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Abstract

We study two-sided matching contests with two sets, each of which includes two heterogeneous players with commonly known types. The agents in each set compete in all-pay contests where they simultaneously send their costly efforts, and then are either assortatively or disassortatively matched. We characterize the players' equilibrium efforts for a general value function that assigns values for both agents who are matched as a function of their types. We then analyze the cross-effects of the players' types on their expected payoffs as well as on their expected total effort. We show that although each player's value function increases (decreases) in the types of the players in the other set, his expected payoff does not necessarily increase (decrease) in these types. In addition, depending on the value function, each player's type might have either a positive or a negative marginal effect on the players' expected total effort.

JEL Classification: N/A

Keywords: N/A

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All-Pay Matching Contests

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Abstract

We study two-sided matching contests with two sets, each of which includes two heterogeneous players with commonly known types. The agents in each set compete in all-pay contests where they simultaneously send their costly efforts, and then are either assortatively or disassortatively matched. We characterize the players' equilibrium efforts for a general value function that assigns values for both agents who are matched as a function of their types. We then analyze the cross-effects of the players' types on their expected payoffs as well as on their expected total effort. We show that although each player's value function increases (decreases) in the types of the players in the other set, his expected payoff does not necessarily increase (decrease) in these types. In addition, depending on the value function, each player's type might have either a positive or a negative marginal effect on the players' expected total effort.

Keywords: Two-sided matching, All-pay contest

JEL classification: D44, J31, D72, D82

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1 Introduction

In one-sided contests, which are also referred to as standard contests, agents exert efforts and then commonly known fixed prizes are awarded to the agents who win according to the contest rules. Each agent knows exactly what his prize will be if he is in first, second, or any other place. Such one-sided contests are applied in rent-seeking, lobbying in organizations, R&D races, political contests, promotions in labor markets, trade wars, and military and biological wars of attrition. In contrast, in two-sided matching contests, agents from two sides exert efforts but the prizes for players are not commonly known and depend on the results from the other set.[1] These types of contests are applied in the academic arena where one side is made up of universities that invest in hiring outstanding researchers and teachers, and the other side includes international student candidates who aspire to be accepted to higher education universities. In other areas, we can mention accounting or law students on one side, and firms on the other, and models, actors, and artists on one side, and talent agencies on the other.

There is an extensive literature on one-sided contests that include Tullock contests (see, Tullock 1980, Skaperdas 1996, and Baye and Hoppe 2003), all-pay contests (see Baye et al. 1993, Che and Gale 1998, Moldovanu and Sela 2001, 2006, and Siegel 2009), and rank-order tournaments (see Lazear and Rosen 1981, and Rosen 1986). However, the literature on two-sided matching contests in which the agents compete according to the rules of the standard contests is sparse and is mainly seen when players compete in rank-order tournaments (see Bhaskar and Hopkins 2016), in all-pay contests (see Hoppe et al. 2009, Hoppe et al. 2011, and Dizdar et al. 2019), and in Tullock contests (see Cohen et al. 2020). A crucial issue in all these models is the uncertain matching between the two sides, the reason being that in the rank-order tournament there is noise in each player's output, in the all-pay contest (auction) under incomplete information, the information of each player is private, and in the Tullock contest, the contest success function is stochastic.

[1] Peters (2007) showed that equilibrium efforts in a very large finite two-sided matching model can be quite different from the equilibrium efforts in the continuum model.

We study a two-sided matching contest in which the players compete in the all-pay contest under complete information. There are two sets of players, each of which includes two heterogeneous players with commonly known types. They simultaneously exert their efforts, and then they are either assortatively matched, namely, the winners from both sets are matched with each other as well as the losers, or they are disassortatively matched, namely, the winners from both sets are matched with the losers from the other sets. A player has a value which is a function of his own type and the type of his match, and a player who is matched has a payoff of his value minus the cost of his effort.

The equilibrium analysis of this all-pay matching contest is tractable since the players' probabilities of winning depend on the players' types in these sets only and not on the players' types in the other sets. On the other hand, the players' mixed strategies as well as their expected payoffs depend on the types of all the players in both sets, but this does not complicate the equilibrium analysis. However, since the players' equilibrium strategies and their expected payoffs depend on the players' types from both sets, the cross-effects of these types on the players' expected payoffs as well as on their expected total effort is not straightforward, which is the motivation for the present study.

We begin the analysis with assortative all-pay matching contests with a multiplicative value function according to which the value of each pair of players who are matched is the product of their types. Then we show that the expected payoff of a player does not necessarily increase in all the other players' types in the other set. This result is not straightforward given that the larger the types of the players in the other set are, the larger are the players' values of winning. The intuition is that when a player has a high probability to be matched with the higher type in the other set, if the lower type in the other set increases, this player's probability to be matched with the higher type decreases and as such his expected payoff decreases as well.

We also study the assortative all-pay matching contest with an additive value function according to which the value of each pair of players who are matched is the sum of their types. We show that a player's expected payoff increases in the types of the players in the other set. The intuition is that the players in each set are actually symmetric although they have different types, since they have the same difference between the prizes which is the difference between the types of the players in the other set. In that case, no player is dominant and therefore each player's expected payoff increases in both types of the players from the other set.

We continue by studying disassortative all-pay matching contests with a separable value function according to which the players' values are the difference of their values plus a constant. We show that similar to the assortative matching contest with an additive value function, the players in each set have prizes with the same difference between them, and therefore each player's expected payoff increases in the types of the players in the other set.

Afterwards, we examine the marginal effects of the players' types on their expected total effort. In the one-sided all-pay contest with two players under complete information the higher type has a negative marginal effect on the players' expected total effort, since when the higher type increases, the balance of the contest (the difference between the players' types) decreases, and this yields a decrease in the total effort. Likewise, the lower type has a positive marginal effect on the players' expected total effort, since increasing the lower type increases the balance of the contest. In our two-sided all-pay matching contest, these marginal effects of the types on the expected total effort depend on the form of the value function such that each type, either the higher or the lower one, might have a positive or a negative marginal effect on the players' expected total effort. The reason is that when we change each of the players' types the balance of the contest changes, but, the values of winning of the players in the other set change as well. These two parallel changes have opposite effects on the expected total effort, and depending on the players' types, each might be either stronger or weaker than the other, such that any change in a player's type might either increase or decrease the players' expected total effort. In sum, in the two-sided all-pay matching contest, a player's type has a much more complex marginal effect on the results, such as the players' expected payoffs and their expected total effort, than in the standard one-sided all-pay contest.

The rest of the paper is organized as follows: in Section 2 we present the assortative matching contest and analyze it with multiplicative and additive value functions. In Section 3 we present the disassortative all-pay matching contest and analyze it with a separable value function. Section 4 concludes. Some of the proofs appear in the Appendix.

2 The assortative all-pay matching contest

We consider two sets, A and B, each of which includes two players. The players' types in set A are a_i, $i = 1, 2$, where $a_1 \geq a_2$, and the players' types in set B are b_j, $j = 1, 2$ where $b_1 \geq b_2$. All of these types are commonly known. The matching contest proceeds as follows: Each player, a_i, $i = 1, 2$, in set A exerts an effort x_i, and each player, b_j, $j = 1, 2$, in set B exerts an effort y_j. Efforts are submitted simultaneously. Then, the players with the highest efforts from both sets are matched with each other, and the players with the lowest efforts from both sets are also matched with each other. If player a_i from set A is matched with player b_j from set B after exerting efforts of x_i and y_j, correspondingly, the utility of player a_i is $f(a_i, b_j) - x_i$, and the utility of player b_j is $g(a_i, b_j) - y_j$, where $f\ , g : R^2 \to R^1$ are the value functions which we assume to be monotonically increasing in the types of both players who are matched, namely, $\frac{d}{da} f(a, b) \geq 0$, $\frac{d}{db} f(a, b) \geq 0$, $\frac{d}{da} g(a, b) \geq 0$, and $\frac{d}{db} g(a, b) \geq 0$. We also assume that the value functions satisfy $\frac{d}{da}(f(a, b_1) - f(a, b_2)) \geq 0$ for all $b_1 \geq b_2$, and $\frac{d}{db}(g(a_1, b) - g(a_2, b)) \geq 0$ for all $a_1 \geq a_2$. These conditions yield that the higher the type of a player is, the higher is the change of his value as a change in his matched type. We say that this assortative all-pay matching contest has an equilibrium if every player chooses an effort that maximizes his expected utility given the efforts of the other players in both sets.

2.1 The equilibrium analysis

The equilibrium analysis of the assortative all-pay matching contest is derived by that of the standard all-pay contest (see Hillman and Riley 1989, and Baye et al. 1996). We denote by

$pa_i, i = 1, 2$, player a_i's probability of winning in set A, and by $pb_j, j = 1, 2$, player b_j's probability of winning in set B. Then, in set A, player a_i's expected values of winning wa_i and losing $la_i, i = 1, 2$, are

$$
\begin{aligned}
wa_i &= f(a_i, b_1)pb_1 + f(a_i, b_2)pb_2 \\
la_i &= f(a_i, b_1)pb_2 + f(a_i, b_2)pb_1.
\end{aligned}
$$

Our assumption that $\frac{d}{da}(f(a, b_1) - f(a, b_2)) \geq 0$ for all $b_1 \geq b_2$ implies that $wa_1 - la_1 \geq wa_2 - la_2$. In that case, player a_1 in set A chooses an effort from the interval $[0, (f(a_2, b_1) - f(a_2, b_2))(2pb_1 - 1)]$ according to the cumulative distribution function $Fa_1(x)$ which is given by

$$
\begin{aligned}
& f(a_2, b_1)\left[Fa_1(x)pb_1 + (1 - Fa_1(x))pb_2\right] \\
& + f(a_2, b_2)\left[Fa_1(x)pb_2 + (1 - Fa_1(x))pb_1\right] - x = ua_2,
\end{aligned}
\tag{1}
$$

where ua_2, the expected payoff of player a_2, is

$$
ua_2 = la_2 = f(a_2, b_1)pb_2 + f(a_2, b_2)pb_1. \tag{2}
$$

Player a_2 in set A chooses an effort from the same interval $[0, (f(a_2, b_1) - f(a_2, b_2))(2pb_1 - 1)]$ according to the cumulative distribution function $Fa_2(x)$ which is given by

$$
\begin{aligned}
& f(a_1, b_1)\left[Fa_2(x)pb_1 + (1 - Fa_2(x))pb_2\right] \\
& + f(a_1, b_2)\left[Fa_2(x)pb_2 + (1 - Fa_2(x))pb_1\right] - x = ua_1,
\end{aligned}
\tag{3}
$$

where ua_1, the expected payoff of player a_1, is

$$
\begin{aligned}
ua_1 &= (wa_1 - la_1) - (wa_2 - la_2) + la_1 = \\
& f(a_1, b_1)pb_1 + f(a_1, b_2)pb_2 - (f(a_2, b_1) - f(a_2, b_2))(2pb_1 - 1).
\end{aligned}
\tag{4}
$$

Player a_1's probability of winning in set A is then

$$
pa_1 = 1 - \frac{wa_2 - la_2}{2(wa_1 - la_1)} = 1 - \frac{f(a_2, b_1) - f(a_2, b_2)}{2(f(a_1, b_1) - f(a_1, b_2))}. \tag{5}
$$

Similarly, in set B, player b_i's expected value of winning wb_i and losing $lb_i, i = 1, 2$, are

$$\begin{aligned} wb_i &= g(a_1, b_i)pa_1 + g(a_2, b_i)pa_2 \\ lb_i &= g(a_1, b_i)pa_2 + g(a_2, b_i)pa_1. \end{aligned}$$

Our assumption that $\frac{d}{db}(g(a_1, b) - g(a_2, b)) \geq 0$ for all $a_1 \geq a_2$ implies that $wb_1 - lb_1 \geq wb_2 - lb_2$. In that case, player b_1 in set B chooses an effort from the interval $[0, (g(a_1, b_2) - g(a_2, b_2))(2pa_1 - 1)]$ according to the cumulative distribution function $Fb_1(x)$ which is given by

$$\begin{aligned} & g(a_1, b_2) \left[Fb_1(x)pa_1 + (1 - Fb_1(x))pa_2\right] \\ & + g(a_2, b_2) \left[Fb_1(x)pa_2 + (1 - Fb_1(x))pa_1\right] - x = ub_2, \end{aligned} \tag{6}$$

where ub_2, the expected payoff of player b_2, is

$$ub_2 = lb_2 = g(a_1, b_2)pa_2 + g(a_2, b_2)pa_1. \tag{7}$$

Player b_2 in set B chooses an effort from the interval $[0, (g(a_1, b_2) - g(a_2, b_2))(2pa_1 - 1)]$ according to the cumulative distribution function $Fb_2(x)$ which is given by

$$\begin{aligned} & g(a_1, b_1) \left[Fb_2(x)pa_1 + (1 - Fb_2(x))pa_2\right] \\ & + g(a_2, b_1) \left[Fb_2(x)pa_2 + (1 - Fb_2(x))pa_1\right] - x = ub_1, \end{aligned} \tag{8}$$

where ub_1, the expected payoff of player b_1, is

$$\begin{aligned} ub_1 &= (wb_1 - lb_1) - (wb_2 - lb_2) + lb_1 = \\ & \quad g(a_1, b_1)pa_1 + g(a_2, b_1)pa_2 - (g(a_1, b_2) - g(a_2, b_2))(2pa_1 - 1). \end{aligned} \tag{9}$$

Player b_1's probability of winning in set B is then

$$pb_1 = 1 - \frac{wb_2 - lb_2}{2(wb_1 - lb_1)} = 1 - \frac{g(a_1, b_2) - g(a_2, b_2)}{2(g(a_1, b_1) - g(a_2, b_1))}. \tag{10}$$

By the above analysis, in the assortative all-pay matching contest with two sets, $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$, there is a mixed strategy equilibrium in which the players' equilibrium efforts in set A are distributed according to the cumulative distribution functions

$$Fa_1(x) = \frac{x}{(f(a_2, b_1) - f(a_2, b_2))(2pb_1 - 1)} \tag{11}$$

$$Fa_2(x) = \frac{x + (f(a_1, b_1) - f(a_1, b_2) - (f(a_2, b_1) - f(a_2, b_2)))(2pb_1 - 1)}{(f(a_1, b_1) - f(a_1, b_2))(2pb_1 - 1)}, \tag{12}$$

and the players' equilibrium efforts in set B are distributed according to the cumulative distribution functions

$$Fb_1(x) = \frac{x}{(g(a_1, b_2) - g(a_2, b_2))(2pa_1 - 1)} \tag{13}$$

$$Fb_2(x) = \frac{x + (g(a_1, b_1) - g(a_2, b_1) - (g(a_1, b_2) - g(a_2, b_2)))(2pa_1 - 1)}{(g(a_1, b_1) - g(a_2, b_1))(2pa_1 - 1)}, \tag{14}$$

where pa_1 and pb_1 are given by (5) and (10), respectively.[2]

The players' expected total effort in set A is

$$\begin{aligned} TE_A &= \frac{wa_2 - la_2}{2}(1 + \frac{wa_2 - la_2}{wa_1 - la_1}) \\ &= (f(a_2, b_1) - f(a_2, b_2))\frac{b_1 - b_2}{2b_1}(1 + \frac{f(a_2, b_1) - f(a_2, b_2)}{f(a_1, b_1) - f(a_1, b_2)}), \end{aligned} \tag{15}$$

and the players' expected total effort in set B is

$$\begin{aligned} TE_B &= \frac{wb_2 - lb_2}{2}(1 + \frac{wb_2 - lb_2}{wb_1 - lb_1}) \\ &= (g(a_1, b_2) - g(a_2, b_2))\frac{a_1 - a_2}{2a_1}(1 + \frac{g(a_1, b_2) - g(a_2, b_2)}{g(a_1, b_1) - g(a_2, b_1)}). \end{aligned} \tag{16}$$

2.2 A multiplicative value function

Assume that the players have the same multiplicative value function $f(a_i, b_j) = g(a_i, b_j) = a_i b_j$, $i = 1, 2$, $j = 1, 2$.[3] By (11) and (12), in set A, player a_1's cumulative distribution function is

$$Fa_1(x) = \frac{b_1 x}{a_2(b_1 - b_2)^2},$$

[2] By Baye et al. (1996) this equilibrium is unique if $\frac{d}{da}(f(a, b_1) - f(a, b_2)) > 0$ for all $b_1 \geq b_2$, and $\frac{d}{db}(g(a_1, b) - g(a_2, b)) > 0$ for all $a_1 \geq a_2$.

[3] Our results in this section can be immediately extended to value functions of the form $f(a, b) = \delta(a)\rho(b)$, where δ and ρ are strictly increasing and differentiable functions.

and player a_2's cumulative distribution function is

$$Fa_2(x) = \frac{b_1 x + (a_1 - a_2)(b_1 - b_2)^2}{a_1(b_1 - b_2)^2}.$$

Similarly, by (13) and (14), in set B, player b_1's cumulative distribution function is

$$Fb_1(x) = \frac{a_1 x}{b_2(a_1 - a_2)^2},$$

and player a_2's cumulative distribution function is

$$Fb_2(x) = \frac{a_1 x + (b_1 - b_2)(a_1 - a_2)^2}{b_1(a_1 - a_2)^2}.$$

By (2) and (4), the expected payoffs of the players in set A are

$$\begin{aligned} ua_1 &= a_1 \frac{2b_1^2 - b_1 b_2 + b_2^2}{2b_1} - a_2 \frac{(b_1 - b_2)^2}{b_1} \\ ua_2 &= \frac{a_2 b_2}{2}(3 - \frac{b_2}{b_1}), \end{aligned} \tag{17}$$

and by (7) and (9), the expected payoffs of the players in set B are

$$\begin{aligned} ub_1 &= b_1(\frac{2a_1^2 - a_1 a_2 + a_2^2}{2a_1}) - b_2 \frac{(a_1 - a_2)^2}{a_1} \\ ub_2 &= \frac{b_2 a_2}{2}(3 - \frac{a_2}{a_1}), \end{aligned} \tag{18}$$

where by (5) and (10), the probability of players a_1 and b_1 to win are

$$\begin{aligned} pa_1 &= 1 - \frac{a_2}{2a_1} \\ pb_1 &= 1 - \frac{b_2}{2b_1} \end{aligned}$$

In the standard all-pay contest with two players, each player's expected payoff increases in his own type, but does not increase in his opponent's type. However, in our all-pay matching contest, since a player's value of winning is based on the players' types in the other set, the marginal effects of the players' types on each of the player's expected payoff are not straightforward as the following result demonstrates.

Proposition 1 *In the assortative all-pay matching contest with two sets* $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$, *and multiplicative value functions* $f(a_i, b_j) = g(a_i, b_j) = a_i b_j$, $i = 1, 2$, $j = 1, 2$*:*

1) The expected payoff of the player with the higher type in his set (a_1*,* b_1*) increases in his own type, decreases in the other player's type in his set, increases in the higher type of the player in the other set, but might either increase or decrease in the lower type of the player in the other set.*

2) The expected payoff of the player with the lower type in his set (a_2*,* b_2*) increases in his own type, does not depend on the type of the other player in his set, and increases in the types of the players in the other set.*

Proof. See Appendix. ■

By Proposition 1, the expected payoff of the player with the higher type in each set increases in his own type and decreases in the type of the other player in his set. Likewise, the payoff of the player with the lower type in each set increases in his own type and does not depend on the type of the other player in his set. These findings are straightforward, and also hold in the one-sided (standard) all-pay contest with two players. However, the marginal effects of the types of the players in one set on the expected payoffs of the players in the other set are not straightforward. To see that, notice that the expected payoff of the player with the lower type in his set increases in the types of the players' types in the other set since these types increase this player's values of winning whether he wins or loses. On the other hand, the expected payoff of the player with the higher type in his set increases in the higher type of the player in the other set. However, it might decrease in the lower type of the player in the other set when player a_1 has a significantly higher type than the other player in his set a_2, and the lower type in the other set b_2 is relatively low. Then, if b_2 increases, the chance of the player with the lower type in the other set b_2 to win increases and since the winning probability of the player with the higher type a_1 is high, his payoff loss from matching with the player with the lower type b_2 instead of b_1 is relatively high, and therefore the expected payoff of the player with type a_1 decreases in the the value of b_2. Hence, we can say that increasing all the players' types in one set of the matching all-pay contest does not necessarily enhance the expected payoff of the player with higher type in the other set, but does enhance the expected payoff of the player with the lower type in the other set.

Next, we examine the marginal effect of the players' types on their expected total effort. By (15) and (16), the players' expected total effort is

$$\begin{aligned} TE &= TE_A + TE_B \\ &= \frac{a_2(b_1-b_2)^2}{2b_1}\left(\frac{a_1+a_2}{a_1}\right) + \frac{b_2(a_1-a_2)^2}{2a_1}\left(\frac{b_1+b_2}{b_1}\right). \end{aligned} \tag{19}$$

In the one-sided all-pay contest with two players, the expected total effort always decreases in the higher type (value of winning) but increases in the lower one. In our matching all-pay contest, however, the marginal effects of the players' types on their expected total effort are more complicated as the following indicates.

Proposition 2 *In the assortative all-pay matching contest with two sets* $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$, *and multiplicative value functions,* $f(a_i, b_j) = g(a_i, b_j) = a_i b_j$, $i = 1, 2$, $j = 1, 2$, *each of the players' types might have either a positive or negative marginal effect on the players' expected total effort.*

Proof. See Appendix. ■

The reason for the above result is that when the value of the higher type increases, the contest is less balanced, which has a negative marginal effect on the players' expected total effort. On the other hand, when the value of the higher type increases, all the players' values of winning in the other set increase, which has a positive marginal effect on the players' expected total effort. Thus, when the difference of the players' types in a set is relatively small such that the contest is balanced, increasing the value of the higher type upsets this balance which has a negative marginal effect on the expected total effort. However, when the difference of the players' types in a set is relatively large such that the contest in this set is already unbalanced, increasing the higher type has a positive marginal effect on the players' expected total effort.

Likewise, by Proposition 2, the lower type in a set might also have a positive or a negative marginal effect on the players' expected total effort. The reason is that when the difference of the players' types in a set is relatively small, increasing the value of the lower type increases the balance of the contest which has a positive marginal effect on the players' expected total effort. On the other hand, when the difference of the players' types in a set is relatively high and the contest is already unbalanced, by increasing the lower type, the difference of these types is reduced which has a negative marginal effect on the expected efforts in the other set. Thus, increasing the lower type might have also a negative marginal effect on the players' expected total effort.

2.3 An additive value function

We now assume that the players have the same additive value function $f(a_i, b_j) = g(a_i, b_j) = a_i + b_j$, $i = 1, 2$, $j = 1, 2$. By (11) and (12), in set A, both players have the same cumulative distribution functions

$$Fa_1(x) = Fa_2(x) = \frac{b_1 x}{(b_1 - b_2)^2}.$$

Similarly, in set B, both players have the same cumulative distribution functions

$$Fb_1(x) = Fb_2(x) = \frac{a_1 x}{(a_1 - a_2)^2}.$$

By (2) and (4), the expected payoffs of the players in set A are

$$ua_i = a_i + \frac{3b_2}{2} - \frac{b_2^2}{2b_1}\ ,\ i = 1, 2, \tag{20}$$

and by (7) and (9), those in set B are

$$ub_i = b_1 + \frac{3a_2}{2} - \frac{a_2^2}{2a_1}\ ,\ i = 1, 2, \tag{21}$$

where by (5) and (10), the probabilities of players a_1 and b_1 to win are

$$\begin{aligned} pa_1 &= 1 - \frac{a_2}{2a_1} \\ pb_1 &= 1 - \frac{b_2}{2b_1}. \end{aligned}$$

In that case, the marginal effects of the players' types on their expected payoffs are as follows:

Proposition 3 *In the assortative all-pay matching contest with two sets,* $A = \{a_1, a_2\}$*,* $B = \{b_1, b_2\}$*, and additive value functions,* $f(a_i, b_j) = g(a_i, b_j) = a_i + b_j$*,* $i = 1, 2$*,* $j = 1, 2$*, the expected payoff of each player increases in his own type, does not depend on the other player's type in his set, and increases in the types of both players in the other set.*

Proof. See Appendix. ■

In that case, with an additive value function, the players are actually symmetric, since they have the same prizes (up to a constant that is equal to the difference of their own types), and therefore the types of the players have the same marginal effect on each of the players' expected payoffs . Furthermore, their expected payoffs increase in the types of the players from the other set.

Similarly, when the value function is additive, the marginal effects of the players' types on their expected total effort are not ambiguous. By (15) and (16), the players' expected total effort is

$$\begin{aligned} TE &= TE_A + TE_B = \\ & \frac{(b_1 - b_2)^2}{b_1} + \frac{(a_1 - a_2)^2}{a_1} \end{aligned} \tag{22}$$

Then, we have

Proposition 4 *In the assortative all-pay matching contest with two sets* $A = \{a_1, a_2\}$*,* $B = \{b_1, b_2\}$*, and additive value functions* $f(a_i, b_j) = g(a_i, b_j) = a_i + b_j$*,* $i = 1, 2$*,* $j = 1, 2$*, the marginal effect of the higher type in each set on the players' expected total effort is positive and that of the lower type is negative.*

Proof. See Appendix. ■

In that case, with an additive value function, the contest in each set will be balanced independent of the players' types, since both players in a set have the same difference between the prizes of winning and losing. Thus, if the higher type increases, the difference between the players' values of winning and losing in the other set increases, and therefore their expected efforts increase as well. On the other hand, if the lower type increases, the difference between the players' values of winning and losing in the other set decreases, and we have the opposite marginal effects on the players' expected efforts.

3 The disassortative matching contest

We consider the same all-pay matching contest as in the previous section except that the players with the highest efforts from both sets are not matched with each other, but instead are matched with the players with the lower efforts from the other sets. If player a_i, $i = 1, 2$, from set A is matched with player b_j, $j = 1, 2$, from set B after exerting efforts of x_i and y_j, correspondingly, the utility of the player from set A is $f(a_i, b_j) - x_i$ and the utility of the player from set B is $g(a_i, b_j) - y_j$, where $f\ , g : R^2 \rightarrow R^1$ are the value functions that satisfy, $\frac{d}{da}f(a, b) \geq 0$, $\frac{d}{db}f(a, b) \leq 0$, $\frac{d}{da}g(a, b) \leq 0$ and $\frac{d}{db}g(a, b) \geq 0$. We assume that $\frac{d}{da}(f(a, b_1) - f(a, b_2)) \geq 0$ for all $b_1 \geq b_2$, and similarly that $\frac{d}{db}(g(a_1, b) - g(a_2, b)) \geq 0$ for all $a_1 \geq a_2$. These conditions yield that the higher the type of a player is, the higher is the change in his value as a change of his matched type.

The equilibrium analysis is the same as for the assortative all-pay matching contest, and the players' expected payoffs are also the same, except that in each of the players' distributions of efforts in set A given by (11) and (12), and in each of the equations describing the players' expected payoffs in set A given by (1), and (3), the term $f(a_i, b_1)$ is replaced by $f(a_i, b_2)$, $i = 1, 2$, and vice versa. Similarly, in each of the players' distributions of efforts in set B given by (13) and (14), and in each of the equations describing the players' expected payoffs in set B given by (6), and (8), the term $g(a_1, b_i)$ is replaced by $g(a_2, b_i)$, $i = 1, 2$, and vice versa. Then we obtain that in the disassortative all-pay matching contest with two sets $A = \{a_1, a_2\}$ and $B = \{b_1, b_2\}$, there is a mixed strategy equilibrium in which the players' equilibrium efforts in set A are distributed according to the cumulative distribution function

$$\widehat{F}a_1(x) = \frac{x}{(f(a_2,b_2) - f(a_2,b_1))(2\widehat{p}b_1 - 1)} \tag{23}$$

$$\widehat{F}a_2(x) = \frac{x + (f(a_1,b_2) - f(a_1,b_1) - f(a_2,b_2) + f(a_2,b_1))(2q_1 - 1)}{(f(a_1,b_2) - f(a_1,b_1))(2\widehat{p}b_1 - 1)}, \tag{24}$$

and the players' equilibrium efforts in set B are distributed according to the cumulative distribution functions

$$\widehat{F}b_1(x) = \frac{x}{(g(a_2,b_2) - g(a_1,b_2))(2\widehat{p}a_1 - 1)} \tag{25}$$

$$\widehat{F}b_2(x) = \frac{x + (g(a_2,b_1) - g(a_1,b_1) - g(a_2,b_2) + g(a_1,b_2))(2p_1 - 1)}{(g(a_2,b_1) - g(a_1,b_1))(2\widehat{p}a_1 - 1)}, \tag{26}$$

where player a_1's probability of winning in set A is

$$\widehat{p}a_1 = 1 - \frac{f(a_2,b_2) - f(a_2,b_1)}{2(f(a_1,b_2) - f(a_1,b_1))}, \tag{27}$$

and player b_1's probability of winning in set B is

$$\widehat{p}b_1 = 1 - \frac{g(a_2,b_2) - g(a_1,b_2)}{2(g(a_2,b_1) - g(a_1,b_1))}. \tag{28}$$

The players' expected total effort in set A is

$$\widehat{TE}_A = (f(a_2,b_2) - f(a_2,b_1))(\frac{b_1 - b_2}{2b_1})(1 + \frac{f(a_2,b_2) - f(a_2,b_1)}{f(a_1,b_2) - f(a_1,b_1)}), \tag{29}$$

and the players' expected total effort in set B is

$$\widehat{TE}_B = (g(a_2,b_2) - g(a_1,b_2))(\frac{a_1 - a_2}{2a_1})(1 + \frac{g(a_2,b_2) - g(a_1,b_2)}{g(a_2,b_1) - g(a_1,b_1}). \tag{30}$$

3.1 A separable value function

We now assume that the players have separable value functions $f(a_i,b_j) = k + a_i - b_j, g(a_i,b_j) = k + b_j - a_i$, $i = 1,2$, $j = 1,2$. We also assume that $k \geq \max |a_i - b_j|$, $i = 1,2$, $j = 1,2$. By (23) and (24), in set A, both players have the same cumulative distribution function

$$\widehat{F}a_1(x) = \widehat{F}a_2(x) = \frac{b_1 x}{(b_1 - b_2)^2}.$$

Similarly, by (25) and (26), in set B, both players have the same cumulative distribution function

$$\widehat{F}b_1(x) = Fb_2(x) = \frac{a_1 x}{(a_1 - a_2)^2},$$

where by (27) and (28), the probability of players a_1 and b_1 to win are

$$\begin{aligned} \widehat{p}a_1 &= 1 - \frac{a_2}{2a_1} \\ \widehat{p}b_1 &= 1 - \frac{b_2}{2b_1}. \end{aligned}$$

Note that the players' distributions of efforts are the same as in the assortative all-pay matching contest with an additive value function.

The expected payoffs of the players in set A are

$$\begin{aligned} \widehat{u}a_1 &= k + a_1 - \frac{3b_2}{2} + \frac{b_2^2}{2b_1} \\ \widehat{u}a_2 &= k + a_2 - \frac{3b_2}{2} + \frac{b_2^2}{2b_1}, \end{aligned} \tag{31}$$

and the expected payoffs of the players in set B are

$$\begin{aligned} \widehat{u}b_1 &= k + b_1 - \frac{3a_2}{2} + \frac{a_2^2}{2a_1} \\ \widehat{u}b_2 &= k + a_1 - \frac{3a_2}{2} + \frac{a_2^2}{2a_1}. \end{aligned} \tag{32}$$

A comparison of the players' expected payoffs given by (31) and (32) with their expected payoffs in the assortative matching contest with an additive value function given by (20) and (21) shows that the marginal effects of the players' types on the players' expected payoffs in the other set have the same values but with the opposite sign. Thus, we have

Proposition 5 *In the disassortative all-pay matching contest with two sets, $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$, and separable value functions, $f(a_i, b_j) = k + a_i - b_j$, $g(a_i, b_j) = k + b_j - a_i$, $i = 1, 2$, $j = 1, 2$, the expected payoff of each player increases in his own type, does not depend on the other player's type in his set, and decreases in the types of both players in the other set.*

Since the players' distribution of efforts are the same as in the assortative matching contest with an additive value function we obtain that

Proposition 6 *In the disassortative all-pay matching contest with two sets* $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$, *and separable value functions* $f(a_i, b_j) = k + a_i - b_j$, $g(a_i, b_j) = k + b_j - a_i$, $i = 1, 2$, $j = 1, 2$, *the marginal effect of the higher type in each set on the players' expected total effort is positive, and the marginal effect of the lower type in each set on the players' expected total effort is negative.*

The last result is quite surprising since the players' types have the same marginal effects on the expected total effort as in the assortative matching contest with an additive value function even though the players' types have the opposite marginal effects on the players' value functions in the other set.

4 Concluding remarks

In the one-sided all-pay contest in which players compete against each other to win one of the fixed prizes on the other side, the players prefer that their types (values of winning) be large and their opponents' types be small. The designer of such a contest who wishes to maximize the players' total effort, prefers large values of the players' types and also that the difference between them will be small. In this paper, we have demonstrated that in the two-sided matching all-pay contest, the players' preferences about their opponents types as well as the designer's preference about these types depend on the players' value function, and each player's type might have either a positive or a negative effect on the players' expected payoffs on the other side. Similarly, each player"s type might also have either a positive or a negative marginal effect on the players' expected total effort. As such, in the two-sided all-pay matching contest, depending on the form of the players' value function, it could be difficult to anticipate the effects of any change of the players' types on the results.

5 Appendix

5.1 Proof of Proposition 1

By (17), we have:

1) The marginal effect of player a_1's type on his own expected payoff is

$$\begin{aligned}\frac{dua_1}{da_1} &= \frac{d}{da_1}(a_1\frac{2b_1^2 - b_1b_2 + b_2^2}{2b_1} - a_2\frac{(b_1-b_2)^2}{b_1}) \\ &= \frac{1}{2b_1}\left(2b_1^2 - b_1b_2 + b_2^2\right).\end{aligned}$$

Since $b_1 \geq b_2$, the expected payoff of the player with the higher type in his set (a_1) increases in his own type.

2) The marginal effect of player a_2's type on player a_1's expected payoff is

$$\begin{aligned}\frac{dua_1}{da_2} &= \frac{d}{da_2}(a_1\frac{2b_1^2 - b_1b_2 + b_2^2}{2b_1} - a_2\frac{(b_1-b_2)^2}{b_1}) \\ &= -\frac{1}{b_1}\left(b_1 - b_2\right)^2.\end{aligned}$$

Since $b_1 \geq b_2$, the expected payoff of the player with the higher type in his set (a_1) decreases in the other player's type in his set (a_2).

3) The marginal effect of player b_1's type on player a_1's expected payoff is

$$\begin{aligned}\frac{dua_1}{db_1} &= \frac{d}{db_1}(a_1\frac{2b_1^2 - b_1b_2 + b_2^2}{2b_1} - a_2\frac{(b_1-b_2)^2}{b_1}) \\ &= \frac{1}{2b_1^2}\left(-a_1b_2^2 + 2a_1b_1^2 - 2a_2b_1^2 + 2a_2b_2^2\right) \\ &= \frac{1}{2b_1^2}\left(2b_1^2(a_1 - a_2) - b_2^2(a_1 - 2a_2)\right).\end{aligned}$$

Since $b_1 \geq b_2$, the expected payoff of the player with the higher type in his set (a_1) increases in the the higher type of the player in the other set (b_1).

4) The marginal effect of player b_2's type on player a_1's expected payoff is

$$\begin{aligned}\frac{dua_1}{db_2} &= \frac{d}{db_2}(a_1\frac{2b_1^2 - b_1b_2 + b_2^2}{2b_1} - a_2\frac{(b_1-b_2)^2}{b_1}) \\ &= \frac{1}{2b_1}\left(-a_1b_1 + 2a_1b_2 + 4a_2b_1 - 4a_2b_2\right).\end{aligned}$$

Note that if the value of b_1 approaches the value of b_2, then the expected payoff of the player with the higher type in his set (a_1) increases in the lower type of the player in the other set (b_1), but, on the other hand, if the value of b_2 approaches zero and $a_1 > 4a_2$, then the expected payoff of player a_1 decreases in b_2.

Likewise, we have:

5) The marginal effect of player a_1's on player a_2's expected payoff is

$$\frac{dua_2}{da_1} = \frac{d}{da_1}(\frac{a_2 b_2}{2}(3 - \frac{b_2}{b_1})) = 0.$$

Thus, the expected payoff of the player with the lower type in his set (a_2) does not depend on the type of the other player in his set (a_1).

6) The marginal effect of player a_2's type on his own expected payoff is

$$\frac{dua_2}{da_2} = \frac{d}{da_2}(\frac{a_2 b_2}{2}(3 - \frac{b_2}{b_1})) = \frac{1}{2b_1} b_2 \left(3b_1 - b_2\right).$$

Since $b_1 \geq b_2$, the expected payoff of the player with the lower type in his set (a_2) increases in his own type.

7) The marginal effect of player b_1's type on player a_2's expected payoff is

$$\frac{dua_2}{db_1} = \frac{d}{db_1}(\frac{a_2 b_2}{2}(3 - \frac{b_2}{b_1})) = \frac{1}{2}\frac{a_2}{b_1^2} b_2^2,$$

8) The marginal effect of player b_2's type on player a_2's expected payoff is

$$\frac{dua_2}{db_2} = \frac{d}{db_2}(\frac{a_2 b_2}{2}(3 - \frac{b_2}{b_1})) = \frac{1}{2}\frac{a_2}{b_1}\left(3b_1 - 2b_2\right).$$

Since $b_1 \geq b_2$, the expected payoff of the player with the lower type in his set (a_2) increases in the types of the players in the other sets (b_1, b_2).

The above analysis about the players' expected payoffs in set A holds for the players' expected payoffs in set B as well. $Q.E.D.$

5.2 Proof of Proposition 2

By (19), the marginal effect of player a_1's type on the players' expected total effort is

$$\begin{aligned}\frac{dTE}{da_1} &= \frac{d}{da_1}\left(\frac{a_2(b_1-b_2)^2}{2b_1}\left(\frac{a_1+a_2}{a_1}\right)+\frac{b_2(a_1-a_2)^2}{2a_1}\left(\frac{b_1+b_2}{b_1}\right)\right) \\ &= \frac{1}{2a_1^2b_1}\left(a_1^2b_1b_2+a_1^2b_2^2-a_2^2b_1^2+a_2^2b_1b_2-2a_2^2b_2^2\right).\end{aligned}$$

We can see that $\frac{dTE}{da_1} \geq 0$ iff $z = \left(a_1^2b_1b_2+a_1^2b_2^2-a_2^2b_1^2+a_2^2b_1b_2-2a_2^2b_2^2\right) \geq 0$. When a_2 approaches zero we obtain that,

$$\lim_{a_2 \to 0} z = a_1^2b_1b_2 + a_1^2b_2^2.$$

Thus, the marginal effect of player a_1's type on the players' expected total effort is then positive. On the other hand, when a_2 approaches a_1 we have

$$\lim_{a_2 \to a_1} z = -a_1^2(b_1-b_2)^2.$$

Thus, the marginal effect of player a_1's type on the players' expected total effort is then negative.

Similarly, the marginal effect of player a_2's type on the players' expected total effort is

$$\begin{aligned}\frac{dTE}{da_2} &= \frac{d}{da_2}\left(\frac{a_2(b_1-b_2)^2}{2b_1}\left(\frac{a_1+a_2}{a_1}\right)+\frac{b_2(a_1-a_2)^2}{2a_1}\left(\frac{b_1+b_2}{b_1}\right)\right) \\ &= \frac{1}{2a_1b_1}\left(a_1b_1^2-a_1b_2^2+2a_2b_1^2+4a_2b_2^2-4a_1b_1b_2-2a_2b_1b_2\right).\end{aligned}$$

We can see that $\frac{dTE}{da_1} \geq 0$ iff $z = \left(a_1b_1^2-a_1b_2^2+2a_2b_1^2+4a_2b_2^2-4a_1b_1b_2-2a_2b_1b_2\right) \geq 0$. When a_2 approaches zero we obtain that

$$\lim_{a_2 \to 0} z = \frac{1}{b_1}((b_1-b_2)^2 - 2b_1b_2,$$

where the last term might be either positive or negative. That is, the marginal effect of player a_2's type on the players' expected total effort is then either positive or negative. On the other hand, when a_2 approaches a_1 we have

$$\lim_{a_2 \to a_1} z = 3a_1(b_1-b_2)^2.$$

Thus, the marginal effect of player a_2's type on the players' expected total effort is then positive. Hence, we can conclude that each of the players' types might have either a positive or a negative marginal effect on the players' expected total effort. *Q.E.D.*

5.3 Proof of Proposition 3

By (20), we have:

1) The marginal effect of player a_1's type on his own expected payoff is

$$\frac{dua_1}{da_1} = \frac{d}{da_1}(a_1 + \frac{3b_2}{2} - \frac{b_2^2}{2b_1}) = 1,$$

2) The marginal effect of player a_2's type on player a_1's expected payoff is

$$\frac{dua_1}{da_2} = \frac{d}{da_2}(a_1 + \frac{3b_2}{2} - \frac{b_2^2}{2b_1}) = 0.$$

3) The marginal effect of player b_1's type on player a_1's expected payoff is

$$\frac{dua_1}{db_1} = \frac{d}{db_1}(a_1 + \frac{3b_2}{2} - \frac{b_2^2}{2b_1}) = \frac{1}{2b_1^2}b_2^2,$$

4) The marginal effect of player b_2's type on player a_1's expected payoff is

$$\frac{dua_1}{db_2} = \frac{d}{db_2}(a_1 + \frac{2b_1 - b_2}{2} + \frac{b_2^2}{2b_1} - \frac{(b_1 - b_2)^2}{b_1}) = \frac{1}{2b_1}\left(3b_1 - 2b_2\right).$$

Since all the players have the same form of expected payoff, we obtain that the expected payoff of every player increases in his own type, does not depend on the other player's type in his set, and increases in the types of both players in the other set. *Q.E.D.*

5.4 Proof of Proposition 4

By (22), since $a_1 \geq a_2$, the marginal effect of player a_1's type on the players' expected total effort is

$$\frac{dTE}{da_1} = \frac{d}{da_1}(\frac{(b_1 - b_2)^2}{b_1} + \frac{(a_1 - a_2)^2}{a_1}) = \frac{1}{a_1^2}\left(a_1^2 - a_2^2\right) > 0$$

and the marginal effect of player a_2's type on the players' expected total effort is

$$\begin{aligned}\frac{dTE}{da_2} &= \frac{d}{da_2}(\frac{(b_1-b_2)^2}{b_1}+\frac{(a_1-a_2)^2}{a_1}) \\ &= -\frac{1}{a_1}(2a_1-2a_2)<0\end{aligned}$$

Thus, the marginal effect of player a_2's type is opposite to that of player a_1's type on the players' expected total effort. *Q.E.D.*

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