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Persuading Large Investors

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A regulator who designs a public stress test to elicit private investment in a distressed bank must account for large investors' private information on the bank's state. We provide conditions for crowding-in (crowding-out) so that the regulator offers more (less) information to better-informed investors. Crowding-in obtains if investors' private information is not too discriminating of the state. We show that the region of the common prior is consequential: if crowding-in occurs for ex-ante optimistic investors then crowding-out follows if they were instead pessimistic. Investors' value from more precise private signals may come from the effect on the public test's precision.

JEL Classification: D83, G21, G28

Keywords: information design, Bayesian persuasion, stress tests, Financial disclosure, Endogenous Public Signal

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Persuading Large Investors*

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1 Introduction

> *We didn't intend to preemptively nationalize major banks, and we didn't intend to let them fail...the centerpiece of our approach was a "stress test"...if an unhealthy firm couldn't raise enough money from private investors, government would forcibly inject the missing capital.* Geithner (2014, p. 11).

Failure of a systemic bank can send tremors across the economy leading to bankruptcies, job losses, and even a recession. During the financial crisis of 2007-08 it was believed that the demise of certain "financial bombs" could even lead to "a reprise of the Depression," substantiating their "too big too fail" moniker (see Geithner (2014, pp. 4–5)). However, even when the regulator stands ready to recapitalize any bank,[1] she is strongly averse to doing so using public funds.[2] A way out of this conundrum is for private, institutional investors to come to the rescue. Direct communication of the regulator with private investors, as well as targeted monetary incentives (e.g., tax cuts) to foster private investment, might appear to violate the public need for transparency and frugality. The channel that then naturally arises (see above quote) is the public disclosure of stress test results by regulators.[3],[4]

However, regulators and private investors do not necessarily have the same objectives. The regulator, in deciding whether to recapitalize a distressed bank, considers all stakeholders, including the bank's own creditors and employees, as well as any spillovers to the economy. Private investors maximize their own shareholders' value. This can create a conflict of interest, which leads to cases where a regulator may prefer that, for example, an insolvent bank is recapitalized because that will safeguard jobs, while private investors disagree.

A key characteristic of private investors who may come to the rescue of a large financial institution is that they are themselves *large*, including other banks,[5] pension funds,[6] and

[1]In the US DFAST programme, any capital gaps were required to be filled with capital plans filed by the banks, privately produced capital, and if that could not be done, then through the Capital Assistance Program (see `http://www.federalreserve.gov/bankinforeg/stresstests-capital-planning.htm`).

[2]Bailouts are perceived by the public as 'free lunches' to the bankers so they can hurt politicians' popularity as well as tarnish banks' brand image (see Gorton (2015, Sec. 3) regarding stigma associated with the use of the Federal Reserve's discount window); can induce moral hazard leading to excessive risk taking; and come at the expense of tax payers who ultimately have to pick up the bill for any state money spent (see Hoshi and Kashyap (2010) for lessons from the Japanese case).

[3]As the Federal Reserve states, disclosure "*provides valuable information to market participants and the public, enhances transparency, and promotes market discipline*" (see `https://www.federalreserve.gov/bankinforeg/stresstests/executive-summary.htm`). For a very comprehensive overview of stress tests see Petrella and Resti (2016).

[4]Empirical evidence suggests that there is information in stress tests and market participants are indeed listening (see Beltratti (2011), Petrella and Resti (2013), Morgan, Peristiani, and Savino (2014), Petrella and Resti (2016, p.409), and Flannery, Hirtle, and Kovner (2017)).

[5]In September 2008 Bank of America bought Merrill Lynch (see `https://www.ft.com/content/0ba5fbd8-82a0-11dd-a019-000077b07658`).

[6]See, e.g., `https://www.ipe.com/pension-funds-invited-for-banking-rescue-talks/30266.article`.

(foreign) sovereign wealth funds.[7] By virtue of their size they can each have a pivotal effect on the success or failure of any recapitalization effort. Moreover, private investors have information on market activities that naturally place them at an advantage to evaluate the health of peer banks, especially when they share common exposures or financial linkages (e.g., via the interbank lending market, affinity of business models, overlapping clients, etc.). Finally, given the public nature of the stress test, the recapitalization effort can, in principle, be driven by more than one large investor. This introduces a coordination motive (i.e., an externality) in investors' recapitalization decisions. In particular, investors may or may not prefer to recapitalize a bank with another private institution (rather than with the regulator).[8]

Against this background, we analyze the optimal stress test design by a regulator. We consider a model (see Section 2) with the following ingredients: a single bank in distress whose financial health can be in one of two possible states *good* (e.g., with liquidity issues) and *bad* (e.g., with solvency issues); a regulator who designs a stress test to elicit private investment, while ready to reluctantly employ public money, if private capital is insufficient; and two investors with imperfect private information on the bank's state—who observe the public stress test—and can take two possible actions *invest* (recapitalize) or *not*;[9] We assume that, regardless of the state, joint investment is necessary to save the bank,[10] and we consider both the cases of positive and negative investment externalities.

We summarize our results as follows. In the absence of a stress test, investors coordinate their decisions on the basis of the precision of their private signals (their "expertise");[11] optimal investment rules take the form of threshold strategies, i.e., investing if private signals exceed some level. Thus, the regulator must account for this expertise when deciding what to disclose about the bank's state. Full disclosure (i.e., performing a test that perfectly reveals the bank's state) is always an option and makes investors' expertise irrelevant: they would disregard their own research and jointly act on the basis of the public test's outcome. In fact, full disclosure is optimal if the regulator shuns private investment to recapitalize a bank in the bad state. Full disclosure is, however, never optimal if the regulator prefers a private bailout

[7]In relation to efforts by EU banks to recapitalize after 2013's stress test results: "*European banks might once again have to go knocking on the doors of Middle Eastern and Asian sovereign wealth funds.*" (see `https://www.euromoney.com/article/b12khttckrxk92/eu-bank-stresstest-fails-to-reassure`).

[8]Investing with the regulator may make the private investor a minority shareholder and hence diminish its effect on governance; while investing with another private investor may prove beneficial if synergies exist (e.g., a retail bank and a sovereign wealth fund investing in another retail bank) or be detrimental (e.g., two retail banks with competing clients investing in another competitor and hence cannibalizing their own business).

[9]Our model is isomorphic to the one where private investors are, for example, wholesale depositors in the distressed bank, who can decide to either run or not following the public stress test.

[10]This may be due to not enough 'deep pockets' of any single private investor, risk-sharing motives, and/or a pursuit of competition policy by the regulator.

[11]The term "expertise" is used in the same context as in the persuasion literature, e.g., see Alonso and Câmara (2018).

to recapitalize distressed banks. In this case, investors' expertise is the main determinant of what information the regulator discloses.

We characterize the regulator's optimal test as a function of investors' expertise and the importance of the coordination motive (see Section 3). Two salient types of tests—with the distinct feature that they dispel any strategic uncertainty for some test outcome—deserve special consideration: Critical-Fault (CF) tests reveal that the bank is in the bad state (e.g., it is insolvent), discouraging private investors from investing, while Coordinated-Investment (CI) tests provide strong evidence of a good state (e.g., it is faced with liquidity issues), prompting investors to disregard their signals and jointly invest. Which type of test the regulator ultimately runs depends on the ex-ante beliefs of market participants (investors): Critical-Fault tests are optimal when investors are (ex-ante) pessimistic, while Coordinated-Investment tests are selected only if investors are moderately (ex-ante) optimistic. We show that if the externality lowers their responsiveness to public news, then both tests become less informative as the externality increases.

The centerpiece of our analysis is the interplay between private and public information in the presence of a coordination motive: how the accuracy of investors' (exogenous) private signals (i.e., their expertise) drives the informational content of the (endogenous) stress test. In particular, we ask: do expertise and public disclosure act as complements—so that better-informed investors "crowd-in" the regulators' test—or instead act as substitutes—leading to "crowding-out" of the regulator's test?

To elucidate if there is crowding-in or out, it is instructive to decompose the regulator's response when facing better-informed investors into a direct effect—in which expertise is allowed to vary but investment thresholds are kept constant—and an indirect effect which accounts for the change in investment thresholds. The direct effect favors performing a more informative test: joint investment will be more responsive to more evidence of a bank in the good state if private signals are more correlated with the state. The indirect effect is, however, ambiguous as it depends on how responsive investment thresholds are to public news and better-informed investors can be more or less responsive to public news. We thus show in Section 4 that optimal CF and CI-tests ultimately depend on investor responsiveness to public news. We also show that the interaction between public and private information depends on investors' common prior: a change in investors' expertise that leads to a more informative test when investors are ex-ante pessimistic (and, thus, the regulator offers a CF-test) would instead lead to a less informative test if investors were ex-ante optimistic (and the regulator offers a CI-test), and vice-versa.

Our results become more transparent when considering the limit case as private signals become perfectly informative of the state. In Section 4.3 we show that, under general conditions, signals then act as complements: the regulator provides a perfectly informative test as investors' signals perfectly reveal the bank's returns–we refer to this case as *asymptotic crowding-in*. This follows as long as investors remain responsive to public news, which in turn requires a bound on how locally informative private signals become. For instance, we show that asymptotic crowding-in always obtains if investors' signals are discrete.

Our interest in the question of crowding-in or -out is twofold. First, the effect of changes in expertise on investors' welfare are indirectly informed by the impact on the (endogenous) stress test. Indeed, we show in Section 5 that better informed investors can benefit more from the test if the regulator preemptively provides a more informative test. Second, private improvements in expertise may increase or decrease the information provided to other potential investors and stakeholders by the public disclosure of the stress test. The regulator in designing her test does not maximize the information provided; if she did, then full disclosure would be optimal—an environment that is both theoretically uninteresting but also practically implausible;[12] her purpose is to prevent financial havoc which makes recapitalization necessary. However, many investors value more rather than less information to reach their optimal decision.[13] Therefore, our results on crowding-in show that, under general conditions, there are positive informational spillovers from improvements in private information of large private investors on other market participants, see Section 5.

In Section 6, we offer a robustness check of our baseline case with conditionally independent signals. In particular, we extend our analysis to the case of investors who use common sources to garner their private information (e.g., the same investment advisor) and so have correlated signals even conditional on the state. We show that allowing for some correlation between investors' signals does not change our main results as the regulator's optimal test is invariant to the presence of "small" correlated mistakes.

As we conclude in Section 7, we offer some testable implications of our analysis. For instance, the sign of the change in the informativeness of the optimal test with respect to how discriminating (over the bank's state) private information is varies with investors' common prior. This implies that any empirical test should necessarily condition on the prior information on the bank's state, otherwise any estimates risk confounding the effect. The Appendix contains all the proofs.

Finally, while motivated by the optimal design of stress tests in a financial setting, our analysis sheds lights on the optimal disclosure policy when persuading large privately in-

[12]Two empirical papers that study regulatory discretion in the design of the stress test are: Bird, Karolyi, Ruchti, and Sudbury (2015), who investigate the possibility that the Federal Reserve might bias the disclosed results of the stress tests in order to promote desirable market outcomes); and Agarwal, Lucca, Seru, and Trebbi (2014), who investigate whether regulatory effectiveness depends on rules and/or regulators' incentives.

[13]The famous exception to this rule is the 'Hirschleifer effect', as discussed in this context by Goldstein and Leitner (2018).

formed players to act in other scenarios as well, e.g., currency crises (Corsetti, Dasgupta, Morris, and Shin (2004)).

1.1 Related Literature

Our paper is related to, and borrows from, different literatures. First is the theoretical literature on Information Design and Bayesian Persuasion (Kamenica and Gentzkow (2011)).[14] The extant papers study how one or multiple senders can further their goals by selectively disclosing information to other players (receivers) (see Bergemann and Morris (2019) and Kamenica (2019) for excellent overviews). We contribute to the strand of this literature that studies persuasion of multiple, privately informed, receivers. There have been many papers studying each of these cases separately.[15] Combining in this paper both features (multiple receivers that observe exogenous private signals of a payoff-relevant state) allows us to address two important issues in information design: the interaction of (endogenous) public and private information, and how a coordination motive among non-atomistic players helps or hinders the efforts of the sender.

A few papers consider, like ours, public persuasion of privately informed receivers.[16] Bergemann and Morris (2016b) and Bergemann and Morris (2016a) provide a characterization of the outcome distributions induced in coordination games by arbitrary information structures. Theirs is a feasibility study; we instead look at the optimal choice of disclosure when a regulator maximizes social welfare. Goldstein and Huang (2016) considers a setting with atomistic investors in which the sender can announce whether or not the fundamentals exceed a given threshold. We allow for more general public disclosure policies and focus on the effect of large investors in stress testing.

The closest paper to ours is Inostroza and Pavan (2018), who analyze the robust design of stress tests in a global game of regime change. While Inostroza and Pavan (2018) also consider financial regulation as their main application,[17] there are two main differences with our paper. First, by considering a robust approach when coordinating atomistic investors they find that the regulator's optimal stress test has the perfect coordination property: all investors select the same action for any realization of the test,[18]—i.e., optimal testing dispels any strategic

[14]See also Brocas and Carrillo (2007), and Rayo and Segal (2010).

[15]For instance, the case with multiple, albeit initially uninformed, receivers is the focus of Alonso and Câmara (2016b), Bardhi and Guo (2018), Mathevet, Perego, and Taneva (2020), Bergemann and Morris (2016b), Che and Hörner (2018), Arieli and Babichenko (2019), Li, Song, and Zhao (2019), and Taneva (2019) among others. In turn, persuasion of a single, privately-informed receiver is explored in Kolotilin, Mylovanov, Zapechelnyuk, and Li (2017), Kolotilin (2018), Alonso and Câmara (2016a), and Guo and Shmaya (2019), among others.

[16]See also Laclau and Renou (2017).

[17]See also Inostroza (2019).

[18]Our analysis also allows for a robust approach, i.e., to look at the worst equilibrium for the regulator.

uncertainty about investors' actions. We, instead, consider persuasion of large investors and optimal tests will seldom eliminate all strategic uncertainty—in fact, we find that investors' reliance on their expertise will typically lead them to an uncoordinated response. Second, we solve for the optimal stress test as a function of investors' private signals which allows us to assess how the information conveyed by the public test depends on these signals.[19]

The second strand of literature we relate to is the theoretical literature on Stress Testing and Financial Disclosures.[20] For an excellent overview of stress test design and regulatory disclosure in the financial system see Goldstein and Sapra (2013). In Gick, Pausch, and Bundesbank (2014) a regulator discloses information over the health of the banking sector to induce an optimal share of investors to invest. Bouvard, Chaigneau, and Motta (2015) study a credit rollover setting where a policy maker must choose between transparency (full disclosure) and opacity (no disclosure) but cannot commit to a disclosure policy. Castro, Martinez, and Philippon (2014) demonstrate that stress tests will be more informative when the regulator has a strong (fiscal) position. Orlov, Zryumov, and Skrzypacz (2018) focus on a regulator who designs a stress tests to reveal information over the value of a risky asset held by multiple banks. Goldstein and Leitner (2018) characterize the optimal stress test when the regulator tries to promote risk-sharing opportunities amongst banks. Williams (2017) analyses the impact of stress tests on banks' portfolio choices. Quigley and Walther (2020) study parametrized stress test design when banks can preemptively engage in costly, certifiable disclosure of the state, and show that reducing the noise in the stress test crowds-out voluntary disclosure by banks. Shapiro and Zeng (2020) focus on capital requirements and banks' endogenous choice of risk as key elements of stress testing.

Our paper differs from all the above in several important aspects: (i) it provides an algorithm for the derivation of the optimal test, without any constrain on its design; (ii) it considers both negative and positive externalities amongst large—i.e., non-atomistic—and privately informed players, and studies their effect on the design of the test; (iii) it identifies conditions for public information to either crowd-in or crowd-out private information.

The third literature strand we relate to is the theoretical literature on Endogenous Public Signals in Global Games.[21] Applications include: a central bank strategically shaping its

However, even then we do not have perfect coordination. For such result the assumption of atomistic investors in Inostroza and Pavan (2018) is key.

[19]See also Basak and Zhou (2020) for an analysis of the optimal dynamic disclosure policy in global games of regime change. Alonso and Câmara (2016a) consider persuasion of non-interacting receivers with heterogeneous priors where the coordination motive studied here is absent.

[20]There is some natural overlap with the first literature category and, to an extend, the separation between the two is artificial.

[21]Two empirical papers that study public information as a coordination mechanism in a financial context are Chen, Goldstein, and Jiang (2010) for a study in mutual fund complementarities) and Hertzberg, Liberti, and Paravisini (2011) for the amplified role of public information in a related context).

public announcements (as in Morris and Shin (2002), Cornand and Heinemann (2008), Angeletos and Pavan (2009), James and Lawler (2011), and Chahrour (2014)); policy makers either signalling their private information through their policy choices (Angeletos, Hellwig, and Pavan (2006)); or directly shaping the informativeness of public announcements (Edmond (2013), Cornand and Heinemann (2008)); or the nonstrategic disclosure of summary statistics of agents' actions, for instance as conveyed by market prices (Vives (2017), Bayona (2018)). Our paper differs both in methodology (information design, which imposes no parametric restrictions on stress test designs) and also focus (the impact of private signals on the informational content of public tests).

2 Model

We consider a stylized model of financial regulation with: a single distressed bank encapsulated by a risky asset, which needs capital to be deployable; a regulator (she) who can disclose information about the bank/asset returns through a stress test, in order to optimize the probability of private investment; and two large private investors that must decide whether to invest in the bank; investment is subject to externalities amongst investors and sensitive to their private information as well as the public information revealed by the stress test. We abstract from many institutional details as we focus on the role of information in coordinating investors. The details of our model are as follows.

Investors Preferences and Private Information: Two expected-utility maximizing investors $i = 1, 2$ simultaneously select $a_i \in \{0, 1\}$—i.e., whether to invest in the bank/risky asset. The individual returns to investing depend both on the state of the bank/asset $\omega \in \{-1, 1\}$ and on the action of the other investor. In particular, investor i's ex-post utility is

$$\begin{aligned} u_i^{ex-post} &= a_i v_i^{ex-post}, \text{ where} \\ v_i^{ex-post} &= \omega + \gamma a_j, \text{ and } \gamma \in (-1, 1). \end{aligned}$$

Investor behavior is easily predicted if they know the state: investing is a dominant action if $\omega = 1$ ("good" state) and is dominated if $\omega = -1$ ("bad" state).[22] Furthermore, we have a positive investment externality when $\gamma > 0$ while the externality is negative if $\gamma < 0$. Hence, γ captures whether investors' underlying businesses are substitutes (e.g., they are both high street retail banks) so that their co-existence (in the case $a_i = a_j = 1$) as shareholders would

[22]Moreover, the unique investment equilibrium in this case is also Pareto efficient (from investors' perspective) whenever $\gamma > -1/2$.

create frictions; or complements (e.g., one of them is a sovereign wealth fund and the other is a private bank) so there are potential synergies via joint investment. We have also normalized the expected utility of non-investment to zero.

All players process information according to Bayes' rule and hold a common prior belief $\mu_0 = \Pr[\omega = 1]$. In the absence of additional private or public information, investors would perfectly coordinate their decisions if their prior belief is sufficiently precise: investing is dominant whenever $\mu_0 > (1 - \min\{\gamma, 0\})/2$ while non-investing is dominant if $\mu_0 < (1 - \max\{\gamma, 0\})/2$.

Each investor $i = 1, 2$ observes a private signal $x_i \in X \subseteq [0, 1]$.[23] We concentrate on the symmetric case, so that signal distributions do not depend on the investor's identity. Hence, define:

$$F(x) \equiv \Pr[x_i \leq x | \omega = 1], \; G(x) \equiv \Pr[x_i \leq x | \omega = -1], \qquad i = 1, 2,$$
$$F(y|x) \equiv \Pr[x_i \leq y | \omega = 1, x_j = x], \; G(y|x) \equiv \Pr[x_i \leq y | \omega = -1, x_j = x], \quad i \neq j.$$

Moreover, f, g denote the corresponding densities, and $\overline{F}$, $\overline{G}$ the complementary distributions. We introduce some additional notation to describe the distributions F and G. Let $\lambda(x) \equiv f(x)/g(x)$ denote the likelihood ratio of a good state, with $\overline{\Lambda}(x) \equiv \overline{F}(x)/\overline{G}(x)$, while $h_F(x) \equiv f(x)/\overline{F}(x)$ and $h_G(x) \equiv g(x)/\overline{G}(x)$ represent the hazard rates of F and G. We will assume that f and g are continuous in X, while $F(y|x)$ and $G(y|x)$ are continuously differentiable in X^2.[24] We further assume the following:

A1. $\lambda(x)$ is strictly increasing for $x \in \{t \in [0, 1] : f(t) > 0 \text{ and } g(t) > 0\}$ (MLRP).

A2-a. For $\gamma > 0$, $\overline{F}(k|x)$ and $\overline{G}(k|x)$ are non-decreasing in x for $x \in [0, 1]$.

A2-b. For $\gamma < 0$, $\overline{F}(k|x)$ and $\overline{G}(k|x)$ are non-increasing in x for $x \in [0, 1]$.

The assumption of a monotone likelihood ratio (Assumption A1) implies that higher realizations of x_i are indicative of a high state, and it is a standard assumption in signaling models. Assumption A2-a implies that when there are positive externalities, investors' signals exhibit positive dependence for each state: conditional on ω, a higher realization of x_i makes it more likely that the other investor's signal exceeds a fixed threshold. Similarly, Assumption A2-b guarantees that signals exhibit negative dependence for each state when there are negative investment externalities. Note that both Assumptions A2-a and A2-b are satisfied in

[23] Any compact subset of $\mathbb{R}$ would suffice.

[24] We depart from these smoothness conditions on distributions in Section 4.3 when we study the case of discrete signals. We will also appeal to stricter smoothness conditions as required for some of our results.

the important case that signals are conditionally independent, so that for all $(x, y) \in X^2$, $\overline{F}(y|x) = \overline{F}(y)$ and $\overline{G}(y|x) = \overline{G}(y)$.

Stress Testing (Strategic Experimentation): The regulator's preferences over investors' choices are given by:

$$u_R^{ex-post} = a_i a_j \left[\eta \mathbb{I}(\omega = -1) + \mathbb{I}(\omega = 1)\right], \, \eta \in [-1, 1],$$

where $\mathbb{I}(\cdot)$ is the indicator function. Hence, the regulator prefers that investors coordinate their actions, with η representing her payoff from joint investment if the state is bad. For instance, if $\eta < 0$ the regulator is fully aligned with players in that she would like them to invest iff $\omega = 1$, while if $\eta > 0$ the regulator is pro-investment regardless of the state. The regulator (government) and investors agree that a bank with $\omega = 1$ (e.g., illiquid but not insolvent) is a worthwhile investment; the disagreement may be for banks with $\omega = -1$ (e.g., with a large proportion of NPLs): investors would not invest if they knew the state, because that would hurt the value to their own shareholders; while the regulator might still want to recapitilize (and save) an $\omega = -1$ bank because she takes into account the value to all stakeholders (e.g., bank's shareholders, creditors, employees, etc) as well as the health of the overall financial system. Parameter η then controls the discrepancy between investors and regulator in terms of investment in the bad state. Moreover, the regulator desires joint investment by both investors. If only one of them invests then the government will "take (the rest of) the bill" when $\eta \mathbb{I}(\omega = -1) + \mathbb{I}(\omega = 1) > 0$.[25] Any money from the regulator (government) will come at the expense of taxpayers and hence such a scenario is to be averted if possible.

To influence investment decisions, the regulator can selectively disclose information about asset returns in the form of a stress test π, which consists of a finite realization space $S(\pi)$ and a family of likelihood functions over $S(\pi)$, $\{\pi(\cdot|\omega)\}_{\omega \in \{-1,1\}}$, with $\pi(\cdot|\omega) \in \Delta(S(\pi))$. Given the common prior, we can without loss take $S(\pi) \subset \Delta(\{-1, 1\})$, so that we can represent $\pi = \{q_i, \Pr[q_i]\}_{i \in I_\pi}$ as a distribution over posterior beliefs $q_i \in S(\pi)$ induced by observing the test's outcome, where I_π indexes all possible outcomes. Importantly, the test's realization is conditionally independent (on the state ω) of investors' signals, so that the test only allows investors to better coordinate their investments by dispelling the uncertainty they face about fundamentals. We say that the regulator "experiments more" when she selects a Blackwell-more informative stress test (see Blackwell and Girshick (1954)).

[25] To which extent an investor desires to invest with another investor or the government (when $\omega = 1$) is controlled by the parameter γ, e.g., for $\gamma = 0$ she is indifferent.

We make two important assumptions regarding the test design. First, as in Kamenica and Gentzkow (2011), the regulator can commit to *any test* that is correlated with the state. Second, we abstract from the costs of designing, implementing and disclosing the test, as the regulator perceives all tests to be costless.[26]

Timing: The regulator publicly selects $\pi = \{q_i, \Pr[q_i]\}_{i \in I_\pi}$. Investor i observes the public realization of π and privately observes x_i, updates his beliefs according to Bayes' rule, and both investors simultaneously make investment decisions. We look for Perfect Bayesian Equilibria (PBE). If there are multiple PBEs we concentrate on the equilibrium that maximizes the regulator's ex-ante expected utility.[27] This will be relevant only if $\eta > 0$, in which case we select the equilibrium that maximizes the probability of joint investment.

3 Optimal Tests

The regulator can ensure that both investors disregard their signals and jointly invest if $\omega = 1$ by providing a perfectly informative test, albeit foregoing at the same time any investment when $\omega = -1$. This is indeed beneficial to the regulator whenever $\eta \leq 0$ as she then seeks to match private investment to the state—see Lemma 2 below. However, when $\eta > 0$ the regulator prefers that investors simultaneously invest regardless of the state. To understand how she can boost joint investment through selective disclosure, we first analyze equilibrium investment behavior when the test leads investors to an interim common posterior $\mu = \Pr[\omega = 1]$.

3.1 Investors' Equilibrium Behavior

Suppose that investor j follows a threshold strategy and invests if his private signal exceeds a threshold, $x_j \geq k_j$. For what values of (k_i, k_j) is this an equilibrium of the investment subgame? Bayesian updating implies that investor-i's odds of a bad state after observing $x_i = x$ are

$$\frac{\Pr[\omega = -1 | x_i = x]}{\Pr[\omega = 1 | x_i = x]} = \frac{1-\mu}{\mu} \frac{1}{\lambda(x)}, \tag{1}$$

[26]In any case, costly experimentation would not be a serious limitation if all experiments impose the same cost as this would not affect the regulator's choice if he decides to experiment.

[27]Our focus on the regulator-preferred PBE is not essential for our analysis. In footnote 30 we show how our framework also allows for an adversarial approach (see Inostroza and Pavan (2018)) in which investors coordinate on the worst equilibrium for the regulator.

and the expected interim value from investing given $j's$ threshold rule is,

$$\begin{aligned} v_i(k_j, x; \mu) &\equiv \mathbb{E}[\omega|x_i = x] + \gamma\mathbb{E}[a_j|x_i = x] \\ &= \left(1 + \gamma\overline{F}\left(k_j|x\right)\right)\Pr\left[\omega = 1|x_i = x\right] - \left(1 - \gamma\overline{G}\left(k_j|x\right)\right)\Pr\left[\omega = -1|x_i = x\right]. \end{aligned}$$

Assumptions A1 and A2-a if $\gamma > 0$ (or A1 and A2-b if $\gamma < 0$) guarantee that $v_i(k_j, x; \mu)$ is strictly single-crossing in x, so that investor's i best response must also be a threshold strategy. Then, any interior equilibrium (k_i, k_j) requires $v_i(k_j, k_i; \mu) = 0$ and $v_j(k_i, k_j; \mu) = 0$, that is,

$$\frac{\Pr\left[\omega = -1|x_i = k_i\right]}{\Pr\left[\omega = 1|x_i = k_i\right]} = \frac{1 + \gamma\overline{F}(k_j|k_i)}{1 - \gamma\overline{G}(k_j|k_i)}, \text{ and } \frac{\Pr\left[\omega = -1|x_j = k_j\right]}{\Pr\left[\omega = 1|x_j = k_j\right]} = \frac{1 + \gamma\overline{F}(k_i|k_j)}{1 - \gamma\overline{G}(k_i|k_j)}.$$

In words, at the equilibrium thresholds, the odds of the bad state equals the ratio of the gains from investing if returns turn out to be high to the investment losses if returns are actually low, given the revised likelihood that the other investor also invests. Using (1) we can succinctly express these conditions as:

$$\lambda(k_i)\frac{1 + \gamma\overline{F}(k_j|k_i)}{1 - \gamma\overline{G}(k_j|k_i)} = \frac{1 - \mu}{\mu} = \lambda(k_j)\frac{1 + \gamma\overline{F}(k_i|k_j)}{1 - \gamma\overline{G}(k_i|k_j)}. \tag{2}$$

For instance, when considering symmetric threshold strategies $k_i = k_j = k$ and defining

$$\widetilde{R}(k) \equiv \overbrace{\lambda(k)}^{\text{Likelihood}} \underbrace{\frac{1 + \gamma\overline{F}(k|k)}{1 - \gamma\overline{G}(k|k)}}_{\text{Coordination}}, \tag{3}$$

then (2) collapses to the condition

$$(1 - \mu)/\mu = \widetilde{R}(k). \tag{4}$$

Any internal symmetric BNE must satisfy (4). The function $\widetilde{R}(\cdot)$ is the product of two terms: the first is the likelihood ratio of a favorable state at the investment threshold—which, by assumption A1, is always increasing—while the second captures the effect of coordination amongst investors. The following lemma characterizes symmetric equilibria $k(\mu)$ of the investment subgame for any interim $\mu = \Pr[\omega = 1]$.

Lemma 1. *Suppose that Assumption 1 and 2-a (or 2-b if $\gamma < 0$) hold. Then,*

(i) If $\lambda'(x)$ is bounded away from zero for $x \in \{t \in [0,1] : f(t) > 0 \text{ and } g(t) > 0\}$, then there exists $\overline{\gamma} > 0$, so that whenever $\gamma < \overline{\gamma}$ there is a unique symmetric equilibrium of the

investment subgame, while for $\gamma > \overline{\gamma}$ there are multiple symmetric equilibria. In the case of multiple equilibria, we select the regulator-preferred equilibrium, which for $\eta > 0$ induces the highest joint investment, i.e., the lowest investment threshold.[28]

(ii) Define the non-decreasing function

$$R(k) \equiv \max_{0 \leq k' \leq k} \widetilde{R}(k'), \tag{5}$$

with $R^{-1}(y) = \min \left\{ k : y = \widetilde{R}(k) \right\}$. Then, for $\mu \in \left(0, \frac{1}{1+R(0)}\right)$, the equilibrium investment threshold is

$$k(\mu) = \begin{cases} R^{-1}\left((1-\mu)/\mu\right), & \text{if } \frac{1}{1+R(1)} \leq \mu \leq \frac{1}{1+R(0)}, \\ 1, & \text{if } \mu \leq \frac{1}{1+R(1)}. \end{cases} \tag{6}$$

If $\mu > 1/\left(1 + \min_{k \in (0,1)} \widetilde{R}(k)\right)$, then $k(\mu) = 0$.

(iii) $k(\mu)$ is non-increasing, and, whenever differentiable, satisfies

$$k'(\mu) = -\frac{1}{\mu^2 \left. R'(k)\right|_{k=k(\mu)}}. \tag{7}$$

To understand the equilibria in the investment subgame, consider first the case of negative externalities $\gamma < 0$. The optimal threshold then decreases with the other investor's threshold; this may lead to asymmetric equilibria in which one investor is more prone to investing than the other. Concentrating on symmetric equilibria, the coordination term in (3) is always increasing, so that the function $\widetilde{R}(k)$ is strictly increasing and there is a unique solution to (4). That is, there is a unique symmetric equilibrium, characterized by $\widetilde{R}(k) = R(k) = (1-\mu)/\mu$. As shown in the proof, this equilibrium maximizes joint investment, and would be the regulator's preferred equilibrium whenever $\eta > 0$; see Lemma 2 for the case of $\eta \leq 0$.[29]

Under positive externalities ($\gamma > 0$), investors would invest more often (i.e., set a lower threshold) if they expect others to increase investment rates. Strategic complementarity implies that all equilibria are symmetric (see Vives (1999, sec. 2.2.3 and fn. 23)), but opens the door for multiplicity of equilibria. This is only possible, however, under a strong coordination motive ($\gamma > \overline{\gamma}$). Figure 1-a shows that multiplicity is linked to non-monotonicity of the function $\widetilde{R}(k)$. As long as $\eta > 0$, the regulator's preferred equilibrium corresponds to the one with the lowest threshold as it maximizes the likelihood of joint investment.[30]

[28] As we show in Lemma 2, if $\eta \leq 0$ a fully informative test is optimal for the regulator.

[29] The proof also provides a stability condition on investors' best responses so that the symmetric equilibrium is the unique equilibrium when $\gamma < 0$.

[30] Our focus on the regulator's preferred equilibrium is not essential for the analysis. For instance, we

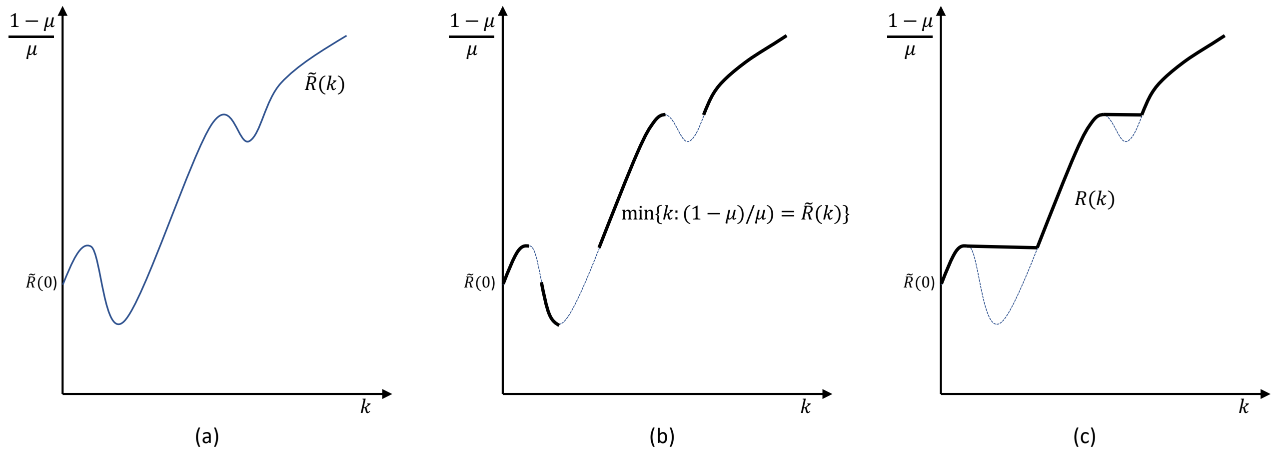

Figure 1: (a) Function $\widetilde{R}(k)$ for $\gamma > 0$, (b) Equilibrium threshold that maximizes joint investment rates $k(\mu) = \min\left\{k : (1-\mu)/\mu = \widetilde{R}(k)\right\}$, (c) Function $R(k)$.

As shown in Figure 1-b, this corresponds to selecting the investment threshold $k(\mu) = \min\left\{k : (1-\mu)/\mu = \widetilde{R}(k)\right\}$. Investors are willing to invest regardless of their private signals if $k(\mu) = 0$—i.e., if $\mu = 1/(1+\widetilde{R}(0))$—and the regulator will never select a stress test with realizations $\mu > 1/(1+\widetilde{R}(0))$ if $\eta > 0$. Therefore, when restricting to undominated stress tests for $\eta > 0$, equilibrium investment can be expressed as $k(\mu) = R^{-1}\left((1-\mu)/\mu\right)$ with R defined by (5); this function is depicted in Figure 1-c.

In summary, $R(k)$ embodies the equilibrium sensitivity of private to public information, as investor's responsiveness to public news—as given by (7)—is inversely related to the slope of $R(k)$. Moreover, when the evidence produced by the test is sufficiently compelling, investors disregard their private signals when making investment decisions: if $\mu = 1/(1+R(0))$ investors always invest, while if $\mu \leq 1/(1+R(1))$ they never invest, irrespective of the realization x_i.

3.2 Characterization of Optimal Tests

Lemma 1 captures the persuasive properties of the stress test: the regulator can induce an investment threshold k by providing suitable evidence that returns are high—indeed, by providing evidence that lead to an interim belief $\mu = 1/(1+R(k))$. Of course, the regulator's

could study an adversarial approach (see Inostroza and Pavan (2018)) in which investors coordinate on the symmetric equilibrium that minimizes the regulator's payoff by instead considering the non-decreasing function $\underline{R}(k) \equiv \min_{k \leq k' \leq 1} \widetilde{R}(k')$. This function would always correspond to the worst-case scenario for the regulator if $\gamma > 0$ as all equilibria of the investment subgame are symmetric, but would be the worst-case if $\gamma < 0$ only if the equilibrium is unique—see proof of Lemma 1 for both claims.

ability to influence investors is limited as the average interim belief must be equal to the prior; that is, the test must be Bayesian plausible.

To study the regulator's optimal test we concentrate on the conditionally independent case and leave a treatment of the case of correlated signals for Section 6. The regulator's interim expected utility when investors follow a k-rule is then:

$$\widetilde{u}_R(k;\mu) \equiv \overline{F}^2(k)\mu + \eta\overline{G}^2(k)(1-\mu)$$

or for the equilibrium $k(\mu)$ in Lemma 1

$$u_R(\mu) \quad \equiv \quad \widetilde{u}_R(k(\mu);\mu) = \overline{F}^2\left(k(\mu)\right)\mu + \eta\overline{G}^2\left(k(\mu)\right)(1-\mu). \tag{8}$$

The following proposition characterizes the regulator's optimal stress test by studying properties of $u_R(\mu)$. In particular, we follow Kamenica and Gentzkow (2011) in obtaining the optimal stress test by computing the concave closure of u_R.

Proposition 1. *The regulator's test can be described with the help of a partition of* $[0,1] = D \cup N$ *into two disjoint sets of intervals defined by signal realizations* $\{\mu_i\}_{i=1,\ldots,I}$ *as follows:*
(i) Interval $[\mu_i,\mu_{i+1}] \subseteq N$ *("non-disclosure") if for prior* $\mu_0 \in [\mu_i,\mu_{i+1}]$ *there exists* $\gamma(\mu_0)$ *such that:*

$$u_R(\mu) \leq u_R(\mu_0) + \gamma(\mu_0)(\mu-\mu_0)\,,\ \mu \in [0,1]. \tag{9}$$

If u_R *is differentiable at* $\mu=\mu_0$*, then* $\gamma(\mu_0) = u_R'(\mu_0)$*. For all priors* μ_0 *in this interval a completely uninformative test is optimal.*
(ii) Interval $(\mu_i,\mu_{i+1}) \subseteq D$ *("disclosure") if*

$$\mu_{i+1} \in \arg\max_{\mu_i<\mu\leq 1} \frac{u_R(\mu)-u_R(\mu_i)}{\mu-\mu_i}, \tag{10}$$

$$\mu_i \in \arg\min_{0\leq\mu<\mu_{i+1}} \frac{u_R(\mu_{i+1})-u_R(\mu)}{\mu_{i+1}-\mu}. \tag{11}$$

For all priors μ_0 *in this interval a test with two realizations that induces public posteriors* μ_i *and* μ_{i+1} *is optimal.*

Proposition 1 can be described with the help of Figure 2, which depicts an example of a u_R and its associated concave closure U_R for $\eta > 0$. For the example of Figure 2 the partition is as follows: the non-disclosure region $N = \{\{0\},[\mu_1,\mu_2],[\mu_3,\mu_4],[\mu_5,1]\}$ and the disclosure region $D = \{(0,\mu_1),(\mu_2,\mu_3),(\mu_4,\mu_5)\}$. For example, if $\mu_0 \in (\mu_2,\mu_3)$ then the regulator optimally provides either bad news driving their common posterior to μ_2 or good news driving the posterior to μ_3 (with probabilities $(\mu_3-\mu_0)/(\mu_3-\mu_2)$ and $(\mu_0-\mu_2)/(\mu_3-$

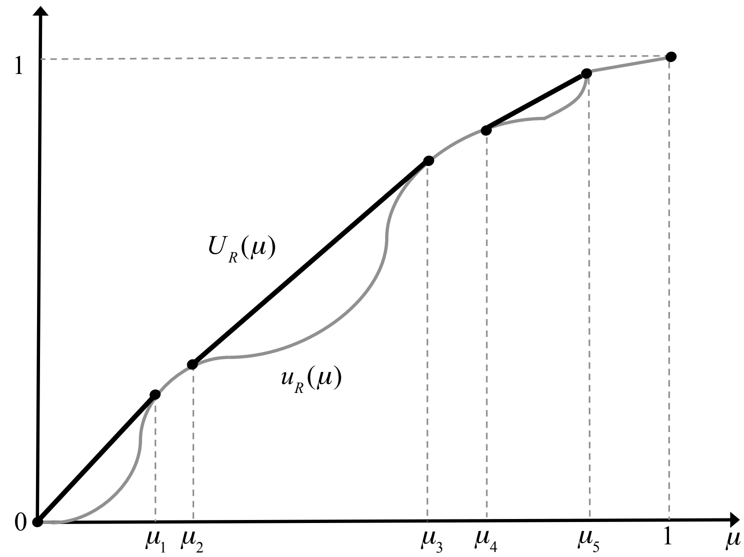

Figure 2: Regulator's indirect utility u_R and associated concave closure U_R, defined via the cutoffs $\{\mu_1, \ldots, \mu_5\}$.

μ_2), respectively), while if $\mu_0 \in [\mu_3, \mu_4]$ the regulator discloses no information to investors. Moreover, Proposition 1 is an algorithm on how to derive the points $\{\mu_1, \ldots, \mu_5\}$, which define the concave closure $U_R(\mu)$. Given the infimum of a disclosure interval (e.g., μ_2), the supremum (i.e., μ_3) is the point on (the graph of) u_R at which the slope of the line from the infimum to that point is maximized. The maximum slope implies that the graph of u_R is below the line connecting the infimum and the supremum. Reversely, given the supremum of a disclosure interval (e.g., μ_5, that is, the first point at which the tangent is above the graph) then the infimum (i.e., μ_4) is the point on (the graph of) u_R at which the slope of the line from the supremum to that point (i.e., the internal angle) is minimized. Now, regarding the non-disclosure region, for example if we start at any point $(\mu_0, u_R(\mu_0))$ such that $\mu_0 > \mu_1$ then until we reach μ_2 all the tangent lines are strictly above the graph of u_R. This is equivalent to saying that for $\mu_0 \in [\mu_1, \mu_2]$ the line between $(\mu_0, u_R(\mu_0))$ and any point $\mu \in (0, \mu_0)$ ($\mu \in (\mu_0, 1)$) has higher (lower) slope than that of the tangent line at μ_0.

While the optimal stress test may vary subtly with the prior belief of investors—as it is the case, for instance, in Figure 2 —there are situations where the regulator strictly prefers to dispel any uncertainty and perfectly disclose the state. For instance, if $\eta \leq 0$, then she would like to induce joint investment only under a good state, so that it is clear that the optimal test fully discloses ω. However, a fully informative test ceases to be optimal if she prefers investors to invest regardless of the state (that is, when $\eta > 0$).

Lemma 2. *(i) For $\eta \in [-1, 0]$ a fully informative test is optimal. (ii) For $\eta \in (0, 1]$ a fully informative test is never optimal, as long as $g(0) > 0$ and $g'(0)$ is bounded.*

3.3 "Critical-Fault" and "Coordinated-Investment" Tests

There are two salient types of optimal stress tests in which, with positive probability, the test leads investors to perfectly coordinate their actions. First, "Critical-Fault" tests (CF-tests) are defined by a realization $\underline{\mu}$ such that for prior $\mu_0 \in (0, \underline{\mu})$ the test either discloses that the state is $\omega = -1$, or improves investors' interim belief to $\underline{\mu}$. That is, the test may conclusively establish that returns are low and discourage investment, regardless of the investors' private signals.[31] These tests afford a simple practical implementation: the regulator specifies a range of scenarios to simulate and the test reveals whether the bank would pass all (in which case investors' posterior belief is $\underline{\mu}$) or would fail some scenario, thus conclusively revealing that $\omega = -1$.

Second, "Coordinated-Investment" tests (CI-tests) are such that investors disregard the realization of their private signal and simultaneously invest after a favorable test outcome. In this case, the test is defined by realizations $\mu_{CI} = 1/(1 + R(0))$ and $\overline{\mu}$ such that for prior $\mu_0 \in (\overline{\mu}, \mu_{CI})$ it leads to the bad news $\overline{\mu}$ or leads to coordinated investment after the favorable news μ_{CI}—see Lemma 1. One possibility is for the test to dispel any fundamental uncertainty and reveal that $\omega = 1$. This is the case if $R(0) = 0$, which requires $\lambda(0) = 0$ so that investors' private signals can detect the occurrence of low returns. If $R(0) > 0$, however, the CI-test eliminates any strategic uncertainty while fundamental uncertainty remains: favorable evidence after outcome $\mu_{CI} < 1$ is so compelling that investors invest regardless of their signals.

In the example of Figure 2 both tests exist with $\underline{\mu} = \mu_1$, $\overline{\mu} = \mu_4$, and $\mu_{CI} = \mu_5$. Trivially, a fully informative stress test would fall under both categories. Below, we inquire on the conditions that lead to the existence of each type of test. In what follows we assume that R is differentiable almost everywhere so that u_R is also.

Existence of Critical-Fault tests. If a CF-test exists, then it is characterized by (10) with $\mu_i = 0$. Let $A_R(k)$ be the regulator's average utility when the test's outcome leads to threshold k,

$$A_R(k) \equiv \frac{\widetilde{u}_R(k; \mu(k))}{\mu(k)} = \overline{F}^2(k) + \eta R(k)\overline{G}^2(k). \tag{12}$$

[31] The regulator could induce investors to perfectly coordinate in not investing even if they remain uncertain about the state by disclosing μ' so that $0 < \mu' < 1/(1 + R(1))$ (see Lemma 1). From Proposition 1, this is never optimal for the regulator as she can increase the likelihood of joint investment by disclosing $\mu = 0$ instead.

Then (10) can be rewritten as:

$$\underline{\mu} \in \arg\max_{0<\mu\leq 1} \frac{u_R(\mu)}{\mu} = A_R(k(\mu)). \tag{13}$$

That is, a critical-fault test is found by maximizing the regulator's average utility. An alternative characterization of (13) will prove useful for comparative statics. Thus, define the elasticity of the regulator's payoff to public news

$$E^{CF}(\mu) = \frac{du_R(\mu)}{d\mu}\frac{\mu}{u_R(\mu)}.$$

Then, the optimal CF-test satisfies the first-order condition,

$$E^{CF}(\underline{\mu}) = 1, \tag{14}$$

where the second-order condition requires $E^{CF}(\mu)$ to be decreasing at $\mu = \underline{\mu}$. Thus, any change in investors' signals that increases this elasticity will compel the regulator to provide a more informative test—i.e., to increase $\underline{\mu}$—while reductions in $E^{CF}(\mu)$ translate into a less informative test.

Lemma 1 shows that for a sufficiently low prior—namely, when $\mu_0 < 1/(1+R(1))$—investors will never invest in the absence of a test. Thus, $u_R(\mu_0) = 0$ and the regulator can always increase average joint investment through a CF-test. However, $1/(1+R(1)) > 0$ requires that private signals are never perfectly informative of $\omega = 1$ (i.e., $g(1) > 0$). Therefore, a CF-test is optimal as long as investors' signals cannot reveal that returns are high.

Lemma 3. *If $\eta > 0$, then a CF-test exists if $\lambda(1)$ is finite.*

Existence of Coordinated-Investment Tests. If a CI-test exists, then it is characterized by (11) with $\mu_{i+1} = \mu_{CI} \equiv 1/(1+R(0))$, which can then be expressed as:

$$\overline{\mu} \in \arg\min_{0\leq\mu<\mu_{CI}} \frac{u_R(\mu_{CI}) - u_R(\mu)}{\mu_{CI} - \mu}. \tag{15}$$

Similar to the case of a CF-test, define the elasticity of the regulator's payoff to public news

$$E^{CI}(\mu) = \frac{d\left(u_R(\mu_{CI}) - u_R(\mu)\right)}{d\mu}\frac{\mu_{CI} - \mu}{u_R(\mu_{CI}) - u_R(\mu)},$$

so that (15) implies that $\overline{\mu}$ satisfies the first-order condition,

$$E^{CI}\left(\overline{\mu}\right)=-1, \tag{16}$$

where the second-order condition requires that $E^{CI}\left(\mu\right)$ is increasing at $\mu=\overline{\mu}$. Thus, any change that increases this elasticity will lead the regulator to decrease $\overline{\mu}$–that is, to a more informative test if μ_{CI} does not decrease. The following lemma provides sufficient conditions for the regulator's test to induce coordinated investment.

Lemma 4. *Suppose that $\eta\in(0,1]$. (i) If f and g are twice-continuously differentiable in $[0,1]$, then the optimal test never reveals $\omega=1$. (ii) If $R(0)>0$, then, a CI-test exists if*

$$R'(0)<2f(0)\left(1+R(0)\right)\left(\frac{1}{\eta}+\frac{1+\gamma}{1-\gamma}\right). \tag{17}$$

Lemma 2 showed that the regulator discloses a high state if $\eta\leq 0$—indeed, in this case, she will provide a fully informative test. Nondisclosure for $\eta>0$ in Lemma 4.i is clear from our discussion when $R(0)>0$ as then $\mu_{CI}<1$ and she can guarantee coordinated investment without disclosing $\omega=1$. If $R(0)=0$, then investors' private signals can reveal $\omega=-1$ and the test must dispel any uncertainty about the bad state to ensure joint investment. This is, however, never optimal for the regulator. In fact, the regulator discloses no information when investors are sufficiently optimistic about the health of the bank and their signals discriminate if returns are low. The proof of the lemma shows that (i) in the absence of a test, investors over-invest relative to the case that they perfectly observe the state, and (ii) the marginal rate of joint investment decreases as the regulator provides more evidence of a good state. These two conditions then guarantee that (9) holds for a sufficiently high prior and non-disclosure is optimal.

If $R(0)>0$, then the regulator can induce joint investment by disclosing $\mu_{CI}=1/(1+R(0))<1$. This is optimal if the rate of increase of the likelihood ratio $d\lambda/dk$ at $k=0$ is sufficiently low—in particular, if (17) holds. Intuitively, investment thresholds are very sensitive to public news when private signals are not very discriminating, i.e., when the likelihood ratios are locally insensitive to changes in the private signal. Then, providing more compelling evidence of the good state leads to a reduction in the investment thresholds and an increase in the probability of joint investment. Thus, even if investors are optimistic about the asset's returns, a high responsiveness to public news still leads to (some) disclosure by the regulator.

3.4 Investment Externalities and Optimal Tests.

The presence of investment externalities can justify the implementation of public stress tests that help investors better coordinate their actions. To study the effect of changes in the externality γ on the optimal test, we focus on the two tests we identified in Section 3.3.

Increasing γ affects the regulator's payoff through changes in investment rules in response to the test, i.e., through changes in both $R(k;\gamma)$ and $k(\mu;\gamma)$ (we make explicit the dependence on γ in the notation of quantities of interest in this segment). In particular, for an increase in the payoff from joint investment we have: $\partial R(x;\gamma)/\partial\gamma > 0$, for all (x,γ)—that is , the regulator can induce more investment by providing the same information; and $\partial k(\mu;\gamma)/\partial\gamma < 0$ for all (μ,γ)—that is, investors reduce their investment thresholds. These are intuitive: a stronger coordination motive in (3) if $\gamma > 0$ (alternatively, a weaker coordination motive if $\gamma < 0$) increases investors' incentive to coordinate investments, and hence they will invest more often.

The overall effect on the informativeness of the CF-test or the CI-test is, however, ambiguous. The reason is that the aforementioned changes (i.e., on R and k) are both level effects, while what is instrumental is the effect on the elasticities: the CF-test satisfies the optimality condition $E^{CF}\left(\underline{\mu}(\gamma);\gamma\right) = 1$ with E^{CF} the elasticity of the regulator's utility with respect to public news—see (14); while the CI-test satisfies $E^{CI}\left(\overline{\mu}(\gamma);\gamma\right) = -1$—see (16). Note that a decrease in the realization of the CF-test $\underline{\mu}$ translates to a less informative test; while, because $d\mu_{CI}/d\gamma < 0$, an increase in the realization of the CI-test $\overline{\mu}$ translates to a less informative test.

Proposition 2. *We have for each optimal test:*

CF-test. Assume the condition of Lemma 3 holds so that $\underline{\mu}$ exists. If

$$\left.\frac{\partial^2 R\left(k;\gamma\right)/\partial k \partial\gamma}{\partial R\left(k;\gamma\right)/\partial\gamma}\right|_{k=\underline{k}} > 2h_G(\underline{k}), \tag{18}$$

where $\underline{k} \equiv k(\underline{\mu}(\gamma);\gamma)$; then $d\underline{\mu}(\gamma)/d\gamma < 0$.

CI-test. Assume the conditions of Lemma 4 hold so that $\overline{\mu}$ exists. If certain regularity conditions on $\overline{F}$, $\overline{G}$ are satisfied (see (A.33) and (A.43) in the Appendix) at $\overline{k} \equiv k(\overline{\mu}(\gamma);\gamma)$, and moreover

$$\left.\frac{\partial R\left(k;\gamma\right)/\partial\gamma}{\partial R\left(k;\gamma\right)/\partial k}\right|_{k=\overline{k}} > \frac{1-\underline{\mu}(\gamma)}{1-\mu_{CI}}\frac{2c}{\mu_{CI}-\underline{\mu}(\gamma)}, \tag{19}$$

$$\left.\frac{\partial}{\partial k}\frac{\partial R\left(k;\gamma\right)/\partial\gamma}{\partial R\left(k;\gamma\right)/\partial k}\right|_{k=\overline{k}} < c, \tag{20}$$

where $c > 0$ is defined in (A.26) (see RHS of (A.26) in the Appendix); then $d\overline{\mu}(\gamma)/d\gamma >$ 0.

Assume that the prior of investors is low so that we are in the CF-test region. Condition (18) provides sufficient conditions for $\partial E^{CF}(\mu;\gamma)/\partial\gamma\big|_{\mu=\underline{\mu}(\gamma)} < 0$. To gain some intution, note that the LHS of (18) is the (semi-)elasticity with respect to k of $\partial R/\partial\gamma$—i.e., the change in the investment threshold via the public test in response to a change in the externality. If this semi-elasticity is large enough (in particular, it exceeds the hazard rate h_G) then increasing the externality makes investors less responsive to public news leading the regulator to offer a less informative test.

Now, assume that the prior of investors is high so that we are in the CI-test region. Conditions (19)–(20) are again local conditions (at $\underline{\mu}$ and $\underline{k}$) which guarantee that $\left(\partial E^{CI}(\mu;\gamma)/\partial\gamma\right)\big|_{\mu=\overline{\mu}(\gamma)} <$ 0. As mentioned, R captures the dependence of investment rules on public information. So (19) postulates that the reaction to public news when γ increases relative to when k increases must exceed a certain threshold; however, the speed of that change relative to k should not be too large, see (20). Both these guarantee that investors become less responsive to public news as γ increases.

3.5 Private Value of Stress Testing

In the absence of a stress test, well-informed investors can coordinate their investment decisions through the precision of their private signals. However, investment externalities and dispersed information still lead to investment frictions when these signals are not perfectly informative of the underlying state.[32] Given these frictions, do investors benefit from the regulator's test? To answer this question, let

$$u_i(\mu) \equiv \mu\left(\overline{F}(k(\mu)) + \gamma\overline{F}^2(k(\mu))\right) + (1-\mu)\left(-\overline{G}(k(\mu)) + \gamma\overline{G}^2(k(\mu))\right),$$

be investor-$i's$ interim equilibrium expected utility after realization μ of the public test. If $\{\{\mu_i\}_{i\in I}, D, N\}$ is the regulator's test in Proposition 1, then the private value of the test when $\mu_0 \in (\mu_l, \mu_{l+1}) \subset D$ is

$$V_i^{test} \equiv \Pr\left[s = \mu_{l+1}\right]\left(u_i(\mu_{l+1}) - u_i(\mu_0)\right) + \Pr\left[s = \mu_l\right]\left(u_i(\mu_l) - u_i(\mu_0)\right). \tag{21}$$

While the regulator always provides a weakly informative test, her focus on promoting joint investment may exacerbate investment distortions and, actually, make investors

[32] Recall that if the state is commonly known by investors, then the unique investment equilibrium maximizes investors joint surplus whenever $\gamma > -1/2$. Thus, when $\gamma > -1/2$ investment inefficiencies are a result of imperfect information.

worse-off—this is clear if $\gamma < 0$ as investors over-invest relative to the joint-investor surplus-maximizing level. However, equilibrium underinvestment when $\gamma > 0$ can be ameliorated by disclosing information that boosts joint investment.

In the Online Appendix we provide a full analysis of the impact on investors' welfare of stress testing by studying conditions for $V_i^{test} > 0$. We now briefly summarize some of the main findings focusing on CF-tests and CI-tests.

First, if $\gamma > 0$, investors benefit from *any* binary test that reveals that the state is $\omega = -1$. This is intuitive: if investors know that $\omega = -1$, then they will refrain from investing which is the investor-optimal response, while if the test reveals $\mu > \mu_0$ it boosts joint investment. Thus, with positive externalities investors always benefit from a CF-test. The same is not true, however, for a CI-test $\{\overline{\mu}, \mu_{CI}\}$ as revealing $\mu = \overline{\mu}$ may exacerbate the underinvestment problem relative to the prior $\mu = \mu_0$.

If $\gamma < 0$, then investors maybe worse off when the regulator runs a CF or CI-test. In both cases, overinvestment may increase after revealing positive news (i.e., $\mu = \underline{\mu}$ in the case of a CF-test and $\mu = \mu_{CI}$ in the case of a CI-test) potentially making investors worse-off.

4 Investors Expertise and Optimal Tests

Our main interest lies in clarifying how the ability of large investors to independently learn about a bank's state affects the regulator's preemptive disclosure of this state. To characterize different levels of investors' expertise, we consider a family of signals $X(\alpha)$ indexed by $\alpha \in [0, 1]$—with distributions $F(x; \alpha)$ and $G(x; \alpha)$—satisfying the following assumption:

A3. Signal $X(\alpha')$ is Blackwell-more informative than signal $X(\alpha)$ whenever $\alpha' > \alpha$, and $X(\alpha)$ fully reveals the state as α tends to 1—in particular, there exists $k \in (0, 1)$ so that $\lim_{\alpha \to 1} \overline{F}(k; \alpha) = 1$ and $\lim_{\alpha \to 1} \overline{G}(k; \alpha) = 0$.

We are interested in the following question: do private and (endogenous) public information act as complements or substitutes? That is, does the regulator provide more or less information to better informed investors? We restrict attention to the case $\eta > 0$[33] and, to make some headway, we analyze these questions in the context of optimal CF-tests $\left\{0, \underline{\mu}(\alpha)\right\}$ and optimal CI-tests $\{\overline{\mu}(\alpha), \mu_{CI}(\alpha)\}$ by studying the effect of α on the informativeness of these tests, i.e., elucidate whether $d\underline{\mu}/d\alpha \gtrless 0$ in the case of CF-tests (see Section 4.1), and whether $d\overline{\mu}/d\alpha \gtrless 0$ when $\mu_{CI}(\alpha)$ remains constant in the case of CI-tests (see Section 4.2).

Our interest in these questions is twofold. First, the effect of changes in expertise on investors' welfare are indirectly informed by the impact on the (endogenous) stress test. Indeed,

[33] If $\eta \leq 0$ then expertise is irrelevant as the regulator's test always reveals the state-see Lemma 2.

we show in Section 5 that better informed investors can benefit more from stress testing if the regulator preemptively provides a more informative test. Second, private improvements in expertise generate informational spillovers on other stakeholders channeled through the regulator's test: improvements in the information of large investors may increase or decrease the public information available to these stakeholders, depending on whether the regulator becomes more or less transparent of the state.

Our analysis in Sections 4.1 and 4.2 shows that the regulator's test depends in subtle ways on how responsive are investors signals to the underlying state. Our results become more transparent, however, when considering the limit case when investors private signals become perfectly informative of the state. In Section 4.3 we show that, under general conditions, signals then act as complements: the regulator provides a perfectly informative test as investors' signals perfectly reveal the bank's state.

4.1 Effect of Investor's Expertise on Critical-Fault Tests.

Consider the optimal CF-test $\left\{0, \underline{\mu}(\alpha)\right\}$ and recall that it satisfies the optimality condition $E^{CF}\left(\underline{\mu}(\alpha); \alpha\right) = 1$–see (14). Thus, any change in investors' signals that increases this elasticity will compel the regulator to provide a more informative test—i.e., $d\underline{\mu}/d\alpha > 0$ if $\left(\partial E^{CF}\left(\mu; \alpha\right)/\partial\alpha\right)\big|_{\mu=\underline{\mu}(\alpha)} > 0$—while reductions in $E^{CF}\left(\mu; \alpha\right)$ translate into a less informative test.

Using (8), we can decompose the change in E^{CF} as follows

$$\frac{\partial E^{CF}(\mu, \alpha)}{\partial\alpha} = \underbrace{\frac{\partial}{\partial\alpha}\left(\mu\frac{\partial \ln \widetilde{u}_R(k;\mu)}{\partial\mu}\right)}_{\text{direct effect}} + \underbrace{\frac{\partial}{\partial\alpha}\left(\mu\frac{\partial \ln \widetilde{u}_R(k;\mu)}{\partial k}k'(\mu)\right)}_{\text{indirect effect}}. \tag{22}$$

There are two basic forces affecting the change in $\underline{\mu}(\alpha)$: a *direct effect* so that, holding constant investment rules, joint investment may be now more sensitive to favorable public news and an *indirect effect* as investment rules of better-informed investors may be more or less responsive to the outcome of the test. The interplay between these two forces—a typically positive direct effect but a positive or negative indirect effect—implies that, in general, the total effect of improving investors' private information on the optimal CF-test is ambiguous.

To see this, consider first the direct effect in (22), which captures the change in the sensitivity of joint investment to *public information* if better informed investors were to follow the same investment rules. Since

$$\mu\frac{\partial \ln \widetilde{u}_R(k;\mu)}{\partial\mu}\bigg|_{k=k(\mu;\alpha)} = \frac{\mu\overline{\Lambda}^2\left(k;\alpha\right) - \eta\mu}{\mu\overline{\Lambda}^2\left(k;\alpha\right) + \eta\left(1-\mu\right)}\bigg|_{k=k(\mu;\alpha)}, \tag{23}$$

the direct effect depends on how discriminating private signals are as measured by the likelihood ratio $\overline{\Lambda}(k;\alpha)=\overline{F}(k;\alpha)/\overline{G}(k;\alpha)$. For instance, improvements in investor's signals that increase $\overline{F}(k;\alpha)$—lowering the rate of Type II errors when an investor fails to invest when the state is high—or that decrease $\overline{G}(k;\alpha)$—lowering the rate of Type I errors when an investor invests even though the state is low— both imply a positive direct effect—i.e., the regulator would then provide a more informative CF-test if investors kept the same investment rules.

The indirect effect in (22) captures the change in the sensitivity of joint investment to *private information* as better informed investors revise their investments rules. We can expand the second term in (22) using (7) and (8) as follows[34]

$$\mu \frac{\partial \ln \widetilde{u}_R(k;\mu)}{\partial k} k'(\mu;\alpha)\Big|_{k=k(\mu;\alpha)} =$$
$$\mu\left(-2\frac{\mu h_F(k;\alpha)\overline{\Lambda}^2(k;\alpha)+(1-\mu)\eta h_G(k;\alpha)}{\mu\overline{\Lambda}^2(k;\alpha)+\eta(1-\mu)}\right)\left(-\frac{1}{\mu^2\partial R(k;\alpha)/dk}\right)\Bigg|_{k=k(\mu;\alpha)}. \tag{24}$$

The first term in (24) is the (semi-)elasticity of joint investment to *private information*: the change in joint investment as investors raise their investment thresholds simply captures the degree of correlation between investor's private signals. The second term represents investors' responsiveness to public news: how much investors lower their investment thresholds in response to a more favorable outcome of the test. Investors' information has an ambiguous effect on this term: increases in α may increase or decrease $|k'(\mu;\alpha)|$ thus leading to more or less responsive investors—see Figure 3. For example, changes in α that increase $\partial R(k;\alpha)/dk$ make investment rules less responsive to public news; this implies a decrease in (24)—a negative indirect effect—and can lead to a less informative CF-test–see Figure 3-b. Conversely, reductions in $\partial R(k;\alpha)/dk$ make investment rules more responsive to public news and can lead to more informative CF-tests–as in Figure 3-c.

Our discussion underscores the dependence of the direct effect on how discriminating private signals become for *fixed* investment rules, and the dependence of the indirect effect on the responsiveness of investment rules to public news. The following proposition provides sufficient conditions for both the direct and the indirect effect to be positive, so that the regulator provides a more informative CF-test to better-informed investors.

Proposition 3. *(i) Suppose that*

$$\partial\overline{G}(k;\alpha)/\partial\alpha\big|_{k=k(\underline{\mu}(\alpha),\alpha)} \leq 0. \tag{25}$$

Then, the direct effect in (22) is positive.

[34] Recall that $h_i(k;\alpha), i=\{F,G\}$, represents the respective hazard rates at the investment threshold.

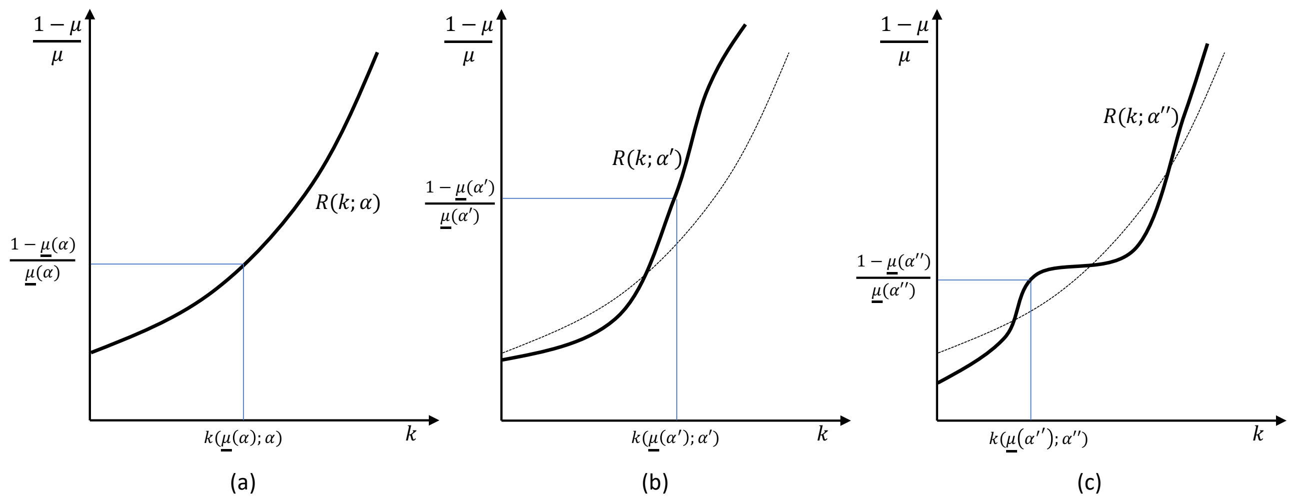

Figure 3: Investor responsiveness and CF-test for $\alpha', \alpha'' > \alpha$.

(ii) Suppose that, in addition,

$$\upsilon \frac{\partial}{\partial \alpha}\left(\frac{h_G(k;\alpha)}{\partial R(k;\alpha)/dk}\right) + (1-\upsilon)\frac{\partial}{\partial \alpha}\left(\frac{h_F(k;\alpha)}{\partial R(k;\alpha)/dk}\right) \geq \frac{\partial \upsilon}{\partial \alpha}\frac{(h_G(k;\alpha) - h_F(k;\alpha))}{\partial R(k;\alpha)/\partial k}, \qquad (26)$$

with $\upsilon = \mu\overline{\Lambda}^2(k;\alpha)/\left(\mu\overline{\Lambda}^2(k;\alpha) + (1-\mu)\eta\right)$*. Then* $d\underline{\mu}(\alpha)/d\alpha > 0$.

An increase in α always leads to more discriminating signals *if* investors can adjust their investment rules: Blackwell-more informative signals allow for tests that either lower Type I or Type II errors, or both.[35] The proof of the proposition shows that if Type I errors are reduced for a *fixed* investment rule, i.e., if $\partial\overline{G}(k;\alpha)/\partial\alpha|_{k=k(\mu,\alpha)} < 0$, then the likelihood ratio $\overline{\Lambda}(k;\alpha)$ increases for Blackwell-more informative signals, leading to a positive direct effect.[36]

If, in addition, (26) holds then the indirect effect is also positive implying that private and public information act as complements: the regulator's test would be more informative when investors' signals improve. The condition (26) depends on the change in responsiveness of investment rules as measured by the change in the ratios $h_i/(\partial R/dk)$, $i = \{F, G\}$. To clarify (26), consider Figure 3. Figures 3-b and -c depict investors' responsiveness when signals $X(\alpha')$ and $X(\alpha'')$ are Blackwell-more informative than signal $X(\alpha)$ in Figure 3-a

[35] That is, written as a function of the $t-$quantile, the likelihood ratio $\overline{F}(\overline{G}^{-1}(t;\alpha);\alpha)/\overline{G}(\overline{G}^{-1}(t;\alpha);\alpha)$ is always non-decreasing in α (see Blackwell and Girshick (1954)).

[36] Condition (25) does not pose a serious constraint in signing the direct effect. Indeed, with the integral probability transform $\widetilde{X} = G(X;\alpha)$, signal $\widetilde{X}|\omega = -1$ is uniformly distributed for all α and (25) is trivially satisfied when investment thresholds are represented in terms of realizations of $\widetilde{X}$.

implying that likelihood ratios $\lambda(k;\alpha')$ and $\lambda(k;\alpha'')$ are more dispersed than $\lambda(k;\alpha)$. More dispersed likelihood ratios, however, do not pin down $\lambda'(k;\alpha')$ and $\lambda'(k;\alpha'')$ at the investment thresholds. Indeed, the indirect effect of moving from α to α'' is positive in Figure 3-c as it leads to a flattening of investors equilibrium response $R(k;\alpha'')$, while it is negative when moving from α to α' in Figure 3-b. In words, better-informed investors in Figure 3-c respond more aggressively to public news when revising their investment rules, which is more conducive to the regulators' disclosure, while investment rules in 3-b are flatter as a function of public news, which leads to a less informative stress test.

4.2 Effect of Investor's Expertise on Coordinated-Investment Tests.

We now turn to the effect of investors' expertise on the optimal CI test $\{\overline{\mu}(\alpha), \mu_{CI}(\alpha)\}$. Changes in α now affect both realizations of this test: the belief which induces coordinated investment $\mu_{CI}(\alpha) = 1/(1+R(0;\alpha))$ and the belief $\overline{\mu}(\alpha)$. Regarding $\mu_{CI}(\alpha)$, a Blackwell-more informative signal leads to more dispersed likelihood ratios, so that $\lambda(0;\alpha') \leq \lambda(0;\alpha)$ for $\alpha' > \alpha$, which, using (3), implies $R(0;\alpha') \leq R(0;\alpha)$ and $\mu_{CI}(\alpha)$ is non-decreasing with α. That is, the regulator needs to raise the belief $\mu_{CI}(\alpha)$ that induces investors to disregard their private signals when their signals become more informative of the state. Thus, to understand the effect of expertise on CI-tests, we only need to focus on the choice of $\overline{\mu}(\alpha)$.

Our discussion will be brief as we follow a similar analysis as in Section 4.1. Recall that the optimal CI-test satisfies $E^{CI}\left(\overline{\mu}(\alpha);\alpha\right) = -1$–see (16). Thus, any change in investors' signals that increases this elasticity will lead the regulator to provide a more informative test—i..e, $d\overline{\mu}/d\alpha < 0$ if $\left(\partial E^{CI}\left(\mu;\alpha\right)/\partial\alpha\right)\big|_{\mu=\underline{\mu}(\alpha)} > 0$. We can decompose the change in the elasticity into a direct and indirect effect. However, noting that

$$E^{CI}\left(\mu;\alpha\right) = \frac{\left(\mu_{CI}(\alpha)-\mu\right)u_R(\mu;\alpha)}{\mu\left(u_R(\mu_{CI}(\alpha);\alpha)-u_R(\mu;\alpha)\right)}\left(-E^{CF}\left(\mu;\alpha\right)\right),$$

our analysis simplifies if $\mu_{CI}(\alpha)$ does not vary with α since then $sign\left(\partial E^{CI}\left(\mu;\alpha\right)/\alpha\right) = -sign\left(\partial E^{CF}\left(\mu;\alpha\right)/\alpha\right)$.

Corollary 1. *Suppose that $\lambda(0;\alpha)$ is constant for $\alpha \in [\underline{\alpha}, \overline{\alpha}]$. If (25) and (26) hold for $\alpha \in [\underline{\alpha}, \overline{\alpha}]$ with $\underline{\mu}(\alpha)$ replaced with $\overline{\mu}(\alpha)$, then the regulator offers a less informative CI-test to better informed investors.*

The condition on the likelihood ratio $\lambda(0;\alpha)$ guarantees that μ_{CI} does not change locally with the expertise of investors. Then, the same conditions that led the regulator to increase $\underline{\mu}(\alpha)$ in Proposition 3.ii also lead now to an increase in $\overline{\mu}(\alpha)$. In contrast, however, this implies that the regulator's test is now less informative.

An important insight from Corollary 1 is that the interplay between private and public information depends not only on the properties of private signals but also on the prior beliefs of investors. Indeed, if investors are very pessimistic about the state, so that a CF-test is optimal, and the regulator offers a more informative test to better informed investors, then she would offer less information if investors were instead very optimistic (so that a CI-test is optimal).

4.3 Asymptotic Crowding-in and Crowding-Out

The analysis in Sections 4.1 and 4.2 showed that the interplay between private and (endogenous) public information hinges crucially on the local properties of investors' private signals. In this section we get a more definitive view of this interplay by studying the regulator's test as investors become privately perfectly informed—i.e., as $\alpha \to 1$. We say that investors' expertise *asymptotically crowds-in* public disclosure if the regulator's test becomes fully informative of the state, while private expertise *asymptotically crowds-out* public disclosure if it becomes completely uninformative, as $\alpha \to 1$.[37] Our main insight is that there are positive informational spillovers (*asymptotic crowding-in*) as long as investors remain sensitive to public news.

A key force driving investors' reaction to public news is the sensitivity of private investment rules to public information, as captured by $|k'(\mu)|$. In particular, from (7) we have that if $\partial R\left(k\left(\mu;\alpha\right);\alpha\right)/\partial k$ becomes small, making $|k'(\mu)|$ large, investors are more likely to aggresively revise their investment rules in response to public news, and the regulator would provide more information if $|k'(\mu)|$ increases with α. We now make this intuition precise by showing that asymptotic crowding-in obtains as long as investors' signals do not become locally very informative.

We analyze separately the case of discrete and continuous signals. First, crowding-in is a general feature of optimal stress tests when investors' signals are finite.

Proposition 4. *Let support*$(X(\alpha)) = \{x_1, \ldots, x_n\}$ *and* $f_i(\alpha) = \Pr\left[x_i|\omega = +1;\alpha\right]$ *and* $g_i(\alpha) = \Pr\left[x_i|\omega = -1;\alpha\right].$ *Suppose that for* $i \in \{1, \ldots, n\}$ *and* $\alpha < 1$ *: (i) either* $f_i(\alpha) > 0$ *or* $g_i(\alpha) > 0$*, and (ii) either* $\lim_{\alpha\to 1} f_i(\alpha)/g_i(\alpha) = 0$ *or* $\lim_{\alpha\to 1} g_i(\alpha)/f_i(\alpha) = 0$*. Then, for every prior* $\mu_0 \in (0,1)$*, the regulator's optimal test converges to a fully informative public signal as* $\alpha \to 1$*.*

[37] Note that if investors' private signals fully reveal the state, then the regulator's test does not affect investment outcomes and any test would be optimal. Our interest lies in understanding how much information the regulator discloses if investors are well informed but not perfectly informed (so $1 - \epsilon < \alpha < 1$ with $\epsilon > 0$ an arbitrarily small constant).

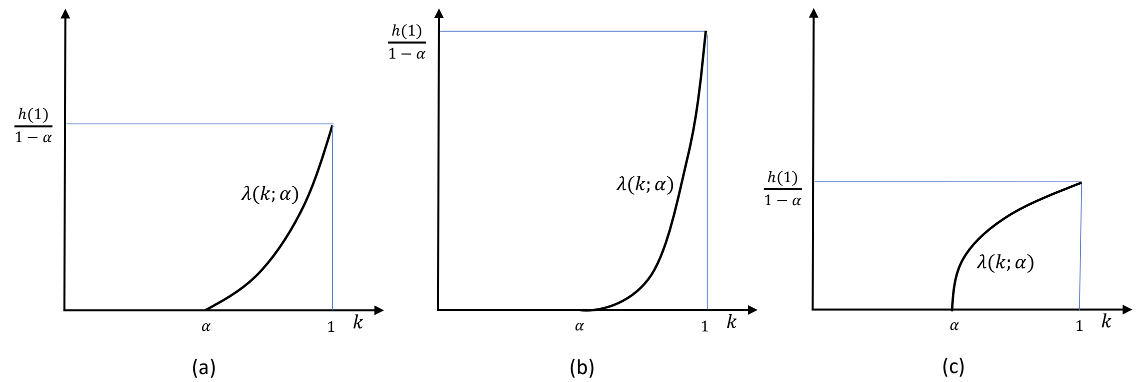

Figure 4: Likelihood ratio $\lambda(k;\alpha)$ when (a) $0 < h'(0) < \infty$, (b) $h'(0) = 0$, (c) $h'(0) = \infty$.

Second, when considering continuous signals, crowding-in follows if the private signal's sensitivity to the state (as measured by the ratio of the rate of change of the likelihood ratio $d\lambda(k,\alpha)/dk$ to $h_G(k,\alpha)$) remains bounded.

Proposition 5. *Suppose that f and g are continuously differentiable in $[0,1]$ and $\frac{d\ln\lambda(k,\alpha)/dk}{h_G(k,\alpha)}$ is uniformly bounded in $\cup_{\alpha>\alpha'}K(\alpha)$ for some $\alpha' < 1$, with $K(\alpha) = \left\{k : \frac{1-\mu}{\mu} = R(k;\alpha) \text{ for } \mu \in (0,1)\right\}$. Then, for any $\mu_0 \in (0,1)$ the regulator's optimal test converges to a fully informative public signal as $\alpha \to 1$.*

Signals on the compact support $[0,1]$ become locally very discriminating of the state at $\widetilde{k}$ when the slope of the likelihood ratio $d\ln\lambda(k,\alpha)/dk$ becomes large. This could owe to one of two reasons: the density $f(\widetilde{k})$ increases rapidly, so a good state makes a higher realization more likely, or the density $g(\widetilde{k})$ decreases rapidly, so a bad state makes a higher realization more unlikely. Proposition 5 shows that crowding-in follows whenever signals become more discriminating owing to the latter effect—i.e., decreasing $g(\widetilde{k})$.

We now present an example in which improvements in investors' private information leads the regulator to provide no public information. In this case, investors coordinate on the basis of the accuracy of their signals with the state, and hence with each other, with no help from the regulator. In fact, we study a family of signals so that as investors' private signals become perfectly informative, the regulator's optimal test takes the form $\{0, \underline{\mu}\}$ when the prior is $\mu_0 < \underline{\mu} < 1$.

Example 1. Fix a function $h(x) \geq 0$ defined in $[0,1]$ such that (i) $h(x)$ is strictly increasing, with $h(0) = 0$ and (ii) letting $\overline{H}(x) = \int_x^1 h(t)dt$, we have

$$\overline{H}(0) = \int_0^1 h(t)dt = 1.$$

Define $X_h(\alpha)$ obtained from h as follows: G is uniform in $[0,1]$ while F has density

$$f(k;\alpha) = \begin{cases} \frac{1}{(1-\alpha)} h(\frac{k-\alpha}{1-\alpha}),\ k \geq \alpha, \\ 0, \quad k < \alpha. \end{cases}$$

In other words, investors learn that $\omega = -1$ upon observing $k < \alpha$, while their posterior belief regarding $\omega = 1$ increases monotonically for $k > \alpha$—Figure 4 shows the function $\lambda(k;\alpha) = f(k;\alpha)$ for three different choices of h. Note also that as α tends to 1, then f becomes a Dirac delta concentrated at $k = 1$.

Remark 1. *Suppose that investors' signals are distributed according to $X_h(\alpha)$ and define*

$$\underline{\mu}_h = \frac{1}{1 + \eta \frac{(1+\gamma)^2}{2} h'(0)}.$$

Then, for $\eta > 0$, the regulator's optimal test converges to $\left\{0, \underline{\mu}_h\right\}$ if $\mu_0 \in \left(0, \underline{\mu}_h\right)$ and is completely uninformative if $\mu_0 \geq \underline{\mu}_h$. In particular, we have investors' asymptotic crowding-in of the public test iff $h'(0) = 0$ while we have asymptotic crowding-out of the public test if $h'(x)$ becomes unbounded as $x \to 0$.

Consistent with our main insight, the informativeness of the public test depends on the local informativeness of investors' private signals at $k = \alpha$, as captured by $h'(0)$. A low value of $h'(0)$ implies that investors posterior beliefs do not vary much around $k = \alpha$–see Figure 4-b–so that changes in the outcome of the test would lead to big revisions of investment thresholds if these are set close to $k = \alpha$—which is the case when $\alpha \to 1$.[38] Conversely, a high value of $h'(0)$–see Figure 4-c–implies that changes in the outcome of the test will have a negligible effect on investment rules if thresholds are set close to $k = \alpha$. In summary, lack of responsiveness of investment rules to public information can lead to crowding-out of public tests.

[38] Note that, in this example, $(d \ln \lambda(k,\alpha)/dk)/h_G(k,\alpha)$ is uniformly bounded if and only if $h'(0) = 0$—cf. Proposition 5.

5 Social and Private Value of Expertise.

In the previous Section we explored the interplay between investors' quality of research (i.e., their expertise) and the regulator's disclosure. We now study how improvements in expertise affect both the private value of the stress test and the social value of the stress test (as captured by the regulator's payoff). We show that expertise has, in general, an ambiguous effect both on social welfare and on investors' payoffs.

Suppose that investors' signals are indexed by α and Assumption A3 holds. Expertise has no impact on equilibrium payoffs when $\eta \leq 0$ as the regulator always provides a perfectly informative test. Thus, we study the case $\eta > 0$ and concentrate on CF-tests. To clarify the effect of expertise, let $A_I(k)$ represent investors' interim equilibrium average expected utility when the test leads to interim belief $\mu(k) = 1/(1 + R(k))$ after which investors select a threshold k,

$$A_I(k) \equiv \frac{u_i(\mu(k))}{\mu(k)} = \overline{F}(k) - R(k)\overline{G}(k) + \gamma \left(\overline{F}^2(k) + R(k)\overline{G}^2(k) \right). \tag{27}$$

Using (12) and (27), the regulator and investors' expected payoff from a CF-test are $U_R^{CF}(\alpha) \equiv \mu_0 A_R(k(\underline{\mu}(\alpha); \alpha); \alpha)$ and $U_I^{CF}(\alpha) \equiv \mu_0 A_I(k(\underline{\mu}(\alpha); \alpha); \alpha)$ and we can use (13) and the properties of $k(\mu)$ in Lemma 1 to evaluate the impact of expertise on welfare.

Consider first the impact of α on U_R^{CF}. Appealing to the envelope theorem,[39] we can write

$$\frac{1}{\mu_0} \frac{dU_R^{CF}}{d\alpha} = \left. \frac{\partial A_R(k; \alpha)}{\partial \alpha} \right|_{k=k(\underline{\mu}(\alpha);\alpha)}. \tag{28}$$

This is intuitive: as the regulator can optimally adjust the test in response to investors' improved information, expertise has only a direct effect on the regulator's payoff. This effect is positive if, and only if, increasing expertise leads investors to raise average joint investment while holding constant the investment threshold $k(\underline{\mu}(\alpha); \alpha)$.

While the sign of (28) is in general ambiguous, it is clear that increases in expertise must be detrimental to the regulator for some parameter values. Indeed, as $\underline{\mu}(\alpha)$ maximizes the regulator's average utility, U_R^{CF} can only decrease when agent's perfectly learn the state—which corresponds to the limit as $\alpha \to 1$. For example, when private signals can detect low returns, increases in expertise are always detrimental to the regulator.

Remark 2. *Consider Example 1. Then* $dU_R^{CF}/d\alpha < 0$ *so that, regardless of the nature of the investment externality, an increase in investors' expertise lowers the regulator's expected utility.*

[39] The optimality condition (13) guarantees that $(\partial A_R(k, \alpha)/\partial k)|_{k(\underline{\mu}(\alpha);\alpha)} = 0$.

Consider now the impact of expertise on investors' equilibrium payoffs. Do better-informed investors benefit more from public disclosure of the stress tests? To answer this question, let $\widetilde{A}_I(\mu;\alpha) \equiv A_I(k(\mu,\alpha);\alpha)$ and consider the marginal impact of expertise on the value of the test V_i^{test} defined in (21),

$$\frac{dV_i^{test}(\alpha)}{d\alpha} = \underbrace{\left(\frac{\partial \widetilde{A}_I(\underline{\mu}(\alpha);\alpha)}{\partial \alpha} - \frac{\partial \widetilde{A}_I(\mu_0;\alpha)}{\partial \alpha}\right)}_{\text{direct effect}} + \underbrace{\frac{\partial \widetilde{A}_I(\underline{\mu}(\alpha);\alpha)}{\partial \mu}\frac{\partial \underline{\mu}(\alpha)}{\partial \alpha}}_{\text{indirect effect}}. \tag{29}$$

The effect of expertise on V_i^{test} is decomposed into two effects. The first term in (29) is the direct effect of increased expertise on V_i^{test} holding constant the regulator's test. Note that the sign of this effect is in principle ambiguous as it depends on the comparison between the effect of expertise on investors average utility at the realization $\underline{\mu}(\alpha)$ and at the prior μ_0. For instance, if investors underinvest (relative to full information) under μ_0 and improvements in private information ameliorate this underinvestment, then the direct effect will be positive.

The second term in (29) describes the indirect effect as improvements in expertise will lead the regulator to preemptively change the public test. This indirect effect depends on whether expertise crowds-in or -out the stress test (i.e., whether $\partial\underline{\mu}(\alpha)/\partial\alpha \gtrless 0\}$, and on the effect of public information on investors' average utility (i.e., on the sign of $\partial\widetilde{A}_I(\mu,\alpha)/\partial\mu$).

While one would expect the test to be less valuable to better-informed investors–as both sources of information on fundamentals may act as substitutes–the sign of the direct effect in (29) depends in general on the functional form of signals $X(\alpha)$. Nevertheless, better-informed investors may benefit more from the test if it compels the regulator to be more transparent. To see this, we study situations in which the direct effect in (29) is small.

Proposition 6. *Let $\eta = 1$ and suppose that a CF-test $\{0, \underline{\mu}(\alpha)\}$ exists. Then, there is $\epsilon > 0$ so that whenever $\underline{\mu}(\alpha) - \epsilon < \mu_0 < \underline{\mu}(\alpha)$, we have $dV_i^{test}/d\alpha > 0$ if and only if $d\underline{\mu}(\alpha)/d\alpha > 0$.*

When the prior belief is close to the realization $\underline{\mu}(\alpha)$, a CF-test is unlikely to change investors views on fundamentals—indeed, the probability that the test reveals that the state is $\omega = -1$ is then $\left(\underline{\mu}(\alpha) - \mu_0\right)/\underline{\mu}(\alpha)$—and investors derive little value from the test. Thus, improvement in private information render the direct effect in (29) as second order, and the total effect is driven by the indirect effect. Proposition 6 shows that the indirect effect is positive if $\underline{\mu}(\alpha)$ increases: better-informed investors gain more from the test if and only if the regulator preemptively discloses more information. Thus, Proposition 6 links our results in Section 4 on the interplay between private and (endogenous) public information to the welfare effect of the test.

The key observation in Proposition OA.1 is that, irrespective of the sign of the investment externality, investors average utility is increasing at the test outcome $\underline{\mu}(\alpha)$. To see this, write (27) as $A_I(k) = \mathcal{E}(k) + \gamma A_{R,1}(k)$ where $\mathcal{E}(k) \equiv \mathbb{E}[\omega 1_{\{x_i \geq k\}}]/\mu(k)$ captures how adapted are investments to the state and $A_{R,1}(k)$ is the regulator's average utility when $\eta = 1$. The fact that the CF-test satisfies $\partial A_R((k(\mu,\alpha);\alpha))/\partial\mu = 0$—see (13)—means that the effect of a more informative CF-test on investor's average utility is given by $\partial\mathcal{E}((k(\underline{\mu}(\alpha),\alpha);\alpha))/\partial\mu$, i.e., whether a more informative test leads to a more adapted decision to the state. The proof then shows that $\partial\mathcal{E}((k(\underline{\mu}(\alpha),\alpha);\alpha))/\partial\mu > 0$. In summary, if $d\underline{\mu}(\alpha)/d\alpha > 0$, then investors benefit from more precise private signals from the increased adaptation to the state that follows a more informative stress test.

6 Investors Correlated Mistakes

Investors often gather information from multiple sources to prepare an investment report. Some of these sources are proprietary to each investor but others are common to all investors — e.g., analysts' forecasts published frequently by investment banks and research by investment advisors. Hence, even though investors privately conduct their own research, reliance on common sources means that their ultimate investment reports are correlated. To capture this in a simple but rich enough setting, we now decompose investors' private information into a conditional (on the state ω) independent signal $\widehat{x}_i$, $i = \{1, 2\}$, and a residual common noise or 'mistake' ε independent of $(\omega, \widehat{x}_1, \widehat{x}_2)$; the residual noise captures reliance on common sources such as using the same investment advisor. Specifically, investor i privately observes:

$$x_i = (1-\beta)\widehat{x}_i + \beta\varepsilon, \tag{30}$$

where $\beta \in [0, 1]$. Let now $\widehat{x}_i|\omega = 1 \sim \widehat{f}$ (distribution $\widehat{F}$), $\widehat{x}_i|\omega = -1 \sim \widehat{g}$ (distribution $\widehat{G}$), and $\varepsilon \sim g_\varepsilon$ all defined in $[0, 1]$. We assume that the investors as well as the regulator know the form of the decomposition of signal x_i in (30), including the value of β, but none observe its constituents. Therefore for $\beta \in (0, 1]$ the private signals of investors are correlated even conditional on the state ω.[40]

Given the special form of (30) the expressions for $F(x;\beta), G(x;\beta)$, $F(y|x;\beta)$, $G(y|x;\beta)$ (and their corresponding densities, survival functions) are easily computed as the convolution of the densities/distributions of the conditionally independent $\widehat{x}_i$ with the density of the noise

[40] The model of this section is an instantiation of the general setup we introduced in Section 2 and analyzed up to and including Section 3.1. For $\beta = 0$ we obtain the conditionally independent case we further covered in Sections 3.2–5.

ε.[41] Now, given A1[42] and A2-a[43] we have that the equilibrium condition for the symmetric threshold strategy is $(1-\mu)/\mu = \widetilde{R}(k;\beta)$ (see (4)), where function $\widetilde{R}(k;\beta)$ is defined in (3). Moreover, for simplicity of exposition we assume here that $\widetilde{R}(k;\beta)$ is strictly increasing in k for all $(k,\beta) \in (0,1)^2$; finally, the equilibrium threshold $k(\mu;\beta)$ is given in Lemma 1 by (6), where now $R(k;\beta) = \widetilde{R}(k;\beta)$—see (5).

So how does correlation affect investment rates and ultimately the regulator's test? Does the existence of common mistakes make the optimal test more or less informative? We provide an answer in the context of 'small' mistakes, i.e., for β positive but very small, and for the CF-test, which has the least requirements for existence. Hence, this is a robustness exercise for our conditionally independent signals baseline case and the ensuing results.

Although we focused in the conditional independent case for Section 3.2 it is straightforward to adapt (8) to the case of correlated signals::

$$\begin{aligned} u_R(\mu;\beta) &\equiv \Pr[\omega=1]\Pr[x_i \geq k(\mu;\beta), x_j \geq k(\mu;\beta)|\omega=1] \\ &+ \eta \Pr[\omega=-1]\Pr[x_i \geq k(\mu;\beta), x_j \geq k(\mu;\beta)|\omega=-1] \\ &= \mu \int \overline{\widehat{F}}^2\left(\frac{k(\mu;\beta)-\beta e}{1-\beta}\right) g_\varepsilon(e)de + \eta(1-\mu)\int \overline{\widehat{G}}^2\left(\frac{k(\mu;\beta)-\beta e}{1-\beta}\right) g_\varepsilon(e)de, \end{aligned} \tag{31}$$

where the last line follows from integrating over ε and recognizing that conditional on (ε,ω) signals (x_i,x_j) are i.i.d. for $i \neq j$. As before $\overline{\widehat{F}} \equiv 1-\widehat{F}$, $\overline{\widehat{G}} \equiv 1-\widehat{G}$. The CF-test is

$$\underline{\mu}(\beta) \in \arg\max_{0<\mu\leq 1} u_R(\mu;\beta)/\mu,$$

and it exists for $\eta > 0$ as long as $\lambda(1;\beta)$ is finite.[44] The main result of this section follows.

Proposition 7. *Assume $\eta > 0$, $\gamma \in (0,\overline{\gamma})$ (see Lemma 1), and $F(x;\beta)$, $G(x;\beta)$ are twice continuously differentiable in $(0,1)^2$. Then, $\lim_{\beta\to 0} d\underline{\mu}/d\beta = 0$; hence, small mistakes have no effect on the design of the CF-test.*

In Proposition 7 we inquire on the effect of small mistakes on the CF-test realization $\underline{\mu}$. This is a robustness exercise as we perturb slightly the conditionally independent signals baseline case.[45] Recall that the regulator will offer a more informative CF-test—i.e.,

[41] Recall that the convolution of two functions $f,g:\mathbb{R}\to\mathbb{R}$ is equal to $\int_{x\in\mathbb{R}} f(y-x)g(x)dx$.

[42] If $\lambda(x;0) = \widehat{f}(x)/\widehat{g}(x)$ is increasing in x then so is $\lambda(x;\beta)$ if we further assume that g_ε is log-concave in $[0,1]$ (see Keilson and Sumita (1982, Thm. 2.1–d)).

[43] Signals in (30) are positively dependent so in this section we focus on the case $\gamma > 0$ (see A2-a and Milgrom and Weber (1982, Theorem 5)).

[44] In the proof of Lemma 3 we do not use the conditional independence of signals.

[45] Similarly to Proposition 4 for $\alpha \to 1$, our interest here lies in understanding how much information the

increase $\underline{\mu}$—if increasing β increases the elasticity E^{CF}—see (14). In turn, in the proof of the proposition, we show that the direct and indirect effects of β (see Lemma A.2) on E^{CF} as $\beta \to 0$ move in opposite direction. The sign of each depends on the sign of the difference $k(\underline{\mu}(\beta);\beta) - \mathbb{E}[\varepsilon]$ as $\beta \to 0$, which informs us on whether the resulting equilibrium threshold for $\beta > 0$ is higher or lower than that when there are no common mistakes. Take first the case when $k(\underline{\mu}(\beta);\beta) - \mathbb{E}[\varepsilon] < 0$, then small correlated mistakes lead investors to increase their investment thresholds (so that, ceteris paribus, they would now invest less often). Then the indirect effect on $\underline{\mu}$ (i.e., stemming from changes in the investment rule, see first term in (A.12)) is negative, while the direct effect (i.e., stemming from variation in β, while investment rules are constant, see second term in (A.12)) is positive. Now, if $k(\underline{\mu}(\beta);\beta) - \mathbb{E}[\varepsilon] > 0$, then the direct (indirect) effect is negative (positive).[46] In either case, both effects exactly cancel each other out, so that the total effect is zero, which solidifies that our results in the case of no mistakes carry over to the case of small mistakes for CF-tests.

7 Concluding Remarks

In this paper we studied the stress test design by a regulator in the presence of large private investors. The regulator would like to elicit investment in order to save a distressed bank, while avoiding the use of public funds to do so. The large private investors, besides the public stress test, rely on the quality of their private research (expertise) to decide on whether to fund banks seeking capital, where coordination motives also play a crucial role.

We characterized the optimal public test and provided conditions for this test to, with positive probability, perfectly coordinate investment choices; nevertheless, perfect coordination is not a generic feature. We showed that if investors' private signals are, at the investment margin, not too discriminating of the distressed bank's state then there is crowding-in of the public stress test: the regulator offers a more informative public test to better-informed investors. For instance, asymptotic crowding-in always obtains if investors' signals are discrete.

Our results have implications on empirical exercises aiming to relate public information released from stress tests to investors' private information. First, in our model, the (exogenous) expertise determines the (endogenous) informativeness of the public stress test: more informative tests can be the result of either less (i.e., when there is crowding-out) or more informed investors (when there is crowding-in), depending on how locally discriminating private information is about the bank's state. On the other hand, in models with investors' information acquisition (see, e.g., Diamond (1985) and Goldstein and Yang (2017)) or banks'

regulator discloses if investors have mostly but not completely uncorrelated (conditional on the state) signals (so $0 < \beta < \epsilon$ with $\epsilon > 0$ an arbitrarily small constant).

[46] The expressions of the indirect and direct effect as $\beta \to 0$ are given in (A.55) in the Appendix.

information release (see, e.g., Quigley and Walther (2020)), there is crowding-out of private by public information. In practice, private information may lead and also be led by public information; therefore, we provide a novel channel that affects their interaction and can explain either a positive or a negative relationship between the two. Second, the slope of the informativeness of the optimal test with respect to how discriminating private information is changes sign depending on whether private investors are ex-ante optimistic or pessimistic (i.e., have high or low prior, respectively) over the distressed bank's state. Hence, any test to measure such slope should condition on the common prior, or otherwise the effect may be confounded and, possibly, muted.

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Appendix

Proof of Lemma 1. Let

$$\widetilde{R}(x;k) \equiv \lambda(x)\widetilde{C}(x;k), \text{ with } \widetilde{C}(x;k) \equiv \left(1+\gamma\overline{F}(k|x)\right) / \left(1-\gamma\overline{G}(k|x)\right). \tag{A.1}$$

We consider first the case $\gamma < 0$ and then analyze the case $\gamma > 0$.

Case i. Suppose that $\gamma < 0$ and Assumptions A1 and A2-b hold. To study best responses, suppose that investor j invests if $x_j \geq k$. Then Assumption A2-b guarantees that $\widetilde{C}(x;k)$ is non-decreasing in x as the product of two positive and non-decreasing functions, therefore

$$v_i(k,x;\mu) = \left(1-\gamma\overline{G}\left(k|x\right)\right) \Pr\left[\omega=-1|x_i=x\right]\left(\lambda(x)\widetilde{C}(x;k)\frac{\mu}{1-\mu}-1\right)$$

is strictly single-crossing in $x \in \{t \in [0,1] : f(t) > 0, g(t) > 0\}$ as the product of a positive function and a strictly increasing function. Therefore, investor i's best response is also a threshold strategy with unique threshold satisfying $v_i(k, x_i(k);\mu) = 0$–i.e., $\widetilde{R}(x_i(k);k) = (1-\mu)/\mu$—for interior values of x_i.

Consider symmetric equilibria, which must satisfy (4). The function $\widetilde{C}(k;k)$ for $\gamma < 0$ is non-decreasing as a product of two positive and non-decreasing functions.[47] Hence, $\widetilde{R}(k) = \widetilde{R}(k;k)$ (see (3)) is strictly increasing and thus $R(k) = \widetilde{R}(k)$ (see (5). We now consider solutions to (4). First, if $(1-\mu)/\mu \leq R(0)$—i.e., if $\mu \geq 1/(1+R(0))$—then an investor would invest regardless of his signal and hence we set his threshold to the lower bound $k(\mu) = 0$. Second, if $(1-\mu)/\mu \geq R(1)$—i.e., if $\mu \leq 1/(1+R(1))$—then an investor would never invest regardless of his signal and hence we set his threshold to the upper bound $k(\mu) = 1$. Finally, for interior values $R(0) < (1-\mu)/\mu < R(1)$—i.e., $1/(1+R(1)) < \mu < 1/(1+R(0))$—there is a unique solution to (4) given by $R^{-1}\left((1-\mu)/\mu\right)$.

We now consider asymmetric equilibria and provide conditions for the unique symmetric equilibrium obtained above to be the unique equilibrium. Let $x_i(k_j)$ be investor i's best response to investor j setting a threshold k_j; i.e., we have $\widetilde{R}(x_i(k_j);k_j) = \mu/(1-\mu)$. Assumption 1 and 2-b imply that for $\gamma < 0$: (i) $\partial\widetilde{R}(x;k)/\partial x > 0$, and (ii) $\partial\widetilde{R}(x;k)/\partial k > 0$, so that $dx_i/dk_j < 0$ and investors play a game with strategic substitutes. The stability condition $\left.\frac{dx_i(k_j)}{dk_j}\right|_{k_j=x_j} \times \left.\frac{dx_j(k_i)}{dk_i}\right|_{k_i=x_i} < 1$ then guarantees a unique equilibrium (see Vives (1999)). Implicitly differentiating, we have that the symmetric equilibrium is the unique equilibrium

[47] Note that $\frac{d}{dk}\left(1+\gamma\overline{F}(k|k)\right) = -\gamma f\left(k|k\right) + \gamma\frac{\partial\overline{F}(k|k)}{\partial k} \geq 0$ and $\frac{d}{dk}\left(1-\gamma\overline{G}(k|k)\right) = \gamma g\left(k|k\right) - \gamma\frac{\partial\overline{G}(k|k)}{\partial k} \leq 0$.

if for $(x_i, x_j) \in (0,1)^2$, we have

$$\frac{\partial \widetilde{R}(x_i; x_j)}{\partial k} \frac{\partial \widetilde{R}(x_j; x_i)}{\partial k} < \frac{\partial \widetilde{R}(x_i; x_j)}{\partial x} \frac{\partial \widetilde{R}(x_j; x_i)}{\partial x}.$$

Case ii. Suppose now that $\gamma > 0$ and Assumptions A1 and A2-a hold. Assumption A2-a guarantees that $\widetilde{C}(x;k)$ is non-decreasing in x as the product of two positive and non-decreasing functions, therefore $v_i(k, x; \mu)$ is again strictly single-crossing in x as the product of a positive function and an strictly increasing function. Therefore, the best response is a threshold strategy with unique threshold satisfying $v_i(k, x; \mu) = 0$. Moreover, all equilibria are symmetric as this is a symmetric game with strategic complementarities (see Vives (1999, sec. 2.2.3 and fn. 23)), .

We now show that the equilibrium is unique for low values of $\gamma > 0$. If $\gamma = 0$ then $\widetilde{R}(k) = \lambda(k)$ is strictly increasing from assumption A1. The assumption that $\lambda'(k)$ is bounded away from zero implies that $\widetilde{R}'(k) > 0$ for γ in a neighbourhood of zero. Therefore, there exists $\overline{\gamma} > 0$ for which $\widetilde{R}(k)$ is still increasing for all k and hence a unique symmetric equilibrium exists for $\gamma < \overline{\gamma}$.

Now, for $\gamma > \overline{\gamma}$, $\widetilde{R}(k)$ is in general non-monotone, so that there are multiple solutions to (4). If we define $R(k) \equiv \max_{0 \leq k' \leq k} \widetilde{R}(k')$ as in (5), then (i) $R(k)$ is non-decreasing, and (ii) for a given $y = (1-\mu)/\mu$, $R^{-1}(y) = \min\left\{k : y = \widetilde{R}(k)\right\}$ provides the symmetric equilibrium corresponding to the highest probability of joint investment, which is the regulator's preferred equilibrium. Therefore, under this equilibrium selection criteria, the equilibrium threshold satisfies $(1-\mu)/\mu = R(k)$ whenever $1/(1+R(1)) < \mu < 1/(1+R(0))$. Moreover, if $\min_{k \in [0,1]} \widetilde{R}(k) = \widetilde{R}(0)$, then whenever $(1-\mu)/\mu \leq R(0)$ (i.e., $\mu \geq 1/(1+R(0))$) an investor would invest for any signal and hence we set her threshold to the lower bound zero. If $(1-\mu)/\mu \geq R(1)$–that is, $\mu \leq 1/(1+R(1))$–then an investor would never invest and we set his threshold to one.

ii-Note that $R^{-1}(y)$ is non-decreasing, so that $k(\mu) = R^{-1}((1-\mu)/\mu)$ must be non-increasing. Differentiating (6) gives (7). □

Lemma A.1. ***(Properties of** $R(k)$ **and** $k(\mu)$**)*** *(i) If f and g are twice continuously differentiable in $[0,1]$ then $k''(1)$ is bounded.*
(ii) We have $\partial R(k;\gamma)/\partial\gamma \geq 0$ for all (k,γ).

Proof. (i) To compute $k''(1)$, we first differentiate each term in

$$k'(\mu) = \frac{1}{\mu} \frac{g(k(\mu))(1-\gamma\overline{G}(k(\mu)))}{(1-\mu)g'(k(\mu))(1-\gamma\overline{G}(k(\mu))) - \mu f'(k(\mu))(1+\gamma\overline{F}(k(\mu))) + \gamma[(1-\mu)g^2(k(\mu)) + \mu f^2(k(\mu))]}, \tag{A.2}$$

and then compute the limit as $\mu \to 1$. We have in turn:

$$\lim_{\mu\to 1} \left[\overbrace{\frac{1}{\mu}}^{I_1(\mu)} \right]' = \lim_{\mu\to 1} -\frac{1}{\mu^2} = \overbrace{-1}^{I_1{}'(1)},$$

$$\lim_{\mu\to 1} \left[\overbrace{g(k(\mu))\left(1-\gamma\overline{G}(k(\mu))\right)}^{I_2(\mu)} \right]' = \lim_{\mu\to 1} g'(k(\mu))\left(1-\gamma\overline{G}(k(\mu))\right) + \gamma g^2(k(\mu)) = \overbrace{g'(0)(1-\gamma) + \gamma g^2(0)}^{I_2{}'(1)},$$

$$\begin{aligned}
&\lim_{\mu\to 1} \left[\overbrace{(1-\mu)g'(k(\mu))(1-\gamma\overline{G}(k(\mu))) - \mu f'(k(\mu))(1+\gamma\overline{F}(k(\mu))) + \gamma\left[(1-\mu)g^2(k(\mu)) + \mu f^2(k(\mu))\right]}^{I_3(\mu)} \right]' = \\
&\lim_{\mu\to 1} -g'(k(\mu))\left(1-\gamma\overline{G}(k(\mu))\right) + (1-\mu)g''(k(\mu))\left(1-\gamma\overline{G}(k(\mu))\right) + (1-\mu)\gamma g'(k(\mu))\, g(k(\mu)) \\
&\quad -f'(k(\mu))\left(1+\gamma\overline{F}(k(\mu))\right) - \mu f''(k(\mu))\left(1+\gamma\overline{F}(k(\mu))\right) + \mu f'(k(\mu))\,\gamma f(k(\mu)) \\
&\quad -\gamma g^2(k(\mu)) + \gamma(1-\mu)2g(k(\mu))\, g'(k(\mu)) + \gamma f^2(k(\mu)) + 2\gamma\mu f(k(\mu))\, f'(k(\mu)) = \\
&-g'(0)(1-\gamma) + 0g''(0)(1-\gamma) + 0\gamma g'(0)g(0) - f'(0)(1+\gamma) - f''(0)(1+\gamma) + f'(0)\gamma 0 - \gamma g^2(0) \\
&\quad +\gamma 0\, 2g(0)g'(0) + \gamma 0 + 2\gamma 0 f'(0) = \\
&\overbrace{-g'(0)(1-\gamma) - f'(0)(1+\gamma) - f''(0)(1+\gamma) - \gamma g^2(0)}^{I_3{}'(1)},
\end{aligned}$$

where the last equality follows from assuming that $f''(0), g''(0) < \infty$. Given that, as defined above, $k'' = I_1{}'I_2/I_3 + I_1(I_2{}'I_3 - I_2 I_3{}')/I_3{}^2$, and all the terms are finite from the above calculations we have $k''(1) < \infty$.

(ii) Recall from (3) that $R(k;\gamma) = \max_{0\le k'\le k} \widetilde{R}(k';\gamma)$. Let:

$$k^*(k,\gamma) \equiv \arg\max_{0\le k'\le k} \widetilde{R}(k';\gamma).$$

Now, pick $\gamma_1 > \gamma_2$. We want to show that for all k:

$$R(k;\gamma_1) = \widetilde{R}(k^*(k,\gamma_1);\gamma_1) \ge \widetilde{R}(k^*(k,\gamma_2);\gamma_2) = R(k;\gamma_2). \tag{A.3}$$

We claim that:

$$\widetilde{R}(k^*(k,\gamma_1);\gamma_1) \geq \widetilde{R}(k^*(k,\gamma_2);\gamma_1). \tag{A.4}$$

This follows from optimality of $k^*(k,\gamma_1)$ for all $k' < k$ and the definition of $k^*(k,\gamma_1)$, which guarantees that it is also less that k.

From the fact that $\partial\widetilde{R}(k;\gamma)/\partial\gamma > 0$ for all (k,γ) (see (3)) we also have that:

$$\widetilde{R}(k^*(k,\gamma_2);\gamma_1) \geq \widetilde{R}(k^*(k,\gamma_2);\gamma_2). \tag{A.5}$$

Then from (A.4) and (A.5) we have $\widetilde{R}(k^*(k,\gamma_1);\gamma_1) \geq \widetilde{R}(k^*(k,\gamma_2);\gamma_2)$ which establishes (A.3). □

Proof of Proposition 1. Since the state is binary and u_R is upper semicontinuous, Kamenica and Gentzkow (2011, p. 2596) guarantee the existence of an optimal test that for every prior μ_0 induces at most two different posterior beliefs. Let $U_R(\mu_0)$ be the regulator's expected payoff for prior μ_0 under an optimal test. As shown in Kamenica and Gentzkow (2011, p. 2596) $U_R(\mu_0)$ is given by the value of the concave closure of $u_R(\mu)$ at $\mu = \mu_0$.

First, for prior μ_0 to belong in a non-disclosure interval $[\mu_i, \mu_{i+1}]$ it must be that $U_R(\mu_0) = u_R(\mu_0)$. Geometrically, this means there is a supporting hyperplane through the point $(\mu_0, u_R(\mu_0))$ that majorizes u_R. Formally, for every $\mu_0 \in [\mu_i, \mu_{i+1}]$, there is $\gamma(\mu_0)$ such that (9) holds for any $\mu \in [0,1]$. Moreover, if u_R is differentiable for $\mu = \mu_0$, then the supporting line is the tangent at μ_0 and so $\gamma(\mu_0) = u_R'(\mu_0)$.

Second, for prior μ_0 to belong to the disclosure interval (μ_i, μ_{i+1}) it must be that $U_R(\mu_0) > u_R(\mu_0)$. Let the optimal test induce two different posteriors μ_i and μ_{i+1} so that $0 \leq \mu_i < \mu_0 < \mu_{i+1} \leq 1$. Bayesian feasibility requires that $\Pr[\mu_i] + \Pr[\mu_{i+1}] = 1$ and

$$\Pr[\mu_i]\,\mu_i + \Pr[\mu_{i+1}]\,\mu_{i+1} = \mu_0 \Rightarrow \Pr[\mu_{i+1}] = \frac{\mu_0 - \mu_i}{\mu_{i+1} - \mu_i}.$$

The value to the regulator of this test is then:

$$\begin{aligned} U_R(\mu_0) &= \overbrace{\frac{\mu_0 - \mu_i}{\mu_{i+1} - \mu_i}}^{\Pr[\mu_{i+1}]} u_R(\mu_{i+1}) + \overbrace{\frac{\mu_{i+1} - \mu_0}{\mu_{i+1} - \mu_i}}^{\Pr[\mu_i]} u_R(\mu_i) \\ &= u_R(\mu_i) + (\mu_0 - \mu_i)\frac{u_R(\mu_{i+1}) - u_R(\mu_i)}{\mu_{i+1} - \mu_i} \qquad \text{(A.6)} \\ &= u_R(\mu_{i+1}) - (\mu_{i+1} - \mu_0)\frac{u_R(\mu_{i+1}) - u_R(\mu_i)}{\mu_{i+1} - \mu_i}. \qquad \text{(A.7)} \end{aligned}$$

The second and third lines above follow from re-arranging terms in the first line, their use will be clear below. Geometrically, in the disclosure interval case the concave closure of u_R coincides with the line that connects the points $(\mu_i, u_R(\mu_i))$ and $(\mu_{i+1}, u_R(\mu_{i+1})$. To make sure all the values of u_R for the signals in the interval are below this line we must pick the infimum and supremum of the interval as follows. Fix μ_i then from (A.6) we have that in order to maximize the value to the regulator we need to pick μ_{i+1} so that the slope of the line from $(\mu_i, u_R(\mu_i))$ to $(\mu_{i+1}, u_R(\mu_{i+1})$ is maximized. In turn, fix μ_{i+1} then from (A.7) we have that in order to maximize the value to the regulator we need to pick μ_i so that the slope of the line from $(\mu_{i+1}, u_R(\mu_{i+1}))$ to $(\mu_i, u_R(\mu_i))$ is minimized. These statements correspond to (10-11). Note also that for $\mu_i > 0$ and $\mu_{i+1} < 1$ there are $\gamma(\mu_i)$ and $\gamma(\mu_{i+1})$, respectively, for which (9) is satisfied, while for any point $\mu_0 \in (\mu_i, \mu_{i+1})$ there is no such $\gamma(\mu_0)$. In words, the infimum and supremum of a disclosure interval are the last and first point, respectively of a non-disclosure interval. □

Proof of Lemma 2. *(i)* For $\eta < 0$ the regulator wants to induce joint investment only if $\omega = 1$. Assume the optimal test leads to an interior interim belief $\mu \in (0, 1)$. Then investors might jointly invest when $\omega = -1$ or refrain from investing when $\omega = 1$–both are possible if their signals do not perfectly reveal the state. Both these possibilities are eliminated by fully revealing the state, in which case partial disclosure cannot be optimal. [48]

(ii) For $\eta > 0$, we look at the value $R(0) = \widetilde{R}(0) = \lambda(0)(1 + \gamma)/(1 - \gamma)$ to show that for all priors in some neighbourhood of 1 the optimal test leads to no disclosure. Then consider two cases:

Case 1. Let $f(0) > 0$ so that $R(0) > 0$. Then for all priors $\mu_0 \in (1/(1 + R(0)), 1)$ either (i) $k(\mu_0) = 0$[49], or (ii) a test that discloses $1/(1 + R(0)), 1$ leads investors to always invest regardless of the test outcome; and so the regulator can achieve utility $\mu_0 + \eta(1 - \mu_0) > \mu_0$, since $\eta > 0$. Note, that μ_0 is the utility of the regulator at prior μ_0 under a fully informative test; given the inequality we just established such a test cannot be optimal.

Case 2. Let $f(0) = 0$ and assume that $g(0) > 0$ so that $R(0) = 0$. Then, for μ_0 close to 1

$$u_R(\mu_0) \;\approx\; u_R(1) + (\mu_0 - 1)u'_R(1). \tag{A.8}$$

Now, $u_R(1) = 1$ and from (8) the regulator's equilibrium marginal indirect utility is

$$u'_R(\mu) = \overline{F}^2(k) - \eta\overline{G}^2(k) - 2k'(\mu)\left(\mu f(k)\,\overline{F}(k) + \eta(1-\mu)g(k)\,\overline{G}(k)\right)\Big|_{k=k(\mu)}, \tag{A.9}$$

[48] If investors' signals perfectly reveal the state, then investment outcomes are independent on the stress test, so any test (and in particular a fully informative one) is optimal.

[49] This would always be the case for $\mu_0 \in (1/(1 + R(0)), 1)$ if $\min_{k\in[0,1]} \widetilde{R}(k) = \widetilde{R}(0)$–see Lemma 1.

so that

$$u'_R(1) \quad = \quad 1 - \eta - 2k'(1)f(0), \tag{A.10}$$

where we used $k(1) = 0$ (see (6)) and $g(0) < \infty$. Now, from (7), $k'(1) < \infty$ iff $R'(0) > 0$. For $R'(0) > 0$ we need $\widetilde{R}'(0) > 0$ (see (5)). Bearing in mind that $f(0) = 0$, $g(0) > 0$, and $g(0) < \infty$ we have that $\widetilde{R}'(0) = \lambda'(0)(1+\gamma)/(1-\gamma)$; given our assumptions, $\lambda'(0) > 0$ obtains as long as we have $f'(0) > 0$ and $g'(0) < \infty$.

The above imply that in (A.10) we have $u'_R(1) = 1 - \eta$, where we used $k'(1) < \infty$ and $f(0) = 0$; this in turn leads from (A.8) to $u_R(\mu_0) \approx 1 + (\mu_0 - 1)(1 - \eta) = \mu_0 + \eta(1 - \mu_0) > \mu_0$, for $\eta > 0$. Hence, as in case 1, a fully informative test is never optimal. □

Proof of Lemma 3. For $0 \leq \mu \leq 1/(1 + R(1))$ investors never invest–see (6)–and so $u_R(\mu) = 0$. Thus, disclosure must be necessarily optimal for any η–i.e., $D_0 \equiv (0, 1/(1 + R(1)) \subseteq D$ for the set D in Proposition 1. For D_0 to be non-empty we need $1/(1 + R(1)) > 0$ or $R(1) = \lambda(1) < \infty$. □

Proof of Lemma 4. *(i)* From Lemma 2 we know that the regulator would never disclose both $\omega = -1$ and $\omega = 1$ if $\eta > 0$. Moreover, for $R(0) > 0$, the regulator can induce both investors to invest by disclosing $\mu_{CI} < 1$ whenever $\mu_0 < \mu_{CI}$, so she never reveals $\omega = 1$. Note that if $R(0) > 0$, then $\tilde{R}(k) > 0$ for $k \in [0, 1]$,[50] and the regulator can induce both investors to invest by disclosing $1/(1 + \min_{k\in[0,1]} \tilde{R}(k)) < 1$or by non-disclosure if $\mu_0 > \mu_{CI}$, so that, again she never reveals $\omega = 1$. Hence, our focus here will be on the case $R(0) = 0$, which for bounded densities in $[0, 1]$ requires $f(0) = 0$ and $g(0) > 0$. We will show that there is a neighbourhood of 1 which belongs to the non-disclosure set in Proposition 1. This requires that (9) must hold for μ_0 close to 1. We will show that it suffices that $u_R(\mu)$ is locally concave at $\mu = 1$ for (9) to hold, i.e., there exists $\mu_{ND} < 1$ so that for $\mu \in (\mu_{ND}, 1)$ the indirect utility $u_R(\mu)$ is concave.

From (8) we have

$$\begin{aligned} u''_R(\mu) \quad = \quad & 2f^2(k(\mu))\left(k'(\mu)\right)^2\mu - 2\overline{F}(k(\mu))\,f'(k(\mu))\left(k'(\mu)\right)^2\mu - 2\overline{F}(k(\mu))\,f(k(\mu))\,k''(\mu)\mu \\ & -2\overline{F}(k(\mu))\,f(k(\mu))\,k'(\mu) - 2\overline{F}(k(\mu))\,f(k(\mu))\,k'(\mu) + 2\eta g^2(k(\mu))\left(k'(\mu)\right)^2(1-\mu) \\ & -2\eta\overline{G}(k(\mu))\,g'(k(\mu))\left(k'(\mu)\right)^2(1-\mu) - 2\eta\overline{G}(k(\mu))\,g(k(\mu))\,k''(\mu)(1-\mu) \\ & +2\eta\overline{G}(k(\mu))\,g(k(\mu))\,k'(\mu) + 2\eta\overline{G}(k(\mu))\,g(k(\mu))\,k'(\mu). \end{aligned}$$

[50]This follows from (3) as the second term is always strictly positive for $k \in [0, 1]$.

Recall that we are considering the case $R(0) = \widetilde{R}(0) = f(0) = 0$ and that as $\mu \to 1$ we have $k(1) = 0$, $\overline{F}\,(k(1)) = \overline{G}\,(k(1)) = 1$. Therefore, if $k'(1), k''(1), f'(0)$ and $g'(0)$ are bounded, then

$$\lim_{\mu \to 1} u_R''(\mu) \quad = \quad -2f'(0)\left(k'(1)\right)^2 + 2\eta g(0)k'(1).$$

The assumptions we made to write the preceding equation already guarantee that $k'(1)$ is bounded. In Lemma A.1 above we also show that $k''(1) < \infty$. Moreover, since $f(0) = 0$ then $\lambda'(0) > 0$–see Assumption 1–implies that $f'(0) > 0$. Therefore, for $\eta > 0$ it follows that $\lim_{\mu\to 1} u_R''(\mu) < 0$; hence, there exists $\mu_{ND} < 1$ so that $u_R(\mu)$ is concave in $(\mu_{ND}, 1)$.

To complete the proof, we will find a neighborhood of 1 in which (9) holds with $\gamma_0 = u_R'(\mu_0)$, so that non-disclosure is optimal for all priors in this neighborhood. Using expression (A.9) in the proof of Lemma 2 we have:

$$\lim_{\mu \to 1} u_R'(\mu) = 1 - \eta < 1.$$

Using this and the fact that $\overline{F}^2, \overline{G}^2 < 1$ for $\mu < 1$ we have

$$u_R(1) + u_R'(1)(\mu - 1) = 1 + (1 - \eta)(\mu - 1) = \mu + \eta(1 - \mu) > u_R(\mu).$$

Now, let $\Delta = \max_{\mu \le \mu_{ND}} \mu + \eta(1 - \mu) - u_R(\mu) > 0$. Then, if u_R is continuously differentiable in $\mu < 1$, then through a continuity argument we can find μ_{ND}^* so that for $\mu_0 > \mu_{ND}^*$

$$\max_{\mu \le \mu_{ND}^*} \left| u_R(1) + u_R'(1)(\mu - 1) - \left(u_R(\mu_0) + u_R'(\mu_0)(\mu - \mu_0)\right)\right| < \Delta.$$

So that for any $\mu_0 > \mu_{ND}^*$, the triangle inequality implies.

$$\max_{\mu \le \mu_{ND}^*} \left| u_R(\mu_0) + u_R'(\mu_0)(\mu - \mu_0) - u_R(\mu)\right| > 0.$$

In summary, for any $\mu_0 \in (\min\{\mu_{ND}, \mu_{ND}^*\}, 1)$, we have that (9) holds and non-disclosure is optimal.

(ii) Now, suppose that $R(0) > 0$ so that $\mu_{CI} < 1$. From the proof of Lemma 1.ii we have that, whenever $k(\mu)$ is differentiable, $k'(\mu) = -1/\mu^2 R'(k(\mu))$. Given the above and using expression (A.9) in the proof of Lemma 2 we have:

$$\lim_{\mu \to \mu_{CI}} u_R'(\mu) = 1 - \eta + \frac{2f(0)}{\mu_{CI}^2 R'(0)}\left(1 + \eta\frac{1+\gamma}{1-\gamma}\right),$$

where we have also used $R(0) = f(0)(1+\gamma)/\left(g(0)(1-\gamma)\right)$ and $\mu_{CI} = 1/(1+R(0))$. Note that the regulator's expected utility from a (non-necessarily optimal) test that discloses $\{0,\mu_{CI}\}$ is $\Pr[s = \mu_{CI}]u_R(\mu_{CI}) = (\mu_0/\mu_{CI})(\mu_{CI} + \eta(1-\mu_{CI}))$ while non-disclosure would generate $u_R(\mu_0) \approx u_R(\mu_{CI}) + (\mu_0 - \mu_{CI})\, u_R'(\mu_{CI})$. A sufficient condition for disclosure to be optimal is that $\Pr[s = \mu_{CI}]u_R(\mu_{CI}) > u_R(\mu_0)$, or in other words $u_R'(\mu_{CI}) > u_R(\mu_{CI})/\mu_{CI}$. Using the expression of $u_R'(\mu_{CI})$, disclosure is optimal if

$$1 - \eta + \frac{2f(0)}{\mu_{CI}^2 R'(0)}\left(1 + \eta\frac{1+\gamma}{1-\gamma}\right) > 1 + \eta R(0),$$

which implies (17). □

Lemma A.2. *(Comp-Statics CF-test). Consider the parametrized functions $R(k;z)$ and*

$$A_R(k;z) \equiv \overline{F}^2(k;z) + \eta\overline{G}^2(k;z)R(k;z). \tag{A.11}$$

The optimal CF-test $\{0, \underline{\mu}(z)\}$ satisfies $d\underline{\mu}/dz > 0$ if, and only if, for $\underline{k} \equiv k(\underline{\mu}(z);z)$

$$\underbrace{-\frac{\partial^2 A_R(k;z)/\partial k^2}{\partial R(k;z)/\partial k}\bigg|_{k=\underline{k}} \frac{\partial R(k;z)}{\partial z}\bigg|_{k=\underline{k}}}_{\text{indirect effect}} + \underbrace{\frac{\partial^2 A_R(k;z)}{\partial k \partial z}\bigg|_{k=\underline{k}}}_{\text{direct effect}} < 0. \tag{A.12}$$

In particular, if $\partial R(\underline{k};z)/\partial z < 0$ and $\partial^2 A_R(\underline{k};z)/\partial k\partial z < 0$, then $d\underline{\mu}/dz > 0$; while if $\partial R(\underline{k};z)/\partial z > 0$ and $\partial^2 A_R(\underline{k};z)/\partial k\partial z > 0$, then $d\underline{\mu}/dz < 0$.

Proof. From (13), the critical-fault test outcome $\underline{\mu}(z)$ satisfies:

$$\underline{\mu}(z) \in \arg\max \frac{u_R(\mu;z)}{\mu} = A_R(k(\mu;z);z),$$

with $k(\mu;z)$ implicitly defined by (6). From the First Order Condition (FOC) we have

$$\frac{\partial A_R(k;z)}{\partial k}\bigg|_{k=\underline{k}} \frac{\partial k(\mu;z)}{\partial \mu}\bigg|_{\mu=\underline{\mu}} = 0,$$

where $\underline{k} = k(\underline{\mu}(z);z)$. Because $\partial k(\mu;z)/\partial\mu|_{\mu=\underline{\mu}} < 0$ (see (7)) the FOC implies that

$$\frac{\partial A_R(k;z)}{\partial k}\bigg|_{k=\underline{k}} = 0. \tag{A.13}$$

Moreover, from the Second Order Condition (SOC) we have:

$$\left.\frac{\partial^2 A_R(k;z)}{\partial k^2}\right|_{k=\underline{k}} \left(\left.\frac{\partial k(\mu;z)}{\partial \mu}\right|_{\mu=\underline{\mu}}\right)^2 + \left.\frac{\partial A_R(k;z)}{\partial k}\right|_{k=\underline{k}} \left.\frac{\partial^2 k(\mu;z)}{\partial \mu^2}\right|_{\mu=\underline{\mu}} < 0,$$

which, given (A.13), is equivalent to $\partial^2 A_R(\underline{k};z)/\partial k^2 < 0$.
Applying the implicit function theorem to (A.13), we have that

$$\frac{d\underline{\mu}}{dz} = -\frac{\left.\frac{\partial^2 A_R(k;z)}{\partial k^2}\right|_{k=\underline{k}} \left.\frac{\partial k(\mu;z)}{\partial z}\right|_{\mu=\underline{\mu}} + \left.\frac{\partial^2 A_R(k;z)}{\partial k \partial z}\right|_{k=\underline{k}}}{\left.\frac{\partial^2 A_R(k;z)}{\partial k^2}\right|_{k=\underline{k}} \left.\frac{\partial k(\mu;z)}{\partial \mu}\right|_{\mu=\underline{\mu}}}. \tag{A.14}$$

The SOC and (7) imply that the denominator of (A.14) is positive. Therefore, $d\underline{\mu}/dz > 0$ if, and only if, the numerator of (A.14) is negative. Implicitly differentiating the equilibrium condition $R(k;z) = (1-\mu)/\mu$ to obtain

$$\frac{\partial k(\mu;z)}{\partial z} = -\left.\frac{\partial R(k,z)/\partial z}{\partial R(k,z)/\partial k}\right|_{k=k(\mu;z)},$$

and replacing this expression in (A.14) implies that the numerator is negative if, and only if, (A.12) holds. Finally, by definition (see (3)) we have $\partial R(k;z)/\partial k > 0$ for all (k,z). Therefore, $-(\partial^2 A_R(\underline{k};z)/\partial k^2)/(\partial R(\underline{k};z)/\partial k) > 0$. Thus, if $\partial R(\underline{k};z)/\partial z < 0$ and $\partial^2 A_R(\underline{k};z)/\partial k \partial z < 0$, then (A.12) holds, implying that $d\underline{\mu}/dz > 0$. If $\partial R(\underline{k};z)/\partial z > 0$ and $\partial^2 A_R(\underline{k};z)/\partial k \partial z > 0$, then the numerator in (A.14) is positive, implying that $d\underline{\mu}/dz < 0$. □

Proof of Proposition 2. **CF-test.** We will apply Lemma A.2 to

$$A_R(k;\gamma) \equiv \overline{F}^2(k) + \eta \overline{G}^2(k) R(k;\gamma), \tag{A.15}$$

where we made the dependence of γ in (12) explicit. First, Lemma A.1.ii above shows that the fact that $\partial \widetilde{R}(k;\gamma)/\partial\gamma > 0$ also implies that $\partial R(k;\gamma)/\partial \gamma > 0$ for all (k,γ). Second, we have:

$$\frac{\partial A_R(k;\gamma)}{\partial k} = -2f(k)\overline{F}(k) - 2\eta g(k)\overline{G}(k)R(k;\gamma) + \eta\overline{G}^2(k)R_k(k;\gamma), \text{ and}$$
$$\frac{\partial^2 A_R(k;\gamma)}{\partial k \partial \gamma} = -2\eta g(k)\overline{G}(k)R_\gamma(k;\gamma) + \eta\overline{G}^2 R_{k\gamma}(k;\gamma) = \eta\overline{G}(k)\left(-2g(k)R_\gamma(k;\gamma) + \overline{G}(k)R_{k\gamma}(k;\gamma)\right),$$

where R_k, $R_{k\gamma}$ are the partial derivatives of R with respect to k and (k, γ), respectively. So, for $\eta > 0$:

$$\frac{\partial^2 A_R(k;\gamma)}{\partial k \partial \gamma} > 0 \quad \Longleftrightarrow \quad -2g(k)R_\gamma(k;\gamma) + \overline{G}(k)R_{k\gamma}(k;\gamma) > 0 \Longleftrightarrow h_G(k) < \frac{1}{2}\frac{R_{k\gamma}(k;\gamma)}{R_\gamma(k;\gamma)}, \tag{A.16}$$

where we also used the fact that $R_\gamma > 0$ (see Lemma A.1.ii) and the definition of h_G . Lemma A.2 then guarantees that (A.16) applied to $k = k(\underline{\mu}, \gamma)$ implies that $d\underline{\mu}/d\gamma < 0$.

CI-test. Define

$$\overline{A}_R(\mu, k; \gamma) \equiv \frac{\mu}{\mu_{CI}(\gamma) - \mu}\left(1 - \overline{F}^2(k)\right) + \eta \frac{1-\mu}{\mu_{CI}(\gamma) - \mu}\left(1 - \overline{G}^2(k)\right), \tag{A.17}$$

where $\mu_{CI}(\gamma) = 1/(1 + R(0))$ and from (3) $R(0) = \widetilde{R}(0) = \lambda(0)(1+\gamma)/(1-\gamma)$. Note that $\overline{A}_R$ depends directly on μ and k; this is in contrast to A_R in (A.15), which depended only on k and not on μ directly. The coordinated-investment test outcome $\overline{\mu}$ satisfies (15), and using (8), we have:

$$\overline{\mu} \in \arg \min_{\mu \in (0, \mu_{CI})} \frac{u_R(\mu_{CI}(\gamma)) - u_R(\mu)}{\mu_{CI}(\gamma) - \mu} = \overline{A}_R(\mu, k(\mu;\gamma);\gamma).$$

And the FOC for optimality is:

$$\left.\frac{\partial \overline{A}_R(\mu, k;\gamma)}{\partial \mu}\right|_{\mu=\overline{\mu},\, k=k(\overline{\mu};\gamma)} + \left.\frac{\partial \overline{A}_R(\mu, k;\gamma)}{\partial k}\right|_{\mu=\overline{\mu},\, k=k(\overline{\mu};\gamma)} \left.\frac{\partial k(\mu;\gamma)}{\partial \mu}\right|_{\mu=\overline{\mu}} = 0. \tag{A.18}$$

While the SOC is (bearing in mind that we are looking for a minimum):

$$\begin{aligned} &\left.\frac{\partial^2 \overline{A}_R(\mu, k;\gamma)}{\partial \mu^2}\right|_{\mu=\overline{\mu},\, k=k(\overline{\mu};\gamma)} + 2 \left.\frac{\partial^2 \overline{A}_R(\mu, k;\gamma)}{\partial k \partial \mu}\right|_{\mu=\overline{\mu},\, k=k(\overline{\mu};\gamma)} \left.\frac{\partial k(\mu;\gamma)}{\partial \mu}\right|_{\mu=\overline{\mu}} \\ + &\left.\frac{\partial^2 \overline{A}_R(\mu, k;\gamma)}{\partial k^2}\right|_{\mu=\overline{\mu},\, k=k(\overline{\mu};\gamma)} \left(\left.\frac{\partial k(\mu;\gamma)}{\partial \mu}\right|_{\mu=\overline{\mu}}\right)^2 + \left.\frac{\partial \overline{A}_R(\mu, k;\gamma)}{\partial k}\right|_{\mu=\overline{\mu},\, k=k(\overline{\mu};\gamma)} \left.\frac{\partial^2 k(\mu;\gamma)}{\partial \mu^2}\right|_{\mu=\overline{\mu}} > 0. \end{aligned} \tag{A.19}$$

To reduce the notational burden for the rest of this proof we adopt the following conventions: we write $k_\mu(\overline{\mu};\gamma)$ for $\partial k(\mu;\gamma)/\partial \mu|_{\mu=\overline{\mu}}$ and letting $\overline{k} \equiv k(\overline{\mu};\gamma)$, we write $A_\mu(\overline{\mu}, \overline{k};\gamma)$ for $\partial \overline{A}_R(\mu, k;\gamma)/\partial \mu|_{\mu=\overline{\mu},\, k=k(\overline{\mu};\gamma)}$. Under this convention, the FOC in (A.18) becomes:

$$A_\mu(\overline{\mu}, \overline{k};\gamma) + A_k(\overline{\mu}, \overline{k};\gamma)k_\mu(\overline{\mu};\gamma) = 0,$$

and the SOC in (A.19) becomes:

$$A_{\mu\mu}(\overline{\mu},\overline{k};\gamma)+2A_{\mu k}(\overline{\mu},\overline{k};\gamma)k_{\mu}(\overline{\mu};\gamma)+A_{kk}(\overline{\mu},\overline{k};\gamma)k_{\mu}^{2}(\overline{\mu};\gamma)+A_{k}(\overline{\mu},\overline{k};\gamma)k_{\mu\mu}(\overline{\mu};\gamma)>0.$$

To figure the sensitivity of $\overline{\mu}$ to γ we differentiate the FOC (A.18):

$$\begin{aligned}
& A_{\mu\mu}(\overline{\mu},\overline{k};\gamma)\frac{d\overline{\mu}}{d\gamma}+A_{\mu k}(\overline{\mu},\overline{k};\gamma)\left(k_{\mu}(\overline{\mu};\gamma)\frac{d\overline{\mu}}{d\gamma}+k_{\gamma}(\overline{\mu};\gamma)\right)+A_{\mu\gamma}(\overline{\mu},\overline{k};\gamma)\\
+ & A_{\mu k}(\overline{\mu},\overline{k};\gamma)\frac{d\overline{\mu}}{d\gamma}k_{\mu}(\overline{\mu};\gamma)+A_{kk}(\overline{\mu},\overline{k};\gamma)\left(k_{\mu}(\overline{\mu};\gamma)\frac{d\overline{\mu}}{d\gamma}+k_{\gamma}(\overline{\mu};\gamma)\right)k_{\mu}(\overline{\mu};\gamma)\\
+ & A_{k\gamma}(\overline{\mu},\overline{k};\gamma)k_{\mu}(\overline{\mu};\gamma)+A_{k}(\overline{\mu},\overline{k};\gamma)\left(k_{\mu\mu}(\overline{\mu};\gamma)\frac{d\overline{\mu}}{d\gamma}+k_{\mu\gamma}(\overline{\mu};\gamma)\right)=0,
\end{aligned}$$

and solving for $d\overline{\mu}/d\gamma$ we get:

$$\frac{d\overline{\mu}}{d\gamma} = -\frac{\begin{array}{c}A_{\mu\gamma}(\overline{\mu},\overline{k};\gamma)+A_{k\gamma}(\overline{\mu},\overline{k};\gamma)k_{\mu}(\overline{\mu};\gamma)\\ +\; A_{\mu k}(\overline{\mu},\overline{k};\gamma)k_{\gamma}(\overline{\mu};\gamma)+A_{kk}(\overline{\mu},\overline{k};\gamma)k_{\gamma}(\overline{\mu};\gamma)k_{\mu}(\overline{\mu};\gamma)+A_{k}(\overline{\mu},\overline{k};\gamma)k_{\mu\gamma}(\overline{\mu};\gamma)\end{array}}{A_{\mu\mu}(\overline{\mu},\overline{k};\gamma)+2A_{\mu k}(\overline{\mu},\overline{k};\gamma)k_{\mu}(\overline{\mu};\gamma)+A_{kk}(\overline{\mu},\overline{k};\gamma)k_{\mu}^{2}(\overline{\mu};\gamma)+A_{k}(\overline{\mu},\overline{k};\gamma)k_{\mu\mu}(\overline{\mu};\gamma)}. \tag{A.20}$$

Note that the denominator in (A.20) is the left-hand-side of the SOC in (A.19) and hence it is positive. Therefore, from (A.20) $d\overline{\mu}/d\gamma>0$ iff the numerator is negative.

Below we compute in closed form all the terms in the numerator of (A.20) to determine their sign. From the expression for $\overline{A}_{R}(\mu,k;\gamma)$ in (A.17) (recalling our notational convention) we have in turn:

$$\begin{aligned}
A_{\mu}(\mu,k;\gamma) &= \frac{\mu_{CI}(\gamma)}{(\mu_{CI}(\gamma)-\mu)^{2}}(1-\overline{F}(k)^{2})+\eta\frac{1-\mu_{CI}(\gamma)}{(\mu_{CI}(\gamma)-\mu)^{2}}(1-\overline{G}(k)^{2}),\\
A_{k}(\mu,k;\gamma) &= \frac{\mu}{\mu_{CI}(\gamma)-\mu}2\overline{F}(k)f(k)+\eta\frac{1-\mu}{\mu_{CI}(\gamma)-\mu}2\overline{G}(k)g(k),
\end{aligned} \tag{A.21}$$

so that:

$$A_{\mu k}(\mu,k;\gamma) = \frac{\mu_{CI}(\gamma)}{(\mu_{CI}(\gamma)-\mu)^{2}}2\overline{F}(k)f(k)+\eta\frac{1-\mu_{CI}(\gamma)}{(\mu_{CI}(\gamma)-\mu)^{2}}2\overline{G}(k)g(k), \tag{A.22}$$

$$A_{kk}(\mu,k;\gamma) = \frac{\mu}{\mu_{CI}(\gamma)-\mu}2\left(-f^{2}(k)+\overline{F}(k)f'(k)\right)+\eta\frac{1-\mu}{\mu_{CI}(\gamma)-\mu}2\left(-g^{2}(k)+\overline{G}(k)g'(k)\right). \tag{A.23}$$

Note that the right-hand-side (RHS) of (A.22) is positive for all (μ,k,γ), while that of (A.23) is indeterminate and depends on the sign of the term in the two parentheses (recall that

$\mu < \mu_{CI}$). Moreover,

$$A_{\mu\gamma}(\mu,k;\gamma) = \Big(-\frac{\mu_{CI}(\gamma)+\mu}{(\mu_{CI}(\gamma)-\mu)^3}(1-\overline{F}(k)^2) - \eta\frac{2-\mu_{CI}(\gamma)-\mu}{(\mu_{CI}(\gamma)-\mu)^3}(1-\overline{G}(k)^2)\Big)\mu'_{CI}(\gamma), \tag{A.24}$$

$$A_{k\gamma}(\mu,k;\gamma) = \Big(-\frac{\mu}{(\mu_{CI}(\gamma)-\mu)^2}2\overline{F}(k)f(k) - \eta\frac{1-\mu}{(\mu_{CI}(\gamma)-\mu)^2}2\overline{G}(k)g(k)\Big)\mu'_{CI}(\gamma), \tag{A.25}$$

where from the definition of $\mu_{CI} = 1/(1+R(0))$ and (3) we have:

$$\mu'_{CI}(\gamma) = -\frac{2\lambda(0)}{(1+\lambda(0)+\gamma(\lambda(0)-1))^2}. \tag{A.26}$$

Since, clearly $\mu'_{CI}(\gamma) < 0$ we have that the RHSs of both (A.24) and (A.25) are positive for all (μ, k, γ).

We will now proceed to calculate k_γ, k_μ and $k_{\mu\gamma}$. From (6) for the range of μ we are considering we have:

$$k(\mu;\gamma) = R^{-1}((1-\mu)/\mu;\gamma) \Rightarrow R(k(\mu;\gamma);\gamma) - (1-\mu)/\mu = 0. \tag{A.27}$$

Hence from (A.27) we have:

$$\frac{\partial R(k;\gamma)}{\partial k}\Big|_{k=k(\mu,\gamma)}\frac{\partial k(\mu;\gamma)}{\partial\gamma} + \frac{\partial R(k;\gamma)}{\partial\gamma}\Big|_{k=k(\mu,\gamma)} = 0 \Rightarrow k_\gamma(\mu;\gamma) = -\frac{R_\gamma(k(\mu;\gamma);\gamma)}{R_k(k(\mu;\gamma);\gamma)}, \tag{A.28}$$

where in the last equality we condensed notation using the conventions we introduced also above. Now, by definition R_k is positive (see (5)) and from Lemma A.1.ii is also positive, both for all (k,γ). Hence, the RHS of the last expression in (A.28) is negative. Again, from (A.27) we have:

$$\frac{\partial R(k;\gamma)}{\partial k}\Big|_{k=k(\mu,\gamma)}\frac{\partial k(\mu;\gamma)}{\partial\mu} + 1/\mu^2 = 0 \Rightarrow k_\mu(\mu;\gamma) = -\frac{1}{\mu^2 R_k(k(\mu;\gamma);\gamma)}, \tag{A.29}$$

so the RHS of the last expression in (A.29) is also negative. And using the first expression in (A.29) we have:

$$\frac{\partial^2 R(k;\gamma)}{\partial k^2}\Big|_{k=k(\mu,\gamma)}\frac{\partial k(\mu;\gamma)}{\partial\mu}\frac{\partial k(\mu;\gamma)}{\partial\gamma} + \frac{\partial R(k;\gamma)}{\partial k}\Big|_{k=k(\mu,\gamma)}\frac{\partial^2 k(\mu;\gamma)}{\partial\mu\partial\gamma} + \frac{\partial^2 R(k;\gamma)}{\partial k\partial\gamma}\Big|_{k=k(\mu,\gamma)}\frac{\partial k(\mu;\gamma)}{\partial\mu} = 0,$$

$$\Rightarrow \quad k_{\mu\gamma}(\mu;\gamma) = -k_\mu(\mu;\gamma)\frac{R_{kk}(k(\mu;\gamma);\gamma)k_\gamma(\mu;\gamma) + R_{k\gamma}(k(\mu;\gamma);\gamma)}{R_k(k(\mu;\gamma);\gamma)}. \tag{A.30}$$

We have that k_μ, k_γ are negative for all (μ, γ) and R_k is positive for all (k, γ). But we don't know the sign of R_{kk} and $R_{k\gamma}$ in general. So the sign of the RHS of (A.30) is indeterminate.

Recall, that the calculations above served the purpose of figuring out the sign of the numerator in (A.20), which determines the sign of $d\overline{\mu}/d\gamma$. Below we write this numerator with the signs we unravelled above and the corresponding equations:

$$\begin{aligned} &\overbrace{A_{\mu\gamma}(\overline{\mu}, \overline{k}; \gamma)}^{+\text{ve (A.24)}} + \overbrace{A_{k\gamma}(\overline{\mu}, \overline{k}; \gamma)}^{+\text{ve (A.25)}} \overbrace{k_\mu(\overline{\mu}; \gamma)}^{-\text{ve (A.29)}} \\ + \quad &\underbrace{A_{\mu k}(\overline{\mu}, \overline{k}; \gamma)}_{+\text{ve (A.22)}} \underbrace{k_\gamma(\overline{\mu}; \gamma)}_{-\text{ve (A.28)}} + \underbrace{A_{kk}(\overline{\mu}, \overline{k}; \gamma)}_{?\text{ (A.23)}} \underbrace{k_\gamma(\overline{\mu}; \gamma)}_{-\text{ve (A.28)}} \underbrace{k_\mu(\overline{\mu}; \gamma)}_{-\text{ve (A.29)}} + \underbrace{A_k(\overline{\mu}, \overline{k}; \gamma)}_{+\text{ve (A.21)}} \underbrace{k_{\mu\gamma}(\overline{\mu}; \gamma)}_{?\text{ (A.30)}} \end{aligned} \tag{A.31}$$

Now, if the expression in (A.31) is negative then the coordinated investment test outcome $\overline{\mu}$ increases with γ. Coupled with the observation we made above that μ_{CI} decreases with γ this means that if (A.31) is negative then an increase in coordination γ will make the coordinated investment test less informative. Below, we derive sufficient conditions so that (A.31) is negative. We have in turn:

1. Pick the first and third term in (A.31): $A_{\mu\gamma}(\mu, k; \gamma) + A_{\mu k}(\mu, k; \gamma) k_\gamma(\mu; \gamma)$. From (A.24), (A.22), and (A.28) we have:

$$\begin{aligned} &\big(-\frac{\mu_{CI}(\gamma) + \mu}{(\mu_{CI}(\gamma) - \mu)^3}(1 - \overline{F}(k)^2) - \eta \frac{2 - \mu_{CI}(\gamma) - \mu}{(\mu_{CI}(\gamma) - \mu)^3}(1 - \overline{G}(k)^2)\big)\mu'_{CI}(\gamma) \\ + \quad &\big(\frac{\mu_{CI}(\gamma)}{(\mu_{CI}(\gamma) - \mu)^2} 2\overline{F}(k)f(k) + \eta \frac{1 - \mu_{CI}(\gamma)}{(\mu_{CI}(\gamma) - \mu)^2} 2\overline{G}(k)g(k)\big) k_\gamma(\mu; \gamma). \end{aligned} \tag{A.32}$$

Now, to proceed we assume that $1 - \overline{F}(k)^2 < 2\overline{F}(k)f(k)$ and $1 - \overline{G}(k)^2 < 2\overline{G}(k)f(k)$ or

$$\big[\ln(1 - \overline{F}(k)^2)\big]' > 1, \ \big[\ln(1 - \overline{G}(k)^2)\big]' > 1. \tag{A.33}$$

Given (A.33) we have that (A.32) is less than:

$$\begin{aligned} &\big(\frac{\mu_{CI}(\gamma)}{(\mu_{CI}(\gamma) - \mu)^2} k_\gamma(\mu; \gamma) - \frac{\mu_{CI}(\gamma) + \mu}{(\mu_{CI}(\gamma) - \mu)^3} \mu'_{CI}(\gamma)\big) 2\overline{F}(k)f(k) \\ + \quad &\big(\frac{1 - \mu_{CI}(\gamma)}{(\mu_{CI}(\gamma) - \mu)^2} k_\gamma(\mu; \gamma) - \frac{2 - \mu_{CI}(\gamma) - \mu}{(\mu_{CI}(\gamma) - \mu)^3} \mu'_{CI}(\gamma)\big) 2\eta\overline{G}(k)g(k). \end{aligned} \tag{A.34}$$

We would like (A.34) to be negative, for this it is sufficient that:

$$\frac{\mu_{CI}(\gamma)}{(\mu_{CI}(\gamma)-\mu)^2}k_\gamma(\mu;\gamma) - \frac{\mu_{CI}(\gamma)+\mu}{(\mu_{CI}(\gamma)-\mu)^3}\mu'_{CI}(\gamma) \quad < \quad 0, \text{ and} \tag{A.35}$$

$$\frac{1-\mu_{CI}(\gamma)}{(\mu_{CI}(\gamma)-\mu)^2}k_\gamma(\mu;\gamma) - \frac{2-\mu_{CI}(\gamma)-\mu}{(\mu_{CI}(\gamma)-\mu)^3}\mu'_{CI}(\gamma) \quad < \quad 0. \tag{A.36}$$

After a bit of algebra also letting $\mu'_{CI}(\gamma)$ in (A.26) be $-c$, and recalling that $k_\gamma = -R_\gamma/R_k$ from (A.28) we have that sufficient conditions for (A.35) and (A.36) are

$$\frac{R_\gamma(k;\gamma)}{R_k(k;\gamma)} > \frac{2c}{\mu_{CI}-\mu}, \; \frac{R_\gamma(k;\gamma)}{R_k(k;\gamma)} > \frac{1-\mu}{1-\mu_{CI}}\frac{2c}{\mu_{CI}-\mu},$$

which can be combined in

$$\frac{R_\gamma(k;\gamma)}{R_k(k;\gamma)} \quad > \quad \frac{1-\mu}{1-\mu_{CI}}\frac{2c}{\mu_{CI}-\mu}. \tag{A.37}$$

2. Now, pick the second and fifth term in (A.31): $A_{k\gamma}(\mu,k;\gamma)k_\mu(\mu;\gamma)+A_k(\mu,k;\gamma)k_{\mu\gamma}(\mu;\gamma)$. From (A.25), (A.29), (A.21), and (A.30) we have:

$$\begin{aligned}&\big(-\frac{\mu}{(\mu_{CI}(\gamma)-\mu)^2}2\overline{F}(k)f(k) - \eta\frac{1-\mu}{(\mu_{CI}(\gamma)-\mu)^2}2\overline{G}(k)g(k)\big)\mu'_{CI}(\gamma)k_\mu(\mu;\gamma)\\ + \quad &\big(\frac{\mu}{\mu_{CI}(\gamma)-\mu}2\overline{F}(k)f(k) + \eta\frac{1-\mu}{\mu_{CI}(\gamma)-\mu}2\overline{G}(k)g(k)\big)k_{\mu\gamma}(\mu;\gamma).\end{aligned} \tag{A.38}$$

Note that:

$$\frac{\mu}{(\mu_{CI}(\gamma)-\mu)^2} > \frac{\mu}{\mu_{CI}(\gamma)-\mu} \text{ and } \frac{1-\mu}{(\mu_{CI}(\gamma)-\mu)^2} > \frac{1-\mu}{\mu_{CI}(\gamma)-\mu}$$

so that (A.38) is smaller than:

$$\begin{aligned}&\frac{\mu}{\mu_{CI}(\gamma)-\mu}2\overline{F}(k)f(k)\big(k_{\mu\gamma}(\mu;\gamma) - \mu'_{CI}(\gamma)k_\mu(\mu;\gamma)\big)\\ + \quad &\eta\frac{1-\mu}{\mu_{CI}(\gamma)-\mu}2\overline{G}(k)g(k)\big)\big(k_{\mu\gamma}(\mu;\gamma) - \mu'_{CI}(\gamma)k_\mu(\mu;\gamma)\big).\end{aligned} \tag{A.39}$$

Hence, for the above to be negative it suffices that:

$$
\begin{aligned}
& k_{\mu\gamma}(\mu;\gamma) < \mu'_{CI}(\gamma) k_\mu(\mu;\gamma) && \text{(A.40)}\\
\iff \quad & \frac{k_{\mu\gamma}(\mu;\gamma)}{k_\mu(\mu;\gamma)} > \mu'_{CI}(\gamma) \\
\iff \quad & -\frac{R_{kk}(k(\mu;\gamma);\gamma) k_\gamma(\mu;\gamma) + R_{k\gamma}(k(\mu;\gamma);\gamma)}{R_k(k(\mu;\gamma);\gamma)} > \mu'_{CI}(\gamma) \\
\iff \quad & -\frac{-R_{kk}(k(\mu;\gamma);\gamma)\dfrac{R_\gamma(k(\mu;\gamma);\gamma)}{R_k(k(\mu;\gamma);\gamma)} + R_{k\gamma}(k(\mu;\gamma);\gamma)}{R_k(k(\mu;\gamma);\gamma)} > \mu'_{CI}(\gamma) && \text{(A.41)}\\
\iff \quad & \frac{R_{kk}(k(\mu;\gamma);\gamma) R_\gamma(k(\mu;\gamma);\gamma) - R_{k\gamma}(k(\mu;\gamma);\gamma)}{R_k(k(\mu;\gamma);\gamma)^2} > \mu'_{CI}(\gamma) \\
\iff \quad & \frac{R_{kk}(k(\mu;\gamma);\gamma) R_\gamma(k(\mu;\gamma);\gamma) - R_{k\gamma}(k(\mu;\gamma);\gamma) R_k(k(\mu;\gamma);\gamma)}{R_k(k(\mu;\gamma);\gamma)^2} > -c \\
\iff \quad & \frac{R_{k\gamma}(k(\mu;\gamma);\gamma) R_k(k(\mu;\gamma);\gamma) - R_{kk}(k(\mu;\gamma);\gamma) R_\gamma(k(\mu;\gamma);\gamma)}{R_k(k(\mu;\gamma);\gamma)^2} < c \\
\iff \quad & \left. \frac{\partial}{\partial k} \frac{R_\gamma(k;\gamma)}{R_k(k;\gamma)} \right|_{k=k(\mu;\gamma)} < c, && \text{(A.42)}
\end{aligned}
$$

where the second line follows for $k_\mu < 0$ (see (A.29)), in the third line we substitute for $k_{\mu\gamma}$ (see (A.30)), in the fourth line we substitute for k_γ (see (A.28)), in the fifth line we multiply the LHS by $R_k > 0$ (see (5)), in the sixth line we substitute $\mu'_{CI} = -c$ (where we define as c minus the RHS of (A.26)), and in the last line we observe that the LHS is the derivative of R_γ/R_k with respect to k, computed at $k = k(\mu;\gamma)$.

3. Finally, we have term $A_{kk}(\mu, k;\gamma) k_\gamma(\mu;\gamma) k_\mu(\mu;\gamma)$ in (A.31). For this to also be negative (given that $k_\gamma < 0$, see (A.28) and $k_\mu < 0$, see (A.29)) we need A_{kk} to be negative. From (A.23) we have that a sufficient condition for this is that

$$-f^2(k) + \overline{F}(k) f'(k) < 0 \text{ and } -g^2(k) + \overline{G}(k) g'(k)$$

or

$$\left(\overline{F}(k)^2\right)'' < 0 \text{ and } \left(\overline{G}(k)^2\right)'' < 0. \qquad \text{(A.43)}$$

Therefore, given (A.20) a sufficient set of conditions for $d\overline{\mu}/d\gamma > 0$ is (A.33), (A.37), (A.42), and (A.43) computed at the optimal $\mu = \overline{\mu}$ and $k = \overline{k}$, as given in Lemma 2. □

Proof of Proposition 3. *(i)* We show that the likelihood ratio $\overline{\Lambda}(k;\alpha) = \overline{F}(k;\alpha)/\overline{G}(k;\alpha)$ must increase in α if a Blackwell-more informative signal decreases the probability of investing for a fixed threshold when $\omega = -1$. This guarantees a positive direct effect in (22) as (23) increases with $\overline{\Lambda}(k;\alpha)$.

Define $k_I(t;\alpha)$ implicitly by $\overline{G}(k_I(t;\alpha),\alpha) = t$, so that $k_I(t;\alpha)$ would be the lower bound of the rejection region of a UMP test of level t that tests the null $H_0 : \omega = -1$, against the alternative $H_1 : \omega = 1$. Whenever differentiable, we have $\partial k_I/\partial\alpha = \left(\partial\overline{G}(k_I;\alpha)/\partial\alpha\right)/g(k_I;\alpha)$. A Blackwell-more informative signal leads to a higher power test of the same level (Blackwell and Girshick (1954)), i.e.,

$$\overline{F}(k_I(t;\alpha');\alpha') \geq \overline{F}(k_I(t;\alpha);\alpha), \tag{A.44}$$

so that the function $\overline{F}(\overline{G}^{-1}(t;\alpha);\alpha)$ must be non-decreasing in α (see also Lehman 1988). Differentiating $\overline{F}(k_I(t;\alpha);\alpha)$ we must have

$$-f(k_I;\alpha)\frac{\partial k_I}{\partial\alpha} + \frac{\partial\overline{F}}{\partial\alpha} = -\frac{f(k_I;\alpha)}{g(k_I;\alpha)}\frac{\partial\overline{G}}{\partial\alpha} + \frac{\partial\overline{F}}{\partial\alpha} \geq 0. \tag{A.45}$$

Note that $\overline{\Lambda}(k;\alpha)$ increases in α iff

$$\frac{\partial\overline{F}}{\partial\alpha} \geq \frac{\overline{F}(k,\alpha)}{\overline{G}(k,\alpha)}\frac{\partial\overline{G}}{\partial\alpha}. \tag{A.46}$$

and MLRP guarantees $\overline{F}(k,\alpha)/\overline{G}(k,\alpha) \geq f(k,\alpha)/g(k,\alpha)$. If $\partial\overline{G}(k,\alpha)/\partial\alpha\big|_{k=k(\underline{\mu}(\alpha),\alpha)} < 0$, (A.45) implies

$$\frac{\partial\overline{F}}{\partial\alpha} \geq \frac{f(k,\alpha)}{g(k,\alpha)}\frac{\partial\overline{G}}{\partial\alpha} \geq \frac{\overline{F}(k,\alpha)}{\overline{G}(k,\alpha)}\frac{\partial\overline{G}}{\partial\alpha}.$$

(ii) We show that if (26) holds, the indirect effect in (22) is positive, so that, together with (25) the total effect is positive and $d\underline{\mu}(\alpha)/d\alpha > 0$. By differentiating (24), we can write

$$\left(\frac{\mu}{2}\right)\frac{\partial}{\partial\alpha}\left(\mu\frac{\partial\ln\widetilde{u}_R(k;\mu)}{\partial k}k'(\mu)\right) = \frac{\partial}{\partial\alpha}\left(\frac{\left(\frac{h_F(k;\alpha)}{\partial R(k;\alpha)/dk}\right)\overline{\Lambda}^2(k;\alpha) + \eta\left(\frac{h_G(k;\alpha)}{\partial R(k;\alpha)/dk}\right)}{\mu\overline{\Lambda}^2(k;\alpha) + \eta(1-\mu)}\right).$$

$$= \frac{\partial\left(\frac{h_F(k;\alpha)}{\partial R(k;\alpha)/\partial k}\right)}{\partial\alpha}\upsilon + \frac{\partial\left(\frac{h_F(k;\alpha)}{\partial R(k;\alpha)/\partial k}\right)}{\partial\alpha}(1-\upsilon) - \frac{\partial\upsilon}{\partial\alpha}\left(\frac{h_G(k;\alpha) - h_F(k;\alpha)}{\partial R(k;\alpha)/\partial k}\right)$$

with $\upsilon = \mu\overline{\Lambda}^2(k;\alpha)/\left(\mu\overline{\Lambda}^2(k;\alpha) + (1-\mu)\eta\right)$. Then (26) guarantees that this derivative is positive. $\square$

Proof of Corollary 1. If $\lambda(0;\alpha)$ is constant in $[\underline{\alpha},\overline{\alpha}]$, then $\mu_{CI}(\alpha) = 1/(1+R(0;\alpha)) = 1/(1+\lambda(0;\alpha)\frac{1+\gamma}{1-\gamma})$ is constant in $[\underline{\alpha},\overline{\alpha}]$. Since

$$E^{CI}(\mu;\alpha) = \frac{(\mu_{CI}(\alpha)-\mu)\, u_R(\mu;\alpha)}{\mu\,(u_R(\mu_{CI}(\alpha);\alpha)-u_R(\mu;\alpha))}\left(-E^{CF}(\mu;\alpha)\right),$$

then we must have $\frac{\partial E^{CF}(\mu;\alpha)}{\partial\alpha} = -\frac{\partial E^{CI}(\mu;\alpha)}{\partial\alpha}$. As (25) and (26) guarantee that $\left.\frac{\partial E^{CF}(\mu;\alpha)}{\partial\alpha}\right|_{\mu=\overline{\mu}(\alpha)} > 0$, then we must have $\left.\frac{\partial E^{CI}(\mu;\alpha)}{\partial\alpha}\right|_{\mu=\underline{\mu}(\alpha)} < 0$ implying $d\overline{\mu}/d\alpha > 0$. □

Proof of Proposition 4. With $k(\mu;\alpha) \in \{x_1,\ldots,x_n\}$ the investment threshold in a symmetric equilibrium, define $\underline{\mu}(i;\alpha)$ as the minimum prior belief so that investors are willing to invest if they observe $x = x_i$,

$$\underline{\mu}(i;\alpha) \equiv \min\{\mu : k(\mu;\alpha) \leq x_i\}.$$

The equilibrium conditions (2) adapted to this discrete case imply that

$$\underline{\mu}(i;\alpha) = \frac{1}{1+\frac{f_i(\alpha)}{g_i(\alpha)}\frac{1+\gamma\overline{F}_i(\alpha)}{1-\gamma\overline{G}_i(\alpha)}}, \tag{A.47}$$

where $\overline{F}_i(\alpha) = \sum_{j=i}^n f_j(\alpha)$ and $\overline{G}_i(\alpha) = \sum_{j=i}^n g_j(\alpha)$.

Fix a prior $\mu_0 \in (0,1)$. Under the conditions of the proposition, there exists $i^* \in \{2,\ldots,n\}$ so that $\lim_{\alpha\to1} f_i(\alpha)/g_i(\alpha) = 0$ for $i < i^*$ and $\lim_{\alpha\to1} g_i(\alpha)/f_i(\alpha) = 0$ for $i \geq i^*$, which using (A.47) implies

$$\lim_{\alpha\to1}\underline{\mu}(i;\alpha) = 1 \text{ for } i < i^* \text{and} \lim_{\alpha\to1}\underline{\mu}(i;\alpha) = 0 \text{ for } i \geq i^*, \tag{A.48}$$

and we can find $\tilde{\alpha} < 1$ so that $\max_{i\geq i^*}\underline{\mu}(i;\alpha) \leq \mu_0 < \min_{i<i^*}\underline{\mu}(i;\alpha)$ for $\alpha \geq \tilde{\alpha}$.

We first show that for $\alpha \geq \tilde{\alpha}$, the optimal test leads to some disclosure—i.e., for the optimal test in Proposition 1 adapted to the discrete case, $\mu_0 \in D$. We then show that the optimal test converges to a fully revealing test as $\alpha \to 1$.

Let $\underline{i}(\alpha)$ and $\overline{i}(\alpha)$ be defined by $\underline{\mu}(\underline{i}(\alpha);\alpha) = \min_{i<i^*}\underline{\mu}(i;\alpha)$ and $\underline{\mu}(\overline{i}(\alpha);\alpha) = \max_{i\geq i^*}\underline{\mu}(i;\alpha)$—note that if $\widetilde{R}(i;\alpha) \equiv \frac{f_i(\alpha)}{g_i(\alpha)}\frac{1+\gamma\overline{F}_i(\alpha)}{1-\gamma\overline{G}_i(\alpha)}$ increases in i then we would immediately have $\overline{i}(\alpha) = i^*$ and $\underline{i}(\alpha) = i^*-1$. Then, for $\alpha \geq \tilde{\alpha}$ we have $k(\mu;\alpha) = x_{\overline{i}(\alpha)}$ and the regulator's payoff if she provides no information to investors is

$$\mu\left(\overline{F}_{\overline{i}(\alpha)}(\alpha)\right)^2 + (1-\mu)\,\eta\left(\overline{G}_{\overline{i}(\alpha)}(\alpha)\right)^2.$$

We now show that the test $\pi_{\left\{\underline{\mu}(\bar{i}(\alpha);\alpha),\underline{\mu}(\underline{i}(\alpha);\alpha)\right\}}$, supported on $\underline{\mu}(\bar{i}(\alpha);\alpha)$ and $\underline{\mu}(\underline{i}\,(\alpha)\,;\alpha)$ strictly outperforms the test $\pi_{\varnothing}$ where the regulator provides no public information. Note that any other potential test that may dominate $\pi_{\left\{\underline{\mu}(\bar{i}(\alpha);\alpha),\underline{\mu}(\underline{i}(\alpha);\alpha)\right\}}$ can only be supported on $\left\{\mu \leq \underline{\mu}(\bar{i}(\alpha);\alpha)\right\} \cup \left\{\mu \geq \underline{\mu}(\underline{i}\,(\alpha)\,;\alpha)\right\}$. Then, from (A.48), we have that any of these tests, including $\pi_{\left\{\underline{\mu}(\bar{i}(\alpha);\alpha),\underline{\mu}(\underline{i}(\alpha);\alpha)\right\}}$, converges to a fully informative test as $\alpha \to 1$, concluding our proof.

Let Δ denote the difference between the regulator's payoff under $\pi_{\left\{\underline{\mu}(\bar{i}(\alpha);\alpha),\underline{\mu}(\underline{i}(\alpha);\alpha)\right\}}$ and $\pi_{\varnothing}$:

$$\begin{aligned}
\Delta \;\equiv\;& \Pr\left[\underline{\mu}(\bar{i}(\alpha);\alpha)\right]\left(\underline{\mu}(\bar{i}(\alpha);\alpha)\left(\overline{F}_{\bar{i}(\alpha)}(\alpha)\right)^2 + \left(1-\underline{\mu}(\bar{i}(\alpha);\alpha)\right)\eta\left(\overline{G}_{\bar{i}(\alpha)}(\alpha)\right)^2\right) + \\
& \Pr\left[\underline{\mu}(\underline{i}\,(\alpha)\,;\alpha)\right]\left(\underline{\mu}(\underline{i}\,(\alpha)\,;\alpha)\left(\overline{F}_{\underline{i}(\alpha)}(\alpha)\right)^2 + \left(1-\underline{\mu}(\underline{i}\,(\alpha)\,;\alpha)\right)\eta\left(\overline{G}_{\underline{i}(\alpha)}(\alpha)\right)^2\right) \\
& -\left(\mu\left(\overline{F}_{\bar{i}(\alpha)}(\alpha)\right)^2 + (1-\mu)\,\eta\left(\overline{G}_{\bar{i}(\alpha)}(\alpha)\right)^2\right) \\
=\;& \Pr\left[\underline{\mu}(\underline{i}\,(\alpha)\,;\alpha)\right]\underline{\mu}(\underline{i}\,(\alpha)\,;\alpha)\left(\left(\overline{F}_{\underline{i}(\alpha)}(\alpha)\right)^2 - \left(\overline{F}_{\bar{i}(\alpha)}(\alpha)\right)^2\right) + \\
& + \Pr\left[\underline{\mu}(\underline{i}\,(\alpha)\,;\alpha)\right]\left(1-\underline{\mu}(\underline{i}\,(\alpha)\,;\alpha)\right)\eta\left(\left(\overline{G}_{\underline{i}(\alpha))}(\alpha)\right)^2 - \left(\overline{G}_{\bar{i}(\alpha)}(\alpha)\right)^2\right).
\end{aligned}$$

By assumption, either $f_i(\alpha) > 0$ for $\alpha < 1$ implying that

$$\left(\left(\overline{F}_{\underline{i}(\alpha)}(\alpha)\right)^2 - \left(\overline{F}_{\bar{i}(\alpha)}(\alpha)\right)^2\right) = \left(\sum_{j=\underline{i}(\alpha)}^{\bar{i}(\alpha)} f_j\,(\alpha)\right)\left(\overline{F}_{\bar{i}(\alpha)}(\alpha) + \overline{F}_{\underline{i}(\alpha)}(\alpha)\right) > 0,$$

or $g_i(\alpha) > 0$ for $\alpha < 1$, implying that

$$\left(\left(\overline{G}_{\underline{i}(\alpha)}(\alpha)\right)^2 - \left(\overline{G}_{\bar{i}(\alpha)}(\alpha)\right)^2\right) = \left(\sum_{j=\underline{i}(\alpha)}^{\bar{i}(\alpha)} g_j\,(\alpha)\right)\left(\overline{G}_{\bar{i}(\alpha)}(\alpha) + \overline{G}_{\underline{i}(\alpha)}(\alpha)\right) > 0.$$

Therefore, $\Delta > 0$ for $1 > \alpha \geq \tilde{\alpha}$. □

Proof of Proposition 5. Let $t = \frac{1-\mu}{\mu}$ so that the equilibrium threshold (6) can be written as $k = R^{-1}(t;\alpha)$ and the regulator's average payoff (12) can be expressed as a function of t

$$A_R(t;\alpha) \equiv A_R(\mu(t);\alpha) = \overline{F}^2(R^{-1}(t;\alpha);\alpha) + \eta\overline{G}^2(R^{-1}(t;\alpha);\alpha)t.$$

We will show that under the conditions of the proposition, there exists $\tilde{\alpha}$ with $dA_R(t;\alpha)/dt < 0$ for $t > 0$ and $\alpha > \tilde{\alpha}$. Thus, $t^* = 0$ maximizes $A_R(t;\alpha)$ implying that the optimal critical-fault test has $\underline{\mu} = 1/(1+t^*) = 1$.

Differentiating $A_R(t;\alpha)$, where we drop the explicit relation with α to ease notation,

$$\begin{aligned}\frac{dA_R}{dt} &= \left.\left(-2\frac{f(k)\overline{F}(k)}{R'(k)} - 2\eta\frac{g(k)\overline{G}(k)}{R'(k)}t + \eta\overline{G}^2(k)\right)\right|_{k=R^{-1}(t)} \\ &= \left.\frac{g(k)}{R'(k)}\left(-2\lambda(k)\overline{F}(k) - 2\eta\overline{G}(k)t + \eta\frac{\overline{G}^2(k)R'(k)}{g(k)}\right)\right|_{k=R^{-1}(t)}\end{aligned}$$

Let $C(k) = \frac{1+\gamma\overline{F}(k)}{1-\gamma\overline{G}(k)}$ so that (i) $R(k) = \lambda(k)C(k)$, (ii) $|C(k)| < \max\left\{\frac{1+\gamma}{1-\gamma}, 1\right\}$ and, (iii)

$$\frac{R'(k)}{R(k)} = \frac{d\ln\lambda(k)}{dk} + \frac{dC(k)/dk}{C(k)}.$$

Then, using $t = R(k)$, we can write

$$\begin{aligned}\frac{dA_R}{dt} &= \left.\frac{g(k)}{R'(k)}\left(-2\frac{t}{C(k)}\overline{F}(k) - 2\eta\overline{G}(k)t + \eta t\frac{\overline{G}^2(k)R'(k)}{g(k)R(k)}\right)\right|_{k=R^{-1}(t)} \\ &= \left.\frac{g(k)}{R'(k)}t\left(-2\frac{\overline{F}(k)}{C(k)} - 2\eta\overline{G}(k) + \eta\frac{\overline{G}^2(k)}{g(k)}\left(\frac{d\ln\lambda(k)}{dk} + \frac{dC(k)/dk}{C(k)}\right)\right)\right|_{k=R^{-1}(t)} \\ &= \left.\frac{g(k)}{R'(k)}t\left(-2\frac{\overline{F}(k)}{C(k)} - 2\eta\overline{G}(k) + \eta\overline{G}(k)\left(\frac{\overline{G}(k)}{g(k)}\frac{d\ln\lambda(k)}{dk} + \frac{\overline{G}(k)}{g(k)}\frac{dC(k)/dk}{C(k)}\right)\right)\right|_{k=R^{-1}(t)} \quad \text{(A.49)}\end{aligned}$$

We make the following three observations regarding the asymptotic behavior of the terms in (A.49):

(i) For $t > 0$, $\overline{F}^2(R^{-1}(t;\alpha),\alpha) \to 1$ and $\overline{G}^2(R^{-1}(t;\alpha),\alpha) \to 0$ as $\alpha \to 1$–see Assumption 3. This captures the idea that investors will always invest if $\omega = 1$ and never invest if $\omega = -1$ as their private signals become perfectly informative of the state. Therefore, $C(k;\alpha) = C(R^{-1}(t;\alpha);\alpha) \to 1+\gamma$, and

$$\lim_{\alpha\to 1} -2\frac{\overline{F}(k)}{C(k)} - 2\eta\overline{G}(k) = -2\frac{1}{1+\gamma} < 0.$$

(ii) We have

$$\left.\frac{\overline{G}(k)}{g(k)}\frac{dC(k)/dk}{C(k)}\right|_{k=R^{-1}(t)} = -\gamma\frac{\overline{G}(k)}{g(k)}\left.\left(\frac{f(k)}{1+\gamma\overline{F}(k)} + \frac{g(k)}{1-\gamma\overline{G}(k)}\right)\right|_{k=R^{-1}(t)} \leq 0.$$

(iii) By assumption, $\frac{\overline{G}(k)}{g(k)}\frac{d\ln\lambda(k)}{dk}$ is uniformly bounded in $\cup_{\alpha>\alpha'}\left\{k : \frac{1-\mu}{\mu} = R(k;\alpha) \text{ for } \mu \in (0,1)\right\}$,

so that

$$\lim_{\alpha\to 1}\overline{G}(k)\left(\frac{\overline{G}(k)}{g(k)}\frac{d\ln\lambda(k)}{dk}\right)\Bigg|_{k=R^{-1}(t)}=0.$$

Using these three observations, we have that the term in parenthesis in (A.49) becomes strictly negative for any $t>0$ and α close to 1. As $\frac{g(k)}{R'(k)}>0$, then we must have that $\frac{dA_R}{dt}<0$ for $t>0$. □

Proof of Remark 1. From (13) and using $A_R(k)$ defined in (12), the regulator's optimal CF-test is $\underline{\mu}=1/\left(1+R\left(\underline{k}\right)\right)$ with $\underline{k}$ satisfying $dA_R(\underline{k})/dk=0$, i.e.,

$$-2f(\underline{k})\overline{F}(\underline{k})-2\eta g(\underline{k})\overline{G}(\underline{k})R(\underline{k})+\eta\overline{G}^2(\underline{k})R'(\underline{k})=0.$$

We will make the change of variables $x=(k-\alpha)/(1-\alpha)$ so that $k=\alpha+(1-\alpha)x$, and for $k\geq\alpha$ we can write

$$\begin{aligned}f(k)&=\frac{1}{(1-\alpha)}h(x),\ \overline{F}(k)=\int_k^1\frac{1}{(1-\alpha)}h(\frac{k'-\alpha}{1-\alpha})dk'=\overline{H}(x),\\ \overline{G}(k)&=(1-\alpha)(1-x).\end{aligned}$$

Let $C\left(x,\alpha\right)\equiv\frac{1+\gamma\overline{H}(x)}{1-\gamma(1-\alpha)(1-x)}$. Then for $k\geq\alpha$,

$$\begin{aligned}R(k)&=\lambda(k)\frac{1+\gamma\overline{F}(k)}{1-\gamma\overline{G}(k)}=\frac{1}{(1-\alpha)}h(x)\frac{1+\gamma\overline{H}(x)}{1-\gamma(1-\alpha)(1-x)}=\frac{h(x)}{(1-\alpha)}C\left(x,\alpha\right),\\ R'(k)&=f'(k)\frac{1+\gamma\overline{F}(k)}{1-\gamma\overline{G}(k)}-\gamma\frac{\left(f(k)\right)^2}{1-\gamma\overline{G}(k)}-\gamma f(k)\frac{1+\gamma\overline{F}(k)}{\left(1-\gamma\overline{G}(k)\right)^2}\\ &=C\left(x,\alpha\right)\left(\frac{1}{(1-\alpha)^2}h'\left(x\right)-\frac{1}{(1-\alpha)^2}\frac{\gamma\left(h\left(x\right)\right)^2}{1+\gamma\overline{H}(k)}-\frac{1}{(1-\alpha)}\frac{\gamma h(x)}{1-\gamma\overline{G}(k)}\right)\\ &=\frac{C\left(x,\alpha\right)}{(1-\alpha)^2}\left(h'\left(x\right)-\frac{\gamma\left(h\left(x\right)\right)^2}{1+\gamma\overline{H}(k)}-\frac{(1-\alpha)\gamma h(x)}{1-\gamma(1-\alpha)(1-x)}\right).\end{aligned}$$

The first order condition for $\underline{k}$, expressed in terms of $\underline{x}$, becomes

$$\begin{aligned}&-2\frac{h(\underline{x})\overline{H}(\underline{x})}{(1-\alpha)}-2\eta(1-\underline{x})h(\underline{x})C\left(\underline{x},\alpha\right)+...\\ &+(1-x)^2C\left(\underline{x},\alpha\right)\eta\left(h'\left(\underline{x}\right)-\frac{\gamma\left(h\left(\underline{x}\right)\right)^2}{1+\gamma\overline{H}(\underline{x})}-\frac{(1-\alpha)\gamma h(\underline{x})}{1-\gamma(1-\alpha)(1-\underline{x})}\right)=0,\end{aligned}$$

which simplifies to

$$-2h(\underline{x})\overline{H}(\underline{x}) - 2\eta\,(1-\alpha)\,(1-\underline{x})C\,(\underline{x},\alpha)\,2h(\underline{x}) + ...$$
$$+\eta\,(1-\alpha)\,(1-\underline{x})C\,(\underline{x},\alpha)\,(1-\underline{x})\left(h'\,(\underline{x}) - \gamma\frac{(h\,(\underline{x}))^2}{1+\gamma\overline{H}(\underline{x})} - \gamma\frac{(1-\alpha)h(\underline{x})}{1-\gamma(1-\alpha)(1-\underline{x})}\right) = 0.$$

To simplify notation let

$$S(x,\alpha) = \left(-2h(x) + (1-x)\left(h'\,(x) - \gamma\frac{(h\,(x))^2}{1+\gamma\overline{H}(x)} - \gamma\frac{(1-\alpha)h(x)}{1-\gamma(1-\alpha)(1-x)}\right)\right)$$

so that the marginal increase in the average sender's payoff is

$$A_R'(x) = \frac{1}{1-\alpha}\left(-2h(x)\overline{H}(x) + \eta\,(1-\alpha)\,(1-x)C\,(x,\alpha)\,S(x,\alpha)\right).$$

We first show that under the assumption of uniform boundedness of $h'(x)$ in $[0,1]$—with right and left derivatives at the extremes—we have $\underline{x} \to 0$ as $\alpha \to 1$. Indeed, uniform boundedness of $h'(x)$ implies that of $(1-x)C\,(x,\alpha)\,S(x,\alpha)$ in $(x,\alpha) \in [0,1]^2$, i.e.,

$$|(1-x)C\,(x,\alpha)\,S(x,\alpha)| \leq M.$$

Then, for each $x \in (0,1)$ there exists $\alpha\,(x) \equiv 1 - \frac{2h(x)\overline{H}(x)}{M} \in (0,1)$ such that

$$A_R'(x) \leq -\frac{2h(x)\overline{H}(x)}{1-\alpha} + M < -\frac{2h(x)\overline{H}(x)}{1-\alpha\,(x)} + M = 0,\ \ \alpha > \alpha\,(x)\,,$$

which follows from $h(x)\overline{H}(x) > 0$ if $x \in (0,1)$. For any interval $[\epsilon, 1-\epsilon]$ we have $A_R'(x) < 0$ for $\alpha > \alpha\,(\epsilon)$ so that $A_R(\epsilon) > A_R(1-\epsilon)$ for any $\epsilon > 0$.

While the investment threshold satisfies $\underline{x} \to 0$ as $\alpha \to 1$, we still need to find the optimal CF-test. Indeed, the optimal test satisfies

$$\underline{\mu} = \frac{1}{1+R(\underline{x})} = \frac{1}{1+\frac{h(\underline{x})}{(1-\alpha)}T\,(\underline{x},\alpha)},$$

and

$$\frac{h(\underline{x})}{(1-\alpha)}C\,(\underline{x},\alpha) = \eta(1-\underline{x})\frac{C^2\,(\underline{x},\alpha)}{2\overline{H}(\underline{x})}\left(-2h(\underline{x}) + (1-\underline{x})\left(h'\,(\underline{x}) - \gamma\frac{(h\,(\underline{x}))^2}{1+\gamma\overline{H}(\underline{x})} - \gamma\frac{(1-\alpha)h(\underline{x})}{1-\gamma(1-\alpha)(1-\underline{x})}\right)\right)$$

Therefore, as $\alpha \longrightarrow 1$ we have $\underline{x} \to 0$, and $C^2\,(\underline{x},\alpha) \longrightarrow (1+\gamma)^2$, $\overline{H}(\underline{x}) \to 1, h(\underline{x}) \to 0$,

so that

$$\lim_{\alpha\to 1}\frac{h(\underline{x})}{(1-\alpha)}T\left(\underline{x},\alpha\right)=\eta\frac{(1+\gamma)^2}{2}h'\left(0\right).$$

Therefore

$$\lim_{\alpha\to 1}\underline{\mu}=\frac{1}{1+\eta\frac{(1+\gamma)^2}{2}h'\left(0\right)}.$$

□

Proof of Remark 2. Using the change of variable $x=(k-\alpha)/(1-\alpha)$ in (12) we can express the regulator's expected average utility as

$$\begin{aligned}A_R(x;\alpha)&=\overline{F}^2(x;\alpha)+\eta\overline{G}^2(x;\alpha)R(x;\alpha)=\overline{H}^2(x)+\eta\left((1-\alpha)(1-x)\right)^2R(x;\alpha)\\&=\overline{H}^2(x)+\eta\frac{(1-\alpha)}{1-\gamma(1-\alpha)(1-x)}h(x)\left(1+\gamma\overline{H}(x)\right)(1-x)^2.\end{aligned}$$

Differentiating with respect to α, we obtain

$$\frac{\partial A_R(x;\alpha)}{\partial\alpha}=-\eta\frac{1}{\left(1-\gamma(1-\alpha)(1-x)\right)^2}h(x)\left(1+\gamma\overline{H}(x)\right)(1-x)^2\le 0.$$

□

Proof of Proposition 6. For $\mu_0\in(\underline{\mu}(\alpha)-\epsilon,\underline{\mu}(\alpha))$ for some $\epsilon>0$, the sign of (29) is

$$sign\left[\frac{dV_i^{test}(\alpha)}{d\alpha}\right]=sign\left[\frac{\partial\widetilde{A}_I(\underline{\mu}(\alpha),\alpha)}{\partial\mu}\frac{\partial\underline{\mu}(\alpha)}{\partial\alpha}\right]$$

We show that, evaluated at $\underline{\mu}(\alpha)$, average investor's utility is increasing, i.e., $\partial\widetilde{A}_I(\underline{\mu}(\alpha)),\alpha)/\partial\mu>0$, so that

$$sign\left[\frac{dV_i^{test}(\alpha)}{d\alpha}\right]=sign\left[\frac{\partial\underline{\mu}(\alpha)}{\partial\alpha}\right].$$

Using $A_I(k;\alpha)=\mathcal{E}(k;\alpha)+\gamma A_{R,1}(k;\alpha)$ where $\mathcal{E}(k;\alpha)\equiv\mathbb{E}_{X(\alpha)}[\omega 1_{\{x_i\ge k\}}]/\mu(k;\alpha)$ captures how adapted are investments to fundamentals and $A_{R,1}(k;\alpha)$ is the regulator's average utility when $\eta=1$, we can write

$$\frac{\partial\widetilde{A}_I(\mu,\alpha)}{\partial\mu}=\frac{\partial\mathcal{E}(k(\mu,\alpha);\alpha)}{\partial\mu}+\gamma\frac{\partial A_R(k(\mu,\alpha);\alpha)}{\partial\mu}.$$

Optimality conditions for CF-tests (13) implies that $\partial A_R(k(\underline{\mu}(\alpha),\alpha);\alpha)/\partial\mu = 0$ so that

$$\frac{\partial \widetilde{A}_I(\underline{\mu}(\alpha),\alpha)}{\partial \mu} = \frac{\partial \mathcal{E}(k(\underline{\mu}(\alpha),\alpha);\alpha)}{\partial \mu},$$

We next show that $\partial \mathcal{E}(k(\mu,\alpha);\alpha)/\partial\mu > 0$ when evaluated at $\mu = \underline{\mu}(\alpha)$. Differentiating $\mathcal{E}(k;\alpha) = \overline{F}(k;\alpha) - R(k;\alpha)\overline{G}(k;\alpha)$,

$$\frac{\partial \mathcal{E}(k(\underline{\mu}(\alpha),\alpha);\alpha)}{\partial \mu} = -f(k;\alpha) + R(k;\alpha)g(k;\alpha) - \left.\frac{\partial R(k;\alpha)}{\partial k}\overline{G}(k;\alpha)\right|_{k=k(\underline{\mu}(\alpha),\alpha)} \frac{\partial k(\underline{\mu}(\alpha),\alpha)}{\partial \mu} \tag{A.50}$$

The necessary FOC for a critical-fault tests—see (13)— requires that at $k = k(\underline{\mu}(\alpha),\alpha)$

$$\frac{\partial R(k;\alpha)}{\partial k}\overline{G}(k;\alpha) = 2f(k;\alpha)\frac{\overline{F}(k;\alpha)}{\overline{G}(k;\alpha)} + 2g(k;\alpha)R(k;\alpha)g(k;\alpha),$$

and replacing $(\partial R(k;\alpha)/\partial k)\,\overline{G}(k;\alpha)$ in (A.50), and recalling from Lemma 1-iii that $\partial k(\mu)/\partial\mu < 0$ we have

$$\frac{\partial \mathcal{E}(k(\underline{\mu}(\alpha),\alpha);\alpha)}{\partial \mu} = -f(k;\alpha)\left(1 + 2\frac{\overline{F}(k;\alpha)}{\overline{G}(k;\alpha)}\right) - R(k;\alpha)g(k;\alpha)\Big|_{k=k(\underline{\mu}(\alpha),\alpha)} \frac{\partial k(\underline{\mu}(\alpha),\alpha)}{\partial \mu} > 0.$$

for all $\gamma \in (-1,1)$, and irrespective of α. □

Proof of Proposition 7. We first derive the equation for the symmetric equilibrium threshold. Given (30) we have:

$$\begin{aligned}\mathbb{P}[\omega = 1|x_i = x] &= \frac{\mathbb{P}[\omega = 1]\mathbb{P}[x_i = x|\omega = 1]}{\mathbb{P}[\omega = 1]\mathbb{P}[x_i = x|\omega = 1] + \mathbb{P}[\omega = -1]\mathbb{P}[x_i = x|\omega = -1]} \\ &= \frac{\mu\int \widehat{f}(\frac{x-\beta\varepsilon}{1-\beta})g_\varepsilon(\varepsilon)d\varepsilon}{\mu\int \widehat{f}(\frac{x-\beta\varepsilon}{1-\beta})g_\varepsilon(\varepsilon)d\varepsilon + (1-\mu)\int \widehat{g}(\frac{x-\beta\varepsilon}{1-\beta})g_\varepsilon(\varepsilon)d\varepsilon} = \frac{\mu f(x;\beta)}{\mu f(x;\beta) + (1-\mu)g(x;\beta)},\end{aligned}$$

where we integrate over ε and use conditional independence; in the last line we note that

$$f(x;\beta) = \int \widehat{f}(\frac{x-\beta\varepsilon}{1-\beta})g_\varepsilon(\varepsilon)d\varepsilon,\; g(x;\beta) = \int \widehat{g}(\frac{x-\beta\varepsilon}{1-\beta})g_\varepsilon(\varepsilon)d\varepsilon,$$

where above and in subsequent quantities we are now explicit on the dependence on the parameter β. Similarly,

$$F(y|x;\beta)=\mathbb{P}[x_j\le y|x_i=x,\omega=1]=\frac{\int\widehat{F}\left(\frac{y-\beta\varepsilon}{1-\beta}\right)\widehat{f}\left(\frac{x-\beta\varepsilon}{1-\beta}\right)g_\varepsilon(\varepsilon)d\varepsilon}{\int\widehat{f}\left(\frac{x-\beta\varepsilon}{1-\beta}\right)g_\varepsilon(\varepsilon)d\varepsilon},$$

$$G(y|x;\beta)=\mathbb{P}[x_j\le y|x_i=x,\omega=-1]=\frac{\int\widehat{G}\left(\frac{y-\beta\varepsilon}{1-\beta}\right)\widehat{g}\left(\frac{x-\beta\varepsilon}{1-\beta}\right)g_\varepsilon(\varepsilon)d\varepsilon}{\int\widehat{f}\left(\frac{x-\beta\varepsilon}{1-\beta}\right)g_\varepsilon(\varepsilon)d\varepsilon},$$

while, also as before $\overline{F}(y|x;\beta)=1-F(y|x;\beta)$, $\overline{G}(y|x;\beta)=1-G(y|x;\beta)$. Hence, the symmetric equilibrium threshold $k(\mu;\beta)$ solves,

$$\frac{\mathbb{P}[\omega=-1|x_i=k]}{\mathbb{P}[\omega=1|x_i=k]}=\frac{1+\gamma\overline{F}(k|k;\beta)}{1-\gamma\overline{G}(k|k;\beta)}\Rightarrow\frac{1-\mu}{\mu}=\underbrace{\underbrace{\frac{f(k;\beta)}{g(k;\beta)}}_{\lambda(k;\beta)}\cdot\underbrace{\frac{1+\gamma\overline{F}(k|k;\beta)}{1-\gamma\overline{G}(k|k;\beta)}}_{r(k;\beta)}}_{\widetilde{R}(k;\beta)},$$

i.e., the counterpart of (2). For this section— to make our point in the most straightforward way— we assume that $\widetilde{R}(k;\beta)$ is strictly increasing in k for all β so that from (5) $R(k;\beta)=\max_{0\le k'\le k}\widetilde{R}(k';\beta)=\widetilde{R}(k;\beta)$.

In this case, the sender's utility is given by (31) and similarly to (A.11) we let

$$A_R(k;\beta)\equiv\frac{u_R(\mu;\beta)}{\mu}=\int\overline{\widehat{F}}^2\left(\frac{k-\beta\varepsilon}{1-\beta}\right)g_\varepsilon(\varepsilon)d\varepsilon+\eta R(k;\beta)\int\overline{\widehat{G}}^2\left(\frac{k-\beta\varepsilon}{1-\beta}\right)g_\varepsilon(\varepsilon)d\varepsilon. \quad \text{(A.51)}$$

Below, we apply Lemma A.2 to show that as $\beta\to 0$ the two part parts of the sum in (A.12) cancel each other out, leading to $d\underline{\mu}(\beta)/d\beta=0$, i.e., the CF-test realization is unaffected by small 'mistakes'.

First, we have

$$\frac{\partial A_R(k;\beta)}{\partial\beta}=-\int\frac{k-\varepsilon}{(1-\beta)^2}2\left(\overline{\widehat{F}}\widehat{f}\right)\left(\frac{k-\beta\varepsilon}{1-\beta}\right)g_\varepsilon(\varepsilon)d\varepsilon+\eta R_\beta(k;\beta)\int\overline{\widehat{G}}^2\left(\frac{k-\beta\varepsilon}{1-\beta}\right)g_\varepsilon(\varepsilon)d\varepsilon$$
$$-\eta R(k;\beta)\int\frac{k-\varepsilon}{(1-\beta)^2}2\left(\overline{\widehat{G}}\widehat{g}\right)\left(\frac{k-\beta\varepsilon}{1-\beta}\right)g_\varepsilon(\varepsilon)d\varepsilon,$$

where for notational brevity in the first integral we let $\left(\overline{\widehat{F}}\widehat{f}\right)\left(\frac{k-\beta\varepsilon}{1-\beta}\right)\equiv\overline{\widehat{F}}\left(\frac{k-\beta\varepsilon}{1-\beta}\right)\widehat{f}\left(\frac{k-\beta\varepsilon}{1-\beta}\right)$, and similarly in the third integral; hence:

$$\begin{aligned}\frac{\partial^2 A_R(k;\beta)}{\partial k\partial\beta} =& -\int \frac{1}{(1-\beta)^2}2\left(\widehat{\overline{F}}\widehat{f}\right)\left(\frac{k-\beta\varepsilon}{1-\beta}\right)g_\varepsilon(\varepsilon)d\varepsilon - \int \frac{k-\varepsilon}{(1-\beta)^3}2(\widehat{\overline{F}}\widehat{f})'\left(\frac{k-\beta\varepsilon}{1-\beta}\right)g_\varepsilon(\varepsilon)d\varepsilon \\ &+\eta R_{k\beta}(k;\beta)\cdot\int \widehat{\overline{G}}^2\left(\frac{k-\beta\varepsilon}{1-\beta}\right)g_\varepsilon(\varepsilon)d\varepsilon - \eta R_\beta(k;\beta)\int\frac{1}{1-\beta}2\left(\widehat{\overline{G}}\widehat{g}\right)\left(\frac{k-\beta\varepsilon}{1-\beta}\right)g_\varepsilon(\varepsilon)d\varepsilon \\ &-\eta R_k(k;\beta)\int\frac{k-\varepsilon}{(1-\beta)^2}2\left(\widehat{\overline{G}}\widehat{g}\right)\left(\frac{k-\beta\varepsilon}{1-\beta}\right)g_\varepsilon(\varepsilon)d\varepsilon \\ &-\eta R(k;\beta)\int\frac{1}{(1-\beta)^2}2\left(\widehat{\overline{G}}\widehat{g}\right)\left(\frac{k-\beta\varepsilon}{1-\beta}\right)g_\varepsilon(\varepsilon)d\varepsilon - \eta R(k;\beta)\int\frac{k-\varepsilon}{(1-\beta)^3}2(\widehat{\overline{G}}\widehat{g})'\left(\frac{k-\beta\varepsilon}{1-\beta}\right)g_\varepsilon(\varepsilon)d\varepsilon.\end{aligned}$$

We will now inquire on the value of $\partial A_R^2/(\partial k\partial\beta)$ in the case of small 'mistakes', i.e., as β tends to zero. The terms within the integral are easily computed in the limit $\beta=0$ and we use that function $R=\widetilde{R}$ and its partials $R_k, R_\beta, R_{k\beta}$, are continuous at $\beta=0$ for all k.[51] Hence,

$$\begin{aligned}\lim_{\beta\to 0}\frac{\partial^2 A_R(k;\beta)}{\partial k\partial\beta} =& -2\left(\widehat{\overline{F}}\widehat{f}\right)(k) - (k-\overline{\varepsilon})2(\widehat{\overline{F}}\widehat{f})'(k) + \eta R_{k\beta}(k,0)\widehat{\overline{G}}^2(k) - \eta R_\beta(k,0)2\left(\widehat{\overline{G}}\widehat{g}\right)(k) \\ &-\eta R_k(k;0)(k-\overline{\varepsilon})2\left(\widehat{\overline{G}}\widehat{g}\right)(k) - \eta R(k;0)2\left(\widehat{\overline{G}}\widehat{g}\right)(k) - \eta R(k;0)(k-\overline{\varepsilon})2\left(\widehat{\overline{G}}\widehat{g}\right)'(k) \\ =&(k-\overline{\varepsilon})\left(-2\left(\widehat{\overline{F}}\widehat{f}\right)'(k) - \eta R_k(k;0)2\left(\widehat{\overline{G}}\widehat{g}\right)(k) - \eta R(k;0)2\left(\widehat{\overline{G}}\widehat{g}\right)' - \eta R_k(k;0)2\left(\widehat{\overline{G}}\widehat{g}\right)(k)\right. \\ &\left.-\eta R_{kk}(k;0)\widehat{\overline{G}}^2(k)\right) - 2\left(\widehat{\overline{F}}\widehat{f}\right)(k) - \eta R_k(k;0)\widehat{\overline{G}}^2(k) - \eta R(k;0)2\left(\widehat{\overline{G}}\widehat{g}\right)(k) \\ =&(k-\overline{\varepsilon})\frac{\partial^2 A_R(k;0)}{\partial k^2} + \frac{\partial A_R(k;0)}{\partial k},\end{aligned} \tag{A.52}$$

where the last line follows from the fact that $A_R(k;0)=\widehat{\overline{F}}^2(k)+\eta\widehat{\overline{G}}^2(k)R(k;0)$. Now, we also have that

$$\begin{aligned}\lim_{\beta\to 0}\left(\left.\frac{\partial^2 A_R(k;\beta)}{\partial k\partial\beta}\right|_{k=\underline{k}(\beta)}\right) &= \left.\left(\lim_{\beta\to 0}\frac{\partial^2 A_R(k;\beta)}{\partial k\partial\beta}\right)\right|_{k=\underline{k}(0)} \\ &= (\underline{k}(0)-\overline{\varepsilon})\left.\frac{\partial^2 A_R(k;0)}{\partial k^2}\right|_{k=\underline{k}(0)} + \left.\frac{\partial A_R(k;0)}{\partial k}\right|_{k=\underline{k}(0)} \\ &= (\underline{k}(0)-\overline{\varepsilon})\left.\frac{\partial^2 A_R(k;0)}{\partial k^2}\right|_{k=\underline{k}(0)},\end{aligned} \tag{A.53}$$

where $\underline{k}(\beta)\equiv k(\underline{\mu}(\beta);\beta)$; note that $\underline{k}(\beta)$ is continuous in β given our assumptions on F, G and γ so that $\underline{k}(0)=k(\underline{\mu}(0);0)$—i.e., the investment threshold if there were no common

[51] This follows from assuming that $F(x;\beta)$, $G(x;\beta)$ are twice continuously differentiable in $(x,\beta)\in(0,1)^2$.

mistakes; then, the second line follows from (A.52); and the third line uses that $\underline{k}(0)$ satisfies the FOC $\partial A_R(k;0)/\partial k = 0$.

Second, following similar steps (and using that $R = \widetilde{R}$ for $\gamma \in (0, \overline{\gamma})$) we can show that:

$$\lim_{\beta \to 0} \left(\left. \frac{\partial R(k;\beta)}{\partial \beta} \right|_{k=\underline{k}(\beta)} \right) = \left. \left(\lim_{\beta \to 0} \frac{\partial R(k;\beta)}{\partial \beta} \right) \right|_{k=\underline{k}(0)} = (\underline{k}(0) - \overline{\varepsilon}) \left. \frac{\partial R(k;0)}{\partial k} \right|_{k=\underline{k}(0)} . \tag{A.54}$$

We now take the limit of the left hand side of (A.12) as $\beta \to 0$, taking into account expressions (A.53) & (A.54) to arrive at:

$$\begin{aligned}
\lim_{\beta \to 0} \frac{d\mu}{d\beta} &= \frac{- \left. \frac{\partial^2 A_R(k;0)}{\partial k^2} \right|_{k=\underline{k}(0)}}{\left. \frac{\partial R(k;0)}{\partial k} \right|_{k=\underline{k}(0)}} (\underline{k}(0) - \overline{\varepsilon}) \left. \frac{\partial R(k;0)}{\partial k} \right|_{k=\underline{k}(0)} + (\underline{k}(0) - \overline{\varepsilon}) \left. \frac{\partial^2 A_R(k;0)}{\partial k^2} \right|_{k=\underline{k}(0)} \\
&= -(\underline{k}(0) - \overline{\varepsilon}) \left. \frac{\partial^2 A_R(k;0)}{\partial k^2} \right|_{k=\underline{k}(0)} + (\underline{k}(0) - \overline{\varepsilon}) \left. \frac{\partial^2 A_R(k;0)}{\partial k^2} \right|_{k=\underline{k}(0)} \\
&= 0.
\end{aligned} \tag{A.55}$$

Finally, in figuring out the signs of the indirect and direct effects in (A.55) we note that from the SOC $\partial^2 A_R(k;0)/\partial k^2$ is negative at $\underline{k}(0)$. □

Online Appendix

Here we analyze the implications of stress test design for investors' equilibrium payoffs. Note that in the absence of a stress test, well-informed investors can coordinate their investment decisions through the quality of their research (i.e., the precision of their private signals). However, investment externalities and dispersed information still lead to investment frictions when these signals are not perfectly informative of the underlying fundamentals.[52] Given these frictions, do investors benefit from the regulator's test? To answer this question, let the regulator's test in Proposition 1 be $\{\{\mu_i\}_{i\in I}, D, N\}$. Then, the private value of the test when $\mu_0 \in (\mu_l, \mu_{l+1}) \subset D$ is

$$V_i^{test} = \Pr[s = \mu_{l+1}](u_i(\mu_{l+1}) - u_i(\mu_0)) + \Pr[s = \mu_l](u_i(\mu_l) - u_i(\mu_0)). \tag{OA.1}$$

While the regulator always provides a weakly informative test, her focus on promoting joint investment may exacerbate investment distortions and, actually, make investors worse-off[53]—this is clear if $\gamma < 0$ as investors over-invest relative to the joint-surplus maximizing level. On the other hand, equilibrium underinvestment when $\gamma > 0$ can be ameliorated by disclosing information that boosts joint investment.

We focus on the symmetric, conditionally independent case and provide conditions for $V_i^{test} > 0$. In this case, investor's i interim expected utility with a symmetric investment threshold k is

$$\widetilde{u}_i(k;\mu) = \underbrace{\mu\overline{F}(k) - (1-\mu)\overline{G}(k)}_{\mathbb{E}[\omega 1_{\{x_i \geq k\}}]} + \gamma\left(\underbrace{\mu\overline{F}^2(k) + (1-\mu)\overline{G}^2(k)}_{\widetilde{u}_S(k;\mu)\, if\, \eta=1}\right), \tag{OA.2}$$

The regulator's test has a dual effect on investors' payoffs. First, it allows them to better estimate the underlying state so that investment decisions can be more adapted to fundamentals. Second, it affects the likelihood of coordinated investment—e.g., raising expected joint investment when $\eta = 1$. The overall impact on investor welfare is however ambiguous. If there are negative externalities and $\eta = 1$, then the regulator discloses information to maximize the term in parenthesis in (OA.2), which is detrimental to investors given $\gamma < 0$. If there are

[52] Recall that if the state is commonly known by investors, then the unique investment equilibrium maximizes investors joint surplus whenever $\gamma > -1/2$. Thus, when $\gamma > -1/2$ investment inefficiencies are a result of imperfect information.

[53] Recent papers offer alternative stories in which stress tests may lead to investment inefficiencies (e.g., Goldstein and Leitner (2018)). As an example in a different context, Alonso and Câmara (2016) show that voters that adopt a project following a simple majority procedure are always (weakly) worse-off by the presence of information designer that is set on adopting the project.

positive externalities, then the regulator is "partially aligned" with investors, even if $\eta = 1$, as the latter benefit from simultaneously investing. However, promoting joint investment can lead to less adapted investment decisions (i.e., reduce the first term in (OA.2)).

To clarify the private value of the test, let $A_I(k)$ represent investors' i interim equilibrium expected utility when the test leads to interim belief $\mu(k) = 1/(1+R(k))$ after which investors select a threshold k,

$$A_I(k) \equiv \frac{\widetilde{u}_i(k;\mu(k))}{\mu(k)} = \overline{F}(k) - R(k)\overline{G}(k) + \gamma\left(\overline{F}^2(k) + R(k)\overline{G}^2(k)\right) = \mathcal{E}(k) + \gamma A_R(k), \tag{OA.3}$$

where $\mathcal{E}(k) \equiv \mathbb{E}[\omega 1_{\{x_i \geq k\}}]/\mu(k)$ captures how adapted are investments to fundamentals and $A_R(k)$ is the regulator's average utility when $\eta = 1$. We can then use Proposition 1 and the properties of $A_I(k)$ to evaluate investors' value from stress testing.

Proposition 8. *For an optimal test* $\left\{\{\mu_l\}_{l\in I}, D, N\right\}$ *with* $k_l = k(\mu_l)$, *we have that* $V_i^{test} \geq 0$ *if and only if (a)* $\mu_0 \in [\mu_l, \mu_{l+1}] \subset N$, *or (b)* $\mu_0 \in (\mu_l, \mu_{l+1}) \subset D$, *and*

$$\frac{A_I(k_l) - A_I(k_0)}{R(k_l) - R(k_0)} \geq \frac{A_I(k_0) - A_I(k_{l+1})}{R(k_0) - R(k_{l+1})}. \tag{OA.4}$$

Proof of Lemma 8. Let the regulator's test be $\left\{\{\mu_l\}_{l\in I}, D, N\right\}$ and set $k_i = k(\mu_i)$. The equilibrium condition $\frac{1-\mu_i}{\mu_i} = R(k_i)$ implies

$$\mu_{i+1} - \mu_i = \frac{1}{1+R(k_{i+1})} - \frac{1}{1+R(k_i)} = \frac{R(k_i) - R(k_{i+1})}{(1+R(k_i))(1+R(k_{i+1}))},$$

and the expected utility of investor i if the test leads to posteriors $\{\mu_l, \mu_{l+1}\}$ is

$$\begin{aligned}
U_i(\mu_l, \mu_{l+1}) &\equiv \Pr[s = \mu_{l+1}]\, u_i(\mu_{l+1}) + \Pr[s = \mu_l]\, u_i(\mu_l) \\
&= \frac{\mu_0 - \mu_l}{\mu_{l+1} - \mu_l} u_i(\mu_{l+1}) + \frac{\mu_{l+1} - \mu_0}{\mu_{l+1} - \mu_l} u_i(\mu_l) \\
&= \left(\frac{R(k_l) - R(k_0)}{R(k_l) - R(k_{l+1})} \frac{1}{1+R(k_0)} A_I(k_{l+1}) + \frac{R(k_0) - R(k_{l+1})}{R(k_l) - R(k_{l+1})} \frac{1}{1+R(k_0)} A_I(k_l)\right) \\
&= \mu_0 \left(\frac{R(k_l) - R(k_0)}{R(k_l) - R(k_{l+1})} A_I(k_{l+1}) + \frac{R(k_0) - R(k_{l+1})}{R(k_l) - R(k_{l+1})} A_I(k_l)\right). \qquad \text{(OA.5)}
\end{aligned}$$

Absent the regulator, investors expected utility is $u_i(\mu_0) = \mu_0 A_I(k_0)$. Therefore, investors benefit from observing the regulator's test if

$$\frac{R(k_l) - R(k_0)}{R(k_l) - R(k_{l+1})} A_I(k_{l+1}) + \frac{R(k_0) - R(k_{l+1})}{R(k_l) - R(k_{l+1})} A_I(k_l) \geq A_I(k_0)$$

in other words, as $R(k)$ is increasing in k, when

$$\frac{A_I(k_l)-A_I(k_0)}{R(k_l)-R(k_0))}\geq\frac{A_I(k_0)-A_I(k_{l+1})}{R(k_0)-R(k_{l+1})}.$$

Letting $y_i=R(k_i)\left(=\frac{1-\mu_i}{\mu_i}\right)$, this expression is equivalent to

$$\frac{A_I(R^{-1}(y_l))-A_I(R^{-1}(y_0))}{y_l-y_0}\geq\frac{A_I(R^{-1}(y_0))-A_I(R^{-1}(y_{l+1}))}{y_0-y_{l+1}},$$

which is always satisfied if $A_I(R^{-1}(y))$ is convex in $y\in[\frac{1-\mu_{l+1}}{\mu_{l+1}},\frac{1-\mu_l}{\mu_l}]$, while the reverse inequality holds if $A_I(R^{-1}(y))$ is concave. □

Clearly, investors derive no value from regulation when the regulator does not release any substantive information, i.e., when $\mu_0\in[\mu_l,\mu_{l+1}]\subset N$. If, $\mu_0\in(\mu_l,\mu_{l+1})\subset D$, however, the test leads investors to interim priors μ_l,μ_{l+1}, and the value of the test depends on the curvature of their average utility with respect to their investment responsiveness—that is, on a comparison of the curvatures of $A_I(k)$ and $R(k)$ as expressed by (OA.4). For $\mu_0\in(\mu_l,\mu_{l+1})\subset D$, for instance, investors benefit from the test if $A_I(R^{-1}(y))$ is strictly convex in $y\in\left(\frac{\mu_{l+1}}{1+\mu_{l+1}},\frac{\mu_l}{1+\mu_l}\right)$ and are strictly worse off if it is strictly concave. We can sharpen our results for the case of a CF-test, $\{0,\underline{\mu}\}$, and CI-test, $\{\overline{\mu},\mu_{CI}\}$, whenever they exist.

Lemma OA.1. *(a) Suppose $\mu_0\in(0,\underline{\mu})$. Then, investors are (weakly) better off in the presence of a regulator if (i) $\gamma>0$, or (ii) $\gamma<0$, and for any $k\geq k(\underline{\mu})$ in the union of the support of F and G we have*

$$\frac{d\lambda}{dk}\geq\frac{|\gamma|}{1-|\gamma|\,\overline{F}\,(k)}\left(f(k)\left(\overline{\Lambda}\,(k)-\lambda(k)\right)\right). \tag{OA.6}$$

Proof of Lemma OA.1. (a) For a CF- test $\{0,\underline{\mu}\}$, we can simplify (OA.5) to

$$U_i\left(0,\underline{\mu}\right)\ \ =\Pr\left[s=\underline{\mu}\right]u_i(\underline{\mu})=\mu_0A_I(k(\underline{\mu})).$$

Therefore, investors are not worse-off by a CF-test iff $U_i\left(0,\underline{\mu}\right)\geq u_i(\mu_0)$, that is iff

$$A_I(k(\underline{\mu}))\geq A_I(k(\mu_0)). \tag{OA.7}$$

Using the expression for $R(k)$, we can express (OA.3) as

$$\begin{aligned}A_I(k)&=\overline{F}(k)\left(1+\gamma\overline{F}(k)\right)-R(k)\overline{G}(k)\left(1-\gamma\overline{G}(k)\right).\\&=\left(1+\gamma\overline{F}(k)\right)\left(\overline{F}(k)-\lambda(k)\overline{G}(k)\right).\end{aligned}$$

Differentiating this expression we have

$$A_I'(k) = -\gamma f(k)\left(\overline{F}(k) - \lambda(k)\overline{G}(k)\right) - \left(1 + \gamma\overline{F}(k)\right)\overline{G}(k)\lambda'(k). \tag{OA.8}$$

If $\gamma > 0$ then Assumption A1 implies that the right hand side is always negative.[54] Thus, $A_I'(k) < 0$ and (OA.7) is satisfied as $k(\underline{\mu}) \leq k(\mu_0)$. To study $\gamma < 0$, rewrite (OA.8) as

$$A_I'(k) = -\overline{G}(k)\lambda'(k)\left(1 + \gamma\left(\overline{F}(k) + \frac{f(k)}{\lambda'(k)}\left(\Lambda(k) - \lambda(k)\right)\right)\right).$$

Since $\overline{G}(k)\lambda'(k) \geq 0$, then whenever

$$\overline{F}(k) + \frac{f(k)}{\lambda'(k)}\left(\Lambda(k) - \lambda(k)\right) \quad \leq \quad \frac{1}{|\gamma|},$$

which is equivalent to (OA.6), then $A_I'(k) \leq 0$ and (OA.7) holds as $k(\underline{\mu}) \leq k(\mu_0)$. □

For a CF-test, (21) simplifies to

$$V_i^{test} = \Pr\left[s = \underline{\mu}\right] u_i(\underline{\mu}) - u_i(\mu_0) = \mu_0\left(A_I(k(\underline{\mu})) - A_I(k(\mu_0))\right), \tag{OA.9}$$

and the test is privately valuable iff the inconclusive realization raises investors' average equilibrium utility. Lemma OA.1-a shows that the sign of V_i^{test} depends on the type of investment externality. First, if $\gamma > 0$ investors' average utility (OA.3) is monotone in the investment threshold, implying that $A_I(k(\mu))$ increases in μ. Thus, if $\gamma > 0$, investors would benefit from *any* test that reveals that the state is $\omega = -1$. This is intuitive: if investors know that $\omega = -1$, then they will refrain from investing which is the investor-optimal response, while if the test reveals $\mu > \mu_0$ it boosts joint investment. Overall, with positive externalities investors always benefit from a CF-test.

If $\gamma < 0$, however, revealing $\mu > \mu_0$ raises joint investment and exacerbates the negative externality between investors. However, disclosure also leads to a higher correlation of investment and state. If private signals are locally very discriminating (as measured by the rate at which the likelihood ratio increases) so that (OA.6) holds, the benefit from improved matching dominates the negative effect from overinvestment and a CF-test is beneficial to investors even when there are negative externalities.

[54] Note that a monotone likelihood ratio implies that $\Lambda(k) \geq \lambda(k)$,so that $\overline{F}(k) - \lambda(k)\overline{G}(k)$.

References

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