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Abstract

We consider the problem faced by a group of players who need to collectively decide what public signal to acquire, and how to share its cost, before voting on whether to take some action, when each player is privately informed about his state-dependent payoffs from the action. We characterize the welfare maximizing mechanism for information acquisition taking into account the subsequent voting game. We identify novel distortions that arise from the information asymmetry and from the fact that after observing the signal realization, the players vote independently of their actions in the mechanism.

JEL Classification: N/A

Keywords: collective decision-making, Mechanism-Design, Information-design, rational inattention, Public Good Provision

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Collective Information Acquisition

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September 2020

Abstract

We consider the problem faced by a group of players who need to collectively decide what public signal to acquire, and how to share its cost, before voting on whether to take some action, when each player is privately informed about his state-dependent payoffs from the action. We characterize the welfare maximizing mechanism for information acquisition taking into account the subsequent voting game. We identify novel distortions that arise from the information asymmetry and from the fact that after observing the signal realization, the players vote independently of their actions in the mechanism.

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1 Introduction

Consider a group of individuals who need to make a collective decision. For instance, the board of directors of a firm who need to vote on a merger or an acquisition, a committee that needs to vote on whether to hire a candidate, or a congressional committee that votes on a proposal for a regulation. The optimal action depends on the state of nature which is uncertain. All group members want to take the right decision, and hence, to reduce their uncertainty, they may wish to have some evidence or analysis presented to them before reaching a decision. For example, the board of directors may hire a consulting firm to collect and analyze information about the market, the hiring committee may invest

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time in evaluating the candidate's prior work, and the congressional committee may call on expert witnesses. These means of reducing uncertainty are all costly in terms of effort, time and money. However, the group members may differ in the cost they incur when a wrong decision is made, and furthermore, this cost may be private information. In light of this, how should the group decide on the amount of information to acquire? How is this decision affected by the fact that the group members will base their collective action on the acquired information?

To address these questions, we study the following model. A group of n players is faced with a binary decision. There are two states of nature, and all players would like the action to match the state. However, they differ in their disutility from a mismatch, and this disutility is private information. Prior to making the binary decision, the players have the opportunity to collectively acquire a costly public signal about the state. The players then proceed in two steps. First, they have to agree on which signal to acquire and how to distribute its cost. Second, they all observe the signal realization and take a supermajority vote on the binary decision. The default action is the one chosen when no signal is acquired. To explore the bounds on the social surplus that the group can achieve, we abstract from the particular details of the bargaining over information in the first step, and take a mechanism-design approach. That is, we characterize the optimal feasible mechanism for deciding which signal to acquire, taking into account the incentive and participation constraints as well as the second stage voting game.

Our group decision problem may be viewed as a variant of rational inattention à la Sims (2003). In contrast to the single player case on which this literature has focused, we study a model of *collective* rational inattention: A group needs to agree on which signal to acquire, taking into account the trade-off between the cost and benefit of more precise information. There are three key differences between the problem we study and the problem of individual rational inattention. First, the final decision following a signal realization is determined by an *equilibrium* in a game. Second, the group members may disagree on the benefit from each signal. Finally, in order to aggregate the individuals' willingness to pay for signals, the individuals need to disclose their private information.

A signal can be represented by a probability distribution over the posteriors it induces. Following the rational inattention literature (in particular, the posterior-based approach of Caplin, Dean and Leahy, 2020), we assume that a signal's cost is proportional to the expected KL-divergence between the prior belief and the induced posteriors (or the mutual information between the state and the signal realization), which represents the reduction in uncertainty caused by the signal. This captures situations where there is an overwhelming amount of information available and the difficulty is in processing and understanding that information (see, e.g., Maćkowiak, Matějka and Wiederholt, 2018). While our analysis focuses on this cost specification, our methodology is applicable to a wider range of cost functions. In particular, we show in the supplementary appendix that our results extend to the case where a signal's cost is proportional to the variance of the induced posteriors.

In a standard mechanism-design problem the designer is free to choose the mapping from reports to outcomes. However, this is not the case in our environment in which the designer cannot control the outcome of the second stage voting game. To overcome this difficulty, we introduce an auxiliary direct revelation mechanism in which the players report their types, the planner decides on the signal and then votes on their behalf according to their reports (in other words, it is as if the players commit to vote according to their reported types). We show that the optimal auxiliary mechanism satisfies a property we call "non-wastefulness". This property means that any acquired signal must be instrumental for decision-making: It must have at least one realized posterior for which the collective action is different than if no signal was acquired. We solve for the optimal auxiliary mechanism and then show that its solution coincides with the solution of the optimal "actual" mechanism (which we refer to as the "second-best") in which the designer does not control the players' actions in the ensuing voting game.

To solve for the optimal auxiliary mechanism we first establish that there is no loss of generality in restricting attention to signals that induce only two posteriors on one of the states: A high posterior, which is weakly above the prior, and a low posterior, which is weakly below the prior, where the expected posterior must equal the prior. Thus, the design problem reduces to choosing the mapping from reported types to the following variables: The high posterior r_H, the probability q of the high posterior being realized and each player's share in the cost. We show that the design problem can be written as a variant of a public-good provision problem (where the signal is the public good), with the new twist that the level of the public good affects actions taken in a subsequent game. In this reformulation of our design problem, q plays the same role as the allocation rule in a standard public-good problem in the sense that a player reports truthfully only if the interim expected probability q is non-decreasing in his type.

In characterizing the optimal mechanism, the crux of the proof is the argument that establishes the monotonicity of the interim expected q, which is necessary for incentive-compatibility. The difficulty arises from the fact that for some type realizations, the optimal signal is determined by the binding non-wastefulness constraint. It turns out that in the optimal auxiliary mechanism, q *itself* (and not only its interim-expected value) is monotonic in the types. Consequently, the truthful equilibrium in the auxiliary mechanism can be attained in *dominant* strategies. This allows us to show that the solution of the optimal auxiliary mechanism coincides with the second-best mechanism.

The second-best mechanism exhibits the following features. In one subset of the type space, no signal is acquired. In a second subset, the acquired signal is at its optimal interior solution. In a third subset of the type space, non-wastefulness is a binding constraint in the sense that the high posterior is at its minimal level that induces the non-default action in the voting game. This last subset illustrates the distortion caused by the presence of a second stage voting game, which is outside the control of the designer.

In comparison to the solution under common knowledge of types (the "first-best" solution), there is *under-provision* of information in the sense that whenever a signal is not acquired in the first-best solution, it is not acquired in the second-best, but the converse is not true. A second distortion that occurs in the second-best mechanism is that for any profile of types, the probability q of the high posterior is *lower*, while the high posterior r_H is *higher*. This observation also means that the distortion due to the second-stage voting game - namely, the non-wastefulness constraint - is *not* exacerbated in the second-best mechanism: It is never the case that this constraint is binding in the auxiliary mechanism but not in the first-best.

Our analysis combines information-design with mechanism-design in the sense that the designer needs to elicit the players' private types in order to implement the optimal signal structure. In a linear environment with a single player, Kolotilin et. al (2017) showed that the optimal signal can be implemented without relying on the player's private information. However, it is well known that in environments with multiple interacting players (as in Bergemann and Morris, 2013, Alonso and Câmara, 2016, Taneva, 2019, and Mathevet, Perego, and Taneva, 2020) ignoring the players' private information is suboptimal.

Several recent works have addressed the problem of designing information for a group of voters. Notable papers include Wang (2013), Schnakenberg (2015), Alonso and Câmara (2016), Bardhi and Guo (2018), Chan et al. (2019) and Arieli and Babichenko (2019). These studies characterize the signal that maximizes the probability that in equilibrium voters vote on the outcome favorable to the sender. They differ in whether the designed signals are private or public, and in the class of voting rules that is considered. There are two key differences between these papers and ours. First, in these papers the voters' state-dependent utilities are *commonly known* (i.e., voters have no private information), and hence, in order to design the optimal signal there is no need to elicit information from the voters. Second, in these papers signals are *costless*, and the problem is to find the signal that induces voters to coordinate on an equilibrium which is best for the sender.

The question we study is also related to the problem of designing voting rules that incentivize the voters to acquire costly information. Persico (2003) characterizes the optimal size and voting threshold that efficiently aggregates information when each voter needs to incur a cost to acquire a private binary signal. Gershkov and Szentes (2009) extend the analysis to a broader class of voting mechanisms. Our approach differs in that voters' willingness-to-pay for information is private and the signal they acquire is public. We fix the voting rule and look for the optimal signal, taking into account that this signal depends on the voters' private information, and taking into account that the signal realization affects voting behavior.

An alternative approach to the study of collective information acquisition is analyzed by Chan et al. (2017). They consider a dynamic model where in each point in time a group receives an exogenous signal and needs to vote on whether to stop and vote on a binary action, or continue and receive additional signals. Unlike us, they study a stopping problem in which the signal is exogenously given and the players' preferences are commonly known.[1]

The remainder of the paper is organized as follows. Section 2 presents the model. The mechanism-design problem is presented and analyzed in Section 3. We begin by describing the optimization problem that stems from the optimal auxiliary mechanism in Section 3.1 and characterize the solution in Section 3.2. In Section 3.3 we show that the optimal auxiliary mechanism coincides with the second-best, which we compare with the first-best solution in Section 3.4. Concluding remarks are presented in Section 4. All proofs are relegated to the appendix.

2 Model

There are n players who have to jointly agree on a decision $a \in A = \{0, 1\}$. Following the literature on strategic voting (most notably, Feddersen and Pesendorfer, 1998), we assume that each player's payoff, u_i, depends on the joint action, on his type $\theta_i \in \Theta$ and on the state of the world $\omega \in \Omega = \{0, 1\}$ as follows:

$$u_i(a, \omega, \theta_i) = \begin{cases} 1 & if \quad a = \omega \\ \theta_i & if \quad a = 1, \omega = 0 \\ 1 - \theta_i & if \quad a = 0, \omega = 1 \end{cases}$$

[1] For additional related works that take a collective search approach to sequential information gathering by a group, see the references in Chan et al. (2017).

We assume that the players do not observe the realization of ω and have the common prior belief that the probability that $\omega = 1$ is p. In addition, each player i privately and independently draws a type θ_i from a common distribution F on the interval $[0, 1-p]$ (we explain below why we assume that $\theta_i < 1-p$). We assume that F admits a density f that is strictly positive, continuously differentiable and bounded over $[0, 1-p]$. Let $v(\theta_i) \equiv \theta_i - \frac{1-F(\theta_i)}{f(\theta_i)}$ denote the virtual valuation of the player's type θ_i. We assume that F is regular, i.e. $v(\theta_i)$ is increasing in θ_i.

Our specification of the utility function u_i implies that player i weakly prefers the joint decision $a = 1$ if and only if, given any information he has, his posterior belief on $\omega = 1$ is at least $1 - \theta_i$ (the reason is that if the posterior belief on $\omega = 1$ is r, then the action $a = 1$ yields an expected payoff of $r \cdot 1 + (1-r) \cdot \theta_i$ while the action $a = 0$ yields an expected payoff of $r \cdot (1-\theta_i) + (1-r) \cdot 1$). From our assumption that $p < 1 - \theta_i$ for every θ_i it follows that without further information on the state each player prefers the action $a = 0$.

Before making the joint decision (in a manner described below), the players have the opportunity to acquire a public signal on the state ω. A signal can be represented by a probability distribution over posterior beliefs on $\omega = 1$, such that the expected posterior belief on $\omega = 1$ equals the prior p. To simplify the exposition we assume that the distribution is discrete, with countably many possible realizations. We denote by q_j the probability that the posterior belief on the state $\omega = 1$ is r_j and by J the total number of posteriors (where J can be infinite). We then have:

$$\sum_{j \in \{1,...,J\}} q_j \cdot r_j = p. \tag{1}$$

where $0 < q_j \leq 1$ and $0 \leq r_j \leq 1$ for all $j \in \{1, ..., J\}$, and $\sum_{j \in \{1,...,J\}} q_j = 1$.

The players can decide to acquire no information. Note that this option is equivalent to choosing the degenerate signal that puts all the probability mass on a single posterior belief which is equal to the prior p (i.e. $J = 1$, $q_1 = 1$ and $r_1 = p$).

Signals are costly. The cost of acquiring a signal $\{(q_j, r_j)\}_{j=1}^J$ is proportional to the expected KL-divergence (or relative entropy) between the posteriors and the prior:[2]

$$c\left(\{(q_j, r_j)\}_{j=1}^J\right) = \kappa \cdot \textstyle\sum_{j=1}^J q_j D_{KL}(r_j, p) \tag{2}$$

[2] While the details of our results depend on the particular form of this cost function, the qualitative features of our characterization are not limited to it. In the supplementary appendix we show that our results extend to an alternative cost function that is proportional to the variance of the induced posteriors on the high state.

where κ is some positive constant, and:[3]

$$D_{KL}(r, r') \equiv r \log \frac{r}{r'} + (1-r) \log \frac{1-r}{1-r'}. \quad (3)$$

The cost needs to be shared among the players. We denote by t_i player i's share in covering the cost of the signal so that $\sum_{i=1}^{n} t_i = c\left(\{(q_j, r_j)\}_{j=1}^{J}\right)$. The net payoff of type θ_i from action a in state ω is therefore given by $u_i(a, \omega, \theta_i) - t_i$.

Player i's share in the signal's cost t_i can be interpreted either as his share in the collective effort of processing the acquired information (e.g., the amount of documents he needs to summarize, or the time involved in organizing the data), or as his share in the monetary cost of the signal (e.g., when different departments in an organization use their budgets to pitch in for the cost of hiring a consultant). As is common in the examples described in the introduction, we assume the group members cannot make any monetary transfers that are conditional on their votes. This can follow from institutional constraints that prohibit such vote buying, or because the votes are secret, or because such monetary arrangements cannot be enforced.

As mentioned in the Introduction, the problem of deciding on which signal to acquire is akin to the problem of choosing the optimal level of a non-excludable public good. Following this literature (see, e.g. Mailath and Postlewaite, 1990, and Hellwig, 2013) we assume that participation is voluntary in the sense that each player can veto the public signal from being provided at all. Alternatively, one can view a player who refuses to participate as one who leaves the group and does not benefit from the collective action. In that case we normalize the outside option's payoff to zero.

After the players agree on the signal to acquire, they all observe its realization. The players then vote on the collective decision using an $m-$majority rule: the action $a = 1$ is chosen if, and only if, at least m players vote for this option. Otherwise, the default action $a = 0$ is chosen. We assume that the players do not choose weakly dominated strategies. Thus, player i votes for $a = 1$ if and only if the realized posterior belief that the state is $\omega = 1$ is above $1 - \theta_i$. Consequently, the alternative $a = 1$ is chosen if and only if the realized posterior belief that the state is $\omega = 1$ is above $1 - \theta^{(n-m+1)}$, where $\theta^{(k)}$ is the k^{th} smallest element in θ. For example, if choosing the non-default action $a = 1$ requires unanimity, i.e. $m = n$, then for this action to be chosen the realized posterior belief has to be larger than $1 - \theta^{(1)}$, where $\theta^{(1)}$ is the smallest element in θ. Note that,

[3] Since there are only two states, we represent a distribution over the states by the probability on $\omega = 1$. Thus, the divergence between two distributions can be written as a function of the probabilities that each distribution puts on the state $\omega = 1$.

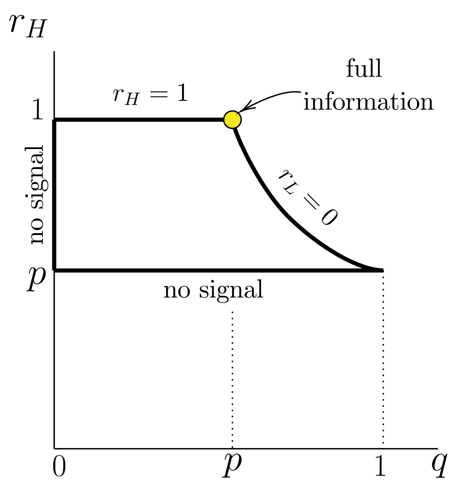

Figure 1: Feasible signals

given $\theta = \{\theta_1, ..., \theta_n\}$, a signal induces a probability distribution over the outcomes of the vote.

The players' objective is to choose the signal that maximizes the ex-ante surplus, taking into account the voting stage that takes place after the signal is realized. Our first observation is that because there are only *two* available collective actions, signals that induce only two posterior beliefs on $\omega = 1$ (one on either side of the prior belief) dominate signals with more posterior beliefs on $\omega = 1$:

Lemma 1 *For any signal that induces more than two posterior beliefs, there exists a signal that induces only two posterior beliefs, generates the same distribution over actions for each realization of state and types and has a strictly lower cost.*

This result is straightforward in standard information design problems where signals are costless. In such settings all that matters is the distribution over the actions in each state, and this distribution can be replicated by signals that induce two posterior beliefs when there are only two actions. In our setup, signals are costly and the cost depends on the entire distribution of posteriors. However, the convexity of our cost function implies that the optimal mechanism does not need to employ signals with more than two posteriors (for an analogous result in a model of individual rational inattention see, e.g., Lemma 1 in Matějka and McKay, 2015).

In light of this result, we restrict attention to signals that induce at most two posterior beliefs. Thus, a signal can be represented by a pair (q, r_H), where $q \in [0, 1]$ is the probability that the posterior belief on $\omega = 1$ is $r_H \geq p$. Equation (1) then implies that with probability $1 - q$ the other posterior belief induced by the signal is $r_L \equiv (p - qr_H)/(1 - q) \leq p$. Thus, when the realized posterior belief is r_L all players agree that the optimal action is $a = 0$. When the realized posterior belief is r_H there are m players who prefer $a = 1$ over $a = 0$ if and only if $r_H \geq 1 - \theta^{(n-m+1)}$. Notice that, since $r_L \geq 0$, then it must be the case that $p \geq q r_H$. Figure (1) illustrates the set of all possible signals, depicted on the plane of q and r_H.

Choosing $q = 0$ or $q = 1$ is equivalent to purchasing no signal (the cost in this case is 0, by Equation 2). We say that a signal is *informative* if $q \in (0, 1)$. We say that a signal is *instrumental* for a type profile θ if at least for one of the signal's realizations there is an $m-$majority for the non-default action $a = 1$. This means that a signal (q, r_H) is instrumental for θ if $r_H \geq 1 - \theta^{(n-m+1)}$.

3 A mechanism for information acquisition

By the revelation principle there is no loss of generality in restricting attention to direct revelation mechanisms in the first stage of the players' interaction, i.e., when they decide on which signal to acquire. We define an *actual* direct mechanism to be a vector of functions $\langle q, r_H, t_1, ..., t_n \rangle$, where $q : \Theta^n \to [0, 1]$, $r_H : \Theta^n \to [p, 1]$ and $t_i : \Theta^n \to \mathbb{R}$ for every $i \in \{1, .., n\}$. Thus, following a profile of reports $\hat{\theta} = (\hat{\theta}_1, ..., \hat{\theta}_n)$, with probability $q(\hat{\theta})$ the players end up with the posterior probability $r_H(\hat{\theta})$ on the state $\omega = 1$ and with probability $1 - q(\hat{\theta})$ they end up with the posterior probability $r_L(\hat{\theta})$ on that state, where $r_L(\hat{\theta}) \equiv (p - q(\hat{\theta}) \cdot r_H(\hat{\theta}))/(1 - q(\hat{\theta}))$. In addition, each player i pays his share $t_i(\hat{\theta})$.

In the actual mechanism the designer cannot directly control the outcome of the second stage voting game. Thus, a player who misreports his true type to the mechanism (say, in order to reduce his share in the cost) retains his ability to vote according to his true preferences in the second stage. As a step towards characterizing the optimal mechanism, we proceed by considering *auxiliary* (direct) mechanisms in which, in addition to choosing which signal to acquire and how to distribute the costs, the mechanism also votes in the name of the players in the second stage. Thus, an auxiliary mechanism effectively chooses the collective action $a = 1$ whenever $r_H > 1 - \theta^{(n-m+1)}$, and the collective action $a = 0$ otherwise. In other words, we assume that the players commit to vote according to their reported types and not their true types. Our focus on direct auxiliary mechanisms follows from the revelation principle which holds in this environment.

Formally, an auxiliary mechanism is an actual mechanism augmented by two decision functions, $a_H(\hat{\theta})$ and $a_L(\hat{\theta})$, which are the collective actions chosen by the mechanism when the posterior beliefs r_H and r_L are realized. We then have:

$$a_H(\hat{\theta}) = \begin{cases} 1 & r_H(\hat{\theta}) > 1 - \hat{\theta}^{(n-m+1)} \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

$$a_L(\hat{\theta}) = 0 \tag{5}$$

Our next result establishes that the surplus-maximizing auxiliary mechanism does at least as well as the surplus-maximizing actual mechanism:

Proposition 1 *The highest expected surplus achievable by an auxiliary mechanism is weakly higher than the highest expected surplus achievable by an actual mechanism.*

This result is not as straightforward as it might seem, because it is not immediately clear that restricting the mechanism to vote according to the first stage reports does not limit the set of implementable outcomes. Nevertheless, in the proof we show that any equilibrium play path in the actual mechanism and the ensuing voting game can be replicated by the auxiliary mechanism. The reason is that deviations from truth-telling in the auxiliary mechanism are more costly than in the actual mechanism.

In light of Proposition 1 we begin by looking for the auxiliary mechanism that attains the highest social surplus. We will then show that the equilibrium that attains this surplus can be replicated by the actual mechanism and the ensuing voting game.

Fix a player i and suppose that the remaining players report their types truthfully. The expected utility of player i of type θ_i who reports $\hat{\theta}_i$ is then given by:

$$\begin{aligned} V\left(\theta_i, \hat{\theta}_i\right) = \mathbb{E}_{\theta_{-i}} \Big[& q(\hat{\theta}_i, \theta_{-i}) \cdot [r_H(\hat{\theta}_i, \theta_{-i}) \cdot u(a_H(\hat{\theta}_i, \theta_{-i}), 1, \theta_i) + (1 - r_H(\hat{\theta}_i, \theta_{-i})) \cdot u(a_H(\hat{\theta}_i, \theta_{-i}), 0, \theta_i)] \\ & + (1 - q(\hat{\theta}_i, \theta_{-i})) \cdot [r_L(\hat{\theta}_i, \theta_{-i}) \cdot u(0, 1, \theta_i) + (1 - r_L(\hat{\theta}_i, \theta_{-i})) \cdot u(0, 0, \theta_i)] \\ & - t_i(\hat{\theta}_i, \theta_{-i}) \Big] \end{aligned} \tag{6}$$

where $\theta_{-i} \in \Theta^{n-1}$ represents the vector of true types of all players other than i, and $\mathbb{E}_{\theta_{-i}}$ is evaluated according to the probability distribution of the true types θ_{-i}.

Since we are interested in the auxiliary mechanism that maximizes the total surplus, it is useful to represent the players' payoffs as the expected *gain* from information (rather than the utility per-se) compared to the case in which the players do not participate in the mechanism and no information is acquired. Note that in the latter case, the default action $a = 0$ is chosen and type θ_i's payoff is $p \cdot (1 - \theta_i) + (1 - p)$. Thus, the gain from information of type θ_i of player i who reports $\hat{\theta}_i$ is given by:

$$U(\theta_i, \hat{\theta}_i) = V(\theta_i, \hat{\theta}_i) - (p \cdot (1 - \theta_i) + (1 - p)) \tag{7}$$

To simplify the exposition, when all players report truthfully, we use the shorter notation $U(\theta_i) \equiv U(\theta_i, \theta_i)$.

The objective of the mechanism is to maximize the total ex-ante expected gain from signals under truthful reporting. Since the players' preferences are quasi linear, this is equivalent to maximizing the sum:

$$\sum_{i=1}^{n} \mathbb{E}_{\theta_i} U(\theta_i) \,. \tag{OBJ}$$

The auxiliary mechanism has to be ex-post budget balanced: the cost of any signal that is acquired has to be fully covered by the players. In what follows we slightly weaken this requirement and allow the auxiliary mechanism to be balanced only *ex-ante*, so that the cost of the acquired signal has to be covered only on average (that is, we allow the mechanism to have a budget deficit in some cases, so long as on average the costs are fully covered):

$$\mathbb{E}_{\theta} \sum_{i=1}^{n} t_i(\theta) = \mathbb{E}_{\theta} \left[c\left(q(\theta), r_H(\theta) \right) \right] \tag{BB}$$

However, as is well known (see, e.g., Borgers, 2015, p.47), if a mechanism is ex-ante budget balanced, one can modify the transfers to satisfy ex-post budget balanceness without affecting the interim expected transfers or the incentives for truthful reporting. That is, if a mechanism is incentive-compatible, individually rational and ex-ante budget balanced, then there is another mechanism that achieves the same allocation of types to signals, and which is also incentive-compatible and individually rational but is ex-post budget balanced. In light of this, we will focus on ex-ante budget-balance in the analysis that follows.

The players cannot be forced to participate in the mechanism. Since no information is acquired when a player opts out, the gain from participation must be non-negative for any type θ_i of any player i:

$$U(\theta_i) \geq 0. \tag{IR}$$

Finally, to guarantee that truthtelling is indeed an equilibrium, the following incentive compatibility condition must hold:

$$U(\theta_i) \geq U(\theta_i, \hat{\theta}_i) \tag{IC}$$

for any type θ_i of any player i, and for any report $\hat{\theta}_i$.

In sum, we look for an auxiliary mechanism that maximizes (OBJ) subject to the constraints (IR), (IC) and (BB).

3.1 The optimization problem

Fix a player i and suppose that all other players $-i$ report truthfully $\theta_{-i} \in \Theta^{n-1}$. If player i's report is such that $r_H(\hat{\theta}) \geq 1 - \hat{\theta}^{(n-m+1)}$, where $\hat{\theta} = (\hat{\theta}_i, \theta_{-i})$, then player i's utility is given by $q(\hat{\theta}) \cdot (\theta_i - (1 - r_H(\hat{\theta}))) - t_i(\hat{\theta})$.[4] If player i's report is such that $r_H(\hat{\theta}) \leq 1 - \hat{\theta}^{(n-m+1)}$ then no signal is acquired and player i's utility is $-t_i(\hat{\theta})$. Thus, we can rewrite the utility of type θ_i of player i who reports $\hat{\theta}_i$ when all other players report truthfully (Equation 7) as follows:

$$U(\theta_i, \hat{\theta}_i) = \int_{\theta_{-i}|r_H(\hat{\theta}_i,\theta_{-i})\geq 1-(\hat{\theta}_i,\theta_{-i})^{(n-m+1)}} q\left(\hat{\theta}_i, \theta_{-i}\right) \cdot \left[\theta_i - \left(1 - r_H\left(\hat{\theta}_i, \theta_{-i}\right)\right)\right] dF^{n-1}\left(\theta_{-i}\right)$$
$$- \int_{\theta_{-i}} t_i(\hat{\theta}_i, \theta_{-i}) dF^{n-1}\left(\theta_{-i}\right)$$

To express $U(\theta_i, \hat{\theta}_i)$ more compactly, we introduce the following notations. Given a report $\hat{\theta}_i$, denote by $Q(\hat{\theta}_i)$ the expected probability that the auxiliary mechanism chooses the action $a = 1$. Denote by $M(\hat{\theta}_i)$ the expected probability that the auxiliary mechanism chooses $a = 1$ but the state is $\omega = 0$ (this is the probability that the auxiliary mechanism deviates from the default action when it shouldn't). Denote by $T_i(\hat{\theta}_i)$ the expected payment of player i. Thus:

$$Q(\hat{\theta}_i) = \int_{\theta_{-i}|r_H(\hat{\theta}_i,\theta_{-i})\geq 1-(\hat{\theta}_i,\theta_{-i})^{(n-m+1)}} q(\hat{\theta}_i, \theta_{-i}) dF^{n-1}\left(\theta_{-i}\right)$$
$$M(\hat{\theta}_i) = \int_{\theta_{-i}|r_H(\hat{\theta}_i,\theta_{-i})\geq 1-(\hat{\theta}_i,\theta_{-i})^{(n-m+1)}} q(\hat{\theta}_i, \theta_{-i}) \cdot \left(1 - r_H(\hat{\theta}_i, \theta_{-i})\right) dF^{n-1}\left(\theta_{-i}\right)$$
$$T_i(\hat{\theta}_i) = \int_{\theta_{-i}} t_i(\hat{\theta}_i, \theta_{-i}) dF^{n-1}\left(\theta_{-i}\right)$$

The expected utility of player i with type θ_i who reports $\hat{\theta}_i$ is then given by:

$$U\left(\theta_i, \hat{\theta}_i\right) = Q(\hat{\theta}_i) \cdot \theta_i - M(\hat{\theta}_i) - T_i(\hat{\theta}_i) \tag{8}$$

Note that our specification of the players' utility has the convenient feature that it is *as if* a player gets a payoff of θ_i every time the collective action 1 is chosen, but he pays a penalty ($M(\hat{\theta}_i)$) that is equal to the probability that this is the *wrong* collective action.

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[4] To see this, plug in equations (4), (5) and (6) into equation (7) and simplify.

The designer's objective function (OBJ) can therefore be written as

$$\sum_{i=1}^{n}\int_{0}^{1-p}\left[Q\left(\theta_{i}\right)\cdot\theta_{i}-M\left(\theta_{i}\right)-T_{i}\left(\theta_{i}\right)\right]dF(\theta_{i}) \tag{9}$$

while incentive compatibility (i.e., Equation IC) requires

$$Q(\theta_{i})\cdot\theta_{i}-M(\theta_{i})-T_{i}(\theta_{i})\geq Q(\hat{\theta}_{i})\cdot\theta_{i}-M(\hat{\theta}_{i})-T_{i}(\hat{\theta}_{i})$$

for all $\hat{\theta}_i$ and θ_i and every player i. Note that $U(\theta_i)$ is the upper envelope of a family of affine functions in θ_i, and is therefore convex. If follows that an auxiliary mechanism satisfies incentive compatibility if and only if $Q(\theta_i)$ is non-decreasing and $U'(\theta_i) = Q(\theta_i)$ (see, e.g. Krishna, 2010, p. 64). Thus $U(\theta_i) = \int_0^{\theta} Q(x)\,dx - M(0) - T_i(0)$ and therefore

$$T_{i}(\theta_{i})=Q(\theta_{i})\cdot\theta_{i}-M(\theta_{i})-\int_{0}^{\theta}Q\left(x\right)dx+M(0)+T_{i}(0). \tag{10}$$

Player i's ex-ante expected utility is given by $\int_0^{1-p} U(\theta_i)dF(\theta_i)$. Applying integration by parts we obtain:

$$\int_{0}^{1-p}U(\theta_{i})dF(\theta_{i})=\int_{0}^{1-p}Q(\theta_{i})[\frac{1-F(\theta_{i})}{f(\theta_{i})}]dF\left(\theta_{i}\right)-T_{i}(0)-M(0)$$

Plugging in $U(\theta_i) = Q(\theta_i)\cdot\theta_i - M(\theta_i) - T_i(\theta_i)$ and rearranging yields:

$$\int_{0}^{1-p}T_{i}(\theta_{i})dF(\theta_{i})=T_{i}(0)+M(0)+\int_{0}^{1-p}\left[v(\theta_{i})\cdot Q(\theta_{i})-M(\theta_{i})\right]dF(\theta_{i}) \tag{11}$$

where $v(\theta_i)$ is the virtual valuation of type θ_i.

Substituting Equation (11) into Equation (9) yields that the designer's problem is to maximize

$$\sum_{i=1}^{n}\int_{0}^{1-p}\left[\frac{1-F(\theta_{i})}{f(\theta_{i})}Q(\theta_{i})\right]dF(\theta_{i})-\sum_{i=1}^{n}T_{i}(0)-\sum_{i=1}^{n}M(0) \tag{12}$$

subject to the following ex-ante budget balance constraint (which is obtained by plugging Equation 11 into Equation BB):

$$\int_{\theta}c(q(\theta),r_{H}(\theta))dF^{n}(\theta)=\sum_{i=1}^{n}T_{i}(0)+\sum_{i=1}^{n}M(0)+\sum_{i=1}^{n}\int_{0}^{1-p}\left[v(\theta_{i})Q(\theta_{i})-M(\theta_{i})\right]dF(\theta_{i}) \tag{13}$$

where individual rationality requires $-M(0) - T_i(0) \geq 0$ for every player i, and therefore

$0 \leq -\sum_{i=1}^{n}[T_i(0)+M(0)]$. Since the constants $T_1(0),\ldots,T_n(0)$ enter the objective function and the constraint only through the aggregate $\sum_{i=1}^{n} T_i(0)$, we can assume that they are all equal. We therefore denote $T(0)=T_1(0)=\ldots=T_n(0)$.

We use the ex-ante budget-balance constraint to substitute for $-\sum_{i=1}^{n} T_i(0)-\sum_{i=1}^{n} M(0)$ in Equation (12) and rewrite the design problem as finding $q(\theta)$ and $r_H(\theta)$ that maximize the aggregate surplus,

$$\sum_{i=1}^{n}\int_{0}^{1-p}[\theta_i\cdot Q(\theta_i)-M(\theta_i)]\,dF(\theta_i)-\int_{\theta} c(q(\theta),r_H(\theta))dF^n(\theta), \tag{14}$$

subject to the following constraints: (i) $Q(\theta_i)$ is monotone and (ii) the aggregate *virtual surplus* is non-negative,

$$\sum_{i=1}^{n}\int_{0}^{1-p}[v(\theta_i)\cdot Q(\theta_i)-M(\theta_i)]\,dF(\theta_i)-\int_{\theta} c(q(\theta),r_H(\theta))dF^n(\theta)\geq 0 \tag{15}$$

The above inequality is a necessary condition for individual rationality and ex-ante budget balance. To show that this is also a sufficient condition, first denote by q^* and r_H^* the solution to the above optimization problem. Let Ψ^* denote the aggregate virtual surplus (the left-hand side of Equation 15) evaluated at q^* and r_H^*. Second, compute $M^*(0)$ using the q^* and r_H^*. Third, set $T^*(0)=-M^*(0)-\frac{1}{n}\Psi^*$. This guarantees ex-ante budget balance according to Equation (13). Since the aggregate virtual surplus Ψ^* is non-negative by Equation (15) then individual rationality is satisfied (i.e. $-T^*(0)-M^*(0)\geq 0$). Finally, to complete the description of the mechanism it remains to define the transfer functions $(t_i^*(\theta))_{i=1}^{n}$ such that for each player i, $\mathbb{E}_{\theta_{-i}}(t_i^*(\theta_i,\theta_{-i}))=T_i^*(\theta_i)$. One way to do this is to simply let $t_i^*(\theta_i,\theta_{-i})=T_i^*(\theta_i)$.

We have therefore transformed the design problem of acquiring the (ex-ante) welfare maximizing signal and sharing its cost into a problem of choosing a welfare maximizing public good and sharing its cost but with the following "twists". First, the public good is multi-dimensional: it is a distribution over posterior beliefs, which can be summarized by a pair of numbers, the high posterior r_H and the probability q of realizing it. Second, unlike a standard problem of public good provision, the characteristics of the public good affect the players' actions in a game that is played after the good is provided. Third, both in our environment and in a standard public good set-up, the marginal utility from the public good is increasing in types. In the latter case, the cost of the optimal level of the public good typically also increases in the types. However, this is not true in our set-up: Even when types are known, the cost of the optimal signal is *not* monotonic in the types.[5]

3.2 Characterization

Assigning a type profile θ to an informative signal that is not instrumental is wasteful: The players' incur a cost, but do not change their behavior relative to having no signal. We therefore introduce the following property:

Definition 1 (Non-wastefulness) *An auxiliary mechanism is non-wasteful if almost every informative signal that it acquires is instrumental, i.e.,* $q(\theta) \in (0,1)$ *implies* $r_H(\theta) \geq 1 - \theta^{(n-m+1)}$ *for almost all* θ*.*

We then have that:

Lemma 2 *The optimal auxiliary mechanism is non-wasteful.*

Note the effect of the supermajority requirement on the distortion of the signal: In one extreme, if one vote for $a = 1$ is enough to make that decision, then non-wastefulness is never binding; in the other extreme, if n is large and a unanimous decision is required, then r_H will be very high such that any information may be too costly.

By Lemma 2, an optimal auxiliary mechanism solves the constrained optimization problem of the previous subsection subject to an additional constraint that the mechanism is non-wasteful. Since $q(\theta) = 0$ whenever $r_H(\theta) < 1 - \theta^{(n-m+1)}$, we can simplify the expressions of $Q(\theta_i)$ and $M(\theta_i)$ as follows:

$$Q(\theta_i) = \int_{\theta_{-i}} q(\theta_i, \theta_{-i}) dF^{n-1}(\theta_{-i}) \tag{16}$$

$$M(\theta_i) = \int_{\theta_{-i}} (1 - r_H(\theta_i, \theta_{-i})) \cdot q(\theta_i, \theta_{-i}) dF^{n-1}(\theta_{-i}) \tag{17}$$

[5] For example, suppose that $n = 20$, $\kappa = 3$, $p = 0.4$ and that the players' types are commonly known. In Section 3.4 we explain how the optimal signal is determined in this case. It is easy to verify that when all players' types are 0.4 then the cost of the optimal signal is ˜1.45, when all players' types are 0.55 then the cost of the optimal signal is ˜1.58 and when all players' types are 0.6 the cost of the optimal signal is ˜1.56 (regardless of the required supermajority). This non-monotonicity is not an artifact of our particular cost specification since it also arises when the cost is equal to the variance of the posterior beliefs.

Consider the Lagrangian associated with maximizing Equation (14) under the constraint (15):[6]

$$\mathcal{L} = \int_{\theta} \left[w\left(\theta,\lambda\right) \cdot q(\theta) - (1 - r_H(\theta)) \cdot q(\theta) - \frac{1}{n} \cdot c(q(\theta), r_H(\theta)) \right] dF^n(\theta) \tag{18}$$

where

$$w(\theta,\lambda) = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{1}{1+\lambda} \theta_i + \frac{\lambda}{1+\lambda} v(\theta_i) \right).$$

The characterization of the optimal solution is given in the following proposition.

Proposition 2 *There exists $\lambda^* > 0$ for which the optimal auxiliary mechanism is characterized as follows. First,*[7]

$$r_H^*\left(\theta,\lambda^*\right) = \max \left\{ \frac{e^{\frac{n}{\kappa}} - e^{\frac{n}{\kappa} w(\theta,\lambda^*)}}{e^{\frac{n}{\kappa}} - 1}, 1 - \theta^{(n-m+1)} \right\} \tag{19}$$

Second, $r_L^*\left(\theta,\lambda^*\right)$ is determined such that

$$D_{KL}(r_H^*\left(\theta,\lambda^*\right), r_L^*\left(\theta,\lambda^*\right)) = \frac{n}{\kappa} \left[r_H^*\left(\theta,\lambda^*\right) - \left(1 - w\left(\theta,\lambda^*\right)\right) \right] \tag{20}$$

provided a solution exists and is in $(0,p)$, otherwise, $r_L^*\left(\theta,\lambda^*\right) = p$.
Third,

$$q^*\left(\theta,\lambda^*\right) = \frac{p - r_L^*\left(\theta,\lambda^*\right)}{r_H^*\left(\theta,\lambda^*\right) - r_L^*\left(\theta,\lambda^*\right)} \tag{21}$$

Finally, both $r_H^*\left(\theta,\lambda^*\right)$ and $r_L^*\left(\theta,\lambda^*\right)$ are decreasing in each player's type, and $q^*\left(\theta,\lambda^*\right)$ is increasing in each player's type.

At an interior solution in which $r_H^*\left(\theta,\lambda^*\right) > 1 - \theta^{(n-m+1)}$ (i.e., non-wastefulness has slack), the low posterior is given by

$$r_L^*\left(\theta,\lambda^*\right) = \min \left\{ \frac{e^{\frac{n}{\kappa}(1 - w(\theta,\lambda^*))} - 1}{e^{\frac{n}{\kappa}} - 1}, p \right\}.$$

[6] To obtain the Lagrangian $\mathcal{L}\left(\lambda\right)$, write the aggregate surplus given by Equation (14) plus λ times the aggregate virtual surplus given by Equation (15). Now plug in the expressions for $Q(\theta_i)$ and $M(\theta_i)$ given by Equations (16) and (17) and divide by $(1+\lambda) \cdot n$.

[7] We do not restrict $r_H(\theta)$ to be at most one if no signal is acquired (i.e., if $q(\theta) = 0$). Indeed, $r_H^*(\theta) > 1$ whenever $w(\theta,\lambda) < 0$. But in this case, $r_L^*(\theta) = p$, and hence, $q(\theta) = 0$. Also notice that since $\theta_i \leq 1 - p$, it follows that $1 - \theta^{(n-m+1)} \geq p$.

Our proof proceed as follows. First, for any $\lambda \geq 0$ and θ, we find $q^*(\theta, \lambda)$ and $r_H^*(\theta, \lambda)$ that maximize $\mathcal{L}$ under the non-wastefulness constraint. Then, we show that the solution $(q^*(\theta, \lambda), r_H^*(\theta, \lambda))$ is unique for every λ and θ and satisfies that $q^*(\theta, \lambda)$ is increasing in any θ_i and that $r_H^*(\theta, \lambda)$ and $r_L^*(\theta, \lambda)$ are decreasing in any θ_i, for any $\lambda \geq 0$. Finally, we establish the existence of some $\lambda^* > 0$ for which $(q^*(\theta, \lambda^*), r_H^*(\theta, \lambda^*))$ is a feasible solution for all θ, which guarantees that q^* and r_H^* are indeed optimal.

One can view the signal purchased by the mechanism as a binary classification test of whether departing from the default action is desirable. Under this view, the ratio $\frac{r_H}{1-r_H}$ is proportional to the *positive likelihood ratio* (PLR), that is, conditional on the test recommending the non-default action, PLR is the ratio between the probability that departing from the default is the right thing to do, to the probability that this is a mistake. Similarly, $\frac{r_L}{1-r_L}$ is proportional to the *negative likelihood ratio* (NLR), that is, conditional on the test recommending the default action, NLR is the ratio between the probability that choosing the default is the right thing to do, to the probability that this is a mistake. Our results show that when types are higher, the two likelihood ratios are lower. Thus, the "quality" of the recommendation to depart from the default action decreases with the types, but the quality of the recommendation to take the default action increases. However, the ratio between the two likelihood ratios, also known as the *diagnostic odds ratio* is constant at an interior solution and equal to $e^{n/\kappa}$. In the statistical literature (mainly in the context of medical experiments) this measure is sometimes considered as a measure of the effectiveness of the classification test (See, e.g. Glas et al., 2003).

Note that Proposition 2 established that $q^*(\theta)$ is increasing in each of its components. An immediate corollary of this is the following (see Mookherjee and Reichelstein, 1992):

Corollary 1 *There exists an optimal auxiliary mechanism in which truthtelling is a dominant strategy equilibrium.*

3.3 From the auxiliary mechanism to the actual mechanism

Up to now we analyzed an auxiliary mechanism in which the players commit to vote for the collective action according to their reported types. When a player considers misreporting in the auxiliary mechanism he takes into account that in the subsequent voting game, his vote will not be cast according to his true preferences. For instance, if a player reports a type *higher* than his true one, any signal that is acquired will be non-wasteful relative to his report. If, in addition, the high posterior r_H realizes, then the mechanism votes for $a = 1$ on his behalf even though according to his true type the player would have preferred to vote for $a = 0$. However, in the actual mechanism that

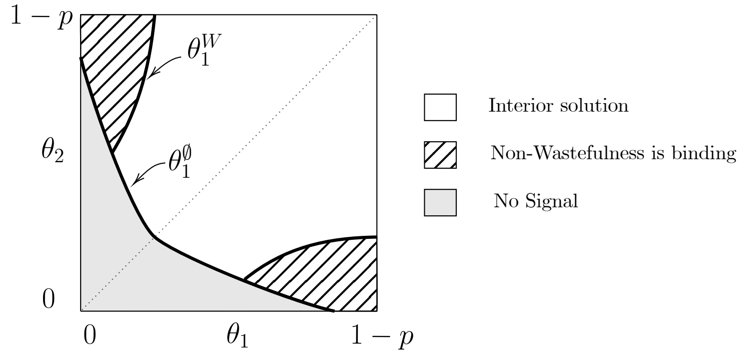

Figure 2: Projection of the second-best mechanism's regions on the type space

will be run, a player is free to vote according to his true preferences. In that case, a player may have an incentive to affect the choice of signal, knowing that he can vote in favour of the default action in the ensuing voting game.

Consider now the "actual" mechanism, which differs from the auxiliary mechanism in that the players' reports affect only the signal acquisition but have no effect on the ensuing voting game. The question is whether the mapping from reports to signals of the optimal auxiliary mechanism, q^* and r_H^*, remain incentive-compatible, budget-balanced and individually rational when players vote on the collective action according to their true preferences?

Proposition 3 *The optimal actual mechanism is characterized by Equations (19)-(21).*

We refer to the optimal actual mechanism as the "second-best". Its characterization highlights three cases. First, there may be realizations of θ in which no signal is purchased, i.e. $q^*(\theta) = 0$. Second, there are realizations of θ for which the non-wastefulness constraint is binding, i.e. $r_H^*(\theta) = 1 - \theta^{(n-m+1)}$. Finally, for other realizations, the signal that is purchased is given by the interior solution to the optimization problem, i.e. $r_H^*(\theta) = \frac{e^{\frac{n}{\kappa}} - e^{\frac{n}{\kappa} w(\theta,\lambda^*)}}{e^{\frac{n}{\kappa}} - 1}$.

To illustrate the projection of these three cases on the players' type space, we consider two players who need a unanimous vote in order to depart from the status-quo and choose $a = 1$ (i.e., $m = n = 2$). We also place additional structure on F by assuming that the inverse failure rate $(1 - F(\theta_i))/f(\theta_i)$ is *concave.*[8] Figure 2 shows how the above three cases map onto three regions in (θ_1, θ_2) space. By symmetry, it is sufficient to consider only the type realizations $\theta_1 \leq \theta_2$, i.e., only the "top-left" triangle. The other triangle is a mirror image. The following observation summarizes the key features of the second-best mapping from type realizations to signals.

[8]For example, the uniform and exponential distributions satisfy this property.

Proposition 4 *Consider the type realizations* (θ_1, θ_2) *for which* $\theta_1 \leq \theta_2$. *In the second-best mechanism, for any type* θ_2 *there exist two unique cutoffs,* $\theta_1^{\emptyset}(\theta_2)$ *and* $\theta_1^{W}(\theta_2)$, *such that:*

(i) *A signal is acquired if and only if* $\theta_1 > \theta_1^{\emptyset}(\theta_2)$,

(ii) *The cutoff* $\theta_1^{\emptyset}(\theta_2)$ *is decreasing in* θ_2,

(iii) *If a signal is acquired and* $\theta_1 < \theta_1^{W}(\theta_2)$ *then the non-wastefulness constraint is binding, but if* $\theta_1 > \theta_1^{W}(\theta_2)$ *then the constraint holds with slack, and*

(iv) *The cutoff* $\theta_1^{W}(\theta_2)$ *is increasing in* θ_2.

The characterization for the case that $\theta_1 > \theta_2$ *is symmetric.*

3.4 Comparison with the efficient mechanism

To better understand the distortions that are introduced by the players' private information, it is instructive to compare the optimal mechanism to the ex-ante efficient acquisition of information when the players' types are known and the only constraint is that they collectively cover the cost of the signal. We maintain the assumption that the players are free to vote on their desired action after they observe the signal realization. An efficient acquisition rule maps every profile of types θ to a signal $q^e(\theta)$, $r_H^e(\theta)$, $r_L^e(\theta)$ that maximizes the objective function that is given by (OBJ) above. Obviously, it cannot be efficient to purchase a signal that leads to the same outcome as acquiring no signal at all. Hence, an efficient rule must also be non-wasteful.

From the proof of Proposition 2 it follows that the efficient acquisition rule is obtained by simply replacing the term $w(\theta, \lambda^*)$ in equations (19)-(21) (which is due to the incentive-compatibility constraint) with $w(\theta, 0)$, which is the average (across players) type realization. Since $w(\theta, \lambda)$ is increasing in each θ_i, decreasing in λ and $w((1-p, ..., 1-p), \lambda) = 1-p$ for any λ, it follows that

$$w(\theta, \lambda^*) < w(\theta, 0) \leq w((1-p, ..., 1-p), 0) = 1-p$$

Since $w(\theta, \lambda^*)$ is a continuous function in each θ_i, there exists $\theta' \geq \theta$ (i.e., $\theta_i' \geq \theta_i$ for all i, with at least one strict inequality) such that $w(\theta', \lambda^*) = w(\theta, 0)$. This has the following implications.

Observation 1. If θ is such that in both the first-best and second-best mechanisms a signal is acquired and the non-wastefulness constraint has slack, then $r_L^e(\theta) < r_L^*(\theta)$, $r_H^e(\theta) < r_H^*(\theta)$ *and* $q^e(\theta) > q^*(\theta)$.

Since $q(\theta)$ is the probability of taking the non-default action, this observation means that in the second-best this action will be taken with a *lower* probability. However, since $r_H^e(\theta) < r_H^*(\theta)$ then whenever the non-default action is taken in the second-best, it is taken with *greater* confidence. On the other hand, since $r_L^e(\theta) < r_L^*(\theta)$ then whenever the default action is taken in the second-best, it is taken with *lower* confidence.

To see why this observation is true, note that when the posterior probabilities are all interior, the fact that $w(\theta, \lambda^*) < w(\theta, 0)$ implies that both the low and high posteriors in the efficient mechanism are lower than the corresponding posteriors in the second-best. In addition, since $w(\theta', \lambda^*) = w(\theta, 0)$ then $q^*(\theta') = q^e(\theta)$. By the monotonicity of q^* in each θ_i, we have that $q^*(\theta) < q^*(\theta')$ and therefore $q^e(\theta) > q^*(\theta)$. This last argument also implies the following:

Observation 2. *If $q^e(\theta) = 0$ then $q^*(\theta) = 0$, but the converse is not true.*

Put differently, there are realizations of θ for which the efficient rule acquires a signal but the second-best rule does not. Hence, the fact that players do not observe each other's type can lead to *under-provision of information* for the collective decision.

Observation 3. *Whenever non-wastefulness is binding in the second-best mechanism, it is also binding in the efficient mechanism.*

The fact that players vote after they observe the realization of the acquired signal introduces an ex-ante distortion even when players' types are known. This occurs when the signal (q, r_H) that maximizes ex-ante welfare subject to *only* budget balance and individual rationality satisfies $q > 0$ and $p < r_H < 1 - \theta^{(n-m+1)}$. In this case, the acquired signal will be distorted such that r_H will increase to $1 - \theta^{(n-m+1)}$. Observation 3 establishes that introducing private types does *not* exacerbate this distortion.

4 Concluding remarks

This paper is concerned with the question of how groups who want to make an informed collective decision bargain over which information to acquire. Instead of committing to a particular bargaining protocol, we took a mechanism-design approach that looks for the signal that maximizes the players' expected sum of utilities, taking into account that (i) players must be willing to participate in the mechanism, (ii) they must be willing to disclose their private willingness-to-pay for information, and (iii) players vote on the outcome after they jointly observe the realization of the acquired signal.

The optimal mechanism exhibits two types of distortions in information acquisition. First, the fact that the group members vote on the basis of the signal realization means that the signal that maximizes the net expected surplus is not necessarily the signal that is acquired (even when types are commonly known). This stems from the fact that it is wasteful to purchase a signal that will not persuade a supermajority to vote against the default. Second, the fact that players need to be incentivized to disclose their types (as this determines what the optimal signal is), further distorts the type of information that is acquired: the probability of acquiring the signal *decreases* while the induced posterior beliefs *increase* (i.e., when the players vote for $a = 1$ they do so with *higher* confidence, but when they vote for $a = 0$ the do so with *lower* confidence).

These distortions suggest that relative to a single individual who acquires costly information before acting, a *group* of individuals (say, committees, boards or households) that collectively chooses what information to acquire, will be *more inattentive*: for the same cost of information, the group is less likely to acquire any signal. Moreover, even when information is acquired, the group is more likely to stick with the default.

In real life there are many situations in which a group of individuals with conflicting interests need to design their information structure and choose how to respond to it. We take a first step in analyzing these situations by identifying novel properties of this set-up relative to the single information designer problem that is studied in the literature. Our paper opens the door to explore additional aspects of this new information-design/acquisition problem.

References

[1] **Alonso, Ricardo and Odilon Câmara** (2016): "Persuading Voters." *American Economic Review* 106(11): 3590-3605.

[2] **Arieli, Itai and Yakov Babichenko** (2019): "Private Bayesian Persuasion." *Journal of Economic Theory* 182, 185-217.

[3] **Bardhi, Arjada and Yingni Guo** (2018): "Modes of Persuasion Toward Unanimous Consent." *Theoretical Economics* 13, 1111–1149.

[4] **Bergemann, Dirk and Stephen Morris** (2013): "Robust Predictions in Games of Incomplete Information." *Econometrica* 81(4), 1251-1308.

[5] **Borgers, Tilman** (2015): An Introduction to Theory of Mechanism Design, Oxford University Press.

[6] **Chan, Jimmy, Seher Gupta, Fei Li and Yun Wang** (2019): "Pivotal Persuasion." *Journal of Economic Theory* 180, 178-202.

[7] **Chan, Jimmy, Alessandro Lizzeri, Wing Suen and Leeat Yariv** (2018): "Deliberating Collective Decisions." *The Review of Economic Studies*, Volume 85(2), 929–963.

[8] **Dean, Mark, Andrew Caplin and John Leahy** (2019): "Rational Inattention, Optimal Consideration Sets and Stochastic Choice." *Review of Economic Studies* 86(3), 1061–1094.

[9] **Feddersen, Timothy and Wolfgang Pesendorfer** (1998): "Convicting the Innocent: The Inferiority of Unanimous Jury Verdicts under Strategic Voting." *American Political Science Review*, 92(1), 23–35.

[10] **Gershkov, Alex and Balázs Szentes** (2009): "Optimal Voting Schemes with Costly Information Acquisition." *Journal of Economic Theory*, 144(1), 36–68.

[11] **Glas Afina S., Jeroen G. Lijmer, Martin H. Prins, Gouke J. Bonsel and Patrick M. M. Bossuyt** (2003): "The Diagnostic Odds Ratio: A Single Indicator of Test Performance." *Journal of Clinical Epidemiolgy* 56(11), 1129-1135.

[12] **Hellwig, Martin F.** (2003): "Public-Good Provision with Many Participants." *The Review of Economic Studies*, 70(3), 589–614.

[13] **Kelly, Frank and Yudovina, Elena** (2014): Stochastic Networks, Cambridge University Press, USA.

[14] **Kolotilin, Anton, Tymofiy Mylovanov, Andriy Zapechelnyuk and Ming Li** (2017): "Persuasion of a Privately Informed Receiver." *Econometrica* 85(6), 1949–1964.

[15] **Krishna, Vijay** (2010): Auction Theory, Academic Press.

[16] **Maćkowiak, Bartosz, Filip Matějka and Mirko Wiederholt** (2018): "Rational Inattention: A Disciplined Behavioral Model." Working Paper.

[17] **Mailath, George J. and Andrew Postlewaite** (1990): "Asymmetric Information Bargaining Problems with Many Agents." *The Review of Economic Studies*, Volume 57(3), 351–367.

[18] **Matějka, Filip and Alisdair McKay** (2015): "Rational Inattention to Discrete Choices: A New Foundation for the Multinomial Logit Model." *American Economic Review*, 105(1), 272–298.

[19] **Mathevet, Laurent, Jacopo Perego and Ina Taneva** (2020): "On Information Design in Games." *Journal of Political Economy* 128(4), 1370-1404.

[20] **Mookherjee, Dilip and Stefan Reichelstein** (1992): "Dominant Strategy Implementation of Bayesian Incentive Compatible Allocation Rules." *Journal of Economic Theory* 56, 378-399.

[21] **Persico, Nicola** (2004): "Committee Design with Endogenous Information." *The Review of Economic Studies*, 71(1), 165–191.

[22] **Schnakenberg, Keith E.** (2015): "Expert Advice to a Voting Body." *Journal of Economic Theory* 160, 102-113.

[23] **Sims, Christopher A.** (2003): "Implications of Rational Inattention." *Journal of Monetary Economics* 50(3), 665-690.

[24] **Taneva, Ina** (2019): "Information Design." *American Economic Journal: Microeconomics* 11(4), 2019, 151-185.

[25] **Wang, Yun** (2013): "Bayesian Persuasion with Multiple Receivers." Working Paper.

5 Appendix: Proofs

Proof of Lemma 1

Let $\theta \in [0,p]^n$ be the players' types. Consider a signal that induces a probability distribution q over a set $R \in [0,1]^J$ of posterior beliefs (on state $\omega = 1$) such that the expected posterior equals p, i.e. $\sum_{r \in R} q(r) \cdot r = p$. Let $\bar{R}$ (respectively, $\underline{R}$) be the set of posterior beliefs above (respectively, below) $1 - \theta^{(n-m+1)}$. Suppose that $\bar{R}$ contains (at least) two distinct elements r' and r'', where $r' < r''$. Both r' and r'' lead to the same collective action $a = 1$ in the voting game.

Consider now a modified signal that induces a distribution $\hat{q}$ over a set of posterior beliefs $\hat{R}$. The set $\hat{R}$ is identical to R, with one difference: The posteriors r' and r'' are replaced by the posterior $\hat{r} \equiv \frac{q(r')}{q(r')+q(r'')} r' + \frac{q(r'')}{q(r')+q(r'')} r''$. The distribution $\hat{q}$ is defined such that $\hat{q}(r) = q(r)$ for all $r \in R \backslash \{r', r''\}$, while $\hat{q}(\hat{r}) = q(r') + q(r'')$. Note that since $\hat{r} \in (r', r'')$, then $\hat{r}$ is above $1 - \theta^{(n-m+1)}$ and so it induces the collective action $a = 1$, which is the same as the collective action induced by r and r'. Thus, the modified signal $\hat{q}$ (over $\hat{R}$) induces the same distribution over outcomes as the original signal q (over R). By construction, the modified signal also satisfies $\sum_{r \in \hat{R}} \hat{q}(r) \cdot r = p$.

To show that the modified signal is cheaper than the original one, we define the function $h(r) \equiv D_{KL}(r, p)$, where D_{KL} is defined in Equation (3). Note that the costs of the two signals, as computed by Equation (2), differ only in the summands $q(r') \cdot h(r') + q(r'') \cdot h(r'')$ that appear in the cost of the original signal (and not in that of the modified one) and the summand $\hat{q}(\hat{r}) \cdot h(\hat{r})$ that appear in the cost of the modified signal (and not in that of the original one). However, since h is a convex function,[9] then we have:

$$\frac{q(r')}{q(r')+q(r'')} h(r') + \frac{q(r'')}{q(r')+q(r'')} h(r'') > h\left(\frac{q(r')}{q(r')+q(r'')} r' + \frac{q(r'')}{q(r')+q(r'')} r''\right)$$

or, equivalently,

$$q(r') \cdot h(r') + q(r'') \cdot h(r'') > \hat{q}(\hat{r}) \cdot h(\hat{r}).$$

Thus, the modified signal induces the same distribution over outcomes as the original one but it is cheaper. The proof for the case in which there are more than two elements in $\underline{R}$ is analogous. ■

[9]To see this note that $\frac{d^2 h}{(dr)^2} = \frac{1}{r(1-r)} > 0$.

Proof of Proposition 1

Consider the two stage game in which players first participate in the actual direct mechanism, and then following the signal realization (if a signal was acquired) they play the voting game. Consider a perfect Bayesian equilibrium of this game in which the players report truthfully in the first stage. Call this the "original" truthful equilibrium. Since the players do not choose weakly dominated actions in the voting game, each player i votes for the action $a = 1$ if and only if $r_H(\theta) \geq 1 - \theta_i$. Hence, in this equilibrium, the collective action 1 is chosen in the second stage if and only if this inequality holds for at least m players.

Consider next an auxiliary mechanism that has the same $\langle q, r_H, t_1, ..., t_n \rangle$ as the actual mechanism, and where $a_H(\hat{\theta})$ and $a_L(\hat{\theta})$ are defined as in (4)-(5). Suppose all players other than i report truthfully in the auxiliary mechanism. If player i also reports truthfully, then his expected payoff would be the same as in the original truthful equilibrium. If i deviates and misreports $\theta_i' \neq \theta_i$, the induced distribution over signals would be exactly the same as if he had deviated in the same way from the original truthful equilibrium. However, the auxiliary mechanism's decision on which collective action to take is weakly suboptimal for player i. This is because in the auxiliary mechanism player i is "forced" to vote for the action $a = 1$ if and only if the realized posterior is above $1 - \theta_i'$ (and not above $1 - \theta_i$, which is the preferred threshold for player i). Thus, deviations from truthtelling are less profitable in the auxiliary mechanism. Therefore, truthtelling must also be an equilibrium in the auxiliary mechanism. It follows that under truthtelling, $a_H(\theta)$ and $a_L(\theta)$ replicate the mapping from types to collective decisions in the original equilibrium. In light of this, any surplus that is attainable in the original equilibrium can also be attained in a truthful equilibrium of the auxiliary mechanism. ■

Proof of Lemma 2

Suppose that $\langle q, r_H, t_1, ..., t_n \rangle$ is an auxiliary mechanism that satisfies incentive compatibility, individual rationality and ex-ante budget balance, but does not satisfy non-wastefulness. We show a modification that increases the expected payoff to the players without affecting the constraints. Therefore, the given mechanism is not optimal.

Since the mechanism does not satisfy non-wastefulness, there exist a non-zero measure of type realizations (θ_i, θ_{-i}) for which $q(\theta) > 0$ and $r_H(\theta) < 1 - \theta^{(n-m+1)}$. Suppose we modify q into q' as follows:

$$q'(\theta)=\begin{cases} q(\theta) & if \quad r_H(\theta)\geq 1-\theta^{(n-m+1)} \\ 0 & if \quad r_H(\theta)<1-\theta^{(n-m+1)} \end{cases}.$$

That is, whenever the original mechanism purchases a non instrumental signal, the modified mechanism does not purchase a signal. Notice that

$$\begin{aligned} Q'(\theta_i) &= \int_{\theta_{-i}|r_H(\theta_i,\theta_{-i})>1-(\theta_i,\theta_{-i})^{(n-m+1)}} q'(\theta_i,\theta_{-i})dF(\theta_{-i}) \\ &= \int_{\theta_{-i}|r_H(\theta_i,\theta_{-i})>1-(\theta_i,\theta_{-i})^{(n-m+1)}} q(\theta_i,\theta_{-i})dF(\theta_{-i}) = Q(\theta_i) \\ M'(\theta_i) &= \int_{\theta_{-i}|r_H(\theta_i,\theta_{-i})>1-(\theta_i,\theta_{-i})^{(n-m+1)}} (1-r_H(\theta_i,\theta_{-i}))\cdot q'(\theta_i,\theta_{-i})dF(\theta_{-i}) \\ &= \int_{\theta_{-i}|r_H(\theta_i,\theta_{-i})>1-(\theta_i,\theta_{-i})^{(n-m+1)}} (1-r_H(\theta_i,\theta_{-i}))\cdot q(\theta_i,\theta_{-i})dF(\theta_{-i}) \\ &= M(\theta_i). \end{aligned}$$

Denote the expected decrease in the cost of purchasing signals by

$$\Delta = \int_{\theta|r_H(\theta)<1-\theta^{(n-m+1)}} c(q(\theta),r_H(\theta))dF(\theta)>0.$$

For every $i\in\{1,\ldots,n\}$ define

$$t_i'(\hat{\theta})=t_i(\hat{\theta})-\frac{\Delta}{n}.$$

The new mechanism satisfies incentive compatibility and individual rationality because $Q'=Q$ and $M'=M$, and the transfers decreased by a constant for all types (so that $-M(0)-T(0)\geq 0$). By construction the mechanism is budget-balanced, and since the expected payment of type 0 decreased, then by Equation (12) the expected surplus increased. ■

Proof of Proposition 2

The proof consists of three parts. First, we find three functions, $r_H^*(\theta,\lambda), r_L^*(\theta,\lambda)$ and $q^*(\theta,\lambda)$, that satisfy non-wastefulness and maximize the Lagrangian that is given by Equation (18), for any multiplier λ and any profile of types θ. Second, we show that for *any* $\lambda\geq 0$, the function $q^*(\theta,\lambda)$ is increasing in each player's type while $r_H^*(\theta,\lambda)$ and $r_L^*(\theta,\lambda)$ are decreasing in each player's type. Hence, the function $Q^*(\theta_i,\lambda)$ that is induced by $q^*(\theta, \lambda)$ (according to Equation 16) is monotone. Third, an argument of Hellwig (2013) guarantees the existence of some $\lambda^* \geq 0$ for which the mechanism defined by $r_H^*(\theta, \lambda^*), r_L^*(\theta, \lambda^*)$ and $q^*(\theta, \lambda^*)$ has a non-negative virtual surplus. Thus, by the Lagrange Sufficiency Theorem (see, e.g., Theorem C.1 in Kelly and Yudovina, 2014), the functions $r_H^*(\theta, \lambda^*), r_L^*(\theta, \lambda^*)$ and $q^*(\theta, \lambda^*)$ define the mechanism that attains the maximal aggregate surplus (Equation 14) subject to (i) $Q(\theta_i)$ is monotone and (ii) the aggregate virtual surplus (Equation 15) is non negative.

PART I. Fix a profile of types θ and a multiplier λ. The part of the Lagrangian (Equation 18) that is affected by q and r_H is:

$$\mathcal{L}(q, r_H \; ; \; w) = q(r_H - (1-w)) - \frac{1}{n} \cdot c(q, r_H) \tag{22}$$

where r_H, w and q are used for brevity instead of $r_H(\theta, \lambda)$, $w(\theta, \lambda)$ and $q(\theta, \lambda)$. Note that λ and θ affect the values of the maximizers q and r_H only through w. Differentiating $\mathcal{L}(q, r_H \; ; \; w)$ with respect to q and r_H and equating to zero yields:

$$\mathcal{L}_1(q, r_H \; ; \; w) = r_H - (1-w) - \frac{1}{n} \cdot c_1(q, r_H) = 0, \tag{FOCq}$$

$$\mathcal{L}_2(q, r_H \; ; \; w) = q - \frac{1}{n} \cdot c_2(q, r_H) = 0. \tag{FOCr}$$

where $c_1(q, r_H)$ is the derivative of the function $c(q, r_H)$ with respect to its first argument q, and $c_2(q, r_H)$ is the derivative of $c(q, r_H)$ with respect to its second argument r_H.

We begin by looking for a pair $(\tilde{q}, \tilde{r}_H)$ that solves (FOCq) and (FOCr). Plugging in $c_2(q, r_H)$ (for the derivation of c_2 and all other partial derivatives of the cost function see the supplementary appendix B) into (FOCr) and simplifying yields:

$$\ln\left(\frac{\tilde{r}_H}{\tilde{r}_L}\frac{1-\tilde{r}_L}{1-\tilde{r}_H}\right) = \frac{n}{\kappa}. \tag{23}$$

where $\tilde{r}_L = \frac{p - \tilde{q}\tilde{r}_H}{1-\tilde{q}}$. Plugging $c_1(q, r_H)$ into (FOCq), and using Equation (23) we obtain:

$$\ln\left(\frac{1-\tilde{r}_L}{1-\tilde{r}_H}\right) = \frac{n}{\kappa}(1-w).$$

We therefore have that:

$$\tilde{r}_L = \frac{e^{\frac{n}{\kappa}[1-w(\theta,\lambda)]} - 1}{e^{\frac{n}{\kappa}} - 1}, \tag{24}$$

$$\tilde{r}_H = \frac{e^{\frac{n}{\kappa}} - e^{\frac{n}{\kappa}w(\theta,\lambda)}}{e^{\frac{n}{\kappa}} - 1}, \tag{25}$$

$$\tilde{q} = \frac{p - \tilde{r}_L(\theta)}{\tilde{r}_H(\theta) - \tilde{r}_L(\theta)} \tag{26}$$

We say that the solution $(\tilde{q}, \tilde{r}_H)$ is interior if $\tilde{q} \in (0, \frac{p}{r_H})$ and $\tilde{r}_H \in (p, 1)$ and we say it is non-wasteful if $\tilde{r}_H \geq 1 - \theta^{(n-m+1)}$.

Since $\mathcal{L}$ is not a concave function in general, it is not a priori guaranteed that $(\tilde{q}, \tilde{r}_H)$ is a maximizer of $\mathcal{L}$. Note, however, that the corner solutions $r_H = 1$ or $q = \frac{p}{r_H}$ (or, equivalently, $r_L = 0$) never maximize $\mathcal{L}$. This is because $\lim_{r_H \to 1} c_2(q, r_H) = \infty$ and therefore $\mathcal{L}_2(q, 1\ ;\ w) < 0$ for any $q \geq 0$ and w. Hence $r_H = 1$ is not a maximizer of $\mathcal{L}$. Similarly, $\lim_{q \to \frac{p}{r_H}} c_1(q, r_H) = \infty$, and therefore $\mathcal{L}_1\left(\frac{p}{r_H}, r_H\ ;\ w\right) < 0$ for any r_H and w. Hence $q = \frac{p}{r_H}$ is not a maximizer of $\mathcal{L}$. Thus, the only candidates for a corner solution are $q = 0$ or $\tilde{r}_H = p$, i.e. solutions in which no signal is acquired.

Our next result establishes that although $\mathcal{L}$ is not concave, if $(\tilde{q}, \tilde{r}_H)$ is interior and non-wasteful then it is indeed a maximizer of $\mathcal{L}$.

Lemma 3 *For any w, if $(\tilde{q}, \tilde{r}_H)$ is interior and non-wasteful, then it maximizes $\mathcal{L}(q, r_H\ ;\ w)$.*

Proof. Given w, suppose that $(\tilde{q}, \tilde{r}_H)$ is interior and that it is non-wasteful (i.e. $\tilde{r}_H \geq 1 - \theta^{(n-m+1)}$). We prove the lemma in two steps. First, we show that $(\tilde{q}, \tilde{r}_H)$ is a local maximizer of $\mathcal{L}(q, r_H\ ;\ w)$. Then, we show that $\mathcal{L}(\tilde{q}, \tilde{r}_H\ ;\ w)$ is greater than the value of $\mathcal{L}$ in the corner solution in which no signal is acquired.

To show that $(\tilde{q}, \tilde{r}_H)$ is a local maximum it suffices to show that $\mathcal{L}_{11}(q, r_H\ ;\ w) < 0$ and that the determinant of the Hessian matrix of $\mathcal{L}(q, r_H\ ;\ w)$ is positive when evaluated at $(\tilde{q}, \tilde{r}_H)$. The former is true because $\mathcal{L}_{11}(\tilde{q}, \tilde{r}_H\ ;\ w) = -\frac{1}{n} \cdot c_{11}(\tilde{q}, \tilde{r}_H) < 0$. To see the latter, note that the determinant of the Hessian matrix of $\mathcal{L}(q, r_H\ ;\ w)$ is given by:

$$\begin{aligned}\mathcal{D} &\equiv \mathcal{L}_{11}(\tilde{q}, \tilde{r}_H\ ;\ w) \cdot \mathcal{L}_{22}(\tilde{q}, \tilde{r}_H\ ;\ w) - (\mathcal{L}_{12}(\tilde{q}, \tilde{r}_H\ ;\ w))^2 \\ &= \left(-\frac{1}{n} \cdot c_{11}(\tilde{q}, \tilde{r}_H)\right) \cdot \left(-\frac{1}{n} \cdot c_{22}(\tilde{q}, \tilde{r}_H)\right) - \left(1 - \frac{1}{n} \cdot c_{12}(\tilde{q}, \tilde{r}_H)\right)^2\end{aligned}$$

Plugging in c_{11}, c_{22} and c_{12}, and using Equation (23), we obtain:

$$\mathcal{D} = \left(\frac{\kappa}{n}\right)^2 \cdot \frac{q}{(1-q)} \cdot \frac{1}{r_H} \cdot \frac{1}{r_L} \cdot \frac{(r_H - r_L)^2}{(1-r_H)(1-r_L)} > 0.$$

It remains to show that $\mathcal{L}(\tilde{q}, \tilde{r}_H \ ; \ w)$ is greater than zero, which is the value of $\mathcal{L}$ in the corner solution in which no information is acquired. For any r_H and w, define $\hat{q}(r_H, w)$ to be the value of q for which (FOCq) is satisfied (whenever such a value exists), and define

$$\hat{r}_L(r_H, w) := \frac{p - r_H \cdot \hat{q}(r_H, w)}{1 - \hat{q}(r_H, w)}$$

to be value of r_L that is uniquely determined by r_H and $\hat{q}(r_H, w)$. By definition we have that $\tilde{q} = \hat{q}(\tilde{r}_H, w)$ and $\tilde{r}_L = \hat{r}_L(\tilde{r}_H, w)$. Let $\hat{\mathcal{L}}(r_H \ ; \ w)$ be the value of the Lagrangian when q is computed according to $\hat{q}(r_H, w)$:

$$\hat{\mathcal{L}}(r_H \ ; \ w) \equiv \mathcal{L}(\hat{q}(r_H, w), r_H \ ; \ w) = \hat{q}(r_H, w) \cdot \frac{1}{n} \cdot c_1(\hat{q}(r_H, w), r_H) - \frac{1}{n} \cdot c(\hat{q}(r_H, w), r_H).$$

Substitute the expressions for $c(\cdot)$ and $c_1(\cdot)$ into the right-hand side and simplify to obtain:

$$\hat{\mathcal{L}}(r_H \ ; \ w) = \frac{1}{n} \cdot \left[p \cdot \ln\left(\frac{p}{\hat{r}_L(r_H, w)} \right) + (1-p) \cdot \ln\left(\frac{1-p}{1 - \hat{r}_L(r_H, w)} \right) \right]. \tag{27}$$

Inspection of Equation (27) reveals that the value of $\hat{\mathcal{L}}(r_H; w)$ is decreasing in $\hat{r}_L(r_H, w)$ when $\hat{r}_L(r_H, w) \leq p$.[10] This means that $\hat{\mathcal{L}}$ attains its minimal value when $\hat{r}_L(r_H, w) = p$, in which case $\hat{\mathcal{L}}$ is zero. Note, however, that since $(\tilde{q}, \tilde{r}_H)$ is interior, then $\tilde{r}_L = \frac{p - \tilde{r}_H \cdot \tilde{q}}{1-q} < p$. This means that $\hat{\mathcal{L}}(\tilde{q}, \tilde{r}_H \ ; \ w) > 0$, which completes the proof. ■

Suppose that $(\tilde{q}, \tilde{r}_H)$ is interior but wasteful. Since we look for maximizers of $\mathcal{L}$ that satisfy non-wastefulness, and since $\mathcal{L}$ is not in general a concave function, the question is whether r_H should be "corrected" so that non-wastefulness exactly binds or should it attain an even higher value. Our next result shows that in this case non-wastefulness should bind, i.e. $r_H^* = 1 - \theta^{(n-m+1)}$. The values of r_L^* and q^* that maximize $\mathcal{L}$ are determined according to (FOCq), whenever possible.

Lemma 4 *Given w, suppose that $(\tilde{q}, \tilde{r}_H)$ is interior but is wasteful. Let $r_H^* = 1 - \theta^{(n-m+1)}$. Let r_L^* be the solution to $D_{KL}(r_H^*, r_L) = \frac{n}{\kappa}(r_H^* - (1-w))$ if a solution exists in $[0, p]$, and $r_L^* = p$ otherwise. Then, r_H^* and $q^* = \frac{p - r_L^*}{r_H^* - r_L^*}$ are maximizers of $\mathcal{L}(q, r_H \ ; \ w)$.*

Proof. Suppose first that r_H^* is given. We begin by finding the value of q^* that maximizes $\mathcal{L}(q, r_H^* \ ; \ w)$. The fact that $\mathcal{L}_{11}(q, r_H^* \ ; \ w) = -\frac{1}{n} \cdot c_{11}(q, r_H^*) < 0$ implies that q^* satisfies

[10] This is because $\frac{d}{dr_L}\left(p \ln\left(\frac{p}{r_L}\right) + (1-p)\ln\left(\frac{1-p}{1-r_L}\right) \right) = -\frac{p - r_L}{r_L(1-r_L)}$.

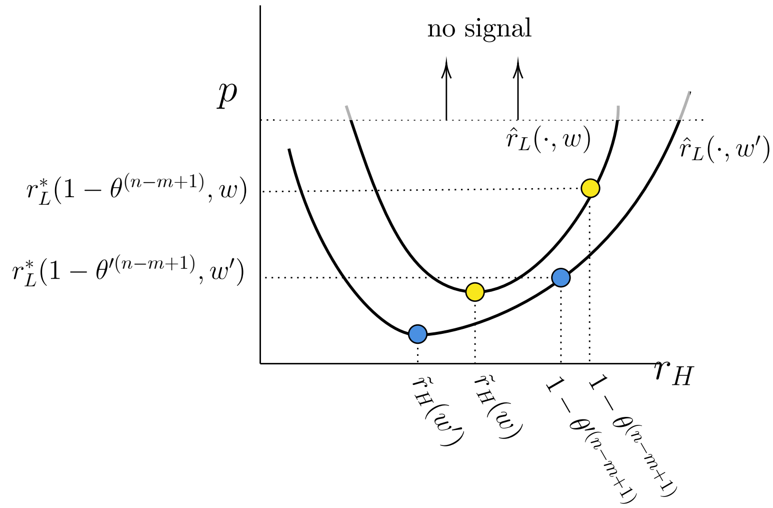

Figure 3: $\hat{r}_L(\cdot, w)$

(FOCq) whenever possible. Since $c_1(q, r_H^*) = \kappa \cdot D_{KL}(r_H^*, r_L)$, we can rewrite (FOCq), when evaluated at r_H^*, as follows:

$$r_H^* - (1-w) - \frac{\kappa}{n} \cdot D_{KL}(r_H^*, r_L) = 0 \tag{28}$$

Thus, r_L^* is the solution to this equation, whenever the solution is in $[0, p]$. The value of q^* is then uniquely determined such that $q^* = \frac{p - r_L^*}{r_H^* - r_L^*}$.

If there is no solution to Equation (28) within the range $[0, p]$, then $r_L^* = p$ and $q^* = 0$. This is because when there is no solution to Equation (28) then it must be the case that its left-hand side is negative for all $r_L \in [0, p]$.[11] In this case, decreasing q increases $\mathcal{L}$, hence $q^* = 0$.

We proceed to show that if $\tilde{r}_H < 1 - \theta^{(n-m+1)}$, then the optimal solution is $r_H^* = 1 - \theta^{(n-m+1)}$. To achieve this, we use the notation defined in the proof of Lemma 3 to show the following three properties of the function $\hat{r}_L(r_H, w)$ (whenever this function is defined), which are depicted in Figure 3.

(P1) *For any w, the function $\hat{r}_L(r_H, w)$ attains a minimum at $\tilde{r}_H$.* Recall that the function $\hat{\mathcal{L}}(r_H; w)$, defined in Equation (27), is decreasing in $\hat{r}_L(\cdot)$. Therefore, for any w, the function $\hat{\mathcal{L}}(r_H; w)$ attains a maximum when $\hat{r}_L(r_H; w)$ is at its minimum value. Since $\hat{\mathcal{L}}(r_H; w)$ is maximized at $\tilde{r}_H$, is follows that $\hat{r}_L(r_H, w)$ attains a minimum at $\tilde{r}_H$.

[11] To see this, note that $D_{KL}(r_H^*, 0) = \infty$ and therefore the left-hand side of Equation (28) is negative for $r_L = 0$. If there is a value $r_L \in [0, p]$ for which the left-hand side is positive, then the fact that D_{KL} is continuous implies, by the intermediate value theorem, that there must also be a solution to the equation within $[0, p]$.

(P2) *For any w, the function $\hat{r}_L(r_H, w)$ is convex in r_H.* Define

$$g(r_H, r_L) \equiv \frac{1}{n} \cdot c_1\left(\frac{p - r_L}{r_H - r_L}, r_H\right) - r_H$$

so that (FOCq) can be written as:

$$g(r_H, r_L) = w - 1.$$

Substituting for c_1 we obtain:

$$g(r_H, r_L) = \frac{\kappa}{n}\left[(1 - r_H)\ln\left(\frac{1 - r_H}{1 - r_L}\right) + r_H \ln\left(\frac{r_H}{r_L}\right)\right] - r_H.$$

The function $g(r_H, r_L)$ is decreasing in its second argument (r_L) and strictly convex. The former property follows from the fact that $g_2 = -\frac{\kappa}{n}\frac{1}{r_L}\frac{r_H - r_L}{1 - r_L} < 0$ while the latter follows from the fact that $g_{11} = \frac{1}{r_H(1-r_H)} > 0$ and that the determinant of the Hessian of g is positive.[12]

To see that $\hat{r}_L(r_H, w)$ is convex in r_H, denote $x \equiv (r_H, \hat{r}_L(r_H, w))$ and $x' \equiv (r'_H, \hat{r}_L(r'_H, w))$ for some two values r_H and r'_H such that $1 \geq r'_H > r_H \geq p$. For any $\alpha \in (0, 1)$, convexity of g implies that:

$$g(\alpha x + (1 - \alpha)x') < \alpha g(x) + (1 - \alpha)g(x')$$

or, equivalently,

$$g(\alpha r_H + (1-\alpha)r'_H, \alpha\hat{r}_L(r_H, w) + (1-\alpha)\hat{r}_L(r'_H, w)) < \alpha g(r_H, \hat{r}_L(r_H, w)) + (1-\alpha)g(r'_H, \hat{r}_L(r'_H, w)). \tag{29}$$

Since $g(r_H, r_L(r_H)) = g(r'_H, r_L(r'_H)) = w - 1$, the right-hand side of (29) equals $w - 1$. Denote $r''_H := \alpha r_H + (1-\alpha)r'_H$ and recall that, by definition, we have that $g(r''_H, r_L(r''_H)) = w - 1$. It therefore follows that

$$g(r''_H, \alpha\hat{r}_L(r_H, w) + (1 - \alpha)\hat{r}_L(r'_H, w)) < g(r''_H, \hat{r}_L(r''_H, w)).$$

And since g is decreasing in its second argument we obtain:

$$\hat{r}_L(r''_H, w) = \hat{r}_L(\alpha r_H + (1 - \alpha)r'_H, w) < \alpha\hat{r}_L(r_H, w) + (1 - \alpha)\hat{r}_L(r'_H, w).$$

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[12] This is because $g_{12} = -\frac{\kappa}{n}\frac{1}{r_L(1-r_L)}$ and $g_{22} = \frac{\kappa}{n}\frac{r_L^2 - 2r_H r_L + r_H}{r_L^2(1-r_L)^2}$ and therefore $(g_{11})(g_{22}) - (g_{21})^2 = \left(\frac{\kappa}{n}\right)^2 \frac{1}{r_H r_L^2}\frac{(r_H - r_L)^2}{(1-r_H)(r_L-1)^2} > 0$

Thus, for any w, the function $\hat{r}_L(r_H, w)$ is convex in r_H.

(P3) *For any r_H, the function $\hat{r}_L(r_H, w)$ is decreasing in w.* Suppose that $w' > w$. Then, by definition, for every r_H we have that $g(r_H, \hat{r}_L(r_H, w)) = w - 1$ and $g(r_H, \hat{r}_L(r_H, w')) = w' - 1$. Therefore:

$$g(r_H, \hat{r}_L(r_H, w')) > g(r_H, \hat{r}_L(r_H, w)).$$

Since g is decreasing in its second argument (as we showed in P2 above) it immediately follows that $\hat{r}_L(r_H, w') < \hat{r}_L(r_H, w)$.

(P1) and (P2) establish that for any w, the function $\hat{r}_L(r_H, w)$ is convex in r_H and attains minimum at $\tilde{r}_H$. We therefore deduce that for all values of $r_H \geq 1-\theta^{(n-m+1)} > \tilde{r}_H$ the function $\hat{r}_L(r_H, w)$ is *increasing* in r_H. Since $\hat{\mathcal{L}}$ is *decreasing* in $\hat{r}_L$ (see Equation 27) it follows that if r_H is restricted to the domain $\left[1-\theta^{(n-m+1)}, 1\right]$, so that non-wastefulness is satisfied, then the maximum of $\hat{\mathcal{L}}$ is attained at $r_H = 1-\theta^{(n-m+1)}$. Thus, $r_H^*(\theta_1, \theta_2) = 1-\theta^{(n-m+1)}$, which completes Part I of the proof. ■

PART II. We now turn to show that $q^*(\theta, \lambda)$ is increasing in each player's type while $r_H^*(\theta, \lambda)$ and $r_L^*(\theta, \lambda)$ are decreasing in each player's type, where $\theta \equiv (\theta_i, \theta_{-i})$. Fix θ_{-i} and λ. Suppose that $\theta_i' > \theta_i$ and denote $w \equiv w(\theta_i, \theta_{-i}, \lambda)$ and $w' \equiv w(\theta_i', \theta_{-i}, \lambda)$ so that $w' > w$. We also denote $\theta' \equiv (\theta_i', \theta_{-i})$.

If $(\tilde{r}_H(\theta, \lambda), \tilde{q}(\theta, \lambda))$ is not interior, then no signal is acquired when the players report θ, i.e $q^*(\theta, \lambda) = 0$ and $r_L^*(\theta, \lambda) = p$. Without loss of generality we can assume that in this case $r_H^*(\theta, \lambda) = 1$, and it immediately follows that $q^*(\theta', \lambda) \geq q^*(\theta, \lambda)$ and $r_L^*(\theta', \lambda) \leq r_L^*(\theta, \lambda)$ and $r_H^*(\theta', \lambda) \leq r_H^*(\theta, \lambda)$.

For the rest of Part II we then assume that $(\tilde{r}_H(\theta, \lambda), \tilde{q}(\theta, \lambda))$ is interior. This also implies that $(\tilde{r}_H(\theta', \lambda), \tilde{q}(\theta', \lambda))$ is interior. To see why, note that assigning the signal $\tilde{r}_H(\theta, \lambda)$ and $\tilde{q}(\theta, \lambda)$ whenever the type realization is θ' yields a total surplus which is higher than the total surplus attained by the same signal when the realization is θ (which is positive). Hence, the corner solution in which no signal is acquired (i.e., the *only* possible corner solution) cannot be optimal when the type realization is θ'.

We divide the analysis into four cases:

Case 1: *Suppose that $\tilde{r}_H(\theta, \lambda) \geq 1-(\theta_i, \theta_{-i})^{(n-m+1)}$ and $\tilde{r}_H(\theta', \lambda) \geq 1-(\theta_i', \theta_{-i})^{(n-m+1)}$.* In this case both $(\tilde{r}_H(\theta, \lambda), \tilde{q}(\theta, \lambda))$ and $(\tilde{r}_H(\theta', \lambda), \tilde{q}(\theta', \lambda))$ are interior and non-wasteful. Since $\tilde{r}_H$ and $\tilde{r}_L$ (as given by Equations 25 and 24) are decreasing in $w(\theta_i, \theta_{-i}, \lambda)$ and

$w(\theta_i, \theta_{-i}, \lambda)$ is increasing in θ_i, then:

$$r_L^*(\theta', \lambda) = \tilde{r}_L(\theta', \lambda) = \frac{e^{1-\frac{n}{\kappa}w\prime} - 1}{e-1} < \frac{e^{1-\frac{n}{\kappa}w} - 1}{e-1} = \tilde{r}_L(\theta, \lambda) = r_L^*(\theta, \lambda),$$
$$r_H^*(\theta', \lambda) = \tilde{r}_H(\theta', \lambda) = \frac{e - e^{\frac{n}{\kappa}w\prime}}{e-1} < \frac{e - e^{\frac{n}{\kappa}w}}{e-1} = \tilde{r}_H(\theta, \lambda) = r_H^*(\theta, \lambda).$$

And since $\tilde{q}$ is decreasing in $\tilde{r}_L$ and decreasing in $\tilde{r}_H$ for all $\tilde{r}_L \leq p \leq \tilde{r}_H$ we obtain:[13]

$$q^*(\theta', \lambda) = \tilde{q}(\theta', \lambda) = \frac{p - \tilde{r}_L(w')}{\tilde{r}_H(w') - \tilde{r}_L(w')} > \frac{p - \tilde{r}_L(w)}{\tilde{r}_H(w) - \tilde{r}_L(w)} = \tilde{q}(\theta, \lambda) = q^*(\theta, \lambda).$$

Case 2: *Suppose that* $\tilde{r}_H(w) \leq 1 - (\theta_i, \theta_{-i})^{(n-m+1)}$ *and* $\tilde{r}_H(w') \leq 1 - (\theta_i', \theta_{-i})^{(n-m+1)}$. In this case both $(\tilde{r}_H(\theta, \lambda), \tilde{q}(\theta, \lambda))$ and $(\tilde{r}_H(\theta', \lambda), \tilde{q}(\theta', \lambda))$ are interior but wasteful. Therefore, by Part I of the proof, we have that $r_H^*(\theta_i', \theta_i, \lambda) = 1 - (\theta_i', \theta_{-i})^{(n-m+1)}$ and $r_H^*(\theta_i, \theta_{-i}, \lambda) = 1 - (\theta_i, \theta_{-i})^{(n-m+1)}$. Since $(\theta_i', \theta_{-i})^{(n-m+1)} \geq (\theta_i, \theta_{-i})^{(n-m+1)}$ we deduce:

$$r_H^*(\theta_i, \theta_{-i}, \lambda) \geq r_H^*(\theta_i', \theta_{-i}, \lambda) \geq \tilde{r}_H(\theta_i', \theta_{-i}, \lambda)$$

and therefore $r_H^*(\theta_i, \theta_{-i}, \lambda)$ is decreasing in each player's type.

Next, recall that in the proof of Lemma (4) we showed that due to (P1) and (P2) the function $\hat{r}_L(r_H, w')$ is *increasing* in r_H when $r_H > \tilde{r}_H(\theta_i', \theta_{-i}, \lambda)$. Hence,

$$\hat{r}_L(r_H^*(\theta_i', \theta_{-i}, \lambda), w') < \hat{r}_L(r_H^*(\theta_i, \theta_{-i}, \lambda), w').$$

In that proof we also established (P3), by which for any r_H the function $\hat{r}_L(r_H, w)$ is *decreasing* in w. Since $w' > w$ we have that $\hat{r}_L(r_H^*(\theta_i, \theta_{-i}, \lambda), w') < \hat{r}_L(r_H^*(\theta_i, \theta_{-i}, \lambda), w)$ and therefore:

$$r_L^*(\theta_i', \theta_{-i}, \lambda) = \hat{r}_L(r_H^*(\theta_i', \theta_{-i}, \lambda), w') < \hat{r}_L(r_H^*(\theta_i, \theta_{-i}, \lambda), w) = r_L^*(\theta_i, \theta_{-i}, \lambda).$$

Finally, since $q^* = \frac{p - r_L^*}{r_H^* - r_L^*}$ is decreasing in r_L^* and decreasing in r_H^* for all $r_L^* \leq p \leq r_H^*$ then $q^*(\theta_i', \theta_{-i}, \lambda) > q^*(\theta_i, \theta_{-i}, \lambda)$.

Case 3: *Suppose that* $\tilde{r}_H(\theta_i, \theta_{-i}, \lambda) < 1 - (\theta_i, \theta_{-i})^{(n-m+1)}$ *and* $\tilde{r}_H(\theta_i', \theta_{-i}, \lambda) \geq 1 - (\theta_i', \theta_{-i})^{(n-m+1)}$. The functions $\tilde{r}_H(x, \theta_{-i}, \lambda)$ and $1 - (x, \theta_{-i})^{(n-m+1)}$ are both continuous

[13] This is because $\frac{d}{dr_L}\left(\frac{p-r_L}{r_H-r_L}\right) = -\frac{r_H-p}{(r_H-r_L)^2}$ and $\frac{d}{dr_H}\left(\frac{p-r_L}{r_H-r_L}\right) = -\frac{p-r_L}{(r_H-r_L)^2}$.

in x. Hence, there must be at least one value $\theta_i'' \in (\theta_i, \theta_i')$ for which

$$\tilde{r}_H(\theta_i'', \theta_{-i}, \lambda) = 1 - (\theta_i'', \theta_{-i})^{(n-m+1)}.$$

According to Case 2 above we know that $q^*(\theta_i'', \theta_{-i}, \lambda) > q^*(\theta_i, \theta_{-i}, \lambda)$. According to Case 1 above we know that $q^*(\theta_i', \theta_{-i}, \lambda) > q^*(\theta_i'', \theta_{-i}, \lambda)$. We thus conclude that $q^*(\theta_i', \theta_{-i}, \lambda) > q^*(\theta_i, \theta_{-i}, \lambda)$. Analogous arguments show that $r_L^*(\theta_i', \theta_{-i}, \lambda) < r_L^*(\theta_i, \theta_{-i}, \lambda)$ and that $r_H^*(\theta_i', \theta_{-i}, \lambda) < r_H^*(\theta_i, \theta_{-i}, \lambda)$.

Case 4: *Suppose that* $\tilde{r}_H(\theta_i, \theta_{-i}, \lambda) > 1-(\theta_i, \theta_{-i})^{(n-m+1)}$ *and* $\tilde{r}_H(\theta_i', \theta_{-i}, \lambda) \leq (\theta_i', \theta_{-i})^{(n-m+1)}$. As in Case 3 above we can find a value $\theta_i'' \in (\theta_i, \theta_i')$ for which

$$\tilde{r}_H(\theta_i'', \theta_{-i}, \lambda) = 1 - (\theta_i'', \theta_{-i})^{(n-m+1)}.$$

According to Case 1 above we know that $q^*(\theta_i'', \theta_{-i}, \lambda) > q^*(\theta_i, \theta_{-i}, \lambda)$. According to Case 2 above we know that $q^*(\theta_i', \theta_{-i}, \lambda) > q^*(\theta_i'', \theta_{-i}, \lambda)$. We thus again conclude that $q^*(\theta_i', \theta_{-i}, \lambda) > q^*(\theta_i, \theta_{-i}, \lambda)$. Analogous arguments show that $r_L^*(\theta_i', \theta_{-i}, \lambda) < r_L^*(\theta_i, \theta_{-i}, \lambda)$ and that $r_H^*(\theta_i', \theta_{-i}, \lambda) < r_H^*(\theta_i, \theta_{-i}, \lambda)$.

Part III. From the above two lemmas, it follows that for any $\lambda \geq 0$ and for each profile of types θ, the values $q^*(\theta, \lambda)$ and $r_H^*(\theta, \lambda)$ that maximize $\mathcal{L}(q, r_H, \lambda)$ are such that $q^*(\theta, \lambda)$ is unique and $r_H^*(\theta, \lambda)$ is unique whenever $q^*(\theta, \lambda) > 0$ (i.e., whenever a signal is purchased). It remains to verify that there exists $\lambda^* \geq 0$ for which $q^*(\theta, \lambda^*)$ and $r_H^*(\theta, \lambda^*)$ induce a non-negative expected aggregate virtual surplus (i.e., there exists $\lambda^* \geq 0$ for which $q^*(\cdot, \lambda^*)$ and $r_H^*(\cdot, \lambda^*)$ are feasible). Let $S(\lambda)$ denote the ex-ante expected virtual surplus (that is given by Equation 15) as a function of λ :

$$S(\lambda) = \mathbb{E}_\theta \left[\sum_{i=1}^{n} \left(v(\theta_i) q^*(\theta, \lambda) - (1 - r_H^*(\theta, \lambda))\, q^*(\theta, \lambda) \right) - c(q^*(\theta, \lambda), r_H^*(\theta, \lambda)) \right]$$

Lemmas 1 and 2 in Hellwig (2003) guarantee that $S(\lambda)$ is continuous in λ and that $S(\lambda) \geq 0$ for a sufficiently large λ. This completes the proof. ■

Proof of Proposition 3

By Corollary 1, there exists an optimal auxiliary mechanism in which truthtelling is a dominant strategy. Consider an actual mechanism with the same functions q^*, r_H^* and r_L^* and the same transfer rules (the only difference between the auxiliary and actual mechanisms is that in the latter the players are not bound by their report in the ensuing voting game). We will show that truthtelling is a dominant strategy also in the actual mechanism.

Assume, by contradiction, that truthtelling is not a dominant strategy in the actual mechanism. This means that there is a type θ_i of player i that prefers to report some $\theta_i' \neq \theta_i$ in the actual mechanism, but not in the auxiliary mechanism, when the other players report some θ_{-i} (which may not coincide with their true types).

It cannot be that $q^*(\theta_i', \theta_{-i}) = 0$ (to simplify the notation we omit throughout this proof the dependence of q^*, r^* and r_L^* on the value of λ^*). To see why, note that when no information is acquired (i.e., $q^*(\theta_i', \theta_{-i}) = 0$) player i prefers the action $a = 0$ in the voting game that follows the actual mechanism. But this is precisely the action that the auxiliary mechanism chooses when $q^*(\theta_i', \theta_{-i}) = 0$. Since player i does not want to deviate and report θ_i' in the auxiliary mechanism, he has no incentive to do so in the actual mechanism.

Suppose that $q^*(\theta_i', \theta_{-i}) > 0$. When the posterior belief $r_L^*(\theta_i', \theta_{-i})$ is realized, the auxiliary mechanism votes for $a = 0$ on player i's behalf. But since $r_L^*(\theta_i', \theta_{-i}) < p$ this is also the action that player i prefers in the voting game that follows the actual mechanism. Suppose then that the posterior $r_H^*(\theta_i', \theta_{-i})$ is realized. Recall that since signals that are purchased in the optimal auxiliary mechanism are non-wasteful then $r_H^*(\theta_i', \theta_{-i}) \geq 1 - (\theta_i', \theta_{-i})^{(n-m+1)} \geq p$. If for such a posterior, player i votes for $a = 1$ in the second stage game following the actual mechanism, then again his action coincides with the action that the auxiliary mechanism chooses for him. Therefore, for i to have a profitable deviation in the actual mechanism but not in the auxiliary mechanism, it must be the case that after $r_H^*(\theta_i', \theta_{-i}) > 1 - (\theta_i', \theta_{-i})^{(n-m+1)}$ player i prefers to vote for $a = 0$. This means that player i of type θ_i strictly gains by increasing the chances of the default action. He may further increase his utility if $m(\theta_i', \theta_{-i}) + t_i(\theta_i', \theta_{-i}) < m(\theta_i, \theta_{-i}) + t_i(\theta_i, \theta_{-i})$. Since by monotonicity of q^* we have $q^*(0, \theta_{-i}) \leq q^*(\theta_i, \theta_{-i})$, and since $m(0, \theta_{-i}) + t_i(0, \theta_{-i}) \leq m(\theta_i, \theta_{-i}) + t_i(\theta_i, \theta_{-i})$ (which immediately follows from the fact that type 0 does not want to report θ_i in the auxiliary mechanism), then the most profitable deviation is to report $\theta_i' = 0$.

If $q^*(0, \theta_{-i}) < q^*(\theta_i, \theta_{-i})$ or $m(0, \theta_{-i}) + t_i(0, \theta_{-i}) < m(\theta_i, \theta_{-i}) + t_i(\theta_i, \theta_{-i})$ then player i has a profitable deviation already in the auxiliary mechanism by reporting that his type is 0. This contradicts truth-telling being a dominant strategy. Otherwise, player i is indifferent between reporting the truth and his most profitable deviation in the actual mechanism, contradicting our initial assumption that player i has a profitable deviation in the actual mechanism. We have therefore established that truthtelling is a dominant strategy in the actual mechanism.

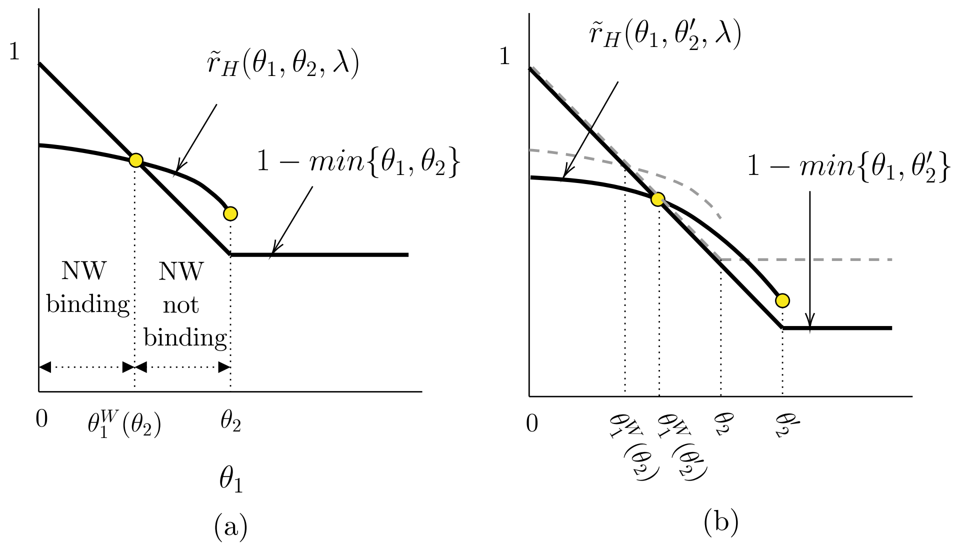

Figure 4: The functions $\tilde{r}_H(\theta_1, \theta_2, \lambda^*)$ and $1 - min\{\theta_1, \theta_2\}$ for some fixed θ_2

Finally, note that in the optimal auxiliary mechanism in which truthtelling is a dominant strategy the budget balance constraint is satisfied only ex-ante. Therefore, the budget of the actual mechanism defined above is also balanced only ex-ante. However, since truthtelling is a dominant strategy in the actual mechanism then it is also a Bayesian Nash equilibrium. Thus, by Borgers (2015, p.47), we can modify the transfers to satisfy ex-post budget balanceness without affecting the interim expected transfers, and hence, truthtelling remains a Bayesian Nash equilibrium. Furthermore, the individual rationality of the auxiliary mechanism also carries over to the real mechanism. Thus, the resulting actual mechanism satisfies incentive compatibility, individual rationality and it is budget-balanced ex-post. Since, by Proposition 1, the expected surplus that is achievable by the optimal actual mechanism is bounded above by the expected surplus that is achievable by the optimal auxiliary mechanism, it follows that the actual mechanism we defined above is the optimal one. ■

Proof of Proposition 4

We use the notation defined in the proof of Proposition 2. Let λ^* be the value that is determined by Proposition 2.

(i) Fix θ_2. For any type θ_1, a signal is purchased in the second best mechanism whenever $r_L^*(\theta_1, \theta_2, \lambda^*) < p$. By Proposition 2, we know that $r_L^*(\theta_1, \theta_2, \lambda^*)$ is decreasing in θ_1.[14]

[14] We are also guaranteed that $r_L^*(\theta_1, \theta_2, \lambda) > 0$ for any θ_1 because, as we discuss in the proof of Proposition (2), the corner solution in which r_L is 0 (or, equivalently, $r_H = p/q$) never maximizes the

Thus, for any $\theta_1' > \theta_1$ we have that $r_L^*(\theta_1', \theta_2, \lambda^*) < r_L^*(\theta_1, \theta_2, \lambda^*) < p$, implying that if a signal is purchased in the second best mechanism for some θ_1 then it is purchased also for $\theta_1' > \theta_1$. It follows that for any θ_2 there is a unique cutoff $\theta_1^{\emptyset}(\theta_2)$ for which a signal is purchased if and only if $\theta_1 > \theta_1^{\emptyset}(\theta_2)$.

(ii) Suppose that θ_1 and θ_2 are such that $r_L^*(\theta_1, \theta_2, \lambda^*) < p$, so that a signal is purchased in the second best mechanism. By Proposition 2, $r_L^*(\theta_1, \theta_2, \lambda^*)$ is decreasing in θ_2, and therefore $r_L^*(\theta_1, \theta_2', \lambda^*) < p$ for any $\theta_2' > \theta_2$. Thus, the cutoff $\theta_1^{\emptyset}(\theta_2)$, below which a signal is not purchased for any $\theta_1 < \theta_1^{\emptyset}(\theta_2)$, is decreasing in θ_2.

(iii) Fix θ_2. Since $w(\theta_1, \theta_2, \lambda^*)$ is increasing in θ_1 then $\tilde{r}_H(\theta_2, \theta_2, \lambda^*) = \frac{e^{\frac{2}{\kappa}} - e^{\frac{2}{\kappa} w(\theta_1, \theta_2, \lambda^*)}}{e^{\frac{2}{\kappa}} - 1}$ is decreasing in θ_1. In addition, our assumption that $\frac{1-F(\theta_1)}{f(\theta_1)}$ is concave in θ_1 implies that $\tilde{r}_H(\theta_2, \theta_2, \lambda^*)$ is also concave in θ_1.[15] Also note that $1 - \min\{\theta_1, \theta_2\}$ is decreasing and convex in θ_1. Figure (4a) illustrates the functions $\tilde{r}_H(\theta_2, \theta_2, \lambda^*)$ and $1 - \min\{\theta_1, \theta_2\}$ for some θ_2, where the values if θ_1 are depicted on the horizontal axis.

When $\theta_1 = \theta_2$ we know that $\tilde{r}_H(\theta_2, \theta_2, \lambda^*) > 1 - \theta_2$. The reason is that by (FOCq) we have $\tilde{r}_H(\theta_2, \theta_2, \lambda^*) - (1 - w(\theta_2, \theta_2, \lambda^*)) = c_1(\tilde{q}(\theta_2, \theta_2, \lambda^*), \tilde{r}_H(\theta_2, \theta_2, \lambda^*))$, and since

$$c_1(\tilde{q}(\theta_2, \theta_2, \lambda^*), \tilde{r}_H(\theta_2, \theta_2, \lambda^*)) = D_{KL}(\tilde{r}_H(\theta_2, \theta_2, \lambda^*), \tilde{r}_L(\theta_2, \theta_2, \lambda^*)) > 0,$$

where $\tilde{r}_L(\theta_2, \theta_2, \lambda^*) = \frac{p - \tilde{q}(\theta_2, \theta_2, \lambda^*) \cdot \tilde{r}_H(\theta_2, \theta_2, \lambda^*)}{1 - \tilde{q}(\theta_2, \theta_2, \lambda^*)}$, then $\tilde{r}_H(\theta_2, \theta_2, \lambda^*) - (1 - w(\theta_2, \theta_2, \lambda^*)) > 0$. Since $w(\theta_2, \theta_2, \lambda^*) \leq \theta_2$ we obtain that $\tilde{r}_H(\theta_2, \theta_2, \lambda^*) > 1 - \theta_2$.

Thus, holding θ_2 fixed, the functions $\tilde{r}_H(\theta_1, \theta_2, \lambda^*)$ and $1 - \min\{\theta_1, \theta_2\}$ cross each other at most once in the range $\theta_1 \in [0, \theta_2]$. If the functions never cross each other, i.e. $\tilde{r}_H(\theta_1, \theta_2, \lambda^*) > 1 - \min\{\theta_1, \theta_2\}$ for all $\theta_1 \in [0, \theta_2]$, then non-wasefulness is never binding and in that case $\theta_1^W(\theta_2) = \theta_1^{\emptyset}(\theta_2)$. If the functions cross each other exactly once, then $\theta_1^W(\theta_2)$ is the value of θ_1 at the point of crossing. Thus, $\tilde{r}_H(\theta_1, \theta_2, \lambda^*) > 1 - \min\{\theta_1, \theta_2\}$ for any $\theta_1 > \theta_1^W(\theta_2)$ and $\tilde{r}_H(\theta_1, \theta_2, \lambda^*) < 1 - \min\{\theta_1, \theta_2\}$ for any $\theta_1 < \theta_1^W(\theta_2)$.

(iv) Suppose that for some θ_2 we have $\theta_1^W(\theta_2) > \theta_1^{\emptyset}(\theta_2)$, so non-wastefulness is binding for some values of θ_1. From (iii) we know that $\theta_1^W(\theta_2) < \theta_2$. Pick some $\theta_2' > \theta_2$. If $\theta_1^W(\theta_2') > \theta_2$ then $\theta_1^W(\theta_2) < \theta_1^W(\theta_2')$ and the proof is complete. Otherwise, in the range $\theta_1 \in [0, \theta_2']$ we have that $1 - \min\{\theta_1, \theta_2'\} = 1 - \theta_1$, as illustrated in Figure (4b). In addition, since $\tilde{r}_H(\theta_1, \theta_2, \lambda^*)$ is decreasing in θ_2 then $\tilde{r}_H\left(\theta_1^W(\theta_2), \theta_2', \lambda^*\right) < \tilde{r}_H\left(\theta_1^W(\theta_2), \theta_2, \lambda^*\right) =$

(partial) Lagrangian that is given by Equation (22).

[15]This is bacuase: (i) $\tilde{r}_H$ is decreasing and concave in w, and (ii) w is convex in θ_1, due to our assumption that $\frac{1-F(\theta_1)}{f(\theta_1)}$ is concave. Thus, $\frac{d^2\tilde{r}_H}{(d\theta_1)^2} = \frac{d^2\tilde{r}_H}{(dw)^2} \cdot \left(\frac{dw}{d\theta_1}\right)^2 + \frac{d^2 w}{(d\theta_1)^2} \cdot \frac{d\tilde{r}_H}{dw} < 0$.

$1-\theta_1^W(\theta_2)$ (note that in Figure (4b) $\tilde{r}_H(\theta_1,\theta_2,\lambda^*)$ and $1-\min\{\theta_1,\theta_2\}$ are the dashed lines). From (iii) we also know that $\tilde{r}_H(\theta_2',\theta_2',\lambda^*)>1-\theta_2'$. Therefore, continuity of $\tilde{r}_H$ implies that there exists $x\in\left(\theta_1^W(\theta_2),\theta_2'\right)$ for which $\tilde{r}_H(x,\theta_2',\lambda^*)=1-x$. The cutoff $\theta_1^W(\theta_2')$ is then given by x, implying that $\theta_1^W(\theta_2')\equiv x>\theta_1^W(\theta_2)$.

For online publication
Supplementary Appendix to Collective Information Acquisition

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1 The variance cost function

In this appendix we show that our analysis extends to the case in which the cost of a signal $\{(q_j, r_j)\}_{j=1}^{J}$ is proportional to the *variance* of the posteriors on the state $\omega = 1$, where the mean posterior is the prior (i.e., $\sum_{j=1}^{J} q_j \cdot r_j = p$.). I.e.,

$$c\left(\{(q_j, r_j)\}_{j=1}^{J}\right) = \kappa \cdot \sum_{j=1}^{J} q_j \left(r_j - p\right)^2 .$$

Note that Lemma 1 extends to this cost function. To see why, define $h(r) \equiv (r-p)^2$ and note that this function is convex in r. Then, the same arguments in the proof of Lemma 1 readily apply to the newly defined function $h(r)$.

It follows that we can restrict attention to signals that are represented by the triplet (q, r_H, r_L) as in the main text. We therefore consider the cost function

$$c(q, r_H, r_L) = \kappa \cdot \left(q \cdot (r_H - p)^2 + (1-q)(r_L - p)^2\right).$$

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Substituting $r_L = \frac{p - q r_H}{1-q}$, we can rewrite the cost as a function of only q and r_H :

$$c(q, r_H) = \kappa \frac{q}{1-q} (r_H - p)^2 .$$

To simplify the exposition, we focus on the case where $\kappa > n$. This guarantees that information is never "too cheap" so that removing all the uncertainty (i.e., $r_H = 1$ and $r_L = 0$) becomes optimal (the KL-divergence cost function in our main text satisfies this for all (n, κ)). The analysis remains essentially the same when $\kappa \leq n$, but the exposition is more cumbersome since we have to take care of more corner solutions in the designer's optimization problem.

To establish that the qualitative analysis in the main text extends to the above cost function, we mimic the steps in the proof of Proposition 2. First, we find three functions, $r_H^*(\theta, \lambda), r_L^*(\theta, \lambda)$ and $q^*(\theta, \lambda)$, that satisfy non-wastefulness and maximize the Lagrangian that is given by Equation (18), for any multiplier λ and any profile of types θ. Second, we show that for *any* $\lambda \geq 0$, the function $q^*(\theta, \lambda)$ is increasing in each player's type while $r_H^*(\theta, \lambda)$ and $r_L^*(\theta, \lambda)$ are decreasing in each player's type. Hence, the function $Q^*(\theta_i, \lambda)$ that is induced by $q^*(\theta, \lambda)$ (according to Equation 16) is monotone. Third, we apply an argument in Hellwig (2013) that guarantees the existence of some $\lambda^* \geq 0$ for which the mechanism defined by $r_H^*(\theta, \lambda^*), r_L^*(\theta, \lambda^*)$ and $q^*(\theta, \lambda^*)$ has a non-negative virtual surplus. It then follows that the functions $r_H^*(\theta, \lambda^*), r_L^*(\theta, \lambda^*)$ and $q^*(\theta, \lambda^*)$ define the mechanism that attains the maximal aggregate surplus (Equation 14) subject to (i) $Q(\theta_i)$ is monotone and (ii) the aggregate virtual surplus (Equation 15) is non negative.

PART I. The first-order condition with respect to q for an interior solution $(\tilde{q}, \tilde{r}_H)$ that maximizes the Lagrangian $\mathcal{L}(q, r_H, \lambda)$ is

$$r_H - (1-w) - \frac{1}{n} \cdot \kappa \cdot \left(\frac{r_H - p}{1-q} \right)^2 = 0 \tag{FOCq}$$

while the first-order condition with respect to r_H is

$$q - \frac{1}{n} \cdot 2 \left(\kappa \frac{q}{1-q} (r_H - p) \right) = 0 \tag{FOCr}$$

where w is as defined in the main text. From the second equation we have that $\frac{r_H - p}{1-q} = \frac{n}{2\kappa}$. Plugging this into the first equation yields:

$$r_H - (1-w) - \frac{1}{n} \cdot \kappa \cdot \left(\frac{n}{2\kappa} \right)^2 = 0$$

Hence,

$$\tilde{r}_H(w) = \frac{n}{4\kappa} + 1 - w \tag{1}$$

Using $\frac{r_H - p}{1-q} = \frac{n}{2\kappa}$ again we can solve for $\tilde{q}$:

$$\tilde{q}(w) = \frac{1}{2} + \frac{2\kappa}{n}(w - (1-p)) \tag{2}$$

Finally, from $r_L = \frac{p - q r_H}{1-q}$ it follows that

$$\tilde{r}_L(w) = 1 - w - \frac{n}{4\kappa} \tag{3}$$

Since $\mathcal{L}$ is not a concave function in general, it is not a priori guaranteed that $(\tilde{q}, \tilde{r}_H)$ is a maximizer of $\mathcal{L}$. Our first result establishes that although $\mathcal{L}$ is not concave, if $(\tilde{q}, \tilde{r}_H)$ is interior and non-wasteful then it is indeed a maximizer of $\mathcal{L}$.

Lemma A0. For any value of r_H and w, define $\hat{q}\left(r_H, w\right)$ to be the value of q that satisfies $\mathcal{L}_1\left(q, r_H; w\right) = 0$ (i.e. FOCq) and define $\hat{r}_L\left(r_H, w\right) \equiv \frac{p - r_H \cdot \hat{q}(r_H, w)}{1 - \hat{q}(r_H, w)}$ to be value of r_L that is determined by r_H and $\hat{q}\left(r_H, w\right)$. Then, $\hat{\mathcal{L}}\left(r_H, w\right) \equiv \mathcal{L}\left(\hat{q}\left(r_H, w\right), r_H\ ;\ w\right) = \kappa(p - r_L)^2$.
Proof. By definition we have that

$$\hat{\mathcal{L}}\left(r_H, w\right) = \hat{q}\left(r_H, w\right) \cdot c_1\left(\hat{q}\left(r_H, w\right), r_H\right) - c\left(\hat{q}\left(r_H, w\right), r_H\right)$$

Substituting

$$\begin{aligned}
\hat{q}\left(r_H, w\right) &= \frac{p - \hat{r}_L\left(r_H, w\right)}{r_H - \hat{r}_L\left(r_H, w\right)} \\
c_1\left(\hat{q}\left(r_H, w\right), r_H\right) &= \kappa \cdot \left(\frac{r_H - p}{1 - \hat{q}\left(r_H, w\right)}\right)^2 \\
&= \kappa \cdot \left(\frac{r_H - p + \hat{q}\left(r_H, w\right) r_H - \hat{q}\left(r_H, w\right) r_H}{1 - \hat{q}\left(r_H, w\right)}\right)^2 \\
&= \kappa \cdot \left(r_H - \hat{r}_L\left(r_H, w\right)\right)^2 \\
c(r_H, \hat{r}_L\left(r_H, w\right)) &= \kappa \cdot \left(\hat{q}\left(r_H, w\right) \cdot \left(r_H - p\right)^2 + \left(1 - \hat{q}\left(r_H, w\right)\right)\left(\hat{r}_L\left(r_H, w\right) - p\right)^2\right) \\
&= \kappa \cdot [\frac{p - \hat{r}_L\left(r_H, w\right)}{r_H - \hat{r}_L\left(r_H, w\right)} \cdot \left(r_H - p\right)^2 + \frac{r_H - p}{r_H - \hat{r}_L\left(r_H, w\right)} \cdot \left(p - \hat{r}_L\left(r_H, w\right)\right)^2] \\
&= \kappa(r_H - p)(p - \hat{r}_L\left(r_H, w\right))
\end{aligned}$$

We obtain that

$$\hat{\mathcal{L}}(r_H, w) = \frac{p - \hat{r}_L(r_H, w)}{r_H - \hat{r}_L(r_H, w)} \cdot \kappa \cdot (r_H - \hat{r}_L(r_H, w))^2 - \kappa(r_H - p)(p - \hat{r}_L(r_H, w))$$
$$= \kappa(p - \hat{r}_L(r_H, w))^2. \qquad \square$$

Lemma A1. *For any w, if $(\tilde{q}, \tilde{r}_H)$ is interior and non-wasteful, then it maximizes $\mathcal{L}(q, r_H\ ;\ w)$.*

Proof. First, we show that $(\tilde{q}, \tilde{r}_H)$ is a local maximum of $\mathcal{L}(q, r_H; w)$. Then, we show that $\mathcal{L}(\tilde{q}, \tilde{r}_H; w)$ is greater than the value of $\mathcal{L}$ in any corner solution.

To show that $(\tilde{q}, \tilde{r}_H)$ is a local maximum it suffices to show that (i) $\mathcal{L}_{11}(q, r_H; w) < 0$ and (ii) the determinant of the Hessian of $\mathcal{L}(q, r_H; w)$ is positive, when evaluated at $(\tilde{q}, \tilde{r}_H)$. To establish (i), note that

$$\mathcal{L}_{11}(q, r_H; w) = \frac{d}{dq}\left(r_H - (1-w) - \frac{1}{n} \cdot \kappa \cdot \left(\frac{r_H - p}{1-q}\right)^2\right) = -\frac{2}{n}\kappa\frac{(p - r_H)^2}{(1-q)^3} < 0$$

To establish (ii) note that

$$\mathcal{L}_{22}(q, r_H; w) = \frac{d}{dr_H}\left(q - \frac{1}{n} \cdot 2\left(\kappa\frac{q}{1-q}(r_H - p)\right)\right) = -\frac{2}{n}q\frac{\kappa}{1-q}$$
$$\mathcal{L}_{12}(q, r_H; w) = \frac{d}{dq}\left(q - \frac{1}{n} \cdot 2\left(\kappa\frac{q}{1-q}(r_H - p)\right)\right) = 1 - \frac{2\kappa(r_H - p)}{n(q-1)^2}$$

The determinant of the Hessian is equal to $\mathcal{L}_{11}(q, r_H; w) \cdot \mathcal{L}_{22}(q, r_H; w) - (\mathcal{L}_{12}(q, r_H; w))^2$, which reduces to

$$\left(\frac{2\kappa}{n} \cdot \frac{r_H - p}{1-q}\right)^2 \cdot \frac{q}{(1-q)^2} - \left(1 - \frac{1}{1-q} \cdot \frac{2\kappa}{n} \cdot \frac{r_H - p}{1-q}\right)^2$$

At $(\tilde{q}, \tilde{r}_H)$ we have $\frac{\tilde{r}_H - p}{1-\tilde{q}} = \frac{n}{2\kappa}$, and hence, the determinant reduces to

$$\frac{\tilde{q}}{(1-\tilde{q})^2} - \frac{\tilde{q}^2}{(1-\tilde{q})^2} > 0.$$

We now turn to show that $\mathcal{L}(\tilde{q}, \tilde{r}_H; w)$ is (weakly) greater than the value of $\mathcal{L}$ in any corner solution. To see this, recall that q can take any value from 0 to $\frac{p}{r_H}$ and that

$$\mathcal{L}\left(q, r_H; w\right) = q[r_H - (1-w)] - \frac{1}{n} \cdot c(q, r_H)$$

If $q = 0$, then no signal is acquired and hence, $\mathcal{L}\left(0, r_H; w\right) = 0$. On the other hand, $\mathcal{L}\left(\tilde{q}, \tilde{r}_H; w\right)$ can be written as

$$\left[\frac{p - \tilde{r}_L}{\tilde{r}_H - \tilde{r}_L} \cdot \frac{\kappa}{n} \cdot (\tilde{r}_H - \tilde{r}_L)^2 - \frac{1}{n}\kappa(\tilde{r}_H - p)(p - \tilde{r}_L)\right] = \kappa(p - \tilde{r}_L)^2 \geq 0 \tag{4}$$

Suppose next that $\mathcal{L}\left(\tilde{q}, \tilde{r}_H; w\right) < \max_{r_H \in (p,1]} \mathcal{L}\left(\frac{p}{r_H}, r_H; w\right)$. Since by (4), $\mathcal{L}\left(\tilde{q}, \tilde{r}_H; w\right) \geq 0$ while

$$\mathcal{L}\left(\frac{p}{r_H}, r_H; w\right) = \frac{p}{r_H}[r_H - (1-w)] - \frac{\kappa}{n} \cdot \frac{p}{r_H - p}\left(r_H - p\right)^2 \tag{5}$$

it must be that any $r'_H \in \arg\max_{r_H} \mathcal{L}\left(\frac{p}{r_H}, r_H; w\right)$ is greater or equal to $1-w$ (otherwise, $\max_{r_H \in (p,1]} \mathcal{L}\left(\frac{p}{r_H}, r_H; w\right) < 0$,a contradiction). The expression on the R.H.S. of (5) has a unique maximizer equal to $\sqrt{\frac{n}{k}(1-w)}$. For this to be greater or equal to $1-w$ it must be that $n \geq \kappa$, a contradiction.

The only remaining corner solution is $r_H = 1$. Recall that by Lemma A0, $\hat{\mathcal{L}}(r_H, w) = \kappa(p - \hat{r}_L(r_H, w))^2$. The fact that $\hat{\mathcal{L}}$ is maximized at $\tilde{r}_H$ implies that $\hat{r}_L(r_H, w)$ attains a minimum at $\tilde{r}_H(w) < 1$. It follows that

$$\hat{\mathcal{L}}(1, w) = \kappa(p - \hat{r}_L(1, w))^2 < \kappa(p - \hat{r}_L(\tilde{r}_H, w))^2 = \hat{\mathcal{L}}(\tilde{r}_H, w).$$

Since $\hat{\mathcal{L}}(\tilde{r}_H, w)$ and $\hat{\mathcal{L}}(1, w)$ are the values of the Lagrangian when r_H attains the values $\tilde{r}_H$ and 1, respectively (where q is optimally determined according to FOCq) the proof is complete. □

Lemma A2. *If an interior solution $(\tilde{q}, \tilde{r}_H)$ exists but is wasteful, then in the optimal solution, $r^*_H = 1 - \theta^{(n-m+1)}$.*

Proof. For any value of r_H and w, define $\hat{q}\left(r_H, w\right)$ to be the value of q that satisfies $\mathcal{L}_1\left(q, r_H; w\right) = 0$ and define $\hat{r}_L\left(r_H, w\right) \equiv \frac{p - r_H \cdot \hat{q}(r_H, w)}{1 - \hat{q}(r_H, w)}$ to be value of r_L that is determined by r_H and $\hat{q}\left(r_H, w\right)$. Therefore,

$$r_H - (1-w) - \frac{\kappa}{n} \cdot \left(\frac{r_H - p}{1 - \hat{q}\left(r_H, w\right)}\right)^2 = 0$$

Solving for $\hat{q}(r_H, w)$ yields

$$\hat{q}(r_H, w) = 1 - \frac{r_H - p}{\sqrt{\frac{n}{\kappa} \cdot (r_H + w - 1)}}$$

Since $\hat{r}_L(r_H, w) = (p - \hat{q}(r_H, w) \cdot r_H)/(1 - \hat{q}(r_H, w))$ the above equation is equivalent to

$$r_H - (1 - w) - \frac{\kappa}{n} (r_H - \hat{r}_L(r_H, w))^2 = 0$$

We can therefore solve for $\hat{r}_L$ to obtain

$$\hat{r}_L(r_H, w) = r_H - \sqrt{\frac{n}{\kappa} [r_H - (1 - w)]} \tag{6}$$

From (6) we can derive the following three properties of the function $\hat{r}_L(r_H, w)$:

(P1) *For any w, the function $\hat{r}_L(r_H, w)$ attains a minimum at $\tilde{r}_H$.* Since

$$\frac{\partial}{\partial r_H} \hat{r}_L(r_H, w) = 1 - \frac{1}{2} \sqrt{\frac{n}{\kappa}} (r_H + w - 1)^{-\frac{1}{2}}$$

we have that

$$\frac{\partial}{\partial r_H} \hat{r}_L(r_H, w) = 0 \Longleftrightarrow 1 = \frac{n}{4\kappa} \cdot \frac{1}{r_H + w - 1} \Longleftrightarrow r_H = \frac{n}{4\kappa} + 1 - w = \tilde{r}_H(w)$$

Since

$$\frac{\partial^2}{\partial r_H \partial r_H} \hat{r}_L(r_H, w) = \frac{1}{4} (r_H + w - 1)^{-\frac{3}{2}} \geq 0$$

we have that $\tilde{r}_H(w)$ is a minimum point.

(P2) *For any w, the function $\hat{r}_L(r_H, w)$ is convex in r_H.* This follows from $\frac{\partial^2}{\partial r_H \partial r_H} \hat{r}_L(r_H, w) \geq 0$.

(P3) *For any r_H, the function $\hat{r}_L(r_H, w)$ is decreasing in w.* This follows from the R.H.S. of (6).

We have thus established that for any w, the function $\hat{r}_L(r_H, w)$ is convex in r_H and attains minimum at $\tilde{r}_H$. Hence, for all values of $r_H \geq 1 - \theta^{(n-m+1)} > \tilde{r}_H$ the function $\hat{r}_L(r_H, w)$ is increasing in r_H. Recall that $\hat{\mathcal{L}}(r_H, w) = \kappa(p - \hat{r}_L(r_H, w))^2$ where $\hat{\mathcal{L}}(r_H, w)$ is the value of the Lagrangian for any r_H, when q is determined according to (FOCq). It follows that when r_H is restricted to the domain $\left[1 - \theta^{(n-m+1)}, 1\right]$ the maximum of $\hat{\mathcal{L}}$ is attained when $r_H = 1 - \theta^{(n-m+1)}$. Thus, $r_H^* (\theta_1, \theta_2) = 1 - \theta^{(n-l+1)}$, which completes the proof. □

PART II. We now turn to show that $q^* (\theta, \lambda)$ is increasing in each player's type while $r_H^* (\theta, \lambda)$ and $r_L^* (\theta, \lambda)$ are decreasing in each player's type. Fix θ_{-i} and λ. Suppose that $\theta'_i > \theta_i$ and denote $w \equiv w (\theta_i, \theta_{-i}, \lambda)$ and $w' \equiv w (\theta'_i, \theta_{-i}, \lambda)$ so that $w' > w$.

If $(\tilde{r}_H (\theta, \lambda), \tilde{q} (\theta, \lambda))$ is not interior, then no signal is acquired when the agents report θ, i.e. $q^* (\theta, \lambda) = 0$ and $r_L^* (\theta, \lambda) = p$. Without loss of generality we can assume that in this case $r_H^* (\theta, \lambda) = 1$, and it immediately follows that $q^* (\theta', \lambda) \geq q^* (\theta, \lambda)$ and $r_L^* (\theta', \lambda) \leq r_L^* (\theta, \lambda)$ and $r_H^* (\theta', \lambda) \leq r_H^* (\theta, \lambda)$.

We therefore assume that $(\tilde{r}_H (\theta, \lambda), \tilde{q} (\theta, \lambda))$ is interior. As we explain in the main text, this also implies that $(\tilde{r}_H (\theta', \lambda), \tilde{q} (\theta', \lambda))$ is interior. Note that: (1) $\tilde{r}_H$ and $\tilde{r}_L$, as given by Equations (1) and (3) are decreasing in $w (\theta_i, \theta_{-i}, \lambda)$, (2) $w (\theta_i, \theta_{-i}, \lambda)$ is increasing in θ_i and (3) $\tilde{q}$ is decreasing in $\tilde{r}_L$ and decreasing in $\tilde{r}_H$ for all $\tilde{r}_L \leq p \leq \tilde{r}_H$. These properties, together with Lemmas A1 and A2, ensure that the remainder of the proof is the the same as in the proof of Proposition 2, with the obvious adjustments to the case of the variance cost.

PART III. From the Lemmas A1 and A2, it follows that for any $\lambda \geq 0$ and for each profile of types θ, the values $q^*(\theta, \lambda)$ and $r_H^*(\theta, \lambda)$ that maximize $\mathcal{L} (q, r_H; w)$ satisfy that $q^*(\theta, \lambda)$ is unique and $r_H^*(\theta, \lambda)$ is unique whenever $q^*(\theta, \lambda) > 0$ (i.e., whenever a signal is purchased). We have also established that $q^*(\theta, \lambda)$ is monotone in any θ_i. It remains to show there exist $\lambda \geq 0$ for which $q^*(\theta, \lambda)$ and $r_H^*(\theta, \lambda)$ induce a non-negative expected aggregate virtual surplus. This follows from the same arguments given in the proof of Proposition 2.

This completes our proof.

2 Properties of the cost function

$$\frac{c(q,r_H)}{\kappa} = q\left(r_H \log\frac{r_H}{p} + (1-r_H)\log\frac{1-r_H}{1-p}\right)$$
$$+ (1-q)\left(\frac{p-q\cdot r_H}{1-q}\log\frac{\frac{p-q\cdot r_H}{1-q}}{p} + \left(1-\frac{p-q\cdot r_H}{1-q}\right)\log\frac{1-\frac{p-q\cdot r_H}{1-q}}{1-p}\right)$$
$$= q\left(r_H \log\frac{r_H}{p} + (1-r_H)\log\frac{1-r_H}{1-p}\right) + (1-q)\left(r_L\log\frac{r_L}{p} + (1-r_L)\log\frac{1-r_L}{1-p}\right)$$

$$\frac{c_1(q,r_H)}{\kappa} = r_H\left(\ln\frac{r_H}{\frac{p-qr_H}{(1-q)}}\right) + (1-r_H)\left(\ln\frac{1-r_H}{1-\frac{p-qr_H}{(1-q)}}\right)$$
$$= r_H\left(\ln\frac{r_H}{r_L}\right) + (1-r_H)\left(\ln\frac{1-r_H}{1-r_L}\right)$$

$$\frac{c_2(q,r_H)}{\kappa} = -q\left(\ln\frac{1}{r_H}(r_H-1)\frac{\frac{p-qr_H}{(1-q)}}{\frac{p-qr_H}{(1-q)}-1}\right) = q\left(\ln\frac{r_H}{r_L}\frac{1-r_L}{1-r_H}\right)$$

$$\frac{c_{11}(q,r_H)}{\kappa} = \frac{(p-r_H)^2}{(1-q)^3\left(1-\frac{p-qr_H}{(1-q)}\right)\frac{(p-qr_H)}{(1-q)}} = \frac{(p-r_H)^2}{(1-q)^3(1-r_L)r_L} = \frac{1}{r_L}\frac{(r_H-r_L)^2}{(1-r_L)(1-q)}$$

$$\frac{c_{22}(q,r_H)}{\kappa} = \frac{q}{r_H(1-r_H)}\frac{-\frac{(p-qr_H)}{(1-q)} - \frac{(1-q)qr_H(1-r_H)}{(1-q)^2} + \frac{(p-qr_H)^2}{(1-q)^2}}{\frac{(p-qr_H)}{1-q}\left(\frac{p-qr_H}{1-q}-1\right)}$$
$$= \frac{q}{r_H(1-r_H)} + \frac{q}{r_L(1-r_L)}\cdot\frac{q}{1-q}$$

$$\frac{c_{12}(q,r_H)}{\kappa} = \left(\ln\frac{r_H\left(\frac{p-qr_H}{(1-q)}-1\right)}{(r_H-1)\frac{(p-qr_H)}{(1-q)}}\right) + \frac{q(p-r_H)}{(1-q)^2\left(\frac{p-qr_H}{(1-q)}\right)\left(\frac{p-qr_H}{(1-q)}-1\right)}$$
$$= \left(\ln\frac{r_H}{r_L}\frac{1-r_L}{1-r_H}\right) + \frac{q(p-r_H)}{(1-q)^2(r_L)(r_L-1)}$$
$$= \left(\ln\frac{r_H}{r_L}\frac{1-r_L}{1-r_H}\right) + \frac{q}{r_L(1-r_L)}\frac{r_H-r_L}{(1-q)}$$