right]\right\}\mathrm{d}t + \sigma b_1(X)\mathrm{d}W.$$

The drift term, in curly brackets, approaches $-\infty$ for $X \to -\infty$ and $\rho - \sigma^2/2$ for $X \to \infty$. The volatility, multiplying the term $\mathrm{d}W$, is bounded and increases with X.

Proposition 1 has three main implications. First, the cash flow Y enters multiplicatively in the value $B(X,Y)$, while X enters via the positive, increasing and convex function $b\,(X)$. The function $b(X)$ is shown in Figure 1 for our benchmark parameters and several values of θ, including $\theta \to \infty$; $b(X) = B(X,Y)/Y$ is the ratio of the property value over the rental price, i.e. the price-to-rent ratio. Figure 1 is consistent with Sinai and Souleles (2005, p. 765), who find empirically that "the price-to-rent ratio [...] increases with expected future rents, just as a price-earnings ratio for stocks should increase with expected future earnings."

The analogy with stocks suggests that function $b(X)$ can also be viewed as the price-dividend ratio. In their seminal work, Campbell and Shiller (1988) assume the logarithm of this ratio to be linear in the dividend growth rate, X; this implies that the function $b(\cdot)$ is approximated by an exponential function. The resulting formula has become known as the Campbell-Shiller log-linear approximate present value formula (e.g. Fairchild et al., 2015). Figure 1 shows that this approximation may be reasonable, where the slope is higher for lower rates of mean reversion. In the limit $\theta \to \infty$, i.e. immediate mean reversion, the

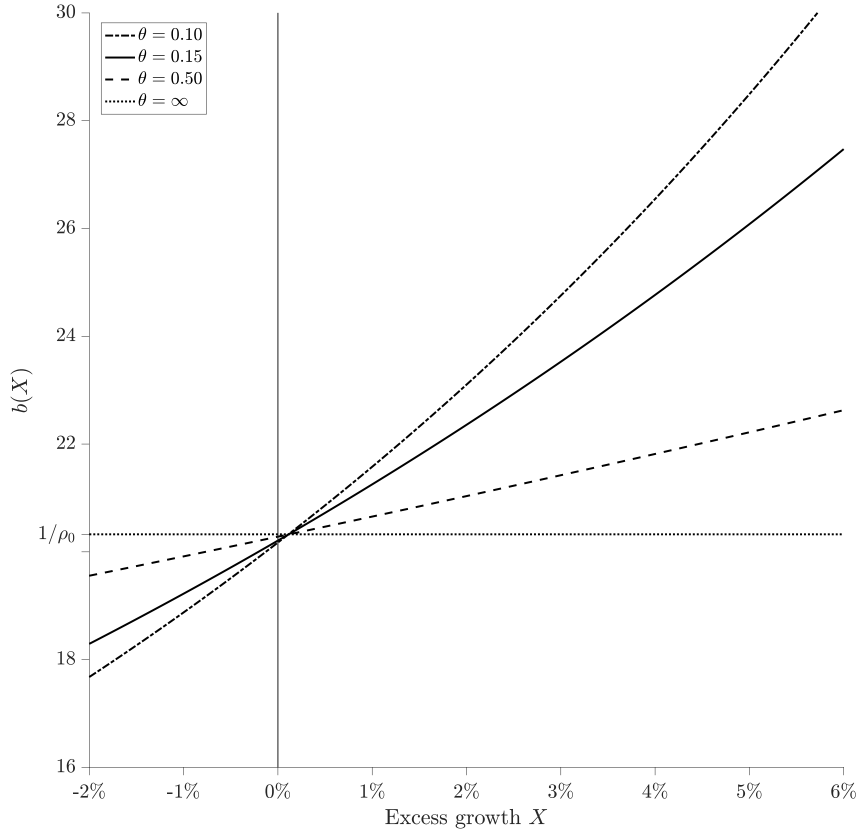


Figure 1: **Function $b(X)$ (as defined in equation 4.3) for different values of θ.**

slope is zero as X loses its predictive power. In the context of real estate, the rental yield is $b(X)^{-1} = Y/B(X,Y)$. In the classic case where $\theta \to \infty$ (see equation 2.4) the rental yield is constant at

$$\lim_{\theta \to \infty} Y/B(X,Y) = \rho_0. \tag{4.8}$$

Conversely, our model predicts that rental yields (or dividend-price ratios) are inversely related to the excess growth rate X, where the slope of this relation is determined by the persistence parameter θ.

Second, the drift of the stochastic process $\{\ln B(X_t, Y_t)\}$ is negative for low and positive for high values of X. Our finding that the growth rate of rents (or dividends) covaries with expected asset returns is consistent with empirical findings for both housing markets (e.g. Campbell et al., 2009) and stock markets (e.g. Vuolteenaho, 2002).[10] The fact that the drift is bounded above but not below (see Part 5 in Proposition 1) may explain why prices tend to increase gradually, but fall quickly. Interestingly, this instability is caused not by the variance term, as it would be for a geometric Brownian motion, but rather by the drift term, which is itself stochastic and may fall deep into negative territory. As noted in Sagi (2021, p. 3648), the key property of a geometric Brownian motion with drift is the "scaling of the mean and variance of log-price changes with the time between changes". It has recently been argued that this scaling property is structurally violated for property prices (e.g. Giacoletti, 2021, Sagi, 2021). In our model, these violations are explained by persistent rent growth, such that changes in property values are (partially)

[10] This literature typically suggests a behavioural explanation, where prices initially underreact to news, leading to a positive correlation of asset returns over time. Interestingly, the same asset-return pattern is implied by our model.

predictable even as the valuation of these properties is unbiased. Price predictability does not, in our model, imply market inefficiency. This is in contrast with e.g. Case and Shiller (1989, p. 125), who observe that "year-to-year changes in prices tend to be followed by changes in the same direction in the subsequent year" and conclude that "the market for single-family homes does not appear to be efficient". Housing prices in our model are similarly predictable, but without implying inefficient markets. We are unaware of other models with this property.

Third, short-term house-price volatility increases with the growth rate X, in line with the positive relation found empirically between the land share, also known as the land value ratio, and volatility of housing prices (e.g. Davis and Heathcote, 2007). This empirical relation is known as the land leverage hypothesis (e.g. Clapp et al., 2020) and remains unexplained in classic real-options models.

4.2 Option Value: Cobb-Douglas Case

For a Cobb-Douglas production function, the value function and investment policy can be found in closed form as presented in Proposition 2.

Proposition 2 (Option Value: Cobb-Douglas Case)**.** *Let* $\phi = 0$ *and* $\alpha \in (0, 1)$. *Define*

$$\rho_1 := \rho - \frac{\mu}{1-\alpha} - \frac{1}{2}\left(\frac{\sigma}{1-\alpha}\right)^2 < \rho_0. \tag{4.9}$$

Assume $\rho_1 > 0$ *and* $0 < \theta < \infty$.

1. *Investment is optimal if and only if* $X \leq X^*$ *for some critical value* $X^* \in \mathbb{R}$.

2. *For* $X > X^*$, *i.e. prior to investment, Bellman's equation (3.2) can be solved in closed form as follows:*

$$V(X, Y) = C\, Y^{1/(1-\alpha)}\, v\,(X)\,, \quad X > X^*, Y \in \mathbb{R}_{\geq 0}, \tag{4.10}$$

where $C > 0$ *is an integration constant. Function* $v(\cdot)$ *is*

$$v(X) := \exp\left(\frac{X}{\theta(1-\alpha)}\right) \mathrm{H}_{-\rho_1/\theta}\left[\frac{1}{\sqrt{\theta}\,\sigma}\left(X - \frac{\sigma^2}{1-\alpha}\right)\right], \quad X \in \mathbb{R}, \tag{4.11}$$

where $\mathrm{H}_n(x)$ *is the generalised Hermite polynomial defined in terms of Kummer's (confluent hypergeometric) function, denoted* $\mathrm{M}(\cdot,\cdot,\cdot)$, *as follows:*

$$\mathrm{H}_n(x) := 2^n \sqrt{\pi}\left[\frac{1}{\Gamma\left(\frac{1-n}{2}\right)} \mathrm{M}\left(-\frac{n}{2}, \frac{1}{2}, x^2\right) - \frac{2\,x}{\Gamma\left(-\frac{n}{2}\right)} \mathrm{M}\left(\frac{1-n}{2}, \frac{3}{2}, x^2\right)\right].$$

3. *It holds that* $X^* < X^\dagger$, *where* $X^\dagger$ *is the unique solution to*

$$1 - \rho\,\alpha\, b(X^\dagger) = \frac{\sigma^2}{2}\frac{\alpha}{1-\alpha} b_1(X^\dagger)^2 b(X^\dagger). \tag{4.12}$$

4. *The critical value* $X^* \in \mathbb{R}$ *is the unique solution to*

$$b_1(X^*) = (1-\alpha) v_1(X^*), \tag{4.13}$$

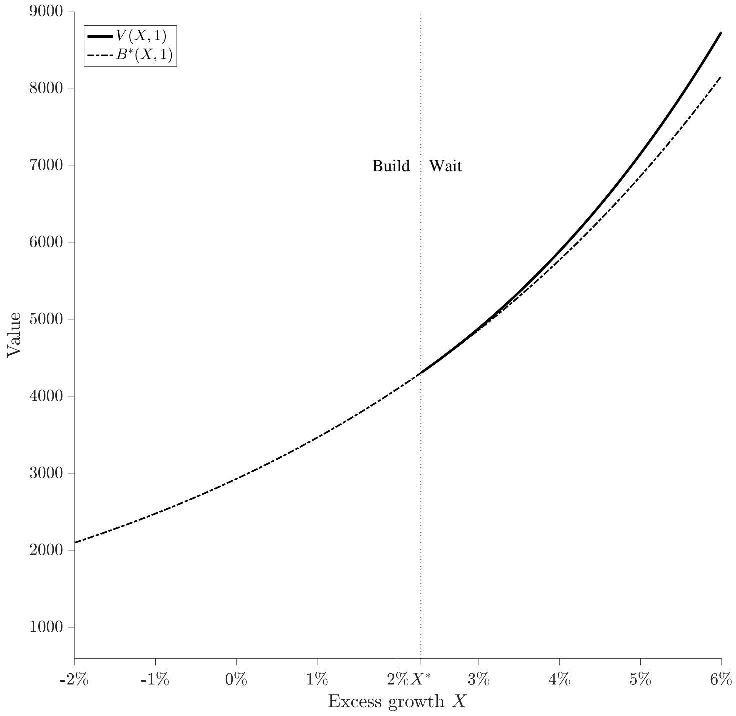


Figure 2: **Value of vacant land with Cobb-Douglas production function and benchmark parameters.** The value of vacant land under the optimal policy, i.e. $V(X,Y)$, is shown as a solid line for $X > X^* \approx 2.28\%$ and fixed $Y = 1$. The (sub-optimal) value obtained by immediately exercising the option to build, i.e. $B^*(X,Y)$, is shown as a dotted line, also for fixed $Y = 1$. Investment is optimal if and only if $X \leq X^* \approx 2.28\%$.

where $v_1(X) := \theta v'(X)/v(X)$ in analogy with equation (4.7). Equation (4.13) can be solved numerically. Conditional on X^*, the value-matching condition (3.6) gives the integration constant C in closed form.

The optimal strategy of the investor presented in Proposition 2 has two key characteristics. First, since $X \leq X^*$ is both a necessary and a sufficient condition for investment, the optimal moment of investment does not depend on the cash flow Y. Second, the optimal moment of investment depends on the growth rate X, but not in the way one would intuitively expect. We have grown accustomed to the classic real-options dictum (e.g Dixit and Pindyck, 1994) that investors should optimally wait until the value of investment is *high enough*. Our finding is diametrically opposed to this commonly held wisdom, suggesting instead that the decision maker ought to delay investment when the growth rate (hence the value) is high, and invest only if the growth rate (hence the value) is *low enough*. This finding—in lay terms: don't build when demand is booming—is, to the best of our knowledge, novel in the real-options literature. To illustrate, Figure 2 displays the option value of vacant land, $V(X,1)$ as a function of X, using the analytic formula in Part 2 of the Proposition. Unlike classic real-options models (e.g. those in the seminal work by Dixit and Pindyck (1994)), the option value $V(X,1)$ matches the value of immediately exercising the option at the point where $V(X,1)$ takes its lowest (rather than its highest) value. We have come across no other real-options models with this specific property.

While seemingly counterintuitive, the conclusion we draw from Proposition 2 (and Figure 2) is driven by two closely related mechanisms. First, higher cash flows Y im-

Table 1: Critical excess growth rates X^* above which investment is postponed in the Cobb-Douglas case.

Critical growth rate X^*	$\rho = 5.5\%$	6.0%	6.5%
$\alpha = 0.65$	3.37%	3.95%	4.45%
0.70	1.67%	**2.28%**	2.80%
0.75	−0.72%	0.36%	1.01%

Table 2: Steady-state probability that $X > X^*$.

$\mathbb{P}(X > X^*)$	$\rho = 5.5\%$	6.0%	6.5%
$\alpha = 0.65$	0.11%	0.02%	0.00%
0.70	6.73%	**1.86%**	0.53%
0.75	74.39%	37.23%	17.72%

ply higher floorspace values $B(X, Y)$, which in turn imply higher levels of investment $K^*(X, Y)$ (see equation 3.4). High growth rates X imply high future cash flows Y, pushing up the value $B(X, Y)$. This higher asset value makes it optimal to build a larger structure today (see equation 3.4). Growth prospects may be so bright that the investor is inclined to invest so much that the interest cost of this investment would not (initially, at least) be covered by the rents generated by the building, i.e. if $\rho\alpha B(X, Y) > Y$, or, equivalently, if $\rho\alpha b(X) > 1$. If this condition holds, the optimal response of the investor is to wait until high growth has sufficiently pushed up rental rates before making the (large and irreversible) investment.

The foregoing argument establishes that $X < X^+$ is necessary (but insufficient) for investment, where X^+ is determined by $\rho\alpha b(X^+) = 1$. This is related to Part 4 of Proposition 2, which establishes an even stricter condition, accounting not only for the interest cost of the investment but also for the information value of a brief delay. While $X < X^\dagger$ presented in Part 3 is a necessary (but insufficient) condition, Part 1 provides the optimal (necessary and sufficient) condition that $X \leq X^*$ (naturally, $X^* < X^\dagger$). Like the (suboptimal) threshold $X^\dagger$, the (optimal) threshold X^* incorporates the benefit of waiting for an infinitesimal time interval $\mathrm{d}t$. Unlike $X^\dagger$, however, the optimal threshold X^* additionally incorporates the option value of being able to delay *even longer* after waiting a short time $\mathrm{d}t$.

Table 1 provides the critical growth rates above which investors should optimally postpone investment for different values of α and ρ. For our benchmark parameters, the investor optimally postpones investment if X exceeds 2.28% (shown in bold). Table 2 reports steady-state (i.e. unconditional) probabilities that $X > X^*$ for various parameter values. These probabilities are implied by the optimal thresholds X^* in Table 1 and the steady-state distribution of X, which is normal with mean zero and standard deviation $\sigma\sqrt{\theta/2} \approx 1.10\%$. While the unconditional probability that X exceeds the critical growth rate is 1.86% for our benchmark parameters, as shown in bold, this number is highly sensitive to the choice of parameters, rising drastically as the cost of capital ρ is lowered or the construction share α is increased.

4.3 Option Value: Stone-Geary and Leontief Cases

For the Stone-Geary production function, which contains the Leontief as a special case for which $\alpha = 0$, we define the function $Y(X)$ to be the investment boundary between the investment region (also known as the exercise region), where the decision maker optimally invests, and the continuation region, where the decision maker optimally postpones investment. While an analytic solution for the the investment boundary is no longer available, the following Conjecture is natural and supported by the numerical results below.

Conjecture 1 (Option Value: Stone-Geary Case)**.** *Let $\phi > 0, \alpha \in (0,1)$ and $\theta < \infty$. Assume $\rho_1 > 0$, as defined in equation (4.9).*

1. *Investment is never optimal if $X \geq X^*$, where X^* is the Cobb-Douglas threshold established in Proposition 2. Hence, $Y(X) = \varnothing$ for $X \geq X^*$.*
2. *In the limit for low and high X, we have*

$$\lim_{X \uparrow X^*} Y(X) = \infty, \tag{4.14}$$

$$\lim_{X \to -\infty} Y(X)(\rho - \mu - X)^{-1} \geq \alpha^{-1}\left(\frac{\phi}{1-\alpha}\right)^{1-\alpha}. \tag{4.15}$$

Part 1 implies that the critical growth rate and the probability that an investor will postpone investment for any level of cash flow Y are identical for the Stone-Geary and the Cobb-Douglas case (see Tables 1 and 2). This is because, compared to the Cobb-Douglas case, the Stone-Geary production function requires an additional fixed expenditure ϕ on top of the variable investment. The resulting exercise region is conjectured to be smaller than for the Cobb-Douglas production function. Further, as the fixed investment ϕ becomes negligible relative to the rental cash flow as $Y \to \infty$, the exercise regions for the Stone-Geary and Cobb-Douglas production functions converge in this limit.

Part 2 indicates that $Y(X)$ converges to a straight line as $X \to -\infty$. For exceedingly negative values of X, the value-matching condition implies $B^*[X, Y(X)] = V[X, Y(X)] \geq 0$, as the option value must be non-negative. Substituting equation (3.5) for $B^*[X, Y(X)]$, equation (4.2) for $B[X, Y(X)]$ and applying the limit (4.4) yields equation (4.15).

For free-boundary problems in more than one dimension, analytic solutions are typically nonexistent. For our numerical procedure, the differential operator (3.1) is parabolic rather than elliptic, due to the absence of a second derivative with respect to Y; as a result, standard (i.e. elliptic) methods for partial differential equations (PDEs) cannot be used. We therefore use a robust numerical procedure called the Poisson optional stopping times (POST) method (Lange et al., 2020). This method assumes that opportunities to exercise the option (also known as *optional stopping times*) are generated by an independent Poisson process with intensity $0 < \lambda < \infty$. By taking λ to be large but finite, we can arbitrarily closely approximate the case $\lambda = \infty$, where opportunities to invest arrive continuously. The Poisson intensity $\lambda = 512$, used in our calculations below, implies that the investor can expect 512 investment opportunities per annum, i.e. more than one per day, and is sufficiently high to closely approximate the limit $\lambda \to \infty$. The POST algorithm as applied to our situation is

$$(\rho + \lambda - \mathrm{L})\, V^{(n+1)}(X, Y) = \lambda \max\left\{B^*(X, Y),\, V^{(n)}(X, Y)\right\}, \qquad n \in \mathbb{N}, \tag{4.16}$$

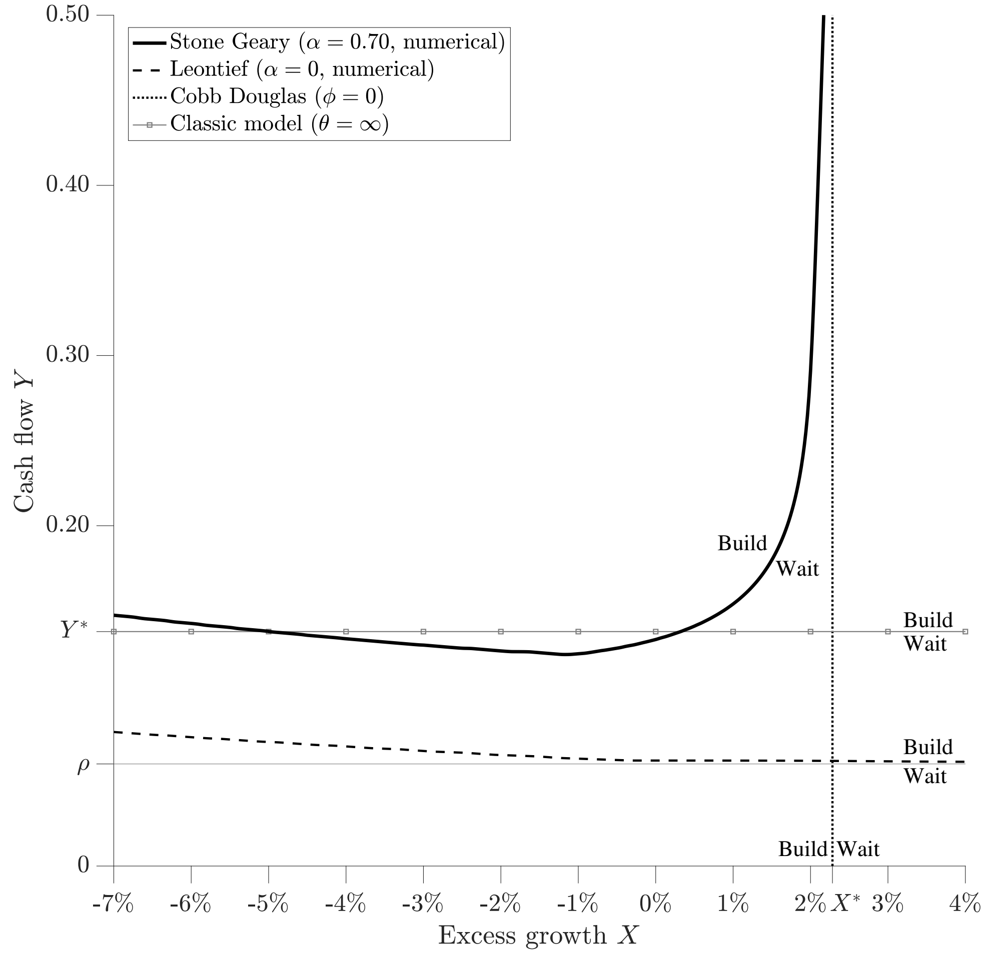


Figure 3: **Optimal exercise policy with Stone-Geary production function and benchmark parameters.** The critical value of the cash flow above which investment is optimal is shown as a solid curve for a Stone-Geary technology and a dotted curve for a Leontief technology. Also shown are the vertical investment locus for the Cobb-Douglas investor ($\phi = 0$), located at $X^* \approx 2.28\%$, and the horizontal investment locus for the classic model ($\theta = \infty$), located at $Y^* \approx 0.14$. Both Cobb-Douglas and Stone-Geary investors optimally postpone investment for any $X > X^*$. For exceedingly negative growth rates, the Stone-Geary investor requires higher and higher rents to be enticed to invest. As a result, the curve $Y(X)$ is U-shaped as a function of X, with a vertical asymptote on the right-hand side at X^*. For high growth rates, investors faced with a Leontief production function will invest as soon as the cash flow exceeds the interest cost of the investment, i.e. as soon as $Y \geq \rho \cdot \phi = \rho = 0.06$. In contrast with the Stone-Geary investment locus, the Leontief investment locus is entirely downward sloping and the counterintuitive finding that waiting is optimal for high growth rates disappears.

where L is the differential operator (3.1), $V^{(n)}(X,Y)$ is the value function $V(X,Y)$ after n iterations of the algorithm, and the algorithm is initialised using $V^{(1)}(X,Y) = 0$. Algorithm (4.16) produces an increasing[11] sequence of functions that converge monotonically and geometrically to the limiting solution $V(X,Y)$. For practical purposes, our numerical analysis is performed in a (finite-dimensional) vector space. The finite-difference stencil used to discretise the differential operator L ensures that the monotonic and geomet-

[11]In the sense that $V^{(n+1)}(X,Y) \geq V^{(n)}(X,Y), \forall X, Y$.

ric convergence properties of POST algorithm (4.16) persist through the discretisation, thereby ensuring that the discretised problem can be reliably solved (for details see Appendix D).

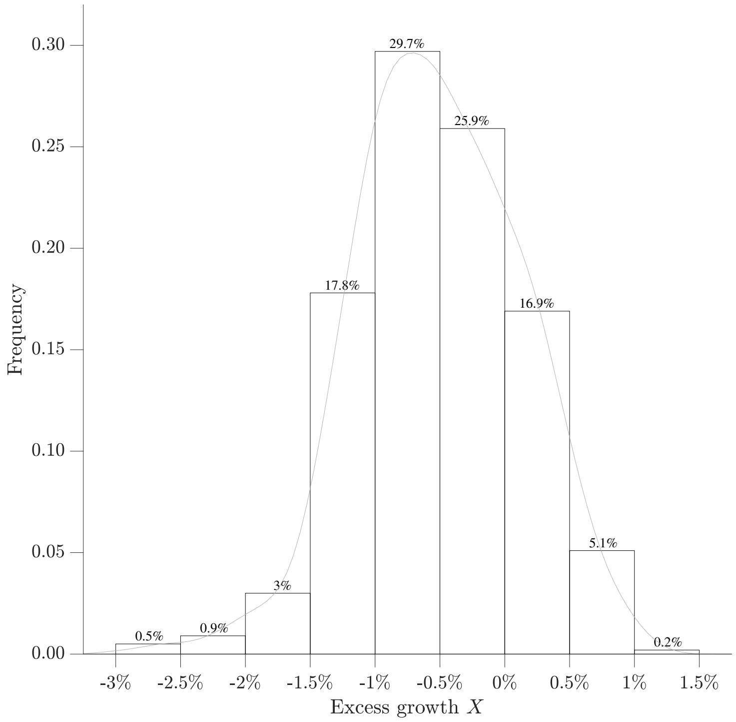
Figure 3 shows the optimal investment region obtained through the above procedure using our benchmark parameters. Consistent with Conjecture 1, the investment boundary $Y(X)$ turns out to be U shaped, where the vertical asymptote at $X = X^*$ corresponds to the critical value from the Cobb-Douglas solution (see Proposition 2). Hence, the Cobb-Douglas finding that investment dries up for high growth rates persists. The investment boundary $Y(X)$ achieves its minimum around $X = -1\%$, and is downward (upward) sloping to the left (right) of this point. Along the downward-sloping part of the investment locus, there is a standard trade-off between the level of the cash flow and its growth, which act as substitutes. Along the upward-sloping part, in contrast, they act as complements in that higher growth implies that a higher cash flow is required in order to invest. The Stone-Geary production function is crucial to this result; when the size of the investment is fixed, e.g. in the case of a Leontief production function, Figure 3 shows that the investment locus is entirely downward sloping. The counterintuitive finding that waiting is optimal for high growth rates then disappears.

Figure 3 also shows the optimal investment strategy in the classic model ($\theta \to \infty$), indicated by a straight horizontal line. In the classic model, the option value and investment decision are independent of X, hence $Y(X) = Y^*$, where Y^* can be derived analytically (see equation (5.9) in Proposition 4 below). Compared to the classic model, introducing persistence in growth rates can make investment more attractive, as evidenced by the fact that, for intermediate values of X, the investment boundary for the Stone-Geary production function falls below the horizontal line which is optimal for the classic real-options model.

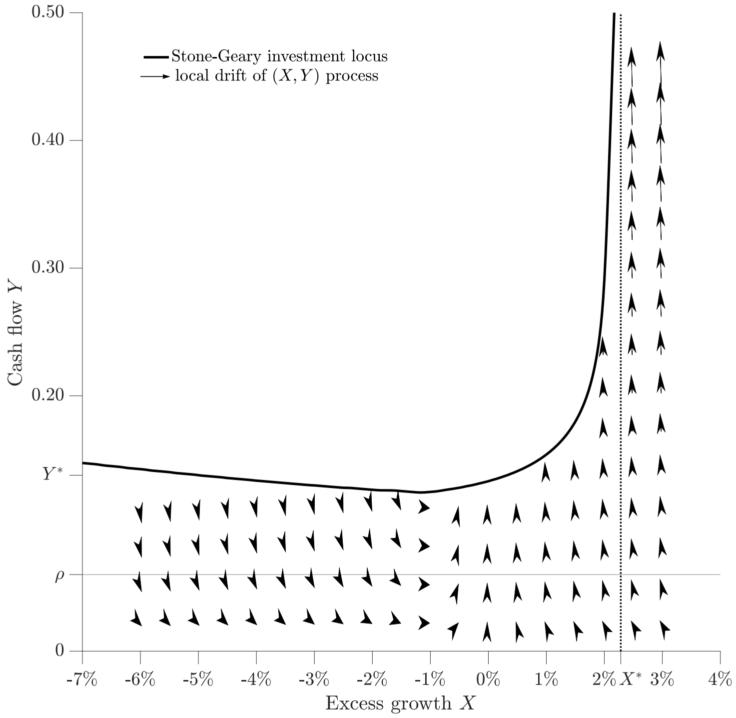
Panel A of Figure 4 shows the distribution of X at the moment of investment for the Stone-Geary production function, obtained by simulating 10,000 trajectories for the bivariate process (X_t, Y_t) starting from the line $Y_0 = \rho$, while X_0 is drawn from its steady-state distribution. We discretise equations (2.1) and (2.2) using time step $\Delta t = 0.01$ and time span $t_{\max} = 2{,}000$, which is sufficiently long to ensure that all simulated paths cross the Stone-Geary investment locus. For each path, we record the value of X at which the Stone-Geary investment locus is crossed. As Panel A shows, $\sim$20% of simulated paths cross the investment locus to the left of $X = -1\%$, i.e. on the downward-sloping part. Investment decisions for $X < -1\%$ correspond to cases where Y is expected to fall; recall that the drift of $\ln Y$ is $\mu + X$, where $\mu = 1\%$. The majority of paths, around 80%, cross the boundary on its upward-sloping part, i.e. for $X > -1\%$, where the drift of $\ln Y$ is positive. Along this upward-sloping part, positive growth shocks move the process (X, Y) away from the investment boundary, making investment less rather than more likely. For $\sim$80% of investment decisions, accelerating growth may thus cause rational investors to delay rather than speed up new investment.

Panel B of Figure 4 provides further insight as to why 80% of investment occurs on the upward-sloping part of the investment boundary by plotting the local drift of the bivariate process (X, Y) in conjunction with the Stone-Geary investment locus. The drift points away from the investment boundary along its downward-sloping part, making crossings in this region less likely, while pointing toward the investment boundary along its upward-sloping part, making crossings more likely.

Proposition 3 below formalises the intuition that on the upward-sloping part of the investment boundary, the size of the investment increases with the growth rate X.



Panel A: Distribution of X at the moment of investment



Panel B: Drift of (X, Y) process along with Stone-Geary investment locus

Figure 4: **Crossings of the Stone-Geary investment locus using benchmark parameters.** Panel A shows the distribution of X at the moment of investment, i.e. the value of X corresponding to the first crossing of the Stone-Geary investment locus, showing that only ∼20% of crossings occur for $X < -1\%$. Panel B shows the local drift of the bivariate process (X, Y), specified in equations (2.1) and (2.2), along with the investment locus. The drift points away from the investment boundary for X smaller than -1%, explaining why the majority of crossings occur to the right of this point, on the upward-sloping part of the boundary.

Proposition 3 (Optimal Investment in the Stone-Geary Case)**.** *Let $\phi > 0, \alpha \in (0,1)$ and $\theta < \infty$. Assume $\rho_1 > 0$, as defined in equation (4.9). Let $Y(X)$ be the investment boundary. If the investment boundary is upward sloping, i.e. $Y'(X) > 0$, then the optimal level of investment, $K^*[X, Y(X)]$ as defined in equation (3.4), increases with X:*

$$\frac{\mathrm{d}K^*[X, Y(X)]}{\mathrm{d}X} > 0.$$

Proposition 3 says that a higher growth rate X at the moment of investment leads to a higher level of investment K, resulting in a higher floorspace density $F(K)$ of the constructed building. Indeed, this finding translates to the level of cities. As we discuss in the next section, the higher floorspace densities in booming cities can be explained not by the size of these cities, but by their growth rate at the moment of the construction of new buildings.

5 Extension to Cities

5.1 City Model

This section extends our results for individual plots of land to the growth of cities. Our analysis uses ideas from standard models of agglomeration in cities (for recent examples, see Lucas and Rossi-Hansberg, 2002; Glaeser and Gyourko, 2005; Rossi-Hansberg and Wright, 2007; Ahlfeldt et al., 2015; Combes et al., 2019). We consider circular cities with a central business district (CBD). Locations within a city are characterised by their distance $r < R$ to the CBD, where $R > 0$ is the city's radius in two spatial dimensions. All jobs are located in the CBD and workers commute from home to the CBD. The CBD itself does not occupy any land; all land is used for residential purposes, as in Rossi-Hansberg and Wright (2007). Workers have a Cobb-Douglas utility function $U(f, c, r)$ with floorspace f, other consumption c and commuting cost as inputs. Commuting cost is an increasing function of the distance r to the CBD. Since workers are homogeneous and perfectly mobile both within and between cities, workers' utility is equal to some exogenous benchmark level U across locations within and between cities. For simplicity, we assume U to be constant over time; without loss of generality, we normalise it to unity. Hence

$$U(f, c, r) = \beta^{-\beta}(1-\beta)^{\beta-1} f^{\beta} c^{1-\beta} r^{-\psi\beta} = U = 1, \tag{5.1}$$

where $0 < \beta < 1$ is the share of floorspace in consumption, and $r^{-\psi\beta}$ with $0 < \psi < 2$ measures the utility cost of commuting.

Let A_{jt} be the productivity of a worker in city j at time t. Since labour is the only factor of production and since labour markets are perfectly competitive, wages are equal to productivity. Workers maximise their utility subject to their budget constraint: labour income should be equal to or greater than rental payments plus other consumption, i.e.

$$A_{jt} \geq f\ Y_{jt}(r) + c, \tag{5.2}$$

where $Y_{jt}(r)$ is the rental rate for a unit of floorspace for j, t, and r.

The production function of floorspace remains as in equation (2.5). All land in and surrounding cities is owned by private investors. They choose the size and time of investment in construction to maximise the value of the property after investment. The market for land is perfectly competitive. Finally, there is no central planner that levies taxes or awards subsidies to internalise agglomeration externalities.

Similar to Rossi-Hansberg and Wright (2007), the actual labour productivity A_{jt} is the product of an exogenous factor A^o_{jt} and the city's population N_{jt} raised to the power χ:

$$A_{jt} = A^o_{jt} N^{\chi}_{jt}, \tag{5.3}$$

where $0 \leq \chi < \psi\beta/2$. The parameter χ measures agglomeration externalities; for $\chi = 0$, there are no agglomeration externalities, in which case labour productivity is fully determined by the exogenous driving force, $A_{jt} = A^o_{jt}$. A violation of the restriction $\chi < \psi\beta/2$ would imply an explosive positive feedback loop, whereby a productivity shock increases both the city's radius (as commuting costs ψ are low) and its population density (as the demand for floorspace is low when β is low), leading to further agglomeration benefits, hence productivity, and so on.

Analogous to equations (2.1)–(2.2), the growth rate of the exogenous driving force A^o_{jt} follows an Ornstein-Uhlenbeck process:

$$\begin{aligned} \mathrm{d}\ln A^o_{jt} &= \beta\left(1 - \psi^o\chi\right)\left(\mu + X_{jt}\right)\mathrm{d}t, & (5.4)\\ \mathrm{d}X_{jt} &= \theta\left(-X_{jt}\mathrm{d}t + \sigma\,\mathrm{d}W_{jt}\right), & (5.5) \end{aligned}$$

where $\mu \geq 0, \theta > 0$ and $\sigma > 0$ are as in Section 2, and where $\psi^o := (2 - \psi\beta)/(\psi\beta)$, which is, due to previous assumptions, positive. We assume $\psi^o\chi < 1$. As in equation (2.4), we recover the classic case where A_{jt} follows a geometric Brownian motion for $\theta \to \infty$. While a more parsimonious specification of equations (5.4) and (5.5) is possible, the presentation above is convenient for our purposes.

The next section starts with an analysis of the classic model without persistence in the growth rate, i.e. $\theta \to \infty$. We show that this model yields the counterfactual prediction that the floorspace density is identical for all cities (irrespective of their size) and for all locations within a city (irrespective of their distance to the CBD). This prediction runs counter to the empirical correlation between the population of a city and its floorspace density, and is subsequently resolved in Section 5.3 by allowing for persistence in the growth rate, i.e. $\theta < \infty$.

5.2 Cities Without Persistence in Growth

Proposition 4 (Private Investment in Cities without Persistence in Growth)**.** *Assume $\phi > 0$ and $\alpha \in (0, 1)$. Define ρ_0 as in equation (4.1) and assume $\rho_0 > 0$ and $\theta \to \infty$ (no persistence in growth). Define the running maxima*

$$A^{\max}_{jt} := \max_{s \leq t} A_{js} \quad \text{and} \quad A^{o\max}_{jt} := \max_{s \leq t} A^o_{js}.$$

1. *The log rental cash flow at location r in city j at time t satisfies*

$$\ln Y_{jt}(r) = \beta^{-1} \ln A_{jt} - \psi \ln r. \tag{5.6}$$

2. *An investor owning vacant land at R_{jt} will invest in construction as soon as $Y_{jt}(R_{jt}) = Y^*$. This investment will result in a building with floorspace density F^*, where both*

Y^* and F^* are positive constants, which do not depend on j, t, or R_{jt}. Locations $r < R_{jt}$ fall within the bounds of the city, while locations $r > R_{jt}$ are vacant.

3. *City j's labour productivity and its population satisfy*

$$\ln A_{jt} = \left(1 - \frac{1-\beta}{\beta}\chi\right)^{-1} \left[\ln A^o_{jt} + \frac{2-\psi}{\psi\beta}\frac{\chi}{1-\chi\psi^o}\ln A^{o\max}_{jt}\right] + c_1, \quad (5.7)$$

$$\ln N_{jt} = \psi^o \ln A^{\max}_{jt} + \frac{1-\beta}{\beta}\left(\ln A_{jt} - \ln A^{\max}_{jt}\right) + c_2, \quad (5.8)$$

where c_1 and c_2 are appropriate constants that depend only on the model's parameters.

4. *In the absence of agglomeration externalities, i.e. $\chi = 0$, the model is exactly equivalent to the model for an individual plot of land discussed in sections 2–4. The law of motion of $\ln Y_{jt}(r)$ is*

$$\mathrm{d}\ln Y_{jt}(r) = \mu \mathrm{d}t + \sigma\,\mathrm{d}W_{jt},$$

as in equation (2.4) for the limiting case $\theta \to \infty$, while Y^* and F^* satisfy

$$Y^* = \frac{\rho_0}{\alpha}\left(\frac{\alpha\eta\phi}{(1-\alpha)\eta - 1}\right)^{1-\alpha}, \quad (5.9)$$

$$F^* = \left[\left(\frac{\alpha\eta\phi}{(1-\alpha)\eta - 1}\right)^{1-\alpha} - \phi\right]^{\alpha/(1-\alpha)}, \quad (5.10)$$

where $\eta := (\sqrt{\mu^2 + 2\rho\sigma^2} - \mu)/\sigma^2$.

Part 1, which specifies the log rental rates at various locations in a city, follows from the solution to the workers' utility maximisation problem and the equality of utility across locations (see equation (5.1)). The equation consists of two terms: (i) a fixed city/time effect that is proportional to the city's log labour productivity, and (ii) a common discount factor ψ per unit of log distance from the CBD that applies to all cities.

Part 2 indicates that the city will be extended to a particular radius R_{jt} whenever the rental rate of a unit of floorspace at that location reaches some threshold Y^*. Neither the threshold Y^* nor the floorspace density F^* depend on the city's radius R_{jt}. While larger cities typically have more floorspace per unit of land, this empirical regularity is not supported by the classic model: conditional on $Y(R_{jt}) = Y^*$, the future evolution of $\{Y_t\}$ is independent of r. While locations $r < R_{jt}$ closer to the CBD command higher rental cash flows, i.e. $Y_{jt}(r) > Y^*$ for $r < R_{jt}$, the cash flows at these locations were equal to Y^* when these structures were built. Conditional on investment occurring, therefore, future cash flows are identical in law. The resulting decision problem and its solution are identical, too, regardless of wherever or whenever investment occurred. This implies—erroneously—that identical structures are built across space and time.

Part 3 specifies the relations between log labour productivity $\ln A_{jt}$ on the one hand and its exogenous driving force $\ln A^o_{jt}$ and log population $\ln N_{jt}$ on the other hand. These equations show that we can use the persistence in the log population of a city as a proxy for the persistence in log rental rates at particular locations within this city. Both relations consist of two terms. The first term captures the effect of $\ln A^o_{jt}$ when equal to its running maximum. At that point, $\ln Y_{jt}(R_{jt}) = \ln Y^*$. Any further increase in $\ln A^o_{jt}$ will lead to new construction at the city's edge that keeps the rental rate there constant at Y^*, while log rental rates at locations within the city will rise due to the induced increase in $\ln A_{jt}$. As shown in equation (5.3), agglomeration externalities result in a multiplier effect: the new construction yields an increase in population, which further increases labour productivity A_{jt} and therefore induces further construction. Due to the irreversibility of construction, the radius of the city depends on $\ln A^{o\max}_{jt}$ rather than on $\ln A^o_{jt}$ itself: a subsequent decline in $\ln A^o_{jt}$ does not lead to a corresponding decline in $\ln R_{jt}$. Nevertheless, $\ln N_{jt}$ declines when $\ln A^o_{jt}$ falls, since log labour productivity and hence log rental rates fall. Lower rental rates drive up demand for floorspace per person and therefore reduce $\ln N_{jt}$, even though the supply of floorspace remains constant (similar to Glaeser and Gyourko, 2005). The decline in $\ln N_{jt}$ feeds back into log labour productivity via agglomeration externalities for $\chi > 0$ (again, see equation (5.3)).

In the absence of agglomeration externalities, i.e. $\chi = 0$ (Part 4), labour productivity is fully driven by its exogenous driving force, $\ln A_{jt} = \ln A^o_{jt}$. The model for an investor owning vacant land near the city is then identical to the model discussed in sections 2–4 with $\theta \to \infty$. In this case, Y^* and F^* can be solved in closed form, since we do not have to account for the difference in the laws of motion of $\ln A_{jt}$ for $\ln A^o_{jt} = \ln A^{o\max}_{jt}$ and $\ln A^o_{jt} < \ln A^{o\max}_{jt}$.

Although we arrive at the counterfactual implication that the floorspace density F^* is constant across the city in its entirety and across cities of different population sizes based on certain specifications in the model (relating to the utility function and cost of commuting), this implication will hold even after myriad adjustments. Given the empirical relevance of the irreversibility of construction (both in the upward and downward direction), the question arises as to what else can account for the empirical correlation between the floorspace density of a city and its population size.

The counterfactual implication of the classic model can be resolved in two ways. First, we could relax the assumption of the irreversibility of construction in the upward direction. The model could allow existing buildings to be demolished and replaced by denser structures. This is neither rare nor ubiquitous—much of Manhattan's real estate is over a century old—such that this extension, while realistic, does not fully resolve the problem. Alternatively, we could allow for persistent growth in productivity, the implications of which are investigated next.

5.3 Cities With Persistence in Growth

Proposition 5 (Private Investment in Cities with Persistence in Growth)**.** *Assume $\phi > 0$ and $\alpha \in (0, 1)$. Define ρ_1 as in equation (4.9) and assume $\rho_1 > 0$, $0 < \theta$ (persistence in growth) and $\chi = 0$ (no agglomeration externalities).*

1. *The log rental cash flow $\ln Y_{jt}(r)$ satisfies equation (5.6) in Proposition 4. Its law of motion is identical to that of $\ln Y$ in Section 2 (see equations (2.1) and (2.2)).*

2. *The problem solved by an investor owning vacant land at R_{jt} is the same as that solved in Section 4.3: she should invest in construction as soon as $Y_{jt}(R_{jt}) = Y(X_{jt})$, where $Y(X)$ is the function solved numerically in Section 4.3. Her decision is otherwise independent of the current radius R_{jt}. Locations $r < R_{jt}$ fall within the bounds of the city, while locations $r > R_{jt}$ are vacant.*

3. *The optimal floorspace density is an increasing function of the growth rate X_{jt} at the moment of construction, as in Proposition 3, but does not depend on the radius R_{jt}.*

To avoid complications caused by the feedback of agglomeration externalities on the city's population and its productivity, Proposition 5 focuses on the special case without agglomeration externalities ($\chi = 0$). As the arguments below demonstrate, the proof of this proposition is straightforward and hence omitted. For $\chi = 0$, labour productivity equals its exogenous driving force, $\ln A_{jt} = \ln A^o_{jt}$. The stochastic process driving the evolution of $Y_{jt}(r)$ in equations (5.3)–(5.6) is then identical to the process in equations (2.1)–(2.2). The optimal strategy of the landowner is therefore identical to that in Section 4.3.

Since all locations in city j share the same growth rate X_{jt}, while each location is characterised by its own rental cash flow $Y_{jt}(r)$, a city can be thought of as a vertical line in the (X, Y) space in Figure 3, with every point on this line corresponding to a particular location with distance r to the CBD.[12] Locations closer to the CBD command higher rental rates and therefore correspond to "higher" points on this vertical line. Suppose that $Y_{jt}(R_{jt}) = Y(X_{jt})$ at time t, where $Y(X)$ denotes the investment boundary derived in Section 4.3. Then, $Y_{jt}(r) > Y(X_{jt})$ for all locations with $r < R_{jt}$, since $Y_{jt}(r)$ is a declining function of r; see equation (5.6). Since construction should be effected as soon as $Y_{jt}(r) > Y(X_{jt})$, all these locations must consist of built area. Conversely, all locations for which $r > R_{jt}$ are agricultural land.

Since almost all investment occurs along the upward-sloping part of the investment boundary $Y(X_{jt})$, as shown in Section 4.3, landowners will pause construction after a positive shock to a city's growth rate X_{jt}, since this shock will shift the point $(X_{jt}, Y_{jt}(R_{jt}))$ into the continuation region of the (X, Y) space. That is—as before—investors at the perimeter of the city halt construction when prospects are bright. This implies that the probability that $X > X^*$ shown in Table 2 can now be interpreted as the proportion of cities that do not invest in new construction irrespective of the rental cash flow at the city's edge. For our benchmark parameters, this probability is $\sim$2%.

Given that $Y'(X) > 0$ for most of the new construction, Proposition 3 implies that the level of investment, and hence the floorspace density, are increasing functions of the growth rate at the moment of investment. This resolves the counterfactual implication of the classic model that the floorspace density is independent of the radius of the city. Although in our model the floorspace density likewise does not depend (directly) on the radius, it does depend on the historical growth rate (which is, naturally, strongly correlated with the current city radius). Interestingly, the model predicts a positive correlation between the city's current size (as measured by its population) and the historical growth rate. In the very long run, $t \to \infty$, X_t and $\ln Y_t - \ln Y_0$ are uncorrelated. However, in the short run, this correlation is as high as $\sqrt{3}/2 \approx 0.87$ and decays very slowly; see equation (2.3). Conditional on our benchmark value $\theta = 0.15$, the correlation between both variables, even after 20 years, is 54%.

At first sight, the implication of our model that the density of construction depends not on the distance to the city centre, but on the growth rate at the moment of construction, seems to be at odds with the typically high floorspace density in CBDs. However, modern high-density business districts are often developed on vacant land outside the historical centre. In NYC, new business districts sprang up around the Empire State Building in the 1930s and around the World Trade Center and Battery Park between 1970 and 1980, the latter on vacant land previously occupied by the docks. In Paris, a new business district

[12] Note that in the CBD, the rental cash flow $Y_{jt}(0)$ is infinite (see equation (F.1)).

emerged between 1970 and 2015 in la Défense, 15 km from the old centre. In Amsterdam around 1990, several locations were presented as candidates for a new business district, including sites in the historical centre, close to Central station, and close to the southern train station (approximately 4 km away). While the local council promoted the central location, businesses pushed for the southern one. Today, the business district near the southern station is flourishing, while the old city centre has remained largely residential. As these examples show, a new high-density business district on a city's perimeter disrupts the standard circular structure. Taking the dependence on the growth rate into account may be particularly important for cities with growth spurts.[13]

We conjecture that an amended version of Proposition 5 holds in the presence of agglomeration externalities, $\chi > 0$. The positive feedback loop from productivity to population and back to productivity (by agglomeration) will affect the investment boundary $Y(X)$. However, we conjecture that it remains U-shaped and that there remains a critical growth threshold X^*, above which new construction is optimally postponed irrespective of the rental cash flow.

6 Conclusion

This paper has presented the counterintuitive finding that investors owning vacant land at the edge of a city should rationally postpone investing in new real estate when a city's growth rate exceeds a critical threshold, irrespective of the current rental cash flow that could be generated by this investment. This is not merely an esoteric mathematical finding; numerical calculations based on realistic parameter values show that it may apply at any given point in time to ∼2% of cities. Moreover, a large part of the investment boundary is upward sloping, implying that the growth and the level of the cash flow act as complements rather than substitutes at the moment of investment; i.e. when growth is high, a higher cash flow is needed in order to invest. For our benchmark parameters, 80% of investment decisions occur in this region. Along this part of the investment boundary, a positive growth shock causes rational investors not to speed up investment but rather to postpone it. Accelerating growth may thus lead to less rather than more investment.

This counterintuitive conclusion is rooted in two forms of persistence. First, many newly erected buildings remain in place for a century or more. While the size of the building is variable at the moment of construction, thereafter it remains almost permanently fixed. Second, the growth rates of the population and the rental cash flow in a city (which are strongly correlated) are highly persistent. This persistence in growth rates implies that, in high-growth areas, investors wish to erect larger structures than is profitable given current rental rates. The optimal response is to delay investment until the expected growth has been realised. This mechanism explains why the price of vacant land at the edge of "superstar" cities vastly exceeds that of other cities, despite spurring little investment. The option to build is so valuable that investors are reluctant to relinquish it.

The above rationale for postponing investment hinges on the investor's capacity to determine the scale of the development. Postponement is less valuable when the size of the building is constrained at the outset (e.g. due to regulation). In this case, the

[13] Lucas and Rossi-Hansberg's (2002) model yields circular cities which do not necessarily have a single CBD, and if there is just one, it is not necessarily located in the city centre. The reasons for that, however, differ for their model cf. ours.

intuitive finding holds that the level and the growth rate of the cash flow are substitutes: to spur investment, at least one should be high. On the contrary, we find that they are complements rather than substitutes for all investment decisions where (i) the growth rate of the cash flow is predictable and (ii) the scale of the development is variable at the moment of construction. This counterintuitive mechanism can be applied to many investment decision in land development and construction.

A related strand of literature has focused on regulatory restrictions (e.g. minimum lot size regulation) imposed by incumbent owners (e.g. Glaeser and Gyourko, 2002; Glaeser et al., 2005; Hsieh and Moretti, 2019). In this interpretation, the high value of vacant land in the vicinity of growing cities is attributable to inefficient regulatory restrictions and collusion by landowners to limit housing supply in an effort to raise property values. However, it is unclear whether new construction on one plot of land imposes negative or positive externalities on the value of neighbouring plots: limited demand for floorspace in a city yields negative externalities, whereas agglomeration spillovers generate positive externalities (e.g. Davis et al., 2014, Knoll et al., 2017). Neighbouring owners benefit from new construction because property prices typically increase with the size of the city, as shown by Combes et al. (2019) for the case of France.

One may wonder whether the sluggish rate of investment in floorspace in attractive locations is a market failure. The answer is a qualified "no". In the absence of agglomeration externalities, the decisions of rational investors are Pareto efficient. The fact that agricultural land remains vacant even as its price soars is less a market failure than an efficient response that optimises its option value. When taking into account agglomeration externalities (which are obviously present in reality), the answer is less clear cut. In general, perfect competition on the land market at the perimeter of cities generates suboptimally low floorspace densities (e.g. Rossi-Hansberg, 2004), leading to urban sprawl. How the optimal timing of investment is affected is an open question. On the one hand, a social planner could raise the minimum floorspace density while leaving the timing of investment to the market. On the other, higher floorspace densities resemble higher construction shares α in the production function of floorspace, which generally increases the value of waiting and hence the option value. A definitive answer would require the model in Section 4.3 to be extended by accounting for agglomeration externalities. This remains a challenge for future research.

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A Distribution of $(X, \ln Y)$

Lemma 1. *Conditional on* $(X_0, \ln Y_0)$*, it follows that*

1. *The distribution of* $(X_t, \ln Y_t)$ *for any* $t \geq 0$ *conditional on* $(X_0, \ln Y_0)$ *is*

$$\left[\begin{array}{c} X_t \\ \ln Y_t \end{array}\right] \sim \mathrm{N}\left(\left[\begin{array}{c} \mathrm{e}^{-\theta t} X_0 \\ \ln Y_0 + \mu t + \frac{1-\mathrm{e}^{-\theta t}}{\theta} X_0 \end{array}\right], \frac{\sigma^2}{2}\left[\begin{array}{cc} \theta\left(1-\mathrm{e}^{-2\theta t}\right) & \left(1-\mathrm{e}^{-\theta t}\right)^2 \\ \left(1-\mathrm{e}^{-\theta t}\right)^2 & 2t - 4\frac{1-\mathrm{e}^{-\theta t}}{\theta} + \frac{1-\mathrm{e}^{-2\theta t}}{\theta} \end{array}\right]\right). \tag{A.1}$$

2. *The steady-state distribution of* X*, i.e. the distribution of* X_t *for* $t \to \infty$ *reads*

$$X \sim \mathrm{N}\left(0, \frac{1}{2}\theta\sigma^2\right).$$

   A steady-state distribution of Y *does not exist.*

3. *The correlation between* X_t *and* $\ln Y_t$ *conditional on* $(X_0, \ln Y_0)$*, reads*

$$\mathrm{Cor}(X_t, \ln Y_t | X_0, Y_0) = \frac{\left(1-\mathrm{e}^{-\theta t}\right)^2}{\sqrt{\theta\left(1-\mathrm{e}^{-2\theta t}\right)\left(2t - 4\frac{1-\mathrm{e}^{-\theta t}}{\theta} + \frac{1-\mathrm{e}^{-2\theta t}}{\theta}\right)}} = \frac{\sqrt{3}}{2}\left(1 - \frac{\theta}{8}t\right) + O(t^2).$$

4. *For* $\theta \to \infty$*, the distribution of* $\ln Y_t$ *conditional on* $\ln Y_0$ *is*

$$\ln Y_t \sim \mathrm{N}\left(\ln Y_0 + \mu\, t, \sigma^2\, t\right), \quad t \geq 0.$$

Proof

1. Equations (2.1)–(2.2) can be written as

$$\mathrm{d}\left(\begin{array}{c} X_t \\ \ln Y_t \end{array}\right) = \left(\begin{array}{c} 0 \\ \mu \end{array}\right)\mathrm{d}t + \left(\begin{array}{cc} -\theta & 0 \\ 1 & 0 \end{array}\right)\left(\begin{array}{c} X_t \\ \ln Y_t \end{array}\right)\mathrm{d}t + \left(\begin{array}{c} \sigma\theta \\ 0 \end{array}\right)\mathrm{d}W_t.$$

   Following Karatzas and Shreve (2012, p. 354), the solution is

$$\left(\begin{array}{c} X_t \\ \ln Y_t \end{array}\right) = \mathbf{M}_t\left[\left(\begin{array}{c} X_0 \\ \ln Y_0 \end{array}\right) + \int_0^t \mathbf{M}_s^{-1}\left(\begin{array}{c} 0 \\ \mu \end{array}\right)\mathrm{d}s + \int_0^t \mathbf{M}_s^{-1}\left(\begin{array}{c} \sigma\theta \\ 0 \end{array}\right)\mathrm{d}W_s\right],$$

   where the matrix $\mathbf{M}_t$ satisfies

$$\frac{\mathrm{d}\mathbf{M}_t}{\mathrm{d}t} = \left(\begin{array}{cc} -\theta & 0 \\ 1 & 0 \end{array}\right)\mathbf{M}_t, \qquad \mathbf{M}_0 = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right).$$

   The solution for $\mathbf{M}_t$ and its inverse are

$$\mathbf{M}_t = \left(\begin{array}{cc} \mathrm{e}^{-\theta t} & 0 \\ \frac{1-\mathrm{e}^{-\theta t}}{\theta} & 1 \end{array}\right), \quad \mathbf{M}_t^{-1} = \left(\begin{array}{cc} \mathrm{e}^{\theta t} & 0 \\ \frac{1-\mathrm{e}^{\theta t}}{\theta} & 1 \end{array}\right).$$

   For a fixed time $t > 0$, the expectation of $(X_t, \ln Y_t)$ conditional on $(X_0, \ln Y_0)$, denoted E_0, is

$$\begin{aligned} \mathrm{E}_0\left(\begin{array}{c} X_t \\ \ln Y_t \end{array}\right) &= \mathbf{M}_t\left[\left(\begin{array}{c} X_0 \\ \ln Y_0 \end{array}\right) + \int_0^t \mathbf{M}_s^{-1}\left(\begin{array}{c} 0 \\ \mu \end{array}\right)\mathrm{d}s\right], \\ &= \left(\begin{array}{c} \mathrm{e}^{-\theta t} X_0 \\ \ln Y_0 + \mu t + \frac{1-\mathrm{e}^{-\theta t}}{\theta} X_0 \end{array}\right). \end{aligned}$$

The covariance matrix of $(X_t, \ln Y_t)$ conditional on $(X_0, \ln Y_0)$ is

$$
\begin{aligned}
&\mathrm{E}_0\left[\left\{\begin{pmatrix} X_t \\ \ln Y_t \end{pmatrix} - \mathrm{E}_0\begin{pmatrix} X_t \\ \ln Y_t \end{pmatrix}\right\}\left\{\begin{pmatrix} X_t \\ \ln Y_t \end{pmatrix} - \mathrm{E}_0\begin{pmatrix} X_t \\ \ln Y_t \end{pmatrix}\right\}'\right], \\
&= \mathbf{M}_t\left[\int_0^t \mathbf{M}_s^{-1}\begin{pmatrix} \sigma\theta \\ 0 \end{pmatrix}\begin{pmatrix} \sigma\theta \\ 0 \end{pmatrix}' \mathbf{M}_s'^{-1}\,\mathrm{d}s\right]\mathbf{M}_t', \\
&= \frac{\sigma^2}{2}\begin{pmatrix} \theta\,(1-\mathrm{e}^{-2\theta t}) & (1-\mathrm{e}^{-\theta t})^2 \\ (1-\mathrm{e}^{-\theta t})^2 & 2t - 4\frac{1-\mathrm{e}^{-\theta t}}{\theta} + \frac{1-\mathrm{e}^{-2\theta t}}{\theta} \end{pmatrix}.
\end{aligned}
$$

2. The proofs of the other three parts are straightforward, hence omitted.

■

B Proof of Proposition 1

1. Define

$$
b(X,t) := \exp\left[-\rho_0\, t + \left(X - \sigma^2\right)\frac{1-\mathrm{e}^{-\theta t}}{\theta} + \frac{1}{2}\sigma^2\frac{1-\mathrm{e}^{-2\theta t}}{2\theta}\right], \tag{B.1}
$$

such that $b(X,0) = 1$ and $b(X,\infty) = 0$. We must prove that equation (4.2), i.e.

$$
B(X,Y) = Y b(X) = Y\int_0^\infty b(X,t)\mathrm{d}t,
$$

using equation (4.3) in the second step, satisfies equation (3.3). Using differential operator (3.1), an explicit computation yields

$$
(\mathrm{L}-\rho)\left[Y\, b(X,t)\right] \;=\; Y\,\frac{\mathrm{d}\, b(X,t),}{\mathrm{d}t}.
$$

Hence

$$
(\rho-\mathrm{L})B(X,Y) \;=\; \int_0^\infty (\rho-\mathrm{L})[Y\, b(X,t)]\,\mathrm{d}t \;=\; Y\left[-b(X,t)\right]_{t=0}^{t=\infty} \;=\; Y,
$$

since $\rho_0 > 0$ by assumption. This confirms Bellman's equation (3.3).

2. (a) For the left-hand tail, take X sufficiently negative to ensure $\rho - \mu - X > 0$. Then, by the variable transformation $s = (\rho-\mu-X)t$, we have

$$
\begin{aligned}
b(X) :=\; &\frac{1}{\rho-\mu-X}\times \\
&\int_0^\infty \exp\left[\frac{-\rho_0\, s}{\rho-\mu-X} + \left(X-\sigma^2\right)\frac{1-\mathrm{e}^{\frac{-\theta\, s}{\rho-\mu-X}}}{\theta} + \frac{\sigma^2}{2}\frac{1-\mathrm{e}^{\frac{-2\,\theta\, s}{\rho-\mu-X}}}{2\theta}\right]\mathrm{d}s.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
&\lim_{X\to-\infty}(\rho-\mu-X)b(X) \\
&= \lim_{X\to-\infty}\int_0^\infty \exp\left[\frac{s}{\rho-\mu-X}\left(-\rho_0 + X - \sigma^2 + \frac{\sigma^2}{2}\right)\right]\mathrm{d}s = \lim_{X\to-\infty}\int_0^\infty \mathrm{e}^{-s}\mathrm{d}s = 1.
\end{aligned}
$$

The first equality holds because $1 - \mathrm{e}^{-x} = x + O(x^2)$, while the second follows from the definition (4.1) of ρ_0.

(b) For large X, extract terms independent of t and consider the variable transformation $s = X\mathrm{e}^{-\theta t}/\theta$ as follows:

$$
\begin{aligned}
b(X) &:= \int_0^\infty \exp\left[-\rho_0 t + \left(X-\sigma^2\right)\frac{1-\mathrm{e}^{-\theta t}}{\theta} + \frac{1}{2}\sigma^2\frac{1-\mathrm{e}^{-2\theta t}}{2\theta}\right]\mathrm{d}t, \\
&= E(X)\int_0^\infty \exp\left[-\rho_0 t - \left(X-\sigma^2\right)\frac{\mathrm{e}^{-\theta t}}{\theta} - \frac{\sigma^2}{2}\frac{\mathrm{e}^{-2\theta t}}{2\theta}\right]\mathrm{d}t, \\
&= E(X)\int_0^{X/\theta} (\theta s)^{-1}\left(\frac{X}{\theta\, s}\right)^{-\rho_0/\theta}\exp\left[-s + \sigma^2\frac{s}{X} - \frac{\theta\,\sigma^2}{4}\frac{s^2}{X^2}\right]\mathrm{d}s, \\
&= \frac{E(X)}{\theta}\left(\frac{X}{\theta}\right)^{-\rho_0/\theta}\int_0^{X/\theta} s^{\rho_0/\theta-1}\exp\left[-s + \sigma^2\frac{s}{X} - \frac{\theta\,\sigma^2}{4}\frac{s^2}{X^2}\right]\mathrm{d}s,
\end{aligned}
$$

where

$$E(X) := \exp\left(\frac{X-\sigma^2}{\theta} + \frac{\sigma^2}{4\theta}\right).$$

Hence

$$\lim_{X\to\infty} \frac{b\,(X)}{E(X)}\left(\frac{X}{\theta}\right)^{\rho_0/\theta} = \lim_{X\to\infty} \frac{1}{\theta}\int_0^{X/\theta} s^{\rho_0/\theta-1}\mathrm{e}^{-s}\mathrm{d}s = \frac{1}{\theta}\Gamma\left(\frac{\rho_0}{\theta}\right).$$

3. Substitution of equation (4.2) for $B\,(X,Y)$ in equation (3.3) yields the result.

4. To prove that $b_1(X)$ and $b_2(X)$ are sigmoid functions, we must show that the following (in)equalities hold:

(a) increasing:	$b_1'(X) > 0, \quad b_2'(X) > 0,$
(b) bounded:	$0 < b_1(X) < b_2(X) < 1,$
(c) limits:	$\lim_{X\to-\infty} b_1(X) = \lim_{X\to-\infty} b_2(X) = 0,$
	$\lim_{X\to\infty} b_1(X) = \lim_{X\to\infty} b_2(X) = 1.$

(a) Equation (4.3) and (B.1) imply

$$b_k\,(X) = \int_0^\infty \left(1-\mathrm{e}^{-\theta t}\right)^k \frac{b(X,t)}{b\,(X)}\mathrm{d}t = \mathrm{E}_b\left[\left(1-\mathrm{e}^{-\theta t}\right)^k\right], \quad k\in\mathbb{N},$$

where $b(X,t)/b\,(X)$ is interpreted as a density function with associated expectation operator $\mathrm{E}_b\,[\cdot]$. Monotonicity of $b_1(X)$ requires $b_2(X) > b_1\,(X)^2$. Then

$$b_2\,(X) = \mathrm{E}_b\left[\left(1-\mathrm{e}^{-\theta t}\right)^2\right] > \left(\mathrm{E}_b\left[1-\mathrm{e}^{-\theta t}\right]\right)^2 = b_1\,(X)^2\,.$$

Monotonicity of $b_2(X)$ requires $b_3(X) > b_2(X)b_1\,(X)$. For a positive random variables $x\in\mathbb{R}_{>0}$, it holds

$$\mathrm{Cov}(x^2,x) = \mathrm{E}[x^3] - \mathrm{E}[x^2]\mathrm{E}[x] > 0.$$

Thus we obtain

$$b_3\,(X) = \mathrm{E}_b\left[\left(1-\mathrm{e}^{-\theta t}\right)^3\right] > \mathrm{E}_b\left[\left(1-\mathrm{e}^{-\theta t}\right)^2\right]\times\mathrm{E}_b\left[\left(1-\mathrm{e}^{-\theta t}\right)\right] = b_2(X)b_1(X).$$

(b) To show $0 < b_2(X) < b_1(X) < 1$ or, what is equivalent, $0 < \theta^2 b''(X) < \theta b'(X) < b\,(x)$, we compute

$$\begin{aligned}0 &< \theta b'\,(X) = \int_0^\infty \left(1-\mathrm{e}^{-\theta t}\right)b(X,t)\mathrm{d}t < \int_0^\infty b(X,t)\mathrm{d}t = b\,(X)\,,\\ 0 &< \theta^2 b''\,(X) = \int_0^\infty \left(1-\mathrm{e}^{-\theta t}\right)^2 b(X,t)\mathrm{d}t < \int_0^\infty \left(1-\mathrm{e}^{-\theta t}\right)b(X,t)\mathrm{d}t = \theta b'\,(X)\,.\end{aligned}$$

(c) Let $b(X,t;\rho_0)$ denote the function $b\,(X,t)$ from equation (B.1), now with ρ_0 added as an explicit argument, and similarly for $b\,(X;\rho_0)$. Hence, $b(X,t;r)|_{r=\rho_0} = b\,(X,t)$ and $b\,(X;r)\,|_{r=\rho_0} = b\,(X)$. The derivative of $b\,(X)$ satisfies

$$\begin{aligned}\theta\, b'(X) &= \int_0^\infty \left(1-\mathrm{e}^{-\theta t}\right)b(X,t)\mathrm{d}t = b(X) - \int_0^\infty \mathrm{e}^{-\theta t} b(X,t;\rho_0)\,\mathrm{d}t\\ &= b(X) - b(X;\rho_0+\theta).\end{aligned}$$

Hence, $b_1(X)$ can be written as

$$b_1(X) := \frac{\theta\, b'(X)}{b(X)} = 1 - \frac{b(X;\rho_0+\theta)}{b(X)}.$$

Using the limits of $b(X;\rho_0)$ for $X\to+\infty$ and $X\to-\infty$, we obtain

$$\frac{b(X;\rho_0+\theta)}{b(X)} \approx \begin{cases} \dfrac{\Gamma\left(\frac{\rho_0}{\theta}+1\right)}{\Gamma\left(\frac{\rho_0}{\theta}\right)}\dfrac{\theta}{X} = \dfrac{\rho_0}{X} & \to 0 \text{ as } X\to\infty,\\ \dfrac{\rho_0+\frac{\sigma^2}{2}-X}{\rho_0+\theta+\frac{\sigma^2}{2}-X} & \to 1 \text{ as } X\to-\infty,\end{cases}$$

using $\Gamma(x+1) = x\Gamma(x)$ in the second equality in the first line. The desired result then follows.

The second derivative of $b(X)$ satisfies

$$\theta^2\, b''(X) = \int_0^\infty \left(1 - \mathrm{e}^{-\theta t}\right)^2 b(X,t)\,\mathrm{d}t = b(X) - 2b(X;\rho_0+\theta) + b(X;\rho_0+2\theta).$$

Hence, $b_2(X)$ can be written as

$$b_2(X) := \frac{\theta^2\, b''(X)}{b(X)} = 1 - 2\frac{b(X;\rho_0+\theta)}{b(X;\rho_0)} + \frac{b(X;\rho_0+2\theta)}{b(X;\rho_0)}$$

Using the limits of $b(X;\rho_0)$ for $X \to +\infty$ and $X \to -\infty$, we obtain

$$\begin{aligned} \frac{b(X;\rho_0+\theta)}{b(X;\rho_0)} &\to \begin{cases} 0, & \text{as } X \to +\infty, \\ 1, & \text{as } X \to -\infty, \end{cases} \\ \frac{b(X;\rho_0+2\theta)}{b(X;\rho_0)} &\to \begin{cases} 0, & \text{as } X \to +\infty, \\ 1, & \text{as } X \to -\infty, \end{cases} \end{aligned}$$

such that the desired result follows.

5. Using Part 1, we obtain

$$\begin{aligned} \mathrm{d}\ln B(X,Y) &= \mathrm{d}\ln Y + \mathrm{d}\ln b(X) \\ &= (\mu+X)\mathrm{d}t + b_1(X)\mathrm{d}X + \frac{1}{2}\left[b_2(X) - b_1(X)^2\right]\mathrm{d}X^2 \\ &= (\mu+X)\mathrm{d}t + b_1(X)\left(-X\mathrm{d}t + \sigma\mathrm{d}W\right) + \frac{1}{2}\left[b_2(X) - b_1(X)^2\right]\sigma^2\mathrm{d}t \\ &= \left(\mu + \left[1 - b_1(X)\right]X + \frac{\sigma^2}{2}\left[b_2(X) - b_1(X)^2\right]\right)\mathrm{d}t + \sigma b_1(X)\,\mathrm{d}W. \end{aligned}$$

Next, consider equation (4.6). Division by $b(X)$ and substitution of $b_1(X)$ and $b_2(X)$ yields

$$b(X)^{-1} + \mu - \rho + \left[1 - b_1(X)\right]X + \frac{1}{2}\sigma^2 b_2(X) = 0.$$

Since $\lim_{X\to\infty} b(X)^{-1} = 0$ (see Part 2) and $\lim_{X\to\infty} b_2(X) = 1$ (see Part 4) and taking the limit $X \to \infty$ and rearranging terms implies

$$\lim_{X\to\infty}\left[1 - b_1(X)\right]X = \rho - \mu - \frac{\sigma^2}{2} = \rho_0.$$

This implies that the drift term remains finite as $X \to \infty$, because

$$\lim_{X\to\infty}\left(\mu + \left[1 - b_1(X)\right]X + \frac{\sigma^2}{2}\left[b_2(X) - b_1(X)^2\right]\right) = \mu + \rho_0 + 0 = \rho - \sigma^2/2.$$

For $X \to -\infty$, the drift approaches $-\infty$, as can be easily established. For the volatility, the monotonicity and the bounds follow directly from Part 4.

■

C Proof of Proposition 2

1. The proof of Part 1 follows via that of the remaining parts.

2. Substituting conjecture (4.10) into Bellman's equation (3.2), we find that function $v(\cdot)$ satisfies the ordinary differential equation (ODE):

$$0 = \left(\frac{\mu + X}{1-\alpha} - \rho\right)v(X) - \theta X v'(X) + \frac{1}{2}\sigma^2\theta^2 v''(X), \quad X \in \mathbb{R}, \tag{C.1}$$

as will be used in Part 4. Next, we define

$$v(X) := \mathrm{e}^{X/\theta/(1-\alpha)}\,\overline{v}(X).$$

Equation (C.1) implies that $\overline{v}(\cdot)$ must satisfy

$$0 = -\rho_1\overline{v}(X) - \theta\left(X - \frac{\sigma^2}{1-\alpha}\right)\overline{v}'(X) + \frac{1}{2}\theta^2\sigma^2\overline{v}''(X), \quad \forall X \in \mathbb{R}, \tag{C.2}$$

where ρ_1 is defined in equation (4.9). By a linear transformation

$$Z(X) := \frac{1}{\sigma\sqrt{\theta}}\left(X - \frac{\sigma^2}{1-\alpha}\right), \qquad \text{(C.3)}$$
$$w\left[Z\left(X\right)\right] := \overline{v}\left(X\right),$$

it follows that equation (C.2) can be written in terms of $w(\cdot)$ as

$$0 = \frac{1}{2}w''(Z) - Z\,w'(Z) + \omega\,w(Z), \quad \forall Z \in \mathbb{R}, \qquad \text{(C.4)}$$

where $\omega := -\rho_1/\theta < 0$. The resulting ODE is known as Hermite's ODE. If ω were a positive integer, the solution would be $\mathrm{H}_\omega(Z)$, where H_ω is the Hermite polynomial of order ω. As we will show below, our solution can still be written as $\mathrm{H}_\omega(Z)$ if the Hermite polynomial is interpreted in a generalised sense, which allows for negative values of ω.

To solve ODE (C.4), we propose the following series expansion as our candidate solution:

$$\begin{aligned} w(Z) &= \sum_{i=0}^{\infty} c_i Z^i, \text{ such that} \\ -Zw'(Z) &= -Z\frac{\mathrm{d}}{\mathrm{d}Z}\sum_{i=0}^{\infty} c_i Z^i = -\sum_{i=1}^{\infty} i\,c_i\,Z^i, \\ \frac{1}{2}w''(Z) &= \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}Z^2}\sum_{i=0}^{\infty} c_i Z^i = c_2 + \frac{1}{2}\sum_{i=1}^{\infty} c_{i+2}\,(i+2)\,(i+1)\,Z^i. \end{aligned}$$

Using these equalities, Hermite's ODE (C.4) becomes

$$\begin{aligned} 0 &= c_2 + \omega\,c_0 + \sum_{i=1}^{\infty}\left[\frac{1}{2}c_{i+2}(i+2)(i+1) - (i-\omega)\,c_i\right] Z^i \\ &= \sum_{i=0}^{\infty}\left[\frac{1}{2}c_{i+2}(i+2)(i+1) - (i-\omega)\,c_i\right] Z^i, \qquad \forall Z \in \mathbb{R}. \end{aligned}$$

This equation holds only if the coefficient in square brackets is zero for *every* single value of $i = 0, 1, 2, 3, \cdots$. Hence we need

$$c_{i+2} = \frac{2(i-\omega)}{(i+2)(i+1)}c_i, \qquad \forall i = 0, 1, 2, 3, \cdots$$

This recursive equation relates c_{i+2} to c_i. Two independent solutions $w_k(Z)$ for $k = 1, 2$ may be obtained by starting with an arbitrary value of c_0 (or c_1) and considering only even (or odd) powers as follows:

$$\begin{aligned} w_1(Z) &= c_0\left[1 + 2\tfrac{-\omega}{2\times 1}Z^2 + 2^2\tfrac{-\omega}{2\times 1}\tfrac{2-\omega}{4\times 3}Z^4 + \cdots\right] =: c_0\,\mathrm{M}\left(-\frac{\omega}{2}, \frac{1}{2}, Z^2\right), \\ w_2(Z) &= c_1\left[Z + 2\tfrac{1-\omega}{3\times 2}Z^3 + 2^2\tfrac{1-\omega}{3\times 2}\tfrac{3-\omega}{5\times 4}Z^5 + \cdots\right] =: \frac{c_1\,Z}{\sigma}\,\mathrm{M}\left(\frac{1-\omega}{2}, \frac{3}{2}, Z^2\right), \end{aligned}$$

where, on the far right-hand side, we use the definition of the confluent hypergeometric function of the first kind, denoted by $\mathrm{M}(\cdot,\cdot,\cdot)$, see e.g. Abramovich and Stegun (1972, p. 504, equation 13.1.2).

In the limit where $Z \to \infty$, these functions behave like

$$\begin{aligned} c_0\,\mathrm{M}\left(-\frac{\omega}{2}, \frac{1}{2}, Z^2\right) &\approx c_0\,\frac{\sqrt{\pi}}{\Gamma\left(-\frac{\omega}{2}\right)}\,1/Z^{1+\omega}\,\exp(Z^2), \quad \text{as } Z \to \infty, \\ c_1\,Z\,\mathrm{M}\left(\frac{1-\omega}{2}, \frac{3}{2}, Z^2\right) &\approx c_1\,\frac{\sqrt{\pi}}{2\Gamma\left(\frac{1-\omega}{2}\right)}\,1/Z^{1+\omega}\,\exp(Z^2), \quad \text{as } Z \to \infty, \end{aligned} \qquad \text{(C.5)}$$

see e.g. Abramovich and Stegun (1972, p. 504, equation 13.1.4). By the approximation sign "≈", we mean that the ratio of the quantities on the left- and right-hand sides approaches to unity as $Z \to \infty$.

We recall from Lemma 1 that the steady-state distribution of X is normal with a variance of $\theta\sigma^2/2$. Since Z is a linear transformation of X with 'slope' coefficient $1/(\sigma\sqrt{\theta})$, see equation (C.3), the steady-state distribution of Z is normal with variance $1/2$. The steady-state probability density of Z decays therefore proportional to $\exp(-Z^2)$ in the limit where $Z \to \infty$. Hence, if we multiply the steady-state density by $w_1(Z)$ or $w_2(Z)$, then as $Z \to \infty$ the product is proportional to $1/Z^{1+\omega}$, which is not an integrable function (recall that $\omega < 0$). Hence, for $c_0, c_1 \geq 0$, we have

$$\mathrm{E}\left[w_1(Z)|Z > 0\right] = \mathrm{E}\left[w_2(Z)|Z > 0\right] = \infty.$$

In such circumstances, Dixit and Pindyck (1994, pp. 181-2) use a 'no-bubble argument' to rule out a solution with undesirable asymptotic properties. In our case, however, this rules out both our candidate solutions. Hence, we must pick c_0 and c_1 so that the combination combination $w(Z) = w_1(Z) + w_2(Z)$ contains only terms behave appropriately as $Z \to \infty$ and, in particular, are integrable with respect to the density $\exp(-Z^2/2)$. From (C.5), this can be achieved by choosing

$$c_0 = 2^{\omega}\sqrt{\pi}\frac{1}{\Gamma\left(\frac{1-\omega}{2}\right)}, \qquad c_1 = -2^{\omega}\sqrt{\pi}\frac{2}{\Gamma\left(-\frac{\omega}{2}\right)},$$

where the factor $2^{\omega}\sqrt{\pi}$ is introduced for later convenience. The full solution then reads

$$\begin{aligned} w(Z) &= w_1(Z) + w_2(Z), \\ &= 2^{\omega}\sqrt{\pi}\left[\frac{1}{\Gamma\left(\frac{1-\omega}{2}\right)}\,\mathrm{M}\left(-\frac{\omega}{2},\frac{1}{2},Z^2\right) - \frac{2Z}{\Gamma\left(-\frac{\omega}{2}\right)}\,\mathrm{M}\left(\frac{1-\omega}{2},\frac{3}{2},Z^2\right)\right], \\ &= \mathrm{H}_{\omega}(Z), \end{aligned} \tag{C.6}$$

where the third equality holds only if the Hermite polynomial is understood in a generalised sense, in which case it is defined as in the second line. The solution (4.11) in the Proposition is obtained by $\overline{v}(X) = w(Z) = \mathrm{H}_{\omega}(Z)$ with $Z := X/\sigma/\sqrt{\theta} - \sigma/(1-\alpha)/\sqrt{\theta}$.

The resulting solution is well behaved as $Z \to \infty$, because $\mathrm{H}_{\omega}(Z) \approx (2Z)^{\omega}$ as $Z \to \infty$, which is decreasing in Z (recall $\omega := -\rho_1/\theta < 0$). As such, we have

$$V(X,Y) \approx C\,Y^{1/(1-\alpha)}\left(\frac{2X}{\sigma/\sqrt{\theta}} - \frac{2\sigma}{(1-\alpha)/\sqrt{\theta}}\right)^{-\rho_1/\theta}\exp\left(\frac{X}{\theta(1-\alpha)}\right), \quad X \to \infty,$$

ensuring the right-hand tail is integrable with respect to the unconditional density of X, as desired. Hence, the derived solution satisfies Bellman's equation (3.2) as well as the required transversality condition, ensuring integrability with respect to the relevant density.

Some computer packages, such as Wolfram's `Mathematica`, automatically compute $\mathrm{H}_{\omega}(\cdot)$ for negative values of ω by using the second line in (C.6) as the definition of the third.[14] Other software packages, notably `Matlab`, return an error message, in which case the second rather than the third line of equation (C.6) must be used.

3. As is standard in the theory of optimal stopping (e.g. Øksendal, 2007, p. 217-18), we have a sufficient condition for continuation as follows:

$$(\mathrm{L}-\rho)B^*(X,Y) > 0 \qquad \Rightarrow \qquad (X,Y) \text{ in continuation region},$$

where L is the differential operator (3.1) and $B^*(X,Y)$ is given in equation (3.5).

A lengthy (but essentially straightforward) algebraic computation, using Bellman's equation for $B(X,Y)$ (see equation 3.3) and the decomposition $B(X,Y) = Y b(X)$ (see equation 4.2), gives

$$(\mathrm{L}-\rho)B^*(X,Y) = Y^{1/(1-\alpha)}\,[\alpha b(X)]^{\alpha/(1-\alpha)}\left[\alpha\,\rho\,b(X) + \frac{1}{2}\sigma^2\frac{\alpha}{1-\alpha}b(X)b_1(X)^2 - 1\right].$$

Hence

$$b(X)\left[\alpha\,\rho + \frac{1}{2}\sigma^2\frac{\alpha}{1-\alpha}b_1(X)^2\right] > 1 \quad \Rightarrow \quad X \text{ in continuation region.} \tag{C.7}$$

The quantity on the left-hand side of the inequality is continuous and strictly increasing in X, as both $b(X)$ and $b_1(X)$ are continuous and strictly increasing in X. Further, it approaches zero as $X \to -\infty$, while being unbounded above as $X \to \infty$. Hence, the level one must be crossed exactly once. This implies that $X^{\dagger}$, obtained by setting the left-hand side equal to unity, exists and is unique. The strict monotonicity of the left-hand side further implies the decision maker should continue for any $X > X^{\dagger}$. For future reference, we also have

$$\frac{1}{b(X)} - \alpha\,\rho + \frac{1}{2}\sigma^2\frac{\alpha}{1-\alpha}b_1(X)^2 \begin{cases} > 0, & X < X^{\dagger}, \\ = 0, & X = X^{\dagger}, \\ < 0, & X > X^{\dagger}, \end{cases} \tag{C.8}$$

as will be used in Part 4 of the proof.

4. Equations (4.2), (3.5) and (4.10) for $B(X,Y)$, $B^*(X,Y)$ and $V(X,Y)$, respectively, imply that the value-matching

[14]See http://functions.wolfram.com/05.01.26.0002.01.

condition (3.6) and smooth-pasting conditions (3.7) can be written as

$$
\begin{aligned}
V(X^*,Y^*) = C\,Y^{*1/(1-\alpha)}v(X^*) &= \frac{1-\alpha}{\alpha}\left[\alpha Y^* b\left(X^*\right)\right]^{1/(1-\alpha)} = B^*(X^*,Y^*), \\
C\,Y^{*1/(1-\alpha)}v'\left(X^*\right) &= Y^{*1/(1-\alpha)}\left[\alpha b\left(X^*\right)\right]^{\alpha/(1-\alpha)} b'\left(X^*\right).
\end{aligned}
$$

Dividing both equations by $Y^{*1/(1-\alpha)}$, the dependence on the variable Y^* is eliminated. Moreover, the integration constant C is eliminated by dividing the second equation by the first, in which case we obtain equation (4.13) in the Proposition, which can be solved numerically for the critical value X^*. Conditional on X^*, the integration constant C can be expressed in closed form using either condition above, e.g. by using the first we have

$$
C = (1-\alpha)\,\alpha^{\alpha/(1-\alpha)}\,\frac{b(X^*)^{1/(1-\alpha)}}{v(X^*)} > 0.
$$

To show existence and uniqueness of X^*, we introduce the function $f(\cdot)$ on $\mathbb{R}$ as

$$
\begin{aligned}
f(X) &:= (1-\alpha)\,v_1(X) - b_1(X), \qquad \text{(C.9)}\\
\theta f'\left(X\right) &= (1-\alpha)\left[v_2\left(X\right) - v_1\left(X\right)^2\right] - b_2\left(X\right) + b_1\left(X\right)^2, \\
\theta^2 f''\left(X\right) &= (1-\alpha)\left[v_3\left(X\right) - 3v_2\left(X\right)v_1\left(X\right) + 2v_1\left(X\right)^3\right] \\
&\quad - b_3\left(X\right) + 3b_2\left(X\right)b_1\left(X\right) - 2b_1\left(X\right)^3.
\end{aligned}
$$

where we provide the first to derivatives of $f\left(X\right)$ for future reference. The critical value X^* is defined in equation (4.13) as the intersection of $f(\cdot)$ with the horizontal axis. We must show this intersection exists and is unique.

Existence. We note that $f(\cdot)$ and its derivatives are continuous on $\mathbb{R}$, as both $b(\cdot)$ and $v(\cdot)$ and their derivatives are continuous on $\mathbb{R}$. Further, using the explicit formulas for $b(\cdot)$ and $v(\cdot)$, it can be shown that $f(X)$ is strictly negative (positive) as X goes to negative (positive) infinity, which means that $f(\cdot)$ must change sign at least once, such that existence of at least one intersection is established.

Uniqueness. Given that at least one intersection with the horizontal axis exists, our strategy will be to *assume* that $f(X) = 0$ for some $X \in \mathbb{R}$. If we can show that $f'(X)$ is strictly positive (negative), then we know that the point under consideration represents an up-crossing (down-crossing) of the horizontal axis. Below, we will show that up-crossings can only occur strictly to the left of $X^\dagger$, where $X^\dagger$ was defined in Part 3 of the Proposition. Conversely, down-crossings, if they exist, can only occur weakly to the right of $X^\dagger$. Because the function $f(\cdot)$ approaches the horizontal axis from below while ending above the horizontal axis, this argument establishes that there must be exactly one up-crossing, which must occur strictly to the left of $X^\dagger$, while down-crossings are ruled out.

To operationalise the above argument, we use the following implication proved below:

$$
f(X) = 0 \;\Rightarrow\; f'(X)\begin{cases} > 0 & \text{if } X < X^\dagger : \text{an `up-crossing' occurs,} \\ = 0 & \text{if } X = X^\dagger : \text{indeterminate,} \\ < 0 & \text{if } X > X^\dagger : \text{a `down-crossing' occurs.} \end{cases} \qquad \text{(C.10)}
$$

Implication (C.10) says that any intersection of $f(\cdot)$ with the horizontal axis, if it exists, is guaranteed to be an up-crossing (down-crossing) when it occurs strictly to the left (right) of $X^\dagger$, in which case the slope of $f(\cdot)$ is strictly positive (negative). The first-derivative test is indeterminate at $X^\dagger$, such that we cannot establish whether an intersection at $X^\dagger$, if it exists, is an up-crossing, down-crossing, tangent from above, or tangent from below.

Next, we show that case left indeterminate by the first-derivative test, must be either a down-crossing or a tangent from below, because

$$
f(X) = 0 \text{ and } f'(X) = 0 \;\Rightarrow\; f''(X) < 0. \qquad \text{(C.11)}
$$

Jointly, implications (C.10) and (C.11) imply that up-crossings can only occur strictly to the left of $X^\dagger$, while no down-crossing weakly to the right of $X^\dagger$ can exist if the function $f(\cdot)$ is to remain positive. Since $f(\cdot)$ changes sign from negative to positive at least once, we must have at least one up-crossing, which must occur strictly to the left of $X^\dagger$.

Proof of implication (C.10). We write the ODEs for $b(\cdot)$ (see equation 4.6) and for $v(\cdot)$ (see equation C.1) as follows:

$$
0 = 1/b(X) + \mu + X - \rho - Xb_1(X) + \frac{1}{2}\sigma^2 b_2(X), \quad X \in \mathbb{R}, \qquad \text{(C.12)}
$$

$$
0 = \mu + X - (1-\alpha)\rho - (1-\alpha)Xv_1(X) + (1-\alpha)\frac{1}{2}\sigma^2 v_2(X), \quad X \in \mathbb{R}. \qquad \text{(C.13)}
$$

Subtraction of the first from the second equation and substitution of equation (C.9) yields

$$
0 = \alpha\rho - 1/b(X) - X\,f\left(X\right) + \frac{1}{2}\sigma^2\left[(1-\alpha)v_2(X) - b_2(X)\right]. \qquad \text{(C.14)}
$$

We evaluate this equation at the point $f(X) = 0$:

$$\frac{1}{b(X)} - \alpha\,\rho = \frac{1}{2}\sigma^2\left[(1-\alpha)v_2(X) - b_2(X)\right]. \tag{C.15}$$

Moreover, by equation (C.9) $v_1(X) = (1-\alpha)^{-1}\,b_1(X)$ for $f(X) = 0$, and hence $f'(X)$ satisfies

$$\begin{aligned}\theta\, f'(X) &= (1-\alpha)v_2(X) - b_2(X) - \frac{\alpha}{1-\alpha}b_1(X)^2 \\ &= \frac{2}{\sigma^2}\left[\frac{1}{b(X)} - \alpha\,\rho - \frac{1}{2}\sigma^2\frac{\alpha}{1-\alpha}b_1(X)^2\right],\end{aligned}$$

using equation (C.15) in the final step. Using equation (C.8), the last line gives the desired result.

Proof of implication (C.11).

Differentiating equation (C.14) with respect to X and evaluating this equation at the point $f(X) = f'(X) = 0$ yields.

$$-\frac{2}{\sigma^2}\frac{b_1(X)}{b(X)} = (1-\alpha)v_3(X) - (1-\alpha)v_2(X)\,v_1(X) - b_3(X) + b_2(X)\,b_1(X). \tag{C.16}$$

For $f(X) = f'(X) = 0$, equation (C.9) implies

$$\begin{aligned}\theta^2\, f''(X) &= (1-\alpha)\left[v_3(X) - v_2(X)v_1(X) - 2v_1(X)\left\{v_2(X) - v_1(X)^2\right\}\right] \\ &\quad - \left[b_3(X) - b_2(X)b_1(X) - 2b_1(X)\left\{b_2(X) - b_1(X)^2\right\}\right] \\ &= -\frac{2}{\sigma^2}\frac{b_1(X)}{b(X)} - 2(1-\alpha)v_1(X)\left[v_2(X) - v_1(X)^2\right] + 2b_1(X)\left[b_2(X) - b_1(X)^2\right] \\ &= -\frac{2}{\sigma^2}\underbrace{\frac{b_1(X)}{b(X)}}_{>0} - 2\frac{\alpha}{1-\alpha}\underbrace{b_1(X)}_{>0}\underbrace{\left\{b_2(X) - b_1(X)^2\right\}}_{=\theta b_1'(X)>0} \quad < \quad 0,\end{aligned}$$

where we use equation (C.16) in the second step and

$$\begin{aligned}(1-\alpha)\,v_1(X) &= b_1(X) \\ (1-\alpha)\left[v_2(X) - v_1(X)^2\right] &= b_2(X) - b_1(X)^2\end{aligned}$$

for $f(X) = f'(X) = 0$ by equation (C.9).■

D Discretisation of Algorithm (4.16)

We discretise a bounded region of the state space $(X, Y) \in \mathbb{R}\times\mathbb{R}_{\geq 0}$ using n grid points. Discretisation converts the function $V(X, Y)$ on $\mathbb{R}\times\mathbb{R}_{\geq 0}$ into a vector of length n, denoted $\mathbf{V}$ (vectors and matrices are bold). Similarly, the function $B^*(X, Y)$ becomes a vector $\mathbf{B}^*$. Discretising the differential operator L yields an $n \times n$ matrix, denoted $\mathbf{L}$. The iterative solution method, which is analogous to the iterative function-space solution method (4.16), reads as follows:

$$\left[(\rho+\lambda)\,\mathbf{I} - \mathbf{L}\right]\,\mathbf{V}^{(i+1)} = \lambda \max\left\{\mathbf{B}^*\,,\,\mathbf{V}^{(i)}\right\}, \qquad i \in \mathbb{N}, \tag{D.1}$$

with initialisation $\mathbf{V}^{(0)} = \mathbf{0}$, where $\mathbf{0}$ is a vector containing zeroes, $\mathbf{I}$ denotes the $n\times n$ identity matrix, and the max-operator is applied elementwise. The discretised POST algorithm (D.1) has the same attractive properties as its function-space equivalent (4.16) if two conditions hold:

1. The matrix $\mathbf{L}$ is a weakly diagonally dominant matrix, and
2. The matrix $\mathbf{L}$ has non-positive diagonal elements and non-negative off-diagonal elements.

If these statements hold, then for any $\rho, \lambda > 0$, algorithm (D.1) results in a non-decreasing sequence of vectors $\{\mathbf{V}^{(n)}\}_{n\in\mathbb{N}}$ that converge to a limiting vector $\mathbf{V}$; see Lange et al. (2020) for the proof.
The only question that remains is how to choose the $n \times n$ matrix $\mathbf{L}$ such it satisfies the above requirements. For the infinitesimal generator L defined in equation (3.1), the five-point stencil below generates such a discretisation:

	$\frac{1}{\mathrm{d}Y}(\mu+X)^+\,Y$	
$\frac{1}{\mathrm{d}X^2}\frac{\theta^2\sigma^2}{2} + \frac{1}{\mathrm{d}X}\theta X^+$	$-\frac{1}{\mathrm{d}X^2}\theta^2\sigma^2 - \frac{1}{\mathrm{d}X}\theta\lvert X\rvert - \frac{1}{\mathrm{d}Y}\lvert\mu+X\rvert\,Y$	$\frac{1}{\mathrm{d}X^2}\frac{\theta^2\sigma^2}{2} + \frac{1}{\mathrm{d}X}\theta X^-$
	$\frac{1}{\mathrm{d}Y}(\mu+X)^-\,Y$	

where $X \in \mathbb{R}$, $Y \in \mathbb{R}_{\geq 0}$, $(\cdot)^+ = \max\{0, \cdot\}$, $(\cdot)^- = \max\{0, -\cdot\}$, while $\mathrm{d}X$ and $\mathrm{d}Y$ denote the horizontal and vertical spacing of the grid in the X and Y directions, respectively. This stencil satisfies the assumptions above, since the center value, which is placed on the diagonal of $\mathbf{L}$, is constructed to be non-positive, while the values corresponding to its four neighbours, which are placed on the off diagonal elements of $\mathbf{L}$, are constructed to be non-negative. Diagonal dominance of $\mathbf{L}$ follows trivially, since the value in the center, which ends up on the diagonal of $\mathbf{L}$, is not exceeded in absolute value by sum of the other values in the stencil.

In the approximation of second derivatives with respect to X, we have used a central difference scheme, which uses grid points to the left and right of the center point. In the approximation of first derivatives with respect to X, in contrast, we use either a 'forward' or 'backward' approximation. This means that, in addition to the center point, we use either the left- or right-hand neighbour, but never both. Which one is chosen depends on the direction of the drift, such that the nearest neighbour on the right gets a positive value if the drift is towards the right (this happens when $X < 0$). The same reasoning is applied to derivative with respect to Y, and this leads to the desirable result that negative values are guaranteed to end up at the center of the stencil. While forward and backward approximations of derivatives are only first-order accurate in the grid spacing, the resulting 'upwind' scheme guarantees numerical stability, which is our main concern here.

We must also consider boundary conditions. When we reach the edge of our grid, some points in our stencil may not be 'available'. One method for dealing with such 'ghost points' besides the grid is simply to ignore the stencil value corresponding to the non-existent neighbour, which leads to Dirichlet boundary conditions. Alternatively, the stencil value corresponding to the non-existent neighbour may be re-assigned to the center value, leading to Neumann boundary conditions. In our numerical analysis, using Dirichlet or Neumann boundary conditions makes no noticeable difference to the optimal policy.

We performed the calculation on a $701 \times 1{,}001$ grid, ranging from $X = -0.08$ to $X = 0.06$ and $Y = 10^{-6}$ to $Y = 0.2$ (implying step sizes $\mathrm{d}X = \mathrm{d}Y = 2 \times 10^{-4}$). This implies $n = 701 \times 1{,}001 \approx 702{,}000$ (recall that the matrix $\mathbf{L}$ is $n \times n$). Only a subset of the range used for our calculation is displayed in the figures. Experiments with finer grids, larger grids and different boundary conditions gave nearly identical results.

Finally, a note on numerical efficiency and convergence. The vector $\mathbf{V}^{(i)}$ in equation (D.1) may be obtained from the vector $\mathbf{V}^{(i-1)}$ by explicitly computing the inverse matrix $[(\rho + \lambda)\,\mathbf{I} - \mathbf{L}]^{-1}$. However, this is computationally inefficient, because the $n \times n$ matrix $(\rho + \lambda)\,\mathbf{I} - \mathbf{L}$ is sparse—containing fewer than $5n$ non-zero entries when generated by the five-point stencil above—while its inverse is dense. Hence, it is computationally more efficient to use 'implicit' sparse linear algebra techniques to solve the n equations in algorithm (D.1).

Furthermore, because the contraction rate is determined by $\lambda/(\rho + \lambda)$, which may be close to unity when λ is large, it is advisable to start with a low value of λ, e.g. $\lambda = 1$, and gradually update the value of λ, e.g. by considering the sequence $\lambda = 2^k$ for $k = 0, 1, 2 \ldots$. Each time λ is doubled, the (final) value function corresponding to the previous problem can be used to initialise the value function for the next problem. The resulting method is numerically stable and converges quickly. After nine doublings, the final Poisson intensity equals $2^9 = 512$, which implies an average of 512 investment opportunities per annum. This number is sufficiently high to closely approximate the solution corresponding to $\lambda = \infty$; indeed, further doublings of λ do not noticeably change the exercise region in Figure 3.

E Proof of Proposition 3

Proposition 3 follows immediately from equation (3.4) and (4.2) because $K^*(X, Y)$ is increasing in $B(X, Y)$, $B(X, Y)$ is increasing in both its arguments and because the Proposition on only considers the increasing part of $Y(X)$.

F Proof of Proposition 4

We conjecture that all past and future investors (for whom $r < R_{jt}$ and $r > R_{jt}$ respectively) have applied/will apply the same investment tresholds $\{Y^*, F^*\}$ as the current investor owning land at R_{jt}. Conditional on this conjecture, we derive the optimal values of $\{Y^*, F^*\}$ at R_{jt}. Finally, we show that the optimal policy of all other investors for whom $r \neq R_{jt}$ is the solution to the same programme. Hence, they have chosen/will choose the same $\{Y^*, F^*\}$, confirming the conjecture.

1. Define $f_{jt}(r)$ to be the demand for floorspace of an individual living at r, j, t. This demand follows from the maximization of the utility function (5.1) subject to the budget constraint (5.2):

$$\begin{aligned} f_{jt}(r) &= \beta \frac{A_{jt}}{Y_{jt}(r)}, \qquad c = (1-\beta)\,A_{jt} \Rightarrow \\ \ln Y_{jt}(r) &= \beta^{-1} \ln A_{jt} - \psi \ln r, && \text{(F.1)} \\ f_{jt}(r) &= \beta A_{jt}^{(\beta-1)/\beta} r^{\psi}, && \text{(F.2)} \end{aligned}$$

where the second line follows from the substitution of the utility equivalence condition $U(f, c, r) = 1$, see equation (5.1).

2. Equation (F.1) implies that $\ln Y_{jt}(r)$ is determined by a time-varying term $\beta^{-1} \ln A_{jt}$ that is common to all locations in the city plus a time-invariant term $-\psi \ln r$. A sufficient condition for investors at every location r to choose the same $\{Y^*, F^*\}$ is that the law of motion for $\beta^{-1} \ln A_{js}$ for $s \geq t$ conditional on $\beta^{-1} \ln A_{jt} - \psi \ln r = Y^*$ is identical for all r.

   Since investors at every $r < R_{jt}$ have invested in floorspace at the first moment t that $Y_{jt}(r)$ reached Y^* and since $Y_{jt}(r)$ is a declining function of r, see equation (F.1), R_{jt} satisfies

$$\ln R_{jt} = (\psi\beta)^{-1} \ln A_{jt}^{\max} - \psi^{-1} \ln Y^*, \tag{F.3}$$

Moreover, all locations with $r < R_{jt}$ are built area since equation (F.3) holds for these locations for a value $\ln A_{js}^{\max} < \ln A_{jt}^{\max}$, which therefore must have been attained for some $s < t$, while all locations with $r > R_{jt}$ are still vacant since equation (F.3) holds for these locations for value of $\ln A_{js}^{\max} > \ln A_{jt}^{\max}$, which therefore will be attained for some $s > t$.

3. The quantity $\ln N_{jt}$ satisfies

$$\begin{aligned}
\ln N_{jt} &= \ln\left(2\pi F^* \int_0^{R_{jt}} \frac{r}{f_{jt}(r)} \mathrm{d}r\right) = \ln\left(2\frac{\pi}{\beta} F^* A_{jt}^{\beta^o} \int_0^{R_{jt}} r^{1-\psi} \mathrm{d}r\right) \\
&= \beta^o \ln A_{jt} + (2-\psi)\ln R_{jt} + \ln\left(\frac{2}{2-\psi}\frac{\pi}{\beta}F^*\right), \\
&= \beta^o \ln A_{jt} + \frac{2-\psi}{\psi\beta} \ln A_{jt}^{\max} + \chi^{-1} c_0 \\
&= \psi^o \ln A_{jt}^{\max} + \beta^o \left(\ln A_{jt} - \ln A_{jt}^{\max}\right) + \chi^{-1} c_0, \\
c_0 &:= \chi\left[\ln\left(\frac{2}{2-\psi}\frac{\pi}{\beta}F^*\right) - \frac{2-\psi}{\psi}\ln Y^*\right],
\end{aligned} \tag{F.4}$$

where $\beta^o := (1-\beta)/\beta$ and where we substitute equation (F.2) for $f_{jt}(r)$ in the first line and (F.3) for $\ln R_{jt}$ in the third line. The fourth line proves equation (5.8).

Substitution of equation (5.3) for $\ln N_{jt}$ and rearranging terms yields

$$(1-\chi\beta^o)\ln A_{jt} = \ln A_{jt}^o + \chi(\psi^o - \beta^o)\ln A_{jt}^{\max} + c_0, \tag{F.5}$$

where $1-\chi\beta^o > 0$ since $\chi < \psi\beta/2 < \beta$ and where $\chi(\psi^o - \beta^o)/(1-\chi\beta^o) < 1$, since

$$\begin{aligned}
\chi(\psi^o - \beta^o) &= \chi\frac{2-\psi}{\psi\beta} < 1 - \chi\beta^o = 1 - \chi\frac{1-\beta}{\beta} \qquad \text{since} \\
2\chi &< \psi\beta < (1+\chi)\psi\beta.
\end{aligned}$$

We want to show that $\ln A_{jt} = \ln A_{jt}^{\max}$ if and only if $\ln A_{jt}^o = \ln A_{jt}^{o\max}$. Let $\ln A_{js} = \ln A_{jt}^{\max}$ for some $s < t$. Equation (F.5) implies that $\ln A_{jt}$ is an increasing function of $\ln A_{jt}^o$. Suppose that $\ln A_{jt}^o > \ln A_{js}^o$. Then, $\ln A_{jt} > \ln A_{js}$, which contradicts $\ln A_{js} = \ln A_{jt}^{\max}$. Hence, $\ln A_{js}^o = \ln A_{js}^{o\max}$: $\ln A_{js}$ and $\ln A_{js}^o$ reach a maximum for the same $s \leq t$.

Setting $\ln A_{jt} = \ln A_{jt}^{\max}$ and hence $\ln A_{jt}^o = \ln A_{jt}^{o\max}$ in equation (F.5)

$$\ln A_{jt}^{\max} = (1-\chi\psi^o)^{-1}\left(\ln A_{jt}^{o\max} + c_0\right), \tag{F.6}$$

in the second line and where $1-\chi\psi^o > 0$ by assumption. The future evolution of $\ln Y_{jt}(R_{jt})$ conditional on equation (F.3) and $\ln A_{jt} = \ln A_{jt}^{\max}$ is fully determined by the future evolution of $\ln A_{jt}$. Equation (F.5) and (F.6) imply that the future evolution of $\ln A_{jt}$ is fully determined by the future evolution of $\ln A_{jt}^o$. Conditional on equation (F.3) holding for r, this future evolution is the same for every r. Hence, investors will choose the same $\{Y^*, F^*\}$ for every r.

Rearranging terms in the first line of equation (F.5) using the second line to substitute to $A_{jt}^{\max}$ yields

$$\begin{aligned}
(1-\chi\beta^o)\ln A_{jt} &= \ln A_{jt}^o + \chi(\psi^o - \beta^o)\ln A_{jt}^{\max} + c_0 \\
&= \ln A_{jt}^o + \frac{2-\psi}{\psi\beta}\frac{\chi}{1-\chi\psi^o}\left(\ln A_{jt}^{o\max} + c_0\right) + c_0,
\end{aligned} \tag{F.7}$$

which is equation (5.7), where c_1 and c_2 are defined by equation (F.5) and (F.7).

4. Equation (5.3) implies that for $\chi = 0$, $\ln A_{jt} = \ln A_{jt}^o$ for all j and t. Then, equation (5.4), (5.5) and (F.1) imply that (analogous to equation (2.4) since $\theta \to \infty$)

$$\mathrm{d}\ln Y_{jt}(r) = \beta^{-1}\mathrm{d}\ln A_{jt}^o = \mu \mathrm{d}t + \sigma \mathrm{d}W_{jt}.$$

Equivalent to equation (3.5), the present value of the rental cash flow $B^*(\cdot)$ minus the cost of investment conditional on its current cash flow Y satisfies

$$B^*(Y) = \frac{1-\alpha}{\alpha}\left(\frac{\alpha Y}{\rho_0}\right)^{1/(1-\alpha)} - \phi,$$

where we leave out the argument X of $B^*(X, Y)$ since $\theta \to \infty$, implying that X has no predictive power. The value of vacant land $V(Y)$ can be written as

$$V(Y) = B^*(Y^*)\left(\frac{Y}{Y^*}\right)^{\eta},$$

for some $\eta > 0$, which automatically satisfies the value-matching condition $V(Y^*) = B^*(Y^*)$. The value function $V(Y)$ satisfies the PDE

$$\rho V(Y) = \left(\mu + \frac{\sigma^2}{2}\right) Y\; V'(Y) + \frac{1}{2}\sigma^2 Y^2 V''(Y).$$

Substituting $V(Y)$ given above, we find that the positive square root of the resulting characteristic equation is

$$\eta = \sigma^{-2}\left(\sqrt{\mu^2 + 2\rho\sigma^2} - \mu\right) \;>\; 0.$$

The derivatives of $B^*(Y)$ and $V(Y)$ are

$$\begin{aligned} V'(Y) &= \eta\frac{B^*(Y^*)}{Y^*}\left(\frac{Y}{Y^*}\right)^{\eta-1}, \\ B^{*\prime}(Y) &= \frac{1}{\rho_0}\left(\frac{\alpha Y}{\rho_0}\right)^{\alpha/(1-\alpha)}. \end{aligned}$$

Evaluating at $Y = Y^*$ and setting the expressions on the right-hand side equal to each other gives

$$\begin{aligned} \eta\frac{B^*(Y^*)}{Y^*} &= \frac{1}{\rho_0}\left(\frac{\alpha Y^*}{\rho_0}\right)^{\alpha/(1-\alpha)}, \\ \frac{\eta}{Y^*}\left[\frac{1-\alpha}{\alpha}\left(\frac{\alpha Y^*}{\rho_0}\right)^{1/(1-\alpha)} - \phi\right] &= \frac{1}{\rho_0}\left(\frac{\alpha Y^*}{\rho_0}\right)^{\alpha/(1-\alpha)}, \end{aligned}$$

where the second line follows by the definition of $B^*(Y^*)$. Solving for Y^* yields equation (5.9). Substitution in equation (2.5), (3.4) and (4.2) and solving for F^* yields equation (5.10).

■