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Abstract

This paper develops a model of automation as an embodied technology. The gradual discovery and obsolescence of technologies gives rise to a distribution of capital with varying degrees of automation. I derive conditions under which, aggregating over heterogeneous production units, output can be represented as a CES production function, the parameters of which are determined endogenously by the distribution of technology. Through the lens of the canonical model, I show how the distribution of automation technology determines its aggregate effects; in the long run, only the distribution of technology matters. The transition dynamics of the economy in response to an increase in frontier automation technology are consistent with notable micro and macro US stylized facts of recent decades: at the firm level, a fall in the labor share driven by reallocation towards low labor share establishments; at the macro level, slowing total factor productivity growth and a fall in the real interest rate.

JEL Classification: O4, E1, E22, E25

Keywords: automation, Factor shares, Vintage capital, aggregation

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1 Introduction

In this paper I develop a model of automation as an embodied technology created via irreversible investments: *putty-clay automation*. Paraphrasing Phelps (1963), in this model, different to the existing literature on automation, new robots (or any other kind of capital that enables automation) are like putty; before they are installed they can be designed to perform any technologically feasible number of tasks. Once robots are created and installed, they are like hard-baked clay; they can perform only the tasks for which they have been designed, even as new and better blueprints for robots are discovered. In the model (as in the world), both new and old types of robots will be created at any given time because old types are cheaper to make than new ones. Thus, in equilibrium, the capital stock of the economy will be composed of many different types of capital, with varying degrees of automation.

To study this economy with heterogeneous capital in a tractable framework, I derive conditions under which aggregate output in this economy can be represented using the canonical constant elasticity of substitution (CES) function with capital and labor aggregates as input, but here the parameters of CES are themselves functions of the level and distribution of automation technology in the economy. This model of the aggregate production function delivers sharp characterizations of the effect of automation on the labor share and on aggregate productivity. The labor share *decreases* if the *most* automated capital in the economy become more automated but *increases* if the *least* automated capital is scrapped. Aggregate productivity, on the other hand, increases if automation increases anywhere in the economy. The aggregation result also yields a theory of the aggregate elasticity of substitution between capital and labor, σ in standard notation, which relates this parameter to the shape of the distribution of automation capital.

Modeling automation as embodied technological progress provides several novel insights. Salient among these is that labor's share of value added does not inevitably decline in response to a permanent increase in automation: if and when technology diffuses through the economy, the labor share returns to a stable long-run level that is independent of the state of automation technology.

Studying this long-run, "natural" level, I show that it is directly related to the speed of technological diffusion, and equivalently, to the ease with which the most productive firms (those with the most automated capital) can expand their scale relative to the least productive firms in the economy. By implication, the "natural" level of the labor share is directly related to the shape of the firm-size (or automation capital) distribution of the economy, and therefore to σ. A fall in the rate of technological diffusion (or a technological change that favors the growth of the most productive firms) leads to a permanent fall in labor's share of value added that comes about through a greater concentration of economic activity among low labor share firms, consistent with the empirical evidence presented by Autor et al. (2020) and Kehrig and Vincent (2021). The theory is also consistent with the work of Akcigit and Ates (2021) who relate the evolution of factor shares (among other macro aggregates) to a slowdown in the rate of technological diffusion.

Furthermore, I show that along the transition path of the economy in response to an exogenous increase in frontier automation technology the model does well in matching the evidence presented by Kehrig and Vincent (2021). I replicate figures from their paper using model-simulated data and show that, as in the data, the fall in the aggregate labor share in the putty-clay model is driven entirely by reallocation. The model, in other words, can rationalize observed cross-sectional trends in the labor share either as arising from a permanent change in technology that affects the long-run firm-size distribution, or as the endogenous response of the economy to an automation shock.

The response of aggregate variables along the transition path also is consistent with salient macro stylized facts of recent decades. The labor share falls along the transition path[1] and returns to its long-run level at a slow rate. The growth rate of total factor productivity, which in the model is determined endogenously by the evolution of the firm-size distribution, and the real interest rate both decrease along the transition path. Thus, in the putty-clay automation model, the "Solow productivity paradox" (Solow (1987), Brynjolfsson et al. (2017)) – the observation that the appearance of new technologies is accompanied by a slowdown in productivity growth – is not a paradox after all.

Related Literature This paper contributes to a growing literature that, starting with the seminal paper by Zeira (1998), models automation as a task-replacing tech-

[1] The timing and magnitude of the fall (if any) in the labor share in the US is inevitably dependent on the definition used. In calibrating the model, I use US industry-level accounts and remove the financial and real estate sectors and also the capitalization of intellectual property products from value added, as in Koh et al. (Forthcoming).

nology and considers its macroeconomic effects. The model in this paper starts from different primitives but shares features with Acemoglu and Restrepo (2018). The latter develop a comprehensive theory of automation that features endogenous automation and the creation of new tasks that only labor can perform (the "race" to which the title refers); when these two forces are evenly matched, the economy has a balanced growth path. For the dynamic model described in Section 3 to have a balanced growth path a closely related condition must hold. In the full Acemoglu and Restrepo (2018) model, the endogenous creation of new tasks acts as a self-correcting force that restores employment and the labor share even as automation technology progresses. In the model of this paper, the diffusion of technology and the scrapping of old technologies act as a self-correcting force that operates to restore long-run stability, even if no new human-only tasks are introduced.

Aghion et al. (2017) present a model of automation that builds on the structure of Zeira (1998). The authors examine the ability of such a model to generate dynamics consistent with the Kaldor growth facts. Among other insights, the model connects automation with the emergence of a Baumol (1967) "cost disease"; the authors also explore the effect of a possible technological singularity leading to an explosion in economic growth. Hémous and Olsen (Forthcoming) develop a model of automation and horizontal innovation that endogenously links automation to the rise in the skilled-unskilled wage premium.

Hubmer and Restrepo (2021) develop a model with firms that are heterogeneous in their degree of automation, although, in contrast to the present paper, in stationary equilibrium all firms have the same degree of automation. Their model generates transition paths in which the aggregate labor share declines and the median labor share increases, consistent with the evidence in Kehrig and Vincent (2021). The authors also extend their model to the case of imperfect competition and provide a sector-level decomposition of the decline in the labor share into technological and pro/anti-competitive channels.

The aggregation procedure I implement was first introduced in Houthakker (1955), who shows that aggregation of micro-production units operating a Leontief technology with Pareto-distributed productivities results in a Cobb-Douglas aggregate production function. In a static setting in which the productivity distribution itself is the primitive, Levhari (1968) and Sato (1975) extend the result to general CES functions. Jones (2005) applies this aggregation result in a growth context to consider how the aggregate production changes over time and how technological change matters for the shape of the production function. León-Ledesma and Satchi (2018) also study aggregation in a model where firms choose technology from a technology frontier. This paper's central concern is similar in spirit but I focus on automation specifically. The model I develop in Section 2 is closest to Lagos (2006), who develops an aggregative model of total factor productivity. My model can be thought of as extending his work to a model of both TFP and the share parameter (α). Oberfield and Raval (2021) use a closely related aggregation result to estimate industry-level elasticities of substitution starting from plant-level elasticities. Dvorkin and Monge-Naranjo (2020) obtain a related aggregation result in production under different assumptions for an economy with endogenous worker assignment.

The dynamic version of the model features a putty-clay vintage capital structure, first introduced by Johansen (1959) and further developed by Massell (1962) and Phelps (1963). The implementation of the putty-clay technology in this model is closely related to Gilchrist and Williams (2000). As in Jovanovic and Yatsenko (2012), and in contrast to most vintage capital models, an important feature of the model in this paper is that investment occurs in both old and new vintages of capital.

The rest of the paper is organized as follows. Section 2 develops a static model that presents, in the simplest possible setting, the main analytical results in the paper. Section 3 embeds this static production structure into a dynamic model that, in aggregate, closely resembles the standard neoclassical growth model and derives the requirements for the existence and properties of a balanced growth path of the economy. Section 4 analyzes the long- and short- run effects of automation. Section 5 concludes.

2 Static Model

I start with a static version of the model to simplify the exposition of technology and the aggregation result that yields a constant elasticity of substitution (CES) macro production function. The static economy consists of a continuum of firms, each operating a firm-specific technology to produce a homogeneous good that is sold in a competitive market. To produce, firms must hire labor, which is supplied by homogeneous workers in a competitive labor market.

2.1 Firms

A continuum of competitive firms produce a homogeneous final good Y. A firm i is characterized by its technology $\tilde{a}_i$ and capital stock k_i. Technology is embodied in the capital stock, so a firm's capital consists of k_i units of capital that embody the technology level $\tilde{a}_i$. I assume that each firm owns exclusive rights to its technology, so there is one firm per technology $\tilde{a}_i$[2].

Technology Production of one final good requires the completion of a continuum of all tasks in an interval $[0, q]$[3]. Both workers and machines can perform tasks. Workers can perform all tasks in $[0, q]$,[4] whereas a machine of type $\tilde{a}$ can perform tasks in $[0, \tilde{a}]$, with $\tilde{a} < q$. So to produce one unit of the final good using a machine of type $\tilde{a}$, a worker must complete the remaining tasks $[\tilde{a}, q]$. If it takes workers one unit of time to complete all tasks in $[0, q]$, a worker-machine pair would produce $1/(q - \tilde{a})$ goods in one unit of time; I further assume that the productivity of a worker-machine pair is increasing in the quantity of tasks, and that the productivity gains from worker specialization are governed by a parameter $\gamma > 1$. With these assumptions, the productivity of a worker-machine pair is given by:

$$z(a) = \left(\frac{q}{q - \tilde{a}}\right)^{\gamma} = \left(\frac{1}{1 - a}\right)^{\gamma}, \tag{1}$$

where I define $a \equiv \frac{\tilde{a}}{q}$ as the fraction of total tasks that the machine can complete, which I call the machine's (and firm's) *degree of automation.* When referring to machine types, I adopt the notational convention that variables with a tilde measure the highest *level* of task that a machine can perform, whereas variables without a tilde measure *the fraction of total production tasks* that a machine can perform. The function $z(a)$ is related to span of control in models in which production is organized

[2] I refer to the locations at which each type of technology is operated as 'firms', but the reader may prefer to think of these as plants or establishments given that the defining characteristic of a firm in the model is that it is a unit of production in which only one particular type of technology may be operated.

[3] Following Becker and Murphy (1992), the production function *in terms of tasks* is the Leontief function,

$$y = \min_{0<x<q} y(x)$$

with the interpretation that producing y goods requires that each task x be performed $y(x)$ times.

[4] The assumption that workers can perform all tasks is not essential to the results, but it simplifies notation and algebra. What is important is that there are some tasks that only workers can perform.

in knowledge hierarchies as in Garicano (2000) and Garicano and Rossi-Hansberg (2006). In these papers, productivity gains arise when more knowledgeable workers are employed at higher levels in the hierarchy of production. Here, productivity gains occur when workers are paired with machines because workers are essential to production (because $\tilde{a} < q$) and machines save workers time by performing a portion of the required tasks.[5] The degree of productivity gains is scaled by the parameter γ, which, in the spirit of Becker and Murphy (1992), I interpret as as capturing returns to worker specialization.

Firm Production Function and Profit Maximization Each firm i is characterized by its technology $\tilde{a}_i$ (equivalently, its degree of automation a_i) and capital stock k_i. At the firm level, I assume that the number of (effective) worker hours per machine is technologically constrained to be in a fixed proportion, resulting in the following (Leontief) production function for firm i:

$$y_i = z\left(a_i\right) \min\left(k_i, H \cdot n_i\right), \tag{2}$$

where k_i and n_i are the firm's capital stock and labor input. H is labor-augmenting productivity, common to all workers, so that $H \cdot n_i$ is the efficiency adjusted number of hours worked by firm i's workers.

Firm-size Distribution I assume that each firm has exclusive use of a technology, so there is a one-to-one correspondence between i and a_i, and I can model the firm-size distribution as the measure of capital at each degree of automation a_i. I denote this measure $k\left(a\right)$, and assume that it has bounded support $a \in [a_\ell, a_h]$, so that a_ℓ and a_h are, respectively, the least and the most automated types of capital in the economy. Since a is a fraction and I have assumed that at least some level of worker-only tasks is needed in production, the bounds satisfy $0 \leq a_\ell < a_h < 1$. Anticipating the discussion of aggregation in the following section, the integral of $k\left(a\right)$ is the aggregate capital stock of the economy, $K = \int_{a_\ell}^{a_h} k\left(a\right)\ \mathrm{d}a$. Normalizing the firm-size distribution by

[5] All things being equal, an increase in q decreases productivity per unit of output: the higher is q, the more a worker-machine pair must work more to produce each good. I interpret q as a quality margin; thus, the same good produced with higher q is of higher quality (more enjoyable to consumers). In the dynamic model presented in Section 3, I make this quality interpretation explicit by introducing q into consumer preferences.

aggregate capital, I define the PDF $f(a) \equiv \frac{k(a)}{K}$, which is the density function of degrees of automation in the economy. I define the density $g(z)$ as the distribution of the function $z(a)$ given that a has density $f(a)$. The density function $g(z)$ is the measure of productivities associated with a given distribution of automation. Since $f(a)$ is bounded, so too is g(z), with lower bound $z_\ell \equiv z(a_\ell)$ and upper bound $z_h \equiv z(a_h)$.

2.2 Equilibrium and Aggregation

I start by solving for aggregate labor demand. The profit maximization problem for a firm with productivity z is:

$$\max_n \; z \min(k(z), H \cdot n) - W \cdot n \tag{3}$$

It follows that the optimal choice of hours as a function of z is:

$$n(z) = \begin{cases} \frac{k(z)}{H} & \text{if } z \geq \frac{W}{H} \\ 0 & \text{if } z < \frac{W}{H} \end{cases}$$

Since z is a function of a, I can rewrite the optimal choice of hours in terms of the degree of automation of the marginal firm.

$$n(a) = \begin{cases} \frac{k(a)}{H} & \text{if } a \geq a_w \\ 0 & \text{if } a < a_w \end{cases},$$

where $a_w = 1 - \left(\frac{W}{H}\right)^{\sigma-1}$. Aggregate labor demand, N, is the integral over firm labor choices, $N = \int_{z_\ell}^{z_h} n(z) \; \mathrm{d}z$. Substituting in the optimal choice of hours and $k(z) = K \cdot g(z)$, the integral becomes $N = \frac{K}{H} \int_{W/H}^{z_h} g(z) \; \mathrm{d}z$. Denoting by $G(\cdot)$ the CDF of z, integrating gives the aggregate labor demand as:

$$N = \frac{K}{H} \left[1 - G\left(\frac{W}{H}\right)\right] \tag{4}$$

Aggregate labor demand is the product of the total capital stock, divided by worker efficiency, times the fraction of active firms (1 minus the CDF at $\frac{W}{H}$, the term in square brackets). I assume an increasing labor supply function $N^s(W)$. Equating

$N^s(W)$ to labor demand as in Equation 4 gives the equilibrium wage. With the equilibrium wage in hand, aggregate output Y can be solved for as the integral over firm-level output, $Y = \int_{W/H}^{z^h} zk(z)\ \mathrm{d}z = K\int_{W/H}^{z^h} zg(z)\ \mathrm{d}z$, which can be expressed as:

$$Y = K\left[1 - G\left(\frac{W}{H}\right)\right]\mathbb{E}_G\left(z|z \geq \frac{W}{H}\right) \tag{5}$$

Aggregate output is the product of three terms: the capital stock, the fraction of active firms, and the average productivity of active firms, given by the third term ($\mathbb{E}_G\left(z|z \geq \frac{W}{H}\right) = \left[1 - G\left(\frac{W}{H}\right)\right]^{-1}\int_{W/H}^{z^h} zg(z)\ \mathrm{d}z$).

2.3 From Automation to a CES Aggregate Production Function

Equation 5 expresses aggregate output as a function of the equilibrium wage. To express aggregate output in a more familiar form – as a function of capital, labor and technology – I start with an assumption about the distribution of degrees of automation $f(a)$. In particular, I assume that $f(a)$ is a 3 parameter beta distribution[6] that is characterized by one shape parameter, ρ, and lower and upper bounds a_ℓ and a_h. Using this assumption, the following lemma characterizes the distribution of $z(a)$.

Lemma 1. *If $a \sim B(1, \rho; a_\ell, a_h)$, so that the PDF of a, $f(a) = \rho\frac{(a_h - a)^{\rho-1}}{(a_h - a_\ell)^\rho}$, then $z(a) = \left(\frac{1}{1-a}\right)^\gamma$ is distributed with CDF $G(z)$:*

$$G(z) = 1 - \left(1 - \frac{1 - \left(\frac{z_\ell}{z}\right)^{\frac{1}{\gamma}}}{1 - \left(\frac{z_\ell}{z_h}\right)^{\frac{1}{\gamma}}}\right)^\rho \tag{6}$$

and $z \in [z_\ell, z_h]$, with $z_\ell = \left(\frac{1}{1-a_\ell}\right)^\gamma$ and $z_h = \left(\frac{1}{1-a_h}\right)^\gamma$.

Proof. See Appendix A.1 □

With $\gamma = 1$, this lemma is equivalent to the derivation in Geerolf (2017) of a Pareto-distributed span of control in an economy that has a simplified Garicano (2000) production function and a beta distribution for worker skills. The distribution $G(z)$,

[6] The standard beta distribution is bounded in $[0, 1]$ and is characterized by two shape parameters. I restrict one of the shape parameters to equal 1 and set the bounds to $0 < a_\ell < a_h < 1$.

known to statisticians as a truncated beta-Pareto distribution[7], is closely related to the Pareto distribution; indeed, with $\rho = 1$, $G(z)$ is exactly a Pareto distribution with tail parameter $\frac{1}{\gamma}$ and lower bound z_ℓ, truncated at an upper bound z_h.

The distribution $G(z)$ generalizes the distribution derived by Sato (1969) (which, in turn, extends the result in Levhari (1968)) for $z_\ell > 0$. Contrastin those papers, the productivity distribution here is not a primitive; instead, I derive it from the underlying distribution firm-size (or automation technology) distribution $F(a)$ and the productivity function $z(a)$. These differences are crucial because they allow me to develop a theoretical link between automation and the aggregate production function. They are also important in proving the existence of a balanced growth path and its associated stationary distribution in the dynamic model that I present in Section 3.

Before proceeding, I place a further restriction on parameters: for the remainder of the paper, I assume that $\gamma = \rho + 1$. This is not an innocuous restriction because, in the logic of the model, γ can be thought of as a deep technological parameter that governs firm-level returns to automation, whereas ρ is the shape parameter of the automation distribution, which means it is closer to an equilibrium object. The gain from imposing this restriction is substantial: as shown in Proposition 1, with this parametrization, aggregate output in the economy can be represented exactly as a canonical CES production function. This knife-edge parametrization of the model makes it possible to solve the integral in Equation 5 in closed form. I have verified numerically that the results presented in this section and the next are not substantially affected by deviations from the knife-edge parametrization. To economize on notation, and to be consistent with the literature, I also define $\rho \equiv \frac{\sigma}{1-\sigma}$ (σ will correspond exactly to the elasticity parameter of the CES). In sum, I make the following assumptions about the distributions $F(a)$ and $G(z)$:

Assumption 1: Distributions $F(a)$ and $G(z)$ *Assume $\gamma = \rho + 1 = \frac{1}{1-\sigma}$ and that the automation distribution $F(a)$ and its associated productivity distribution $G(z)$, with $z(a) = \left(\frac{1}{1-a}\right)^{\frac{1}{1-\sigma}}$, are:*

[7] Akinsete et al. (2008) and Lorenzutti et al. (2012) derive moments, MGF, etc., for this distribution.

$$F(a) = 1 - \left(\frac{a_h - a}{a_h - a_\ell}\right)^{\frac{\sigma}{1-\sigma}} \qquad \text{and} \qquad G(z) = 1 - \left(1 - \frac{1 - \left(\frac{z_\ell}{z}\right)^{1-\sigma}}{1 - \left(\frac{z_\ell}{z_h}\right)^{1-\sigma}}\right)^{\frac{\sigma}{1-\sigma}}$$

The following proposition shows that, given this assumption, aggregate output can be represented by a CES production function with elasticity of substitution $0 < \sigma < 1$.

Proposition 1. *Using Assumption 1, aggregate output for this economy can be represented as the CES production function with elasticity of substitution* $0 < \sigma < 1$*,*

$$Y = A\left(\alpha K^{\frac{\sigma-1}{\sigma}} + (1-\alpha)(H \cdot N)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} \tag{7}$$

where H *is labor-augmenting productivity and the capital* (K) *and labor aggregates* (N) *are:*

$$K = \int_Z k(z)\ dz \qquad \text{and} \qquad N = \frac{K}{H}\left[1 - G\left(\frac{W}{H}\right)\right]$$

where W *is the market-clearing wage that solves* $N^s(W) = N$*. The total factor productivity term* A *and the capital distribution parameter* α *are functions of parameters of the productivity distribution* $G(z)$ *or, equivalently, of the automation distribution* $F(a)$*:*

$$A = (z_h)^{1-\sigma}(z_\ell)^{\sigma} = \frac{1}{(1-a_h)(1-a_\ell)^{\frac{\sigma}{1-\sigma}}} \tag{8}$$

$$\alpha = 1 - \left(\frac{z_\ell}{z_h}\right)^{1-\sigma} = \frac{a_h - a_\ell}{1 - a_\ell} \tag{9}$$

Proof. See Appendix A.3. □

Proposition 1 contains 3 linkages between the parameters of the automation distribution $F(a)$ and the aggregate production function. I discuss each of these in turn.

Elasticity of Substitution σ The elasticity of substitution σ of the aggregate production function is related to the shape parameter of the underlying firm-size distribution[8]. Inspecting the CDF of a $B\left(1, \frac{\sigma}{1-\sigma}\right)$ distribution, $F(x) = 1 - (1-x)^{\frac{\sigma}{1-\sigma}}$, we see that the effect of increasing σ is to shift mass towards the left tail of the distribution, which in the model means that firms with relatively low automation are relatively larger. The intuition for why this translates into a higher elasticity of substitution is that the higher σ is, the larger the mass of firms in the neighborhood of the cutoff level of automation a_w, and, consequently, the higher the mass of firms that go from active to idle in response to a small increase in the cutoff a_w (corresponding to a small increase in the wage). Because firms in the neighborhood of this cutoff require more labor per unit of output, the economy in aggregate becomes more capital intensive when these firms shut down. In other words, the aggregate economy substitutes towards capital in response to an increase in the wage, and this effect is stronger the higher σ is.

The aggregation result in Proposition 1 requires that σ be smaller than 1. The discussion in the previous paragraph hints why this is the case: $\sigma > 1$ would imply that the economy becomes *more* labor intensive as the wage increases. Sato (1969) shows that this is a general existence requirement in this class of aggregative models (Mangin (2015) proves a similar result using a different approach). This is an important theoretical insight in the long-running debate about the value of σ in macro models[9].

A and α The TFP parameter A and the distribution parameter α are functions of moments of the productivity distribution $G(z)$; as shown in Online Appendix O5, A is the mean of z and as such is increasing in both the lower and upper bounds of the productivity distribution. TFP, therefore, is increasing in both a_h and a_ℓ: the more automated the capital stock, the higher the Total Factor Productivity. Substituting for z_h and z_ℓ, the distribution parameter α can be expressed in terms of a_h and a_ℓ, $\alpha = \frac{a_h - a_\ell}{1 - a_\ell}$. The distribution parameter, therefore, is increasing in a_h and decreasing in a_ℓ. Since the parameters of the distribution $F(a)$ determine A and α, they also determine factor shares in this economy, as the following proposition establishes.

[8] I have imposed $\gamma = \frac{1}{1-\sigma}$ so that, strictly speaking, σ is also related to the returns to automation at the firm level. In describing the effect of varying σ I have in mind a local approximation leaving γ unchanged and varying the shape parameter of the beta distribution. The resulting aggregate production function is not an exact CES function, but it is very close to CES for small deviations.

[9] Furthermore, as I show in Online Appendix O3.5, the dynamic version of the model can reconcile the disconnect between estimates of $\sigma < 1$ in cross-sectional data (for example, Oberfield and Raval (2021)) and $\sigma > 1$ in time series data (notably Karabarbounis and Neiman (2014)).

Proposition 2. *The labor share is given by*

$$LS = \frac{1-a_h}{1-a_w} \tag{10}$$

Proof. The labor share with a CES production function can be expressed as: $LS = (1-\alpha)^{\sigma} A^{\sigma-1} \left(\frac{W}{H}\right)^{1-\sigma}$ Substituting in for A and α from Equations 8 and 9, and replacing $z_h = \left(\frac{1}{1-a_h}\right)^{\frac{1}{1-\sigma}}$ and $\frac{W}{H} = \left(\frac{1}{1-a_w}\right)^{\frac{1}{1-\sigma}}$ gives the result after algebraic manipulation. □

Proposition 2 provides an easily interpretable expression for the labor share in terms of the automation distribution: the labor share is the ratio of the fraction of non-automated tasks at the most automated firm $1-a_h$ over the fraction of non-automated tasks at the least automated (active) firm $1-a_w$. a_h and a_w are measures of automation that capture different margins. An increase in a_h can be interpreted as an *extensive* margin of automation because an increase in a_h implies that new automation technologies have been discovered and embodied as capital goods. On the other hand, an increase in a_w can be interpreted as an increase along an *intensive* margin: some fraction of the least automated technologies installed in the economy is left idle or scrapped, and so the average automation level of firms that are still active increases.

This interpretation is related to the dynamics at work in models of innovation and diffusion. Increasing a_h is an innovation: a jump in the technology frontier. Increasing a_w corresponds to diffusion because the technology of the lowest active firm is closer to the frontier following an increase in a_w. Benhabib et al. (2020) develop a model in which the firm productivity distribution has a finite upper bound; thus, both innovation (growth at the frontier) and diffusion (catch-up from firms in the left tail of the distribution) are present. They find that the presence of a finite upper bound affects the key properties of the BGP equilibrium. By analogy, the model in this paper uses the same firm level technology as Lagos (2006), but in the models of the latter, the firm productivity distribution is unbounded (so only the "diffusion" margin is active). Introducing the upper bound here allows me to derive model of both the TFP term A *and* capital share parameter α of the aggregate production function.

The expression for the labor share makes explicit my model's connection to David Ricardo's theory of rent (Ricardo (1821)). In his original formulation, the profits of landowners are determined by the difference between the productivity of their own land and that of the best available rent-free land; this is so because the productivity of the worst land determines the wages of landless laborers. This economic rent is not competed away because new land cannot be readily created. By analogy, in my model, the productivity of the least automated capital determines the wages of labor, and owners of supra-marginal machines earn *Ricardian rents* (or *quasi-rents* in the language of Phelps (1963)) from automation. These are profits that arise not from a lack of competition in product markets, but from frictions that impede the free reallocation of resources to their most efficient use. In the static model shown above, this impediment to reallocation is assumed; in the dynamic model presented below, it is sustained in equilibrium by capital irreversibility and adjustment costs that are specific to each technology.

3 Dynamic Model and Balanced Growth

In this section, I embed the static structure of the previous section into a dynamic model that, in aggregate, closely resembles a standard neoclassical growth model. I introduce households that supply labor and make consumption and savings decisions that determine the evolution of the capital stock and its distribution.

I introduce growth into the model by assuming exogenous growth paths for three variables: the number of tasks necessary for the production of the final good, q_t; the frontier level of automation technology $\tilde{a}_{h,t}$; and labor augmenting technological change H_t. The endogenous aggregate states of the dynamic economy are the aggregate capital stock K_t and the distribution of automation technology (or firm-size distribution) $F_t(a)$.

The production and investment technologies in the dynamic model are closely related to models with putty-clay technology and irreversible investment, introduced by Johansen (1959) and further developed by Massell (1962) and Phelps (1963). The vintage capital structure in a putty-clay model of irreversible investment is a natural way to model an economy in which different types of capital coexist in the economy at any given time. In this model, a type or vintage of capital corresponds to capital with different levels of automation. The implementation of putty-clay technology in this model is closely related to that of Gilchrist and Williams (2000): the firm-level Leontief production function leads to variable utilization, and old (less automated) capital is scrapped endogenously as newer (more automated) capital is installed. Importantly, as in Chari and Hopenhayn (1991) and Jovanovic and Yatsenko (2012) (but unlike most vintage capital models), here investment optimally occurs in both old and new vintages of capital. Automation technologies diffuse gradually in the economy - there is a lag between the introduction of a new automation technology and its peak usage - and it is incomplete technological diffusion that gives rise to the firm-size distribution $F_t(a)$, which is equivalently the distribution of quantities of capital of different degrees of automation.

3.1 Firm Life Cycle

I start by describing the production and investment decision of an incumbent firm, followed by the endogenous scrapping and entry decisions.

Incumbent Firms A firm is characterized by its degree of automation $a_{i,t} \equiv \frac{\tilde{a}_i}{q_t}$ and capital stock $k_{i,t}$ (for simplicity I refer to a unit of capital as a machine) and produces final output by combining capital and labor, $n_{i,t}$, in the Leontief production function $y_{i,t} = z(a_{i,t}) \min[k_{i,t}, H_t \cdot n_{i,t}]$, where $z(a_{i,t}) = \left(\frac{1}{1-a_{i,t}}\right)^{\frac{1}{1-\sigma}}$. As in Section 2.2, the firms' optimal choice of hours is given by a cutoff rule: firms with productivity $z(a_{i,t}) < \frac{W_t}{H_t}$ do not hire labor in period t, and so their capital remains idle. The firm's profit function is therefore given by:

$$\pi(a_{i,t}, k_{i,t}) = k_{i,t}\pi(a_{i,t}) = k_{i,t} \max\left[z(a_{i,t}) - \frac{W_t}{H_t}, 0\right] \tag{11}$$

A firm's technology $\tilde{a}_i$ is fixed at birth (firm entry is described below). However, because the number of tasks required to produce final output, q_t, changes over time, a firm's automation, $a_{i,t}$ is not constant over time. In particular, the degree of automation of a machine that can perform tasks up to $\tilde{a}_i$ decreases over time as q_t grows because that machine becomes less automated relative to the total measure of tasks required to produce a unit of output. This gives rise to technological obsolescence and endogenous scrapping (described below). The law of motion for $a_{i,t}$ is:

$$a_{i,t+1} = \frac{\tilde{a}_i}{q_{t+1}} = \frac{q_t}{q_{t+1}} a_{i,t} \tag{12}$$

At the end of each period, the firm must pay a maintenance cost $\kappa > 0$ per machine to keep a machine operational. If the firm does not pay the maintenance cost, the machine depreciates completely at the end of the period. In addition, machines fail with i.i.d. probability δ. Incumbent firms can invest in firm specific capital to grow their stock; investment is irreversible. I denote the value of a firm with capital $k_{i,t}$ and automation $a_{i,t}$ by $V(a_{i,t}, k_{i,t})$:

$$V(k_t, a_t) = \max_{\iota_t} k_t \pi_t(a_t) + \mathbb{1}\left[-\kappa k_t - \bar{\phi}\iota_t - \phi(\iota_t, a_t) + \Lambda_{t,t+1} V(k_{t+1}, a_{t+1})\right], \tag{13}$$

$$\text{s.t.} k_{t+1} = (1-\delta) k_t + \iota_t, \tag{14}$$

$$\iota_t \geq 0, \tag{15}$$

where ι_t is the firm's choice of investment; $\bar{\phi}$ parameterizes the cost of investment in terms of final goods; $\phi(\iota_t, a_t)$ is an increasing function in both arguments that determines the marginal cost of investment; the indicator function reflects the firm's choice to either pay the maintenance cost κk_t or scrap its capital; and $\Lambda_{t,t+1}$ is the time t discount factor of consumption at time $t+1$. For simplicity, following Abel (1983), I assume $\bar{\phi} = 0$ and $\phi(\iota_t, a_t) = \frac{1}{2\Upsilon_t}\phi(a_t)\iota_t^2$, with $\phi(0) > 0$, $\phi' > 0$ and $\phi'' > 0$; and $\Upsilon_t > 0$ is a scaling factor that grows at the same rate as aggregate output to ensure the existence of a balanced growth path. As shown below, this assumption ensures the firm's investment is optimally always weakly positive; thus, the irreversibility constraint never binds. Assuming for now that the firm does not scrap its capital, the first order condition for investment is

$$\frac{\phi(a_t)}{\Upsilon_t}\iota_t = \Lambda_{t,t+1} V_k(k_{t+1}, a_{t+1}). \tag{16}$$

Following Abel and Eberly (1997), I hypothesize that the solution to the value function $V(\cdot)$ is a linear function of the capital stock,

$$V(k_t, a_t) = v(a_t) k_t + \psi(a_t) \tag{17}$$

where $v(a)$ and $\psi(a)$ are unknown functions. With this hypothesis, the solution to the firm's investment problem is

$$\iota_t = \Upsilon_t \frac{\Lambda_{t,t+1} v\left(a_{t+1}\right)}{\phi\left(a_t\right)}, \tag{18}$$

which, since every term on the right hand side is positive, confirms that investment is always weakly positive. To solve for the unknown functions $v(a)$ and $\psi(a)$, I substitute Equation 17 into Equation 13, assuming for now that the firm chooses not to scrap its capital:

$$\begin{aligned} v\left(a_t\right) k_t + \psi\left(a_t\right) = & \left(\pi_t\left(a_t\right) - \kappa + \Lambda_{t,t+1}\left(1-\delta\right) v\left(a_{t+1}\right)\right) k_t \\ & - \frac{1}{2\Upsilon_t} \phi\left(a_t\right) \iota_t^2 + \Lambda_{t,t+1}\left(v\left(a_{t+1}\right) \iota_t + \psi\left(a_{t+1}\right)\right) \end{aligned}$$

This equation must hold for all values of k_t, so the term multiplying k_t on the left hand side must equal the sum of the terms multiplying k_t on the right hand side, and similarly for the terms not multiplying k_t. These equalities give

$$v\left(a_t\right) = \pi_t\left(a_t\right) - \kappa + \left(1-\delta\right) \Lambda_{t,t+1} v\left(a_{t+1}\right), \tag{19}$$

and, substituting in the solution to the investment problem (Equation 18),

$$\psi\left(a_t\right) = \frac{\Upsilon_t \left(\Lambda_{t,t+1} v\left(a_{t+1}\right)\right)^2}{2\phi\left(a_t\right)} + \Lambda_{t,t+1} \psi\left(a_{t+1}\right). \tag{20}$$

The function $v\left(a_t\right)$ is the value of each installed machine of type a_t, whereas $\psi\left(a_t\right)$ captures the present value of the rents accruing to the automation technology. For convenience I also define the firm-specific price of installed capital as the discounted continuation value of a machine of type a_t:

$$p_{\iota,t}\left(a_t\right) \equiv \Lambda_{t,t+1} v_t\left(a_{t+1}\right). \tag{21}$$

Scrapping If firms do not pay the maintenance cost κ their capital fully depreciates, but a firm that scraps its capital still owns its technology (and associated rents $\psi(a)$). The firm therefore chooses not to scrap if the continuation value of a machine exceeds the maintenance cost, $\left(1-\delta\right) \Lambda_{t,t+1} v\left(a_{t+1}\right) > \kappa$. The machine value function, incorporating the scrapping decision, is:

$$v\left(a_t\right)=\pi_t\left(a_t\right)+\mathbb{1}_{(1-\delta)p_{\iota,t}(a_t)>\kappa}\left[\left(1-\delta\right)\Lambda_{t,t+1}v\left(a_{t+1}\right)-\kappa\right]. \tag{22}$$

It follows that the scrapping threshold at the end of the period is defined implicitly by:

$$\left(1-\delta\right)p_{\iota,t}\left(a_t\right)=\kappa. \tag{23}$$

The automation level of the lowest machine installed at the beginning of time t is $a_{\ell,t}$, so the scrapping threshold condition implies that machines in $(a_{\ell,t},a_{\ell,t+1}]$ are scrapped each period. In terms of the CDF of the automation distribution, the fraction of capital that is scrapped at the end of time t is $\varsigma_t=F_t\left(a_{\ell,t+1}\right)$.

New Automation Technologies and Firm Entry In each period, new automation technologies $\left(\tilde{a}_{h,t},\tilde{a}_{h,t+1}\right)$ are discovered. Potential entrants draw technologies from the entire set and enter; consequently. a measure $\tilde{a}_{h,t+1}-\tilde{a}_{h,t}$ new firms are born each period with zero initial capital stock. The value function of a new firm is:

$$\begin{aligned} V\left(0,a_t\right)&=\max_{k_{t+1}}-\frac{1}{2\Upsilon_t}\phi\left(a_t\right)\left(k_{t+1}\right)^2+\Lambda_{t,t+1}V\left(k_{t+1},a_{t+1}\right), \\ &\text{s.t.} k_{t+1}\geq 0 \end{aligned} \tag{24}$$

and (similar to the above) initial investment is given by $k_{t+1}=\Upsilon_t\frac{\Lambda_{t,t+1}v(a_{t+1})}{\phi(at)}$.

3.2 Households

The representative household chooses consumption C_t, savings S_{t+1} and labor supply N_t every period to solve the following intertemporal problem:

$$\max_{C_t,S_{t+1},N_t}\sum_{t=0}^{\infty}\beta^t\frac{\left(q_tC_te^{-\nu(N_t)}\right)^{1-\eta}-1}{1-\eta}, \tag{25}$$

where C is consumption; N is labor supply; and q is the task input into production (which I interpret as the quality of production, such that consuming goods that require a higher task input to produce gives households higher utility)[10] and $\nu\left(N\right)$

[10] I include q_t explicitly in agent's preferences so as to introduce into the model the notion that higher task input increases the quality of production. In the context of this model, omitting q

is the utility cost of supplying labor[11]. This functional form for preferences satisfies the conditions required for balanced growth (see King et al. (1988)). The household is subject to the following budget constraint:

$$C_t = W_t N_t + R_{S,t} S_t P_{S,t-1} - P_{S,t} S_{t+1}, \tag{26}$$

with $R_{S,t} = \frac{D_t + P_{s,t}}{P_{s,t-1}}$. The household saves in a mutual fund that owns the shares S_t of all firms in the economy. D_t (defined below) is the aggregate dividend (per share) of all firms, and $P_{s,t}$ is the time t price of a unit of this mutual fund; W_t is the wage. The equilibrium conditions from household optimization are the standard intratemporal labor supply condition and intertemporal Euler equation:

$$\nu'(N_t) = \frac{W_t}{C_t} \tag{27}$$

$$1 = \Lambda_{t,t+1} R_{s,t+1}, \tag{28}$$

where $\Lambda_{t,t+1}$ is the discount factor between t and $t+1$, $\Lambda_{t,t+1} \equiv \beta \frac{\left(q_{t+1} e^{-\nu(N_{t+1})}\right)^{1-\eta} C_{t+1}^{-\eta}}{\left(q_t e^{-\nu(N_t)}\right)^{1-\eta} C_t^{-\eta}}$.

3.3 Aggregation

Similar to the static model of Section 2, aggregate output, labor input and the capital stock are given by $Y_t = \int_{a_{\ell,t}}^{a_{h,t}} z_t(a) k_t(a) \ \mathrm{d}a$, $N_t = \int_{a_{\ell,t}}^{a_{h,t}} n_t(a) \ \mathrm{d}a$ and $K_t = \int_{a_{\ell,t}}^{a_{h,t}} k_t(a) \ \mathrm{d}a$, respectively, whereas aggregate investment and the aggregate price of investment are:

$$I_t = \int_{a_{\ell,t}}^{a_{h,t}} \iota_t(a) \ \mathrm{d}a \tag{29}$$

and

$$P_{I,t} = \int_{a_{\ell,t+1}}^{a_{h,t}} p_{\iota,t}(a) j_t(a) \ \mathrm{d}a \tag{30}$$

where $j_t(a)$ is the time-t distribution of investment, $j_t(a) \equiv \left(\frac{\iota_t(a)}{I_t}\right)$. The dividend is the part of aggregate output that is not paid to workers in wages or used to invest in

from preferences would imply that increases in q result in technological regression (conceptually, producing the same good requires higher effort the higher q is). An alternative modelling choice is that of Acemoglu and Restrepo (2018), in which increasing q raises labor productivity because human labor is assumed to be more productive in higher index tasks.

[11] ν is assumed to be continuously differentiable, increasing, and convex.

new capital, per unit of capital[12]:

$$D_t = \frac{Y_t - W_t N_t - \frac{P_{I,t}}{2} I_t}{K_t}. \tag{31}$$

The aggregate (ex-dividend) value of all firms in the economy is the sum of the intangible plus tangible value of assets in the economy:

$$P_{S,t} S_{t+1} = P_{K,t} K_{t+1} + \Psi_{t+1}, \tag{32}$$

where

$$P_{K,t} K_{t+1} = \int_{a_{\ell,t+1}}^{a_{h,t}} p_{\iota,t}(a)\, k_{t+1}(a) \ \mathrm{d}a \tag{33}$$

is the aggregate value of installed capital (tangible value), and

$$\Psi_{t+1} = \int_{a_{\ell,t+1}}^{a_{h,t}} \psi_{t+1}(a) \ \mathrm{d}a \tag{34}$$

is the aggregate value of rents accruing to technology (intangible value). The aggregate capital stock of the economy evolves as follows:

$$K_{t+1} = (1-\delta)(1-\varsigma_t) K_t + I_t, \tag{35}$$

where δ is exogenous depreciation and ς_t is the fraction of capital that is scrapped every period.

3.4 Dynamics of the Automation Distribution $F_t(a)$

Unlike in the static model of Section 2, here the firm-size distribution $f_t(a)$ is an endogenous state variable, determined by the entry, ongoing investment and scrapping

[12]Note that the aggregate resources used up in investment are given by the integral over goods used for investment at each firm, $\frac{1}{2\Upsilon_t}\int_{a_{\ell,t+1}}^{a_{h,t}} \phi(a_t)\, \iota_t(a)^2 \ \mathrm{d}a = \frac{1}{2}\int_{a_{\ell,t+1}}^{a_{h,t}} p_{\iota,t}(a)\, \iota_t(a) \ \mathrm{d}a = \frac{P_{I,t}}{2} I_t$, where the second equality follows from firms' first order condition for investment and the second from Equation 30).

decisions of firms. The law of motion for $f(a)$ is

$$f_{t+1}(a) = \begin{cases} \frac{K_t}{K_{t+1}}(1-\delta) f_t(a) + \frac{I_t}{K_{t+1}} j_t(a) & a_{\ell,t+1} < a < a_{h,t+1} \\ 0 & a \leq a_{\ell,t+1} \end{cases} \tag{36}$$

The first line is the evolution of the part of the capital stock that is not scrapped; it comes from aggregating the law of motion of capital at the firm level (Equation 14). The value of $f_{t+1}(a)$ is the weighted average of surviving capital plus new investment, where the weights are given by $\frac{K_t}{K_{t+1}}(1-\delta)$ and $\frac{I_t}{K_{t+1}}$, respectively. The second line corresponds to the part of the capital stock that is scrapped. The focus of analysis in this paper is the equilibria of the economy that admit a CES representation of the aggregate production function; per Proposition 1, the CES representation obtains if the distribution $F(a)$ is in the beta distribution family specified in Assumption 1. I make the following assumption:

Assumption 2. Cost function $\phi(a)$ and investment distribution $j_t(a)$. *The marginal cost function $\phi(a) \in \Phi^*$, where Φ^* is the set of functions such that $j(a)$ is in the beta family specified in Assumption 1, $j(a) = \frac{\sigma}{1-\sigma} \frac{(a_{h,t}-a)^{\frac{\sigma}{1-\sigma}-1}}{(a_{h,t}-a_{\ell,t})^{\frac{\sigma}{1-\sigma}}}$.*

In other words, I parametrize the marginal cost of investment at each point in the automation distribution such that the distribution of investment $j_t(a)$ is in the beta family. The first-order condition for firm investment (Equation 18) shows that $\phi(a)$ determines investment as a function of a, and, thereby, it determines its distribution. I derive the functional form of $\phi(a)$ in Online Appendix O4. This assumption guarantees that, in a stationary equilibrium, when $f_{t+1} = f_t = j_t$, the distribution $f(a)$ is in the beta family.

3.5 Equilibrium and Balanced Growth Path

An equilibrium of the dynamic model consists of sequences of: (i) aggregate allocations $\{C_t,\ I_t,\ N_t\}$; (ii) prices $\{W_t,\ P_{I,t},\ P_{K,t},\ \Psi_{t+1}\}$; (iii) firm allocations $\{\iota_{i,t},\ n_{i,t}\}$; and (iv) firm scrapping decisions, such that household and firm equilibrium conditions are satisfied and markets for the final good savings and labor are cleared. Market clearing in final goods requires that all production is either consumed or invested, $Y_t = C_t + \frac{P_{I,t}}{2} I_t$, while savings market clearing requires that savings equal the aggregate capital stock, $K_{t+1} = S_{t+1}$. The state variables of the model are the exogenous states H_t, q_t and $\tilde{a}_{h,t}$ (labor augmenting technology, tasks and frontier automation technology, respectively) and the endogenous states K_t, $a_{\ell,t}$ and $f_t(a)$ (the aggregate capital stock, the lower bound of the automation distribution, and the firm-size (or automation) distribution, respectively).

Stationary Equilibrium and the Balanced Growth Path I introduce growth into the economy by assuming deterministic growth paths for labor-augmenting technology H_t, automation technology $\tilde{a}_{h,t}$ and tasks q_t, which grow at rates g, g_a and g_q respectively. I also assume that $a_{h,t} = \frac{\tilde{a}_{h,t}}{q_t} < 1 \; \forall \; t$; thus, the economy never reaches a state in which the frontier technology is such that all tasks are fully automated (see Aghion et al. (2017) for a discussion of the economics of that possibility). I show that in a stationary equilibrium the model has a steady state with constant growth that satisfies the Jones and Scrimgeour (2008) definition of a balanced growth path. To the usual requirements for existence of the balanced growth path I add that the automation distribution $F(a)$ is stationary; in fact, as shown below, stationarity of $F(a)$ is a necessary and sufficient condition for a balanced growth path.

Proposition 3. *Balanced Growth. In a stationary equilibrium such that $f_{t+1} = f_t = j_t$, and with $g_a = g_q$, the economy has a balanced growth path that satisfies the Jones and Scrimgeour (2008) definition: aggregate quantities $\{Y_t, C_t, I_t, K_t\}$ grow at rate g and factor shares are constant and strictly positive.*

Proof. According to Assumption 2 and Proposition 1, aggregate output in this economy can be represented as a CES production function with elasticity of substitution $\sigma < 1$, $Y_t = A\left(\alpha K_t^{\frac{\sigma-1}{\sigma}} + (1-\alpha)(H_t \cdot N_t)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}$. The conditions for the existence of a balanced growth path follow from the Uzawa (1961) growth theorem, which states that the balanced growth path exists only if all technological growth is labor augmenting, and that in that steady state $\{Y_t, C_t, I_t, K_t\}$ grow at the rate of labor augmenting technical progress, g. This requires that the TFP term $A_t = (z_{h,t})^{1-\sigma}(z_{\ell,t})^{\sigma}$ is constant in the steady state. Substituting in the definitions for z_h and z_ℓ in terms of the bounds of the automation distribution $F(a)$ gives $A_t = \left(\frac{1}{1-a_{h,t}}\right)\left(\frac{1}{1-a_{\ell,t}}\right)^{\frac{\sigma}{1-\sigma}}$. Constant TFP therefore requires that $a_h = \frac{\tilde{a}_{h,t}}{q_t}$ and $a_\ell = \frac{\tilde{a}_{\ell,t}}{q_t}$ are both constant in the steady state, so q_t, $\tilde{a}_{h,t}$, and $\tilde{a}_{\ell,t}$ must grow at the same rate; and that σ is constant. The constancy of σ follows from Assumption 2, and q_t and $\tilde{a}_{h,t}$ grow at the same rate

g_q by assumption. I show in Appendix A.2 that $\tilde{a}_\ell$ also grows at rate g_q. Since a_h, a_ℓ and σ fully characterize the distribution $F(a)$, the conditions for constant TFP in the steady state and stationarity of the distribution $F(a)$ are the same. □

Proposition 3 establishes that the economy has a continuum of balanced growth paths, which are indexed by the parameters of the stationary firm-size distribution $F(a)$: $\{\sigma, a_h, a_\ell\}$. The distribution itself is an endogenous state variable determined by the marginal cost of investment function $\phi(a)$. It follows that there exists a balanced growth consistent with a CES representation of the aggregate production function for any $\phi(a)$ that satisfies Assumption 2.

According to Proposition 3, improvements in automation technology that are matched by growth in tasks do not lead to long-run growth, and the distribution of automation $F(a)$ does not affect the steady state growth rate[13]. The automation distribution does, however, determine factor shares and TFP in any balanced growth path.

4 Long- and Short- Run Effects of Increased Automation

Concerns about labor's share of national income have a long history in macroeconomics (the "principal problem of Political Economy", according to David Ricardo), and this interest that has been revived due to the decline in recent decades in labor's share of US value added. The adoption of automation technology has been cited as one of the causes of this decline[14]. In this section I use a calibrated version of the dynamic model to study comparative statics and transition dynamics of factor shares and other macro and micro moments in response to an increase in frontier automation technology a_h.

As discussed above, the cost function $\phi(a)$ is crucial in determining the distribution of automation technology in the model, which in turn determines the effect of

[13]However, because q grows in the steady state, output measured in *quality* terms grows at rate $g + g_q$. Since growth in q that is not matched by growth in $\tilde{a}_h$ reduces productivity in quantity terms, growth in automation technology in the balanced growth path enables growth in quality.

[14]Bergholt et al. (2019), Autor and Salomons (2018), Hubmer and Restrepo (2021), among others. The connection between corporate tax rates and the labor shares explored by Kaymak and Schott (2019) is also consistent with automation. Moll et al. (2021) study the effects of automation on inequality in a setting with heterogeneous skills and returns to wealth.

automation on productivity and factor shares. The most interesting comparative statics in this model, therefore, require comparisons of different calibrations of the function $\phi(a)$. To make comparative statics tractable and sharpen the focus on the distribution of aggregate income, I specialize the model in two ways.

No-scrapping Limit First, I derive the limit of the economy described in the previous section as $\kappa \to 0$ and $g_q \to 1$, which is the *no-scrapping limit* (the full equilibrium conditions are provided in Online Appendix O1). As the maintenance cost κ and the growth rate of tasks (and, therefore, the growth rate of automation technology) g_q go to zero, both the firm exit and firm entry margins are shut down. Therefore, in the limit, all firms are active in equilibrium, which implies that $a_w \to a_\ell$; that the aggregate capital labor ratio converges to H_t ($K_t/\bar{N} \to H_t$); and that $Y_t \to A_t K_t = A_t H_t \bar{N}$.

Capitalist-Worker Model Second, I study a version of the model which, in a formal sense, is the least favorable to workers. Rather than a representative household, the economy is here populated by: i) a representative capitalist who owns the technology and capital of the economy and consumes dividends; and ii) a representative worker who supplies labor and consumes but does not save. I solve a planning problem for this economy with a planner who only values the representative capitalist's utility (in this sense the equilibria of this version of the model are the least favorable to workers). The planner, who faces the same technological constraints to investment as firms, chooses the optimal investment distribution, subject to the constraint that the distribution is in the beta family specified in Assumption 2. The investment distribution is fully characterized by three parameters: a_h, σ and a_w. Since a_h and σ are technological parameters, the planner's choice simplifies to that of the optimal a_w, which is the lower bound of the distribution and the degree of automation of the least automated capital in the economy. I summarize the existence and properties of the equilibrium in the following proposition:

Proposition 4. *Existence and uniqueness of the capitalist-worker equilibrium. There exists a unique stationary equilibrium of the capitalist-worker, no-scrapping economy. At the stationary equilibrium, the degree of automation of the least automated capital* *is given by:*

$$a_w^* = \max\left[\frac{a_h + \sigma - 1}{\sigma}, 0\right]. \tag{37}$$

Proof. See Appendix A.4. The full model and equilibrium conditions are in Online Appendix O2. □

Because here workers consume their wages every period, income and wealth effects on labor supply cancel out and workers supply a constant amount of labor, $N_t = \bar{N}$. Since the focus of this paper is not on the long run effects of automation on labor supply, this simplifies the exposition of the model considerably at negligible cost in terms of intuition. I present the results for a version of the model with elastic labor supply in Online Appendix O2.

4.1 Long-Run Effect of an Increase in a_h

Having solved the stationary equilibrium of the capitalist-worker model, I turn to the analysis of comparative statics with respect to an increase in the frontier automation level a_h.

Proposition 5. *Comparative statics with respect to frontier automation a_h. In the no-scrapping, capitalist-worker equilibrium the long run labor share is*

$$LS^* = \begin{cases} \sigma & \text{if } a_h + \sigma > 1 \\ 1 - a_h & \text{if } a_h + \sigma \leq 1 \end{cases}$$

Long run TFP is

$$TFP^* = \begin{cases} \sigma^{\frac{\sigma}{1-\sigma}} \left(\frac{1}{1-a_h}\right)^{\frac{1}{1-\sigma}} & \text{if } a_h + \sigma > 1 \\ \frac{1}{1-a_h} & \text{if } a_h + \sigma \leq 1 \end{cases}$$

Proof. From Proposition 4, the optimal $a_w^* = \max\left[\frac{a_h+\sigma-1}{\sigma}, 0\right]$. Substituting a_w^* into Equation 8 (noting that $a_\ell \to a_w$ in the no-scrapping equilibrium) and Equation 10 gives the result. □

As automation a_h increases from zero (such that $a_h + \sigma < 1$), the labor share declines and productivity increases with frontier automation technology; thus, there is a trade-off between the productivity of the economy and the distribution of aggregate income. In this part of the parameter space, in equilibrium, the least automated firms have zero automation, $a_w^* = 0$. Intuitively, low productivity translates into low labor demand and wages, such that firms with zero automation can profitably stay in the market.

For sufficiently high a_h ($a_h + \sigma > 1$), the level of frontier technology has no further long-run effect on the labor share, and only the distribution of automation technology, as parametrized by σ, matters. Intuitively, an increase in automation increases productivity and therefore labor demand, which, in pushing up wages, forces low automation firms out of the market, thereby increasing wages and productivity. In the long run, the diffusion of technological innovation through the cross-section of firms undoes the trade-off between productivity and the aggregate distribution of income.

Proposition 5 suggests a theoretical explanation for the long-run stability of the labor share, since as long as the parameter σ is constant, the labor share will tend to return to this natural level. Using long time series data for the US, France and the UK, Charpe et al. (2019) find evidence of long-term cycles in the labor share around a stable long-run level, lending empirical support to the notion of a natural resting point for the labor share. In addition to parametrizing the (short-run) elasticity of substitution of the aggregate production function, the parameter σ has two additional interpretations in the model, and they connect the theory in this paper to two salient explanations for the decline in labor's share of income.

First, σ parametrizes the curvature of the cost function $\phi(a)$. In particular, the second derivative of the cost function $\phi(a)$ with respect to a is increasing in σ[15], which implies that the higher σ is, the higher the cost of producing the most automated types of capital in the economy relative to less automated types of capital. In the stationary equilibrium, a lower σ, therefore, translates into a distribution $f(a)$ that is more right (negatively) skewed[16]. This also implies that the lower σ is, the higher the concentration of value added in the firms that have the highest degrees of automation, and, therefore, the lowest labor shares.

The theory in this paper, therefore, posits a link between the cross-sectional distribution of technology and the labor share. Two recent, influential papers examine

[15] $\frac{\partial\left(\frac{\partial^2 \phi(a)}{\partial a^2}\right)}{\partial \sigma} > 0, \ \forall\ a, \sigma \in (0,1)$

[16] The skewness of the PDF of a $B\left(1; \frac{\sigma}{1-\sigma}\right)$ is $\frac{2-4\sigma}{(2\sigma-3)\sqrt{\frac{\sigma}{2-\sigma}}}$, an increasing function of σ.

the connection between the cross-sectional distributions of the labor share and value added and the aggregate labor share, Kehrig and Vincent (2021) (hereafter KV21) and Autor et al. (2020). The empirical evidence in KV21 shows that, in the case of manufacturing in the US, the aggregate labor share fell while establishment level labor shares rose slightly. This was due to the reallocation of value added away from high labor share establishments towards low labor share establishments. In the stationary equilibrium of the model, the x^{th} percentile of the firm-level labor share distribution is given by

$$LS_x = (x + \sigma (1 - x))^{\frac{1}{1-\sigma}} , \tag{38}$$

an increasing function of σ but with a slope smaller than one for all x and σ in $(0, 1)$, such that as σ changes, the aggregate labor share changes by more than the firm-level labor shares, consistent with the evidence in KV21. Autor et al. (2020) argue that the concentration of value added in larger establishments ('superstars') underlies the fall in sectoral labor shares. In the model, the value added share of the top $x\%$ of firms can be calculated as $\mathcal{C}_{x,t} = \frac{\int_{a_h - x(a_h - a_w)}^{a_h} y_t(a) \, \mathrm{d}a}{Y_t}$. After simplification, in the stationary equilibrium this reduces to:

$$\mathcal{C}_x = \left(\frac{x}{x + \sigma(1 - x)} \right)^{\frac{\sigma}{1-\sigma}} , \tag{39}$$

a decreasing function of σ for any $x \in (0, 1)$. The putty-clay automation model can, therefore, rationalize the observation that an increased concentration of value added in low labor share establishments has led to a fall in the aggregate labor share caused by a fall in σ. Intuitively, a fall in σ is a change to technology that favors the growth of firms that have the most automated capital, to the detriment of firms that have less automated capital.

The parameter σ also has an interpretation as parameterizing the speed of technological diffusion in the model. In the general model, with growth in tasks and automation, technology diffuses gradually through the economy and there is a time lag between the time a technology is first introduced and its peak usage. For $\sigma > 0.5$, the mode of $f(a)$ is at a_w, which corresponds to the peak usage of a technology. Using Equation 12 I can solve for the diffusion lag τ_d in an economy with technology growing at rate g_q as $a_h (1 + g_q)^{-\tau_d} = a_w$, with the interpretation that τ_d is the number of periods it takes for the degree of automation of a technology to fall from a_h to a_w, due to task growth at rate g_q. Substituting in for the equilibrium a_w^* and solving for τ_d gives:

$$\tau_d = -\frac{\log\left(\frac{a_h+\sigma-1}{\sigma a_h}\right)}{\log\left(1+g_q\right)}, \tag{40}$$

which is a decreasing function of σ. Moreover, note that in this model the ratio of the labor productivity of the frontier firm and the technological laggard is $\frac{z(a_h)}{z(a_w)}$,which, substituting in for equilibrium values, gives

$$\frac{\text{Productivity at frontier}}{\text{Productivity of laggard}} = \sigma^{\frac{1}{\sigma-1}}, \tag{41}$$

again, a decreasing function of σ. These two equations connect the putty-clay automation model to the theory of Akcigit and Ates (2021), in which a decrease in the rate of technological diffusion leads to an increase in concentration and a fall in the labor share. In this model, a decrease in σ results in a fall in the rate of technological diffusion, increased concentration, a fall in the labor share, and a widening of the labor productivity gap between frontier and laggard firms.

To summarize, the putty-clay automation model can rationalize empirical findings regarding the aggregate labor share and its distribution in the cross-section of firms as a fall in σ to a new stationary equilibrium. In the model, a fall in σ has three interpretations: i) a decrease in the aggregate elasticity of substitution between capital and labor; ii) a change in technology that makes it less costly for more productive firms to increase their scale relative to less productive firms; and iii) a slowing down of the diffusion of automation technology in the cross-section of firms. Figure 1 summarizes these findings.

4.2 Transition Dynamics in Response to an Increase in a_h

I now turn to the transition dynamics between steady states in response to a one-time, unanticipated increase in the frontier technology a_h. The transition path in the capitalist-worker framework is the solution to a dynamic planning problem in which the planner chooses the investment distribution along the transition path to maximize the consumption of the representative capitalist.

As in the stationary problem, the planner's choice of distribution reduces to choosing

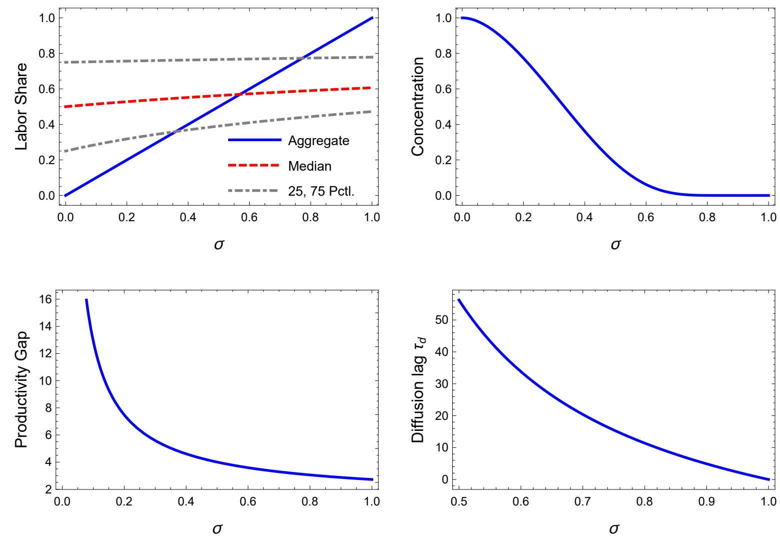

Figure 1: The role of σ in the putty-clay automation model

The top left panel plots the aggregate labor share σ and the median, the 25th percentile, and the 75th percentile firm-level labor shares from Equation 38. The top right panel plots the share of value added of the top 10% of firms in the stationary equilibrium, Equation 39. The bottom left panel plots the productivity of the frontier firm/productivity of the laggard firm, Equation 41. The bottom right panel plots the technology diffusion lag τ_d, measured in periods of the model, Equation 40, for $g_q = 1\%$.

an increasing sequence $\{a_{w,t}\}_{t=1}^{T}$ for the lower bound of the automation distribution, with boundary values $a^*_{w,0}$ and $a^*_{w,T}$ the initial and final steady values of a_w, respectively. Conceptually, in each period the planner chooses how much of the existing capital stock to scrap (an increase in a_w), and scrapped capital is replaced in the capital stock by investment in non-obsolete capital. The more capital is scrapped, the higher the investment required to replace it, and the larger the reduction in dividends to finance increased investment; on the other hand, the faster capital is scrapped, the faster the transition to the new steady state automation distribution, with higher productivity and, therefore, dividends. Consistent with this intuition, there are two broad classes of equilibria, and these lead to relatively fast or relatively slow convergence to the steady state. I focus on the class of slow convergence equilibria because equilibria with fast convergence exhibit empirically implausible dynamics for real interest rates, productivity growth and investment, among other variables (I show transition dynamics for both the slow and fast convergence equilibrium in Online Appendix O3).

To calibrate the main parameters of the model, I use data from the BEA/BLS Integrated Industry-Level Production Accounts for the United States. I set σ to match labor's share of value added from industry-level accounts at the start of the sample (1987) and I size the automation shock to match the fall in the labor share over the sample period – a decline of approximately 6 percentage points to the recent minimum (2013)[17]. I use standard values for all other parameters ($\eta = 1$, $\delta = 0.075$, $\beta = 0.99$). The left panel of Figure 2 shows the transition path of the labor share in response to the automation shock. Both the fall in the labor share and the return to its long run level are gradual, but the increase from the minimum is much slower. Consistent with the evidence presented by Charpe et al. (2019), the putty-clay framework generates persistent deviations in the labor share from its long-run level. The dynamics of the labor share result from two offsetting forces. On the one hand, scrapping obsolete capital raises wages and increases the labor share; on the other hand, installing more automated capital thickens the right tail of the automation distribution, increasing output but not (directly) wages. The labor share falls initially because the latter effect dominates. This is illustrated in the right panel of Figure 2, which plots the initial (solid black), terminal (dotted red), and intermediate (dotted gray) values of $f(a)$. The distribution travels rightwards due to scrapping of obsolete capital, and the right tail of the distribution thickens as newly introduced types of capital are installed. At the start of the transition (the leftmost of the gray dotted lines), the thick right tail of the distribution is visible, but the left boundary is almost the same as in the steady state. Productivity, therefore, increases more than wages, and the labor share falls.

4.2.1 Macro Trends: Slowing TFP, Declining Real Interest Rates

Along the transition path of the economy, the model can qualitatively replicate two important stylized facts in US macro data of the past decades: a slowdown in total factor productivity growth and a fall in real interest rates. Figure 2 shows the transition path of these two variables and the labor share over the part of the transition path when the labor share is declining.

[17] As noted in several studies, the financial and real estate sectors play a significant and specific role in driving the decline in labor's share of GDP. Since these sectors are absent from the model economy, I exclude them from both the denominator and numerator. Additionally, as pointed out by Koh et al. (Forthcoming), the capitalization of intellectual property investment in value added markedly changes the evolution of the labor share. To address this point, I exclude IPP investment (from BEA table 3.7I) from the measure of value added. Finally, I use the sum of college and non-college labor compensation as the measure of total labor compensation and obtain a labor share in

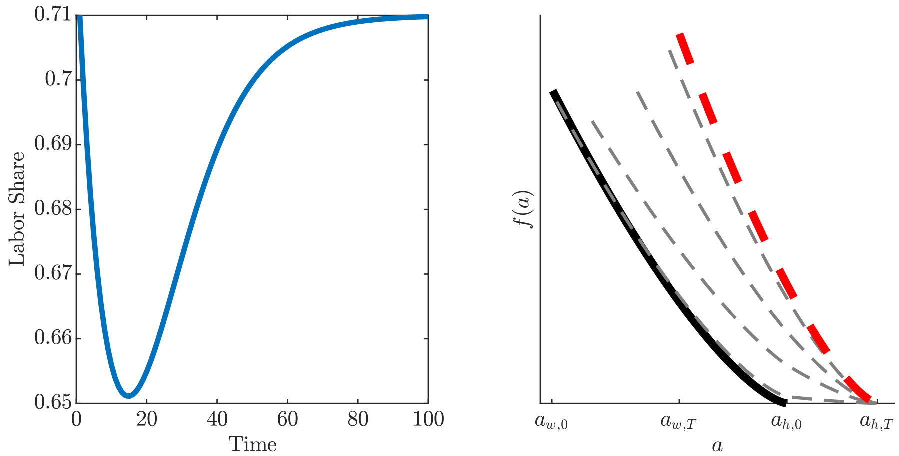

Figure 2: Transition paths of the labor share and automation distribution $f(a)$

Transition path of the labor share (left panel) and $f(a)$ (right panel) in response to an unanticipated increase in a_h at time zero. Parameters: $a_{h,0} = 0.3$, $a_{h,T} = 0.65$, $\delta = 0.075$, $\beta = 0.99$, $\eta = 1$, $\sigma = 0.71$. In the right panel the solid black line is the initial distribution, the red dashed line is the final distribution, and the gray dashed lines are distributions along the transition path.

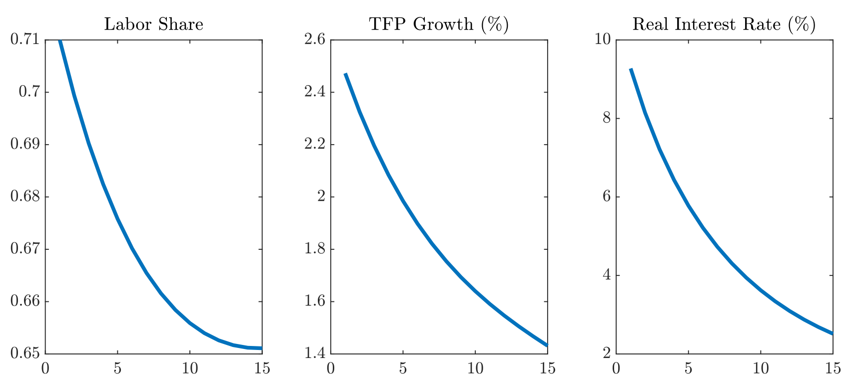

Figure 3: Transition path of macro stylized facts

Transition paths for macro stylized facts in response to an unanticipated increase in a_h at time zero. Parameters: $a_{h,0} = 0.3$, $a_{h,T} = 0.65$, $\delta = 0.075$, $\beta = 0.99$, $\eta = 1$, $\sigma = 0.71$.

Falling Total Factor Productivity Growth and Real Interest Rate As shown in the top right panel of Figure 2, TFP growth decreases monotonically along the transition path. It is true of any transition path of the model that TFP growth must eventually fall because TFP grows due to the introduction and diffusion of new automation technology. After a one-time shock to $a_{h,t}$ the diffusion process eventually stops. As shown by Proposition 1, both improvements in technology at the frontier and scrapping of obsolete capital raise TFP. Intuitively, the increase in TFP is largest at the start of the transition path because, at that point, the productivity difference between scrapped capital and the new capital introduced to replace it is maximal.

Similarly, the real interest rate decreases monotonically along the transition path. The interest rate must increase initially to compensate the capitalist for the cut in dividends required to finance increased investment, undertaken to replace obsolete with new capital. As TFP (and, hence, output) growth slows along the transition path, the required compensation falls. By implication, the value of the "stock market" (capital plus technology, $P_{K,t}K_{t+1}+\Psi_t$, defined in Equations 33 and 34) falls when the automation shock hits; this is so because of the impending obsolescence of a fraction of the capital stock and the reduced rents that accrue to the non-obsolete part of the capital stock. The stock market rises, at a decreasing rate, along the transition path as productivity grows. The model can, therefore, generate joint dynamics of factor shares and asset prices that are qualitatively consistent with the findings of Greenwald et al. (2019). This is a promising direction for future research.

4.2.2 Micro Trends: Reallocation Toward Low Labor Share Firms

In this section I show the performance of the model in matching notable stylized facts with regards to the distribution of value-added, firm-level labor shares and aggregate industry-level shares that are analyzed by Hartman-Glaser et al. (2019), Autor et al. (2020) and KV21, among others.

To do so, I calibrate the model to match the evidence for the manufacturing sector in KV21. Thus, I set the following: $\sigma = 0.61$ to match the labor share in 1967; the size of the shock so that, at the lowest point in the transition period, the labor share falls to the level observed at the end of the KV21 sample, 0.41; and $\delta = 0.1125$ so that the model reaches the low point in the labor share in the same number of periods as

1987 of 0.71, which falls to 0.65 by 2013.

in the KV21 data. The remaining parameters are as above. Note that the plots in this section show transition paths from the initial period to the period in which the labor share reaches its minimal value along the transition path. Consistent with the results of the previous sections, the labor share eventually returns to its steady state level.

Figure 4 plots the transition of the aggregate labor share and of the firm-level quantiles, and it is the model analog to Figure 2 in KV21. The aggregate labor share (solid blue line) falls from 0.61 to 0.41 over the course of 46 periods, as in the KV21 data. At the firm level, the median and the top and bottom quartiles are essentially unchanged during the transition. As in KV21, the fact that the aggregate labor share declines sharply while the firm-level quantiles are constant shows that the decline in the labor share is driven entirely by reallocation in value added towards low labor-share establishments.

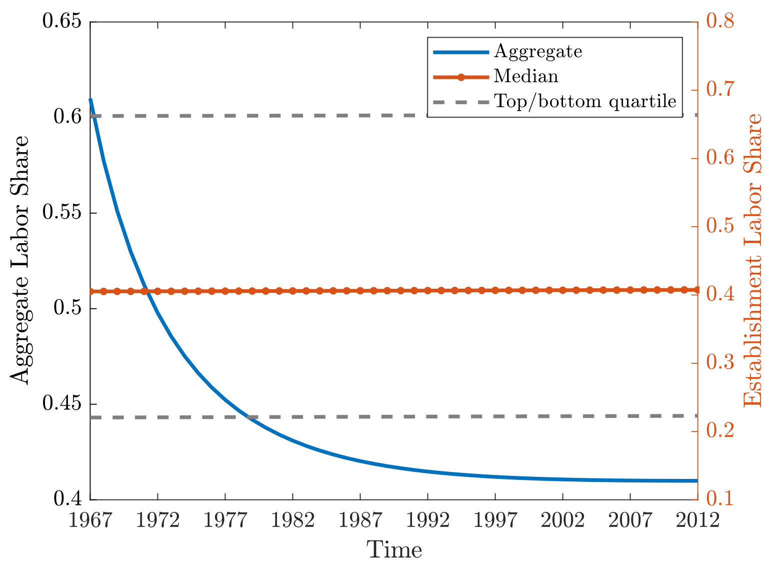

Figure 4: Aggregate and firm-level labor shares in the model

Transition paths for aggregate and firm-level labor shares in response to an unanticipated increase in a_h at time zero. Parameters: $a_{h,0} = 0.65$, $a_{h,T} = 0.77$, $\delta = 0.1125$, $\beta = 0.99$, $\eta = 1$, $\sigma = 0.61$.

Figure 5 plots the model cross-firm distributions of labor shares in the initial period ("1967") and the end period ("2012"); it is also the model analog to Figure 3 in KV21. The leftmost panels plot the share of establishments by labor share, the middle panels the share of labor input, and the rightmost panels the share of value added. The share of firms by labor share (leftmost panels) is skewed in the model (unlike in the data) because the equilibrium firm-size distribution is a beta distribution (per Assumption 2). Consistent with the evidence in KV21, there is no significant change in the distribution of establishments by labor share between the start and the end periods. Comparing the top and bottom middle panels, there is, as in KV21, some reallocation in labor input towards low labor share establishments. However, the rightmost panels show that the fall in the aggregate labor share in the model is driven overwhelmingly by reallocation in value added towards low labor share establishments, as in the data.

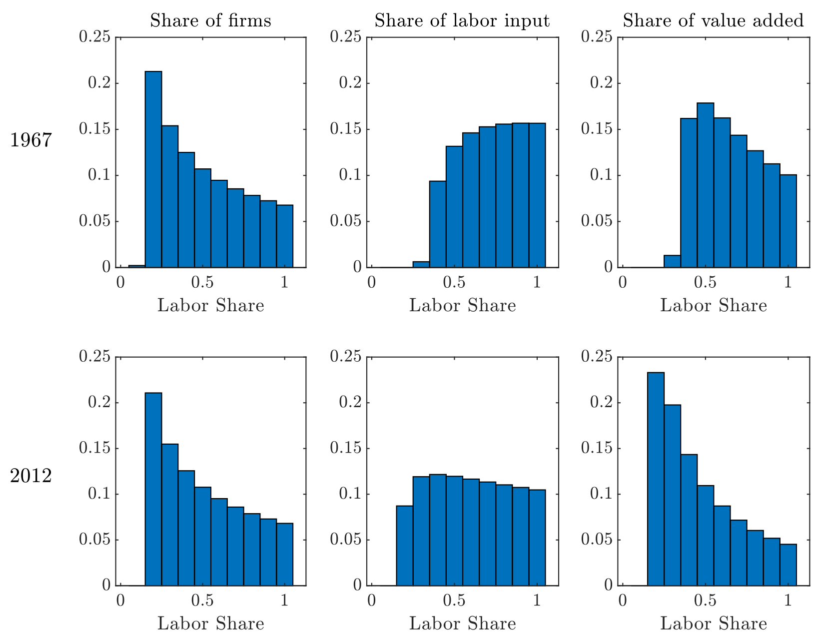

Figure 5: Model distribution of establishments, labor input and value-added conditional on the labor share

Histograms of share of firms, labor input and value added by labor share for the initial ("1967") and the 46th ("2012") period of the transition path in response to an unanticipated increase in a_h at time zero. Parameters: $a_{h,0} = 0.65$, $a_{h,T} = 0.77$, $\delta = 0.1125$, $\beta = 0.99$, $\eta = 1$, $\sigma = 0.61$.

KV21 further decompose the reallocation effect, summarized by the change in the covariance between firm-level labor shares (LS_i) and firm-level value added shares (ω_i) as $\Delta Cov\left(LS_i, \omega_i\right) = Cov\left(\Delta LS_i, \omega_i\right) + Cov\left(LS_i, \Delta\omega_i\right) + Cov\left(\Delta LS_i, \Delta\omega_i\right)$, which they term, respectively, the "Big Player", "Superstar" and "Rising Star" scenarios. The mapping from firms in the model to establishments in the data is imprecise because firms in the model operate only one type of technology and the entrance of new technologies into the economy occurs only through the creation of new "firms". This also means that entry and exit play a much bigger role in driving the dynamics of the labor share in the model than in the data. Moreover, in the model, unlike in the data, the labor share *increases* at the firm level as wages rise along the transition path. These important caveats aside, I construct a model analog to the KV21 covariance decomposition and find that the fall in $Cov\,(LS_i, \omega_i)$ is driven by the "Superstar" and "Rising Star" components, with the former playing a larger role in the model than in the KV21 data.

4.3 Discussion

The putty-clay automation model suggests two possible explanations for the evolution of factor shares and other macro aggregates over the last decades. The first is a technological change that has permanently (or very persistently) tilted the balance in favor of firms that operate more advanced technology and have higher productivity. In the model, this is a decrease in σ,which decreases the marginal cost of creating the most automated types of capital relative to the least automated types of capital. Consistent with the theory of Akcigit and Ates (2021), a decrease in σ also translates into a slowdown in the diffusion of automation technology in the cross-section of firms. Using an aggregation methodology closely related to the one I use in this paper, Oberfield and Raval (2021) estimate the elasticity of substitution in US manufacturing (using NAICS industry definitions) from 1987 to 2007, and find that σ decreased from 0.72 to 0.54, lending further empirical support to this explanation.

The second explanation is that the economy is on a slow transition path back to the steady state equilibrium following a shock (or perhaps a series of shocks) to frontier automation technology. Due to the putty-clay structure of the model, this technological shock propagates gradually through the economy, resulting in slow-moving dynamics that, in practice, might be hard to disentangle from permanent changes. The long-run labor share data analyzed by Charpe et al. (2019) lends empirical support to this explanation, as it exhibits large long-run fluctuations around a relatively stable level, even as successive waves of automation technology have diffused through economies. A possible challenge to this explanation is that one might expect that automation technology has to some extent diffused to most industrialized countries, and yet the observed evolution of factor shares differs markedly across space, as shown by Gutiérrez and Piton (2020).

Naturally, the two explanations are not mutually exclusive, and disentangling the two is an important question for future research.

The results in this section are derived under the implicit assumption that labor of different types is in perfectly elastic supply, such that workers can freely reallocate across the task spectrum. As in Caselli and Manning (2019), this implies that workers always gain from technological progress in absolute terms. Furthermore, the long-run analysis in this paper shows that the diffusion of technology means that eventually "workers'" position *relative* to "capitalists'" is also unchanged by technology. Yet the transition to a new steady state may be very slow, and an important question that this paper leaves unanswered is how workers of different types might be affected along that transition path, given that in reality human capital investment is also at least partially irreversible. The work done by Hémous and Olsen (Forthcoming), Jaimovich et al. (2020) and Bonfiglioli et al. (2021), shows that differential impacts of automation across occupations are a first-order concern.

5 Conclusion

This paper develops a model of automation as an embodied technology that diffuses via irreversible investments. The theory in the paper highlights the critical role that the distribution of automation technology (equivalently, in the model, the firm-size distribution) plays in mediating the effect of technology on macroeconomic aggregates. In the long-run, the level of automation technology matters for productivity but not for factor shares, which are determined entirely by the distribution. As long as human labor is required in production, the process of technological diffusion acts as a self-correcting force that raises wages and productivity and returns the labor share to its natural level.

The "short-run", however, is very long, suggesting that in practice it may be difficult to determine whether the recent drop in labor's share of income is a permanent change or perhaps the result of a wave of automation sweeping through the economy, following which the labor share will rise back to its long-run level as the diffusion process plays out. To the extent that policy makers are concerned about fluctuations in factor shares (for example, because a skewed distribution of asset ownership means that factor share fluctuations translate into income and wealth inequality), distinguishing between the two candidate explanations is critical for appropriate policy design. Furthermore, the paper in this model assumes that workers are able to seamlessly reallocate to different tasks. Studying the interaction of irreversibility in both automation and human capital is an important direction for future research.

The putty-clay automation model also suggests that the Solow productivity paradox – the observation that productivity growth has decreased, even as seemingly revolutionary technologies have appeared in the economy – may not be a paradox after all. Productivity growth is fastest at the start of a wave of technological diffusion because at that point the productivity difference between new technologies and the ones that are scrapped is at its widest. As that productivity gap inevitably shrinks, so too does productivity growth.

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A Proofs

A.1 Proof of Lemma 1

Start with the CDF $G(z)$, $G(z) = 1 - \left(\frac{\left(\frac{z_\ell}{z}\right)^{\frac{1}{\gamma}} - \left(\frac{z_\ell}{z_h}\right)^{\frac{1}{\gamma}}}{1 - \left(\frac{z_\ell}{z_h}\right)^{\frac{1}{\gamma}}} \right)^{\rho}$. The random variable $\tilde{u} = G(z)$ is uniformly distributed in $[0,1]$ (by the probability integral transformation).

$$\tilde{u} \;=\; 1 - \left(\frac{\left(\frac{z_\ell}{z}\right)^{\frac{1}{\gamma}} - \left(\frac{z_\ell}{z_h}\right)^{\frac{1}{\gamma}}}{1 - \left(\frac{z_\ell}{z_h}\right)^{\frac{1}{\gamma}}} \right)^{\rho}$$

If $\tilde{u} \sim U[0,1]$, then $u = 1 - \tilde{u}$ is also $u \sim U[0,1]$. Now, express z as a function of u:

$$z \;=\; \left(\left(\frac{1}{z_h}\right)^{\frac{1}{\gamma}} + \left(\left(\frac{1}{z_\ell}\right)^{\frac{1}{\gamma}} - \left(\frac{1}{z_h}\right)^{\frac{1}{\gamma}} \right) u^{\frac{1}{\rho}} \right)^{-\gamma}$$

By the probability integral transform, the random variable $\tilde{\chi} = u^{\frac{1}{\rho}}$ is distributed $B(\rho, 1)$, $\tilde{\chi} \in [0,1]$. It follows that the random variable $\chi = z_h^{-\frac{1}{\gamma}} + \left(z_\ell^{-\frac{1}{\gamma}} - z_h^{-\frac{1}{\gamma}} \right) \tilde{\chi}$ is distributed $B(\rho, 1)$ with support $\chi \in \left[z_h^{-\frac{1}{\gamma}}, z_\ell^{-\frac{1}{\gamma}} \right]$. So $z = \chi^{-\gamma}$ has distribution $G(z)$. Now, let $a \equiv 1 - \chi$. Since $\chi \sim B\left(\rho, 1; z_h^{-\frac{1}{\gamma}}, z_\ell^{-\frac{1}{\gamma}}\right)$, $a \sim B(1, \rho; a_\ell, a_h)$, with $a_\ell \equiv 1 - z_\ell^{-\frac{1}{\gamma}}$ and $a_h \equiv 1 - z_h^{-\frac{1}{\gamma}}$. Finally, let $\tilde{a} = qa$. Then, $\tilde{a} \sim B(1, \rho; \tilde{a}_\ell, \tilde{a}_h)$, with $a_\ell = \frac{\tilde{a}_\ell}{q}$ and $a_h = \frac{\tilde{a}_h}{q}$[18], and $z = \left(\frac{q}{q - \tilde{a}}\right)^{\gamma} = \left(\frac{1}{1-a}\right)^{\gamma}$

[18]Proof: start with $f(\chi) = \frac{\rho\left(\chi - H^{-\frac{1}{\gamma}}\right)^{\rho - 1}}{\left(L^{-\frac{1}{\gamma}} - H^{-\frac{1}{\gamma}}\right)^{\rho}}$ and substitute in $a = 1 - \chi$

$$f(x) = \frac{\rho\left(1 - a - H^{-\frac{1}{\gamma}}\right)^{\rho-1}}{\left(L^{-\frac{1}{\gamma}} - H^{-\frac{1}{\gamma}}\right)^{\rho}} = \frac{\rho\left((1 - a_h) - a\right)^{\rho - 1}}{\left((1 - a_h) - \left(1 - L^{-\frac{1}{\gamma}}\right)\right)^{\rho}} = \frac{\rho\left(a_h - a\right)^{\rho-1}}{\left(a_h - a_\ell\right)^{\rho}}$$

So $x \sim B(1, \rho; \ell, h, 1 - a_h)$. Now, let $\tilde{a} = qx$.

$$f(\tilde{a}) = \frac{1}{q} \frac{\rho\left(a_h - \frac{\tilde{a}}{q}\right)^{\rho-1}}{\left(a_h - a_\ell\right)^{\rho}} = \frac{1}{q^{\rho}} \frac{\rho\left(q a_h - \tilde{a}\right)^{\rho-1}}{\left(a_h - a_\ell\right)^{\rho}} = \frac{\rho\left(\tilde{a}_h - \tilde{a}\right)^{\rho-1}}{\left(\tilde{a}_h - \tilde{a}_\ell\right)^{\rho}}$$

A.2 Proof of Proposition 3

To complete the proof of Proposition 3, I prove that the lower bound of the automation distribution $\tilde{a}_\ell$ grows at rate $1 + g_q$. In stationary equilibrium, the value of the marginal machine that isn't scrapped at the end of period t is $\Lambda v(a_\ell) = \kappa$. Replacing for $v(a_{\ell,t+1})$,

$$\kappa = \Lambda \left(\left(\frac{1}{1 - a_\ell} \right)^{1-\sigma} - \frac{W_{t+1}}{H_{t+1}} \right)$$

Solving for a_ℓ and since $\tilde{W} \equiv \frac{W_t}{H}$ is constant in the balanced growth path, $a_\ell = 1 - \left(\tilde{W} + \frac{\kappa}{\Lambda} \right)^{\sigma-1}$. Replacing $a_\ell = \frac{\tilde{a}_{\ell,t+1}}{q_{t+1}}$, $\tilde{a}_{\ell,t+1} = q_{t+1} \left(1 - \left(\tilde{W} + \frac{\kappa}{\Lambda} \right)^{\sigma-1} \right)$. Likewise, $\tilde{a}_{\ell,t+2} = q_{t+2} \left(1 - \left(\tilde{W} + \frac{\kappa}{\Lambda} \right)^{\sigma-1} \right)$. Since q grows at rate $(1 + g_q)$, $\tilde{a}_{\ell,t+2} = (1 + g_q) q_{t+1} \left(1 - \left(\tilde{W} + \frac{\kappa}{\Lambda} \right)^{\sigma-1} \right)$. Substituting completes the proof:

$$\tilde{a}_{\ell,t+2} = (1 + g_q) \tilde{a}_{\ell,t+1}.$$

A.3 Proof of Proposition 1

I show below how to go from the distribution $G(z)$ to the CES aggregate production function.

Aggregate output Y is:

$$Y = \int_{W/H}^{z_h} z k(z) \, \mathrm{d}z = K \int_{W/H}^{z_h} z g(z) \, \mathrm{d}z$$

Divide both sides by aggregate capital K to obtain:

$$
\begin{aligned}
y &= \int_{W/H}^{z_h} zg(z)\ \mathrm{d}z = \int_{W/H}^{z_h} \sigma \frac{\left(\frac{z_\ell}{z}\right)^{1-\sigma}}{1-\left(\frac{z_\ell}{z_h}\right)^{1-\sigma}} \left(1 - \frac{1-\left(\frac{z_\ell}{z}\right)^{1-\sigma}}{1-\left(\frac{z_\ell}{z_h}\right)^{1-\sigma}}\right)^{\frac{\sigma}{1-\sigma}-1} \mathrm{d}z \\
&= (z_h)^{1-\sigma}(z_\ell)^{\sigma} \left(\frac{1-\left(\frac{z_\ell}{z_h}\right)^{1-\sigma}}{1-\left(\frac{W/H}{z_h}\right)^{1-\sigma}}\right)^{\frac{\sigma}{\sigma-1}}
\end{aligned}
$$

Note

$$
1-G(W/H) = \left(\frac{\left(\frac{z_\ell}{W/H}\right)^{1-\sigma} - \left(\frac{z_\ell}{z_h}\right)^{1-\sigma}}{1-\left(\frac{z_\ell}{z_h}\right)^{1-\sigma}}\right)^{\frac{\sigma}{1-\sigma}} = \frac{\alpha}{\alpha + (1-\alpha)(H\cdot n)^{\frac{\sigma-1}{\sigma}}}
$$

Substitute in

$$
y = (z_h)^{1-\sigma}(z_\ell)^{\sigma} \left(\frac{1-\left(\frac{z_\ell}{z_h}\right)^{1-\sigma}}{1-\left(\frac{W/H}{z_h}\right)^{1-\sigma}}\right)^{\frac{\sigma}{\sigma-1}} = A\left(\alpha + (1-\alpha)(H\cdot N)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}}
$$

Multiply both sides by K:

$$
Y = A\left(\alpha K^{\frac{\sigma-1}{\sigma}} + (1-\alpha)(H\cdot N)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}},
$$

which is the CES production function with $A = (z_h)^{1-\sigma}(z_\ell)^{\sigma}$ and $\alpha = 1-\left(\frac{z_\ell}{z_h}\right)^{1-\sigma}$. Or, expressed in terms of the parameters of the automation distribution $f(a)$, $A = \frac{(1-a_\ell)^{-\frac{\sigma}{1-\sigma}}}{1-a_h}$ and $\alpha = \frac{a_h - a_\ell}{1-a_\ell}$.

A.4 Proof of Proposition 4

The full description of the capitalist worker economy is provided in Online Appendix O2. The planner's problem is to choose the optimal stationary, no-scrapping equilibrium. Since the planner only values the consumption of the capitalist (C_K), they solve:

$$\max_{C_{K,t},I_t,a_{w,t}} \sum_{t=0}^{\infty} \beta^t \frac{(C_{K,t})^{1-\eta}-1}{1-\eta}$$

s.t.

$$C_{K,t} = R_{K,t}K_t - \frac{P_{I,t}}{2} I_t$$

In the no-scrapping equilibrium, the capital-labor ratio is constant, $K_t = H_t N_t$ and the steady state investment rate$\frac{\bar{I}}{\bar{K}} = g + \delta$. Additionally, because workers work and consume but do not save, the labor supply is perfectly inelastic in equilibrium ($N_t = \bar{N}$). Using these two facts and substituting in for prices from the equilibrium conditions of the no-scrapping equilibrium (see O1),

$$C_{K,t} = \left(\frac{a_h - a_w}{1-a_h}\right)(1-a_w)^{-\frac{1}{1-\sigma}} H_t\bar{N} - \frac{1}{2}\frac{1}{r+\delta}\left(\frac{a_h - a_w}{1-a_h}\right)(1-a_w)^{-\frac{1}{1-\sigma}}(g+\delta) H_t\bar{N}$$

So everything on the RHS is a parameter except a_w. The planner's problem therefore reduces to choosing the optimal a_w^*, $a_w^* = \arg\max_{a_w \in (0,a_h)} C_K(a_h, a_w, \sigma)$, which has the unique solution: $a_w^* = \max\left[\frac{a_h+\sigma-1}{\sigma}, 0\right]$. The SOC at a_w^* is < 0 as long as $g - 2r - \delta < 0$.

Putty-Clay Automation - Online Appendix

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O1 Equilibrium Conditions

O1.1 Baseline Model

O1.1.1 Aggregate

1. Euler Equation

$$\left(q_t e^{-\nu(N_t)}\right)^{1-\eta} C_t^{-\eta} P_{S,t} = \beta \left(q_{t+1} e^{-\nu(N_{t+1})}\right)^{1-\eta} C_{t+1}^{-\eta} \left(P_{s,t+1} + D_{t+1}\right) \tag{1}$$

2. Labor supply

$$\nu'\left(N_t\right) = \frac{W_t}{C_t} \tag{2}$$

3. Production function

$$Y = A_t \left(\alpha_t K^{\frac{\sigma-1}{\sigma}} + \left(1-\alpha_t\right)\left(HN\right)^{\frac{\sigma-1}{\sigma}}\right)^{\frac{\sigma}{\sigma-1}} \tag{3}$$

where

$$A_t = \left(\frac{1}{1-a_{h,t}}\right)\left(\frac{1}{1-a_{\ell,t}}\right)^{\frac{\sigma}{1-\sigma}} \tag{4}$$

$$\alpha_t = \frac{a_{h,t} - a_{\ell,t}}{1-a_{\ell,t}} \tag{5}$$

4. Rental rate of capital

$$R_{K,t} = \alpha_t A_t \left(\alpha_t + (1-\alpha_t) \left(\frac{H_t N_t}{K_t} \right)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{1}{\sigma-1}} \tag{6}$$

5. Labor demand

$$W_t = (1-\alpha_t) A_t H_t \left(\alpha_t \left(\frac{K_t}{H_t N_t} \right)^{\frac{\sigma-1}{\sigma}} + (1-\alpha_t) \right)^{\frac{1}{\sigma-1}} \tag{7}$$

6. Dividend

$$D_t = \frac{Y_t - W_t N_t - \frac{P_{I,t}}{2} I_t}{K_t} \tag{8}$$

7. Market clearing

$$Y_t = C_t + \frac{P_{I,t}}{2} I_t \tag{9}$$

8. Capital LOM

$$K_{t+1} = (1-\delta)(1-\varsigma_t) K_t + I_t \tag{10}$$

9. Scrapping

$$\varsigma_t = F_t(a_{\ell,t+1}) = 1 - \left(\frac{a_{h,t} - a_{\ell,t+1}}{a_{h,t} - a_{\ell_t}} \right)^{\frac{\sigma}{1-\sigma}} \tag{11}$$

10. LOM for distribution

$$f_{t+1}(a) = \begin{cases} \frac{K_t}{K_{t+1}} (1-\delta) f_t(a) + \frac{I_t}{K_{t+1}} j_t(a) & a_{\ell,t+1} < a < a_{h,t+1} \\ 0 & a \leq a_{\ell,t+1} \end{cases} \tag{12}$$

O1.1.2 Firms

1. Value function

$$V(k_t, a_t) = v(a_t) k_t + \psi(a_t) \tag{13}$$

where,

$$v(a_t) = \pi_t(a_t) + \mathbb{1}_{(1-\delta)\Lambda_{t+1} v(a_{t+1}) > \kappa} \left[(1-\delta) \Lambda_{t+1} v(a_{t+1}) - \kappa \right] \tag{14}$$

$$\psi\left(a_t\right)=\frac{\Upsilon_t\left(\Lambda_{t+1}v\left(a_{t+1}\right)\right)^2}{2\phi\left(a_t\right)}+\Lambda_{t+1}\psi\left(a_{t+1}\right) \tag{15}$$

$$\pi_t\left(a_t\right)=\max\left[z\left(a_t\right)-\frac{W_t}{H_t},0\right] \tag{16}$$

$$z\left(a_t\right)=\left(\frac{1}{1-a_t}\right)^{\frac{1}{1-\sigma}} \tag{17}$$

2. Labor demand

$$n_t\left(a\right)=\begin{cases}\frac{k_t(a)}{H_t} & \text{if } a\geq a_{w,t}\\ 0 & \text{if } a<a_{w,t}\end{cases} \tag{18}$$

where

$$a_{w,t}=1-\left(\frac{W_t}{H_t}\right)^{\sigma-1} \tag{19}$$

3. Scrapping

$$\text{Scrap}=\begin{cases}1 & \text{if } \left(1-\delta\right)\Lambda_{t+1}v\left(a_{t+1}\right)\leq\kappa\\ 0 & \text{otherwise}\end{cases} \tag{20}$$

4. Investment

$$\iota_t=\Upsilon_t\frac{\Lambda_{t+1}v\left(a_{t+1}\right)}{\phi\left(a_t\right)} \tag{21}$$

5. New firm investment

$$\iota_t^N=\Upsilon_t\frac{\Lambda_{t+1}v\left(a_{t+1}\right)}{\phi\left(a_t\right)} \tag{22}$$

O1.1.3 Aggregation

1. Aggregate output

$$Y_t=K_t\int_{a_{w,t}}^{a_{h,t}}z\left(a\right)\ dF_t\left(a\right) \tag{23}$$

where

$$F_t\left(a\right)=1-\left(\frac{a_{h,t}-a}{a_{h,t}-a_{\ell,t}}\right)^{\frac{\sigma}{1-\sigma}} \tag{24}$$

2. Labor demand

$$N_t=\frac{K_t}{H_t}\int_{a_{w,t}}^{a_{h,t}}f\left(a\right)\ d\left(a\right) \tag{25}$$

3. Investment

$$I_t = \int_{\Gamma_{q,t} a_{\ell,t}}^{\Gamma_{q,t} a_{h,t}} \iota_t (a) \ \mathrm{d}a \tag{26}$$

4. Price of investment

$$P_{I,t} = \left(\int_{\Gamma_{q,t} a_{\ell,t}}^{\Gamma_{q,t} a_{h,t}} p_{\iota,t} (a) \, j_t (a) \ \mathrm{d}a \right) \tag{27}$$

where

$$p_\iota (a_t) \equiv \Lambda_{t+1} v (a_{t+1}) \tag{28}$$

and

$$j_t (a) \equiv \frac{\iota_t (a)}{I_t} \tag{29}$$

5. Price of capital

$$P_{K,t} = \int_{a_{\ell,t+1}}^{a_{h,t}} p_{\iota,t} (a) \, f_{t+1} (a) \ \mathrm{d}a \tag{30}$$

6. Aggregate rents

$$\Psi_{t+1} = \int_{a_{\ell,t+1}}^{a_{h,t}} \psi_{t+1} (a) \ \mathrm{d}a \tag{31}$$

O1.2 Stationary Equilibrium

Variables with bars grow at the rate g in the stationary balanced growth path, so, for example $K_t = (1+g)^t \bar{K}$. Variables with no bars are constant in the stationary balanced growth path.

O1.2.1 Aggregate

1. Euler Equation

$$\frac{D}{P_S} = \frac{(\Gamma_q)^{\eta-1} (\Gamma)^\eta}{\beta} \tag{32}$$

2. Labor supply

$$\nu' (N) = \frac{\bar{w}}{\bar{C}} \tag{33}$$

3. Rental rate of capital

$$R_K = \alpha A \left(\alpha + (1-\alpha) \left(\frac{\bar{H}N}{\bar{K}} \right)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{1}{\sigma-1}} \tag{34}$$

4. Labor demand

$$\bar{W} = (1-\alpha) A \left(\alpha \left(\frac{\bar{K}}{\bar{H}N} \right)^{\frac{\sigma-1}{\sigma}} + (1-\alpha) \right)^{\frac{1}{\sigma-1}} \tag{35}$$

5. Market clearing

$$\bar{Y} = \bar{C} + \frac{P_I}{2} \bar{I} \tag{36}$$

6. Production function

$$\bar{Y} = \bar{A} \left(\alpha \bar{K}^{\frac{\sigma-1}{\sigma}} + (1-\alpha) \left(\bar{H}N \right)^{\frac{\sigma-1}{\sigma}} \right)^{\frac{\sigma}{\sigma-1}} \tag{37}$$

7. Capital LOM

$$\frac{\bar{I}}{\bar{K}} = 1 + g - (1-\delta)(1-\varsigma) \tag{38}$$

8. Scrapping

$$\varsigma = F(\Gamma_q a_\ell) = 1 - \left(\frac{a_h - \Gamma_q a_\ell}{a_h - a_\ell} \right)^{\frac{\sigma}{1-\sigma}} \tag{39}$$

9. Price of Investment

$$P_I = \frac{1}{2} \left(\int_{\Gamma_q a_\ell}^{\Gamma_q a_h} p_\iota(a) \, j(a) \ \mathrm{d}a \right) \tag{40}$$

where,

$$j(a) = \left(\frac{\Gamma}{\Gamma - (1-\delta)(1-\varsigma)} \right) \begin{cases} 0 & \forall\ a < \Gamma_q a_\ell \\ f(a) \left(g_k(a) - \frac{(1-\delta)}{\Gamma} \right) & \forall\ a \in (\Gamma_q a_\ell, a_h) \\ f\left(\frac{a}{\Gamma_q} \right) & \forall\ a \in (a_h, \Gamma_q a_h) \end{cases} \tag{41}$$

and,

$$g_k(a) \equiv \left(\frac{a_h - \frac{a}{\Gamma_q}}{a_h - a}\right)^{\rho-1} \tag{42}$$

10. Price of Capital

$$P_K = P_I \tag{43}$$

O1.3 No-scrapping limit

This is the limit of the stationary equlibrium of the model as $\kappa \to 0$ and $\Gamma_q \to 0$. In this case, $a_w \to a_\ell$ and $\frac{\bar{K}}{N} \to \bar{H}$, and the distribution F is constant over time. The omitted equilibrium conditions are as above.

1. Rental rate of capital

$$R_K = \left(\frac{a_h - a_\ell}{1 - a_h}\right)\left(\frac{1}{1 - a_\ell}\right)^{\frac{1}{1-\sigma}} \tag{44}$$

2. Labor demand

$$\bar{W} = (1 - \alpha) A\bar{H} \tag{45}$$

3. Production function

$$\bar{Y} = A \min\left(\bar{K}, \bar{H}N\right) \tag{46}$$

4. Capital LOM

$$\frac{\bar{I}}{\bar{K}} = g + \delta \tag{47}$$

5. Price of Investment

$$P_I = \frac{\alpha A}{r + \delta} \tag{48}$$

O1.4 Capitalist-Worker Economy

In the capitalist-worker equilibrium, $a_w^* = \max\left[\frac{a_h+\sigma-1}{\sigma}, 0\right]$. The equilibrium allocations and prices of the model if $a_h + \sigma > 1$ so $a_w^* > 0$ are:

1. Aggregate output

$$Y_t = \frac{1}{\sigma}\left(\frac{\sigma}{1 - a_h}\right)^{\frac{1}{1-\sigma}} H_t\bar{N} \tag{49}$$

2. Workers' consumption

$$C_{N,t} = \left(\frac{\sigma}{1-a_h}\right)^{\frac{1}{1-\sigma}} H_t \bar{N} \tag{50}$$

3. Wage

$$W_t = \left(\frac{\sigma}{1-a_h}\right)^{\frac{1}{1-\sigma}} H_t \tag{51}$$

4. Capitalist's consumption/Aggregate Dividend

$$C_{K,t} = D\bar{K} = \frac{1}{2}\left(1 + \frac{r-g}{r+\delta}\right)\left(\frac{1-\sigma}{\sigma}\right)\left(\frac{\sigma}{1-a_h}\right)^{\frac{1}{1-\sigma}} H_t \bar{N} \tag{52}$$

5. Rental rate of capital

$$R_K = \left(\frac{1-\sigma}{\sigma}\right)\left(\frac{\sigma}{1-a_h}\right)^{\frac{1}{1-\sigma}} \tag{53}$$

6. Price of investment

$$P_I = \frac{1}{r+\delta}\left(\frac{1-\sigma}{\sigma}\right)\left(\frac{\sigma}{1-a_h}\right)^{\frac{1}{1-\sigma}} \tag{54}$$

7. Labor share

$$LS = \sigma \tag{55}$$

O2 Capitalist-Worker Model

Consider an economy populated by a representative capitalist and worker. The capitalist invests in capital, receives returns from capital and consumes; the worker works and consumes but does not save.

Worker The worker solves

$$\max_{C_{N,t},N_t} \sum_{t=0}^{\infty} \beta^t \frac{\left(C_{N,t} e^{-\nu(N_t)}\right)^{1-\eta} - 1}{1-\eta}$$

s.t.

$$C_{N,t} = W_t N_t$$

So labor supply is given by $\nu'(N_t) = \frac{W_t}{C_{N,t}} = \frac{W_t}{W_t N_t}$. It follows that $N_t \nu'(N_t) = 1$ and the labor supply is fixed, $N_t = \bar{N}$. From 46, it follows that the capital stock is $K_t = H_t \bar{N}$.

Capitalist The capitalist saves in shares, earns returns and consumes.

$$\max_{C_{K,t}, S_t} \sum_{t=0}^{\infty} \beta^t \frac{(C_{K,t})^{1-\eta} - 1}{1-\eta}$$

s.t.

$$C_{K,t} = (D_t + P_{s,t}) S_t - P_{S,t} S_{t+1},$$

where $D_t = \frac{Y_t - W_t N_t - \frac{P_{I,t}}{2} I_t}{K_t}$ and $P_{S,t} = P_{K,t} + \frac{\Psi_{t+1}}{K_t}$ so that the price of "shares" S_t is the sum of the price of installed capital $P_{K,t}$ and the per-unit-of-capital value of rents Ψ_{t+1}.

Planner Now consider the problem of a planner that places welfare weight of one on the capitalist's utility (zero weight on the worker's). I write the planner's problem as a choice of investment in each type of capital, $\iota_t(a)$, $\forall\ a < a_{h,t}$, and the least automated type of capital to invest in, $a_{w,t+1}$.

$$\max_{a_{w,t+1}, \iota_t(a)} \sum_{t=0}^{\infty} \beta^t \frac{(C_{K,t})^{1-\eta} - 1}{1-\eta}$$

s.t. consumption of capitalist:

$$C_{K,t} = \underbrace{\int_{a_{w,t}}^{a_{h,t}} z(a)\, \mathrm{d}F(a)}_{=Y_t/K_t} - \underbrace{\left(\frac{1}{1-a_{w,t}}\right)^{\frac{1}{1-\sigma}}}_{=W_t} - \underbrace{\frac{1}{2} \int_{a_{w,t+1}}^{a_{h,t+1}} \phi_t(a)\, \iota_t(a)^2\ \mathrm{d}a}_{=\left(P_{I,t} I_t\right)/2}$$

LOM of the aggregate capital stock:

$$K_{t+1} = (1-\delta)\left(1 - F_t(a_{w,t+1})\right) K_t + \int_{a_{w,t+1}}^{a_{h,t+1}} \iota_t(a)\ \mathrm{d}a$$

LOM for non-scrapped part of capital stock ($a > a_{w,t+1}$):

$$f_{t+1}(a) = \frac{K_t}{K_{t+1}}(1-\delta) f_t(a) + \frac{\iota_t(a)}{K_{t+1}}, \ \forall a > a_{w,t+1}$$

LOM for scrapped part of capital stock:

$$f_{t+1}(a) = 0, \ \forall a \leq a_{w,t+1}$$

Since $\phi_t(a) \in \Phi^*$ by assumption, the function $j(a)$ to belong to the Beta family specified in Assumption 2, and the problem can be written in much simpler form. The distribution has three parameters, σ, $a_h$and a_w. I treat σ as a deep technological parameter and $a_h$as an exogenous state, which means that the choice of distribution reduces to the optimal choice of $a_{w,t+1}$. Furthermore, note that since $N_t = \bar{N}$, $K_t = H_t\bar{N}$. Abstracting from growth in H_t, $K_{t+1} = K_t$ and rearranging the LOM of the aggregate capital stock, $I_t = \bar{N}(1-(1-\delta)(1-F_t(a_{w,t+1})))$. In the stationary equilibrium, the problem reduces to

$$a_w^* = \arg\max_{a_w \in (0,a_h)} C_K(a_h, a_w, \sigma)$$

In the stationary equilibrium (with $K_t = H_t\bar{N}$,and (from Equation 47) $I_t = (g+\delta)K_t = (g+\delta)H_t\bar{N}$), the planner therefore choses a_w to maximize consumption:

$$\begin{aligned} C_{K,t} &= \left(\frac{a_h - a_w}{1-a_h}\right)(1-a_w)^{-\frac{1}{1-\sigma}}\left(H_t\bar{N}\right) - \frac{1}{2}\frac{1}{r+\delta}\left(\frac{a_h - a_w}{1-a_h}\right)(1-a_w)^{-\frac{1}{1-\sigma}}(g+\delta)H_t\bar{N} \\ &= \frac{1}{2}\left(1+\frac{r-g}{r+\delta}\right)\left(\frac{a_h - a_{w,t}}{1-a_h}\right)(1-a_{w,t})^{-\frac{1}{1-\sigma}}H_t\bar{N} \end{aligned} \tag{56}$$

The first order condition w.r.t. $a_{w,t}$ gives:

$$0 = \frac{1}{2}\left(1+\frac{r-g}{r+\delta}\right)H_t\bar{N}\left(\frac{(1-a_w)^{-\frac{1}{1-\sigma}-1}(\sigma(1-a_w)-(1-a_h))}{(1-a_h)(1-\sigma)}\right)$$

Solving for a_w,

$$a_w^* = \max\left[\frac{a_h + \sigma - 1}{\sigma}, 0\right], \tag{57}$$

where the max enters because $a_w > 0$. Substituting for a_w^* (assuming $a_h + \sigma > 1$, so $a_w^* > 0$) into the equilibrium conditions of the no-scrapping stationary equilibrium (presented in Appendix O1.3) gives the full characterization of the solution of the stationary equilibrium of the capitalist-worker economy.

O2.1 Capitalist-worker model with elastic labor supply

In the capitalist-worker economy presented in the previous section, the labor supply is constant across stationary balanced growth paths (since workers consume their income every period, the income and wealth effects on labor supply cancel out). In that special case, a_w^* has a particularly simple form: the labor share is constant and equal to σ. In this subsection I show that the result is not dependent on the labor supply being constant. I assume the following the standard constant-elasticity form for preferences over labor supply:

$$\nu\left(N_t\right) = \frac{N_t^{1+\frac{1}{\xi}}}{1+\frac{1}{\xi}},$$

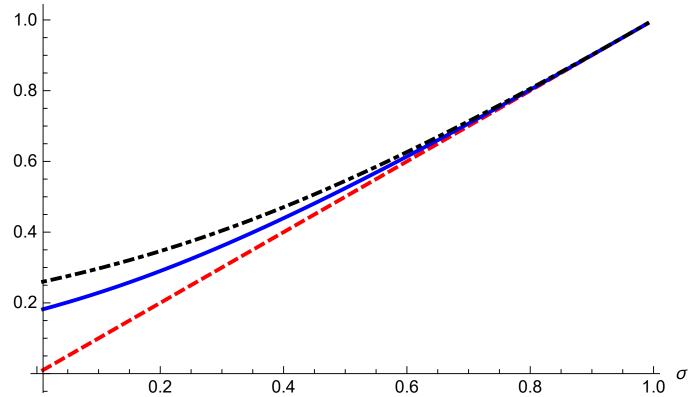
where ξ is the Frisch elasticity of labor supply. With this formulation, the first order condition for labor supply reads $N_t^{\frac{1}{\xi}} = \frac{W_t}{C_t}$. In the stationary, no-scrapping equilibrium, $\frac{W}{C} = \frac{(1-\alpha)}{N\left(1-\left(\frac{g+\delta}{r+\delta}\right)\alpha\right)}$. Substituting into Equation 56 gives:

$$C_K = H_t \frac{1}{2}\left(1+\frac{r-g}{r+\delta}\right)\left(\frac{a_h-a_w}{1-a_h}\right)\left(1-a_w\right)^{-\frac{1}{1-\sigma}}\left(\frac{1-a_h}{1-a_w-\left(\frac{g+\delta}{r+\delta}\right)\left(a_h-a_w\right)}\right)^{\frac{\xi}{1+\xi}}$$

The FOC with respect to a_w has a closed form solution. Substituting into the expression for the labor share, the labor share is independent of a_h but depends on all other parameters. Figure 1 compares the labor share in the model with inelastic labor supply to the one with perfectly elastic labor supply, and an intermediate case of unit elasticity ($\xi = 1$).

O3 Transition path

In this section I describe the solution algorithm for the transition path of the model in response to an unanticipated increase in a_h from $a_{h,0}$ to $a_{h,T}$ at $t = 0$. I show first the law of motion of the distribution $f_t\left(a\right)$.



Comparison of the labor share as a function of σ for the no-scrapping, capitalist-worker equilibrium of the model with inelastic ($\xi = 0$; red dashed) labor supply, unit elasticity ($\xi = 1$; solid blue) and perfectly elastic labor supply ($\xi \to \infty$, dash dotted black) . In both cases the labor supply is independent of frontier automation a_h. Other parameters: $g = 0,\ \delta = 0.075,\ \beta = 0.99,\ \eta = 1$.

Figure 1: Labor share as a function of σ

O3.1 Law of motion for distribution

The firm-size/automation distribution is the key state variable of the model. The law of motion (given that the aggregate capital stock is constant, per the solution to the capitalist-worker model, above) is:

$$f_{t+1}(a) = \begin{cases} (1-\delta) f_t(a) + \frac{I_t}{\bar{K}} j_t(a) & a_{w,t+1} < a < a_{h,t+1} \\ 0 & a \leq a_{w,t+1} \end{cases} \tag{58}$$

Along the transition path, the distribution of investment at time t, $j_t(a)$, is in the Beta family by assumption, and has support $(a_{w,t+1}, a_{h,T})$. The initial steady state distribution, f_1 is also in the Beta family and has support $(a_{w,1}, a_{h,0})$:

$$j_t(a) = \frac{\sigma}{1-\sigma} \frac{(a_{h,T} - a)^{\frac{\sigma}{1-\sigma}-1}}{(a_{h,T} - a_{w,t+1})^{\frac{\sigma}{1-\sigma}}} \quad \text{and} \quad f_1(a) = \frac{\sigma}{1-\sigma} \frac{(a_{h,0} - a)^{\frac{\sigma}{1-\sigma}-1}}{(a_{h,0} - a_{w,1})^{\frac{\sigma}{1-\sigma}}}$$

With associated CDFs $J_t(a) = 1 - \left(\frac{a_{h,T}-a}{a_{h,T}-a_{w,t+1}}\right)^{\frac{\sigma}{1-\sigma}}$ and $F_t(a) = 1 - \left(\frac{a_{h,0}-a}{a_{h,0}-a_{w,t}}\right)^{\frac{\sigma}{1-\sigma}}$ At any point along the transition path, $f_t(a)$ for $t \geq 2$ is

$$f_t(a) = (1-\delta)^{t-1} f_1(a) + (1-\delta)^{t-2} \frac{I_1}{\bar{K}} j_1(a) + \cdots + \frac{I_{t-2}}{\bar{K}} (1-\delta) j_{t-2}(a) + \frac{I_{t-1}}{\bar{K}} j_{t-1}(a) \tag{59}$$

where the support of $f_t(a)$ is $(a_{w,t}, a_{h,T})$. Note that, defining $f_{1,t} = \frac{\sigma}{1-\sigma} \frac{\left(a_{h,0}-a\right)^{\frac{\sigma}{1-\sigma}-1}}{\left(a_{h,0}-a_{w,t}\right)^{\frac{\sigma}{1-\sigma}}}$ as the truncation of the initial steady state distribution, $f_1 = (1 - F_1(a_{w,t})) f_{1,t}$ and similarly, for $\tau > t$, $j_t = (1 - J_t(a_{w,\tau})) j_\tau$. Substituting Equations O3.1 and O3.1 into Equation 59 gives:

$$f_t(a) = (1-\delta)^{t-1} (1 - F_1(a_{w,t})) f_{1,t}(a) + \sum_{i=1}^{t-1} (1-\delta)^{i-1} \frac{I_i}{\bar{K}} (1 - J_i(a_{w,t})) j_{t-1}(a) \tag{60}$$

Note that the distribution $j_{t-1}(a)$ can be taken out of the summation on the RHS since it is not indexed by i. Integrating both sides of Equation 60 from $a_{w,t}$ to $a_{h,T}$ gives

$$1 = \left[(1-\delta)^{t-1} (1 - F_1(a_{w,t}))\right] + \left[\sum_{i=1}^{t-1} (1-\delta)^{i-1} \frac{I_i}{\bar{K}} (1 - J_i(a_{w,t}))\right]$$

since the three PDFs integrate to 1 by definition. I define

$$\upsilon_{1,t} \equiv (1-\delta)^{t-1} (1 - F_1(a_{w,t})), \tag{61}$$

so $\upsilon_{1,t}$ is the share of time the initial capital stock that remains in the total capital stock at time t, when a fraction $(1-\delta)^{t-1}$ remains after random depreciation and a fraction $F_1(a_{w,t})$ has been scrapped. Writing the LOM of the distribution in terms of $\upsilon_{1,t}$,

$$f_{t+1}(a) = \upsilon_{1,t} f_{1,t+1}(a) + (1 - \upsilon_{1,t}) j_t(a) \tag{62}$$

so $\upsilon_{1,t}$ and $a_{w,t}$ jointly characterize the law of motion of the distribution, which reduces the state space from the entire distribution to two scalars.

O3.2 Variables along transition path

Total Factor Productivity and Output Starting with Equation 60, multiply left and right by $z(a)$ and integrating from $(a_{w,t}, a_{h,T})$ gives:

$$\int_{a_{w,t}}^{a_{h,T}} z(a) f_t(a) \, \mathrm{d}a = \upsilon_{1,t} \int_{a_{w,t}}^{a_{h,T}} z(a) f_{1,t}(a) \, \mathrm{d}a + (1 - \upsilon_{1,t}) \int_{a_{w,t}}^{a_{h,T}} z(a) j_{t-1}(a) \, \mathrm{d}a$$

Integrating

$$A_t = \upsilon_{1,t} A(a_{w,t}, a_{h,0}) + (1 - \upsilon_{1,t}) A(a_{w,t}, a_{h,T}), \tag{63}$$

where $A(a_w, a_h) = \frac{1}{(1-a_h)(1-a_w)^{\frac{\sigma}{1-\sigma}}}$ per Proposition 1 in the paper. So TFP at time t is the weighted average of the TFP of the initial capital stock and the new capital stock. Note also that $Y_t = A_t \bar{K}$ so

$$Y_t = (\upsilon_{1,t} A(a_{w,t}, a_{h,0}) + (1 - \upsilon_{1,t}) A(a_{w,t}, a_{h,T})) \bar{K}. \tag{64}$$

Investment Given that $K_t = H_t \bar{K} \ \forall \ t$,

$$I_t = 1 - (1 - \delta)(1 - \varsigma_t) \tag{65}$$

where $\varsigma_t \equiv 1 - (\upsilon_{1,t}(1 - F_t(a_{w,t+1})) + (1 - \upsilon_{1,t})(1 - J_t(a_{w,t+1})))$ is the fraction of capital scrapped between times t and $t+1$.

Wages Wages along the transition path are (assume $H_t = 1$ wlog):

$$W_t = \left(\frac{1}{1 - a_{w,t}}\right)^{\frac{1}{1-\sigma}} \tag{66}$$

Price of investment and dividends Dividends/capitalist's consumption are given by:

$$C_{K,t} = \bar{K} D_t = A_t - W_t - \frac{P_{I,t}}{2} \frac{I_t}{\bar{K}} \tag{67}$$

And the price of investment is defined as:

$$P_{I,t} = \Lambda_{t,t+1} \int_{a_{\ell,t+1}}^{a_{h,t}} v_t(a) j_t(a) \, \mathrm{d}a \tag{68}$$

Where the discount factor in the capitalist-worker model is $\Lambda_{t,t+1} = \left(\frac{C_{K,t}}{C_{K.t+1}}\right)^{\eta}$. Substituting Equation 68 into 67,

$$C_{K,t} = A_t - W_t - \beta \left(\frac{C_{K,t}}{C_{K.t+1}}\right)^{\eta} \left(\int_{a_{\ell,t+1}}^{a_{h,t}} \upsilon_t(a)\, j_t(a)\ \mathrm{d}a\right) \tag{69}$$

Note that if $\{C_{K,t+1}, \dots, C_{K,T}\}$ is known, the above can be solved for $C_{K.t}$. In the computational algorithm I make use of this equation by iterating backwards from the (known) terminal value of $C_{K,T}$[1].

O3.3 Computational algorithm

1. Guess a path for $\{a_{w,t}\}_{t=0}^{T}$
2. Solve for $\upsilon_{1,t}$, Y_t, W_t, I_t from Equations 61, 63, 64, 65 and 66.
3. Starting from the terminal (final steady state) values $C_{K,T}$ and $P_{I,T}$, iterate backwards on Equation 68, solving the polynomial for $\{C_{K.T-1}, C_{K,T-2}, \dots C_{K,1}\}$ and $\{P_{I,T-1}, P_{I,T-2}, \dots P_{I,1}\}$.
4. Compute capitalist's utility along transition path as $U_K = \sum_{t=0}^{T} \beta^t u(C_{K,t})$, store U_K.
5. Update guess of $\{a_{w,t}\}_{t=0}^{T}$, repeat

I parameterize the path for $a_{w,t}$ as a logistic function, a flexible two-parameter specification:

$$a_{w,t} = \frac{a_{w,0}\left(1+\exp\left(-k\left(1-t_0\right)\right)\right) - a_{w,T}}{\exp\left(-k\left(1-t_0\right)\right)} + \frac{a_{w,T} - \frac{\left(a_{w,0}(1+\exp(-k(1-t_0))) - a_{w,T}\right)}{\exp(-k(1-t_0))}}{1+\exp\left(-k\left(t-t_0\right)\right)}, \tag{70}$$

where the initial ($a_{w,0}$) and terminal ($a_{w,T}$) values are known, and the parameters t_0 (the midpoint of the sigmoid curve) and k (the logistic growth rate) control the speed

[1]Note that $\int_{a_{\ell,t+1}}^{a_{h,t}} \upsilon_t(a)\, j_t(a)\ \mathrm{d}a = \Lambda_{t,t+1}\left(A\left(a_{w,t+1}, a_{h,T}\right) - W_{t+1}\right) + (1-\delta)\,\Lambda_{t,t+2}\left(1 - J_t\left(a_{w,t+1}\right)\right)\left(A\left(a_{w,t+2}, a_{h,T}\right) - W_{t+2}\right) + \dots + (1-\delta)^T \Lambda_{t,T}\left(1 - J_t\left(a_{w,T}\right)\right)\left(A\left(a_{w,T}, a_{h,T}\right) - W_T\right) + \dots$, with $\Lambda_{t,t+j} = \left(\frac{C_{K,t}}{C_{K.t+j}}\right)^{\eta}$ and $A\left(a_{w,t}, a_{h,t}\right) = \left(\frac{1}{1-a_{h,t}}\right)\left(\frac{1}{1-a_{w,t}}\right)^{\frac{1}{1-\sigma}}$ and $W_t = H\left(\frac{1}{1-a_{w,t}}\right)^{\frac{1}{1-\sigma}}$. Note that given $\{a_{w,t}\}_{t=0}^{T}$, $a_{h,T}$ and $\{C_{K,i}\}_{i=t+1}^{T}$, the only unknown in this expression is $C_{K,t}$.

of adjustment. I implement the algorithm by searching over the $\{k, t_0\}$ parameter space for the values that maximize U_K. Finally, I divide the $\{k, t_0\}$ space into pairs that result in fast and slow convergence (defined below). I report below the paths that result in maximum utility within each set. In the paper, I report the results for the U_K-maximal path within the slow convergence set.

O3.4 Solution to the transition path

There are two broad classes of equilibria, that I call *fast* and *slow* convergence equilibria. I define fast convergence equilibria as ones in which the labor share converges to its steady state value from above - i.e. the labor share overshoots it's steady state value and falls along the transition path. Conversely, I define slow convergence equilibria as ones where the labor share converges to its steady state value from below. In the main body of the paper I focus on the best slow convergence equilibrium, that is to say the equlibrium with the highest utility among the ones in which convergence of the labor share is from below. I do so because, even though nothing rules out these equilibria a priori, fast transitions are characterized by implausible magnitudes for the variables of interest. Study of fast convergence equilibria of the model, and their existence in practice, is an interesting subject for further research.

Figure 2 plots the dynamics of 8 variables in the best fast and slow transition paths of the model, calibrated as in the main body of the paper. Clockwise from top-left, the first two panels show the difference in the speed of convergence: in the fast equilibrium, old capital is scrapped quickly as shown by the fast increase in $a_{w,t}$ and the fast fall in the share of year 0 capital in the capital stock ($\upsilon_{1,t}$, Equation 61). As a result, the bottom right panel shows that even though the labor share falls initially, it rises quickly and overshoots the steady state level, falling gradually to its long-run equilibrium value of $LS = \sigma$. The fast scrapping and replacement of capital requires a large jump in investment - which increases to almost 50% of output at the peak - and therefore a large cut in dividends (and therefore consumption), which fall much more than in the slow equilibrium. As a result, the real interest rate increases to 40%, before falling sharply. Similarly, TFP growth increases sharply (to 15%) at the start of the transition, and falls thereafter. Unlike in the slow equilibrium, the quality-adjusted price of investment increases along the transition path. Figure 3 plots slow transition paths for a number of the variables in the model. In addition to the variables

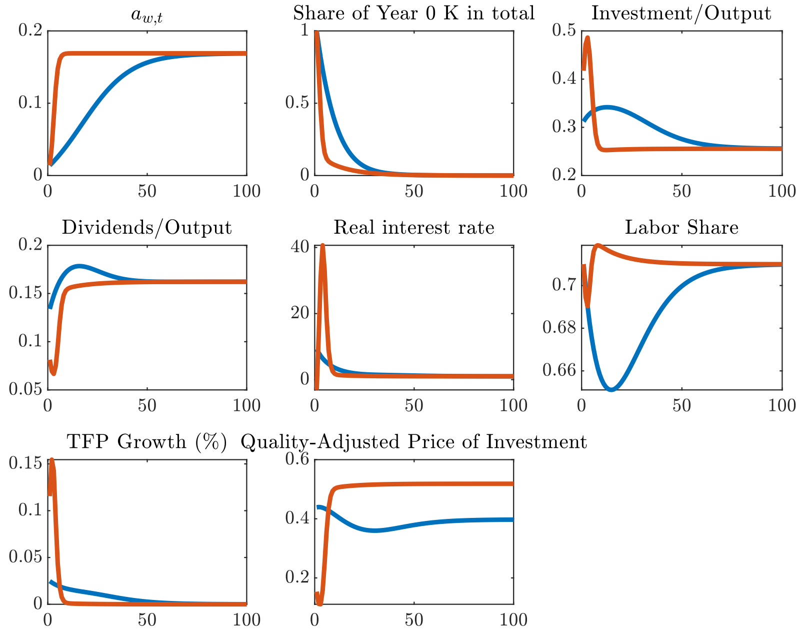


Figure 2: Comparison of fast and slow transition paths

Comparison of slow and fast transition paths in response to an unanticipated increase in a_h at time zero. Parameters: $a_{h,0} = 0.3$, $a_{h,T} = 0.65$, $\delta = 0.075$, $\beta = 0.99$, $\eta = 1$, $\sigma = 0.71$.

discussed above and in the main body in the text, an interesting feature of the putty-clay automation model is that it delivers a rising value of intangibles/GDP along the transition path (bottom right panel of Figure 3), in accordance with evidence (Crouzet and Eberly (2018)), and with recent theories that link the rise in intangibles to the decline in productivity growth and the labor share (de Ridder (2020)).

O3.5 Quality-Adjusted Price of Investment

The model has a vintage capital structure, which allows me to define a meaningful notion of a quality-adjusted price of investment. I show that with the chosen calibration, the quality-adjusted price of investment decreases endogenously along the transition path as the labor share falls, consistent with empirical evidence. In an influential paper, Karabarbounis and Neiman (2014) show that for the canonical CES model to reconcile a decreasing relative price of investment with declines in the labor share, it must be the case that $\sigma > 1$, so that capital and labor are gross substitutes.

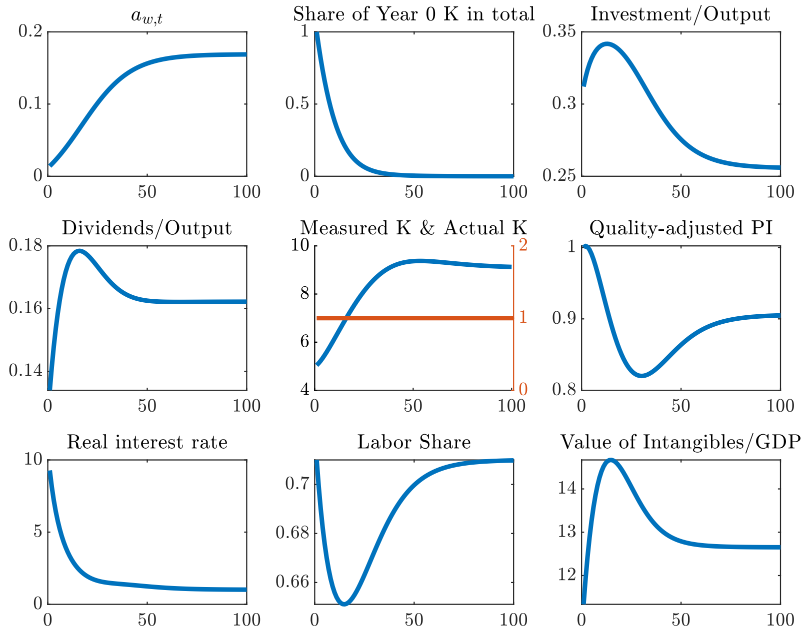


Figure 3: Slow transition paths

Slow transition paths in response to an unanticipated increase in a_h at time zero. Parameters: $a_{h,0} = 0.3$, $a_{h,T} = 0.65$, $\delta = 0.075$, $\beta = 0.99$, $\eta = 1$, $\sigma = 0.71$.

However, the weight of evidence from empirical estimates points to $\sigma < 1$[2]. In an exercise similar in spirit to León-Ledesma and Satchi (2018), I run the Karabarbounis and Neiman (2014) regressions using model-simulated data and find that, even though in the model $\sigma < 1$, the *estimated* elasticity of substitution exceeds unity. Thus, the putty-clay model can reconcile the disparate findings regarding σ in the literature.

In the model, as in the world, new types of capital that have a higher degree of automation enter the economy and older types of capital become obsolete and eventually disappear altogether. This allows me to calculate a meaningful quality-adjusted price of investment in the model, following a procedure akin to the hedonic regression approach used by statistical agencies. Taking the initial steady state of the model as the reference point, the challenge in computing changes in the price of investment along the transition path is that the automation shock introduces into the economy new types of capital that by definition do not exist in the initial steady state. However, I

[2]See Chirinko (2008) for a survey of estimates of σ.

can use the structure of the model to price these types of capital *as if* they existed in the initial steady state and calculate price changes using time zero prices as base prices. Aggregating over these deflated prices of investment gives the model analog to the quality-adjusted price of investment.

The initial steady-state price of capital with automation degree a, $\bar{p}_{\iota,0}(a)$, is the discounted present value of profits,$\bar{p}_{\iota,0}(a) = \frac{\bar{\pi}_0(a)}{r+\delta}$, where $\pi_0(a) = z(a) - \frac{\bar{W}_0}{\bar{H}_0}$. (Because I am studying the no-scrapping limit, the stream of profits is discounted only by the real interest rate r and exogenous depreciation rate δ). I define the constant-quality, time t price of type a capital as:

$$\tilde{p}_{\iota,t}(a) \equiv \frac{p_{\iota,t}(a)}{\bar{p}_{\iota,0}(a)}, \tag{71}$$

that is, the time t price of type a capital divided by its price in the base period (the initial steady state). Thus, the aggregate quality-adjusted price of investment at time t is:

$$\tilde{P}_{I,t} = \int \tilde{p}_{\iota,t}(a)\, j_t(a)\ \mathrm{d}a, \tag{72}$$

where, because $j_t(a)$ is the distribution of investment at time t, $\tilde{P}_{I,t}$ is the average quality-adjusted price of the basket of investment goods.

The middle right panel of Figure 3 shows the evolution of $\tilde{P}_{I,t}$ along the transition path. Intuitively, two forces drive the price of investment: the prices of each unit of capital, and the changing composition of the investment bundle $j_t(a)$. The prices of investment goods fall relative to the initial steady state because as the automation shock diffuses through the economy, wages increase, which decreases the profits (Ricardian rents) earned by capital. On the other hand, relative to the steady state basket of investment goods, the new basket includes more automated, and, therefore, more expensive, types of capital. Along the transition path, for the chosen calibration, the first effect dominates and $\tilde{P}_{I,t}$ falls.

O3.6 A Karabarbounis and Neiman (2014) Regression

In an influential paper, Karabarbounis and Neiman (2014) (KN) estimate the elasticity of substitution of the CES production function by running (in their baseline specification) a cross-country regression of long-run changes in the labor share projected onto changes in the quality-adjusted price of capital. The authors show that the coefficient of interest in this regression is an estimate of $\hat{\sigma} - 1$, where $\hat{\sigma}$ is the estimated elasticity of substitution. They find that $\hat{\sigma} > 1$, suggesting that capital and labor are gross substitutes on aggregate. This contrasts a large number of studies that have found that labor and capital are gross complements, $\sigma < 1$ (e.g. Oberfield and Raval (2021)). To show that the putty-clay automation model can reconcile these disparate findings, I run the KN regression using model-generated data. I simulate transition paths for N identical economies, calibrated as above, each of which is hit by the automation shock at a random time between $t = 1$ and $t = 50$. I then run the following set of KN-type regressions:

$$\frac{LS_i}{1 - LS_i} \Delta_k LS_i = \alpha_k + \beta_k \Delta_k \log \tilde{P}_{I,i} + u_{k,i}, \tag{73}$$

for a range of values of k, where Δ_k denotes the change between time zero and time k in the labor share (LS) or in the log of the quality-adjusted price of investment ($\tilde{P}_{I,i}$), defined in Equation 72. Figure 4 plots the average $\hat{\beta}_k$ from a large number of repetitions of the simulation. The estimated $\hat{\beta}_k$ are greater than zero for a large range of k length differences, which leads to the conclusion that $\hat{\sigma} > 1$, even though the true elasticity of substitution is less than 1 in the model. As in Gilchrist and Williams (2000), in my model, the capital-labor substitutability along the transition path is greater than the short-run substitutability, since along the transition path the economy substitutes away from less to more automated capital.

O4 Characterization of $j(a)$ and $\phi(a)$

In this section I characterize the investment side of the model in the stationary equilibrium of the dynamic model presented in the paper. Throughout I solve for an equilibrium in which frontier automation technology $\tilde{a}_h$ and task input q_t grow at constant rate $1 + g_q \equiv \frac{q_{t+1}}{q_t} = \frac{\tilde{a}_{h,t+1}}{\tilde{a}_{h,t}}$ and labor augmenting technology H_t grows at constant rate $1 + g \equiv \frac{H_{t+1}}{H_t}$. In the stationary equilibrium, the firm size distribution is invariant and given by $F(a) = 1 - \left(\frac{a_h - a}{a_h - a_\ell}\right)^{\frac{\sigma}{1-\sigma}}$.

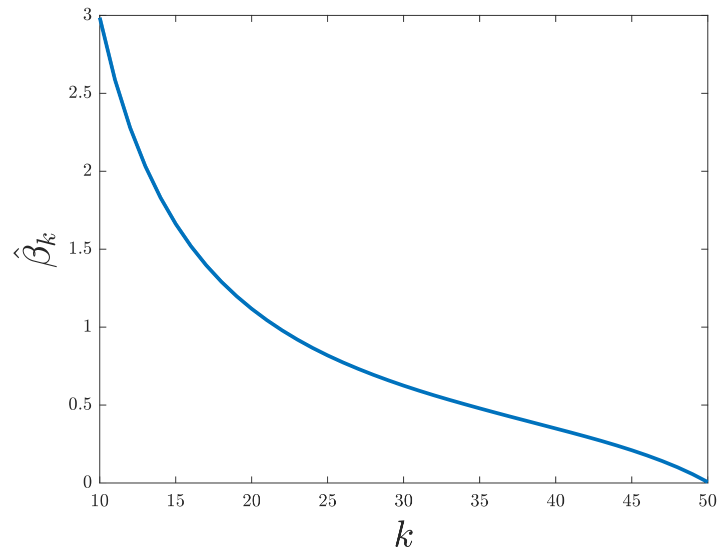


Figure 4: Coefficients from a Karabarbounis and Neiman (2014)-style regression for length k differences

Estimated coefficients for regressions in Equation 73 for $k = 10$ *to* 50. *Parameters:* $a_{h,0} = 0.3$, $a_{h,T} = 0.65$, $\delta = 0.075$, $\beta = 0.99$, $\eta = 1$, $\sigma = 0.71$.

O4.1 Solving for $v(a)$ in a balanced growth path

In this section I solve for the value function $v(a)$ in a stationary equilibrium. The value function of an installed machine with automation a_t is

$$v(a_t) = \pi(a_t) + \mathbb{1}_{(1-\delta)R_{t+1}v(a_{t+1})>\kappa}\left[(1-\delta)\Lambda_{t+1}v(a_{t+1}) - \kappa\right],$$

with profit function $\pi(a_t) = \max\left[\left(\frac{1}{1-a_t}\right)^{\frac{1}{1-\sigma}} - \frac{W_t}{H_t}, 0\right]$ and the law of motion of a_t is $a_{t+1} = \frac{\tilde{a}}{q_{t+1}} = \frac{q_t}{q_{t+1}}a_t = \frac{a_t}{1+g_q}$.

In the BGP, the wage w_t grows at the same rate as H_t, so the effective wage $\tilde{W} \equiv \frac{W_t}{H_t}$ is constant, as is the discount rate Λ. As q grows, incumbent firms' automation falls, as therefore do productivity and profits. After a certain number of periods, profits are too low to cover the maintenance cost κ and the firm scraps its capital. I denote the remaining time to scrapping for any level of automation by $T(a)$. Writing the problem sequentially and substituting in the law of motion for automation:

$$v(a) = \sum_{j=0}^{T(a)} ((1-\delta)\Lambda)^j \left(\left(\frac{1}{1-\frac{a}{(1+g_q)^j}}\right)^{\frac{1}{1-\sigma}} - \tilde{W} - \kappa\right).$$

The scrapping time $T(a)$ solves $\left(\frac{1}{1-\frac{a}{(1+g_q)^{T(a)}}}\right)^{\frac{1}{1-\sigma}} - \tilde{W} - \kappa = 0$, so $T(a) = \frac{\log(a) - \log\left(1-\left(\tilde{W}+\kappa\right)^{\sigma-1}\right)}{\log(1+g_q)}$. Note that $a_\ell = 1 - \left(\tilde{W}+\kappa\right)^{\sigma-1}$, so

$$T(a) = \frac{\log(a) - \log(a_\ell)}{\log(1+g_q)} \tag{74}$$

O4.2 Distribution of investment $j(a)$ and the cost function $\phi(a)$ in stationary equilibrium

Incumbent firms The capital stock of a firm with automation a at time t, expressed relative to the aggregate capital stock using the firm size distribution $f(a) = \frac{\sigma}{1-\sigma} \frac{(a_h - a)^{\frac{\sigma}{1-\sigma}-1}}{(a_h - a_\ell)^{\frac{\sigma}{1-\sigma}}}$ is:

$$k_t(a) = K_t \left(\frac{\sigma}{1-\sigma}\right) \frac{(a_h - a)^{\frac{\sigma}{1-\sigma}-1}}{(a_h - a_\ell)^{\frac{\sigma}{1-\sigma}}}.$$

Since in the BGP a firm with automation a at time t has automation $\frac{a}{1+g_q}$ at time $t+1$, the same firm's capital stock at time $t+1$ is:

$$k_{t+1}\left(\frac{a}{1+g_q}\right) = K_{t+1}\left(\frac{\sigma}{1-\sigma}\right) \frac{\left(a_h - \frac{a}{1+g_q}\right)^{\frac{\sigma}{1-\sigma}-1}}{(a_h - a_\ell)^{\frac{\sigma}{1-\sigma}}}.$$

Solving for the growth rate of the firms capital stock, and noting that in BGP $\frac{K_{t+1}}{K_t} = \frac{H_{t+1}}{H_t} = 1+g$,

$$\frac{k_{t+1}}{k_t} = \frac{K_{t+1}\left(\frac{\sigma}{1-\sigma}\right) \frac{\left(a_h - \frac{a}{1+g_q}\right)^{\frac{\sigma}{1-\sigma}-1}}{(a_h - a_\ell)^{\frac{\sigma}{1-\sigma}}}}{K_t\left(\frac{\sigma}{1-\sigma}\right) \frac{(a_h - a)^{\frac{\sigma}{1-\sigma}-1}}{(a_h - a_\ell)^{\frac{\sigma}{1-\sigma}}}} = \Gamma\left(\frac{a_h - \frac{a}{1+g_q}}{a_h - a}\right)^{\frac{\sigma}{1-\sigma}-1}$$

For convenience I define the following function:

$$g_k(a) \equiv \left(\frac{a_h - \frac{a}{1+g_q}}{a_h - a}\right)^{\frac{\sigma}{1-\sigma}-1} \tag{75}$$

The law of motion for any firm's capital stock (assuming the firm does not scrap its capital) is:

$$k_{t+1} = (1-\delta)\, k_t + \iota_t$$

Expressed using the firm size distribution as above, the law of motion becomes

$$K_{t+1}\left(\frac{\sigma}{1-\sigma}\right)\frac{\left(a_h - \frac{a}{1+g_q}\right)^{\frac{\sigma}{1-\sigma}-1}}{(a_h - a_\ell)^{\frac{\sigma}{1-\sigma}}} = (1-\delta)\, K_t\left(\frac{\sigma}{1-\sigma}\right)\frac{(a_h - a)^{\frac{\sigma}{1-\sigma}-1}}{(a_h - a_\ell)^{\frac{\sigma}{1-\sigma}}} + \iota_t(a)$$

Solving for $\iota(a)$,

$$\begin{aligned}
\iota_t(a) &= K_t\frac{\left(\frac{\sigma}{1-\sigma}\right)}{(a_h - a_\ell)^{\frac{\sigma}{1-\sigma}}}\left((1+g)\left(a_h - \frac{a}{1+g_q}\right)^{\frac{\sigma}{1-\sigma}-1} - (1-\delta)(a_h - a)^{\frac{\sigma}{1-\sigma}-1}\right) \\
&= K_t f(a)\left((1+g)\left(\frac{a_h - \frac{a}{1+g_q}}{a_h - a}\right)^{\frac{\sigma}{1-\sigma}-1} - (1-\delta)\right)
\end{aligned}$$

Dividing both sides by I_t (and given that in the stationary equilibrium $\frac{I_t}{K_t} = \Gamma - (1-\delta)(1-\varsigma)$), the stationary distribution of investment for incumbent firms that don't scrap their capital is:

$$j(a) \equiv f(a)\left(\frac{(1+g)\, g_k(a) - (1-\delta)}{(1+g) - (1-\delta)(1-\varsigma)}\right), \ \forall\ a \in ((1+g_q)\, a_\ell, a_h),$$

where $g_k(a)$ is as defined in Equation 75. Note that firms that scrap their capital have zero investment by definition, so $j(a) = 0, \ \forall\ a < \Gamma_q a_\ell$.

New entrants New entrants are those that enter in $(\tilde{a}_{h,t}, (1+g_q)\, \tilde{a}_{h,t})$. Since they enter with zero capital, the LOM for capital in the first period is simply $k_{t+1} = \iota_t$. Expressed in relative terms,

$$\iota_t(a) = K_{t+1} f\left(\frac{a}{(1+g_q)}\right) = \frac{\sigma}{1-\sigma} \frac{\left(a_h - \frac{a}{(1+g_q)}\right)^{\frac{\sigma}{1-\sigma}-1}}{(a_h - a_\ell)^{\frac{\sigma}{1-\sigma}}}$$

Dividing both sides by I_t gives the stationary distribution of investment for firms that enter at the end of period t:

$$j(a) = \left(\frac{\Gamma}{\Gamma - (1-\delta)(1-\varsigma)}\right) f\left(\frac{a}{(1+g_q)}\right), \ \forall \ a \in (a_h, (1+g_q) a_h)$$

O4.2.1 Investment rate distribution

To summarize, the investment rate distribution is the following step wise function:

$$j(a) = \left(\frac{1+g}{1+g-(1-\delta)(1-\varsigma)}\right) \begin{cases} 0 & \forall \ a < (1+g_q) a_\ell \\ f(a) \left(g_k(a) - \frac{(1-\delta)}{1+g}\right) & \forall \ a \in ((1+g_q) a_\ell, a_h) \\ f\left(\frac{a}{(1+g_q)}\right) & \forall \ a \in (a_h, (1+g_q) a_h) \end{cases}$$

With $g_k(a) = \left(\frac{a_h - \frac{a}{(1+g_q)}}{a_h - a}\right)^{\frac{\sigma}{1-\sigma}-1}$.

O4.2.2 Cost function $\phi(a)$

The investment decision of incumbent and new firms imply a distribution of investment over a_t,

$$\iota(a_t) = \Upsilon_t \frac{\Lambda_{t+1} v(a_{t+1})}{\phi(a_t)}.$$

Rearranging gives

$$\phi(a_t) = \Upsilon_t \frac{\Lambda_{t+1} v(a_{t+1})}{\iota(a_t)}$$

Plugging in for $v(a)$ and $\iota(a)$,

$$\phi\left(a\right)=\begin{cases}\epsilon>0 & \forall\ a\leq\left(1+g_{q}\right)a_{\ell}\\ \frac{\Upsilon_{t}\Lambda_{t+1}\sum_{j=0}^{T(a)}((1-\delta)\Lambda)^{j}\left(\left(\frac{1}{1-\frac{a}{(1+g_{q})^{1+j}}}\right)^{\frac{1}{1-\sigma}}-\left(\frac{1}{1-a_{w}}\right)^{\frac{1}{1-\sigma}}-\kappa\right)}{K_{t}\frac{\frac{\sigma}{1-\sigma}}{(a_{h}-a_{\ell})^{\frac{\sigma}{1-\sigma}}}\left((1+g)\left(a_{h}-\frac{a}{(1+g_{q})}\right)^{\frac{\sigma}{1-\sigma}-1}-(1-\delta)(a_{h}-a)^{\frac{\sigma}{1-\sigma}-1}\right)} & \forall\ a\in\left(\left(1+g_{q}\right)a_{\ell},a_{h}\right)\\ \frac{\Upsilon_{t}\Lambda_{t+1}\sum_{j=1}^{T(a)}((1-\delta)\Lambda)^{j-1}\left(\left(\frac{1}{1-\frac{a}{(1+g_{q})^{1+j}}}\right)^{\frac{1}{1-\sigma}}-\left(\frac{1}{1-a_{w}}\right)^{\frac{1}{1-\sigma}}-\kappa\right)}{K_{t+1}f\left(\frac{a}{(1+g_{q})}\right)} & \forall\ a\in\left(a_{h},\left(1+g_{q}\right)a_{h}\right)\end{cases},$$

where $T\left(a\right)$ is as defined in Equation 74. For completeness I specify that the cost of investment for technologies below the scrapping threshold is some $\epsilon>0$ (conceptually these technologies exist and could be invested in, but their productivity is too low for it to profitable to do so at any positive cost).

O5 Moments of firm-level distribution

O5.1 Mean and Hazard Rate of $G\left(z\right)$

With $\rho=\frac{\sigma}{1-\sigma}$, the mean of z is $\mathbb{E}_{G}\left(z\right)=\left(z_{h}\right)^{1-\sigma}\left(z_{\ell}\right)^{\sigma}$. The hazard rate of G is $\frac{g(z)}{1-G(z)}=\frac{\sigma}{z\left(1-\left(\frac{z}{z_{h}}\right)^{1-\sigma}\right)}$. So the hazard rate of G is equal to a constant times the hazard rate of the truncated Pareto distribution with tail parameter $1-\sigma$. If $\sigma>\frac{1}{2}$ ($\sigma<\frac{1}{2}$), $G\left(z\right)$ has higher (lower) hazard rate than $F\left(z\right)$ for all z, in which it has thinner (fatter) tails. For $\sigma=\frac{1}{2}$, the hazard rates (and hence distributions) are identical, so $G\left(z\right)$ is exactly the truncated Pareto for $\sigma=\frac{1}{2}$.

O5.2 Percentiles of labor share distribution

The x^{th} percentile of the labor share distribution is $LS_{x}=\left(\frac{1-((1-x)a_{w}+xa_{h})}{1-a_{w}}\right)^{\frac{1}{1-\sigma}}$. In the no-scrapping, capitalist-worker equilibrium the x^{th} percentile is $LS_{x}^{*}=\left(x+\sigma(1-x)\right)^{\frac{1}{1-\sigma}}$

O5.3 Concentration

Output share of top $x\%$ of active firms is given by $\mathcal{C}_{x,t} = \frac{\int_{a_h - x(a_h - a_w)}^{a_h} y_t(a)\ \mathrm{d}a}{Y_t}$. With full utilization ($Y = AK$) this becomes: $\mathcal{C}_{x,t} = \left(\frac{x}{\frac{1-a_h}{1-a_W} + x\left(\frac{a_h - a_w}{1-a_W}\right)} \right)^{\frac{\sigma}{1-\sigma}}$. In the stationary no-scrapping economy this becomes $\mathcal{C}_{x,t} = \left(\frac{x}{x + \sigma(1-x)} \right)^{\frac{\sigma}{1-\sigma}}$.

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