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PERCEIVED COMPETITION IN NETWORKS

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Abstract

Agents compete for the same resources and are only aware of their direct neighbors in a network. We propose a new equilibrium concept, referred to as peer-consistent equilibrium (PCE). In a PCE, each agent chooses an effort level that maximizes her subjective perceived utility and the effort levels of all individuals in the network need to be consistent. We develop an algorithm that breaks the network into communities. We use this decomposition to completely characterize peer-consistent equilibria by identifying which sets of agents can be active in equilibrium. An agent is active if she either belongs to a strong community or if few agents are aware of her existence. We show that there is a unique stable PCE. We provide a microfoundation of eigenvector centrality, since, in any stable PCE, agents' effort levels are proportional to their eigenvector centrality in the network.

JEL Classification: C72, D85

Keywords: Social Networks, eigenvector centrality, policies

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Perceived Competition in Networks*

Olivier Bochet[†] Mathieu Faure[‡] Yan Long[§] Yves Zenou[¶]

December 18, 2020

Abstract

Agents compete for the same resources and are only aware of their direct neighbors in a network. We propose a new equilibrium concept, referred to as *peer-consistent equilibrium* (PCE). In a PCE, each agent chooses an effort level that maximizes her subjective perceived utility and the effort levels of all individuals in the network need to be consistent. We develop an algorithm that breaks the network into communities. We use this decomposition to completely characterize peer-consistent equilibria by identifying which sets of agents can be active in equilibrium. An agent is active if she either belongs to a strong community or if few agents are aware of her existence. We show that there is a unique stable PCE. We provide a behavioral foundation of eigenvector centrality, since, in any stable PCE, agents' effort levels are proportional to their eigenvector centrality in the network.

Keywords: Social network, incomplete network knowledge, peer-consistent equilibrium, eigenvector centrality, policies.

JEL Classification: C72, D85, Z13.

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1 Introduction

Competition in economics is usually viewed as "reciprocal." That is, if firm i considers firm j as a competitor, then firm j will also consider firm i as a competitor. However, in the real world, firms may *perceive* other firms as competitors when the reverse is not necessarily true. Furthermore, even though competition is *global*, in many cases firms only care about their *local* competitors. For example, when estimating demand and thus deciding on investment, bars or restaurants may only account for their perceived local competitiors, even though, in reality, they are competing with a larger set of bars or restaurants.

In this paper, we model this idea of "perceived" competition using a network in which agents are only aware of the activity of their direct neighbors; a directed link from agent i to agent j means that the former perceives the latter as a competitor while the reverse is not necessarily true.[1] We study a standard contest game in which agents compete in terms of efforts to capture a given resource available in the economy. However, because agents are locally sighted, they are only aware of the resources in their neighborhood, i.e., the resources shared with their direct neighbors.

We propose a new equilibrium concept, referred to as *peer-consistent equilibrium* (PCE), which captures both the agents' local sightedness—each agent chooses an effort level that maximizes her *perceived* utility—and the fact that the effort levels of all individuals in the network are consistent in equilibrium. Indeed, at a PCE, individual i's perceived subjective utility is equal to her objective payoff. Therefore, although individuals initially start with a *wrong* perception of the resource attainable in their neighborhood, this wrong perception induces an interaction pattern that eventually leads to a *correct* perception of the resource at equilibrium. That is, their wrong perception is peer-consistent. This is our first contribution.

We then explore the role of the network's architecture in determining which agent is active (i.e., exerts strictly positive effort) and which agent is not, at peer-consistent equilibria. Our second contribution is to give an exact prediction by providing an endogenous and unique partitioning of the agents in a network. Agents within a given layer-k perfect sub-network[2] are strongly connected to each other and are either all active or inactive

[1] In many situations involving networks, information is "local" in the sense that agents only know the activity of their direct neighbors but do not know anything about the activity of the other agents in the network. For instance, using network data among several departments at the University of Chicago and Columbia University, Friedkin (1983) showed that a respondent i was *less* likely to know about another faculty member j's current research if the respondent was *further* from j in the network. Similarly, Alatas et al. (2016) found that subjects are much less likely to know about the wealth status of individuals in their village who were further apart in the network. In a recent study using network data from 75 villages, Breza et al. (2018) asked 4,554 individuals to assess whether five randomly chosen pairs of households in their village were linked through financial, social, and informational relationships. They found that network knowledge was low and highly localized, declining steeply with the pair's network distance to the respondent. They conclude that the assumption of full network knowledge may serve as a poor approximation of the real world.

[2] A perfect sub-network is a sub-network that (i) contains at least two agents, (ii) is strongly connected

at a PCE.[3] We develop an algorithm called the Network Layer Decomposition (NLD) to fully characterize the equilibrium partitions using the network topology. We show that the lower is the layer k, i.e., the smaller k is, the less other agents are aware of agents belonging to these perfect sub-networks. In particular, agents belonging to a layer-1 perfect sub-network are "hiding" from the other agents in the network, since there is no path from a strongly connected community that can reach them. However, these agents may be aware of all other agents in the network. This gives them a sizable advantage in terms of competition and they are thus more likely to grab the resources available in the economy. We show that, to be a "root" of a PCE, that is, the highest layer perfect sub-network of all agents with positive efforts, the spectral radius of the adjacency matrix of the layer-k perfect sub-network has to be larger than that of the maximum of all the communities (or lower-layer perfect sub-networks) that can reach this sub-network. This condition is automatically satisfied for the layer-1 perfect sub-network but less likely to be true the higher the layer. There are typically multiple peer-consistent equilibria. In some of them only some agents are active, while in others all agents are active.

The multiplicity of peer-consistent equilibria is interesting as it underlines the behavioral richness of our equilibrium concept. Yet, one may want to make more precise predictions for any given network. To address this question, we study the stability of the peer-consistent equilibria. Our third contribution is to provide a very simple and intuitive characterization of the stable PCE. We show that a layer-k perfect sub-network is at the root of a *stable* PCE if the spectral radius of its adjacency matrix is the largest one in the network (i.e., larger than any other perfect sub-network) and thus equal to that of the whole network. This implies that there is always a *unique* stable PCE. In this PCE, all agents can be active or not. For example, if the spectral radius of the adjacency matrix of a layer-1 perfect sub-network is the largest one in the network, then this is the unique stable PCE for which only agents belonging to this sub-network are active. Hence, it is possible to have a unique stable peer-consistent equilibrium in which only some agents are active, but not all.

The last contribution of our paper is to show that peer-consistent equilibria provide a behavioral foundation of the eigenvector centrality measure.[4] More precisely, we prove that the effort level of agents at the stable PCE is proportional to their eigenvector centrality in the whole network. This result is very general and holds beyond strongly connected networks, for which eigenvector centrality is usually defined. This property is based on the consistency requirement of a PCE: the agents' local perception of resources has to be correct in equilibrium, i.e., their local share of the total resources has to be consistent with that of all other agents. There are other papers that have provided a microfoundation of eigenvector centrality. For example, Golub and Jackson (2010, 2012) develop models on

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and (iii) is such that there is no path from any agent belonging to another strongly connected sub-network that can reach it. The layer k indicates at which step the algorithm defines a sub-network as perfect.

[3] It is, however, important to point out that, in a given layer-k perfect sub-network, all agents being active at a peer-consistent equilibrium does not imply that they all exert the same equilibrium effort.

[4] Some papers have also provided an axiomatic foundation of eigenvector centrality; see e.g., Palacios-Huerta and Volij (2004); Dequiedt and Zenou (2017); Bloch et al. (2019).

DeGroot updating in which eigenvector centrality is the right way to characterize an agent's influence. However, this arises from a heuristic learning process; it's not about behavior in a game. Banerjee et al. (2013) provide a microfoundation of eigenvector centrality by showing that it is the limit of diffusion centrality. In Sadler (2020), there is a theorem that shows that there exists a network game of strategic complements with an equilibrium in which actions are ordered according to eigenvector centrality. Our model is different in the sense that it provides a behavioral foundation for eigenvector centrality measure based on a contest network game and PCE. Moreover, in all these models, in equilibrium, all agents exert strictly positive effort; this means that the eigenvector centrality is well defined by the Perron-Frobenius theorem and that the network is assumed to be strongly connected. Our model solves for a more general framework in which the network is directed and weakly connected.[5] In our unique stable equilibrium, some agents may exert zero effort and still the eigenvector centrality is well defined. So, basically we provide a behavioral foundation of eigenvector centrality for any weakly connected directed network.

More generally, in order to determine which agents grab the most resources, our model puts forward the tension between the size and density of strongly connected communities and other agents' "awareness" of these communities, i.e., how many agents perceive them as competitors. The less agents are aware of certain communities and the tighter and denser these communities are, the more likely they will grab most of the resources in the economy. For example, if we think about conflicts in Africa between different local ethnicities (König et al., 2017; Amarasinghe et al., 2020), then clearly those that (i) are of large size, (ii) are very dense and connected, but (iii) are also difficult to "reach", will have a significant advantage over the others in terms of seizing the available resources.

In the last part of the paper, we study the policy implications of our model. We first examine the impact of adding a directed link between two agents. We show that, when a link is added, the sender becomes more central and more aware of others' activities, while the reverse is not true. This creates a new path in the network that makes others more likely to be reached by this agent, which, in turn, may lower their status in terms of layer perfect sub-network. As a result, adding a link may have the counter-intuitive result that it decreases the number of active agents in the network. We then study the key-player policy (Ballester et al., 2010) and highlight another counter-intuitive result. By removing an agent in the network, we may make several inactive agents (for example, in terms of criminal effort) active. Finally, we show that, by merging two different connected networks (i.e., social mixing), the total activity is higher than the sum of total activity in each disconnected network.

Contribution to the literature

Our paper contributes to the games-on-network literature.[6] First, Ballester et al. (2006) have demonstrated a direct relationship between Nash equilibrium effort in any network and Katz-Bonacich centrality. Here, we provide another direct link between PCE in any

[5] Clearly, a strongly connected network is a particular case.

[6] For overviews, see Jackson (2008), Jackson and Zenou (2015), Bramoullé et al. (2016), Jackson et al. (2017). Two prominent papers from this literature are Ballester et al. (2006) and Bramoullé et al. (2014).

network and eigenvector centrality, which complements the findings of Ballester et al. (2006).

Second, this literature has mostly focused on models with perfect information about the network and linear best-reply functions. As in our model, there are some recent works with non-linear best-reply functions[7] but the network is always assumed to be perfectly observed. Sundararajan (2007), Galeotti et al. (2010), Fainmesser and Galeotti (2016), Belhaj and Deroian (2019), and Jackson (2019) are exceptions; as in our model, they assume that agents only observe the actions of their neighbors. However, their models differ in that they use a standard concept of equilibrium (Bayesian-Nash equilibrium) and do not provide a general characterization of all equilibria.

There are also network games with imperfect information about the network that introduce new equilibrium concepts related to our PCE. In particular, McBride (2006), Lipnowski and Sadler (2019), and Battigalli et al. (2020) consider *self-confirming and peer-confirming equilibria*. However, their concept of equilibrium is very different from ours. The closest is that of Lipnowski and Sadler (2019), who apply self-confirming equilibria (SCE) and rationalizable SCE to games where feedback about the actions of others is described by a network topology: agents *observe only the actions of their peers (i.e., neighbors)*, but their payoffs may depend on everybody's actions and are not observed ex-post. The main difference to our PCE is that Lipnowski and Sadler (2019) allow agents to make conjecture on agents who are not their neighbors;[8] in our model, we assume that agents do not even know the existence of these agents. The peer-confirming equilbrium concept of Lipnowski and Sadler (2019) is such that adding links in the network restricts the set of permissible profiles/conjectures and thus the set of equilibria.[9] This is not true in our model. Moreover, none of these papers provide a complete characterization of all equilibria for any network structure as we do here.

More generally, we believe that there is a trade-off between obtaining general results in terms of the characterization of equilibria and using specific utility function and network games. Indeed, our PCE is specific to our utility function, which is based on the Tullock contest function; in the network-game literature with imperfect network knowledge, the results are more general and can be applied to many more network games. However, we are able to provide more general results in terms of equilibrium characterization. Thus, we view our work as complementary to this literature.

Our equilibrium characterization in terms of layer perfect sub-networks is also related to other network models, which also partition agents into endogenous community structures, including risk sharing (Ambrus et al., 2014), interaction between market and community

[7]See, in particular, Baetz (2015), Allouch (2015), Melo (2019), Parise and Ozdaglar (2019), Zenou and Zhou (2020).

[8]Indeed, a strong assumption that is implicit in the definition of peer-confirming equilibrium in Lipnowski and Sadler (2019) is that players know the network structure.

[9]When the network is complete, the set of peer-confirming equilibria coincides with the set of Nash equilibria. For the empty network, peer-confirming equilibria coincide with rationalizable equilibria. Increasing the number of links reduces the number of equilibria. In contrast, the set of PCE may very well increase when links are added.

(Gagnon and Goyal, 2017), behavioral communities (Jackson and Storms, 2019), information resale and intermediation (Manea, 2020), and technology adoption (Leister et al., 2020). However, the driving forces and policy implications are very different. In particular, all these papers assume a perfect knowledge of the network and use standard equilibrium concepts.

Our paper also contributes to the conflict literature,[10] especially the more recent literature on conflicts in networks.[11] In this literature, the structure of local conflicts is modeled as a network where rivals invest in conflict-specific technology to attack their respective neighbors. This literature assumes that the network is undirected (which is a particular case of our network) and that agents know the network, and solves the model using standard Nash equilibrium concept. Further, they usually do not provide a general characterization of all possible equilibria.

The rest of the paper unfolds as follows. In the next section, we describe our model and introduce our new concept of equilibrium. In Section 3.2, we focus on strongly connected networks and determine the unique equilibrium of this game. Section 3 provides a general characterization of all peer-consistent equilibria (PCEs) for any network by first introducing our NLD algorithm, then by providing the exact condition under which each equilibrium exists, and, finally, by determining the unique stable PCE. In Section 4, we study the policy implications of our model, while in Section 5 we investigate the economic implication of our results. Finally, Section 6 offers concluding remarks.

In Appendix A, we provide some useful results on nonnegative matrices and propose a definition of eigenvector centrality in weakly connected (directed) networks. All the proofs of the results in the main text can be found in Appendix B. We provide additional results in Appendix C and additional examples in Appendix D. Finally, Appendix E deals with the case when the network is not layer-generic.

2 The model

2.1 Underlying contest game

Consider a finite set of agents, denoted by $N = \{1, 2, \cdots, n\}$. There is a given resource, available in a fixed amount $V \in \mathbb{R}_+ = [0, \infty)$ to be shared between the n agents. The agents play a *contest game*, described as follows. Each agent $i \in N$ exerts some action (effort) $x_i \in \mathbb{R}_+$ before the resource V is distributed. An action profile $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ determines, for each agent i, her share of the resource V using the following *proportional rule*:

$$v_i(\mathbf{x}) = \begin{cases} \frac{x_i}{\sum_{j \in N} x_j} V & \text{if } \sum_{i \in N} x_i > 0, \\ \frac{1}{n} V & \text{if } \sum_{i \in N} x_i = 0. \end{cases} \qquad (1)$$

[10]See Corchón (2007), Konrad (2009), and Jensen (2016) for overviews.

[11]See e.g., Goyal and Vigier (2014), Franke and Öztürk (2015), Hiller (2017), König et al. (2017), Kovenock and Roberson (2018), Xu et al. (2019). For overviews, see Kovenock and Roberson (2012) and Dziubiński et al. (2016).

Equation (1) corresponds to the well-known "Tullock contest function" from the contest literature (Skaperdas, 1996; Kovenock and Roberson, 2012).[12] One important difference is that we do not interpret $\frac{x_i}{\sum_{j \in N} x_j}$ as the *probability* of agent i getting V, but as the *percentage* of resource V that agent i can obtain, given her and the other agents' effort choices. We assume that the resource V is exogenously given and that the sharing rule (1) is symmetric and takes a proportional form.

For each agent i, the total cost of exerting effort x_i is equal to cx_i, where $c > 0$ is the (constant) marginal cost of effort. Let $\pi_i : \mathbb{R}^n_+ \to \mathbb{R}$ be agent i's payoff function, which is given by:

$$\pi_i(\mathbf{x}) = v_i(\mathbf{x}) - cx_i \tag{2}$$

2.2 Networks: Locally-sighted individuals

We embed the contest played by agents into a network, by assuming that agents are *locally-sighted* in the sense that they are only aware of people to whom they are directly linked, but do not know anything else about the network. In other words, local-sightedness implies that each individual only perceives resources and interactions of her *local environment*, i.e., her direct links.

Formally, given the set of agents N, a (*directed*) *network* is a pair $(N, \mathbf{G})$ where $\mathbf{G}$ is an $n \times n$ adjacency matrix, with entry $g_{ij} \in \{0, 1\}$. For each pair $i, j \in N$, agent i is linked to j if and only if $g_{ij} = 1$. Since the perception of a link is not necessarily reciprocal, it is possible that $g_{ji} = 0$, i.e., the network is *directed.* That is, we allow for the situation in which an individual is considered as a neighbor (contender) of another, but not vice versa.[13] To be precise, for each $i \in N$, let $\mathcal{N}_i = \{j \in N : g_{ij} = 1\}$ be the (directed) *neighborhood* of agent i. This will be clearer when we introduce the notion of "perceived competition."

There is a *(directed) path* from individual i to individual j in the network if there is a sequence $\{j_1, j_2, \cdots, j_m\} \subseteq N$ with $j_1 = i$, $j_m = j$ and such that $g_{j_\ell j_{\ell+1}} = 1$ for each $\ell \in \{1, ..., m-1\}$. In this case, agent i is said to be connected to j through a path. In order to indicate that such a path exists between i and j, we use the notation $i \rightrightarrows j$. A (directed) *cycle* is a (directed) path from some agent $i \in N$ to herself.

Definition 1. *Let $(N, \mathbf{G})$ be a directed network.*

- $(N, \mathbf{G})$ *is* ***weakly connected*** *if the underlying undirected graph (i.e., ignoring the directions of edges) is connected. Accordingly, a directed network is* ***disconnected*** *if it is not weakly connected.*

[12] The theoretical foundations of the Tullock contest function are well established. In particular, the Tullock contest function can be derived using a stochastic, axiomatic, optimally-derived, and microfounded approach (Skaperdas, 1996; Jia, 2008; Jia et al., 2013).

[13] A network is *undirected* if for each pair i, $j \in N$, $g_{ij} = g_{ji}$. Note that undirected networks are special cases of directed networks.

- $(N, \mathbf{G})$ *is* ***semi-connected*** *if, for any pair of agent* $i, j \in N$*, there is a path from* i *to* j *(i.e.,* $i \rightrightarrows j$*) or a path from* j *to* i *(i.e.,* $j \rightrightarrows i$*).*
- $(N, \mathbf{G})$ *is* ***strongly connected*** *if each node can reach every other node by a path, that is, for any pair of agents* $i, j \in N$*, there is a path from* i *to* j *(i.e.,* $i \rightrightarrows j$*).*
- $(N, \mathbf{G})$ *satisfies* ***no-isolation*** *if, for each* $i \in N$*,* $\mathcal{N}_i \neq \emptyset$*.*

Throughout the paper, we consider directed networks $(N, \mathbf{G})$ that are (at least) weakly connected and satisfy no-isolation. Note that strongly connected networks are semi-connected and also satisfy no-isolation. In turn, semi-connected networks are weakly connected. Note also that a weakly connected network that satisfies no-isolation necessarily contains at least one directed cycle.

Given the exogenous resource V, let W_i be the resource *perceived* to be "owned" by the local environment of agent i, i.e., by agent i as well as her neighbors. Each individual $i \in N$ competes for W_i, with agents in $\mathcal{N}_i$ (i.e., her neighbors). Indeed, given an effort profile $\mathbf{x} \in \mathbb{R}^n_+$, for each $i \in N$, let $\mathbf{x}_{-i}$ be the effort sub-profile of agents $j \in \mathcal{N}_i$. Given W_i, the *perceived utility* of agent i is then equal to

$$u_i(x_i, \mathbf{x}_{-i}; W_i) = \begin{cases} \frac{x_i}{x_i + \sum_{j \in \mathcal{N}_i} x_j} W_i - c x_i & \text{if } x_i + \sum_{j \in \mathcal{N}_i} x_j > 0, \\ \frac{1}{1 + |\mathcal{N}_i|} W_i & \text{if } x_i + \sum_{j \in \mathcal{N}_i} x_j = 0. \end{cases} \tag{3}$$

Remark 1. *Apart from their position in the network, all agents are identical.*

The aim of our paper is to study how the individual's network position affects the effort and outcome of each agent. This is why we assume that all agents are ex-ante identical, i.e., $c_i = c$ for all $i \in N$. In other words, the only source of heterogeneity stems from the agents' network position and, thus, the set of agents they perceive as competitors. Differences in *perceived competition* are therefore the main source of heterogeneity.

2.3 Peer-consistent equilibrium

The critical assumption of our model is that, in order to choose an effort level, each individual i considers the resource attainable in her neighborhood $\{i\} \cup \mathcal{N}_i$ as given, while in reality, it is determined by the fraction of effort exerted in this neighborhood with respect to the whole network. The following equilibrium concept captures this idea.

Definition 2. *A* ***Peer-Consistent Equilibrium*** *(PCE) is a vector* $\mathbf{x}^* \in \mathbb{R}^n_+$ *such that,*

(i) *for each* $i \in N$ *and each* $x_i \in \mathbb{R}_+$*,*

$$u_i(x_i^*, \mathbf{x}^*_{-i}; W_i) \geq u_i(x_i, \mathbf{x}^*_{-i}; W_i),$$

(ii) *for each* $i \in N$*,*

$$W_i = \frac{x_i^* + \sum_{j \in \mathcal{N}_i} x_j^*}{\sum_{j \in N} x_j^*} V, \text{[14]}$$

[14]this quantity being equal to $V(1 + |\mathcal{N}_i|)/n$ when $\mathbf{x}^* = \mathbf{0}$, by convention.

Condition (i) states that, given her perception of the total resource share in her neighborhood, W_i, and her neighbors' effort levels, $\mathbf{x}^*_{-i}$, each individual i chooses an effort that maximizes her preceived utility. In other words, each agent i takes W_i as given, and chooses the effort level x_i that maximizes $u_i(x_i, \mathbf{x}_{-i}, W_i)$.

Condition (ii) states that the effort levels of all individuals in the network will, in turn, determine the share of each individual (and the share of each neighborhood), according to the sharing rule (1). This is a *consistency* requirement imposed in equilibrium. Indeed, at equilibrium, the vector $\{W_i\}_i$ of local resources and the vector of efforts $\mathbf{x}^*$ have to be consistent with each other, and this has to be true for each neighborhood.[15]

Moreover, note that, in equilibrium, we have

$$u_i\left(x_i^*, \mathbf{x}^*_{-i}; W_i\right) = \frac{x_i^*}{x_i^* + \sum_{j \in \mathcal{N}_i} x_j^*} W_i - c x_i^* = \frac{x_i^*}{\sum_{j \in N} x_j^*} - c x_i^* = \pi_i(\mathbf{x}^*).$$

Indeed, at a peer-consistent equilibrium, individual i's perceived *subjective* utility is equal to her *objective* payoff, i.e., her payoff function in the underlying contest game. Therefore, although individuals initially start with a "wrong" perception of the resource attainable in their neighborhood, this wrong perception induces an interaction pattern that eventually leads to a correct perception of the resource at equilibrium.[16] This is why we call it a "peer-consistent equilibrium."

Remark 2. *Peer-consistent equilibria and Nash equilibria coincide if and only if the network is complete, in which case the unique PCE is the Nash equilibrium of the contest game.*

Indeed, a peer-consistent equilibrium of our contest game on a *complete network* is simply a Nash equilibrium on a contest game, since all agents observe the whole network and the notion of local sightedness disappears. As soon as at least one link is missing, the coincidence disappears: at least one agent is not aware of the existence of some other agents. Observe that the PCE is neither a refinement of the concept of Nash equilibrium, nor a superset (such as correlated equilibrium or peer-confirming equilibria). Rather, it is the outcome of a decentralized optimization problem where each agent must play the effort that maximizes their perceived utility, and where each perceived utility must be ex-post consistent with the realized outcome.

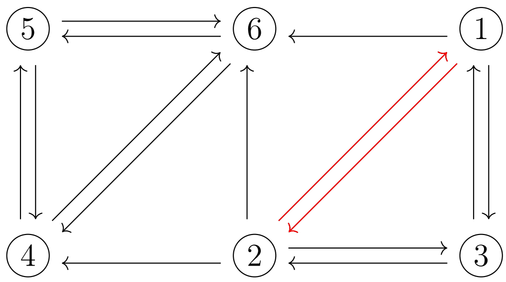
[15] Condition (ii) is similar to the condition imposed for a Walrasian equilibrium in which aggregate supply equals aggregate demand. Here, aggregate supply is V and aggregate demand is the sum of the local resources W_i.

[16] The equality of subject utility and objective payoff derives from the fact that the proportional sharing function of the contest is consistent, a property extensively studied in the literature of fair division (Moulin, 2003). A sharing function is *consistent* if, after a share allocation is chosen, some individuals are left with their shares and, in the reduced problem with the remaining individuals and the remaining resource, each of the remaining individuals is assigned the same amount as initially. As implied by this argument, consistency of the sharing function is required in order to make sense of the equilibrium.

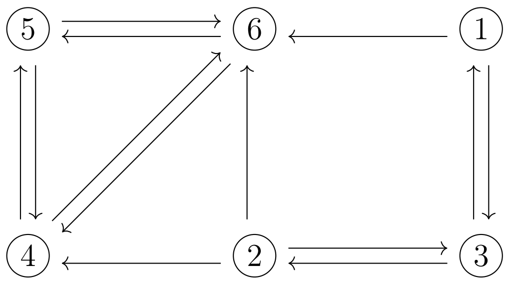
2.4 Peer-consistent equilibrium: An illustration

To understand our peer-consistent equilibrium concept (Definition 2) and the role played by the consistency requirement, let us consider the following two-part example in which we compare the set of peer-consistent equilibria in two closely related networks. This will be part of our leading examples in this paper.

Example 1. Consider the two networks displayed in Figure 1. In order to illustrate our definition of a peer-consistent equilibrium (PCE), let us focus on agents 1 and 6.



(a) A dense 2-layer network



(b) A less dense 2-layer network

Figure 1: Two similar networks with different densities

Same local competitors, same perceived utility: First, as in Definition 2(i), each agent i maximizes her utility (3) by taking W_i as given. Clearly, in both networks displayed in Figures 1(a) and 1(b), agent 6 has the same perceived utility function, namely $\frac{x_6}{x_4+x_5+x_6}W_6 - cx_6$. Indeed, agent 6's perception of her own environment does not change across the two networks. On the other hand, agent 1's perceived utility function switches from $\frac{x_1}{x_1+x_2+x_3+x_6}W_1 - cx_1$ to $\frac{x_1}{x_1+x_3+x_6}W_1 - cx_1$, which implies that agent 1 will *decrease* her effort when the link from 1 to 2 is removed. At the same time, one may think that agent 6's equilibrium level of effort would be unchanged, since the perception of her environment is unchanged across the two networks. This intuition is wrong, as we show next.

Same perceived utility, different equilibrium efforts: Now, by imposing the consistency requirement of Definition 2(*ii*), we can show that agent 6 will produce *more effort* in the network of Figure 1(b) compared to that of Figure 1(a), even though she faces exactly the same competitors (and thus the same perceived utility), namely 4 and 5. This is due to the fact that agent 1's perceived competitors reduce from three (agents 2, 3, 6) to two (agents 3, 6), triggering a decrease in agent 1's effort. This decrease, in turn, implies that, in the network of Figure 1(b), there are more resources left to grab for agent 6, as a side effect of the *consistency requirement* of the PCE. ◇

3 General analysis: Peer-consistent equilibria

This section aims to present a complete analysis of peer-consistent equilibria in weakly connected networks. We first provide a general algebraic characterization in Section 3.1. Then, in Section 3.2, we focus on strongly connected networks, for which we show that PCE provides a microfoundation of eigenvector centrality. The characterization provided so far being implicit, Sections 3.3 and 3.4 are devoted to providing an alternative characterization, by constructing a decomposition algorithm that allows the easy identification of all PCEs in any network. Finally, we show in Section 3.5 that the set of PCEs can be refined to a unique *stable* PCE. Importantly, this particular PCE provides a microfoundation of eigenvector centrality in the case where the network is no longer strongly connected.

3.1 General characterization of peer-consistent equilibria

To begin with, we show that an effort profile is a peer-consistent equilibrium if and only if it is a (properly normalized) nonnegative *eigenvector* of $\mathbf{G}$. For each effort vector $\mathbf{x} \in \mathbb{R}^n_+$, let $X = \sum_{j \in N} x_j$ be the sum of efforts of all agents in the *whole network.*

Theorem 1. *Let* $(N, \mathbf{G})$ *be a weakly connected network. Then* $\mathbf{x}^*$ *is a peer-consistent equilibrium if and only if*

$$\mathbf{G}\mathbf{x}^* = \frac{cX^*}{V - cX^*}\mathbf{x}^*, \quad \text{and} \quad \mathbf{x}^* \in \mathbb{R}^n_+ \setminus \{\mathbf{0}\}.$$

There always exists at least one PCE, because a nonnegative matrix always admits a nonnegative eigenvector (see Lemma A2 in Appendix A.1). However, there might be more than one PCE. It is also worth mentioning that, since we only assume that the network is weakly connected, there might exist peer-consistent equilibria where some components are equal to zero.[17] Let us distinguish between agents who are *active* and *inactive* in equilibrium. Given $(N, \mathbf{G})$ and a peer-consistent equilibrium $\mathbf{x}^*$, let $N_+(\mathbf{x}^*)$ be the set of agents who are *active* at equilibrium x^*. That is, $N_+(\mathbf{x}^*) = \{i \in N : x^*_i > 0\}$. Observe that, if $\mathbf{x}^*$ is a PCE, then its set of active agents is *closed*,[18] which means that

$$i \rightrightarrows j \text{ and } j \in N_+(\mathbf{x}^*) \implies i \in N_+(\mathbf{x}^*). \tag{4}$$

[17] It can even be the case that there is no PCE where every agent is active.
[18] We prove this formally in Lemma B6 in Appendix B.

In other words, if there is a path from i to j and j is active at the PCE x^*, then i is also necessarily active. We also prove a stronger statement below (Proposition 2), after showing how a network can be decomposed in a convenient manner in order to find equilibria.

For any network $(N, \mathbf{G})$ and any subset of agents $M \subseteq N$, let $\mathbf{G}_M$ denote the restriction of matrix $\mathbf{G}$ to M. If $\mathcal{N}_i = \emptyset$, then agent i is irrelevant to the equilibrium analysis. Indeed, agent i's effort is zero in any PCE and $\mathbf{x}^*$ is a PCE for network $(N, \mathbf{G})$ if and only if $\mathbf{x}^*_{-i}$ is a PCE for the network $(N \setminus \{i\}, \mathbf{G}_{N\setminus\{i\}})$. Consequently, in the rest of this paper, we will always assume that the network satisfies the *no-isolation* property (Definition 1).

3.2 Strongly connected networks

First, let us focus on *strongly connected networks* (see Definition 1).[19] Let $\rho(\mathbf{G})$ be the *spectral radius* of the matrix $\mathbf{G}$. That is, $\rho(\mathbf{G}) = \max\{|\lambda_1|, \ldots, |\lambda_n|\}$, where $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of $\mathbf{G}$. By the Perron-Frobenius theorem (see Lemma A1 in Appendix A.1), $\rho(\mathbf{G})$ is an eigenvalue of $\mathbf{G}$, associated to a positive eigenvector, uniquely defined up to multiplication by a positive constant.

Proposition 1 (Existence and uniqueness of PCE in strongly connected networks.)**.** *Let $(N, \mathbf{G})$ be a strongly connected network. Then, there exists a unique peer-consistent equilibrium $\mathbf{x}^*$ in which*

$$\mathbf{G}\mathbf{x}^* = \rho(\mathbf{G})\mathbf{x}^* \quad \text{and} \quad X^* = \frac{V\rho(\mathbf{G})}{c\,[1+\rho(\mathbf{G})]}. \tag{5}$$

That is, $\mathbf{x}^*$ is a positive eigenvector of $\mathbf{G}$ associated with $\rho(\mathbf{G})$.

Proposition 1 provides a *microfoundation of eigenvector centrality* in such networks. It shows that, for any strongly connected network, at the unique PCE, the effort of each agent is proportional to her eigenvector centrality.[20] This characterization relies on both requirements in Definition 2. First, given W_i, each agent i chooses her effort x_i^* that maximizes her perceived utility (3). This leads to:

$$\frac{\sum_{j\in\mathcal{N}_i} x_j^*}{\left(x_i^* + \sum_{j\in\mathcal{N}_i} x_j^*\right)^2} W_i = c. \tag{6}$$

[19] A special case of a network that is strongly connected is the *complete network*. A contest on a complete network is identical to the standard contest game (Skaperdas, 1996). In particular, a peer-consistent equilibrium of the contest game on a complete network is simply a Nash equilibrium on a contest game, since all agents observe the whole network (Remark 2).

[20] In strongly connected networks, eigenvector centrality is a well-defined measure of centrality captured by the Perron-Frobenius vector associated to the adjacency matrix (Jackson, 2008). In Section A.2 of Appendix A, we provide a more general definition of eigenvector centrality for networks that are not necessarily strongly connected.

Second, Definition 2(ii) requires that all efforts are consistent at the PCE by imposing that

$$W_i = \frac{x_i^* + \sum_{j \in \mathcal{N}_i} x_j^*}{\sum_{j \in N} x_j^*} V.$$

By plugging this value into (6), we obtain:

$$\frac{\sum_{j \in \mathcal{N}_i} x_j^*}{x_i^* + \sum_{j \in \mathcal{N}_i} x_j^*} \frac{V}{X^*} = c,$$

where $X^* = \sum_{j \in N} x_j^*$ is the equilibrium aggregate effort. By solving this equation, we get:

$$\mathbf{G}\mathbf{x}^* = \frac{cX^*}{V - cX^*} \mathbf{x}^*, \tag{7}$$

which is equal to (5). Either equation (5) or (7) show that, for any PCE, the effort of each agent is determined by her eigenvector centrality. This is mainly due to the consistency requirement (*ii*) of Definition 2 (and also to the Tullock contest utility function), since (6) by itself does not deliver this result. This is a new result, complementing that of Ballester et al. (2006), who show that, for any network, in a game with strategic complementarities, for each agent who chooses effort that maximizes a linear-quadratic utility function, her equilibrium effort is equal to her Katz-Bonacich centrality. Here, we show that, if each agent chooses her effort that maximizes a utility based on the Tullock contest function, then, at any PCE, her effort will be proportional to her eigenvector centrality. Example D1 in Appendix D.1, in which the network displayed in Figure D1 is strongly connected, illustrates this result.

Below, we show that this characterization in terms of eigenvector centrality carries over to any weakly connected network, after refining the set of peer-consistent equilibria in a meaningful way.

Observe that the peer-consistent equilibrium $\mathbf{x}^*$ of any strongly connected network $(N, \mathbf{G})$ exhibits the so-called "neighborhood effect." That is, individuals whose neighbors have a higher share of the resources obtain themselves a higher share of the resources. Indeed, for each agent i, $\sum_{k \in \mathcal{N}_i} x_k^* = \rho(\mathbf{G}) x_i^*$. Thus, we have, for any i, j:

$$x_i^* + \sum_{k \in \mathcal{N}_i} x_k^* > x_j^* + \sum_{k \in \mathcal{N}_j} x_k^* \iff \sum_{k \in \mathcal{N}_i} x_k^* > \sum_{k \in \mathcal{N}_j} x_k^* \iff x_i^* > x_j^*$$

The emergence of a neighborhood effect obtained at the peer-consistent equilibrium is, in fact, not obvious. Indeed, there are two opposite effects of having connections to neighbors who are expected to obtain a higher share of the resources. On the one hand, the expectation of a higher share of resources by her neighbors encourages an individual to exert higher effort (*resource effect*). On the other hand, the expectation of a higher effort exerted by neighbors discourages an individual to exert higher efforts (*competition effect*). Proposition 1 shows that, in equilibrium, the resource effect dominates the competition effect. Hence, we observe a neighborhood effect in equilibrium, that is, individuals connected to high-effort neighbors also exert high effort. The reason for this effect is traced back to the proportionality of agents' efforts to their eigenvector centrality (Proposition 1). Indeed, a node is more (eigenvector) central the more it is connected to (eigenvector) central nodes (Jackson, 2008; Jackson et al., 2017).

3.3 General network decomposition: An algorithm approach

3.3.1 General results

Let us start with some important definitions.

Definition 3. *Let $M \subset N$. $(M, \mathbf{G}_M)$ is a strongly connected component of $(N, \mathbf{G})$ if:*

- (i) *it is strongly connected*
- (ii) *it is **maximal** in the sense that, for each $i \in N \setminus M$, $(M \cup \{i\}, \mathbf{G}_{M\cup\{i\}})$ is not a strongly connected network.*

To illustrate this definition, consider the (weakly connected) network in Figure 1(a) with $N = \{1, 2, 3, 4, 5, 6\}$ and the sub-networks $(M_1, \mathbf{G}_{M_1})$ and $(M_2, \mathbf{G}_{M_2})$ with $M_1 = \{1, 2, 3\}$ and $M_2 = \{4, 5, 6\}$. We can see that both $(M_1, \mathbf{G}_{M_1})$ and $(M_2, \mathbf{G}_{M_2})$ are strongly connected components: each is a strongly connected sub-network, and it is not possible to enlarge any of these two sub-networks to form a larger strongly connected network. The analysis prevails for the network in Figure 1(b). Consider, now, the network displayed in Figure D1 in Appendix D.1 with $N = \{1, 2, 3, 4\}$. There is no strongly connected component other than the whole network itself. Indeed, consider the sub-network $\{1, 2, 3\}$. We can always add agent 4 and obtain a larger strongly connected network, which is $N = \{1, 2, 3, 4\}$. The same applies for the sub-network $\{2, 3, 4\}$. Using the same argument, $\{1, 2\}$ and $\{3, 4\}$ are not strongly connected components.

Definition 4. *Let $M \subset N$. Agent $i \in N\backslash M$ is an **adjunct** to the sub-network $(M, \mathbf{G}_M)$ of $(N, \mathbf{G})$ if i is connected to some agent $j \in M$ through a path. The **adjunct set** of M, denoted by $\bar{M}$,*[21] *is therefore defined as the set of all agents that are adjuncts to M, that is*

$$\bar{M} = \{i \in N \setminus M :\ \exists j \in M \text{ with } i \ \rightrightarrows\ j\}.$$

*The sub-network $(M, \mathbf{G}_M)$ is **adjunct cycle-free** if the adjunct sub-network $(\bar{M}, \mathbf{G}_{\bar{M}})$ has no cycle.*

Note that it is possible to have $\bar{M}_l = \emptyset$. In Figure 1(a), agents 1, 2 and 3 are adjuncts to the sub-network $\{4, 5, 6\}$, while agents 4, 5 and 6 are not adjunct to the sub-network $\{1, 2, 3\}$. Thus, the adjunct set of $M_1 = \{1, 2, 3\}$ is $\bar{M}_1 = \emptyset$, while that of $M_2 = \{4, 5, 6\}$ is

[21] Note that $\bar{M}$ is defined with respect to N; hence N should appear in the notation. However, to ease presentation, we omit it here.

$\bar{M}_2 = \{1, 2, 3\}$. Therefore, $M_2 \cup \bar{M}_2 = N$. Observe that the sub-network $M_1 = \{1, 2, 3\}$ is adjunct cycle-free, whereas $M_2 = \{4, 5, 6\}$ is not. Clearly, the exact same analysis applies for Figure 1(b).

Definition 5. *Let $M \subset N$. Then, $(M, \mathbf{G}_M)$ is a **perfect sub-network** of $(N, \mathbf{G})$ if it is: (i) a strongly connected component of $(N, \mathbf{G})$; (ii) adjunct cycle-free; and (iii) $|M| \geq 2$.*

Observe that the adjunct set of a perfect sub-network is acyclic. The sub-network $\{1, 2, 3\}$ in Figures 1(a) and (b) is a perfect sub-network, while $\{4, 5, 6\}$ is not. The only perfect sub-network of the network $(N, \mathbf{G})$ in Figure D1 in Appendix D is $(N, \mathbf{G})$ itself.

As it turns out, perfect sub-networks play a key role in determining the sets of active individuals at a peer-consistent equilibrium. Such sub-networks are the focus of the decomposition algorithm that we introduce next. The algorithm has different steps. Each step is denoted by a superscript $k = 1, 2, 3, \cdots$; thus, a sub-network found at step k will be denoted by M_l^k. At each step k, the algorithm finds the perfect sub-networks, but in the new network that has been netted out of all the previously identified perfect subnetworks, as well as the subnetworks of all their respective adjunct sets.

We prove in Appendix B (see Lemma B7) that any weakly connected network admits a perfect sub-network. This enables us to initiate the following procedure.

Algorithm 1 (Network Layer Decomposition, NLD)**.**
Step 1. *Let $\{M_l^1\}_{l=1,\ldots,m^1}$ be the collection of perfect sub-networks of network $(N, \mathbf{G})$. Denote $M^1 := \cup_{l=1}^{n^1} \left(M_l^1 \cup \bar{M}_l^1\right)$. If $N^1 := N \backslash M^1 = \emptyset$, the algorithm stops; otherwise, go to step 2.*

Step k** ($k \geq 2$)**. *Consider the network $\left(N^{k-1}, \mathbf{G}_{N^{k-1}}\right)$, where $N^{k-1} := N \backslash M^k$ and $M^k := \cup_{l=1}^{m^k} \left(M_l^k \cup \bar{M}_l^k\right)$.[22] Let $\{M_l^k\}_{l=1,\ldots,m^k}$ be the collection of perfect sub-networks of network $\left(N^{k-1}, \mathbf{G}_{N^{k-1}}\right)$. If $N^k := N \backslash M^k = \emptyset$, the algorithm stops; otherwise, go to step $k+1$.*

Definition 6. *Suppose that the NLD algorithm stops at step T. We call the network $(N, \mathbf{G})$ a T-layer network. Moreover, for each $k \leq T$ and each $l \leq m^k$, we call $(M_l^k, \mathbf{G}_{M_l^k})$ a layer-k perfect sub-network of $(N, \mathbf{G})$.*

The Network Layer Decomposition algorithm partitions the network into communities or sub-networks called layer-k perfect sub-networks, where, within each community, all agents have the same propensity to exert positive effort. Indeed, the NLD algorithm will first search for the perfect sub-networks M_l^1 or layer-1 perfect sub-networks. These are sub-networks or communities, which have at least two members, are strongly connected, and cannot be reached by other strongly connected communities (Definition 5). In other words, few agents are "aware" of these communities, which can therefore grab a large share of resources within their neighborhood. After knowing M^1, i.e., all the layer-1 perfect sub-networks and their adjuncts, the algorithm will remove them from the network as well

[22]Note that the adjunct set $\bar{M}_l^k$ is defined relative to network $\left(N^{k-1}, \mathbf{G}_{N^{k-1}}\right)$. Hence, $\bar{M}_l^k = \{i \in N^k \setminus M_l^k : \exists j \in M_l^k \text{ with } i \rightrightarrows j\}$.

as their links. In Step 2, the algorithm will search for the communities that become perfect sub-networks after the removal of M^1. These are the layer-2 perfect sub-networks. The NLD algorithm then continues until (finite) Step T is reached, at which point all agents in the network have been selected at some step, and all the perfect sub-networks have been identified.

Remark 3. *The NLD algorithm ends in finite time T and the decomposition*

$$\{M_l^1\}_{l=1,...,m^1}, \{M_l^2\}_{l=1,...,m^2}, ..., \{M_l^k\}_{l=1,...,m^k}$$

is unique up to re-labelling.

In Lemma B8 in Appendix B, we derive a series of properties of the decomposition obtained through the application of the NLD algorithm. These implications are useful in understanding the topology of the many layer perfect sub-networks that typical networks contain, and the connections that exist between these different layers.

Example D2 in Appendix D.2 explains how the adjacency matrix $\mathbf{G}$ of a network $(N, \mathbf{G})$ is affected by the NLD algorithm. We consider a network that has three layer-1 perfect sub-networks, two layer-2 perfect sub-networks, and two layer-3 perfect sub-networks. The matrix $\mathbf{G}$ of this network is decomposed in three steps, as displayed in Figure D2. We show how each step of the NLD algorithm affects the adjacency matrix by removing the set of all agents and their adjuncts belonging to the corresponding layer-k perfect sub-networks.

3.3.2 Examples

Let us illustrate the network decomposition using simple examples, one for two 2-layer networks and one for a 3-layer network.

Example 1: A 2-layer network I
Consider the network $(N, \mathbf{G})$ in Figure 1(a) with $N = \{1, 2, 3, 4, 5, 6\}$. In Step 1 of the NLD algorithm, the sub-network $(M_1^1, \mathbf{G}_{M_1^1})$ with $M_1^1 = \{1, 2, 3\}$ is the only perfect sub-network (Definition 5) of network $(N, \mathbf{G})$ with adjunct set $\bar{M}_1^1 = \emptyset$, so that $M^1 = M_1^1 \cup \bar{M}_1^1 = \{1, 2, 3\}$. Therefore, the sub-network $M_1^1 = \{1, 2, 3\}$ is a layer-1 perfect sub-network (Definition 6). Let us remove $M^1 = \{1, 2, 3\}$ and its links. We end up with the network $(N^1, \mathbf{G}_{N^1})$, with $N^1 = N \backslash M^1 = \{4, 5, 6\}$. We are now in Step 2. In N^1, we can see that the sub-network $M_1^2 = \{4, 5, 6\}$ is *now* a perfect sub-network or a layer-2 perfect sub-network with adjunct set $\bar{M}_1^2 = \emptyset$, so that $M^2 = M_1^2 \cup \bar{M}_1^2 = \{4, 5, 6\}$. Since $N^2 = N^1 \backslash M^2 = \emptyset$, the NLD algorithm stops at Step 2.

Example 1: A 2-layer network II
Consider now the network $(N, \mathbf{G})$ displayed in Figure 1(b) with $N = \{1, 2, 3, 4, 5, 6\}$, a variation of Figure 1(a) in which we *deleted* the directed edges between agents 1 and 2. It is easily verified that the NLD algorithm proceeds in exactly the same way as in the network of Figure 1(a) and stops at Step 2 with $M_1^1 = \{1, 2, 3\}$ being a layer-1 perfect sub-network and $M_1^2 = \{4, 5, 6\}$ a layer-2 perfect sub-network.

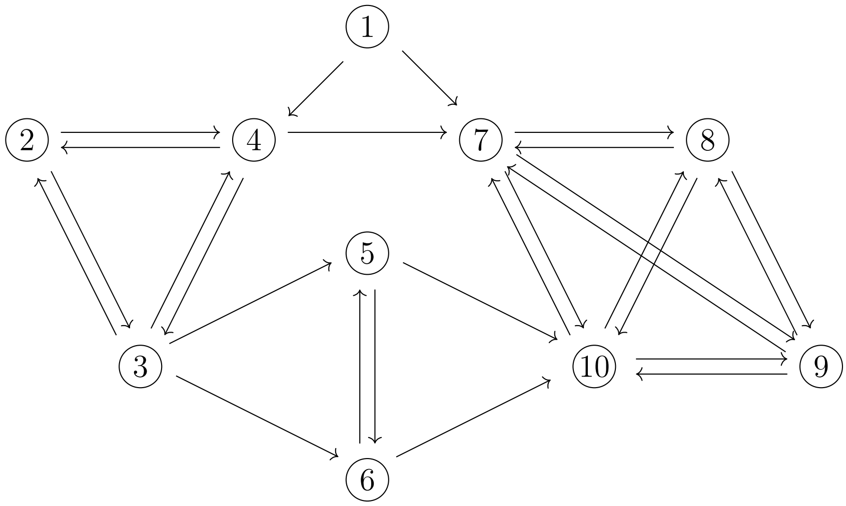


Figure 2: Network structure in Example 2

Example 2. A 3-layer network
Consider now the network $(N, \mathbf{G})$ displayed in Figure 2 with $N = \{1, 2, \cdots, 10\}$. Let us run the NLD algorithm. In Step 1, we can see that only the sub-network $M_1^1 = \{2, 3, 4\}$ is a perfect sub-network with adjunct set $\bar{M}_1^1 = \{1\}$, so that $M^1 = M_1^1 \cup \bar{M}_1^1 = \{1, 2, 3, 4\}$. Thus, $M_1^1 = \{2, 3, 4\}$ is a layer-1 perfect sub-network. Indeed, $\{5, 6\}$ is not a perfect sub-network in Step 1 because it is not adjunct-cycle free, since its adjunct set is $\{1, 2, 3, 4\}$ and contains cycles. Similarly, $\{7, 8, 9, 10\}$ is not adjunct-cycle free, since its adjunct set is $\{1, 2, 3, 4, 5, 6\}$ and contains cycles. Let us remove $M^1 = \{1, 2, 3, 4\}$ and its links. We end up with the network $(N^1, \mathbf{G}_{N^1})$, with $N^1 = N \backslash M^1 = \{5, 6, 7, 8, 9, 10\}$.

We are now in Step 2. In N^1, we can see that the sub-network $M_1^2 = \{5, 6\}$ is *now* a perfect sub-network with adjunct set $\bar{M}_1^2 = \emptyset$, so that $M^2 = M_1^2 \cup \bar{M}_1^2 = \{5, 6\}$. Thus, M_1^2 is a layer-2 perfect sub-network. It is easily verified that, at Step 2, $\{7, 8, 9, 10\}$ cannot be a perfect sub-network because its adjunct set is $\{5, 6\}$, which contains a cycle. Let us remove $M^2 = \{5, 6\}$ and its links. We end up with the network $(N^2, \mathbf{G}_{N^2})$, with $N^2 = N^1 \backslash M^2 = \{7, 8, 9, 10\}$.

We are now in Step 3. In N^2, the sub-network $M_1^3 = \{7, 8, 9, 10\}$ is a perfect sub-network. It is therefore a layer-3 perfect sub-network, with adjunct set $\bar{M}_1^3 = \emptyset$. Hence, $M^3 = M_1^3 \cup \bar{M}_1^3 = \{7, 8, 9, 10\}$. Then $N^3 = N^2 \backslash M^3 = \emptyset$, and the NLD algorithm stops.

3.4 Layer-based characterization of peer-consistent equilibria

So far, we have shown that any weakly connected network $(N, \mathbf{G})$ can be decomposed into multiple layer perfect sub-networks. In each layer k, several layer-k perfect sub-networks may exist. With the help of a new definition, we will be ready to provide a characterization of the set of peer-consistent equilibria.

Definition 7. *Let $(N, \mathbf{G})$ be a $T-$layer network. For each layer-k perfect sub-network $(M_l^k, \mathbf{G}_{M_l^k})$, $l \leq m^k$, denote by Q_l^k the adjunct set of M_l^k in N. That is,*

$$Q_l^k = \{i \in N \backslash M_l^k : \exists j \in M_l^k, \text{ with } i \rightrightarrows j\}.$$

Let $D_l^k := M_l^k \cup Q_l^k$. We call D_l^k a ***candidate set with root $\boldsymbol{M_l^k}$****.*

A *candidate set* is a set of agents that could naturally be the set of active players at equilibrium. Indeed, if there is one agent i who is active in M_l^k, then all agents in M_l^k as well as those belonging to the adjunct set Q_l^k will be active. This is because all agents who are path-connected to i are necessarily active, since the set of active agents at a peer-consistent equilibrium is closed (see (4)).

Note the difference between the definition of the adjunct sets $\bar{M}_l^k$ and Q_l^k, in which the latter is defined with respect to N, irrespective of the Step k at which M_l^k is computed. Note also that, in Step 1 of the NLD algorithm, $\bar{M}_l^1 = Q_l^1$. For each step k of the NLD algorithm, there may be several layer-k perfect sub-networks. Consider Example 2 with the network displayed in Figure 2. Then $M_1^1 = \{2, 3, 4\}$, $M_1^2 = \{5, 6\}$, and $M_1^3 = \{7, 8, 9, 10\}$. In Step 1, the adjunct set $\bar{M}_1^1 = Q_1^1 = \{1\}$. In Sep 2, we have $\bar{M}_1^2 = \emptyset$ while $Q_1^2 = \{1, 2, 3, 4\}$. In Step 3, $\bar{M}_1^3 = \emptyset$ while $Q_1^3 = \{1, 2, 3, 4, 5, 6\}$. Thus, $D_1^1 = M^1 = \{1, 2, 3, 4\}$, $D_1^2 = M_1^2 \cup Q_1^2 = \{1, 2, 3, 4, 5, 6\}$, and $D_1^3 = M_1^3 \cup Q_1^3 = N$.

Definition 8. *Let $(N, \mathbf{G})$ be a T-layer network. For each layer-k perfect sub-network $(M_l^k, \mathbf{G}_{M_l^k})$, $k \leq t$, $l \leq m^k$, let D_l^k be the associated candidate set. A* ***peer-consistent equilibrium*** *$\mathbf{x}^*$ of $(N, \mathbf{G})$ such that $N_+(\mathbf{x}^*) = D_l^k$ for some $k \leq t$ and $l \leq m^k$ is called an* ***equilibrium with root $\boldsymbol{M_l^k}$****. We refer to any equilibrium that admits a root as a* ***simple equilibrium****.*

We have the following key result:

Proposition 2. *Let $(N, \mathbf{G})$ be a T-layer network. There is at most one peer-consistent equilibrium $\mathbf{x}^*$ with root M_l^k. It exists if and only if $\rho(\mathbf{G}_{M_l^k}) > \rho(\mathbf{G}_{Q_l^k})$. In particular, there always exists an equilibrium with root M_l^1, for any $l = 1, ..., m^1$.*

This proposition is simple but very powerful in terms of characterizing the equilibrium for each perfect sub-network.[23] The NLD algorithm determines the different layers of the perfect sub-networks. Proposition 2 states that, for each layer, there can be *at most* one PCE with root M_l^k. This provides a necessary and sufficient condition in terms of the largest eigenvalues comparisons for this equilibrium to exist. This condition is automatically satisfied for layer-1 perfect sub-networks, because $\rho(\mathbf{G}_{Q_l^1}) = \rho(\mathbf{G}_{\bar{M}_1^1}) = 0$ (by construction, $\bar{M}_1^1$ is acyclic). For $k \geq 2$, however, finding out if a PCE with root M_l^k exists requires checking the non-trivial[24] inequality $\rho(\mathbf{G}_{M_l^k}) > \rho(\mathbf{G}_{Q_l^k})$.

[23]To ease presentation, we refer to any layer-k perfect sub-network as a perfect sub-network.

[24]For $k \geq 2$, Q_l^k necessarily contains cycles, by construction, and thus $\rho(\mathbf{G}_{Q_l^k}) > 0$.

We proceed in the same way for $k > 2$ until the last layer T of perfect sub-networks is determined by the NLD algorithm. In summary, we need to check the eigenvalue condition for each layer $k = 1, 2, \cdots, T$ and each $l = 1, ..., m_l^k$. For a given k and m_l^k, if $\rho(\mathbf{G}_{M_l^k}) > \rho(\mathbf{G}_{Q_l^k})$ holds, then there exists a peer-consistent equilibrium $\mathbf{x}^*$ with root M_l^k.

Remark 4 (Equilibrium payoffs)**.** *Pick any peer-consistent equilibrium* $\mathbf{x}^*$ *with root* M_l^k *for some* $k \leq T$ *and* $l \leq m_l^k$*. Then, the payoff of each active agent* i *is given by*

$$u_i(x_i^*) = \frac{c}{\rho(\mathbf{G}_{M_l^k})} x_i^* \tag{8}$$

where $u_i(x_i^*) := u_i(x_i^*, \mathbf{x}_{-i}^*; W_i)$*. Moreover, the sum of utilities of active agents at* $\mathbf{x}^*$ *is given by:*

$$\sum_i u_i(x_i^*) = \frac{V}{1 + \rho(\mathbf{G}_{M_l^k})}$$

This remark shows that the equilibrium utility of each active agent is proportional to her equilibrium effort. It also shows that the equilibrium payoff (8) is decreasing in $\rho(\mathbf{G}_{M_l^k})$. Therefore, in denser perfect sub-networks, agents obtain relatively lower utility. Further, the more agents are active and the denser the network is, the lower the aggregate utility.

Let us now illustrate Proposition 2 with the following examples.[25]

Example 1: Peer-consistent equilibria

A 2-layer network: I

Consider the network $(N, \mathbf{G})$ in Figure 1(a) with $N = \{1, 2, \cdots, 6\}$. In Step 1 of the NLD algorithm, the sub-network $(M_1^1, \mathbf{G}_{M_1^1})$ with $M_1^1 = \{1, 2, 3\}$ is the only perfect sub-network with adjunct set $\bar{M}_1^1 = \emptyset$. In Step 2, $(M_1^2, \mathbf{G}_{M_1^2})$, with $M_1^2 = \{4, 5, 6\}$, is the only perfect sub-network of $(N^1, \mathbf{G}_{N^1})$, with $\bar{M}_1^2 = \emptyset$ and $Q_1^2 = \{1, 2, 3\}$. Thus, we have two candidate sets: $D_1^1 = \{1, 2, 3\}$ and $D_1^2 = \{1, 2, 3, 4, 5, 6\} = N$.

(a) Equilibrium with root $\boldsymbol{M_1^1}$: Since $\rho(\mathbf{G}_{M_1^1}) = 2 > \rho(\mathbf{G}_{Q_1^1}) = 0$, there is an equilibrium with root $M_1^1 = \{1, 2, 3\}$, where only agents 1, 2 and 3 are active.

(b) Equilibrium with root $\boldsymbol{M_1^2}$: Since $\rho(\mathbf{G}_{M_1^2}) = 2$ and it is not strictly greater than $\rho(\mathbf{G}_{Q_1^2}) = \rho(\mathbf{G}_{M_1^1}) = 2$, there is *no* equilibrium with root M_1^2.

As a result, there is a unique peer-consistent equilibrium such that $x_1^* = x_2^* = x_3^* = \frac{2V}{9c}$ and $x_4^* = x_5^* = x_6^* = 0$.

A 2-layer network: II

Consider now the network $(N, \mathbf{G})$ displayed in Figure 1(b) with $N = \{1, 2, \cdots, 6\}$, a variation of Figure 1(a) in which we *deleted* the directed edges between agents 1 and 2, so that the layer-1 perfect sub-network M_1^1 is now less dense and, thus, its spectral radius

[25] We also illustrate Proposition 2 with an additional example, i.e., Example D3 in Appendix D.3. In this example, there is no peer-consistent equilibrium in which all agents are active.

is smaller. The NLD algorithm proceeds in the same way as in Figure 1(a), and thus we have two candidate sets: $D_1^1 = \{1, 2, 3\}$ and $D_1^2 = \{1, 2, 3, 4, 5, 6\} = N$.

(a) Equilibrium with root M_1^1: Since $\rho(\mathbf{G}_{M_1^1}) = \sqrt{2} > \rho(\mathbf{G}_{Q_1^1}) = 0$, there is an equilibrium with root $M_1^1 = \{1, 2, 3\}$, where only agents $1, 2$ and 3 are active.

(b) Equilibrium with root M_1^2: Since $\rho(\mathbf{G}_{M_1^2}) = 2 > \sqrt{2} = \rho(\mathbf{G}_{Q_1^2}) = \rho(\mathbf{G}_{M_1^1})$, there is an equilibrium with root M_1^2, where all agents in the network are active. At such an equilibrium, note that efforts are not symmetric. Indeed, $x_1^* = \frac{V}{9c}$, $x_2^* = \frac{7V}{45c}$, $x_3^* = \frac{2V}{15c}$, and $x_i^* = \frac{4V}{45c}$ for each $i \in \{4, 5, 6\}$.

There are now *two* peer-consistent equilibria. Therefore, by removing the links between agents 1 and 2, we enlarged the set of peer-consistent equilibria from a *unique* equilibrium to *two* PCE. ◇

Example 2: Peer-consistent equilibria

Consider now the 3-layer network displayed in Figure 2 with $M_1^1 = \{2, 3, 4\}$, $M_1^2 = \{5, 6\}$, and $M_1^3 = \{7, 8, 9, 10\}$. Despite the existence of three perfect sub-networks, let us use Proposition 2 to show that there are 'only' two peer-consistent equilibria. The sub-matrices of each perfect sub-network are given by

$$\mathbf{G}_{M_1^1} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \quad \mathbf{G}_{M_1^2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \mathbf{G}_{M_1^3} = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

We can verify that the spectral radius are such that: $\rho(\mathbf{G}_{M_1^2}) = 1 < \rho(\mathbf{G}_{M_1^1}) = 2 < \rho(\mathbf{G}_{M_1^3}) = 3$. Let us now apply Proposition 2 to find all the simple peer-consistent equilibria.

(a) Equilibrium with root M_1^1: Since $\rho(\mathbf{G}_{M_1^1}) > \rho(\mathbf{G}_{Q_1^1}) = 0$, there is an equilibrium with root $M_1^1 = \{2, 3, 4\}$, which, using Theorem 1, is given by:

$$x_1^* = \frac{2V}{21c},\; x_i^* = \frac{4V}{21c} \text{for } i \in \{2, 3, 4\},\; x_j^* = 0 \text{ for } j \in \{5, 6, 7, 8, 9, 10\}.$$

The equilibrium payoffs are $u(x_1^*) = \frac{1}{21}V$, $u(x_i^*) = \frac{2}{21}$ for $i = 2, 3, 4$.

(b) No equilibrium with root M_1^2: Since $\rho(\mathbf{G}_{M_1^2}) < \rho(\mathbf{G}_{Q_1^2}) = \rho(\mathbf{G}_{M_1^1})$, there is no equilibrium with root M_1^2 such that the active individual set is $D_1^2 = M_1^1 \cup M_1^2 = \{1, 2, 3, 4, 5\}$.

(c) Equilibrium with root M_1^3: Since $\rho(\mathbf{G}_{M_1^3}) > \rho(\mathbf{G}_{Q_1^3}) = \max\{\rho(\mathbf{G}_{M_1^1}), \rho(\mathbf{G}_{M_1^2})\}$, there is an equilibrium with root M_1^3 such that the set of active individuals is $D_1^3 = M_1^1 \cup M_1^2 \cup M_1^3 = N$. This equilibrium is given by

$$x_1^* = \frac{21V}{364c},\; x_i^* = \frac{18V}{364c} \text{for } i \in \{2, 5, 6\},\; x_j^* = \frac{27V}{364c} \text{ for } j \in \{3, 4\}, \text{ and } x_k^* = \frac{36V}{364c}, \text{ for } k \geq 7.$$

The equilibrium payoffs are

$$u(x_1^*) = \frac{7V}{364},\ u(x_i^*) = \frac{6V}{364},\ \text{ for } i \in \{2,5,6\},\ u(x_j^*) = \frac{9V}{364} \text{ for } j \in \{3,4\},\ u(x_k^*) = \frac{12V}{364} \text{ for } k \geq 7.$$

In summary, in the network in Figure 2, there are two PCE: one in which only agents 1, 2 and 3 are active, and one in which all agents are active. ◇

More generally, Proposition 2 provides us with some simple principles to assess whether a given agent i is active in equilibrium. Indeed, an agent i is active at a PCE if either:
(i) she belongs to a layer-1 perfect sub-network, since few agents are aware of individuals from this sub-network; or

(ii) she belongs to higher-layer perfect sub-networks in which agents are strongly connected to each other, and can be reached by agents from lower-level perfect sub-networks who are not strongly-connected to each other.

Our notion of *peer-consistent equilibrium* is key to understanding this result because agents are locally-sighted and are only aware of the activity of their direct neighbors. As a result, links and paths in a network reflect our idea of "perceived competition" and affect the effort of the agents in the network. Consider the two networks in Figures 1(a) and (b) (Example 1). Nobody is aware of the community $M_1^1 = \{1,2,3\}$, i.e., nobody in the network perceives them as competitors. Thus, in both cases, there is a PCE in which only agents 1, 2 and 3 are active. However, agents in $M_1^1 = \{1,2,3\}$ are aware of the community $M_1^2 = \{4,5,6\}$, since agent 1 perceives agent 6 as a competitor, while agent 2 perceives both agents 4 and 6 as competitors. Moreover, the community $M_1^1 = \{1,2,3\}$ is less dense than $M_1^2 = \{4,5,6\}$ in Figure 1(b) than in Figure 1(a). As a result, there is an additional PCE in which all agents are active in Figure 1(b). Importantly, this additional PCE (in which all agents are active) is not a symmetric equilibrium: while agents 4, 5 and 6 all exert the same effort, the differences in perceived competition of agents in the community $M_1 = \{1,2,3\}$ lead agent 1 and 2 to exert different levels of effort.

Consider now the network displayed in Figure 2 (Example 2). Aside from the adjunct agent 1, no agent is aware of the community $\{2,3,4\}$, i.e., no direct link or path can reach $M_1^1 = \{2,3,4\}$ except the singleton $\{1\}$. In other words, only agent 1 perceives agents 2, 3 and 4 as her competitor; nobody else in the network does. Thus, there is an equilibrium where the only active agents in the network are those belonging to $M_1^1 \cup \bar{M}_1^1$. On the contrary, all agents in the network can reach community $M_1^3 = \{7,8,9,10\}$, either directly or through a path; thus, there cannot be an equilibrium where only agents in M_1^3 are active.

Proposition 2 demonstrates that there is a direct link between the *network topology* (who can reach whom, i.e., the direction of the links between agents), how *close-knit communities are*, i.e., the spectral radius of perfect sub-networks, and any *Peer-Consistent Equilibrium* (PCE).

In order to complete our characterization of peer-consistent equilibria, let us now consider an interesting superset of the set of semi-connected networks.

Definition 9. *A weakly connected network* $\mathbf{G}$ *is a* ***layer-generic network*** *if* $\rho(\mathbf{G}_{M_l^k}) = \rho(\mathbf{G}_{M_{l'}^{k'}}) = \rho$ *implies that* $\max\left\{\rho(\mathbf{G}_{Q_l^k}), \rho(\mathbf{G}_{Q_{l'}^{k'}})\right\} \geq \rho$.[26]

Hence, layer-generic networks are such that, for any two perfect sub-networks that have the same spectral radius, there must exist a perfect sub-network in one of their adjunct sets with a greater or equal spectral radius. In other words, (i) we exclude weakly connected networks for which two PCE with different roots have the same spectral radius, but (ii) we allow two perfect sub-networks with the same spectral radius if one of them is not part of a PCE. In particular, we exclude a network for which two layer-1 perfect sub-networks have the same spectral radius. For example, the network displayed in Figure E6 in Appendix E.2 is not layer-generic because the two layer-1 perfect sub-networks $M_1^1 = \{2, 3\}$ and $M_2^1 = \{4, 5\}$ have the same spectral radius (i.e., $\rho(\mathbf{G}_{M_1^1}) = \rho(\mathbf{G}_{M_2^1}) = 1$) and the same adjunct set $\bar{M}_1^1 = \bar{M}_2^1 = \{1\}$. On the contrary, the networks in Figures 1(a) and (b) (Example 1) and in Figure 2 (Example 2) are layer-generic. Observe, in particular, that, in the network displayed in Figure 1(a), the perfect sub-networks $M_1^1 = \{1, 2, 3\}$ and $M_1^2 = \{4, 5, 6\}$ have the same spectral radius (i.e., $\rho(\mathbf{G}_{M_1^1}) = \rho(\mathbf{G}_{M_1^2}) = 2$) but, because $M_1^2 = \{4, 5, 6\}$ is not a PCE, this network is layer-generic.

The following inclusions summarize the relative strength of all four notions of connectedness we consider here (Definitions 1 and 9): Strongly Connected (StrCN), Semi-Connected (SemiCN), Layer-Generic (LGN) and Weakly Connected (WCN):

$$\text{StrCN} \subset \text{SemiCN} \subset \text{LGN} \subset \text{WCN}.$$

If $\mathbf{G}$ is a semi-connected network, then there is at most one element per layer. Hence, the layer decomposition writes $(M_1^k)_{k=1,\ldots,t}$, which implies that SemiCN $\subset$ LGN.[27]

Corollary 1. *Let* $(N, \mathbf{G})$ *be a layer-generic network, and let* $\mathbf{x}^*$ *be a peer-consistent equilibrium of* $(N, \mathbf{G})$*. Then* $\mathbf{x}^*$ *is a simple equilibrium. Moreover, equilibrium efforts are proportional to eigenvector centrality in the sub-network of active players.*

The last statement of Corollary 1 must be understood as follows: if $\mathbf{x}^*$ is a PCE, then the effort of active agents are proportional to their eigenvector centrality, *in the sub-network to which they belong*. It is important to understand that this result does not say anything about the eigenvector centrality of agents in the whole network, since inactive agents are not taken into account. A direct consequence of Corollary 1 is that there is a *finite* number of equilibria, because, for any k, l, there is at most one PCE with root M_l^k. Indeed, the set of peer-consistent equilibria is actually finite if and only if the network is layer-generic.

When the network is *not* layer-generic, there may exist *non-simple equilibria*, i.e., equilibria such that the set of active agents is not a candidate set, but instead a union of candidate sets. In this case, the set of peer-consistent equilibria is *infinite*. In Section E of

[26]In other terms, $\rho \leq \max_{M_s^t : M_s^t \subset Q_l^k \cup Q_{l'}^{k'}} \rho(\mathbf{G}_{M_s^t})$.

[27]In Section C.1 of the Appendix, we show that if the network is a semi-connected T-layer network, then there are at most T peer-consistent equilibria (Corollary C2).

the Appendix, we deal with this case and show that we can still describe the set of peer-consistent equilibria in a simple way. Indeed, for each layer k, the set of PCE is always a finite union of convex sets, where each set is a simple unique PCE with root M_l^k (see Proposition E8).

Proposition 3. *Let* $(N, \mathbf{G})$ *be a weakly connected network. The following are equivalent:*

(i) The set of peer-consistent equilibria is finite.

(ii) For any pair $(\mathbf{x}^{1*}, \mathbf{x}^{2*})$ *of peer-consistent equilibria,* $\rho\left(\mathbf{G}_{N_+(\mathbf{x}^{1*})}\right) \neq \rho\left(\mathbf{G}_{N_+(\mathbf{x}^{2*})}\right)$.

(iii) $(N, \mathbf{G})$ *is a layer-generic network.*

In this section, we have first introduced a novel and intuitive decomposition algorithm that breaks down the network into communities, which we refer to as layer-k perfect sub-networks. We were then able to pin down all peer-consistent equilibria in layer-generic networks by comparing the spectral radius of these perfect sub-networks and that of their adjunct set in the whole network. In several of our examples, we have seen that there were multiple peer-consistent equilibria (e.g., the networks in Figure 1(b) and in Figure 2 had two PCEs while the network in Figure D3 had three PCEs). Multiplicity of peer-consistent equilibria is a salient feature of our model. Further, and most importantly, in many such equilibria, some agents end up being inactive.[28]

Remark 5. *Our results are not qualitatively affected if we consider* ***weighted*** *networks, i.e.,* $g_{ij} \in [0, 1]$*, instead of networks in which* $g_{ij} = \{0, 1\}$.

Indeed, suppose that $\mathbf{G}$ is a $n \times n$ matrix, where entry $g_{ij} \in [0, 1]$ represents the intensity (or strength) of the directed link from i to j. Then, the results of this section still hold. Note that layer-generic networks are actually generic in the space of weighted networks.

3.5 Stability and eigenvector centrality

This section is devoted to refining the set of equilibria by characterizing those peer-consistent equilibria that are stable, in a way that will be explicated below. Such an approach has two fundamental objectives. First, it allows us to identify which equilibria are robust to perturbations and it provides a *dynamic microfoundation* to the concept of peer-consistent equilibrium. Second, refining the set of equilibria is necessary if one wants to extend the eigenvector centrality microfoundation to general networks. Indeed, as noted

[28] Observe that we have assumed that all agents were ex ante identical and their only heterogeneity stemmed from their network position. If we relax this assumption and allow for agents to have different costs of effort, i.e., $c = c_i$ for agent i, the NLD algorithm decomposition will be exactly the same but the link between spectral radius and PCE (Proposition 2) will no longer hold true. There will be a trade-off between belonging to a densely connected community and the cost of effort. Similarly, if we assume a more general sharing rule than the one defined in (1), the NLD algorithm will deliver the same result but Proposition 2 will be affected. This is because the NLD algorithm does not rely on any parameter of the model, only on the network topology.

above, efforts of active agents are proportional to their eigenvector centrality in the subgraph of active players. However, this raises a natural question: *is there a link between the eigenvector centrality in the whole network and PCE?* The answer is positive and we show that, in any layer-generic network, exactly one PCE is proportional to the eigenvector centrality of the whole network, and it is precisely the PCE we identify as the stable one.

As usual, stability of equilibria is defined through a meaningful dynamical system, the rest point of which is the equilibria we want to consider. A stable equilibrium is then defined as a stable rest point of the dynamics, i.e., a rest point to which, starting from conditions close enough to it, the system asymptotically stabilizes back to it. For this purpose, we introduce *perceived best-response dynamics*. This captures the idea that agents smoothly adapt their actions in the direction of their *best possible action*, given the information available to them.

3.5.1 The perceived best-response dynamics

We now present the continuous-time dynamics with respect to which we characterize stability. Even though it is very close—in terms of interpretation—of the classical continuous-time best-response dynamics,[29] we explain how it is related to a simple discrete-time model. Consider a discrete-time sequence of effort profiles, in which, after observing their neighbors' effort level as well as the local resources in the previous period, agents adapt their effort levels at each period of time. Specifically, before choosing her effort level at period t, agent i observes the effort of her neighbors $\mathbf{x}_{-i}^{t-1}$ as well as the realized local resource W_i^{t-1} at period $t-1$. She can then compute her optimal effort level with respect to quantity W_i^{t-1} by maximizing the map[30]

$$b_i \in [0, +\infty[\mapsto \frac{b_i}{b_i + (\mathbf{G}\mathbf{x}^{t-1})_i} W_i^{t-1} - cb_i.$$

Since $W_i^{t-1} = \frac{x_i^{t-1} + (\mathbf{G}\mathbf{x}^{t-1})_i}{X^{t-1}} V$, the maximizer is equal[31] to $Br_i(\mathbf{x}^t)$, where $Br_i(\cdot)$ is defined by

$$Br_i(\mathbf{x}) = \max\left\{ -(\mathbf{G}\mathbf{x})_i + \left(\frac{V}{cX} (\mathbf{G}\mathbf{x})_i \left(x_i + (\mathbf{G}\mathbf{x})_i \right) \right)^{1/2}, 0 \right\}. \tag{9}$$

Then, agent i chooses an effort level equal to a convex combination of her last effort level and the perceived best response with respect to what she observed at the last time period:

$$x_i^t = (1-\epsilon) x_i^{t-1} + \epsilon Br_i(\mathbf{x}^{t-1}) \tag{10}$$

When ϵ is small, the sequence generated by (10) is related to the solution curves of the continuous-time system

$$\dot{\mathbf{x}}(t) = \mathbf{B}(\mathbf{x}(t)), \tag{11}$$

[29]See Fisher (1961), Gilboa and Matsui (1991), Matsui (1992) and, more recently, Bramoullé et al. (2014), Bervoets and Faure (2019).

[30]Observe that $(\mathbf{G}\mathbf{x})_i = \sum_{j \in \mathcal{N}_i} x_j^*$. We use this more compact notation whenever it is convenient.

[31]This holds if and only if the action profile $\mathbf{x}$ is such that $(\mathbf{G}\mathbf{x})_i = 0 \Rightarrow x_i = 0$.

where $B_i(\mathbf{x}) = -x_i + Br_i(\mathbf{x}),\ i = 1, ..., N$. Indeed, system (10) is a so-called *Cauchy-Euler scheme*, designed to approximate the solutions of (11), by choosing a small ϵ. In other words, system (11) can be interpreted as a smooth *limit* version of (10).

Choosing the appropriate state space, the stationary points of this ordinary differential equation are precisely the peer-consistent equilibria of our problem. We now consider the stability notion to be naturally associated to the dynamics (11). Stability for a given PCE $\mathbf{x}^*$ means that the solutions of (11) starting from initial conditions close enough to $\mathbf{x}^*$ converge back to $\mathbf{x}^*$. Formally,

Definition 10. *A peer-consistent equilibrium* $\mathbf{x}^*$ *is said to be* ***asymptotically stable*** *for (11) if there exists an open neighborhood* U *of* $\mathbf{x}^*$ *such that*

$$\lim_{t \to +\infty} \sup_{\mathbf{x}_0 \in U \cap \mathbf{S}} \|\phi(\mathbf{x}_0, t) - \mathbf{x}^*\| = 0,$$

where $\mathbf{S}$, defined in (B.6) in Section B.1.5 of the Appendix, contains all the relevant states of the problem we consider, and $(\phi(\mathbf{x}, t))_{\mathbf{x} \in \mathbf{S}, t \geq 0}$ is the semi-flow associated to (11) on $\mathbf{S}$. Specifically, $\phi(\mathbf{x}, t)$ is equal to the position of the (unique) solution of (11) starting at $\mathbf{x}$.

Definition 10 states that a PCE $\mathbf{x}^*$ is asymptotically stable if it *uniformly* attracts all solutions starting in an open neighborhood of itself. This is a standard concept of stability used in economics (Benaïm and Hirsch, 1999; Weibull, 2003), and in network games in particular (Bramoullé et al., 2016; Bervoets and Faure, 2019).

3.5.2 Stable PCE: A simple characterization

We now characterize the PCEs that are asymptotically stable with respect to the best-response dynamics (11). It turns out that being asymptotically stable depends entirely on the sub-network of active players in this PCE, in a very simple and intuitive way.

Let $(N, \mathbf{G})$ be a layer-generic network.[32] Given a PCE $\mathbf{x}^*$, we call $\rho(\mathbf{x}^*)$ the largest eigenvalue of the subnetwork $(N_+(\mathbf{x}^*), \mathbf{G}_{N_+(\mathbf{x}^*)})$.

Theorem 2. *Let* $(N, \mathbf{G})$ *be a layer-generic network. Then, there is a unique asymptotically stable equilibrium* $\mathbf{x}^*$*. It is such that* $\rho(\mathbf{x}^*) = \rho(\mathbf{G})$*. Moreover, agents' effort levels at a stable PCE are proportional to their eigenvector centrality in the network* $(N, \mathbf{G})$*.*

The intuition behind the characterization in terms of largest eigenvalues is as follows. Since the network is layer-generic, there is exactly one PCE for which the largest eigenvalue of the set of active players is equal to $\rho(\mathbf{G})$. We must show that it is the only asymptotically stable equilibrium. Suppose that $\mathbf{x}^*$ is a PCE such that $\rho(\mathbf{x}^*)$ is strictly smaller than $\rho(\mathbf{G})$. Then, one can find a perfect sub-network M_l^k in which agents are inactive at $\mathbf{x}^*$, while having $\rho(\mathbf{G}_{M_l^k}) = \rho(\mathbf{G})$. Now, suppose that we slightly perturb $\mathbf{x}^*$ so that, instead of playing zero, agents in M_l^k play $\epsilon \mathbf{u}_i$, where $\mathbf{u}$ is the normalized positive eigenvector

[32] Our main result (Theorem 2) holds under the less restrictive assumption that $(N, \mathbf{G})$ has a *unique dominant component*, as properly defined in condition (UDC) in Section A.2 of Appendix A. In fact, eigenvector centrality is well defined if and only if the network satisfies the condition (UDC).

associated to $\rho(\mathbf{G})$. Since, for agents in M_l^k, this initial condition is associated to an eigenvalue that is strictly larger than the eigenvalue associated to $\mathbf{x}^*$, the agents in M_l^k will want to increase their effort, and not come back to zero. Thus, it is clear that $\mathbf{x}^*$ cannot be stable.[33] We conclude the proof by showing that the (unique) PCE for which $\rho(\mathbf{x}^*) = \rho(\mathbf{G})$ is stable using standard methods. The last part of the theorem directly follows from the definition of eigenvector centrality.

Theorem 2 provides a simple and efficient analytic method for checking which PCEs are stable. It suffices to check which perfect sub-network has the highest spectral radius in the network.

First, consider Example 1 with the two 2-layer networks displayed in Figure 1(a) and Figure 1(b) with $N = \{1, 2, \cdots, 6\}$. In both networks, there is one layer-1 network $M_1^1 = \{1, 2, 3\}$ and one layer-2 network $M_1^2 = \{4, 5, 6\}$. The only difference between these two networks is that the one in Figure 1(a) has two extra links between agents 1 and 2 compared to the network in Figure 1(b). This is an important difference because the largest eigenvalue of the layer-1 perfect subnetwork changes: it is equal to 2 in Figure 1(a), whereas it is equal to $\sqrt{2}$ in Figure 1(b). In Figure 1(a), there is a unique equilibriumc that is clearly stable, in which only agents 1, 2 and 3 are active. In Figure 1(b), we have seen that there were two PCEs, one with root $M_1^1 = \{1, 2, 3\}$ and one with root $M_1^2 = \{4, 5, 6\}$. Since $\rho(\mathbf{G}_{M_1^1}) = \sqrt{2} < \rho(\mathbf{G}_{M_1^2}) = 2 = \rho(\mathbf{G})$, there is a unique stable PCE for which all agents are active. Thus, disconnecting agents 1 and 2 has a dramatic impact on the stable peer-consistent equilibria. The fact that the layer-1 perfect sub-network in Figure 1(b) is less dense than in Figure 1(a) prevents agents 1, 2 and 3 from capturing the entire resource V and thus obliges them to share V with the other players in the PCE.

Second, consider Example 2 with the 3-layer network depicted in Figure 2. We have seen that there were two PCE with roots $M_1^1 = \{1, 2, 3\}$ and $M_1^3 = \{6, 7, 8, 9\}$, respectively. Since $\rho(\mathbf{G}_{M_1^1}) = 2 < \rho(\mathbf{G}_{M_1^3}) = 3 = \rho(\mathbf{G})$, the only stable PCE is the equilibrium with root M_1^3, where all agents are active. Note that in both examples where there exists a peer-consistent equilibrium x^* with $N_+(x^*) = N$, then x^* must be the stable equilibrium. This is actually always true:

Corollary 2. *Pick a layer-generic network. If there exists a peer-consistent equilibrium* $\mathbf{x}^*$ *with* $N_+(\mathbf{x}^*) = N$*, then* $\mathbf{x}^*$ *is the asymptotically stable PCE.*

In summary, for any (layer-generic) network, we can determine the unique stable peer-consistent equilibrium. First, we run the NLD algorithm that defines the different layer perfect sub-networks (Algorithm 1). Second, we determine the different peer-consistent equilibria by checking, for each PCE, that the spectral radius of the corresponding perfect sub-network is strictly greater than that of its adjunct set in the whole network (Proposition 2). For each PCE, we can ascertain the effort of each agent, which is equal to her eigenvector centrality (Theorem 1). Finally, the unique stable peer-consistent equilibrium in the network is the PCE for which the corresponding perfect sub-network has the same largest eigenvalue as the whole network (Theorem 2).

[33] For ease of presentation, *asymptotically stable* PCEs are referred to as *stable* PCEs.

4 Policy interventions

4.1 Adding links

We now consider the policy implications of our model. We start with the simplest intervention: given a network and its unique stable peer-consistent equilibrium, what would happen if we added a link between two agents?

In Appendix C.2.1, we consider the case when the network is *strongly connected.* Proposition C2 shows that, by adding a link from individual i (the "sender") to individual j (the "receiver"), the sender's effort as well as her resource share always increase. However, the effect on the receiver is ambiguous. Proposition C2 also shows that the effort increase of the sender is sufficiently large to compensate for the ambiguous effect of the receiver, so that total effort increases. Indeed, by adding a link between i and j, the sender i increases her eigenvector centrality (Theorem 1) and, as a result, increases her effort.

Next, we consider *layer-generic* networks. We only focus on *stable* peer-consistent equilibria, i.e., equilibria for which the largest eigenvalue of the corresponding perfect sub-network is equal to that of the whole network (Theorem 2). This is equivalent to considering PCEs for which the corresponding perfect sub-network has the *largest total effort* in the network.

Compared to the strongly connected networks, the situation is more complicated due to the possible change in layer perfect sub-network. However, we are able to show that adding a link from an active individual i (the "sender") to another active individual j (the "receiver") has the same effect as in strongly connected networks.

Proposition 4. *Pick a layer-generic network* $(N, \mathbf{G})$ *with* $\mathbf{x}^*$ *the (unique) asymptotically stable peer-consistent equilibrium for which* $i, j \in N_+(\mathbf{x})$*, and* $g_{ij} = 0$*. Let* $\widehat{\mathbf{G}}$ *be the network obtained from* $\mathbf{G}$ *by adding a link from* i *to* j*. Then* $\widehat{\mathbf{G}}$ *admits an asymptotically stable peer-consistent equilibrium* $\widehat{\mathbf{x}}^*$ *that has the following properties:*

(i) $N_+(\widehat{\mathbf{x}}^*) \subseteq N_+(\mathbf{x}^*)$,

(ii) $\widehat{X}^* \geq X^*$,

(iii) $\widehat{x}_i^* > x_i^*$.

Indeed, by adding a link between i and j, the sender i becomes more central in terms of eigenvector centrality and thus increases her effort, i.e., $\widehat{x}_i^* > x_i^*$. This increases the spectral radius in the network, i.e. $\rho(\widehat{\mathbf{G}}) \geq \rho(\mathbf{G})$. Since total effort is increasing in $\rho(\mathbf{G})$ (see (5)), it raises total effort, i.e., $\widehat{X}^* \geq X^*$. Result *(i)* is more complex. Indeed, when a link is added, the sender becomes more central and more aware of others' activities, while the reverse is not true: nobody is more aware about i's activity. This creates a new path in the network that makes others more likely to be reached by i and thus i is more likely to become part of the adjunct set of j and her neighbors. This, in turn, may lower their status in terms of layer perfect sub-network, since they are more likely to go down the ladder in terms of layers. On the contary, i is only positively affected by the added link since there is no new link pointing in her direction and thus she will stay in the same layer. This is why, when a link is added between agents i and j, the number of active agents is either the same or lower, i.e., $N_+(\widehat{\mathbf{x}}^*) \subseteq N_+(\mathbf{x}^*)$.

In Proposition C3 in Appendix C.2.1, we investigate the effect of adding a link pointing to an agent in the root. We show that, if we add a link from an i_0 agent in the adjunct set of the root M_l^k of $\mathbf{x}^*$ to an agent in the root of $\mathbf{x}^*$, then any agent k who is unaware of i_0 is negatively impacted (because $\widehat{x}_k^* < x_k^*$). On the contrary, i_0 unsurprisingly benefits from this new link. The outcome is ambiguous for other agents. It may be the case that agents in lower layers see their payoff drop. We illustrate this possibility in Example D5 in Appendix D.4, where the network is displayed in Figure D5. We can see that agent 1 is connected through a path to agent 3, who benefits from the addition of a link to agent 4. Nevertheless, agent 1 is harmed by the addition of this link.

4.2 Key players

We consider another possible intervention, that is, removing one agent as well as all links from the network. This is known as the *key-player* policy (Zenou, 2016) and it is particularly relevant in the crime application (Ballester et al., 2006, 2010) but also in the conflict application (König et al., 2017; Amarasinghe et al., 2020) because governments want to target these individuals (the key players) in order to reduce total activity X (total crime or total conflict).

Proposition C4 in Appendix C.2.2 shows that, when removing a player, total effort will never increase. This is because the largest eigenvalue either stays the same or is reduced; the latter decreases total effort. However, the distribution of efforts may be greatly altered, as shown in the following example.

Example 3. Key players and the spread of efforts across local neighborhoods Consider the network displayed in Figure 1(a) (Example 1). We have shown that there is a unique stable peer-consistent equilibrium where the only active agents belong to the layer-1 perfect sub-network $M_1^1 = \{1, 2, 3\}$ with $x_1^* = x_2^* = x_3^* = \frac{2V}{9c}$ and thus the total effort is $X^* = \frac{2V}{3c}$.

Let us now remove the active agent 1 from the network as well as all of her links. It is easily verified that the unique stable PCE $\mathbf{x}^{[-1]*}$ is such that *now* $\{2, 3, 4, 5, 6\} \subseteq N_+(\mathbf{x}^*)$ even though the total effort remains the same at $\frac{2V}{3c}$. Indeed, by removing agent 1, the spectral radius of $M_1^1 = \{1, 2, 3\}$ decreases from 2 to 1 and becomes strictly smaller than the spectral radius of $M_1^2 = \{4, 5, 6\}$, which is equal to 2. As a result, the only stable PCE is now such that agents 2, 3, 4, 5 and 6 are active. Thus, removing an agent can have the *counter-productive effect* of making inactive agents active. In the standard key-player policy (Zenou, 2016), this is not possible since total effort always decreases as *all* agents reduce their individual effort. ◇

4.3 Social mixing

We conclude this section with a brief look at the issue of *social mixing.* To address this issue, we need to depart slightly from our initial model in which there was one (layer-generic) network. Suppose, instead, that we start with two disconnected (layer-generic) networks $(N^1, \mathbf{G}^1)$ and $(N^2, \mathbf{G}^2)$, each of which has a unique stable PCE. We can think of this situation as two fully segregated neighborhoods, each endowed with their own resources V^1 and V^2, respectively. An important question for the planner is whether merging these two neighborhoods (social mixing) into a connected network $(N, \mathbf{G})$, with $N = N^1 + N^2$, $V = V^1 + V^2$, leads to an increase in total activity and resources.

Proposition C5 in Appendix C.2.3 shows that the total effort in any new stable PCE of the connected network $(N, \mathbf{G})$ is higher than the sum of total efforts in each disconnected neighborhood. Hence, linking the two neighborhoods is beneficial to aggregate effort. On the other hand, the distribution of resources between agents in N^1 and N^2 is less clear. Indeed, the distribution in the new equilibrium depends on the specific connections that are formed between the two groups. It is therefore possible to have some agents who are worse-off following the mixing of the two neighborhoods.

5 Economic implications of our model

5.1 The concept of perceived competition

We would now like to illustrate our results and to highlight our concept of "perceived" competition. Understanding how layer decomposition works is crucial to discovering who is active and who is not in a network, as well as to determining how much effort agents exert when they are active. However, these two questions need to be answered separately.

Let us first focus on the question: *Who is active in a network?* For a given agent, being active or not in equilibrium will be determined by a combination of the two following elements: (i) the layer of the perfect sub-network to which she belongs (indeed, agents belonging to lower layers are more aware of others' activities in the network, while fewer people are aware of them; this gives them an advantage in terms of competition); and (ii) the spectral radius of this perfect sub-network. Agents belonging to more connected and denser sub-networks have more "power" in terms of competition over others.

Indeed, according to Proposition 2, in order to be active in equilibrium, the perfect sub-network to which a given agent belongs must satisfy (at least) *one of the two* following conditions:

(a) it must exhibit the largest spectral radius among all perfect sub-networks and be "hidden" from all other perfect sub-networks that have the same property;[34]

[34] This means that a path must not exist from another perfect sub-network having the same spectral radius.

(b) it must be aware of the sub-networks with the largest spectral radius.[35]

For obvious reasons, being "denser" makes it more likely for a perfect sub-network to satisfy condition (a), while belonging to a lower layer makes condition (b) more likely because, by construction, perfect sub-networks in lower layers are aware of a larger region of the network.

Once the set of active players is established, we can turn to the second question: *How active is an agent among the set of active agents?* Here, the answer is simpler, since the effort level of an active player is determined by her relative position in the sub-network of active players, which is fully captured by her eigenvector centrality. Consequently, the more aware of other active players an agent is, the more active she is. Also, the more "hidden" from other active agents she is, the more active she is. However, this does not mean that removing links from other active agents will necessarily increase her effort level because, by doing so, it might be that the perfect sub-network to which she belongs is not dense enough, rendering this perfect sub-network inactive in equilibrium.

In summary, our algorithm breaks down a network of agents into communities or layer perfect sub-networks. Agents belonging to lower layers are "hiding" from other agents in the network and this gives them an advantage in terms of competition; they are thus more likely to be active. Indeed, when each agent decides how much effort to exert, they consider their "perceived" competitors as this will determine how many resources they will have to share with other agents in the network. Agents then need to belong to large and dense communities to be able to grab resources. If the lowest layers have the largest and most dense communities, then they will be the only active agents in the whole network. On the contrary, if the highest layers contain the most dense communities, then the number of active agents will increase. In equilibrium, the activity level of each agent will depend on her position in the network, i.e., on her eigenvector centrality.

5.2 Applications

Our model uses a standard proportional rule (see (1)), which corresponds to the well-known Tullock contest function from the contest literature (Skaperdas, 1996; Kovenock and Roberson, 2012). This is a standard contest game in which a price (here V) is allocated among the contestants. Each contestant exerts effort. These efforts determine which contestant will receive which prize. As pointed out by Corchón (2007) and Konrad (2009), this model can be applied to many situations. The most obvious one is the tournament theory (Rosen, 1988), with applications such as examinations, college admission, elections, auctions, R&D races, etc. Other applications are advertising and other types of promotional competition, rent seeking, and appropriation conflict (e.g., civil wars) in which players compete for the allocation of property rights.

Our model adds two aspects to this literature. First, we embed players into a network, where links capture with whom each agent is in direct conflict or competition. Second,

[35] This means that there exists a path from this perfect sub-network to the sub-networks having the largest spectral radius.

we assume that agents have limited information about the network and are only aware of their direct competitors. Let us now provide two applications that are well suited to our model.

Application 1: Static competition

Consider a model of *popularity competition* within a friendship network. Assume that the perception of friends is not necessarily reciprocal,[36] so it differs between every agent. The network is the school each student attends and V is the total benefit level of popularity in the school given its ranking. Given her perception of her number of friends, each student exerts *socialization* effort x_i to increase her popularity among her friends. The ranking of popularity for i with her friends leads to a benefit of W_i. We have local sightedness because, in terms of popularity, each agent only cares about her relative ranking with respect to her perceived friends.

Our model predicts the following. First, the higher the number of *perceived friends*, i.e., the number of competitors for popularity, the higher the effort x_i of each agent i (since effort is equal to eigenvector centrality). Second, belonging to *dense and large* communities of friends implies that an agent is more likely to be active because she perceives more competitors for popularity. Third, belonging to lower layers implies that an agent is more likely to be active (i.e., exerts positive effort) because few connected individuals perceive this agent as a friend and, thus, as a competitor for popularity.

Consider, for example, the network displayed in Figure 1(a) (Example 1). Agents 1, 2 and 3 form a close-knit community and compete against each other to be the most popular. However, while agent 3 only perceives 1 and 2 as her friends and thus competitors for popularity, agent 1 perceives not only 2 and 3 as friends but also 6, while agent 2 perceives $\{1, 3, 4, 6\}$ as her set of competitors. This induces agents 1 and 2 to exert the highest effort in the network because of the many competitors they face for popularity. Agent 3 also exerts effort because she is linked to the high-effort agents, 1 and 2. On the contrary, agents 4, 5 and 6 "ignore" agents 1 and 2 and only perceive as friends those in their own community. Thus, in the unique stable PCE, only agents 1, 2 and 3 will be active and exert positive effort. The other agents do not require effort to be popular because what matters to them is to be popular among themselves. However, when the links between 1 and 2 are removed (Figure 1(b)), agents 1 and 2 perceive fewer individuals as friends and competitors because they form a less connected community. The other community, which includes agents 4, 5 and 6 now becomes relatively more connected compared to the community $\{1, 2, 3\}$. Agents 4, 5 and 6 therefore have an incentive to become active in order to improve their popularity within their community.

More generally, our model predicts that the agents belonging to lower layer sub-networks need to channel more effort because they are initially less popular since few agents perceive

[36]There is plenty of evidence that links are not always reciprocal, especially for friendships. For example, using AddHealth data, Calvó-Armengol et al. (2009) have shown that 14% of (self-reported) friendship relationships are not reciprocal. Using data from college students in France, Algan et al. (2020) found that only about half of the nominated friends reciprocate; while Leider et al. (2009), using an online experiment of Harvard undergraduate students, found an even smaller proportion of reciprocal friendships.

them as friends. Further, the relative density and size of a community is important to determine who will become active at a PCE. If unpopular agents form a strong and large connected community, then they will be the only ones to exert effort to become more popular. In this interpretation of the model, the "jocks" belong to higher layer sub-networks, while the "nerds" belong to lower layer sub-networks. To become popular, the latter group needs to put in much more (socialization) effort than the former. The "jocks" do not necessarily need to make an effort to be popular since they do not compare themselves with the "nerds". However, if their community is large enough, then they need to make more of an effort because the competition between them is fierce.

Application 2: Dynamic Competition

The previous application to popularity competition was static. We now provide another application that is more dynamic and thus in accordance with Section 3.5 on stability.

Consider a university that has a budget of V to allocate between the departments of physics and chemistry. To resolve the situation, consider the network displayed in Figure 1(a) (Example 1) in which the department of physics has three researchers, i.e., agents 1, 2 and 3, whereas the department of chemistry also has three researchers, i.e., agents 4, 5 and 6. In other words, we assume that chemists 4, 5 and 6 are not aware that they are in competition with physicists 1, 2 and 3, but physicists 1 and 2 are aware that they are in competition with chemist 6. The budget V will be distributed according to the sharing rule (1) (Tullock contest function), i.e., it is proportional to the (research) effort of each researcher. At the end of the year, the budget V is allocated according to this rule and everybody observes the (research) efforts. In other words, the university allocates a budget of $V/2$ to each department. From the viewpoint of the chemists $M_1^2 = \{4, 5, 6\}$, they perceive a revenue of $V/2$, which they will share between them; thus, each chemist perceives that she will obtain $V/6$. On the contrary, the physicists $M_1^1 = \{1, 2, 3\}$ believe that they are in competition with both the other physicists and the chemists. In particular, since physicists 1 and 2 perceive that they are in competition with 3 agents (two physicists and chemist 6), both perceive that her resources are equal to $4V/6$, since they know that each agent has received $V/6$. Since $4V/6 > 3V/6$, compared to the chemists, physicists 1 and 2 will exert more effort the following year to obtain a larger share of V. This pattern continues and reinforces itself over time, so that physicists 1 and 2 make more and more effort, which induces physicist 3 (who has no information about the chemists) to also increase her effort. On the contrary, the chemists who only observe the other chemists see their share of V decreasing over time without knowing why. Indeed, they believe that there is less and less budget in the university over time. After some time, the chemists will end up exerting no effort and all the budget will go to the physicists. This is the unique stable PCE.

Consider, now, the network displayed in Figure 1(b) where the links between physicists 1 and 2 have been removed. Now, in the first period, agents 1 and 2 perceive that they are in competition with (i) only 2 agents for agent 1, and (ii) only 3 agents for agent 2. The perceived resources of agent 1 are equal to $V/2$, which is different from those of the chemists who also perceive, like agent 1, that they have two competitors. Indeed, the chemists perceive that their resources are equal to $2V/5$. Thus, in the following year, the physicists and the chemists will exert different levels of effort but these efforts are all positive. This will persist over time, so the only stable PCE is such that all six researchers (both physicists and chemists) will be active even though they exert different research efforts.

More generally, any other story that involves a dynamic adjustment with different perceived competitors will fit our model. For example, consider different restaurants; each must decide how much effort (or investment) to exert in order to attract customers. Each restaurant only cares about the *local* competition and thus her perceived competitors and perceived demand. However, because competition is *global*, their "local" perception is wrong at the outset. They make investments (efforts) and then discover that their perceived demand (or resource) is not correct. In the next period, they change their beliefs about their demand and adjust their investments. This pattern continues until we reach a stable peer-consistent equilibrium. What we show in this paper is that the key determinant of effort of each agent is her position in the network and whether or not she belongs to a large and dense community.

6 Conclusion

In this paper, we consider a contest game of resource competition in which agents have an imperfect knowledge of the network, i.e., they only have information on the activities of their direct neighbors. Thus, their perceived resources and perceived competition are based on this limited knowledge. We develop a new concept of equilibrium, which we refer to as peer-consistent equilibrium (PCE). Each agent chooses an effort level that maximizes her perceived utility. However, at the PCE, effort levels of all agents have to be consistent, i.e., for each agent, her *perceived subjective* utility and resource has to be equal to her *objective* payoff and resource.

We first show that, at any PCE, the effort of an active agent is proportional to her eigenvector centrality. This is true for any network. We believe that this is the first model that provides a microfoundation of network eigenvector centrality.

Then, we develop an algorithm (the Network Layer Decomposition or NLD) that partitions the network into communities or sub-networks called layer-k perfect sub-networks, where, within each community, all agents have the same propensity to exert positive effort. The lower layer perfect sub-networks are selected first because few agents are "aware" of them and agents in these communities can therefore grab a significant amount of resources within their neighborhood. The NLD algorithm stops when all agents in the network have been selected and all perfect sub-networks have been identified. Then, we determine all peer-consistent equilibria by comparing the spectral radius of these perfect sub-networks and that of their adjunct set (i.e., agents that can reach them through a path) in the whole network. We show that, to be active in equilibrium, one needs to belong either to a layer-1 perfect sub-network because few agents are aware of these individuals or to higher-level perfect sub-networks of large size in which agents are strongly connected to each other.

Finally, we demonstrate that there is a unique stable PCE in each network. This PCE corresponds to the perfect sub-network that has the largest spectral radius in the network. Depending on the network structure, at the unique stable PCE, either all agents are active or only a subset of them are.

Lastly, we study the policy implications of our model. We show that adding a link can reduce the number of active agents in the network because it creates a new path that makes some agents more likely to be reached; in turn, this may lower their status in terms of layer perfect sub-network. We also study the key-player policy and show that, by removing an agent from the network, we may make several inactive agents active. Further, we examine social mixing by merging two different disconnected networks and show that total activity is higher than the sum of total activity in each network.

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Appendix

A Non-negative matrices and eigenvector centrality

A.1 The Frobenius normal form

A matrix is called **nonnegative** if all its elements are nonnegative. Here we consider only nonnegative square matrices of order n, i.e., matrices that have n rows and n columns. A nonnegative matrix A is called **irreducible** if the associated directed graph is strongly connected. For convenience any one-by-one matrix is regarded as irreducible.

Lemma A1. *(Perron-Frobenius Theorem) Let* $\mathbf{A}$ *be an irreducible matrix. Then*

(i) $\mathbf{A}$ *has a positive eigenvalue* $\rho(\mathbf{A})$ *such that the value of* $\rho(\mathbf{A})$ *is not less than the absolute value of any other eigenvalue of* $\mathbf{A}$*;*

(ii) the eigenvalue $\rho(\mathbf{A})$*is simple, and corresponds to a positive eigenvector* $\mathbf{x}(\mathbf{A})$*;*

(iii) any non-negative eigenvector is a multiple of $\mathbf{x}(\mathbf{A})$*.*

The vector $\mathbf{x}(\mathbf{A})$ *and the number* $\rho(\mathbf{A})$ *that appear in this lemma are called the* ***Perron-Frobenius vector*** *and the* ***Perron-Frobenius eigenvalue*** *of* $\mathbf{A}$*, respectively.*

The following lemma extends some conclusions of the Perron-Frobenius Theorem to non-negative matrices (not necessarily irreducible).

Lemma A2. *Let* $\mathbf{A}$ *be a nonnegative matrix; then*

a) $\mathbf{A}$ *has a nonnegative eigenvalue* $\rho(\mathbf{A})$ *such that the value of* $\rho(\mathbf{A})$ *is not less than the absolute value of any other eigenvalue of* $\mathbf{A}$*.*

b) To eigenvalue $\rho(\mathbf{A})$ *corresponds a nonnegative eigenvector* $\mathbf{x}(\mathbf{A})$*.*

c) If there exists a positive eigenvector, then it is necessarily associated to eigenvalue $\rho(\mathbf{A})$*.*

Note that if $\mathbf{x}$ is a non-negative eigenvector of $\mathbf{A}$, $\mathbf{x}$ is not necessarily associated with $\rho(\mathbf{A})$. Also there could exist eigenvectors with both negative and positive entries, associated to $\rho(\mathbf{A})$.

Lemma A3. *Any nonnegative matrix* $\mathbf{A}$ *can be put in an upper-triangular block form as follows:*[1]

$$\mathbf{A} = \begin{bmatrix} A_1 & A_{12} & ... & ... & ... & ... & ... & A_{1r} \\ 0 & A_2 & A_{23} & ... & ... & ... & ... & A_{2r} \\ ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & ... & 0 & A_s & A_{ss+1} & ... & ... & A_{sr} \\ 0 & ... & ... & 0 & A_{s+1} & 0 & ... & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & ... & ... & ... & ... & 0 & A_{r-1} & 0 \\ 0 & ... & ... & ... & ... & ... & 0 & A_r \end{bmatrix} \tag{A.1}$$

such that:

(*i*) *each block matrix* A_i *is square and irreducible;*

(*ii*) *for any* $i = 1, ...s$, *there exists* $j \in \{i+1, ..., r\}$ *such that the block matrix* A_{ij} *is not zero.*

This upper triangular block form is known as the **Frobenius normal form**. It is unique up to a permutation. We have $\rho(\mathbf{A}) = \max_{i=1...r} \rho(A_r)$. We call V_i the set of nodes corresponding to the block matrix A_i.

Definition A1. *A nonnegative matrix* $\mathbf{A}$ *is* ***strongly nonnegative*** *if we have*

$$\rho(A_r) = \rho(A_{r-1}) = ... = \rho(A_{s+1}) > \max_{i=1,...,s} \{\rho(A_i)\}$$

Obviously, any irreducible matrix is strictly nonegative because the Frobenius normal form then consists of one block. The next results can be found in Rothblum (2014) or Hu and Qi (2016).

Lemma A4. *A nonnegative matrix* $\mathbf{A}$ *admits a positive eigenvector if and only if* $\mathbf{A}$ *is strongly nonnegative.*

Note that, if $\mathbf{A}$ is an irreducible nonnegative matrix, then the conclusion of Lemma A4 directly implies point (*ii*) of Lemma A1, i.e., the Perron Frobenius Theorem.

We illustrate the Frobenius normal form for the 3-layer network $(N, \mathbf{G})$ displayed in Figure 2 (Example 2) with $N = \{1, 2, \cdots, 10\}$ and with three sub-networks: $M_1^1 = \{2, 3, 4\}$, $M_1^2 = \{5, 6\}$, and $M_1^3 = \{7, 8, 9, 10\}$, which are all strongly connected components of $(N, \mathbf{G})$.

[1] Up to a permutation of indices.

Let $\mathbf{C}(m)$ be the adjacency matrix of the complete m-agents network.[2] Keeping the indexing of agents as it is, we have

$$\mathbf{G} = \begin{bmatrix} 0&0&0&1&0&0&1&0&0&0\\ 0&0&1&1&0&0&0&0&0&0\\ 0&1&0&1&1&1&0&0&0&0\\ 0&1&1&0&0&0&1&0&0&0\\ 0&0&0&0&0&1&0&0&0&1\\ 0&0&0&0&1&0&0&0&0&1\\ 0&0&0&0&0&0&0&1&1&1\\ 0&0&0&0&0&0&1&0&1&1\\ 0&0&0&0&0&0&1&1&0&1\\ 0&0&0&0&0&0&1&1&1&0 \end{bmatrix} = \begin{bmatrix} A_1 & A_{12} & 0 & A_{14}\\ 0 & A_2 & A_{23} & A_{24}\\ 0 & 0 & A_3 & A_{34}\\ 0 & 0 & 0 & A_4 \end{bmatrix},$$

where A_1 captures the links of group 1, i.e., player 1, A_2 the links of group 2, i.e., $\{2,3,4\}$, A_3 the links of group 3, i.e., $\{5,6\}$, and A_4 the links of group 4, i.e., $\{7,8,9,10\}$, that is,

$$A_1 = \mathbf{C}(1) = 0,\ A_2 = \mathbf{C}(3),\ A_3 = \mathbf{C}(2),\ A_4 = \mathbf{C}(4),$$

while A_{ij} captures the link between group i and j; for example, $A_{12} = [0\,0\,1]$, $A_{13} = [0\,0]$, $A_{14} = [1\,0\,0\,0]$, etc. In particular, A_{ij} is distinct from the null matrix, except for A_{13} (there is no link from group 1, i.e., agent 1, to group 3, i.e., agents $\{5,6\}$). Consequently we have $s = 3$ and $r = 4$ and $\rho(A_4) = 3$ while $\rho(A_1) = 0, \rho(A_2) = 2$ and $\rho(A_3) = 1$. Hence, $\mathbf{G}$ is strongly nonnegative and thus admits a positive eigenvector.

Now, remove agent 9 from this network. Then, the Frobenius normal form has the same structure, except that $\rho(A_4) = 2 = \max_{i=1,\dots,3} \rho(A_i)$. Hence, the matrix is no longer strictly nonnegative and, thus, there is no positive eigenvector.

Consider, now, the 2-layer network with 10 agents displayed in Figure D3 (Example D3). Then

$$\mathbf{G} = \begin{bmatrix} A_1 & A_{12} & A_{13} & 0\\ 0 & A_2 & 0 & A_{24}\\ 0 & 0 & A_3 & 0\\ 0 & 0 & 0 & A_4 \end{bmatrix}, \text{ with } A_1 = \mathbf{C}(1),\ A_2 = \mathbf{C}(2),\ A_3 = \mathbf{C}(4) \text{ and } A_4 = \mathbf{C}(3).$$

We have $r = 4$ and $s = 2$. Since $\rho(A_3) \neq \rho(A_4)$, the matrix is not strictly nonnegative. and there is therefore no positive eigenvector.

It might be useful to clarify the relationship between the Frobenius normal form and our NLD algorithm. In the Frobenius normal form of $\mathbf{G}$, any A_i corresponds to the submatrix of either a layer-k perfect sub-network, or to a singleton. Note that, by the no-isolation assumption, A_i cannot be a size one matrix for $i = s{+}1, ..., r$; it then necessarily corresponds to a perfect sub-network for these indexes. If M_l^k is contained in the adjunct set of $M_{l'}^{k'}$,

[2] That is, $C(m)_{ii} = 0$, $C(m)_{ij} = 1$ for $i \neq j$

then there exists some i, i' such that $i' > i$, $\mathbf{G}_{M_l^k} = A_i$ and $\mathbf{G}_{M_{l'}^{k'}} = A_{i'}$. Carefully note that there is nevertheless no precise relationship between the index of the layer of a perfect sub-network and its index in the Frobenius normal form. Indeed, it might be the case that A_i corresponds to a higher layer perfect sub-network than A_{i+1}. The Frobenius normal form does not help us to characterize the peer-consistent equilibria (i.e., which agents are active and which are not) but will be very useful for some of our proofs because of Lemma A4, which can be applied to any perfect sub-network.

A.2 Eigenvector centrality in weakly connected networks

Eigenvector centrality has been informally introduced by Bonacich (1972) to measure popularity in friendship networks. Given a weighted network $(N, \mathbf{G})$, it was originally defined as any non-negative vector $\mathbf{e}$ having the property that the centrality of agent i is proportional to the average centrality of her neighbors:

$$\lambda e_i = \sum_j \mathbf{G}_{ij} e_j, \ \forall i. \tag{A.2}$$

In the particular case of strongly connected networks, this vector is well-defined because there is a unique solution to the system (A.2), given by the eigenvector associated to the largest eigenvalue λ of $\mathbf{G}$. More generally, there is a consensus consisting in regarding eigenvector centrality as being the normalized[3] eigenvector associated to the largest eigenvalue of the network (see e.g., Jackson (2008)).

In weakly connected networks, however, eigenvector centrality cannot be defined in the same way because the largest eigenvalue of a weakly connected network is not always simple. For instance, consider the network in Figure E6 in Appendix E, where $\rho(\mathbf{G}) = 1$. The eigenspace associated to $\rho(\mathbf{G})$ is generated by normalized vectors $(1/3, 1/3, 1/3, 0, 0)$ and $(1/3, 0, 0, 1/3, 1/3)$. Hence, any convex combination of these two vectors is a non-negative eigenvector, which means that eigenvector centrality is not defined for this network.

Consequently, we focus on an (arguably large) subset of weakly connected graphs, in which the notion of eigenvector centrality can be naturally extended. A weakly connected network has *a unique dominant component* if

$$\rho(\mathbf{G}_{M_l^k}) = \rho(\mathbf{G}_{M_{l'}^{k'}}) = \rho(\mathbf{G}) \Rightarrow M_l^k \subset Q_{l'}^{k'} \text{ or } M_{l'}^{k'} \subset Q_l^k. \tag{UDC}$$

Obviously any *layer-generic network* has a unique dominant component. A simple adaptation of the proof of Proposition 3 shows that a weakly connected network admits a unique normalized eigenvector associated to $\rho(\mathbf{G})$ if and only if it has a unique dominant component.

Definition A2 (Eigenvector centrality)**.** *Suppose that $(N, \mathbf{G})$ has a unique dominant component. Then, the eigenvector centrality of agent i is the i-th component of the normalized eigenvector associated to $\rho(\mathbf{G})$.*

[3]meaning the eigenvector whose components sum to one.

In some networks, it may be the case that some agents in the network exhibit a null eigenvector centrality, and one may wonder what it means, and whether or not this definition makes sense when this happens. As we show now, this definition is indeed meaningful, because our definition of eigenvector centrality is robust to any small perturbations of the network, in the following sense:

Lemma A5. *Suppose that* $(N, \mathbf{G})$ *has a unique dominant component and call* $\mathbf{e}$ *the normalized eigenvector associated to* $\rho(\mathbf{G})$*. Let* $(\mathbf{G}^n)_n$ *be a sequence of irreducible matrices such that* $\lim_{n\to+\infty} \mathbf{G}^n_{ij} = \mathbf{G}_{ij}$*. Then* $\mathbf{e}^n \to \mathbf{e}$*, where* $\mathbf{e}^n$ *is the normalized eigenvector associated to* $\rho(\mathbf{G}^n)$*.*

In other words, the sequence of centrality measures always converge to the same vector, regardless of *how* $\mathbf{G}^n$ converges to $\mathbf{G}$. The implication of this observation is that eigenvector centrality is unambiguously defined in networks having a unique dominant component.

Observe that the network $(N, \mathbf{G})$ depicted in Figure E6 in Appendix E does not exhibit such a property; thus, defining an eigenvector centrality for such a network would imply making an arbitrary choice. Indeed, it can be shown that, for any $\lambda \in [0, 1]$, one can find a sequence of strongly connected weighted networks $(N, \mathbf{G}^n)$ such that $\mathbf{e}^n$ converges to $\frac{1}{3}(1, \lambda, \lambda, 1 - \lambda, 1 - \lambda)$.

B Proofs of all results in the main text

B.1 Proof of results in Section 3

B.1.1 Proof of results in Section 3.1

Proof of Theorem 1. We first prove the "only if" part. Let $\mathbf{x}^* \in \mathbb{R}_+^n$ be an equilibrium effort vector of $(N, \mathbf{G})$. We first show that $X^* > 0$. Assume, by contradiction, that $X^* = 0$, i.e. $\mathbf{x}^* = \mathbf{0}$. Then for each $i \in N$, agent i's subjective utility is equal to

$$\frac{1}{1 + |\mathcal{N}_i|} W_i = \frac{V}{n}.$$

Now consider the situation where some agent i deviates and exerts some effort $\epsilon > 0$, for ϵ small, while the others $j \neq i$ exert $x_j = 0$. Then agent i's new subjective utility is equal to $W_i - c \cdot \epsilon = (1 + |\mathcal{N}_i|)\frac{V}{n} - c\epsilon$. Since $1 + |\mathcal{N}_i| \geq 2$, for small enough ϵ, this is a favorable deviation. Therefore, $\mathbf{x}^*$ is not an equilibrium effort vector. We conclude that $X^* > 0$.

We now show that $\mathbf{G}\mathbf{x}^* = \frac{cX^*}{V - cX^*}\mathbf{x}$. To do so, we consider the following two cases in which agent i's neighbors either exhibit collective positive level of effort (Case 1) or no effort at all (Case 2):

Case 1. Agent $i \in N$ is such that $(\mathbf{G}\mathbf{x}^*)_i > 0$.
We consider agent i's optimization problem,

$$\max_{x_i \geq 0} \frac{x_i}{x_i + (\mathbf{G}\mathbf{x}^*)_i} W_i - cx_i.$$

Since x_i^* is an optimal interior solution, we have that x_i^* satisfies the first order condition,

$$\frac{(\mathbf{G}\mathbf{x}^*)_i}{(x_i^* + (\mathbf{G}\mathbf{x}^*)_i)^2} W_i = c. \tag{B.1}$$

Note that in equilibrium, $W_i = \frac{x_i^* + (\mathbf{G}\mathbf{x}^*)_i}{X^*}V$, hence we have that,

$$\frac{(\mathbf{G}\mathbf{x}^*)_i}{x_i^* + (\mathbf{G}\mathbf{x}^*)_i}\frac{V}{X} = c.$$

Therefore,

$$x_i^* = \left(\frac{V}{cX^*} - 1\right)(\mathbf{G}\mathbf{x}^*)_i.$$

Case 2: Agent $i \in N$ is such that $(\mathbf{G}\mathbf{x}^*)_i = 0$.
We show that $x_i^* = 0$. Assume, by contradiction, that $x_i^* > 0$. Then $W_i > 0$. Since $(\mathbf{G}\mathbf{x}^*)_i = 0$, we have that $u_i(x_i, \mathbf{x}_{-i}^*; W_i) = W_i - cx_i$ for any $x_i > 0$, contradicting the fact that x_i^* maximizes $x_i \mapsto u_i(x_i, \mathbf{x}_{-i}, W_i)$. Hence

$$x_i^* = 0 = \left(\frac{V}{cX^*} - 1\right)(\mathbf{G}\mathbf{x}^*)_i.$$

Combining Cases 1 and 2, we obtain that a peer-consistent equilibrium effort level $\mathbf{x}^*$ satisfies the condition

$$\mathbf{G}\mathbf{x}^* = \frac{cX^*}{V - cX^*}\mathbf{x}^*.$$

Suppose now that that $\mathbf{x}^*$ satisfies this identity. For each agent i for whom $(\mathbf{G}\mathbf{x}^*)_i > 0$, the first order condition (B.1) solves the optimization problem of agent i, for $W_i = \frac{x_i^* + (\mathbf{G}\mathbf{x}^*)_i}{X^*}V$. Meanwhile, for each agent i for whom $(\mathbf{G}\mathbf{x}^*)_i = 0$, $x_i^* = 0$ solves the optimization problem with $W_i = 0$. This proves the reverse implication. □

Lemma B6. *If* $\mathbf{x}$ *is a peer-consistent equilibrium then* $N_+(\mathbf{x})$ *is a closed set of* $(N, \mathbf{G})$.

Proof. Let $j \in N_+(\mathbf{x})$ and i be connected to j through a path: there exists $p \in \mathbb{N}^*$ such that $\mathbf{G}_{ij}^p > 0$. By Theorem 1 there exists $\rho > 0$ such that $\mathbf{G}\mathbf{x} = \rho\mathbf{x}$. We then have

$$x_i = \frac{1}{\rho^p}(\mathbf{G}^p\mathbf{x})_i \geq \frac{1}{\rho^p}\mathbf{G}_{ij}^p x_j > 0.$$

This concludes the proof. □

B.1.2 Proof of results in Section 3.2

Proof of Proposition 1. Since Proposition 1 is a special case of Theorem 1, we will prove Proposition 1 as the following corollary of Theorem 1.

Corollary B1. *Let* $(N, \mathbf{G})$ *be a strongly connected network. Then, there exists a unique peer-consistent equilibrium.*

Proof. Suppose that $(N, \mathbf{G})$ is a strongly connected network. Then $\mathbf{G}$ is irreducible and, by Perron-Frobenius Theorem, there exists a positive eigenvector $\mathbf{y}$ associated to $\rho(\mathbf{G})$. Moreover any non-negative eigenvector of $\mathbf{G}$ is a multiple of $\mathbf{y}$. By Theorem 1, $\mathbf{x}^*$ is a PCE if and only if it is a non-negative eigenvector of $\mathbf{G}$, associated to eigenvalue $\frac{cX^*}{V-cX^*}$. Hence $\mathbf{x}^*$ is a PCE if and only if $\mathbf{x}^*$ is a multiple of $\mathbf{y}$ and $\rho(\mathbf{G}) = \frac{cX^*}{V-cX^*}$. Such a vector exists and is uniquely defined. □

B.1.3 Proof of results in Section 3.3

We start with the following lemma.

Lemma B7. *Any network* $(N, \mathbf{G})$ *that satisfies no-isolation has at least one perfect sub-network.*

Proof of Lemma B7. Consider the Frobenius normal form (A.1) associated to $\mathbf{G}$. Since $\mathbf{G}$ satisfies the no-isolation assumption, the matrix A_r is of size at least two, so that the set $\{i \geq 1 : |V_i| \geq 2\}$[4] is not empty. Now let $i_0 := \min\{i \geq 1 : |V_i| \geq 2\}$. The matrix A_{i_0} is irreducible and therefore (V_{i_0}, A_{i_0}) is a strongly connected component of $(N, \mathbf{G})$. For $p = 1, ..., i_0 - 1$ we have $V_p = \{i_p\}$. The adjunct set of (V_{i_0}, A_{i_0}) is therefore contained in the set of nodes $\{i_1, ..., i_0 - 1\}$, in which there can be no cycle.

Proof of Remark 3. At step 1, by Lemma B7, a perfect sub-network of $(N, \mathbf{G})$ exists. Hence N^1 is uniquely defined as $N^1 = N \setminus M^1$, and $|N^1| < |N|$. Now let $k \geq 2$ be such that N^{k-1} is not empty. The network $(N^{k-1}, \mathbf{G}_{N^{k-1}})$ satisfies no isolation: pick any $i \in N^{k-1}$. By construction i cannot be connected to any $j \in \cup_{l=1}^{k-1} M^l$. Thus there must exist $j \in N \setminus \cup_{l=1}^{k-1} M^l = N^{k-1}$ such that i is connected to j. Hence, by Lemma B7, a perfect sub-network of $(N^{k-1}, \mathbf{G}_{N^{k-1}})$ exists. Again N^k is uniquely defined as $N^k = N^{k-1} \setminus M^k$ and $|N^k| < |N^{k-1}|$. The processus stops after a uniquely defined finite number of steps. $\square$

The following lemma provides some useful properties about the NLD algorithm.

Lemma B8. *The following statements about the NLD algorithm are true:*[5]
(i) For any $k \leq T$, any $l, l' \leq m^k$, if we have that $M_l^k \rightrightarrows M_{l'}^k$, then $l = l'$.
(ii) For any $k, t \leq T$, any $i \in M^k$, and $j \in M^t$, if we have that $i \rightrightarrows j$, then $k \leq t$.
(iii) For any k, $2 \leq k \leq T$, any $l \leq m^k$ and any $k' < k$, there exists $l' \leq m_{k'}$ such that $M_{l'}^{k'} \rightrightarrows M_l^k$.

Proof of Lemma B8. (i) Suppose that $l' \neq l$. If $M_l^k \rightrightarrows M_{l'}^k$ then $M_l^k \subset \bar{M}_{l'}^k$. This contradicts the fact that $M_{l'}^k$ is adjunct cycle-free. Hence $l = s$.
(ii) By construction, the adjunct set of M^t is $M^1 \cup ... \cup M^{t-1}$, which gives (ii).
(iii) Fix k, $2 \leq k \leq T$. Let $l \leq m_k$. Note that $M_l^k \subseteq N^{k-1}$. Suppose for the sake of contradiction that for any $j \in M_m^k$, there is no $s \leq m_{k-1}$ and $i \in M_s^{k-1}$ such that i is connected to j through a path. Then M_l^k is a layer-$(k-1)$ perfect sub-network, a contradiction. $\square$

B.1.4 Proof of results in Section 3.4

In the following, we suppose that $(N, \mathbf{G})$ is a T-layer network, and for each $k \leq T$, it has m^k layer-k perfect sub-networks. We start by providing some insights on the relationship between the layer decomposition and the Frobenius normal form.

Lemma B9. *Let $(N, \mathbf{G})$ be a network satisfying no isolation. Consider its Frobenius normal form (A.1). For any $i = 1, ..., r$ either $|V_i| = 1$ or (V_i, A_i) is a layer-k perfect sub-network of $(N, \mathbf{G})$, for some k. As a consequence*

$$\rho(\mathbf{G}) = \max_{i=1,...,r} \rho(A_i) = \max_{k \leq T} \max_{l \leq m^k} \rho\left(\mathbf{G}_{M_l^k}\right) \tag{B.2}$$

[4]Recall that V_i is the set of nodes corresponding to matrix A_i in Frobenius normal form.

[5]Since $(M_l^k, \mathbf{G}_{M_l^k})$ is strongly connected, either there is no path from an element of M_l^k to an element $M_{l'}^{k'}$ or there is a path from any element of M_l^k to any element of $M_{l'}^{k'}$. In the latter case we write $M_l^k \rightrightarrows M_{l'}^{k'}$.

Proof. Suppose that $|V_i| > 1$. By construction of the Frobenius normal form, (V_i, A_i) is a strongly connected component of $(N, \mathbf{G})$. Since $(M^k)_{k=1,...,T}$ is a partition of N and by point (ii) of Proposition B8, there exists $k \in \{1, ..., T\}$ such that $V_i \subset M^k$. Since $|V_i| \geq 2$, the set V_i cannot intersect any of the $\bar{M}_l^k$, because it would contradict the fact that the M_l^k are layer-k perfect subgraphs of $(N, \mathbf{G})$. Hence $V_i \subset \cup_{l=1}^{m^k} M_l^k$. This cannot happen because two strongly connected components of $(N, \mathbf{G})$ cannot intersect. Since $\rho(A_i) = 0$ if $|V_i| = 1$ this concludes the proof of (B.2). □

For any closed set $N' \subset N$ and any k, l, either M_l^k is contained in N', or M_l^k does not intersect N'. Hence the layer decomposition of $(\mathbf{N}', \mathbf{G}_{N'})$ coincides with the intersection of the layer decomposition of $(N, \mathbf{G})$ and N'. As a consequence we have

$$\rho(\mathbf{G}_{N'}) = \max_{t \leq T, s \leq m^t : M_s^t \subseteq N'} \rho(\mathbf{G}_{M_s^t}) \tag{B.3}$$

Lemma B10. *For any $k \leq T$ and $l \leq m^k$ we have $\rho(\mathbf{G}_{Q_l^k}) = \max_{\{t,s: M_s^t \subseteq Q_l^k\}} \rho(\mathbf{G}_{M_s^t})$;*

Proof. The network $(Q_l^k, \mathbf{G}_{Q_l^k})$ is closed, so that we can apply (B.3). □

Lemma B11. *Given any collection of pairwise disconnected perfect sub-networks $\{M_{l_i}^{k_i}\}_{i=1,...,n}$, the matrix $\mathbf{G}_{\cup_{i=1}^n D_{l_i}^{k_i}}$ admits a Frobenius normal form as follows*

$$\mathbf{G}_{\cup_{i=1}^n D_{l_i}^{k_i}} = \begin{bmatrix} A_1 & A_{12} & ... & ... & ... & ... & ... & A_{1s+n} \\ 0 & A_2 & A_{23} & ... & ... & ... & ... & A_{2s+n} \\ ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & ... & 0 & A_s & A_{ss+1} & ... & ... & A_{ss+n} \\ 0 & ... & ... & 0 & \mathbf{G}_{M_{l_1}^{k_1}} & 0 & ... & 0 \\ 0 & ... & ... & .... & ... & ... & ... & ... \\ 0 & ... & ... & ... & ... & 0 & \mathbf{G}_{M_{l_{n-1}}^{k_{n-1}}} & 0 \\ 0 & ... & ... & ... & ... & ... & 0 & \mathbf{G}_{M_{l_n}^{k_n}} \end{bmatrix}. \tag{B.4}$$

and we have

$$\max_{i=1,...,s} \rho(A_i) = \max_{i=1,...,n} \rho(\mathbf{G}_{Q_{l_i}^{k_i}});$$

Proof. The decomposition (B.4) comes from the fact that all the only strongly connected components that are connected to no (other) nodes in $\cup_{i=1}^n D_{l_i}^{k_i}$ are the elements of the family $\{M_{l_i}^{k_i}\}_{i=1,...,n}$. Now for the second statement, note that $\cup_{i=1}^n Q_{l_i}^{k_i}$ is a closed set and thus

$$\max_{i=1,...,n} \rho(\mathbf{G}_{Q_{l_i}^{k_i}}) = \max_{i=1,...,n} \max_{\{t,s: M_s^t \subseteq Q_{l_i}^{k_i}\}} \rho(\mathbf{G}_{M_s^t}) = \max_{\{t,s: M_s^t \subseteq \cup_{i=1}^n Q_{l_i}^{k_i}\}} \rho(\mathbf{G}_{M_s^t}) = \max_{i=1,...,s} \rho(A_i),$$

where the second equality comes from the fact that M_s^t is either included in $Q_{l_i}^{k_i}$ or does not intersect it; and the last equality follows from the fact that the Frobenius normal from associated to $\mathbf{G}_{\cup_{i=1}^n Q_{l_i}^{k_i}}$ is simply the $s \times s$ upper left blocks of (B.4). □

Lemma B12. *Suppose that* $\mathbf{A}$ *is a nonnegative matrix that admits a Frobenius normal form (A.1) with* $r = s+1$ *and* $\rho(A_{s+1}) > \max_{i=1,\dots,s}\{\rho(A_i)\}$*. Then* $\mathbf{A}$ *admits a* ***unique*** *positive eigenvector.*[6]

Proof. We only need to show that, if $\mathbf{x}$ and $\mathbf{y}$ are two positive eigenvector of $\mathbf{A}$ then $\mathbf{x} = \alpha\mathbf{y}$ for some $\alpha > 0$. We can write $\mathbf{A}$ as follows:

$$\mathbf{A} = \begin{bmatrix} A' & B \\ 0 & A_{s+1} \end{bmatrix}, \text{ where } A' = \begin{bmatrix} A_1 & A_{12} & \dots & \dots & A_{1s} \\ 0 & A_2 & A_{23} & \dots & A_{2s} \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & A_s \end{bmatrix} \text{ and } B = \begin{bmatrix} A_{1s+1} \\ A_{2s+1} \\ \dots \\ \dots \\ A_{ss+1} \end{bmatrix}.$$

Let us write $\mathbf{x}$ as $(\mathbf{x}', \mathbf{x}_{[s+1]})$, according to the decomposition of $\mathbf{A}$ we just wrote and let $\rho := \rho(A_{s+1}) = \rho(\mathbf{A})$. We have

$$\begin{bmatrix} \mathbf{x}' \\ \mathbf{x}_{[s+1]} \end{bmatrix} = \rho^{-1} \begin{bmatrix} \mathbf{A}' \cdot \mathbf{x}' + \mathbf{B} \cdot \mathbf{x}_{[s+1]} \\ \mathbf{A}_{s+1} \cdot \mathbf{x}_{[s+1]} \end{bmatrix},$$

so that, in particular, $(\mathbf{I} - \rho^{-1}\mathbf{A}')\,\mathbf{x}' = \rho^{-1}\mathbf{B}\mathbf{x}_{[s+1]}$. Since $\rho(\mathbf{A}') < \rho$ by construction, the matrix $\mathbf{I} - \rho^{-1}\mathbf{A}'$ is invertible and we have

$$\mathbf{x}' = \rho^{-1}\left(\mathbf{I} - \rho^{-1}\mathbf{A}'\right)^{-1}\mathbf{B}\mathbf{x}_{[s+1]} \tag{B.5}$$

Now the matrix $\mathbf{A}_{s+1}$ being irreducible and $\mathbf{x}_{[s+1]}, \mathbf{y}_{[s+1]}$ both being positive eigenvectors of $\mathbf{A}_{s+1}$ we must have $\mathbf{x}_{[s+1]} = \alpha\mathbf{y}_{[s+1]}$ Since identity (B.5) holds for both $\mathbf{x}$ and $\mathbf{y}$, we obtain that $\mathbf{x}' = \alpha\mathbf{y}'$, concluding the proof. ☐

Proof of Proposition 2. First note that if $\mathbf{x}$ is a PCE with root M_l^k then its restriction to D_l^k is a positive eigenvector of $\mathbf{G}_{D_l^k}$. By definition of D_l^k, the matrix $\mathbf{G}_{D_l^k}$ admits a Frobenius normal form of the form

$$\mathbf{G}_{D_l^k} = \begin{bmatrix} A_1 & A_{12} & \dots & \dots & A_{1s+1} \\ 0 & A_2 & A_{23} & \dots & A_{2s+1} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & A_s & A_{ss+1} \\ 0 & \dots & \dots & 0 & A_{s+1} \end{bmatrix}, \text{ with } A_{s+1} = \mathbf{G}_{M_l^k}.$$

If $\rho(\mathbf{G}_{M_l^k}) > \rho(\mathbf{G}_{Q_l^k})$ then

$$\rho(A_{s+1}) = \rho(\mathbf{G}_{M_l^k}) > \rho(\mathbf{G}_{Q_l^k}) = \max_{i=1,\dots,s} \rho(A_i),$$

and we are in the conditions of Lemma B12. Thus $\mathbf{G}_{D_l^k}$ then admits a unique positive eigenvector $\mathbf{y} = (y_i)_{i \in D_l^k}$, such that $\sum_{i \in D_l^k} y_i = \frac{V}{c}\frac{\rho}{1+\rho}$. Let then $\mathbf{x}$ be defined as $x_i = y_i$ if

[6] Uniqueness is up to multiplication by a constant.

$i \in D_l^k$ and $x_i = 0$ if $i \in N \setminus D_l^k$. By construction, $\mathbf{x}$ is a PCE with root M_l^k and there can be no other one.
Now suppose that $\rho(\mathbf{G}_{M_l^k}) \leq \rho(\mathbf{G}_{Q_l^k})$. Then

$$\rho(A_{s+1}) = \rho(\mathbf{G}_{M_l^k}) \leq \rho(\mathbf{G}_{Q_l^k}) = \max_{i=1,\dots,s} \rho(A_i),$$

meaning that $\mathbf{G}_{D_l^k}$ admits no positive eigenvector, by Lemma A4. This concludes the proof. □

Proof of Corollary 1. Suppose that $\mathbf{x}$ is a PCE with $N_+(\mathbf{x}) = \cup_{i=1}^n D_{l_i}^{k_i}$, with $\{M_{l_i}^{k_i}\}_{i=1,\dots,n}$ pairwise disconnected, and $n \geq 2$. Then, by Proposition E7, we have

$$\rho\left(\mathbf{G}_{M_{l_i}^{k_i}}\right) = \rho > \rho\left(\mathbf{G}_{Q_{l_i}^{k_i}}\right), \ \forall i = 1, ..., n,$$

which contradicts the fact that $\mathbf{G}$ is layer-generic. □

Proof of Proposition 3. $(i) \Rightarrow (ii)$: suppose that (ii) does not hold. Then there exists two PCE $\mathbf{x}_1, \mathbf{x}_2$ such that $\rho(\mathbf{x}_1) = \rho(\mathbf{x}_2) =: \rho$. For $\lambda \in [0,1]$ and define $\mathbf{x}^\lambda := \lambda\mathbf{x}_1 + (1-\lambda)\mathbf{x}_2$. Then $X^\lambda = X_1 = X_2$. Hence

$$\mathbf{G}\mathbf{x}^\lambda = \lambda\mathbf{G}\mathbf{x}_1 + (1-\lambda)\mathbf{G}\mathbf{x}_2 = \lambda\rho\mathbf{x}_1 + (1-\lambda)\rho\mathbf{x}_2 = \frac{cX}{V - cX}\mathbf{x}^\lambda,$$

and $\mathbf{x}^\lambda$ is a PCE. Thus there is a continuum of PCE, contradicting (i).
$(ii) \Rightarrow (i)$: this implication follows from the fact that the set of eigenvalues of subgraphs of $\mathbf{G}$ is finite.
$(ii) \Rightarrow (iii)$: Suppose that (iii) does not hold. Then there exists k, l, k', l' such that $\rho(\mathbf{G}_{M_l^k}) = \rho(\mathbf{G}_{M_{l'}^{k'}})$, $\rho(\mathbf{G}_{Q_l^k}) < \rho(\mathbf{G}_{M_l^k})$ and $\rho(\mathbf{G}_{Q_{l'}^{k'}}) < \rho(\mathbf{G}_{M_{l'}^{k'}})$. The last two strict inequalities mean that there exists a PCE with root M_l^k, and a PCE with root $M_{l'}^{k'}$, contradicting (ii).
$(iii) \Rightarrow (ii)$: Assume that (ii) does not hold, and let M_l^k (resp. $M_{l'}^{k'}$) be the root of $\mathbf{x}_1$ (resp. $\mathbf{x}_2$). Being both PCE, it follows that we have $\rho(\mathbf{G}_{Q_l^k}) < \rho(\mathbf{G}_{M_l^k})$ and $\rho(\mathbf{G}_{Q_{l'}^{k'}}) < \rho(\mathbf{G}_{M_{l'}^{k'}})$, contradicting (iii).
Finally we obtain $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$ and the proof is complete. □

B.1.5 Proof of the results in Section 3.5

Let us show that the system (11) is well-behaved on the set

$$\mathbf{S} := \left\{\mathbf{x} \neq \mathbf{0} : \, x_i \geq 0 \; \forall i, \; X \leq \frac{V}{c}\right\} \tag{B.6}$$

in the sense that, for any initial condition in $\mathbf{S}$, there exists a unique solution $(\mathbf{x}(t))_{t\geq 0}$ which forever remains in $\mathbf{S}$.

Indeed, given a PCE $\mathbf{x}^*$, we have $X^* = \frac{V}{c}\frac{\rho}{1+\rho} < \frac{V}{c}$, where ρ is a positive eigenvalue of $\mathbf{G}$. As a consequence $\mathbf{S}$ contains all the relevant states of the problem we consider. We denote by $(\phi(\mathbf{x},t))_{\mathbf{x}\in\mathbf{S},t\geq 0}$ the semi-flow associated to (11) on $\mathbf{S}$. Namely $\phi(\mathbf{x},t)$ is equal to the position of the (unique) solution of (11) starting in $\mathbf{x}$.

We first give a short justification that system (11) induces a semi-flow on $\mathbf{S}$.

Lemma B13. *System (11) induces a semiflow on* $\mathbf{S} := \left\{\mathbf{x} \neq \mathbf{0} : x_i \geq 0\ \forall i,\ X \leq \frac{V}{c}\right\}$. *On the positively invariant set* $\mathbf{S}$, *system (11) writes*

$$\dot{x}_i(t) = -x_i(t) - (\mathbf{Gx})_i(t) + \left(\frac{V}{cX(t)}(\mathbf{Gx})_i(t)\,(x_i(t) + (\mathbf{Gx})_i(t))\right)^{1/2} \quad \textit{for } i = 1, ..., N.$$

Proof. First note that, if $\mathbf{x} \in \mathbf{S}$, then

$$-(\mathbf{Gx})_i + \left(\frac{V}{cX}(\mathbf{Gx})_i\,(x_i + (\mathbf{Gx})_i)\right)^{1/2} \geq 0,$$

meaning that $B_i(\mathbf{x}) = -x_i - (\mathbf{Gx})_i + \left(\frac{V}{cX}(\mathbf{Gx})_i\,(x_i + (\mathbf{Gx})_i)\right)^{1/2}$ for $\mathbf{x} \in \mathbf{S}$.

We need to check that the vector field B points inward on the boundary of $\mathbf{S}$. Suppose that $\mathbf{x} \in \mathbf{S}$, with $X = \frac{V}{c}$. Then

$$\begin{aligned} \dot{X} &= -X - \sum_i (\mathbf{Gx})_i + \sum_i \left((\mathbf{Gx})_i\,(x_i + (\mathbf{Gx})_i)\right)^{1/2} \\ &< -X - \sum_i (\mathbf{Gx})_i + \sum_i (x_i + (\mathbf{Gx})_i) = 0 \end{aligned}$$

Moreover, if $x_i = 0$ then

$$B_i(\mathbf{x}) = -(\mathbf{Gx})_i + \left(\frac{V}{cX}(\mathbf{Gx})_i\,(x_i + (\mathbf{Gx})_i)\right)^{1/2} \geq 0.$$

This concludes the proof. ■

The following result will be useful to prove that a point is not asymptotically stable. It directly follows from the definition of asymptotic stability.

Lemma B14. *Suppose that there exists an open neighborhood* U_0 *of* $\mathbf{x}^*$ *with the property that, for any open neighborhood* U *of* $\mathbf{x}^*$ *and any* $T > 0$, *there exists* $\mathbf{x} \in U$ *such that* $\phi(\mathbf{x},t) \notin U_0$, *for any* $t \geq T$. *Then* $\mathbf{x}^*$ *is not asymptotically stable.*

If $\mathbf{B}(.)$ in (11) is differentiable in an open neighborhood of a PCE, then a simple sufficient condition for an interior equilibrium (i.e., all agents in the network are active) to be asymptotically stable is the following:

Lemma B15. *Suppose that* $\mathbf{x}^*$ *is an interior equilibrium and that the eigenvalues of the Jacobian matrix of* $B(.)$, *evaluated at* $\mathbf{x}^*$, *have negative real parts. Then,* $\mathbf{x}^*$ *is asymptotically stable.*

Unfortunately the map $\mathbf{B}$ is not differentiable at a non-interior PCE, and we then cannot use this result. However we have the following:

Proposition B1. *Let $\mathbf{x}^*$ be a PCE, and let $\mathbf{u} \neq \mathbf{0}$ be such that $u_i \geq 0\ \forall i$. Then the directional derivative of $\mathbf{B}$ in $\mathbf{x}^*$ along $\mathbf{u}$, namely the quantity*

$$D_{\mathbf{u}}B(\mathbf{x}^*) := \lim_{h\to 0, h>0} \frac{B(\mathbf{x}^* + h\mathbf{u})}{h}$$

exists and satisfies $D_{\mathbf{u}}B(\mathbf{x}^*) = \frac{1}{2}\left(-I_N + \frac{1+\rho(\mathbf{x}^*)}{X^*}L(\mathbf{x}^*) + \frac{1}{\rho(\mathbf{x}^*)}\mathbf{G}\right) \cdot \mathbf{u}$, where $\mathbf{L}(\mathbf{x}^*)$ is the matrix where every column is equal to $\mathbf{x}^*$.

Proof of Proposition B1. Let $\mathbf{u} \neq \mathbf{0}$ be such that $u_i \geq 0$ for all i and $h > 0$. Then

$$B_i(\mathbf{x}^*+h\mathbf{u}) = -(\mathbf{x}_i^*+hu_i+(\mathbf{G}(\mathbf{x}^*+h\mathbf{u}))_i)+\left(\frac{V}{c(X^*+hU)}(\mathbf{G}(\mathbf{x}^*+h\mathbf{u}))_i)(x_i^*+hu_i+(\mathbf{G}(\mathbf{x}^*+h\mathbf{u}))_i)\right)^{1/2}$$

The term in the square root can be written

$$\begin{aligned}
& \frac{V}{cX^*}\left(1-h\frac{U}{X^*}\right)\left[(\mathbf{G}\mathbf{x}^*)_i(x_i^*+(\mathbf{G}\mathbf{x}^*)_i)+h\left[(\mathbf{G}\mathbf{x}^*)_i(u_i+(\mathbf{G}\mathbf{u})_i)+(\mathbf{G}\mathbf{u})_i(x_i^*+(\mathbf{G}\mathbf{x}^*)_i)\right]\right]+\mathcal{O}(h^2)\\
=\ & \frac{V}{cX^*}(\mathbf{G}\mathbf{x}^*)_i(x_i^*+(\mathbf{G}\mathbf{x}^*)_i)\left(1-h\frac{U}{X^*}\right)\left[1+h\left[\frac{u_i+(\mathbf{G}\mathbf{u})_i}{x_i^*+(\mathbf{G}\mathbf{x}^*)_i}+\frac{(\mathbf{G}\mathbf{u})_i}{(\mathbf{G}\mathbf{x}^*)_i}\right]\right]+\mathcal{O}(h^2)\\
=\ & \frac{V}{cX^*}(\mathbf{G}\mathbf{x}^*)_i(x_i^*+(\mathbf{G}\mathbf{x}^*)_i)\left[1+h\left[-\frac{U}{X^*}+\frac{u_i+(\mathbf{G}\mathbf{u})_i}{x_i^*+(\mathbf{G}\mathbf{x}^*)_i}+\frac{(\mathbf{G}\mathbf{u})_i}{(\mathbf{G}\mathbf{x}^*)_i}\right]\right]+\mathcal{O}(h^2)
\end{aligned}$$

Observing that $(\frac{V}{cX^*}(\mathbf{G}\mathbf{x}^*)_i(x_i^*+(\mathbf{G}\mathbf{x}^*)_i))^{1/2} = x_i^*+(\mathbf{G}\mathbf{x}^*)_i$, the square root of the above quantity is equal to

$$(x_i^*+(\mathbf{G}\mathbf{x}^*)_i)\left[1+\frac{h}{2}\left[\frac{-U}{X^*}+\frac{u_i+(\mathbf{G}\mathbf{u})_i}{x_i^*+(\mathbf{G}\mathbf{x}^*)_i}+\frac{(\mathbf{G}\mathbf{u})_i}{(\mathbf{G}\mathbf{x}^*)_i}\right]\right]+\mathcal{O}(h^2)$$

Hence, since $(x_i^*+(\mathbf{G}\mathbf{x}^*)_i) = \frac{V}{V-cX^*}x_i^*$, we obtain

$$\begin{aligned}
B_i(\mathbf{x}^*+h\mathbf{u}) &= -(hu_i+h(\mathbf{G}\mathbf{u})_i)+\frac{h}{2}\left[\frac{-UV}{X^*(V-cX^*)}x_i^*+(u_i+(\mathbf{G}\mathbf{u})_i)+\frac{V}{cX^*}(\mathbf{G}\mathbf{u})_i\right]+\mathcal{O}(h^2)\\
&= \frac{h}{2}\left[\frac{-UV}{X^*(V-cX^*)}x_i^*-u_i+\frac{V-cX^*}{cX^*}(\mathbf{G}\mathbf{u})_i\right]+\mathcal{O}(h^2)
\end{aligned}$$

Consequently

$$\lim_{h\to+\infty, h>0}\frac{B_i(\mathbf{x}^*+h\mathbf{u})}{h} = \frac{1}{2}\left[\frac{-UV}{X^*(V-cX^*)}x_i^*-u_i+\frac{V-cX^*}{cX^*}(\mathbf{G}\mathbf{u})_i\right] = \frac{1}{2}\left(\mathbf{D}F(\mathbf{x}^*)\mathbf{u}\right)_i,$$

which proves the result. □

Proof of Theorem 2. Recall that $\mathbf{x}^*$ is an eigenvector of $\mathbf{G}$, associated to eigenvalue $\rho(\mathbf{x}^*)$, given by

$$\rho(\mathbf{x}^*) = \frac{cX^*}{V - cX^*}.$$

In what follows, let $\rho^* := \rho(\mathbf{x}^*)$ and $\rho := \rho(\mathbf{G})$.

• Suppose first that $\rho^* < \rho$, and let $M_{l^*}^{k^*}$ be the root of $\mathbf{x}^*$. For any l, k such that $M_l^k \subset Q_{l^*}^{k^*}$ we necessarily have $\rho(\mathbf{G}_{M_l^k}) < \rho^*$. However, By Lemma B.2, $\rho = \max_{k=1,\dots,T; l=1,\dots,m^k} \rho(\mathbf{G}_{M_l^k})$. Hence there exists $k_0 \le T, l_0 \le m^{k_0}$ such that $\rho_{l_0}^{k_0} = \rho$. Let $C := N \setminus D_{l^*}^{k^*}$. By construction, $\mathbf{G}_C$ is a nonnegative matrix with largest eigenvalue ρ, and we call $\mathbf{u}$ an eigenvector associated to ρ. Let $g(\cdot)$ be defined as

$$g(X) := (1+\rho)\left(-1 + \left(\frac{V}{cX}\frac{\rho}{\rho+1}\right)^{1/2}\right).$$

Then

$$g(X^*) := (1+\rho)\left(-1 + \left(\frac{V}{cX^*}\frac{\rho}{\rho+1}\right)^{1/2}\right) = (1+\rho)\left(-1 + \left(\frac{\rho^*+1}{\rho^*}\frac{\rho}{\rho+1}\right)^{1/2}\right) > 0,$$

so that we can choose $X_0 > X^*$ such that $g(X_0) > 0$. Now let U_0 be defined as

$$U_0 := \{\mathbf{x} \in \mathbf{S} : \; X < X_0\}.$$

Then U_0 is an open neighborhood of $\mathbf{x}^*$ by construction.

We now construct a family of solutions curves and use it to show that $\mathbf{x}^*$ is not asymptotically stable by using Lemma B14. Let $(\mathbf{x}^\epsilon(t))_{t\ge 0}$ be the solution of (11) with initial condition $\mathbf{a}^\epsilon$ defined as follows:

$$\mathbf{a}(\epsilon)_i = \epsilon u_i \; \forall i \in C, \;\text{ and }\; \mathbf{a}(\epsilon)_i = x_i^* \; \forall i \in D_{l^*}^{k^*},$$

where ϵ is a positive number. Then we have, for any $i \notin D_{l^*}^{k^*}$,

$$B_i(\mathbf{a}(\epsilon)) = ((\mathbf{G}+\mathbf{I})\mathbf{a}(\epsilon))_i\left(-1 + \left(\frac{V}{c(X^*+\epsilon U)}\frac{(\mathbf{G}\mathbf{a}(\epsilon))_i}{((\mathbf{G}+\mathbf{I})\mathbf{a}(\epsilon))_i}\right)^{1/2}\right)$$

By definition of C as the complementary of the candidate set with root $M_{l^*}^{k^*}$, we have $g_{ij} = 0$ for any $i \in C$ and any $j \in D_{l^*}^{k^*}$. Consequently

$$(\mathbf{G}\mathbf{a}(\epsilon))_i = \sum_{j\in C} g_{ij}\mathbf{a}(\epsilon)_j = (\mathbf{G}_C\mathbf{a}(\epsilon))_i = \rho\epsilon u_i.$$

Thus, for $i \in C$,

$$\begin{aligned} B_i(\mathbf{a}(\epsilon)) &= (\rho+1)\epsilon u_i\left(-1 + \left(\frac{V}{c(X^*+\epsilon U)}\frac{\rho}{\rho+1}\right)^{1/2}\right) \\ &= (\rho+1)\left(-1 + \left(\frac{V}{c(X^*+\epsilon U)}\frac{\rho}{\rho+1}\right)^{1/2}\right)\mathbf{a}(\epsilon)_i \\ &= g(A(\epsilon))\mathbf{a}(\epsilon)_i, \end{aligned}$$

where $A(\epsilon) = \sum_i a(\epsilon)_i$. Let $X^\epsilon(t) := \sum_i x_i^\epsilon(t)$. Then the we have, for any $i \in C$,

$$x_i^\epsilon(t) = x_i^\epsilon(0) \exp\left(\int_0^t g(X^\epsilon(s))ds\right), \ \forall i \in C.$$

Suppose that the solution curve $\mathbf{x}^\epsilon$ remains in U_0 for any $t \geq 0$ then

$$x_i^\epsilon(t) > x_i^\epsilon(0) \exp\left(g(X_0)t\right), \forall t \geq 0.$$

Since $g(X_0) > 0$, the right-hand side of the last inequality goes to infinity with t. Hence there exists some postive time T^ϵ such that $\mathbf{x}^\epsilon(t) \notin U_0$, for any $t \geq T^\epsilon$. This concludes the proof that $\mathbf{x}^*$ is not asymptotically stable for dynamics (11).

• Suppose now that $\rho^* = \rho$. Let $\mathbf{D}(\mathbf{x}^*) := \frac{1}{2}\left(-I_N + \frac{1+\rho(\mathbf{x}^*)}{X^*}L(\mathbf{x}^*) + \frac{1}{\rho(\mathbf{x}^*)}\mathbf{G}\right)$. We first show that all eigenvalues of $\mathbf{D}(\mathbf{x}^*)$ have a negative real part.
Suppose that $\mathbf{D}(\mathbf{x}^*) \cdot \mathbf{u} = \lambda \cdot \mathbf{u}$, with $\mathbf{u} \neq 0$. Call $U := \sum_{i \in N} u_i$. Then we have

$$-\mathbf{u} - \frac{1+\rho}{X^*}U\mathbf{x}^* + \frac{1}{\rho}\mathbf{G}\mathbf{u} = 2\lambda\mathbf{u}$$

which gives

$$\left(\mathbf{I}_N - \frac{1}{\rho(1+2\lambda)}\mathbf{G}\right)\mathbf{u} = -\frac{1+\rho}{X^*(1+2\lambda)}U\mathbf{x}^*.$$

Suppose that $Re(\lambda) > 0$ or that λ is pure imaginary. Then $|1+\lambda| > 1$ and the matrix $\mathbf{G}/(\rho(1+2\lambda))$' spectral radius is strictly smaller than one. As a consequence $\mathbf{I}_N - \frac{1}{\rho(1+2\lambda)}\mathbf{G}$ is invertible and

$$\left(\mathbf{I}_N - \frac{1}{\rho(1+2\lambda)}\mathbf{G}\right)^{-1} = \sum_{p=0}^{+\infty} \frac{1}{\rho^p(1+2\lambda)^p}\mathbf{G}^p.$$

Consequently

$$\begin{aligned}
\mathbf{u} &= -\frac{1+\rho}{X^*(1+2\lambda)}U\left(\mathbf{I}_N - \frac{1}{\rho(1+2\lambda)}\mathbf{G}\right)^{-1}\mathbf{x}^* \\
&= -\frac{1+\rho}{X^*(1+2\lambda)}U\sum_{p=0}^{+\infty}\frac{1}{\rho^p(1+2\lambda)^p}\mathbf{G}^p\mathbf{x}^* \\
&= -\frac{1+\rho}{X^*(1+2\lambda)}U\sum_{p=0}^{+\infty}\frac{1}{(1+2\lambda)^p}\mathbf{x}^* \\
&= -\frac{1+\rho}{2X^*\lambda}U\mathbf{x}^*
\end{aligned}$$

Since $\mathbf{u} \neq 0$, this equality implies that $U \neq 0$ and summing the coordinates of $\mathbf{u}$ we obtain that $2\lambda = -(1+\rho) < 0$, a contradiction.

Suppose now that $\lambda = 0$. Then we have

$$\left(\mathbf{I}_N - \frac{1}{\rho}\mathbf{G}\right)\mathbf{u} = -\frac{1+\rho}{X^*}U\mathbf{x}^*.$$

Suppose that $U \neq 0$. Then, multiplying both sides of the equality by $\sum_{k=0}^{K}\frac{1}{\rho^k}\mathbf{G}^k$, we obtain the identity

$$\left(\mathbf{I}_N - \frac{1}{\rho^{K+1}}\mathbf{G}^{K+1}\right)\mathbf{u} = -\frac{1+\rho}{X^*}U\sum_{k=0}^{K}\frac{1}{\rho^k}\mathbf{G}^k\mathbf{x}^* = -\frac{1+\rho}{X^*}UK\mathbf{x}^*$$

The modulus of the left-hand is bounded above by $2|\mathbf{u}|$, while the modulus of the right-hand side term grows to infinity with K, which is a contradiction. Hence $U = 0$. This means that

$$\mathbf{G}\mathbf{u} = \rho\mathbf{u},$$

i.e. that $\mathbf{u}$ is in fact an eigenvector associated to the largest eigenvalue of $\mathbf{G}$. Since $\sum_i u_i = 0$, this contradicts the fact that $(N, \mathbf{G})$ is a layer-generic network.
We proved that the real part of every eigenvalue of $\mathbf{D}F(x^*)$ is strictly negative.

Now by Proposition B1, for $\mathbf{x} \in \mathbf{X}$ we have

$$B(\mathbf{x}) = \mathbf{D}(\mathbf{x}^*)\cdot(\mathbf{x}-\mathbf{x}^*) + \|\mathbf{x}-\mathbf{x}^*\|^2 g(\|\mathbf{x}-\mathbf{x}^*\|)$$

Denote by $(\lambda_1, ..., \lambda_1, \lambda_2, ..., \lambda_2, ..., \lambda_P, ..., \lambda_P)$ the eigenvalues of $\mathbf{D}(\mathbf{x}^*)$, and call n_p the multiplicity of eigenvalue λ_p. Let us first put $\mathbf{D}(\mathbf{x}^*)$ in its Jordan form:

$$\mathbf{D}F(\mathbf{x}^*) = \mathbf{P}\mathbf{J}\mathbf{P}^{-1},$$

where $\mathbf{J}$ is diagonal by blocks, i.e.

$$\mathbf{J} = Diag\left(\mathbf{J}_1, ..., \mathbf{J}_P\right) := \begin{pmatrix} \mathbf{J}_1 & \mathbf{0} & ... & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2 & ... & \mathbf{0} \\ ... & ... & ... & ... \\ \mathbf{0} & ... & \mathbf{0} & \mathbf{J}_P \end{pmatrix}, \quad \text{with } \mathbf{J}_p = \begin{pmatrix} \lambda_p & 1 & 0 & ... & 0 \\ 0 & \lambda_p & 1 & ... & 0 \\ ... & ... & ... & ... & \\ 0 & ... & 0 & \lambda_p & 1 \\ 0 & ... & ... & 0 & \lambda_p \end{pmatrix}$$

Define now $\mathbf{Q} := Diag\left(\mathbf{Q}_1, ..., \mathbf{Q}_P\right)$, with $\mathbf{Q}_p = Diag(1, \epsilon, ..., \epsilon^{n_p-1})$. We then have

$$\mathbf{Q}_p^{-1}\mathbf{J}_p\mathbf{Q}_p = \begin{pmatrix} \lambda_p & \epsilon & 0 & ... & 0 \\ 0 & \lambda_p & \epsilon & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & ... & 0 & \lambda_p & \epsilon \\ 0 & ... & ... & 0 & \lambda_p \end{pmatrix}$$

Thus, defining $\mathbf{R} := \mathbf{P}\mathbf{Q}$ we obtain

$$\mathbf{R}^{-1}\mathbf{D}(\mathbf{x}^*)\mathbf{R} = \mathbf{Q}^{-1}\mathbf{J}\mathbf{Q} = \mathbf{D}(\lambda) + \epsilon\mathbf{B},$$

where $\mathbf{D}(\lambda)$ is the diagonal matrix filled with the eigenvalues of $\mathbf{D}(\mathbf{x}^*)$.
Now define $V : \mathbf{S} \to \mathbb{R}^+$ as follows:

$$V(\mathbf{x}) := \left|\mathbf{R}^{-1}(\mathbf{x}-\mathbf{x}^*)\right|^2 = \left\langle \mathbf{R}^{-1}(\mathbf{x}-\mathbf{x}^*) \mid \overline{\mathbf{R}^{-1}}(\mathbf{x}-\mathbf{x}^*)\right\rangle$$

We have

$$\begin{aligned}\dot{V}(\mathbf{x}) &= \left\langle \mathbf{R}^{-1}\dot{\mathbf{x}} \mid \overline{\mathbf{R}^{-1}}(\mathbf{x}-\mathbf{x}^*)\right\rangle + \left\langle \overline{\mathbf{R}^{-1}}\dot{\mathbf{x}} \mid \mathbf{R}^{-1}(\mathbf{x}-\mathbf{x}^*)\right\rangle \\ &= \left\langle (\mathbf{D}(\lambda)+\epsilon\mathbf{B})\mathbf{R}^{-1}(\mathbf{x}-\mathbf{x}^*) \mid \overline{\mathbf{R}^{-1}}(\mathbf{x}-\mathbf{x}^*)\right\rangle + \left\langle (\overline{\mathbf{D}(\lambda)}+\epsilon\overline{\mathbf{B}})\overline{\mathbf{R}^{-1}}(\mathbf{x}-\mathbf{x}^*) \mid \mathbf{R}^{-1}(\mathbf{x}-\mathbf{x}^*)\right\rangle \\ &+ \|\mathbf{x}-\mathbf{x}^*\|^2 h(\|\mathbf{x}-\mathbf{x}^*\|),\end{aligned}$$

where $h(a) \to_{a\to 0} 0$. Hence we have

$$\begin{aligned}\dot{V}(\mathbf{x}) &= \left\langle (\mathbf{D}(\lambda)+\overline{\mathbf{D}(\lambda)})\mathbf{R}^{-1}(\mathbf{x}-\mathbf{x}^*) \mid \overline{\mathbf{R}^{-1}}(\mathbf{x}-\mathbf{x}^*)\right\rangle + 2\epsilon Re\left(\left\langle \mathbf{B}\mathbf{R}^{-1}(\mathbf{x}-\mathbf{x}^*) \mid \overline{\mathbf{R}^{-1}}(\mathbf{x}-\mathbf{x}^*)\right\rangle\right) \\ &+ \|\mathbf{x}-\mathbf{x}^*\|^2 h(\|\mathbf{x}-\mathbf{x}^*\|)\end{aligned}$$

Let $\alpha := \max_{p=1,\dots,P} Re(\lambda_p) < 0$. We have

$$\left\langle (\mathbf{D}(\lambda)+\overline{\mathbf{D}(\lambda)})\mathbf{R}^{-1}(\mathbf{x}-\mathbf{x}^*) \mid \overline{\mathbf{R}^{-1}}(\mathbf{x}-\mathbf{x}^*)\right\rangle \leq 2\alpha|\mathbf{R}^{-1}(\mathbf{x}-\mathbf{x}^*)|^2 = 2\alpha V(\mathbf{x}).$$

As a consequence, choosing ϵ small enough and $\mathbf{x}$ close enough of $\mathbf{x}^*$ we obtain that

$$\dot{V}(\mathbf{x}) \leq \alpha V(\mathbf{x}),$$

which proves that $V(\mathbf{x}(t))$ goes to zero exponentially fast, as t goes to infinity, and this concludes the proof. □

B.2 Proofs of results in Section 4

Proof of Proposition 4. We start by proving the more general following lemma:

Lemma B16. *Suppose that $\mathbf{x}^*$ is a simple equilibrium of $(N,\mathbf{G})$, $i,j \in N_+(\mathbf{x}^*)$ and $g_{ij}=0$. Let $\hat{\mathbf{G}}$ be such that $\hat{\mathcal{N}}_i = \mathcal{N}_i \cup \{j\}$, and for each $l \neq i$, $\hat{\mathcal{N}}_l = \mathcal{N}_l$. Then there exists a unique equilibrium $\hat{\mathbf{x}}^*$ of $(N,\hat{\mathbf{G}})$ such that*

(i) $N_+(\hat{\mathbf{x}}^*) \subseteq N_+(\mathbf{x}^*)$

(ii) $\hat{X}^* \geq X^*$,

(iii) $\hat{x}_i^* > x_i^*$.

Proof of Lemma B16. Let M_l^k be the root of $\mathbf{x}^*$, meaning that $N_+(\mathbf{x}^*) = D_l^k$. Denote by $(D_l^k, \hat{\mathbf{G}}_{D_l^k})$ the sub-network where i is now connected to j. Let $\rho := \rho(\mathbf{G}_{D_l^k})$ and $\rho^* := \rho(\hat{\mathbf{G}}_{D_l^k})$. We have $\rho^* \geq \rho$.
Let $\{V_1, ..., V_r\}$ be a decomposition of D_l^k corresponding to Frobenius normal form (A.1). Then $r = s+1$ and $V_r = M_l^k$. Moreover $\rho(\mathbf{G}_{D_l^k}) = \rho(\mathbf{G}_{M_l^k}) > \rho(A_i)$ for $i = 1, ..., s$. When adding a link from i to j, the decomposition either remains the same (if there is no path from j to i), or the decomposition becomes $\left\{\hat{V}, \{V_i\}_{i\in I}\right\}$, where $I \subset \{1, ..., r\}$. There are three possibilities here:

(*a*) If $\rho(\mathbf{G}_{\hat{V}}) < \rho(\mathbf{G}_{M_l^k})$, then $r \in I$ and $\rho^* = \rho$. There is then a PCE with root M_l^k in $\hat{\mathbf{G}}_{D_l^k}$. In particular $N_+(\hat{\mathbf{x}}^*) = N_+(\mathbf{x}^*)$.

(*b*) If $r \notin I$ then $M_l^k \subseteq \hat{V}$ and there is a PCE with root $\hat{V}$. Again $N_+(\hat{\mathbf{x}}^*) = N_+(\mathbf{x}^*)$.

(*c*) $\rho(\hat{\mathbf{G}}_{\hat{V}}) \geq \rho$ and $i \in I$. Then $\rho^* \geq \rho$ and there is a PCE with root $\hat{V}$. Then $N_+(\hat{\mathbf{x}}^*) \subsetneq N_+(\mathbf{x}^*)$, because agents in M_l^k are inactive in PCE $\hat{\mathbf{x}}^*$.

In all three cases, there is a unique PCE $\hat{\mathbf{x}}^*$ in $\hat{\mathbf{G}}_{D_l^k}$, associated to ρ^*. Points (*i*) and (*ii*) hold by construction, using the fact that $\hat{X}^* = \frac{\rho^*}{\rho^*+1}\frac{V}{c} \geq \frac{\rho}{\rho+1}\frac{V}{c} = X^*$. Also note that $\hat{\mathbf{x}}^* \neq \mathbf{x}^*$: suppose by contradiction that $\mathbf{x}^* = \hat{\mathbf{x}}^*$. Let $v \neq i$. Then $\rho x_v^* = (\mathbf{G}\mathbf{x}^*)_v = (\hat{\mathbf{G}}\mathbf{x}^*)_v = \rho^* x_v$. Hence $\rho = \rho^*$. Thus $\rho x_i^* = (\mathbf{G}\mathbf{x}^*)_i = (\hat{\mathbf{G}}\mathbf{x}^*)_i - x_j^* = \rho x_i^* - x_j^*$, a contradiction.
We now prove that (*iii*) holds. Note that $j \in N_+(\hat{\mathbf{x}}^*)$. Consider the following subsets of agents:

$$V_+ := \left\{v \in D_l^k : \frac{\hat{x}_v^*}{x_v^*} \geq \frac{\hat{x}_l^*}{x_l^*} \ \forall l \in D_l^k\right\}, \ V_- := \left\{v \in D_l^k : \frac{\hat{x}_v^*}{x_v^*} \leq \frac{\hat{x}_l^*}{x_l^*} \ l \in D_l^k\right\}.$$

Note that if $v \neq i$ and $v \in V_+$ then $\frac{\hat{x}_v^*}{x_v^*} = \frac{\sum_{w\in\mathcal{N}_v}\hat{x}_w^*}{\sum_{w\in\mathcal{N}_v}x_w^*}$. Hence $w \in V_+$ for all $w \in \mathcal{N}_v$. By a recursive argument this implies that, if v is connected to w through a path then $w \in V_+$. The same property also holds for V_-. As a consequence $i \in V_+ \cup V_-$. If this were not the case there would exist two nodes $v_+ \neq i$ and $v_- \neq i$ such that $v_+ \in V_+$ and $v_- \in V_-$, which would imply that elements of M_l^k belong to both V_+ and V_-, a contradiction.
Suppose first that we are in the case where $\rho^* > \rho$, and let $v \neq i$. Suppose that $v \in V_+$. Then

$$\frac{1}{\rho^*} = \frac{\hat{x}_v^*}{(\hat{\mathbf{G}}\hat{\mathbf{x}}^*)_v} = \frac{\hat{x}_v^*}{\sum_{w\in\mathcal{N}_v}\hat{x}_w^*} \geq \frac{x_v^*}{\sum_{w\in\mathcal{N}_v}x_w^*} = \frac{x_v^*}{(\mathbf{G}\mathbf{x}^*)_v} = \frac{1}{\rho},$$

a contradiction. Hence $V_+ = \{i\}$.
Suppose finally that $\rho^* = \rho$. Showing that $i \in V_+$ is equivalent to showing that $i \notin V_-$. Suppose by contradiction that $i \in V_-$. Then

$$\frac{1}{\rho} = \frac{\hat{x}_i^*}{(\hat{\mathbf{G}}\hat{\mathbf{x}}^*)_i} = \frac{\hat{x}_i^*}{\sum_{w\in\mathcal{N}_i}\hat{x}_w^* + \hat{x}_j^*} < \frac{\hat{x}_i^*}{\sum_{w\in\mathcal{N}_i}\hat{x}_w^*} \leq \frac{x_i^*}{\sum_{w\in\mathcal{N}_i}x_w^*} = \frac{x_i^*}{(\mathbf{G}\mathbf{x}^*)_i} = \frac{1}{\rho},$$

a contraction. Thus $i \in V_+$ and this concludes the proof of point (iii). $\square$

We now prove Proposition 4 by showing that $\hat{\mathbf{x}}^*$ is asymptotically stable. Obviously $\rho(\mathbf{G}) = \rho^*$. Hence we only need to prove that there is no other simple PCE associated to eigenvalue ρ^*. Since $\mathbf{x}^*$ is the only PCE associated to eigenvalue ρ, it means that, if k', l' are such that $\rho(\mathbf{G}_{M_{l'}^{k'}}) = \rho$, then $M_l^k \subseteq D_{l'}^{k'}$. Hence $\hat{\mathbf{x}}^*$ is the only PCE associated to ρ^* in $\hat{\mathbf{G}}$. $\square$

C Additional results

C.1 Peer-confirming equilibria in semi-connected networks

Corollary C2. *If the network $(N, \mathbf{G})$ is a semi-connected T-layer network then there are at most T peer-consistent equilibria.*

Proof of Corollary C2. The number of PCE is equal to

$$Card\left\{s = 1, ..., T : \ \rho(\mathbf{G}_{M_1^s}) > \max_{k=1,\dots,s-1} \rho(\mathbf{G}_{M_1^k})\right\}.$$

□

C.2 Policy interventions

C.2.1 Adding links

Let us consider strongly connected networks. We know from Proposition 1 that there is a unique equilibrium, that every player is active in equilibrium, and that the individual effort is proportional to her eigenvector centrality.

Proposition C2. *Let $(N, \mathbf{G})$ be a strongly connected network in which $g_{ij} = 0$ for some $i, j \in N$, and $\mathbf{x}^*$ is the unique peer-consistent equilibrium of $(N, \mathbf{G})$. Let $\widehat{\mathbf{G}}$ be such that $\widehat{\mathcal{N}}_i = \mathcal{N}_i \cup \{j\}$, where i is the "sender" and j is the "receiver", and for each $l \neq i$, $\widehat{\mathcal{N}}_l = \mathcal{N}_l$. Then, in the unique equilibrium $\widehat{\mathbf{x}}^*$ of $(N, \widehat{\mathbf{G}})$, total effort strictly increases, i.e., $\widehat{X}^* > X^*$, and both the sender's effort and her resource share increase, i.e., $\widehat{x}_i^* > x_i^*$.*

Proof of Proposition C2. We show that $X' > X$. The rest of the statement is a direct consequence of Lemma B16, which is stated and proved as part of the proof of Proposition 4. By Proposition 1, $X = \frac{V\rho(\mathbf{G})}{c[1+\rho(\mathbf{G})]}$ and $X' = \frac{V\rho(\mathbf{G}')}{c[1+\rho(\mathbf{G}')]}$. Hence $X' > X$, because $\rho(\mathbf{G}) < \rho(\mathbf{G}')$. □

Let us now consider a weakly connected network.

Proposition C3 (Increasing unilateral knowledge)**.** *Let $(N, \mathbf{G})$ be a weakly connected network, $\mathbf{x}^*$ a PCE with root $M_l^k, (k \geq 2)$, and $(N, \widehat{\mathbf{G}})$ the network obtained by adding a link from agent $i_0 \in Q_l^k$ to agent $j_0 \in M_l^k$. Denote by $\widehat{\mathbf{x}}^*$ the PCE with root M_l^k in $(N, \widehat{\mathbf{G}})$. Then,*

$$\frac{\widehat{x}_i^*}{\widehat{x}_j^*} > \frac{x_i^*}{x_j^*}, \ \textit{and hence} \ \frac{u_i(\widehat{x}_i^*)}{u_j(\widehat{x}_j^*)} > \frac{u_i(x_i^*)}{u_j(x_j^*)}, \ \ \textit{for any} \ \ i \in \mathcal{N}^-(i_0), j \notin \mathcal{N}^-(i_0),$$

where $\mathcal{N}^-(i_0) := \{i_0\} \cup \{v \in D_l^k : \ v \rightrightarrows i_0\}$;

Proof of Proposition C3. As in the proof of Lemma B16, define the set of nodes V_+ and V_-. Let us show that $V_- = \{v \in D_l^k : v \notin \mathcal{N}^-(i_0)\}$. As stated above, $i_0 \in V_+$. Also, if $v \neq i_0$ belongs to V_- and if v is connected to w through a path then $w \in V_-$. Hence elements of $\mathcal{N}^-(i_0)$ cannot belong to V_-. This also implies that $M_l^k \subset V_-$, because V_- cannot be empty. Now let $v \neq i_0$. If $\mathcal{N}_v \subset V_-$ then necessarily $v \in V_-$. Thus, by a simple recursive argument, any agent that is not connected to i_0 through a path must belong to V_-. Hence $\{v \in D_l^k : v \notin \mathcal{N}^-(i_0)\} = V_-$. □

C.2.2 Key players

Proposition C4. *Let $\mathbf{x}^*$ be the (unique) asymptotically stable equilibrium of the layer-generic network $(N, \mathbf{G})$ and $\widehat{\mathbf{x}}^*$ the (unique) asymptotically stable equilibrium of the layer-generic network $(N\backslash\{i\}, \mathbf{G}_{N\backslash\{i\}})$. Then, $\widehat{X}^* \leq X^*$.*

Proof of Proposition C4. We have $X^* = \frac{V\rho(\mathbf{G})}{c[1+\rho(\mathbf{G})]}$ and $\widehat{X}^* \leq \frac{V\rho(\mathbf{G}_{N\backslash\{i\}})}{c[1+\rho(\mathbf{G}_{N\backslash\{i\}}]}$. By standard results, $\rho(\mathbf{G}) \geq \rho(\mathbf{G}_{N\backslash\{i\}})$. Hence $\widehat{X}^* \leq X^*$. □

C.2.3 Social mixing

Proposition C5. *Let $(N^1, \mathbf{G^1})$ and $(N^2, \mathbf{G}^2)$ be two layer-generic networks endowed with resources equal to V_1 and V_2, respectively. Let $\mathbf{x}^{1*}$ (resp. $\mathbf{x}^{2*}$) be the unique stable PCE of $(N^1, \mathbf{G^1})$ (resp. $(N^2, \mathbf{G}^2)$), with root $M_{l^1}^{k^1}$ (resp. $M_{l^2}^{k^2}$).Let also $(N, \mathbf{G})$ be the network obtained from $(N^1, \mathbf{G^1})$ and $(N^2, \mathbf{G}^2)$ in which $N = N^1 \cup N^2$, $V = V^1 + V^2$, with $g_{ij} = 1$ and $g_{k\ell} = 1$ for some $(i,\ \ell) \in M_{l^1}^{k^1}$, $(j,\ k) \in M_{l^2}^{k^2}$. Then, there is a unique stable PCE $\mathbf{x}^*$ of $(N, \mathbf{G})$ satisfying $\rho(\mathbf{x}^*) = \rho(\mathbf{G})$, and $X^* > X^{1*} + X^{2*}$.*

Proof of Proposition C5. We have

$$X^1 = \frac{V^1}{c}\frac{\rho(\mathbf{G}^1)}{\rho(\mathbf{G}^1)+1};\ \ X^2 = \frac{V^2}{c}\frac{\rho(\mathbf{G}^2)}{\rho(\mathbf{G}^2)+1};\ \ X = \frac{V^1+V^2}{c}\frac{\rho(\mathbf{G})}{\rho(\mathbf{G})+1}$$

We have $\rho(\mathbf{G}) = \rho(M_{l^1}^{k^1} \cup M_{l^2}^{k^2}) > \max\{\rho(\mathbf{G}^1), \rho(\mathbf{G}^2)\}$. Hence

$$X^1 + X^2 = \frac{V^1}{c}\frac{\rho(\mathbf{G}^1)}{\rho(\mathbf{G}^1)+1} + \frac{V^2}{c}\frac{\rho(\mathbf{G}^2)}{\rho(\mathbf{G}^2)+1}\frac{V^1+V^2}{c} < \frac{\rho(\mathbf{G})}{\rho(\mathbf{G})+1} = X.$$

□

D Additional examples

D.1 Strongly connected networks

Example D1. Strongly connected networks and brokers
Let $N = \{1, 2, 3, 4\}$. Consider the strongly connected network displayed in Figure D1. The network is composed of two groups $N_1 = \{1, 2\}$ and $N_2 = \{3, 4\}$. Agents 2 and 3 are said to be *brokers* as they connect the two groups.

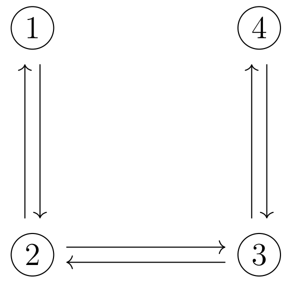

Figure D1: A strongly connected network with brokers

If we solve the system of equations (5) for the strongly connected network displayed in Figure D1, we obtain:

$$x_1^\star = x_4^\star = \frac{\rho}{2(\rho+1)^2}\frac{V}{c} \approx 0.118\frac{V}{c} \qquad u_1^\star = u_4^\star = \frac{1}{2(\rho+1)^2}V \approx 0.073V$$
$$x_2^\star = x_3^\star = \frac{\rho^2}{2(\rho+1)^2}\frac{V}{c} \approx 0.191\frac{V}{c} \qquad u_2^\star = u_3^\star = \frac{\rho}{2(\rho+1)^2}V \approx 0.118V$$

where $\rho = \frac{\sqrt{5}+1}{2}$ is the spectral radius of $\mathbf{G}$. Note that the brokers 2 and 3 have symmetric efforts and payoffs in equilibrium –agents 1 and 4 both display a symmetric effort, but lower than that of agents 2 and 3. A feature of the differences in terms of eigenvector centrality between the brokers and agents 1 and 4 is that brokers exert higher efforts and obtain higher utilities. Indeed, agents 2 and 3 have a *higher eigenvector centrality* than the other two agents.

◇

D.2 NLD algorithm

Example D2. Adjacency matrix decomposition and the NLD algorithm
Consider the matrix decomposition displayed in Figure D2. In this figure, the matrix $\mathbf{G}$ is decomposed in three steps.

Indeed, $(M_1^1, \mathbf{G}_{M_1^1})$, $(M_2^1, \mathbf{G}_{M_2^1})$, and $(M_3^1, \mathbf{G}_{M_3^1})$ are all the perfect sub-networks of $(N, \mathbf{G})$ in step 1, and M^1 is the set of all agents and their adjunct belonging to these perfect sub-networks. The adjacency sub-matrices $\mathbf{G}_{M_1^1}$, $\mathbf{G}_{M_2^1}$ and $\mathbf{G}_{M_3^1}$ are represented by the square sub-matrices in the upper-left part of the matrix $\mathbf{G}$ displayed in Figure D2. Note that these three sub-matrices are "separated" in the sense that no player in one sub-matrix is connected to any player in one of the other two sub-matrices. Observe that, by definition of perfect sub-networks (Definition 5), the sub-network of $\bar{M}_1^1 \cup \bar{M}_2^1 \cup \bar{M}_3^1$ is acyclic. The sub-matrix $\mathbf{S}_1$ represents links from the adjunct sets $\bar{M}_1^1$, $\bar{M}_2^1$, and $\bar{M}_3^1$ to the sets M_1^1, M_2^1, and M_3^1. Hence $\mathbf{S}_1 \neq \mathbf{0}$.

By removing M^1 from the network, we start step 2 and find $(M_1^2, \mathbf{G}_{M_1^2})$ and $(M_2^2, \mathbf{G}_{M_2^2})$ as perfect sub-networks of the remaining network. Again the two sub-matrices $\mathbf{G}_{M_1^2}$ and $\mathbf{G}_{M_2^2}$ are "separated", the sub-network of $\bar{M}_1^2 \cup \bar{M}_2^2$ is acyclic, and $\mathbf{S}_2 \neq \mathbf{0}$. The matrix $\mathbf{W}_1$ represents the links from the sets M_1^1, M_2^1, and M_3^1 to M^2. Since M_1^2 and M_2^2 are not included in M^1, $\mathbf{W}_1 \neq \mathbf{0}$.

Further removing M^2, we start step 3 and find $(M_1^3, \mathbf{G}_{M_1^3})$ and $(M_2^3, \mathbf{G}_{M_2^3})$ as perfect sub-networks of the remaining network. Again the two sub-matrices $\mathbf{G}_{M_1^3}$ and $\mathbf{G}_{M_2^3}$ are "separated", the sub-network of $\bar{M}_1^3 \cup \bar{M}_2^3$ is acyclic, $\mathbf{S}_3 \neq \mathbf{0}$, and $\mathbf{W}_2 \neq \mathbf{0}$. $\diamond$

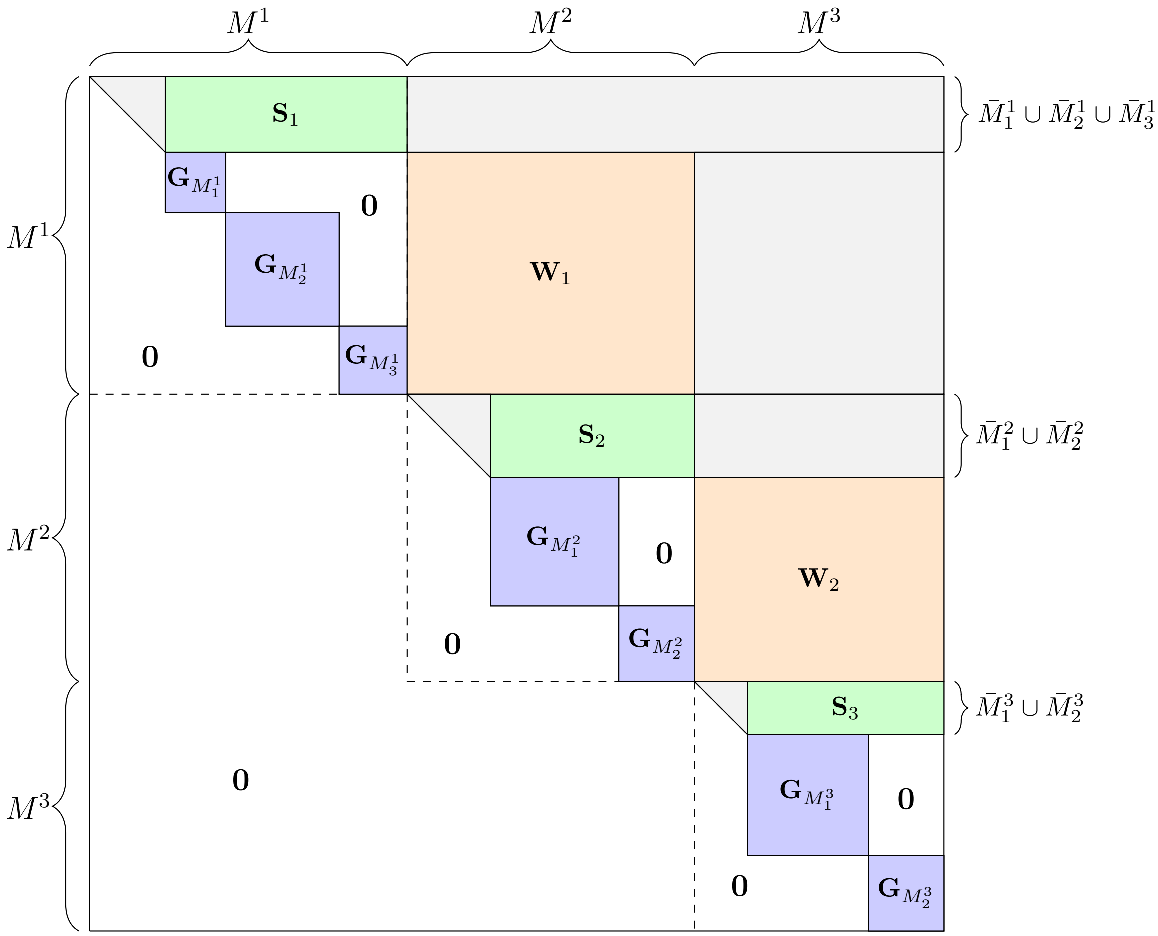

Figure D2: The structure of the adjacency matrix $\mathbf{G}$. The blue sub-matrices are irreducible matrices. The green and yellow sub-matrices are non-zero matrices. An element in the grey area of the matrix can be zero or non-zero. All other elements are zero-elements.

D.3 Peer-consistent equilibria

Let us illustrate Proposition 2 with the following example.

Example D3. A 2−layer network

Consider the network $(N, \mathbf{G})$ in Figure D3 with $N = \{1, 2, \cdots, 10\}$.

Both $(M_1^1, \mathbf{G}_{M_1^1})$ with $M_1^1 = \{4, 5, 6, 7\}$, and $(M_2^1, \mathbf{G}_{M_2^1})$ with $M_2^1 = \{2, 3\}$ are layer-1 perfect sub-networks of $(N, \mathbf{G})$. Moreover, their adjunct sets coincide with $\bar{M}_1^1 = \bar{M}_2^1 = \{1\}$. Note that the sub-network associated to $\{8, 9, 10\}$ cannot be a layer-1 perfect sub-network since $\{1, 2, 3\}$ is not adjunct cycle-free. Hence there is a layer-2 perfect sub-network with $M_1^2 = \{8, 9, 10\}$, $\bar{M}_1^2 = \emptyset$, and $Q_1^2 = \{5, 6, 7\}$. Observe that $\rho(\mathbf{G}_{M_1^1}) = 3 > \rho(\mathbf{G}_{M_1^2}) = 2 > \rho(\mathbf{G}_{M_2^1}) = 1$.

We have three candidate sets: $D_1^1 = \{1, 4, 5, 6, 7\}$, $D_2^1 = \{1, 2, 3\}$ and $D_1^2 = \{1, 2, 3, 8, 9, 10\}$.

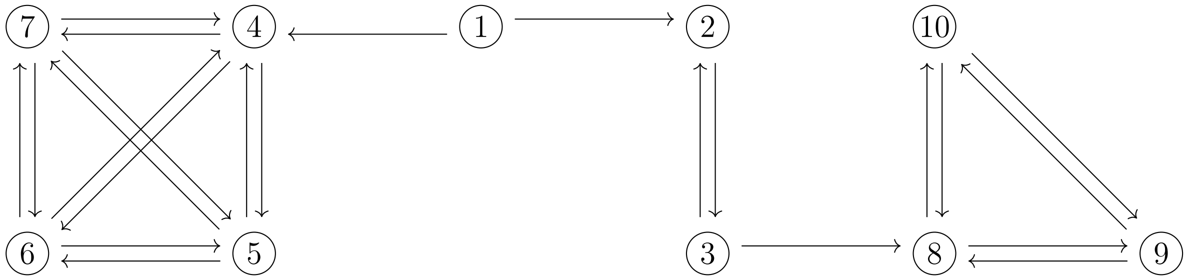

Figure D3: A 2-layer network

(a) Equilibrium with root M_1^1: Since $\rho(\mathbf{G}_{M_1^1}) = 3 > \rho(\mathbf{G}_{Q_1^1}) = 0$, there is an equilibrium with root $M_1^1 = \{4, 5, 6, 7\}$, where only agents $1, 4, 5, 6, 7$ are active.

(b) Equilibrium with root M_2^1: Since $\rho(\mathbf{G}_{M_2^1}) = 1 > \rho(\mathbf{G}_{Q_2^1}) = 0$, there is an equilibrium with root $M_2^1 = \{2, 3\}$, where only agents $1, 2, 3$ are active.

(c) Equilibrium with root M_1^2: We have $\rho(\mathbf{G}_{M_1^2}) < \rho(\mathbf{G}_{M_1^1})$. However this comparison is not relevant here, because M_1^1 is not contained in the adjunct set of M_1^2. According again to Proposition 2, there exists a peer-consistent equilibrium $\mathbf{x}^*$ with $N_+(\mathbf{x}^*) = D_1^2$ if and only if $\rho(\mathbf{G}_{M_1^2}) > \rho(\mathbf{G}_{Q_1^2})$; where $\rho(\mathbf{G}_{Q_l^k}) \equiv \max_{\{M_s^t : M_s^t \subseteq Q_l^k\}} \rho(\mathbf{G}_{M_s^t})$. This reads as $\rho(\mathbf{G}_{M_1^2}) > \rho(\mathbf{G}_{M_2^1})$, which is obviously satisfied. Hence there is a third peer-consistent equilibrium in which the set of active agents is $\{1, 2, 3, 8, 9, 10\}$.

In summary, in the network of Figure D3, there are three PCE. In all three, the effort of each active agent is equal to her eigenvector centrality. Interestingly, there is no equilibrium in which all agents are active, which agrees with the fact that the matrix $\mathbf{G}$ is not strictly nonnegative. $\diamond$

Thus, contrary to Example 2, there is no equilibrium in which all agents are active. Indeed, in Example 2 (Figure 2), the lowest-layer perfect sub-network M_1^3 has the highest spectral radius *and* all agents in the other layer perfect sub-networks (i.e., layer-1 and layer-2 perfect sub-networks) can reach M_1^3 either directly or through a path. As a result, an equilibrium with root M_1^3 exists $-\rho(\mathbf{G}_{M_1^3})$ has the largest eigenvalue– and encompasses all players in the network. On the contrary, in the network displayed in Figure D3, an equilibrium with root M_1^2 (the lowest layer perfect sub-network) exists because it can only be reached by the perfect sub-network $M_2^1 = \{2, 3\}$, which has a lower spectral radius. All agents cannot be active in equilibrium because the lowest-level perfect sub-network (i.e., M_1^1) cannot reach the highest-level perfect sub-network (i.e., M_1^2). Observe that agent 1 has a key position in the network (i.e., middleman) and is active in *any* equilibrium. This is because she is the only player that can reach anybody in the network. Agent 1 is therefore part of the adjunct set of any candidate set and thus belongs to the set of active agents at any equilibrium.

D.4 Adding links

Example D4. Consider the network $(N, \mathbf{G})$ in Figure D4. The network has two layers, and one perfect subnetwork for each layer, which are $M_1^1 = \{2,3,4\}$ and $M_2^1 = \{5,6,7\}$, with $D_1^1 = \{1,2,3,4\}$ and $D_2^1 = \{1,2,3,4,5,6,7\} = N$. Since $\rho(\mathbf{G}_{M_1^1}) \approx 1.325 < \rho(\mathbf{G}_{M_1^2}) = \sqrt{2} = 1.414$, there is a unique PCE in which the set of active agents is $D_1^2 = N$:

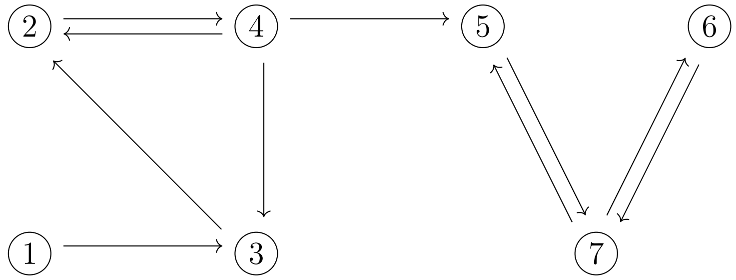

Figure D4

We now consider three different link additions:

(a) First, we add a link from agent 1 to agent 4 and consider the modified network $(N, \widehat{\mathbf{G}})$. The layer structure does *not* change since nobody from the layer-1 perfect sub-network, i.e., $\widehat{M_1^1} = \{2,3,4\}$, can reach agent 1. The layer-2 perfect sub-network is also the same, i.e. $\widehat{M_2^1} = \{5,6,7\}$. Thus, since the largest eigenvalues of each perfect subnetwork remains unchanged, the set of active players at the unique stable equilibrium is still N.

(b) Second, we add a link from agent 5 to agent 4. The layer structure completely changes, as there is, now, only one layer-1 perfect sub-network, which is given by $\widehat{M_1^1} = \{2,3,4,5,6,7\}$, with $\widehat{D}_1^1 = N$. However, the unique stable PCE stays qualitatively the same as it involves the same set of active agents, that is, N.[7]

(c) Finally, we add a link from agent 2 to agent 1. There are still two layers but the layer-1 perfect sub-network is slightly modified since it increases from $M_1^1 = \{2,3,4\}$ to $\widehat{M_1^1} = \{1,2,3,4\}$ while the layer-2 perfect sub-network stays the same, i.e. $M_2^1 = \{5,6,7\} = \widehat{M_2^1} = \{5,6,7\}$. However, there is a substantial difference since, now, $\rho(\widehat{\mathbf{G}}_{\widehat{M_1^1}}) \approx 1.521 > \sqrt{2} = \rho(\widehat{\mathbf{G}}_{\widehat{M_1^2}})$. As a result, the set of active players at the unique stable PCE *reduces* from $N = \{1,2,3,4,5,6,7\}$ to $\widehat{D}_1^1 = \{1,2,3,4\}$.

Example D5. Consider the network $(N, \mathbf{G})$ in Figure D5, without the red link. The network has two layers, and one perfect sub-network for each layer, which are $M_1^1 = \{2,3\}$ and $M_2^1 = \{4,5,6\}$, with $D_1^1 = \{1,2,3\}$ and $D_2^1 = \{1,2,3,4,5,6\} = N$. Since $\rho(\mathbf{G}_{M_1^1}) <$

[7] Of course, the effort of each agent will change since their eigenvector centrality changes.

$\rho(\mathbf{G}_{M_1^2})$, there is a unique PCE $\mathbf{x}^*$, where all agents are active. Their efforts are given by:

$$x_1^* = \frac{1}{6}\frac{V}{c},\ x_2^* = \frac{1}{12}\frac{V}{c},\ x_3^* = \frac{1}{24}\frac{V}{c},\ x_i^* = \frac{1}{8}\frac{V}{c} \text{ for } i = 4, 5, 6.$$

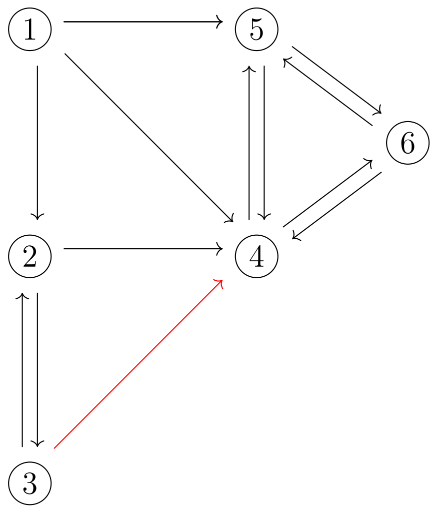

Figure D5

Now, let us illustrate Proposition C3. For that, consider the network which includes the red link from agent 3 to agent 4. The unique PCE is then given by

$$x_1^* = \frac{6}{39}\frac{V}{c},\ x_i^* = \frac{4}{39}\frac{V}{c},\ \forall i = 2, 3, 4, 5, 6.$$

Here $\mathcal{N}^-(i_0) = \{1, 2, 3\}$. Denote $u_i^* := u_i(x_i^*)$ and $\widehat{u}_i^* := u_i(\widehat{x}_i^*)$. Then, the payoff ratios are equal to

$$\frac{u_1^*}{u_i^*} = \frac{4}{3} < \frac{3}{2} = \frac{\widehat{u}_1^*}{\widehat{u}_i^*},\ \frac{u_2^*}{u_i^*} = \frac{3}{4} < 1 = \frac{\widehat{u}_2^*}{\widehat{u}_i^*},\ \frac{u_3^*}{u_i^*} = \frac{1}{3} < 1 = \frac{\widehat{u}_3^*}{\widehat{u}_i^*}, \text{ for } i = 4, 5, 6.$$

However, not everyone in $\mathcal{N}^-(i_0) = \{1, 2, 3\}$ beneficiate of an increase of their payoff when adding the link from 3 to 4. If we compare the relative payoffs, we have

$$\frac{\widehat{u}_1^*}{u_1^*} = \frac{36}{39} < 1,\ \frac{\widehat{u}_2^*}{u_2^*} = \frac{48}{39} > 1,\ \frac{\widehat{u}_3^*}{u_3^*} = \frac{96}{39} > 1,\ \frac{\widehat{u}_i^*}{u_i^*} = \frac{32}{39} < 1 \text{ for } i = 4, 5, 6.$$

E Beyond layer-generic graphs

In this section we assume that $\mathbf{G}$ is a weakly connected directed graph satisfying the no isolation assumption. As mentioned in the text, the set of PCE can be infinite if we drop the layer-genericity assumption.

E.1 Structure of the equilibrium set

We say that two perfect sub-networks are **disconnected** if there is no path from one to the other.

Proposition E6. *Let $\mathbf{x}^*$ be a PCE. Then, there exists a family of pairwise disconnected perfect sub-networks $\{M_{l_i}^{k_i}\}_{i=1,...,n}$ such that*

$$N_+(\mathbf{x}^*) = \cup_{i=1}^n D_{l_i}^{k_i}. \tag{E.1}$$

Proof of Proposition E6. Since $N_+(\mathbf{x})$ is a closed set of $\mathbf{G}$, we have that $\mathbf{x}$ is a positive eigenvector of $\mathbf{G}_{N_+(\mathbf{x})}$, associated to eigenvalue $\rho > 0$. By Lemma A4, that implies that $\mathbf{G}_{N_+(\mathbf{x})}$ is strongly nonnegative, and thus can be written

$$\mathbf{G}_{N_+(\mathbf{x})} = \begin{bmatrix} A_1 & A_{12} & ... & ... & ... & ... & ... & A_{1r} \\ 0 & A_2 & A_{23} & ... & ... & ... & ... & A_{2r} \\ ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & ... & 0 & A_s & A_{ss+1} & ... & ... & A_{sr} \\ 0 & ... & ... & 0 & A_{s+1} & 0 & ... & 0 \\ ... & ... & ... & ... & ... & ... & ... & ... \\ 0 & ... & ... & ... & ... & 0 & A_{r-1} & 0 \\ 0 & ... & ... & ... & ... & ... & 0 & A_r \end{bmatrix} \tag{E.2}$$

where $r > s$, $\rho(A_r) = ... = \rho(A_{s+1}) = \rho$, and $\rho(A_i) < \rho$ for $i = 1, ..., s$. Each A_{s+i} being such that $|V_{s+i}| \geq 2$ for $i = 1, ..., r-s$, each V_{s+i} corresponds to a layer-k_i perfect sub-network of $(N, \mathbf{G})$. Hence, taking $n := r - s$, there exists $k_1, ..., k_n, l_1, ..., l_n$ such that $A_{s+i} = \mathbf{G}_{M_{l_i}^{k_i}}$ for $i = 1, ..., n$.

We now show that $N_+(\mathbf{x}) = \cup_{i=1}^n D_{l_i}^{k_i}$. Since $N_+(\mathbf{x})$ is closed and $M_{l_i}^{k_i} \subset N_+(\mathbf{x})$ we have $D_{l_i}^{k_i} \subset N_+(\mathbf{x})$. Hence $\cup_{i=1}^n D_{l_i}^{k_i} \subset N_+(\mathbf{x})$. Now pick $j \in N_+(\mathbf{x})$. By property (ii) of the Frobenius normal form (see Definition A3), there exists some $i \in \{1, ..., n\}$ such that $j \rightrightarrows M_{l_i}^{k_i}$, meaning that $j \in D_{l_i}^{k_i}$. This concludes the proof. □

Proposition E7. *Let $(N, \mathbf{G})$ be a T-layer network and $\{M_{l_i}^{k_i}\}_{i=1...n}$ be pairwise disconnected perfect sub-networks. There exists a peer-consistent equilibrium (PCE) $\mathbf{x}^*$ with $N_+(\mathbf{x}) = \cup_{i=1}^n D_{l_i}^{k_i}$ if and only if there exists $\rho > 0$ such that*

$$\rho\left(\mathbf{G}_{M_{l_i}^{k_i}}\right) = \rho > \rho\left(\mathbf{G}_{Q_{l_i}^{k_i}}\right), \ \forall i = 1, ..., n. \tag{E.3}$$

Proof of Proposition E7. Again we use the Frobenius normal form to prove this result. Let $D := \cup_{i=1}^n D_{l_i}^{k_i}$ The matrix $\mathbf{G}_{\cup_{i=1}^n D_{l_i}^{k_i}}$ can be written as (B.4). By Lemma A4, this matrix hence admits a positive eigenvector if and only if

$$\rho(\mathbf{G}_{M_{l_n}^{k_n}}) = ... = \rho(\mathbf{G}_{M_{l_1}^{k_1}}) > \max_{i=1,...,s} \rho(A_i) = \max_{i=1,...n} \max_{i=1,...n} \rho\left(\mathbf{G}_{Q_{l_i}^{k_i}}\right).$$

this concludes the proof. □

In full generality, although the set of peer-confirming equilibria is no longer finite, we can still describe it in a simple way; it is always a finite union of convex sets. Recall that the set of simple equilibria is finite: there is at most one PCE with root M_l^k, for $k = 1, ..., T$ and $l = 1, ..., n_k$. Let $\{\rho_1, ..., \rho_P\}$ be the set of positive eigenvalues of $\mathbf{G}$. The set of simple equilibria can be written as

$$\bigcup_{p=1}^P S_p, \quad \text{where } S_p := \left\{\mathbf{x}^* : \ \mathbf{x}^* \text{ is a simple PCE with root } M_l^k \text{ such that } \rho(\mathbf{G}_{M_l^k}) = \rho_p\right\},$$

Proposition E8. *Given any network* $\mathbf{G}$ *the set of peer-consistent equilibria can be written as*

$$PCE = \bigcup_{p=1}^P \Lambda_p,$$

where Λ_p *is the convex polytope generated by* S_p*:* $\Lambda_p = Conv\,(S_p)$.

Proof of Proposition E8. We first show that $\bigcup_{p=1}^P \Lambda_p \subset PCE$. It amounts to showing that, if $S_p = \{\mathbf{x}^1, ..., \mathbf{x}^n\}$, and $\lambda_1, ..., \lambda_p$ are nonnegative numbers that sum to one then $\mathbf{x} := \sum_{j=1} \lambda_j \mathbf{x}^j$ is a PCE. We have

$$\mathbf{G}\mathbf{x} = \sum_{j=1}^n \lambda_j \mathbf{G}\mathbf{x}^j = \sum_{j=1}^p \lambda_j \rho_p \mathbf{x}^j = \rho_p \mathbf{x}.$$

Moreover $X = \sum_i x_i = \sum_i \sum_{j=1}^n \lambda_j x_i^j = \sum_{j=1}^n \lambda_j \sum_i s_i^j = \sum_{j=1}^n \lambda_j \frac{\rho_p}{\rho_p+1} = \frac{\rho_p}{\rho_p+1}$. Hence $\rho_p = \frac{cX}{V-cX}$ and this concludes this implication.

We now turn to the other inculsion. Let $\mathbf{x}$ be a PCE. Then, by Proposition E7, there exists $p \in \{1, ..., P\}$ and a family of pairwise disconnected perfect sub-networks $\{M_{l_i}^{k_i}\}_{i=1,...,n}$ such that $N_+(\mathbf{x}) = \cup_{j=1}^n D_{l_j}^{k_j}$, and $\rho\left(\mathbf{G}_{M_{l_j}^{k_j}}\right) = \rho_p > \rho\left(\mathbf{G}_{Q_{l_j}^{k_j}}\right)$, $\forall i = j, ..., n$. Call $\mathbf{x}^j$ the simple equilibrium associated to candidate set $D_{l_j}^{k_j}$, for $j = 1, ..., n$. We first define the following objects:

$$\tilde{M}_j := D_{l_j}^{k_j} \setminus \left(\cup_{m \neq j} D_{l_m}^{k_m}\right); \ \ \tilde{M} := \cup_{j=1}^n D_{l_j}^{k_j} \setminus \left(\cup_{j=1}^n \tilde{M}_j\right); \ \ \lambda_j := \frac{\sum_{i \in \tilde{M}_j} x_i}{\sum_{i \in \tilde{M}_j} x_i^j}.$$

Note that, by construction, the family $\left\{\tilde{M}, \tilde{M}_1, ..., \tilde{M}_n\right\}$ constitutes a partition of $\cup_{j=1}^n D_{l_j}^{k_j}$. Call $\mathbf{A}_j := \mathbf{G}_{\tilde{M}_j}$ and $\mathbf{A} := \mathbf{G}_{\tilde{M}}$. Then we can write

$$\mathbf{G}_{\cup_{j=1}^n D_{l_j}^{k_j}} = \begin{bmatrix} \mathbf{A} & \mathbf{B}_1 & ... & ... & \mathbf{B}_n \\ 0 & \mathbf{A}_1 & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & ... & 0 & \mathbf{A}_{n-1} & 0 \\ 0 & ... & ... & 0 & \mathbf{A}_n \end{bmatrix}$$

Be aware that this is not a Frobenius normal form because matrices $\mathbf{A}$ and $\mathbf{A}_j$ are in general not irreducible. However we know the following: $\rho(\mathbf{A}_j) = \rho_p$ for $j = 1, ..., n$ and $\rho(\mathbf{A}) < \rho_p$. Moreover, for $j = 1, ..., p$, $\mathbf{x}^j_{|\tilde{M}_j}$ is, by definition, a positive eigenvector of matrix $\mathbf{A}_j$. This is also true for $\mathbf{x}_{|\tilde{M}_j}$. The Frobenius normal form of $\mathbf{A}_j$ verifies the conditions of Lemma B12, (A_{s+1} corresponding here to $M_{l_j}^{k_j}$). As a result $\mathbf{x}_{|\tilde{M}_j}$ and $\mathbf{x}^j_{|\tilde{M}_j}$ are proportionnal:

$$\mathbf{x}_{|\tilde{M}_j} = \alpha_j \mathbf{x}^j_{|\tilde{M}_j}. \tag{E.4}$$

Since $\mathbf{x}_{|\cup_{j=1}^n D_{l_j}^{k_j}}$ is an eigenvector of $\mathbf{G}_{\cup_{j=1}^n D_{l_j}^{k_j}}$ associated to ρ_p we have

$$\rho_p \mathbf{x}_{|\tilde{M}} = \mathbf{A}\mathbf{x}_{|\tilde{M}} + \sum_{j=1}^n \mathbf{B}_j \mathbf{x}_{|\tilde{M}_j},$$

and thus, since $\mathbf{I} - \rho_p^{-1}\mathbf{A}$ is invertible,

$$\rho_p \mathbf{x}_{|\tilde{M}} = \left(\mathbf{I} - \rho_p^{-1}\mathbf{A}\right)^{-1} \sum_{j=1}^n \mathbf{B}_j \mathbf{x}_{|\tilde{M}_j} = \left(\mathbf{I} - \rho_p^{-1}\mathbf{A}\right)^{-1} \sum_{j=1}^n \alpha_j \mathbf{B}_j \mathbf{x}^j_{|\tilde{M}_j}.$$

On the other hand $\mathbf{x}^j_{|\tilde{M}\cup\tilde{M}_j}$ is an eigenvector of $\mathbf{G}_{|\tilde{M}\cup\tilde{M}_j}$ associated to ρ_p. hence

$$\rho_p \mathbf{x}^j_{|\tilde{M}} = \mathbf{A}\mathbf{x}^j_{|\tilde{M}} + \mathbf{B}_j \mathbf{x}_{|\tilde{M}_j},$$

that is,

$$\rho_p \mathbf{x}^j_{|\tilde{M}} = \left(\mathbf{I} - \rho_p^{-1}\mathbf{A}\right)^{-1} \mathbf{B}_j \mathbf{x}_{|\tilde{M}_j}.$$

Finally we get

$$\rho_p \mathbf{x}_{|\tilde{M}} = \sum_{j=1}^n \alpha_j \rho_p \mathbf{x}^j_{|\tilde{M}_j},$$

i.e. $\mathbf{x}_{|\tilde{M}} = \sum_{j=1}^n \alpha_j \mathbf{x}^j_{|\tilde{M}_j}$. Combining this equality with (E.4) and the fact that $\mathbf{x}^m_{|\tilde{M}_j} = 0$ when $j \neq m$, we obtain that

$$\mathbf{x} = \sum_{j=1}^n \alpha_j \mathbf{x}^j$$

Now $\mathbf{x}$ and $\mathbf{x}^j$ being all associated to the same eigenvalue ρ_p we necessarily have $X = X^j = \frac{\rho_p}{\rho_p+1}$ for $j = 1, ..., n$. As a result $\sum_{j=1}^n \alpha_j = 1$ and this concludes the proof. □

Remark E1. *When* $\mathbf{G}$ *is a layer-generic network, then every component is degenerate, i.e., they reduce to a singleton. In full generality, in a given component, the largest eigenvalue of the subgraph of active players is invariant.*

E.2 Example

We illustrate this in the following example.

Example E6. Non-finiteness of equilibria

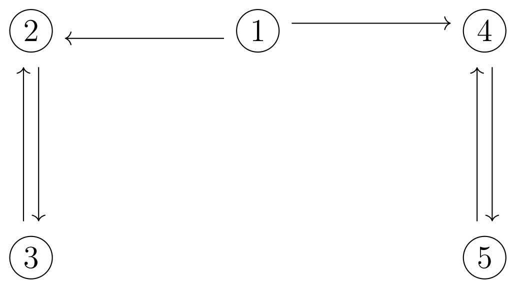

Figure E6: Infinite set of PCE in a 1-layer network

Consider the network $(N, \mathbf{G})$ in Figure E6 with $N = \{1, 2, \cdots, 5\}$. Both $(M_1^1, \mathbf{G}_{M_1^1})$ with $M_1^1 = \{1, 2\}$, and $(M_2^1, \mathbf{G}_{M_2^1})$ with $M_2^1 = \{3, 4\}$ are layer-1 perfect sub-networks of $(N, \mathbf{G})$. Moreover we have $\rho \mathbf{G}_{M_1^1}) = \rho(\mathbf{G}_{M_2^1}) = 1$. Consequently, the set of peer-consistent equilibria is not finite since the network is non layer-generic. More precisely:

$$PCE = \left\{ \frac{V}{12c}(1, \lambda, \lambda, 1-\lambda, 1-\lambda) : \ \lambda \in [0, 1] \right\}.$$

◇