

DISCUSSION PAPER SERIES

DP15614

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under Individual and Aggregate
Stochastic Volatility**

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MONETARY ECONOMICS AND FLUCTUATIONS

CEPR

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Discussion Paper DP15614
Published 28 December 2020
Submitted 26 December 2020

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Household Savings and Monetary Policy under Individual and Aggregate Stochastic Volatility

Abstract

In this paper, we study household consumption-saving and portfolio choices in a heterogeneous-agent economy with sticky prices and time-varying total factor productivity and idiosyncratic stochastic volatility. Agents can save through liquid bonds and illiquid capital and shares. With rich heterogeneity at the household level, we are able to quantify the impact of uncertainty across the income and wealth distribution. Our results help us in identifying who wins and who loses when during periods of heightened individual and aggregate uncertainty. To study the importance of heterogeneity in understanding the transmission of economic shocks, we use a deep learning algorithm. Our method preserves non-linearities, which is essential for understanding the pricing decisions for illiquid assets.

JEL Classification: N/A

Keywords: Machine Learning, deep learning, neural network, HANK, Heterogeneous Agents

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Acknowledgements

The authors are grateful to Marc Maliar for his help with writing the TensorFlow code.

Household Savings and Monetary Policy under Individual and Aggregate Stochastic Volatility*

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December 25, 2020

Abstract

We study a heterogeneous-agent model with sticky-prices in which total factor productivity and individual productivity are subject to stochastic volatility shocks. Agents save through liquid bonds and illiquid capital and shares. To construct equilibrium, we use a deep learning algorithm. Our method preserves non-linearities, which is essential for understanding portfolio choices. With rich heterogeneity at the household level, we are able to quantify the impact of uncertainty across the income and wealth distribution. We find that persistent high levels of uncertainty increase wealth inequality, and that in response to a contractionary monetary policy shock, illiquid wealth inequality decreases and liquid wealth inequality increases

Key Words : machine learning, deep learning, neural network, HANK, heterogeneous agents

*The authors are grateful to Marc Maliar for his help with writing the TensorFlow code.

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1 Introduction

In recent years, the field of macroeconomics has seen substantial development in the understanding of four major topics. First, models now incorporate heterogeneity in income or productivity and heterogeneity in assets with differing liquidity (e.g. liquid checking accounts and illiquid retirement accounts). Second, macroeconomists are studying the effects of uncertainty shocks either at the aggregate level in a representative-agent setup or at the individual level in a heterogeneous-agent setup. Third, in the aftermath of the Great Recession, economists became more interested in computing global solutions to properly account for the effective lower bound on nominal interest rates, borrowing constraints and other types of nonlinearities. Finally, with heterogeneity on the household side, they study redistribution due government fiscal transfers. Unfortunately, these four developments are usually done in isolation. In the present paper, we do all four in one integrated framework suitable for analyzing distributional implications of macroeconomic shocks and policies.

To this purpose, we build an incomplete market heterogeneous-agent new Keynesian model (HANK) that includes agents who face idiosyncratic labor productivity shocks, a non-trivial portfolio choice problem and borrowing constraints. Households decide on how many hours to supply to labor markets. To capture uncertainty at the individual level, the variance of the innovations to labor productivity is time varying. An individual agent can save through three assets that are divided into two categories: liquid and illiquid assets. Liquid assets, which we model as bonds, are freely tradeable at no cost. Trading illiquid assets, which, in the model, are machines and shares of production firms, is subject to an adjustment cost. The distinction between illiquid and liquid assets allows the model to replicate the empirical fact, first documented in Kaplan, Violante and Weidner (2014), that a sufficient share of households have little to no liquid wealth but positive amounts of illiquid wealth. These households, which have been described as the "wealthy hand-to-mouth", make consumption decisions similar to those with no aggregate savings. Therefore, even though they have sufficient wealth on paper to adjust spending and smooth consumption in response to adverse economic events, these individuals instead leave savings unadjusted and allow consumption to fluctuate. On the production side, the model features monopolistically competitive firms which are subject to price adjustment costs. The firms share a common level of total factor productivity. Similar to individual labor productivity, the variance of the total-factor-productivity (TFP) innovations is time varying, which allows us to capture the fact that uncertainty tends to vary over the business cycle.¹ Therefore, the model features uncertainty both at the household (individual) level and at the firm (aggregate) level. To close the model, we introduce a central bank that follows a Taylor rule with a zero lower bound (ZLB) on nominal interest rates.

Our main focus is on (re)distributive effects of aggregate uncertainty shocks. There were huge increases in uncertainty in recent years (due to, e.g., Brexit, political uncertainty associated with Trump's presidency, and COVID-19 pandemics). As ample evidence suggests, there are strong "first-order" (or level) effects from uncertainty shocks. Therefore, the main question we address: Who gains/loses from changes in uncertainty? To the best of our knowledge, we are the first to analyze predictions of a HANK model with fully-specified aggregate risk and uncertainty shocks (i.e., without making any simplifying assumptions on the properties of shocks or on the types of approximations). When studying the impact of aggregate risk and uncertainty, the previous literature restricts attention to one-time, aggregate MIT shocks, and thus, it does not allow one to study business cycle fluctuations. The models without uncertainty shocks but with aggregate MIT risk shocks appear, e.g., in Kaplan, Moll and Violante (2018), Alves, Kaplan, Moll and Violante (2020), while those with aggregate MIT uncertainty shocks are considered in Bayer, Luetticke, Pham-Dao and Tjaden (2019) and Schabb (2020).

The richness of our HANK model allows us to provide a meaningful quantitative assessment of the business cycle cost of individual and aggregate uncertainty. In particular, with rich heterogeneity at the household level, we are able to quantify the impact of uncertainty across the income and wealth

¹See, e.g., 'Measuring Economic Policy Uncertainty' by Scott Baker, Nicholas Bloom and Steven J. Davis at www.PolicyUncertainty.com.

distribution. Furthermore, the model allows us to study the importance of heterogeneity in understanding the transmission of economic shocks. In a representative-agent model, nearly all of the response to monetary policy shocks is driven by intertemporal substitution. Auclert (2019) and Kaplan, Moll and Violante (2018) both show that in heterogeneous-agent models, most of the response is driven by general-equilibrium (indirect) effects. Such effects work through changes in labor and asset income which arise due to changes in the consumption behavior of other agents in the model. The indirect effects often amplify the effects of shocks. Therefore, by working within a heterogeneous agent framework, our model is able to capture these effects.

HANK incomplete-market models with aggregate uncertainty are of the Krusell and Smith (1998) type. They are computationally intense as aggregate dynamics depend on the evolution of distribution. The recent literature has developed new numerical methods that allow to solve HANK models with distribution more efficiently. The examples include Ahn et al. (2018), Boppart et al. (2018), Bayer and Luetticke (2019), and Auclert et al. (2020). The common feature of these methods is that they find perturbation solutions at the aggregate level, expanding around the steady state level of aggregate TFP. As a result, one cannot generally extend these methods to allow for TFP dynamics over time.

Unlike the previous literature, we rely on a solution method that solves the model globally with both individual and aggregate uncertainty. Global solutions are needed as many traditional methods are not well suited for handling uncertainty shocks. For example, low order perturbation methods do not capture the effect of the volatility on the decision rules. As de Groot (2020) shows, even third order perturbation methods may not be sufficient. Since our solution algorithm solves the model globally, we preserve the effect time varying volatility has on the decision rules. Therefore, we are able to quantify the impact first and second moment innovations have on individuals across the income and wealth distribution over economic cycles.

Specifically, in this paper, we use deep learning techniques to solve the model. We build on a deep learning algorithm developed in Maliar, Maliar and Winant (2019) for solving the standard Krusell and Smith (1998) model. Deep learning methods are well suited for handling high-dimensional applications. In our model, agents need to know the cross sectional distribution of agents across states when making decisions. In other words, one of the arguments of the agents' policy function is an infinite dimensional object. To deal with this, the literature has developed methods to approximate the distribution. Our technique allows us to avoid the step of approximating the distribution. Using neural networks to approximate the policy functions, we are able to use the entire cross section of agents as an input to the neural network.

Our results help us identify who wins and who loses during periods of heightened individual and aggregate uncertainty. In the model, innovations to aggregate and individual uncertainty are negatively correlated with the wealth Gini; however, the level of aggregate uncertainty is positively correlated with it. Therefore, while innovations to aggregate uncertainty generate a decline in wealth inequality, persistent high levels of uncertainty increase wealth inequality. The result is driven by the fact that the excess return of illiquid assets increases in periods of high uncertainty, so agents with high levels of wealth benefit proportionally more. Additionally, due to differences in the marginal cost of investment, we find that illiquid wealth inequality declines following a monetary policy shock, while liquid wealth inequality increases. That is, agents with high levels of illiquid assets take advantage of the temporarily high rate of return on liquid assets while the higher marginal cost of investment for agents with low levels of illiquid assets prevents them from doing the same. Overall, our results highlight the importance of distinguishing between liquid and illiquid assets in studying the response of the economy to shocks.

We are related to the literature about the effects of economic uncertainty on the economy. Uncertainty shocks are modeled as temporary but persistent changes to the conditional variance of traditional economic shocks. Most of the research has considered representative agent setups; see, e.g., Basu and Bundick (2017), Born and Pfeifer (2014), Fernández-Villaverde, Guerrón-Quintana, Kuester and Rubio-Ramírez (2015). In addition to uncertainty about aggregate productivity, this literature considers other diverse sources of uncertainty; see Born and Pfeifer (2014) – monetary and fiscal policy, Kelly, Pastor and Veronesi (2016) and Pastor and Veronesi (2012, 2013) – political factors, Basu and Bundick (2017) – preference shocks,

Fernández-Villaverde et al. (2015) – fiscal instruments, Nodari (2014) – financial regulation policy, Stokey (2015) – future tax rates; VAR models with uncertainty shocks are estimated, e.g., in Mumtaz and Surico (2017), Mumtaz and Zanetti (2013).

Fluctuations in idiosyncratic uncertainty on the production side are studied in, e.g., Arellano, Bai and Kehoe (2019), Bloom, Floetotto, Jaimovich, Saporta-Ekstein and Terry (2018), Gilchrist, Sim and Zakrajšek (2014), Bahmann and Bayer (2013, 2014). For example, Bloom et al. (2018) study how increases in microeconomic uncertainty affect firm expansion and contraction and show that heightened uncertainty makes firms less likely to invest in productive capital or hire more workers and is a significant factor driving business cycles. One limitation of these heterogeneous-firm models, however, is that they do not consider how uncertainty affects households of different income and wealth levels since the consumer side features a representative household. We fill in this gap in the present paper.

Few papers thus far have considered stochastic volatility in heterogenous households models. Two notable exceptions are Bayer et al. (2019) and Schabb (2020). Bayer et al. (2019) consider a model in which households are subject to idiosyncratic labor productivity shocks where the variance of the innovation to labor productivity is time varying. However, in their model, the only aggregate risk that is allowed is a one time, probability zero shock (MIT shock). While this approach is useful for understanding the transmission mechanisms of a shock, it is ill suited for studying how individuals across the income and wealth distribution are affected by economic cycles. Schabb (2020) also considers a model with uncertainty at the household level. Unlike Bayer et al. (2019), Schabb (2020) allows for aggregate risk. In his model, agents are subject to common discount factor shocks. While Schabb (2020) solves the model globally with aggregate risk, when considering an uncertainty shock (modelled as an innovation to the variance of the discount factor process), he only considers perturbation solutions. That is, he models the innovation as an MIT shock. Therefore, while his solution method can be used to study how first moment innovations affect individuals over economic cycles, his method is unable to quantify the impact of second moment innovations.

The rest of the paper is organized as follows: Section 2 describes the HANK model with individual and aggregate stochastic volatility. Section 3 discusses the ability of the existing solution methods to solve HANK models with stochastic volatility. Section 4 outlines the methodology of our numerical analysis. Section 5 presents our numerical results, and finally, Section 6 concludes.

2 The HANK model

In this section, we outline our heterogeneous-agent new Keynesian model. The economy consists of households, perfectly competitive final-good producers, monopolistically competitive intermediate-good producers, government and a central bank. We build on a HANK model of Kaplan et al. (2018). We derive the model’s FOCs in Appendix A.

Households. As in Kaplan et al. (2018), households can save through liquid bonds and two illiquid assets, capital and shares. There is a continuum of heterogeneous households who are identical in fundamentals but differ in productivity and asset holdings. A household $j \in [0, 1]$ solves

$$\max_{c_t, \ell_t, i_t, s_t, \tilde{b}_t, k_t} E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t(j)^{1-\gamma} - 1}{1-\gamma} + \psi \frac{(1 - \ell_t(j))^{1-\vartheta} - 1}{1-\vartheta} \right] \quad (1)$$

$$\text{s.t. } P_t c_t(j) + P_t i_t(j) + \tilde{b}_t(j) + P_t \Psi(i_t(j), Q_t s_{t-1}(j) + k_{t-1}(j)) = R_{t-1} \tilde{b}_{t-1}(j) + P_t \zeta_t(j), \quad (2)$$

$$\zeta_t(j) = w_t \ell_t(j) \exp(\eta_{\ell,t}(j)) - \tau_t(j) + G_t + \delta \Pi_t, \quad (3)$$

$$Q_t s_t(j) + k_t(j) = \left[1 - d + r_t^k \right] k_{t-1}(j) + i_t(j) + [Q_t + (1 - \delta) \Pi_t] s_{t-1}(j), \quad (4)$$

$$\Psi(i_t(j), Q_t s_{t-1}(j) + k_{t-1}(j)) = \Gamma_1 |i_t(j)| + \frac{\Gamma_2}{\Gamma_3} \left(\frac{|i_t(j)|}{[Q_t + (1 - \delta)\Pi_t] s_{t-1}(j) + [1 - d + r_t^k] k_{t-1}(j) + \varepsilon} - \xi \right)^{\Gamma_3} \times \left([Q_t + (1 - \delta)\Pi_t] s_{t-1}(j) + [1 - d + r_t^k] k_{t-1}(j) + \varepsilon \right), \quad (5)$$

$$Q_t s_t(j) \geq 0, \quad k_t(j) \geq 0, \quad b_t(j) \geq -\bar{b}, \quad (6)$$

where initial condition $(s_{-1}(j), \tilde{b}_{-1}(j), k_{-1}(j), \eta_{\ell,0}(j))$ is given. Here, E_t is an expectation operator conditional on information of period t ; $\beta \in (0, 1)$ is the discount factor; $\gamma > 0$ are $\vartheta > 0$ are the utility-function parameters; $d \in (0, 1]$ is the depreciation rate of capital; $\Gamma_1, \Gamma_2, \Gamma_3, \xi > 0$ are parameters of the adjustment-cost function $\Psi(\cdot, \cdot)$; $c_t(j), \ell_t(j)$ and $i_t(j)$ are consumption, labor, investment in illiquid assets, respectively; $s_t(j)$ and $k_t(j)$ are illiquid assets – shares and machines (physical capital), and $b_t(j) \equiv \frac{\tilde{b}_t(j)}{P_t}$ is a liquid asset – real bonds; Π_t is the real dividends; δ is the share of profits that are non-traded; P_t, Q_t, R_{t-1} and w_t are a consumption-good price, real price of shares, nominal interest rate (the inverse of the price of a bond), and real wage, respectively; $\zeta_t(j)$ is non-asset income, which includes stochastic labor income $w_t \ell_t(j) \exp(\eta_{\ell,t}(j))$, taxes $\tau_t(j)$, government transfers G_t and transfers of untraded dividends $\delta \Pi_t$; \bar{b} is a borrowing limit on the liquid asset (nominal bonds). The tax function is

$$\tau_t(j) = \Upsilon_t \left[y_t(j) - (y_t(j)^{-\tau_2} + \tau_3)^{-\frac{1}{\tau_2}} \right] + \tau_0 y_t(j), \quad (7)$$

where $y_t(j) \equiv w_t \ell_t(j) \exp(\eta_{\ell,t}(j))$. The term Υ_t adjusts each period so that government spending as a share of output is equal to $\varpi \in (0, 1)$.

Agents hold capital and therefore, pay adjustment costs on capital, described by a function $\Psi(i_t(j), Q_t s_{t-1}(j) + k_{t-1}(j))$, which includes a linear and quadratic components. The purpose of the linear component is to generate a "wealthy" hand to mouth agents who have illiquid assets but no liquid assets and consume their entire labor income each period. The convex component includes a the term ε in the denominator so that we can match the fraction of agents who move from zero liquid assets to positive liquid assets. Without this component, the adjustment cost associated with going from zero illiquid assets to a positive amount of illiquid assets would be infinite. The parameter ξ is needed to ensure that steady state investment implies no change in the illiquid asset account; the latter differs from dK_t as it includes shares and capital.

The exogenous shock of labor endowment follows an exogenous stochastic process,

$$\eta_{\ell,t}(j) = \rho^\ell \eta_{\ell,t-1}(j) + \exp(\sigma_{\ell,t-1}) \varepsilon_{\ell,t}(j), \quad \varepsilon_{\ell,t} \sim \mathcal{N}(0, 1), \quad (8)$$

$$\sigma_{\ell,t} = \rho^{\sigma_\ell} \sigma_{\ell,t-1} + \sigma_{\sigma_\ell} \varepsilon_{\sigma_\ell,t}, \quad \varepsilon_{\sigma_\ell,t} \sim \mathcal{N}(0, 1). \quad (9)$$

That is, the variance of the innovations to labor endowment is time varying and the same across agents.

Production side. To accommodate sticky prices, we assume a standard two-tier production structure: perfectly competitive final-good firms aggregate intermediate goods, indexed by $h \in [0, 1]$, which are produced by monopolistically competitive firms into final goods. Final-good firms aggregate intermediate goods using a Dixit-Stiglitz aggregator with a constant elasticity of substitution $\frac{\mu}{\mu-1}$, i.e., $Y_t = \left(\int_0^1 Y_t(h)^{\frac{\mu-1}{\mu}} dh \right)^{\frac{\mu}{\mu-1}}$, $\mu \geq 1$. There is a mass one of identical intermediate-good firms, whose shares are traded and owned by households. Technology of each producer h is Cobb-Douglas in labor and capital,

$$Y_t(h) = \exp(\eta_{\theta,t}) K_{t-1}(h)^\alpha H_t(h)^{1-\alpha},$$

where $H_t(h)$ is efficiency labor; $K_{t-1}(h)$ is capital; $\exp(\eta_{\theta,t})$ is the productivity level that follows an exogenous stochastic process

$$\eta_{\theta,t} = \rho^\theta \eta_{\theta,t-1} + \exp(\sigma_{\theta,t-1}) \varepsilon_{\theta,t}, \quad \varepsilon_{\theta,t} \sim \mathcal{N}(0, 1), \quad (10)$$

$$\sigma_{\theta,t} = \rho^{\sigma_\theta} \sigma_{\theta,t-1} + \sigma_{\sigma_\theta} \varepsilon_{\sigma_\theta,t}, \quad \varepsilon_{\sigma_\theta,t} \sim \mathcal{N}(0, 1). \quad (11)$$

Therefore, volatility of the productivity shock is a stochastic variable, following the first-order autoregressive process.

Each intermediate-good firm i chooses a price $P_t(h)$ in period t subject to Rotemberg (1982) adjustment costs,

$$\xi(P_t(h), P_{t-1}(h)) \equiv \frac{\mu}{2\kappa^p(\mu-1)\pi^*} \left(\log \left(\frac{P_t(h)}{P_{t-1}(h)} \frac{1}{\pi^*} \right) \right)^2,$$

where $\kappa^p > 0$.

An intermediate-good firm maximizes its present discounted value of future profits subject to demand for its goods,

$$\max_{P_t(h), K_t(h), L_t(h)} E_t \sum_{l=0}^{\infty} \beta^l \tilde{\Lambda}_{t,t+l} [P_{t+l}(h) Y_{t+l}(h) - mc_{t+l} P_{t+l} Y_{t+l}(h)] \quad (12)$$

$$\left. - P_{t+l} \exp(\eta_{\theta,t}) K_{t+l-1}^\alpha H_{t+l}^{1-\alpha} \frac{\mu}{2\kappa^p(\mu-1)\pi^*} \left(\log \left(\frac{P_{t+l}(h)}{P_{t+l-1}(h)} \frac{1}{\pi^*} \right) \right)^2 \right] \\ \text{s.t. } Y_{t+l}(h) = \left(\frac{P_{t+l}(h)}{P_{t+l}} \right)^{\frac{\mu}{1-\mu}} \exp(\eta_{\theta,t}) K_{t+l-1}^\alpha H_{t+l}^{1-\alpha}, \quad (13)$$

where $\tilde{\Lambda}_{t,t+l}$ is the stochastic discount factor, $\tilde{\Lambda}_{t,t+l} \equiv \frac{E_t \left[\int_{j \in U_{t+l}^{illiq}} c_{t+l}^{-\gamma}(j) dj \right]}{\int_{j \in U_t^{illiq}} c_t^{-\gamma}(j) dj}$; mc_t is real marginal cost; U_t^{illiq} is the set of agents at time t who are not against their illiquid asset borrowing constraint. Deriving the first-order condition (FOC) with respect to $P_t(h)$ and imposing $P_t(h) = P_t$, we obtain the Phillips curve

$$\log \left(\frac{\pi_t}{\pi^*} \right) - \frac{\kappa^p}{\mu} (mc_t \mu - 1) = \beta E_t \tilde{\Lambda}_{t,t+1} \log \left(\frac{\pi_{t+1}}{\pi^*} \right) \frac{Y_{t+1} \Delta_t^p}{Y_t \Delta_{t+1}^p}. \quad (14)$$

The marginal costs of intermediate-good production are obtained from the standard cost-minimization problem

$$\min_{K_{t-1}(h), L_t(h)} TC(Y_t(h)) = \left\{ R_t^k K_{t-1}(h) + W_t H_t(h) \right\} \quad (15)$$

$$\text{s.t. } Y_t(h) = \exp(\eta_{\theta,t}) K_{t-1}^\alpha H_t^{1-\alpha}, \quad (16)$$

where W_t is the nominal wage rate. This problem implies that the real marginal cost is not firm specific, $mc_t(h) = mc_t$ (see the appendix for details).

Government. The government runs a balanced budget each period. The government controls the tax parameter Υ_t in (7) so that the following holds period by period

$$\varpi Y_t = \int_0^1 \tau_t(j) dj. \quad (17a)$$

Monetary authority. The monetary authority sets the nominal interest rate according to the following Taylor rule:

$$R_t \equiv \max \left\{ 1, R_* \left(\frac{R_{t-1}}{R_*} \right)^\mu \left[\left(\frac{\pi_t}{\pi_*} \right)^{\phi_\pi} \left(\frac{Y_t}{Y_*} \right)^{\phi_y} \right]^{1-\mu} \exp(\eta_{R,t}) \right\}, \quad (18)$$

where R_t and R_* are the gross nominal interest rate at t and its long-run value, respectively; π_* is the inflation target; Y_* is the output target; Y_t the level of output; $\mu \in [0, 1]$; $\eta_{R,t}$ is a monetary policy shock following the standard exogenous stochastic process

$$\eta_{R,t} = \rho^R \eta_{R,t-1} + \sigma_R \varepsilon_{R,t}, \quad \varepsilon_{R,t} \sim \mathcal{N}(0, 1). \quad (19)$$

Output. Aggregate output Y_t is

$$Y_t = \exp(\eta_{\theta,t}) K_{t-1}^\alpha H_t^{1-\alpha} \Delta_t^p,$$

where $\Delta_t^p \equiv 1 - \frac{\mu}{2\kappa^p(\mu-1)\pi_*} \left(\log \left(\frac{\pi_t}{\pi_*} \right) \right)^2$ is a share of production remaining after paying price adjustment costs and $H_t = \int_0^1 \ell_t(j) \exp(\eta_{\ell,t}(j)) dj$.

Aggregate resource constraint. Using the government budget constraint and the household's budget constraint, we obtain the aggregate resource constraint

$$C_t + K_t - (1-d)K_{t-1} + \int_0^1 \text{AC}(h) dh = Y_t, \quad (20)$$

where $\int_0^1 \text{AC}(h) dh = \int_0^1 \Psi(i_t(j), Q_t s_{t-1}(j) + k_{t-1}(j)) dj$ is aggregate adjustment cost; $C_t = \int_0^1 c_t(j) dj$ and $K_{t-1} = \int_0^1 k_{t-1}(j) dj$ are aggregate consumption and capital, respectively.

3 Discussion

Without aggregate risk, the split between shares and capital is indeterminate. The previous HANK literature accommodates this feature of the model by exogenously specifying how much of the illiquid asset account is held in shares and how much is held in capital; see Kaplan et al. (2018), Auclert et al. (2020) and Bayer and Luettkie (2019). Since in the present analysis, there is aggregate risk, and the risk properties of the two assets differ, agents in the model face a non degenerate portfolio choice problem within their illiquid account.

Determinacy of the split between shares and machines. To understand why in our analysis the split between the shares and capital is determinate, we derive a formula that relates expected returns of shares $E_t [R_{t+1}^s]$ and capital $E_t [R_{t+1}^k]$. Specifically, let $R_{t+1}^s \equiv \frac{Q_{t+1} + (1-\delta)\Pi_t}{Q_{t+1}}$ and $R_{t+1}^k \equiv [1-d + r_{t+1}^k]$. The household's FOCs (omitting the household index) imply the following:

$$E_t [R_{t+1}^s] - E_t [R_{t+1}^k] = \left\{ \frac{v_t^2 - v_t^1}{\beta} + \sigma_{t+1}^s \left[\sigma_{t+1}^{R^k} \text{Corr} \left(R_{t+1}^k, \frac{\lambda_{t+1}}{\lambda_t} [\varkappa_{t+1} + q_{t+1}] \right) - \sigma_{t+1}^{R^s} \text{Corr} \left(R_{t+1}^s, \frac{\lambda_{t+1}}{\lambda_t} [\varkappa_{t+1} + q_{t+1}] \right) \right] \right\} \cdot \left[E_t \left[\frac{\lambda_{t+1}}{\lambda_t} [\varkappa_{t+1} + q_{t+1}] \right] \right]^{-1}, \quad (21)$$

where $\sigma_{t+1}^s \equiv \sqrt{\text{Var} \left(\frac{\lambda_{t+1}}{\lambda_t} [\varkappa_{t+1} + q_{t+1}] \right)}$, $\sigma_{t+1}^{R^s} \equiv \sqrt{\text{Var} (R_{t+1}^s)}$, $\sigma_{t+1}^{R^k} \equiv \sqrt{\text{Var} (R_{t+1}^k)}$; see Appendix B for details. In (21), v_t^1 and v_t^2 are the Lagrange multipliers associated with the borrowing constraints on illiquid assets, shares and capital, respectively; \varkappa_{t+1} is defined in (35) and represents a derivative of the

adjustment cost function with respect to illiquid assets (multiplied by a gross return on illiquid assets); q_{t+1} is the shadow value of one additional unit of illiquid assets.

Equation (21) defines a split between shares and machines since the correlation between the returns on the two assets and the term $\frac{\lambda_{t+1}}{\lambda_t} [\varkappa_{t+1} + q_{t+1}]$ depends on the agents portfolio. In the absence of adjustment costs, the latter term reduces to the standard stochastic discount factor, $\frac{\lambda_{t+1}}{\lambda_t}$. Therefore, we can view the term $\varkappa_{t+1} + q_{t+1}$ as adjusting the stochastic discount factor due to the illiquid nature of the asset.

Suppose that the expected returns on illiquid assets are exogenously given and equal. Then, the left side of (21) is zero. If the borrowing constraints on the illiquid assets are non-binding, $v_t^2 = v_t^1 = 0$, we have

$$\sigma_{t+1}^{R^k} \text{Corr} \left(R_{t+1}^k, \frac{\lambda_{t+1}}{\lambda_t} [\varkappa_{t+1} + q_{t+1}] \right) = \sigma_{t+1}^{R^s} \text{Corr} \left(R_{t+1}^s, \frac{\lambda_{t+1}}{\lambda_t} [\varkappa_{t+1} + q_{t+1}] \right).$$

In the presence of MIT shocks, the returns on capital and shares are constant so that the two correlations above are trivially zero: any split between the shares and capital satisfies this equation. In the presence of aggregate uncertainty, both correlations are nonlinear functions of $\frac{\lambda_{t+1}}{\lambda_t} [\varkappa_{t+1} + q_{t+1}]$. Re-adjusting the illiquid-asset portfolio by increasing the holdings of one asset and reducing those of the other, without affecting the total holdings, would affect \varkappa_{t+1} but not q_{t+1} . Given that $\text{Corr} \left(R_{t+1}^k, \frac{\lambda_{t+1}}{\lambda_t} [\varkappa_{t+1} + q_{t+1}] \right)$ is a nonlinear function of \varkappa_{t+1} , we would expect a split between two types of illiquid assets to be determinate. If the split were to be indeterminate, then after increasing $Q_{t+1} s_t(j)$ and decreasing $k_t(j)$, the change in $\text{Corr} \left(R_{t+1}^k, \frac{\lambda_{t+1}}{\lambda_t} [\varkappa_{t+1} + q_{t+1}] \right)$ would have to be equal to $\frac{\sigma_{t+1}^{R^s}}{\sigma_{t+1}^{R^k}}$ times the change in $\text{Corr} \left(R_{t+1}^s, \frac{\lambda_{t+1}}{\lambda_t} [\varkappa_{t+1} + q_{t+1}] \right)$. That is, if the correlation between $\frac{\lambda_{t+1}}{\lambda_t} [\varkappa_{t+1} + q_{t+1}]$ and R_{t+1}^s were to increase with the fraction of illiquid assets held in shares, then the correlation between $\frac{\lambda_{t+1}}{\lambda_t} [\varkappa_{t+1} + q_{t+1}]$ and R_{t+1}^k would need to decrease with the fraction of illiquid assets held in machines. Therefore, we would need it to be that a higher concentration of one type of illiquid asset in total illiquid assets reduces the correlation between the return on that asset and $\frac{\lambda_{t+1}}{\lambda_t} [\varkappa_{t+1} + q_{t+1}]$. We conclude, therefore, that in general, a split between shares and capital in the illiquid-asset portfolio is determinate under the assumption of aggregate uncertainty.

4 Methodology of our numerical analysis

We rely on deep learning framework introduced in Maliar, Maliar and Winant (2019). A distinctive feature of this framework is that it allows us to solve the model by working directly with the actual state space. Specifically, we parameterize some individual and aggregate policy rules with a neural network.

In the model, there are nineteen aggregate variables,

$$\left\{ C_t, H_t, K_t, I_t, Y_t, \Delta_t^p, i_t, \pi_t, w_t, r_t^k, mc_t, Q_t, \Pi_t, \Upsilon_t, R_{t-1}, \eta_{R,t}, \eta_{\theta,t}, \sigma_{\theta,t}, \sigma_{\ell,t} \right\},$$

and eleven individual variables,

$$\left\{ c_t(j), \ell_t(j), k_t(j), i_t(j), s_t(j), b_t(j), q_t(j), \eta_{\ell,t}(j), v_t^1(j), v_t^2(j), \varphi_t(j) \right\},$$

where the latter three are Lagrange multipliers associated with the borrowing constraints, and $q_t(j)$ is the value of an additional unit of illiquid assets (equal to a ratio of the Lagrange multipliers, associated with the constraints (2) and (5)). Among these variables, there are seven state variables in total: one endogenous aggregate state variable $\{R_{t-1}\}$, four exogenous aggregate state variables $\{\eta_{R,t}, \eta_{\theta,t}, \sigma_{\theta,t}, \sigma_{\ell,t}\}$, three endogenous individual state variables $\{k_{t-1}(j), s_{t-1}(j), b_{t-1}(j)\}$ and one exogenous individual state variable $\{\eta_{\ell,t}(j)\}$. Let J denote the number of agents used in constructing the solution. Therefore the state space includes $4J + 5$ dimensions.

Neural networks. We parameterize the model with two neural networks, each of which includes four hidden layers. Each layer has 32 neurons. We use leaky relu as activation functions. To update the gradient, we use the ADAM optimization algorithm. Finally, we use a batch size of 10. From the two neural networks, the first one has three outputs corresponding to four aggregate variables: marginal costs mc_t , inflation π_t , the tax parameter Υ_t and the share price Q_t . The second neural network has seven outputs corresponding to seven individual variables: labor $\ell_t(j)$, share of illiquid assets out of income net of consumption and the borrowing limit (denoted by $\xi_t^a(j)$), share of capital in illiquid assets $\xi_t^k(j)$ and the three multipliers on the inequality constraints $v_t^1(j)$, $v_t^2(j)$, $\varphi_t(j)$. Therefore, with symmetric agents, as in Krusell and Smith (1998), we need to approximate just seven $4J + 5$ -dimensional decision function to characterize the labor choices of all J agents.

Given the weights of the neural network, we compute the remaining aggregate variables in the following order:

$$\begin{aligned} \ell_t(j) &\rightarrow H_t, \\ k(j) &\rightarrow K_t, \\ \pi_t &\rightarrow \Delta_t^p \rightarrow Y_t \\ mc_t &\rightarrow r_t^k \text{ and } w_t \rightarrow \Pi_t, \\ \pi_t, Y_t \text{ and } R_{t-1} &\rightarrow R_t. \end{aligned}$$

To find individual variables, we first use labor to find consumption, $\ell_t(j) \rightarrow c_t(j)$. Given the individual states and the known values of aggregate variables, we can find the agent's budget constraint, which we denote by $M_t(j)$

$$\begin{aligned} M_t(j) \equiv \frac{R_{t-1}}{\pi_t} b_{t-1}(j) + w_t \ell_t(j) \exp(\eta_{\ell,t}(j)) + \delta \Pi_t + G_t - \tau_t(j) \\ + \left[1 - d + r_t^k\right] k_{t-1}(j) + [Q_t + (1 - \delta) \Pi_t] s_{t-1}(j). \end{aligned} \quad (22)$$

Hence, $M_t(j) \equiv b_t(j) + a_t(j) + \Psi(a_t(j)) + c_t(j)$, where $\Psi(j)$ denotes adjustment costs paid by agent j . We choose

$$a_t(j) = \max(\xi_t^a(j) \cdot [M_t(j) - \bar{b} - c_t(j)], 0.0). \quad (23)$$

Then, we find

$$\begin{aligned} a_t(j) &\rightarrow i_t(j) \rightarrow \Psi(a_t(j)) \rightarrow b(j), \\ i_t(j) &\rightarrow q_t(j). \end{aligned}$$

where $b_t(j) = \max(M_t(j) - a_t(j) - \Psi(a_t(j)) - c_t(j), \bar{b})$. Finally, we find

$$k_t(j) = \xi_t^k(j) \cdot a_t(j). \quad (24)$$

We form the objective function that minimizes the Euler residuals in individual and aggregate FOCs. For the list of such FOCs, see Appendix B.

Relation to the literature. Our deep learning algorithm is capable of dealing with both aggregate and individual risk, as well as with aggregate and individual uncertainty. Note that the previous literature could not address both idiosyncratic and aggregate stochastic volatility. Auclert et al. (2020) build on the work of Reiter (2009) by using the sequence space formulation to solve the model. The method solves the household problem under idiosyncratic risk, and, would be amenable to the introduction of individual uncertainty.² However, their method does not allow for general aggregate risk. Additionally, since it uses

² As Farmer and Toda (2017) show, one can discretize an AR(1) process with stochastic volatility. The discretization results in a state dependent transition matrix. Since this has no effect on the average idiosyncratic productivity, and, hence, doesn't affect the firm's decisions, the method proposed by Auclert et al. (2020) could be used to solve a model with uncertainty at the household level.

a first order approximation of the path of aggregates, one could not use the method to study the response to an increase in aggregate uncertainty.

Bayer and Luetticke (2019) propose another method for solving heterogenous agent models with incomplete markets. Similar to Auclert et al. (2020), their method solves the household problem without aggregate risk. Once solving for the value functions and stationary distributions, the authors approximate them with a family of basis functions. To reduce the dimensionality of the problem, when linearizing, they allow only a reduced number of basis function coefficients to vary. The perturbed system is then solved using first or second order perturbation methods. Since all non-linearities are preserved at the individual level, one could presumably use the method to study the case where there is stochastic volatility at the household level.³ However, as is well documented, for example, in Fernandez-Villaverde (2016), second order perturbation methods are not sufficient for capturing the effect of time varying volatility on decision rules. Therefore, their method would not be applicable for a model that includes aggregate stochastic volatility.

Calibration procedure. The period in the model is one quarter. When choosing parameters for the illiquid asset adjustment cost function, we target the moments of the wealth distribution documented by Kaplan, Violante and Weidner (2014). Table 1 summarizes our choices of the parameter values. The rest of the model’s parameters are computed from the first order conditions; see Appendix B for details.

5 Numerical results

In this section, we present the results of our numerical experiments. In the benchmark case, we solve the model with 50 agents. We present some additional numerical results in Appendix D.

5.1 Decision rules

Figure 1 illustrates how agents choose to divide their savings between the three assets. In the figure, each decision rule corresponds to a different level of individual productivity, $\eta_{\ell,t}(j)$; we use five realizations of $\eta_{\ell,t}(j)$ evenly spaced between -0.8846 and 0.8846 for five levels of individual productivity. The level of the variable is on the vertical axis, while the horizontal axis is machines. The other state variables are set to the following values:

State variable	R	b	s	η_R	η_θ	σ_ℓ	σ_θ
Value	R_*	0.0	0.0	0.0	0.0	$\ln \sigma_\ell^*$	$\ln \sigma_\theta^*$

Here, R_* is a long-run nominal interest rate; σ_ℓ^* and σ_θ^* are standard deviations of idiosyncratic productivity and TFP shocks, respectively, in the absence of uncertainty shocks; see Table 1 for the values assumed.

³Their general method does not consider the case of time varying volatility; see their equation (2).

<i>Parameter</i>	<i>Description</i>	<i>Target/Source</i>
$\gamma = 1.55$	Risk aversion	standard
$\vartheta = 2$	Labor supply elasticity	standard
$\beta = .975$	Discount factor	standard
$d = 0.0135$	Depreciation rate	standard
$\alpha = 0.36$	Capital share	standard
$\mu = \frac{10}{9}$	Elasticity of substitution among goods	profits share of 10%
$\kappa^p = 0.1$	Price adjustment cost	average price duration of 1 year
$\delta = .36$	Fraction of untraded profits	Kaplan et al. (2018)
$\Gamma_1 = 0.05$	Illiquid asset adjustment cost	
$\Gamma_2 = 0.95$	Illiquid asset adjustment cost	
$\Gamma_3 = 2.0$	Illiquid asset adjustment cost	
$\xi = 0.0$	Illiquid asset adjustment cost	
$\varepsilon = 4.0$	Illiquid asset adjustment cost	
$\bar{b} = -0.1$	Liquid asset borrowing constraint	75% of people have liquid assets Kaplan et al. (2014)
$\rho^\ell = 0.9$	Persistence of idiosyncratic shocks	persistence of annual wage =.92
$\sigma_\ell^* = 0.1928$	Standard deviation of idios.-level shocks (in the absence of uncertainty shocks)	standard deviation of annual wage=0.7
$\rho^{\sigma_\ell} = 0.84$	Persistence of idios.-volatility shocks	Based on Bayer et al. (2019)
$\sigma_{\sigma_\ell} = 0.27$	Standard deviation of idios.-volatility shocks	Based on Bayer et al. (2019)
$\rho^\theta = 0.9$	Persistence of TFP-level shocks	standard
$\sigma_\theta^* = 0.016$	Standard deviation of TFP-level shocks (in the absence of uncertainty shocks)	standard
$\rho^{\sigma_\theta} = 0.73$	Persistence of TFP-volatility shocks	Based on Fernández-Villaverde et al. (2015)
$\sigma_{\sigma_\theta} = 0.23$	Standard deviation of TFP-volatility shocks	Based on Fernández-Villaverde et al. (2015)
$\rho^R = 0.25$	Persistence of monetary-policy shocks	standard
$\sigma_R = 0.016$	Standard deviation of monetary-policy shocks	standard
$\tau_0 = 0.0525$	Tax function parameter	Bachmann et al. (2020)
$\tau_2 = 0.768$	Tax function parameter	Bachmann et al. (2020)
$\tau_3 = 1.776$	Tax function parameter	Bachmann et al. (2020)
$\varpi = 0.1$	Government spending as a share of output	

Table 1: Calibration of the model's parameters

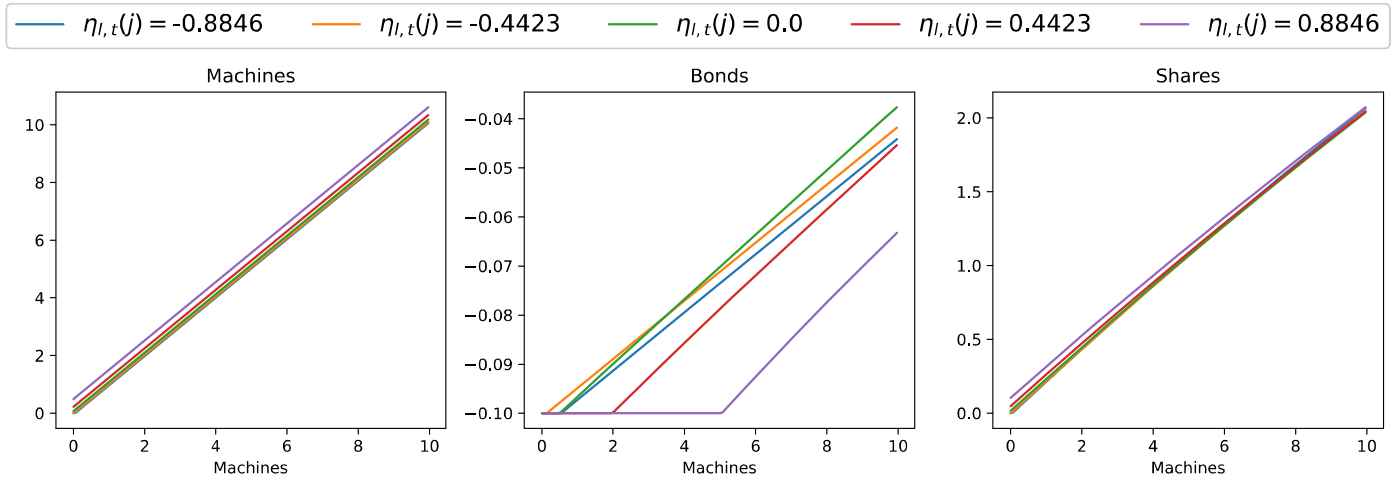


Figure 1. Decision functions for different types of assets.

From the figure, and consistent with the discussion, we see that the shapes of the policy functions for machines and shares differ. The difference between the choice of next period machines for agents of different productivity levels is essentially constant for all levels of machine holdings. For shares, however, we see that the difference between the share holdings for the highest productivity agent and the lowest productivity agent is much greater for lower levels of machine holdings than for higher levels of machine holdings. The result is driven by the difference in the risk of the two assets and the covariance between the asset returns and $\frac{\lambda_{t+1}}{\lambda_t} [z_{t+1} + q_{t+1}]$, as we discuss in Section 3. The bonds decision function has a kink at all levels of individual productivity plotted. Note that the agent with the kink furthest to the right on the machines axis is the highest productivity agent. The high productivity agent prefers to accumulate more capital by borrowing through liquid assets (other things being equal).

5.2 Distribution of assets

We now present the results about wealth and income distributions.

Wealth Gini coefficients. Table 2 shows the Gini coefficients of total wealth, liquid wealth and illiquid wealth in the model and in the data. In particular, for the model, the first row shows the average Gini coefficient, and the second row reports the lower and upper limits of 95% confidence intervals. For the U.S. data, we report the Gini coefficient of total wealth, taken from Krusell and Smith (1998).

	Gini Coefficients		
	Total Wealth	Liquid Wealth	Illiquid Wealth
	Model		
Average	0.573	0.737	0.561
95% CI	(0.569,0.576)	(0.729,0.742)	(0.557,0.564)
	Data		
	0.79	–	–

Table 2: Wealth Gini coefficients.

As we can see, the total wealth Gini coefficient in the model is somewhat lower than that in the data; however, it is substantially higher than what standard models like Ayagari (1994) and Krusell and Smith (1998) (without discount-factor shocks) are able to generate. In our calibration, we did not target the

Distribution Percentile	0-10	10-25	25-50	50-75	75-90	90-100
Income Distribution						
Mean	0.00544	0.01101	0.07964	0.51338	1.48973	97.90080
95% CI	(0.00513, 0.00576)	(0.01044, 0.01158)	(0.07758, 0.08170)	(0.45351, 0.57326)	(1.36091, 1.61856)	(97.71914, 98.08246)
Wealth Distribution						
Mean	-0.00109	0.00474	0.07149	0.50816	1.49878	97.91791
95% CI	(-0.00115, -0.00103)	(0.00435, 0.00514)	(0.06942, 0.07356)	(0.44743, 0.56890)	(1.36839, 1.62918)	(97.73354, 98.10228)

Table 3: Wealth and income shares of the selected groups of the distribution.

size of the Gini coefficient but fractions of households on the borrowing constraints for liquid and illiquid assets; as a result, we could have obtained larger wealth Gini coefficient if we modify the other targets. Furthermore, in the table, liquid wealth is distributed more unequally than illiquid wealth: the Gini of liquid wealth is 0.74 versus 0.56 for that of illiquid wealth. This is because there are more households on the borrowing constraint for liquid wealth than for illiquid wealth (see Table 4).

Wealth and income shares held by selected percentiles of the distribution. Table 3 shows shares of income and wealth held by the bottom 10 percent, 10 to 25 percent, 25 to 50 percent, 50 to 75 percent, 75 to 90 percent and the top 10 percent of the population. In the data, households in the top 10 percent of the wealth distribution hold 64 percent of total wealth; see Krusell and Smith (1998). In turn, our model predicts a substantially higher share and thus, overstates the wealth holdings of wealthy households. As a result, the model generates too small wealth holdings at the bottom of the wealth distribution. Again, we should emphasize that standard models of wealth inequality (Ayagari, 1994, Krusell and Smith, 1998) severely underpredict wealth inequality and the wealth share.

Fractions of the population on the borrowing constraints. Table 4 shows the fractions of households with zero illiquid wealth – shares and machines – taken together and separately, and the fraction of households on the borrowing constraint for liquid wealth (bonds); we provide both the means and confidence intervals of the corresponding statistics. In the model, 11 percent of the population does not hold illiquid assets. The result is roughly consistent with the empirical observation on the U.S. economy reported in Kaplan et al. (2014) that the share of households who do not hold illiquid wealth (referred to as poor hand-to-mouth) varies between 8 and 13 percent of the US population.

	Fraction of Agents			
	No Illiquid Wealth	No Machines	No Shares	Bond Constrained
Mean	0.112	0.112	0.112	0.176
95% CI	(0.108,0.115)	(0.108,0.115)	(0.108,0.115)	(0.171,0.180)

Table 4: Fractions of households on the borrowing constraints.

Furthermore, the model predicts that 18 percent of the population borrows liquid assets (bonds) up to the borrowing limit. In our solution, the household who hold zero illiquid wealth happen to be also on the borrowing constraint for liquid assets. Consequently, there are 7 percent of households who hold machines and shares but are still on the borrowing constraint for bonds.

5.3 Aggregates and uncertainty

Table 5 shows the correlations between aggregate variables and different types of shocks. We focus on four different types of shocks, and four different types of corresponding innovations (we exclude the level of idiosyncratic shocks and their innovations from consideration because the average productivity across agents is time invariant).

Shock	Aggregate Variables					
	Consumption	Labor	Investment	Net Output	Wages	Risk Premium
	Levels					
TFP Level	0.6429	-0.0030	0.3093	0.5446	0.7155	-0.0432
MP Level	0.0032	0.0044	-0.0054	0.0077	-0.0079	-0.1272
TFP Level Innov.	-0.0629	-0.0055	-0.0360	-0.0527	-0.0665	0.0071
MP Level Innov.	0.0012	0.0019	-0.0085	0.0042	-0.0027	0.0346
	Uncertainty					
TFP Uncertainty	0.0045	0.0110	0.0100	0.0044	0.0058	0.0108
Indv. Uncertainty	0.0111	0.0066	0.0036	0.0104	0.0070	0.0008
TFP Uncertainty Innov.	-0.0039	-0.0056	-0.0055	-0.0046	0.0006	0.0050
Indv. Uncertainty Innov.	0.0060	0.0031	0.0034	0.0063	0.0032	-0.0000

Table 5: Correlations between aggregate variables and shocks.

As is seen from the table, both types of uncertainty shocks we consider increase all aggregate variables in the table. That is, in periods of heightened uncertainty, agents consume, invest, work and produce more at the aggregate level, which might seem surprising but it only reflects aggregate effects and curtains the changes at the individual level. What is surprising TFP uncertainty innovations do correlate negatively with aggregate consumption, labor, investment and output, which implies that TFP uncertainty correlates positively with these aggregate variables because of the past values of the innovations. TFP level shocks tend to decrease labor because wages are significantly positively correlated with the TFP level. Additionally, a positive monetary policy shock tends to decrease investment. This is because contractionary monetary policy decreases the risk premium (see the strongly negative correlation between the monetary policy shock and risk premium), so that less wealth is invested into machines. In Appendix D, we provide a table on standard deviations of aggregate variables generated by the model.

To understand how inequality responds to distinct types of shocks, in Table 6, we document the correlations between the shocks and the Gini coefficients of wealth, income and consumption; for wealth, we consider Gini coefficients of the distributions of total assets, illiquid assets, and liquid assets.

Shock	Wealth Gini			Income Gini	Consumption Gini
	Total	Liquid	Illiquid		
	Levels				
TFP Level	-0.0575	0.0123	-0.0447	-0.0010	-0.1142
MP Level	-0.0493	0.0050	-0.0181	0.0041	-0.0516
TFP Level Innov.	0.0467	0.0025	0.0453	-0.0061	0.0529
MP Level Innov.	-0.0390	0.0015	-0.0135	0.0028	-0.0439
	Uncertainty				
TFP Uncertainty	0.0005	0.0036	-0.0029	0.0077	0.0039
Indv. Uncertainty	-0.0033	0.0007	-0.0010	0.0070	-0.0093
TFP Uncertainty Innov.	-0.0027	-0.0039	-0.0004	-0.0045	-0.0026
Indv. Uncertainty Innov.	-0.0009	0.0009	-0.0007	0.0019	-0.0017

Table 6: Correlations between Gini coefficients and shocks.

From the table, we see that the illiquid wealth Gini coefficient responds to shocks similarly to the total wealth Gini coefficient, and the response of the liquid wealth Gini goes in the opposite direction (except for innovations to TFP level shocks and TFP uncertainty shocks which act in the same direction). Thus, when a shock/innovation generally affects illiquid wealth inequality in one direction, the effect on liquid wealth inequality is opposite, i.e., liquid and illiquid assets are substitutes in households' portfolios. A decrease in the illiquid wealth Gini coupled with an increase in the liquid wealth Gini implies that low and middle wealth agents are adjusting their portfolios so that more of their wealth is held in illiquid assets.

Regarding the impacts of different types of shocks/innovations, we observe the following tendencies: First, the TFP-level shock is positively correlated with the liquid-wealth Gini coefficient but negatively correlated with the illiquid-wealth Gini coefficient. An improvement in technology leads to less inequality

in illiquid assets for the following reasons: All else equal, wages and the return on capital rise. Since the second part of the adjustment cost function makes changes in one's asset position relative to the beginning-of-the-period value of one's illiquid assets, an increase in TFP reduces the contribution of this part of the cost of adjusting one's illiquid asset holdings. For agents with low levels of illiquid assets, the decline in the marginal cost of investment is greater than the decline for agents with high levels of illiquid assets. Therefore, agents with low levels of illiquid wealth invest at a higher rate than those with high levels of illiquid wealth, leading to a decline in the illiquid wealth Gini.

It is not surprising that better technology decreases wealth inequality. An increase in TFP increases wages and the return on capital. Higher wages allow low wealth individuals to accumulate more wealth while maintaining or increasing consumption levels. Additionally, the higher returns on illiquid assets incentivize agents to accumulate more illiquid wealth. Therefore, we see lower wealth agents accumulate more illiquid wealth and less liquid wealth. Since agents with high levels of wealth are essentially permanent income consumers, there is little adjustment in their behavior. Therefore, the increase in wealth accumulation by lower wealth agents results in a decline in the wealth Gini.

Second, similarly to an expansionary TFP level shock, a contractionary monetary policy leads to a higher inequality in liquid wealth and a lower inequality in illiquid wealth. The fact that a contractionary monetary policy shock affects inequality similarly to an expansionary TFP-level shock is surprising at first glance. This outcome is again driven by the cost of adjusting between liquid and illiquid assets. When there is a contractionary monetary policy shock, the nominal rate rises, and, so long as inflation does not increase substantially, the real return on bonds increases. Therefore, in the absence of adjustment costs, agents would want to hold more bonds and fewer illiquid assets. The adjustment cost makes the portfolio reallocation less attractive and, since the marginal adjustment cost is greater for agents with low levels of illiquid assets, even less so for low wealth agents. The higher marginal cost of (dis) investment along with the fact that monetary policy shocks have low persistence (so that the rate will only be elevated for a short period) results in a greater portfolio reallocation from illiquid assets to liquid assets for high-wealth individuals than for low-wealth individuals. Therefore, we see an increase in the liquid wealth Gini but a decrease in the illiquid wealth Gini.

Third, uncertainty shocks affect the Gini coefficients of the liquid and illiquid wealth in opposite directions: both positive TFP-volatility shock and positive idiosyncratic volatility shock increase inequality in liquid assets but decrease inequality in illiquid assets. In periods of heightened uncertainty the agent's labor productivity and wages have a higher chance of being low tomorrow than in periods of depressed uncertainty. When labor income is low, agents smooth consumption by drawing down savings. Since illiquid assets are costly to access, agents increase their holdings of liquid assets. That way, if low wages or individual productivity is realized, the agent is able to smooth consumption without incurring an additional cost. Again, since the marginal cost of investment is lower for high-wealth individuals, such individuals reallocate a larger share of their total wealth to liquid assets than low wealth individuals. Therefore, we see a decline in the illiquid wealth Gini and increase in the liquid wealth Gini. However, the source of uncertainty matters for overall wealth inequality, which increases in response to TFP uncertainty but decreases in response to idiosyncratic uncertainty. That is, the periods of heightened idiosyncratic uncertainty tend to be associated with periods of lower wealth inequality, while the opposite happens with aggregate uncertainty (however, the correlations are small). From Table 5, we see that the correlation between the risk premium and TFP uncertainty is much larger than the correlation between the risk premium and individual uncertainty. Recall that the illiquid wealth Gini declines in periods of heightened TFP uncertainty. Given that wealthy agents have larger stocks of illiquid wealth, the increase in the risk premium mutes the decline in the illiquid wealth Gini. In turn, the total wealth Gini increases. Since individual uncertainty shocks have a much smaller correlation with the risk premium, the total wealth Gini declines.

Furthermore, uncertainty innovations decrease wealth inequality, so that in response to an innovation, the total wealth Gini declines (this is also consistent with the impulse responses presented in Figures 2 and 3). But if there were persistent periods of high uncertainty, the table suggests that the Gini would increase if it was aggregate uncertainty and the Gini would decrease if it was individual uncertainty.

Finally, for the income and consumption inequality, the consumption Gini is correlated most with the TFP-level shock, while the income Gini – with the TFP uncertainty shock; better technology decreases inequality in consumption but more uncertainty in technology is detrimental for income equality.

5.4 Impulse-response functions

To construct impulse responses in our model with multiple shocks, we follow Koop, Pesaran and Potter (1996); see Appendix D for additional details on this numerical procedure. The idea of Koop’s at al. (1996) methodology is to compare solutions for two sequences of innovations, which are identical except of one specific innovation in one period.

In our experiments, an innovation that we vary is that to uncertainty, and we consider a two standard-deviation innovation. Figure 2 plots a response of aggregate variables (machines, output, inflation, consumption and the wealth Gini coefficient) to an increase in aggregate uncertainty. All variables are measured in deviations from the level in the case of no impulse to aggregate uncertainty. In Appendix D, we present the impulse response for some other variables.

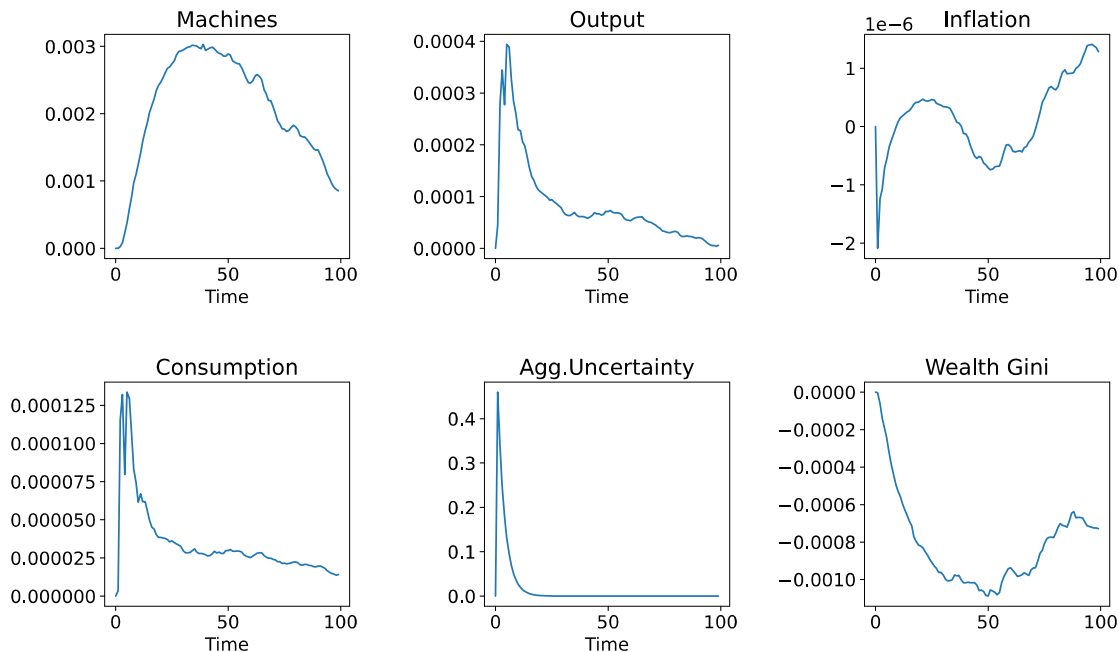


Figure 2. Impulse response functions for aggregate stochastic volatility.

As the figure shows, an increase in aggregate uncertainty increases investment in machines, increases output and decreases the wealth Gini. The decline in the wealth Gini coupled with the increase in machines suggests that some of the increase in investment is driven by the investment of low- and middle-wealth agents. If the increase in investment were driven solely by high-wealth individuals, then we would expect to see a rise in the wealth Gini. Why does an increase in uncertainty incentivize lower-wealth individuals to accumulate capital? All else equal, an increase in TFP uncertainty today increases the spread between the return on illiquid assets and the return on liquid assets. Recall that the agent pays a cost to adjust his illiquid asset savings but does not pay a cost to adjust his liquid asset savings. Due to the increase in the excess return that illiquid assets pay over liquid assets, lower wealth agents now find it optimal to incur the cost of investing in illiquid assets. Therefore, we see an increase in machines and a decline in the wealth Gini.

The results for individual uncertainty are parallel for aggregate uncertainty; see Figure 3. The impact of the idiosyncratic uncertainty shock is similar to that of the aggregate TFP shocks, except that the same

size of shock leads to significantly (about 10-times) larger effects on all the aggregate variables plotted. In particular, the wealth Gini coefficient responds stronger, implying that individual uncertainty is more powerful in decreasing wealth inequality. Our result that uncertainty decreases the wealth Gini might seem counterintuitive, however, it can be explained by the fact that an increase in individual uncertainty induces agents to save more. Indeed, from the figure, we see that much of the increase in savings comes from accumulation of machines; additionally, the effect is highly persistent.

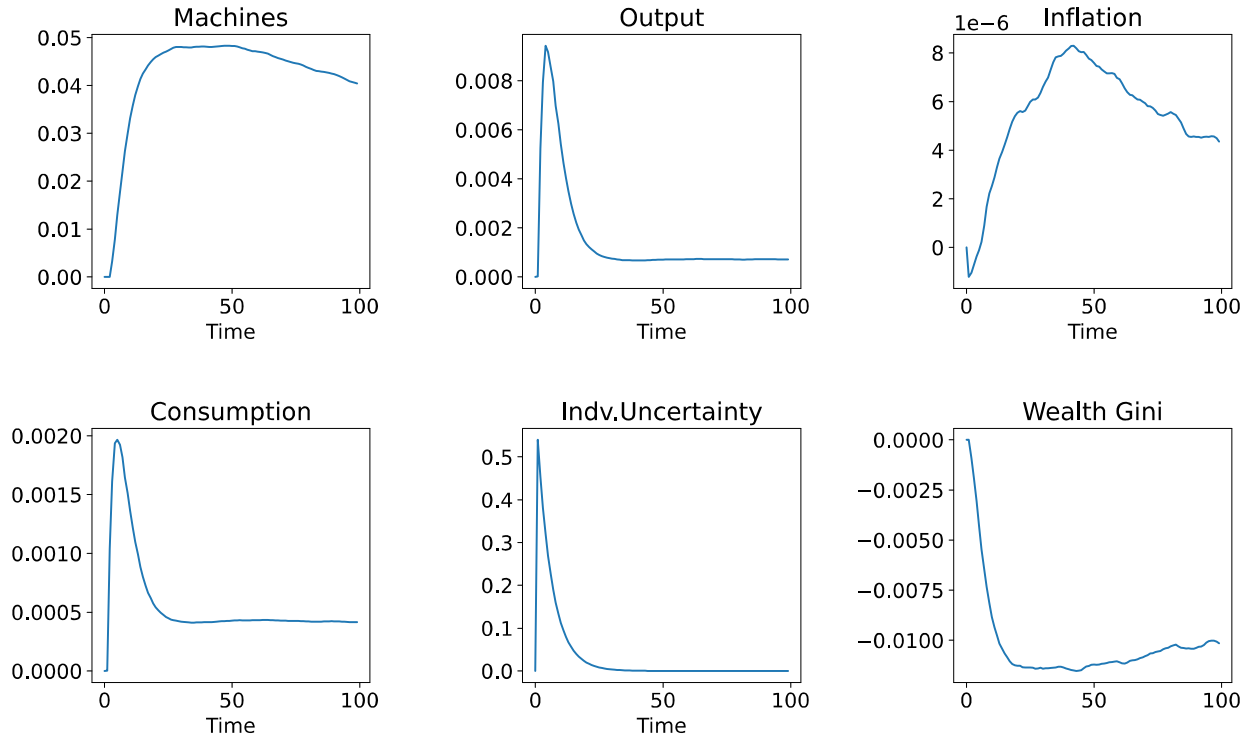


Figure 3. Impulse response functions for idiosyncratic stochastic volatility.

6 Conclusion

In this paper we use deep learning techniques to quantify the effect periods of heightened individual and aggregate uncertainty have on individuals across the wealth distribution. Our results show that it is important to distinguish between uncertainty shocks at the individual level and aggregate level. Periods of heightened individual uncertainty tend to exacerbate wealth inequality in terms of liquid wealth while heightened aggregate uncertainty has the opposite effect. Since liquid wealth such as checking and savings accounts play a key role in helping individuals smooth through adverse economic events, understanding how this form of wealth fluctuates over the business cycle is crucial.

Additionally, this paper serves as a first step towards a more complete quantification of the effect uncertainty has on the economy. In future work, the model could be extended to include correlation between the individual and aggregate uncertainty shocks. Finally, the model could be used to study the effectiveness of government interventions that aim to reduce aggregate or individual uncertainty such as the interventions by the Federal Reserve done at the start of the COVID pandemic.

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Appendix A. Deriving the FOCs

In this section, we present the first-order conditions of the model.

Households. Let $\lambda_t(j)$ and $\mu_t(j)$ be Lagrange multipliers associated with the household's budget constraint (2) and the law of motion for illiquid assets, respectively, and let η_t , \tilde{v}_t^1 and \tilde{v}_t^2 be the Lagrange multipliers associated with the borrowing constraints in (6); we implicitly take into account that the adjustment cost function is given by the function Ψ in (5). The Lagrangian of the household problem (1)–(??) is

$$\begin{aligned} \mathcal{L} = & E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{c_t(j)^{1-\gamma} - 1}{1-\gamma} + \psi \frac{(1-\ell_t(j))^{1-\vartheta} - 1}{1-\vartheta} \right] \\ & + E_0 \sum_{t=0}^{\infty} \beta^t \lambda_t(j) \left\{ R_{t-1} \frac{b_{t-1}(j)}{\pi_t} + w_t \ell_t(j) \exp(\eta_{\ell,t}(j)) + \tau_t(j) - c_t(j) - i_t(j) \right. \\ & \quad \left. - b_t(j) - \Psi(i_t(j), Q_t s_{t-1}(j) + k_{t-1}(j)) \right\} \\ & + E_0 \sum_{t=0}^{\infty} \beta^t \mu_t \left\{ [1-d+r_t^k] k_{t-1}(j) + i_t(j) \right. \\ & \quad \left. + [Q_t + (1-\delta)\Pi_t(j)] s_{t-1}(j) - Q_t s_t(j) - k_t(j) \right\} \\ & + E_0 \sum_{t=0}^{\infty} \beta^t \varphi_t [b_t(j) + \bar{b}] + E_0 \sum_{t=0}^{\infty} \beta^t \tilde{v}_t^1 [Q_t s_t(j)] + E_0 \sum_{t=0}^{\infty} \beta^t \tilde{v}_t^2 [K_t(j)] \end{aligned}$$

Omitting index j , we obtain the following FOCs:

$$\begin{aligned} \text{wrt } c_t : & \quad \lambda_t(j) = c_t^{-\gamma}, \\ \text{wrt } \ell_t : & \quad \psi(1-\ell_t)^{-\vartheta} = \lambda_t(j) \left(\exp(\eta_{\ell,t}(j)) w_t (1 - (\Upsilon_t + \tau_0)) + \Upsilon_t \left((\ell_t \exp(\eta_{\ell,t}(j)) w_t)^{-\tau_2} + \tau_3 \right)^{-\frac{1}{\tau_2}-1} (\ell_t \exp(\eta_{\ell,t}(j)) w_t)^{-\tau_2} \right), \\ \text{wrt } b_t : & \quad c_t^{-\gamma} - \varphi_t = \beta R_t E_t \left[\frac{c_{t+1}^{-\gamma}}{\pi_{t+1}} \right], \\ \text{wrt } i_t : & \quad \mu_t = c_t^{-\gamma} \left[1 + \left\{ \Gamma_1 \text{sign}(i_t) + \Gamma_2 \text{sign}(i_t(j)) \left(\frac{|i_t|}{[Q_t + (1-\delta)\Pi_t] s_{t-1} + [1-d+r_t^k] k_{t-1} + \varepsilon} - \xi \right)^{\Gamma_3-1} \right\} \right], \\ \text{wrt } s_t : & \quad (\mu_t - \tilde{v}_t^1) Q_t \\ & = \beta E_t (-\lambda_{t+1} Q_{t+1}) \left[\frac{\Gamma_2}{\Gamma_3} \left(\frac{|i_{t+1}|}{Q_{t+1} s_t + k_t + \varepsilon} - \xi \right)^{\Gamma_3} - \Gamma_2 \left(\frac{|i_{t+1}|}{[Q_{t+1} + (1-\delta)\Pi_{t+1}] s_t + [1-d+r_{t+1}^k] k_t + \varepsilon} - \xi \right)^{\Gamma_3-1} \right. \\ & \quad \left. + \beta E_t \mu_{t+1} [Q_{t+1} + (1-\delta)\Pi_{t+1}] \right], \quad (29) \\ \text{wrt } k_t : & \quad \mu_t - \tilde{v}_t^2 \\ & = \beta E_t (-\lambda_{t+1}) \left[\frac{\Gamma_2}{\Gamma_3} \left(\frac{|i_{t+1}|}{[Q_{t+1} + (1-\delta)\Pi_{t+1}] s_t + [1-d+r_{t+1}^k] k_t + \varepsilon} - \xi \right)^{\Gamma_3} - \Gamma_2 \left(\frac{|i_{t+1}|}{[Q_{t+1} + (1-\delta)\Pi_{t+1}] s_t + [1-d+r_{t+1}^k] k_t + \varepsilon} - \xi \right)^{\Gamma_3-1} \right. \\ & \quad \left. + \beta E_t \mu_{t+1} [1-d+r_{t+1}^k] \right], \quad (30) \end{aligned}$$

where $\pi_{t+1} \equiv \frac{P_{t+1}}{P_t}$ is the gross inflation rate between t and $t+1$.

After substituting (25) into (26), we get

$$\psi(1-\ell_t)^{-\vartheta} = c_t^{-\gamma} \left(\exp(\eta_{\ell,t}(j)) w_t (1 - (\Upsilon_t + \tau_0)) + \Upsilon_t \left((\ell_t \exp(\eta_{\ell,t}(j)) w_t)^{-\tau_2} + \tau_3 \right)^{-\frac{1}{\tau_2}-1} (\ell_t \exp(\eta_{\ell,t}(j)) w_t)^{-\tau_2} \right)$$

Let $q_t \equiv \frac{\mu_t}{\lambda_t}$, $v_t^1 \equiv \frac{\tilde{v}_t^1}{\lambda_t}$ and $v_t^2 \equiv \frac{\tilde{v}_t^2}{\lambda_t}$. Therefore, FOCs (28), (29) and (30), respectively, become

$$\text{wrt } i_t : \quad q_t(j) = 1 + \left\{ \Gamma_1 \text{sign}(i_t(j)) + \Gamma_2 \text{sign}(i_t(j)) \left(\frac{|i_t(j)|}{[Q_t + (1-\delta)\Pi_t] s_{t-1}(j) + [1-d+r_t^k] k_{t-1}(j) + \varepsilon} - \xi \right) \right. \\ \left. \cdot \text{sign}(i_t(j)) \right\},$$

$$\text{wrt } s_t : \quad (q_t - v_t^1) Q_t \\ = \beta E_t \frac{\lambda_{t+1}}{\lambda_t} \left\{ [Q_{t+1} + (1-\delta)\Pi_{t+1}] \left[\Gamma_2 \left(\frac{i_{t+1}}{[Q_{t+1} + (1-\delta)\Pi_{t+1}] s_t + [1-d+r_{t+1}^k] k_t + \varepsilon} - \xi \right)^{\Gamma_3-1} \right. \right. \\ \left. \left. \frac{i_{t+1}}{[Q_{t+1} + (1-\delta)\Pi_{t+1}] s_t + [1-d+r_{t+1}^k] k_t + \varepsilon} - \frac{\Gamma_2}{\Gamma_3} \left(\frac{i_{t+1}}{[Q_{t+1} + (1-\delta)\Pi_{t+1}] s_t + [1-d+r_{t+1}^k] k_t + \varepsilon} - \xi \right)^{\Gamma_3} \right] \right\} \\ + \beta E_t \frac{\lambda_{t+1}}{\lambda_t} q_{t+1} [Q_{t+1} + (1-\delta)\Pi_{t+1}], \quad (31)$$

$$\text{wrt } k_t : \quad q_t - v_t^2 \\ = \beta E_t \frac{\lambda_{t+1}}{\lambda_t} [1-d+r_{t+1}^k] \left[\Gamma_2 \left(\frac{i_{t+1}}{[Q_{t+1} + (1-\delta)\Pi_{t+1}] s_t + [1-d+r_{t+1}^k] k_t + \varepsilon} - \xi \right)^{\Gamma_3-1} \right. \\ \left. \frac{i_{t+1}}{[Q_{t+1} + (1-\delta)\Pi_{t+1}] s_t + [1-d+r_{t+1}^k] k_t + \varepsilon} - \frac{\Gamma_2}{\Gamma_3} \left(\frac{I_{t+1}}{[Q_{t+1} + (1-\delta)\Pi_{t+1}] s_t + [1-d+r_{t+1}^k] k_t + \varepsilon} - \xi \right)^{\Gamma_3} \right] \\ + \beta E_t \frac{\lambda_{t+1}}{\lambda_t} q_{t+1} [1-d+r_{t+1}^k]. \quad (32)$$

Note that for agents whose borrowing constraint on liquid asset is non-binding, the individual FOC with respect to bonds $c_t(j)^{-\gamma} = \beta R_t E_t \left[\frac{c_{t+1}(j)^{-\gamma}}{\pi_{t+1}} \right]$. Integrate over all agents to obtain $\int c_t(j)^{-\gamma} dj - \int \varphi_t(j) dj = \beta R_t E_t \left[\frac{\int c_{t+1}(j)^{-\gamma} dj}{\pi_{t+1}} \right]$.

Intermediate-good production. In real terms, after substituting the constraint into the problem (12), (13), we get the following intermediate-firm problem

$$\max_{P_t(h)} E_t \sum_{l=0}^{\infty} \beta^l \tilde{\Lambda}_{t,t+l} \left[\frac{P_{t+l}(h)}{P_{t+l}} \left(\frac{P_{t+l}(h)}{P_{t+l}} \right)^{\frac{\mu}{1-\mu}} \bar{Y}_{t+l} - m c_{t+l} \left(\frac{P_{t+l}(h)}{P_{t+l}} \right)^{\frac{\mu}{1-\mu}} \bar{Y}_{t+l} \right. \\ \left. - \frac{\mu}{2\kappa^p (\mu-1) \pi^*} \bar{Y}_{t+l} \left(\log \left(\frac{P_{t+l}(h)}{P_{t+l-1}(h)} \frac{1}{\pi^*} \right) \right)^2 \right],$$

where $\bar{Y}_{t+l} \equiv \exp(\eta_{\theta,t}) K_{t-1}^\alpha H_t^{1-\alpha}$.

The FOC with respect to $P_t(h)$

$$\tilde{\Lambda}_{t,t} \left[\frac{1}{1-\mu} \frac{P_t(h)^{\frac{\mu}{1-\mu}}}{P_t^{1-\mu}} \bar{Y}_t - m c_t \left(\frac{\mu}{1-\mu} \right) \left(\frac{P_t(h)}{P_t} \right)^{\frac{\mu}{1-\mu}-1} \frac{\bar{Y}_t}{P_t} \right. \\ \left. - \frac{\mu}{\kappa^p (\mu-1)} \left(\log \left(\frac{P_{t+l}(h)}{P_{t+l-1}(h)} \frac{1}{\pi^*} \right) \right) \frac{\bar{Y}_t}{P_t(h)} \right] \\ + \beta E_t \tilde{\Lambda}_{t,t+1} \frac{\mu}{\kappa^p (\mu-1)} \left(\log \left(\frac{P_{t+l+1}(h)}{P_{t+l}(h)} \frac{1}{\pi^*} \right) \right) \frac{\bar{Y}_{t+1}}{P_t(h)} = 0.$$

Impose $P_t(h) = P_t$, multiply by P_t , κ^p , Y_t and $1 - \mu$

$$\begin{aligned} \tilde{\Lambda}_{t,t} \left[\frac{1}{1-\mu} \frac{\bar{Y}_t}{P_t} - mc_t \left(\frac{\mu}{1-\mu} \right) \frac{\bar{Y}_t}{P_t} - \frac{\mu}{\kappa^p (\mu-1)} \log \left(\frac{\pi_t}{\pi^*} \right) \frac{\bar{Y}_t}{P_t} \right] \\ + \beta E_t \tilde{\Lambda}_{t,t+1} \frac{\mu}{\kappa^p (\mu-1)} \log \left(\frac{\pi_{t+1}}{\pi^*} \right) \frac{\bar{Y}_{t+1}}{P_t} = 0. \end{aligned}$$

Rearranging the terms and noting that $\tilde{\Lambda}_{t,t} = 1$, we obtain the Phillips curve (14) in the main text.

Cost-minimization problem. Consider now the cost-minimization problem (15)–(16) of the intermediate-good producer. Note that nominal marginal cost is

$$MC_t(h) = \frac{dTC(Y_t(h))}{dY_t(h)} = \Theta_t(h), \quad (33)$$

where $\Theta_t(h)$ is the Lagrange multiplier associated with (16). The FOC with respect to $K_{t-1}(h)$ is

$$R_t^k = \Theta_t(h) \alpha \exp(\eta_{\theta,t}) K_{t-1}(h)^{\alpha-1} H_t(h)^{1-\alpha}.$$

The FOC with respect to $H_t(h)$ is

$$W_t = \Theta_t(h) (1 - \alpha) \exp(\eta_{\theta,t}) K_{t-1}(h)^\alpha H_t(h)^{-\alpha}. \quad (34)$$

Therefore, we have

$$\frac{K_{t-1}(h)}{H_t(h)} = \frac{\alpha}{1 - \alpha} \frac{W_t}{R_t^k},$$

implying that $\frac{K_{t-1}(h)}{H_t(h)}$ is not firm specific. Furthermore, we have

$$K_{t-1}(h) = \frac{\alpha}{1 - \alpha} \frac{W_t}{R_t^k} \cdot H_t(h).$$

Integrating over h , we get

$$K_{t-1} \equiv \int_0^1 K_{t-1}(h) dh = \frac{\alpha}{1 - \alpha} \frac{W_t}{R_t^k} \cdot \int_0^1 H_t(h) dh = \frac{\alpha}{1 - \alpha} \frac{W_t}{R_t^k} \cdot H_t,$$

where $H_t = \int_0^1 H_t(h) dh$. Thus, we have

$$\frac{K_{t-1}(h)}{H_t(h)} = \frac{K_{t-1}}{H_t},$$

which implies

$$\frac{K_{t-1}}{H_t} = \frac{\alpha}{1 - \alpha} \frac{W_t}{R_t^k} = \frac{\alpha}{1 - \alpha} \frac{w_t}{r_t^k}.$$

From (33) and (34), we have

$$\Theta_t(h) = \frac{W_t}{(1 - \alpha) \exp(\eta_{\theta,t}) \left[\frac{K_{t-1}(h)}{H_t(h)} \right]^\alpha} = MC_t(h) = MC_t.$$

The real marginal cost is

$$mc_t(h) = \frac{r_t^k}{\alpha \exp(\eta_{\theta,t})} \left[\frac{K_{t-1}}{H_t} \right]^{1-\alpha} = mc_t,$$

i.e., it is the same for all intermediate-good producers.

Aggregate resource constraint. Recall the household's budget constraint (2):

$$\begin{aligned} P_t c_t(j) + P_t i_t(j) + \tilde{b}_t(j) + \Psi(i_t(j), Q_t s_{t-1}(j) + k_{t-1}(j)) \\ = R_{t-1} \tilde{b}_{t-1}(j) \\ + P_t \ell_t(j) \exp(\eta_{\ell,t}(j)) w_t + P_t \tau_t(j). \end{aligned}$$

Let $C_t = \int_0^1 c_t(j) dj$, $I_t = \int_0^1 i_t(j) dj$, $B_t = \int_0^1 b_t(j) dj$, $T_t = \int_0^1 \tau_t(j) dj$, $K_{t-1} = \int_0^1 k_{t-1}(j) dj$, $H_t = \int_0^1 \ell_t(j) \exp(\eta_{\ell,t}(j)) dj$, $\Pi_t = \int_0^1 \Pi_t(j) dj$. Integrate the household's budget constraint across households:

$$\begin{aligned} P_t C_t + P_t I_t + B_t + \int_0^1 \Psi(i_t(j), Q_t s_{t-1}(j) + k_{t-1}(j)) dj \\ = R_{t-1} B_{t-1} + P_t w_t H_t + \delta \Pi_t + P_t T_t. \end{aligned}$$

Integrate the household's illiquid asset-accumulation equation across households (noting that $\int_0^1 s_t(j) dj = 1$):

$$K_t = \left[(1-d) + r_t^k \right] K_{t-1} + I_t + (1-\delta) \Pi_t.$$

Combining the last two conditions together with (17a), we have

$$\begin{aligned} P_t C_t + P_t \left[K_t - (1-d) K_{t-1} + \int_0^1 \text{AC}(j) dj \right] \\ = R_t^k K_{t-1} + P_t w_t H_t + \Pi_t \end{aligned}$$

where $\int_0^1 \text{AC}(j) dj = \int_0^1 \Psi(i_t(j), Q_t s_{t-1}(j) + k_{t-1}(j)) dj$.

Note that

$$\begin{aligned} \Pi_t &= \int_0^1 \Pi_t(h) dh = \int_0^1 P_t(h) Y_t(h) dh - \int_0^1 \xi(P_t(h), P_{t-1}(h)) dh - R_t^k \int_0^1 k_{t-1}(j) dh - P_t w_t \int_0^1 H_t(h) dh \\ &= P_t Y_t - R_t^k K_{t-1} - P_t w_t H_t. \end{aligned}$$

Also, note that $L_t \equiv \int_0^1 L_t(h) dh = \int_0^1 \ell_t(j) \exp(\eta_{\ell,t}(j)) dj \equiv H_t$. Hence, we obtain

$$P_t C_t + P_t K_t - (1-d) P_t K_{t-1} + P_t \int_0^1 \text{AC}(j) dj = P_t Y_t,$$

or

$$C_t + K_t - (1-d) K_{t-1} + \int_0^1 \text{AC}(j) dj = Y_t.$$

The latter result follows because all unconstrained agents holding capital are the same.

Appendix B.

In this section, we derive formula (21) used in the main text to show that a split between shares and capital in the illiquid-asset portfolio is determinate under the assumption of aggregate uncertainty.

Let us define the derivative of the adjustment-cost function as

$$\begin{aligned} \varkappa_{t+1}(j) &\equiv \Gamma_2 \left(\frac{|i_{t+1}(j)|}{[Q_{t+1} + (1-\delta) \Pi_{t+1}] s_t(j) + [1-d+r_{t+1}^k] k_t(j) + \varepsilon} - \xi \right)^{\Gamma_3} \\ &\quad \times \frac{|i_{t+1}(j)|}{[Q_{t+1} + (1-\delta) \Pi_{t+1}] s_t(j) + [1-d+r_{t+1}^k] k_t(j) + \varepsilon} \\ &\quad - \frac{\Gamma_2}{\Gamma_3} \left(\frac{|i_{t+1}(j)|}{[Q_{t+1} + (1-\delta) \Pi_{t+1}] s_t(j) + [1-d+r_{t+1}^k] k_t(j) + \varepsilon} - \xi \right)^{\Gamma_3}. \quad (35) \end{aligned}$$

We start from conditions (31) and (32), which after subtracting one from another yields

$$v_t^1 + \beta E_t \left\{ \frac{\lambda_{t+1} [Q_{t+1} + (1-\delta)\Pi_{t+1}]}{\lambda_t} [\mathcal{z}_{t+1} + q_{t+1}] \right\} - v_t^2 - \beta E_t \left\{ \frac{\lambda_{t+1}}{\lambda_t} [1-d+r_{t+1}^k] [\mathcal{z}_{t+1} + q_{t+1}] \right\} = 0.$$

Using our definitions of R_{t+1}^s and R_{t+1}^k we can rewrite this as

$$v_t^1 + \beta E_t \left\{ \frac{\lambda_{t+1}}{\lambda_t} R_{t+1}^s [\mathcal{z}_{t+1} + q_{t+1}] \right\} - v_t^2 - \beta E_t \left\{ \frac{\lambda_{t+1}}{\lambda_t} R_{t+1}^k [\mathcal{z}_{t+1} + q_{t+1}] \right\} = 0.$$

Since $E[XY] = E[X]E[Y] + Cov(X, Y)$ for two random variables X and Y , we get

$$\begin{aligned} v_t^1 + \beta E_t \{ R_{t+1}^s \} E_t \left\{ \frac{\lambda_{t+1}}{\lambda_t} R_{t+1}^s [\mathcal{z}_{t+1} + q_{t+1}] \right\} - v_t^2 - \beta E_t \{ R_{t+1}^k \} E_t \left\{ \frac{\lambda_{t+1}}{\lambda_t} R_{t+1}^k [\mathcal{z}_{t+1} + q_{t+1}] \right\} \\ + \beta Cov \left(R_{t+1}^s, \frac{\lambda_{t+1}}{\lambda_t} [\mathcal{z}_{t+1} + q_{t+1}] \right) - \beta Cov \left(R_{t+1}^k, \frac{\lambda_{t+1}}{\lambda_t} [\mathcal{z}_{t+1} + q_{t+1}] \right) = 0. \end{aligned}$$

Introducing $\sigma_{t+1}^{R^s}$, $\sigma_{t+1}^{R^k}$ and σ_{t+1}^ζ , defined as in the main text, and rearranging, we obtain equation (21).

7 Appendix C. Further details on the calibration procedure

We choose the inflation target in the Taylor rule (18), $\pi^* = 1.02$. Steady state in the bond Euler equation implies $\frac{\beta R}{\pi} = 1$, so that

$$R = \frac{\pi}{\beta} = 1.05426.$$

Note that

$$\Delta^p \equiv 1 - \frac{\mu}{2\kappa^p (\mu - 1) \pi^*} \left(\log \left(\frac{\pi_t}{\pi^*} \right) \right)^2 = 1.$$

Furthermore, since $\mu = \frac{10}{9}$, we have

$$mc = \frac{1}{\mu} = 0.9.$$

It is possible to derive the remaining parameters analytically in the absence of adjustment costs but with the adjustment costs, we use the following iterative procedure. Use the FOC with respect to capital to write steady state interest rate as

$$r = \frac{1}{\beta} - (1-d) - \chi,$$

where χ summarizes the effect of adjustment cost in the interest rate

$$\chi = \frac{\Gamma_2 \left(\frac{|i|}{[Q+(1-\delta)\Pi]_{s+[1-d+r^k]_{k+\varepsilon}}} - \xi \right)^{\Gamma_3-1} \frac{|i|}{[Q+(1-\delta)\Pi]_{s+[1-d+r^k]_{k+\varepsilon}}} - \frac{\Gamma_2}{\Gamma_3} \left(\frac{|i|}{[Q+(1-\delta)\Pi]_{s+[1-d+r^k]_{k+\varepsilon}}} - \xi \right)^{\Gamma_3}}{1 + \Gamma_1 \text{sign}(i) + \Gamma_2 \left(\frac{|i|}{[Q+(1-\delta)\Pi]_{s+[1-d+r^k]_{k+\varepsilon}}} - \xi \right)^{\Gamma_3-1} \text{sign}(i)}. \quad (36)$$

Our iterative cycle is as follows:

– Assume that we have no adjustment costs (i.e., fix $\chi = 0$) and find

$$r^k = \frac{1}{\beta} - (1-d) = 0.074.$$

- Since we calibrate $H = 1/3$, we have $mc = \frac{r^k}{\alpha} \left[\frac{K}{H}\right]^{1-\alpha}$.
- Combining these results, we get steady-state values

$$\begin{aligned}
K &= \left[\frac{\alpha \cdot mc}{r^k}\right]^{\frac{1}{1-\alpha}} H, \\
w &= \frac{(1-\alpha)}{H\alpha} Kr^k, \\
Y &= K^\alpha H^{1-\alpha} \Delta^p, \\
C &= Y - dK - \int_0^1 \Psi(j) dj, \\
s &= 1.
\end{aligned}$$

- The utility-function parameter ψ is given by

$$\psi = \frac{C^{-\gamma} \left(w(1 - (\Upsilon + \tau_0)) + \Upsilon ((Hw_t)^{-\tau_2} + \tau_3)^{-\frac{1}{\tau_2}-1} (Hw)^{-\tau_2-1} w \right)}{(1-H)^{-\vartheta}}.$$

- Profit, the share price and investment are

$$\begin{aligned}
\Pi &= Y - rK - wH, \\
Q &= \frac{(1-\delta)\Pi}{r-d}, \\
I &= -[(r-d)K + (1-\delta)\Pi].
\end{aligned}$$

Recompute $\hat{\chi}$ from (36) and update it for the next iteration using weighted average $\chi\lambda + \hat{\chi}(1-\lambda)$.

Appendix D. Numerical analysis

In this section, we present some additional information about our numerical analysis.

Standard deviations. In the table below, we provide standard deviations of selected aggregate variables; we compare the statistics produced by our HANK model with those observed in the U.S. economy. We reproduce the statistics on the U.S. economy from Sims (2010).

$\frac{\sigma(\text{Variable})}{\sigma(\text{Output})}$	Consumption	Labor	Investment	Share Price	Wage	Risk Premium
Model	0.3790	0.6695	0.5475	0.0030	1.2682	0.3780
Datal	0.53	1.12	2.76	-	0.53	-

Construction of impulse-response functions using Koop’s et al. (1996) methodology. For ease of exposition, we will describe how we construct the impulse response functions for the case of one specific shock η_{σ_ℓ} . Using the constructed decision rules, we first simulate the model for 1000 periods. Then, we use the final period of the simulation as the initial condition for the impulse response.

- With this initial condition, we draw realizations of length $N_{\mathcal{T}} + 1$ innovations for shock η_{σ_ℓ} for N_R times. For the other exogenous variables, we draw a series of innovations of length $N_{\mathcal{T}}$ for N_R times.
- For all exogenous variables, we draw one initial innovation which is used in all N_R iterations for all variables except η_{σ_ℓ} . We denote these innovations as Ω_0 .
- Next, we simulate the model under two different realizations of innovations.
- In the first simulation, we do the following: In period 0, we use Ω_0 for all variables including η_{σ_ℓ} ; for the

remaining $N_{\mathcal{T}}$ periods, we use the previously drawn innovations for all variables except η_{σ_ℓ} , and for η_{σ_ℓ} , we use the last $N_{\mathcal{T}}$ innovations.

– In the second simulation, for exogenous variables except for η_{σ_ℓ} , the innovations used are the same as those used in the first. For η_{σ_ℓ} , instead of using the innovation from Ω_0 in period 0, we use the first realization from the time series of length $N_{\mathcal{T}} + 1$. Therefore, from period 1 onward, the innovations used in the two simulations are the same.

– We repeat the process N_R times. Then, for all variables of interest, we take an average over the N_R simulations.

– We repeat the process N_O times, where each time we start with a new initial condition Ω_0 .

– Finally, we average over the N_O iterations and take the difference between the two time series.

This gives us our impulse response. We assume $N_R = 25$, $N_{\mathcal{T}} = 100$ and $N_O = 5$.

Additional impulse response functions. In this section, we display the impulse response functions for some additional variables like share price, profits, wages, and nominal interest rate (as well as aggregate uncertainty and wealth Gini coefficient); the two figures correspond to aggregate and idiosyncratic uncertainty shocks.

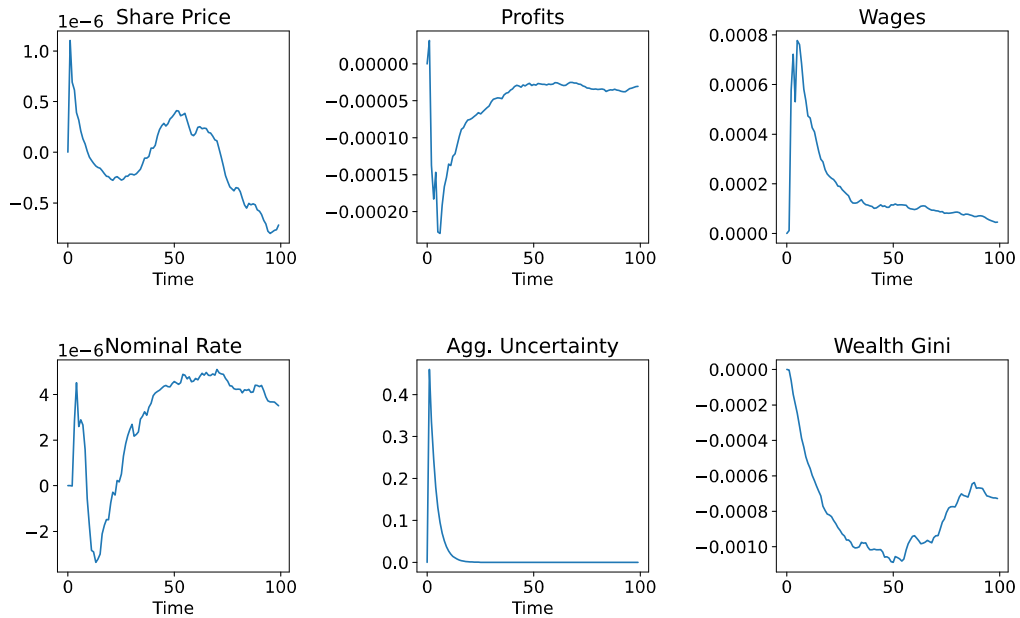


Figure 4. Additional impulse responses for aggregate stochastic volatility.

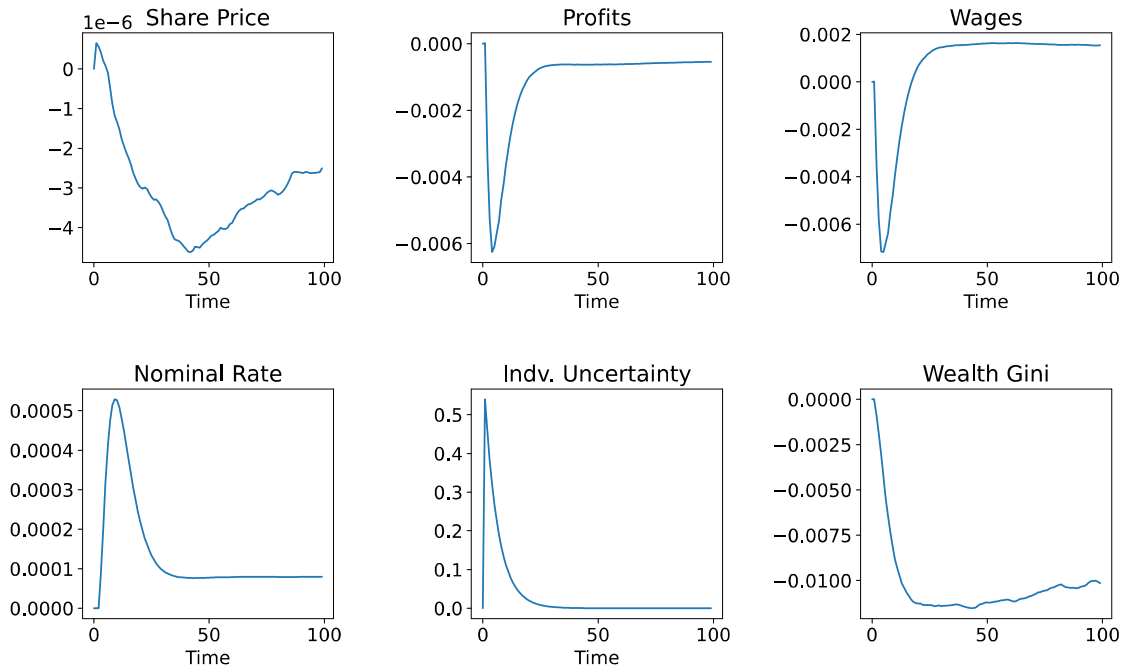


Figure 5. Additional impulse response functions for idiosyncratic stochastic volatility.

As we can see, in response to an aggregate uncertainty shock, wages go up, while they go down in response to an idiosyncratic uncertainty shock of the same size. This discrepancy in the wage behavior explains why the sizes of the effects in Figures 2 and 3 are so different.