# DISCUSSION PAPER SERIES

DP15282

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MONETARY ECONOMICS AND FLUCTUATIONS

ISSN 0265-8003

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Discussion Paper DP15282
Published 11 September 2020
Submitted 11 September 2020

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## Abstract

Conditions of secular stagnation—low output growth g and low interest rates r—have counteracting effects on the cost of servicing public debt, r − g. Using data for ad- vanced economies, we document that r is often less than g, but r − g exhibits substan- tial variability over the medium-term. We build a continuous-time model in which the debt-to-GDP ratio is stochastic and r < g on average. We analytically characterize the distribution of the debt-to-GDP ratio, showing how two candidate explanations for low interest rates, slower trend growth and higher output risk, can lower the debt- to-GDP ratio. When the primary surplus is bounded above, we characterize a fiscal limit, above which default occurs, and a debt tipping point, above which the pub- lic debt is on an unsustainable path, but default does not occur immediately. Slower trend growth and higher output risk can paradoxically improve debt sustainability. A conservative calibration suggests a fiscal limit for the US of 184 percent of GDP and a tipping point of 115 percent of GDP.



JEL Classification: E43, E62, H68

Keywords: secular stagnation, public debt, debt sustainability, Low interest rates, government default



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# Debt Sustainability in a Low Interest Rate World*

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September 9, 2020

**Abstract**

Conditions of secular stagnation—low output growth $g$ and low interest rates $r$—have counteracting effects on the cost of servicing public debt, $r - g$. Using data for advanced economies, we document that $r$ is often less than $g$, but $r - g$ exhibits substantial variability over the medium-term. We build a continuous-time model in which the debt-to-GDP ratio is stochastic and $r < g$ on average. We analytically characterize the distribution of the debt-to-GDP ratio, showing how two candidate explanations for low interest rates, slower trend growth and higher output risk, can *lower* the debt-to-GDP ratio. When the primary surplus is bounded above, we characterize a fiscal limit, above which default occurs, and a debt tipping point, above which the public debt is on an unsustainable path, but default does not occur immediately. Slower trend growth and higher output risk can paradoxically improve debt sustainability. A conservative calibration suggests a fiscal limit for the US of 184 percent of GDP and a tipping point of 115 percent of GDP.

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*The views expressed in this paper are those of the author and do not represent the Federal Reserve System or the Federal Reserve Bank of New York. This paper subsumes an earlier working paper with the same title written for the Hutchins Center on Fiscal and Monetary Policy at the Brookings Institution. We thank Ben Bernanke, Emmanuel Farhi, and Louise Sheiner for discussion on the earlier draft and seminar and conference participants at Brown University, the Center for Equitable Growth, the European Meetings of the Econometric Society, the European Central Bank, the Federal Reserve Bank of St. Louis, the IMF, and the SED (Mexico City). We also thank Manuel Amador, Sushant Acharya, Isaac Baley, Javier Bianchi, Anmol Bhandari, Olivier Blanchard, John Cochrane, Keshav Dogra, Gauti Eggertsson, Francois Gourio, Kyle Herkenhoff, Guido Lorenzoni, Guido Menzio, Fabrizio Perri, Carmen Reinhart, Tom Sargent, Christopher Sims, Guido Tabellini, and Pierre-Olivier Weill for helpful discussions. Neil Mehrotra also gratefully acknowledges the Brookings Institution for financial support.

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# 1 Introduction

A decade after the onset of the Great Recession, trend growth in advanced economies remains historically slow, with economies saddled with increased levels of public debt. The ongoing coronavirus pandemic is likely to push advanced country debt-to-GDP ratios rapidly higher. Figure 1 shows the debt-to-GDP ratios for 19 OECD countries in 2000 and 2018. The majority of these countries exhibit a substantial increase in debt-to-GDP ratios over this period. Japan has almost doubled its debt-to-GDP ratio to over 200 percent. The increase is similarly stark in the US and UK, where debt-to-GDP ratios have increased from 51 and 41 percent to 99 and 108 percent of GDP, respectively.[1] By historical standards, many advanced economies hold high levels of public debt. For these 19 countries, the median level of debt has risen from 55 to 84 percent of GDP between 2000 and 2018.

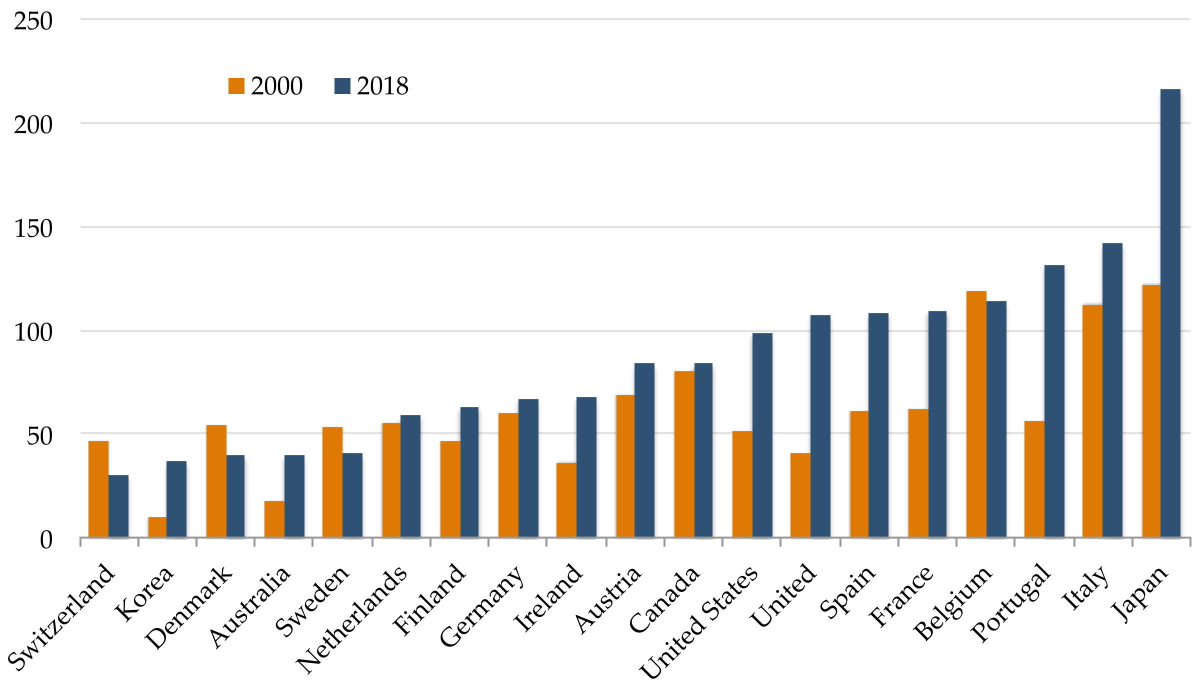


**Figure 1:** Core government debt-to-GDP ratio in percent for 19 OECD countries. Data sources: Bank for International Settlements.

At face value, the combination of high levels of public debt and low GDP growth would seem to be quite problematic for debt sustainability. However, advanced economies have also benefited from historically low real interest rates. Before the pandemic, nominal interest rates lifted off the zero lower bound. Still, short and long-term rates stayed

[1] In the case of the US, these figures include public debt owed to the Social Security Trust Fund. The rise is similarly sharp if restricted to debt held by the public.

relatively low, keeping the cost of debt servicing low for most advanced economies. This combination of low growth and low real interest rates—sometimes labeled the secular stagnation hypothesis (Summers, 2013; Eggertsson, Mehrotra, and Robbins, 2018)—carry contradictory implications for debt sustainability.

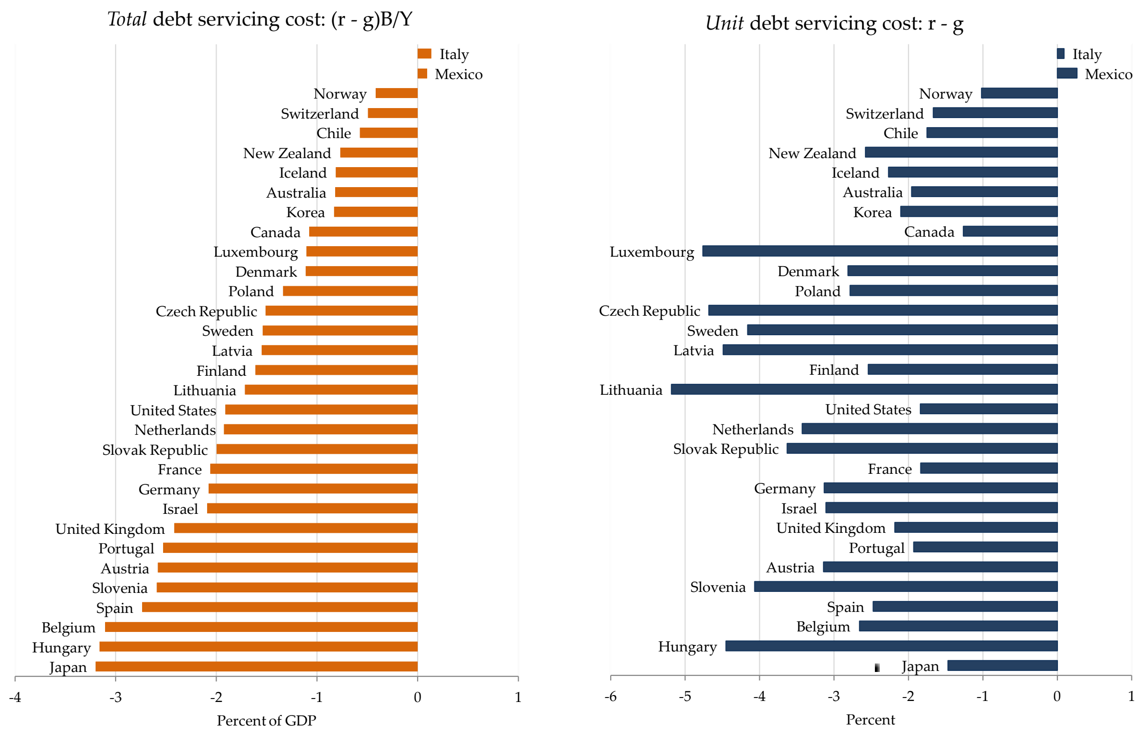


**Figure 2:** The right panel presents the unit public debt servicing costs, while the left panel shows the total costs. Data sources: the OECD.Stat and the Bank for International Settlements.

A typical metric for judging debt sustainability is the gap between the real interest rate paid on government debt $r$ and the growth rate of real GDP $g$. For the US and most other advanced economies, the cost of servicing public debt $r - g$ is negative. Figure 2 shows both the unit cost of maintaining a stable debt-to-GDP ratio $r - g$ and the total cost of servicing the debt.[2] For the US, real yields on 10-year government bonds averaged about 0.5 percent between 2015-19, while underlying GDP growth rates have averaged between 1.5 and 2 percent.

Suppose the interest rate on government debt is permanently below the growth rate of the economy. In this case, the government can run a primary deficit of any size in perpetuity or, equivalently, a government running in primary balance would see its debt-to-GDP ratio shrink to zero. However, interest rates are not constant and, as emphasized in Ball, Elmendorf, and Mankiw (1998), a Ponzi strategy of continuous rolling over the

[2]The latter is simply the unit cost multiplied by the debt to GDP ratio; it represents the real resources needed to keep the debt-to-GDP ratio constant. We will refer to the unit cost of servicing debt as just the cost of servicing debt.

public debt is risky. A sudden rise in interest rates relative to growth with a large stock of debt could quickly result in explosive debt dynamics. This paper aims to provide a theoretical framework for assessing debt sustainability in an environment when, on average, $r < g$.

**Empirical facts.** We start by summarizing some basic facts about the historical relationship between interest rates on government debt and economic growth. Using the macrohistory database of 17 advanced economies recently assembled in Jordà, Schularick, and Taylor (2016), we find that cases in which the real interest rate is less than GDP growth and, hence, the cost of servicing the public debt is negative, are fairly common. Taking five year averages, for all countries, roughly half the observations carry a negative debt-servicing cost. These episodes are not driven by the World Wars, the interwar years and the Great Depression, and, in the case of the US, the cost is negative nearly 70 percent of the time in the postwar period. While the observation that debt servicing costs are negative in the US is not new (Ball, Elmendorf, and Mankiw, 1998; Blanchard and Weil, 2001), this dataset shows that this fact holds for a much wider set of countries and, if anything, is stronger in the postwar period.[3]

Despite the fact that $r < g$ in the postwar period, the cost of servicing the debt still exhibits substantial variability. The interquartile range for $r - g$ is approximately five percentage points, and the dynamics of $r - g$ are characterized by substantial and rising persistence in the postwar period. Given the high levels of public debt shown in Figure 1, a sustained reversion of $r - g$ to levels of even 1 or 2 percentage points would require a large swing in the primary surplus to stabilize debt to GDP ratios.

**Model.** Next, we study public debt sustainability in a continuous-time general equilibrium endowment economy that is consistent with the empirical facts above, and that produces closed-form solutions even with the aggregate risk. In the baseline version of the model, output is modeled as an exogenous geometric Brownian motion. The government is assumed to follow a fiscal rule (e.g., Bohn, 1995), and we separately consider rules that ensure the safety of public debt at all future dates and rules that result in default in certain states.

The public debt is considered sustainable if fiscal policy ensures that the household's transversality condition is satisfied and the net present value of the government's primary surpluses equals the outstanding value of the public debt. We also consider a more heuristic notion of sustainability (along the lines of official institutions like the IMF and World Bank) where the public debt is sustainable so long as the debt-to-GDP ratio con-

[3] Mauro and Zhou (2020) extend this observation to a set of 55 countries over 200 years.

verges to a stationary distribution. As we show, these notions of debt sustainability do not neatly overlap; there are fiscal policies that are sustainable under the first definition, but fail the second definition and vice versa.

**No default.** The transversality condition and the government budget constraint are satisfied so long as the primary surplus responds at least linearly to the debt-to-GDP ratio.[4] This requirement on fiscal policy turns out to be quite mild—the size of the (positive) primary surplus can be arbitrarily small at any particular debt-to-GDP ratio but must nevertheless scale linearly, implying unboundedly large primary surpluses as the debt-to-GDP ratio becomes very large. Whether $r > g$ or $r < g$ does not matter for debt sustainability in this case.

Assuming a particular fiscal rule in which fiscal surplus reacts in a stronger than linear fashion to the debt-to-GDP ratio, we find that the debt-to-GDP ratio evolves as an Ornstein-Uhlenbeck process, the continuous-time analogue to a discrete-time AR(1) process. In this benchmark case, the stationary distribution of the log debt-to-GDP ratio is normal, where the mean depends on population growth, productivity growth, and the variance of the output process in addition to the fiscal parameters. The variance of this distribution depends only on the variance of the output process and the fiscal parameters. This result allows us to perform comparative statics analytically.

The mean debt-to-GDP ratio is *decreasing* in population growth and *increasing* in productivity growth when the elasticity of intertemporal substitution is less than one. Population growth and productivity growth have opposing effects on the debt-to-GDP ratio due to their opposing effects on the real interest rate. Lower population growth leaves the borrowing rate unchanged while directly lowering output growth, shifting the average debt-to-GDP ratio higher. By contrast, when the elasticity of intertemporal substitution is less than one, a decline in productivity growth has a more than a one-for-one effect on the real interest rate, lowering the cost of servicing the debt and thereby reducing the average debt-to-GDP ratio.

To the extent that higher uncertainty accounts for low real interest rates, we find that the variance of the log debt-to-GDP ratio unambiguously increases with higher output uncertainty. However, uncertainty also has an effect on the mean debt-to-GDP ratio that depends on the coefficient of relative risk aversion. Higher uncertainty lowers the real interest rate but this effect may be outweighed by an Ito's lemma term due to Jensen's inequality that works in the opposite direction. For an elasticity of intertemporal substitution that is close to unity (i.e. log preferences) and a coefficient of relative risk aversion that rationalizes the equity risk premium, the interest rate effect dominates and higher

[4] This result has been noted in alternative models in Bohn (1998) and Woodford (1998).

uncertainty results in a lower average debt-to-GDP ratio.

With a weaker fiscal policy that only scales linearly with the debt-to-GDP ratio, we can still derive a well-defined stationary distribution if the debt-to-GDP ratio has a negative drift term (roughly the case of $r < g$) and a lower reflecting barrier. In this case, the log debt-to-GDP ratio follows an ordinary Brownian motion. The resulting stationary distribution is an exponential—in levels, the debt-to-GDP ratio follows a Pareto distribution.

Following Wachter (2013), we extend the model to feature rare disasters to generate downward jumps in output and, therefore, upward jumps in the debt-to-GDP ratio; such jumps are characteristic of debt dynamics in the Great Recession and the 2020 coronavirus pandemic. When the surplus is linear in the debt-to-GDP ratio and the drift term of the debt-to-GDP is sufficiently negative, the debt-to-GDP ratio admits a stationary distribution that is Pareto. A higher probability of a rare disaster (or a larger output loss) have competing effects on the Pareto tail; the direct effect of a higher probability of a disaster fattens the Pareto tail, but the effect on the real interest rate reduces $r - g$ thereby thinning the tail.

We calibrate the baseline version of our model as well as its two extensions to the US, fitting basic asset-pricing facts including the low safe real interest rate and matching the size of the equity premium. We show that slow productivity growth can improve debt sustainability by lowering real interest rates by more than its direct effect on growth. Interestingly, a rise in output uncertainty that generates a 2 percentage point rise in risk premia, actually lowers the mean debt-to-GDP ratio by nearly 10 percent in the baseline model. This substantial quantitative effect comes from the effect of rising uncertainty in depressing the safe interest rate. While the effects of slower population and productivity growth on the mean debt-to-GDP ratio are quantitatively modest in the baseline model, quantitatively realistic increases in risk premia have larger effects on the distribution of the debt-to-GDP ratio.

**Default.** The assumption that the primary surplus can be unboundedly large is a strong one, thus we consider the case that the primary fiscal surplus is bounded above, a property that Ghosh, Kim, Mendoza, Ostry, and Qureshi (2013) labeled as fiscal fatigue. In the presence of a maximum primary surplus, an endogenous maximum public debt limit emerges, which we call a *fiscal limit.* The households refuse to purchase government debt beyond this limit since the transversality condition can no longer be satisfied. For levels of debt below the limit, the interest rate on government debt rises with the debt-to-GDP ratio reflecting default risk. As before, slower productivity growth or higher disaster risk can have counterintuitive effects on debt sustainability. Lower trend growth relaxes the fiscal limit, while higher disaster risk increases the probability of default (without changing the fiscal limit) but reduces the drift of the debt-to-GDP ratio by making government debt a more attractive hedge against disasters. Central to this result is the competing effects of higher disaster risk on the risk-free rate and sovereign default premia.

Fiscal fatigue also gives rise to a *tipping point* level of debt along the lines of Reinhart and Rogoff (2009) and Lorenzoni and Werning (2019), which is distinct from the fiscal limit. Above this tipping point, the debt-to-GDP ratio grows towards the fiscal limit at which default occurs. However, in this region, default does not occur immediately because investors understand that disaster risk may resolve in such a way as to ensure that the debt-to-GDP ratio does not cross the fiscal limit. As before, changes in disaster risk have competing effects on debt dynamics; the risk-free rate drops as the insurance component of government debt becomes more valuable, but sovereign default risk rises. We show that a higher arrival rate of disasters reduces the tipping point level of debt. By contrast, an increase in the size of disasters can move the tipping point in either direction. Moreover, reductions in trend growth *improve* debt sustainability by raising both the debt limit and the tipping point level of debt.

Lastly, our default model also gives rise to a *flipping point* level of debt—a point at which $r$ becomes greater than $g$. Elevated disaster risk has *opposing* effects on the flipping point and tipping point levels of debt, raising the former and lowering the latter. In a world where disaster risk rises, the tipping point level of debt falls compressing fiscal space. However, because the flipping point has *risen,* the policymaker may observe a greater range over which $r < g$. Calibrating our model to 2019 US data and making somewhat pessimistic assumptions on disaster risk (a high probability of 6.5 percent annually) and a low maximum fiscal surplus (2 percent of GDP), we nevertheless find substantial fiscal space. The fiscal limit is 184 percent of GDP, the tipping point is 115 percent of GDP, and the flipping point is 99 percent of GDP. Furthermore, our calibration highlights the potential benefits of higher population growth. Reverting back to the postwar average rate of population growth would push these various thresholds well past 200 percent of GDP.

It is worth emphasizing that the object $r - g$ is not a sufficient statistic for the optimal level of debt, and this paper does not attempt to address this issue.[5] In our model, the policymaker follows a fixed fiscal rule. In particular, the class of fiscal rules we consider allows for a fairly weak fiscal response to rising public debt consistent with a policymaker that narrowly wishes to minimize the direct tax burden for a given level of government expenditure and prefers to raise revenues via government debt, particularly when r<g. Policymakers may favor debt financing because they perceive high economic efficiency

[5] See Barro (1979) and Lucas and Stokey (1983) for optimal debt policy with distortionary taxes. See Bhandari, Evans, Golosov, and Sargent (2017) for optimal debt policy with heterogeneous agents and incomplete markets.

costs or political costs to taxes or sharp reductions in government spending. Policymakers preference for debt financing over taxation seems clear for many advanced economies that have accumulated large debts over the last several decades. US fiscal policy in particular has shown a clear preference for deficit financing since 2000.

The paper is laid out as follows. Section 2 presents basic statistics on the average cost of servicing the public debt and its variability. Section 3 analytically characterizes the relationship between debt dynamics and productivity and population growth as well as output risk in the model where fiscal policy prevents default. Section 4 analyzes the model with fiscal fatigue and sovereign default. Section 5 concludes.

**Related Literature.** This paper builds on a recent emerging literature on low interest rates and secular stagnation. In particular, this paper is closely related to theories of low interest rates that emphasize elevated safety premia. Our work is also related to a smaller academic literature thinking about debt sustainability and budget deficits in the US.

The secular stagnation hypothesis was resurrected by Summers (2013) and formalized using an OLG model with downward nominal wage rigidity in Eggertsson and Mehrotra (2014). Recent work by Eggertsson, Mehrotra, and Robbins (2018) analyzes the sources of low real interest rates in a quantitative life-cycle model.[6] While Eggertsson, Mehrotra, and Robbins (2018) emphasize factors like low population and productivity growth in accounting for low interest rates, Caballero and Farhi (2017) stress a shortage of safe assets and an elevated risk premium in accounting for low interest rates. The papers takes a view that low interest rates on government debt reflect the safety premium—investors desire for a riskless asset. Thus, this paper is closer to the literature that emphasizes a rising risk premium as the chief factor behind low interest rate.[7]

Our paper is also related to a literature that considers the implications of low real interest rates for debt policy and its implications for dynamic efficiency. Abel, Mankiw, Summers, and Zeckhauser (1989) argue that low interest rates on safe assets are not inconsistent with dynamic efficiency. Blanchard and Weil (2001) provide a series of analytical examples to show that even in cases when $r < g$ due to the effects of uncertainty, it may not be feasible for the government to play a debt Ponzi game. Bohn (1995) shows that empirical tests of debt sustainability that rely on $r < g$ are not appropriate in stochastic economies—a finding that we build on in our analytical framework.

This paper is also related to a literature prominent in the late 1980s and 1990s concern-

[6]See Gagnon, Johannsen, and Lopez-Salido (2016) and Jones (2016) for quantitative models of low interest rates due to demographic factors.

[7]See, for example, Del Negro, Giannone, Giannoni, and Tambalotti (2017) and Farhi and Gourio (2018) for a quantitative analysis of how risk and liquidity premia account for low interest rates on government debt.

ing the sustainability of large US deficits and rising US debt. See, for example, Auerbach (1994) for a discussion of the large US deficits in the late 1980s and early 1990s and its implications for the medium term fiscal outlook; Elmendorf and Sheiner (2016) assess the US budget outlook due to population aging. In contrast to the conventional wisdom, Woodford (1990) showed how high levels of debt may be welfare improving and may crowd-in capital in the presence of financial frictions, while stressing the empirical fact of low $r$ relative to $g$ for US government debt in the historical record. Angeletos, Collard, and Dellas (2016) consider optimal policy when interest rates on public debt are low due to financial frictions.

## 2 Stylized Facts on $r - g$

In this section, we briefly summarize some basic facts on the relationship between the return on government debt and the growth rate of the economy. We show that, for advanced countries, $r$ is frequently less than $g$ and $r - g$ exhibits substantial variability over time.[8]

The importance of $r$ relative to $g$ for debt sustainability can easily be seen by inspecting the government's flow budget constraint, expressed here in continuous time and featuring no stochastic disturbances:

$$dB_t = [r_t B_t + (G_t - T_t)]dt, \tag{1}$$

where $T_t$ is real tax revenue (net of any transfers), $G_t$ is real government expenditures, $B_t$ is real government debt, and $r_t$ is the effective real interest rate paid on government debt. $Y_t$ is gross domestic product (GDP), hence the debt-to-GDP ratio is $B_t/Y_t$. Along the balanced growth path with constant interest rate and output growth, equation (1) gives an expression for the primary surplus that keeps the debt-to-GDP ratio constant

$$\frac{T_t - G_t}{Y_t} = (r - g)\,\frac{B_t}{Y_t}.$$

The difference between the return on public debt $r$ and output growth rate $g$ represents the unit fiscal cost of servicing the public debt. If this difference is negative, then issuing public debt raises real resources—higher levels of public debt reduce the tax revenues needed to finance a given level of government spending or, equivalently, *any* primary deficit can be sustained with a sufficiently large stock of public debt. However, if $r > g$ as

[8] Mehrotra (2018) provides further empirical analysis on the risk of reverting to periods of $r > g$ and the historical relationship of $r - g$ with both productivity and population growth.

**Table 1:** Moments of macro variables

| | 17 Advanced Countries | | | United States | | |
|---|---|---|---|---|---|---|
| | Median | 25th perc. | 75th perc. | Median | 25th perc. | 75th perc. |
| Long-term nominal interest rate | 4.61 | 3.62 | 6.38 | 3.92 | 3.32 | 5.48 |
| Inflation rate | 2.14 | 0.11 | 4.39 | 1.75 | 0.00 | 3.51 |
| Real interest rate | 2.71 | 1.17 | 4.82 | 2.66 | 1.52 | 4.29 |
| Real GDP per capita growth | 2.01 | 0.28 | 3.82 | 1.89 | -0.45 | 3.75 |
| Population growth | 0.80 | 0.44 | 1.17 | 1.39 | 0.97 | 1.91 |
| Debt to GDP ratio | 44.2 | 24.3 | 68.6 | 36.4 | 15.1 | 59.0 |
| No. of observations | 2145 | | | 134 | | |

Real interest rate is the long-term nominal interest rate less a three-year moving average of inflation rates. All variables expressed as percentage points. Statistics based on data set after observations with fiscal cost more than 10 percent or less than -10 percent are dropped.

typically assumed, the government must run a primary surplus to stabilize the debt-to-GDP ratio.

## 2.1 Dataset

To analyze the behavior of $r - g$, we draw on the historical macroeconomic dataset of Jordà, Schularick, and Taylor (2016).[9] This dataset provides macroeconomic and financial variables for seventeen advanced economies including the US from 1870 to 2016.

To compute measures of the cost of servicing the debt $r - g$, we need a measure of the ex-ante real interest rate. A three-year moving average of inflation is used as a proxy for expected inflation in line with the approach in Hamilton, Harris, Hatzius, and West (2016). The real interest rate is then the nominal measure from Jordà, Schularick, and Taylor (2016) less the three-year moving average of inflation. When using annual data, we drop extreme observations of $r - g$ above ten percent and below negative ten percent. By using a measure of long-term nominal rates, we are adopting a conservative measure of cost of servicing the public debt; short-term nominal rates are considerably lower and the effective interest rate on government debt will likely be lower than the long-term nominal rate.[10] The resulting dataset is an unbalanced panel of 2145 observations.

Table 1 provides basic summary statistics for the real interest rate, the population growth rate and the real GDP per capita growth rate in the dataset. Values are shown for all countries and for just the US. For all countries, the median nominal long-term rate is 4.6 percent with a median inflation rate of 2.1 percent. For the US, both interest rates

[9]Their data set is available from `http://www.macrohistory.net/data`.

[10]Jordà, Schularick, and Taylor (2016) provide a single long-term and short-term nominal rate. Interest rates for all maturities or an effective interest rate on public debt is not available.

and inflation rates are slightly lower than the global median. Population growth is somewhat higher in the US, as is per capita real GDP growth. Debt to GDP ratios are, on average, slightly lower in the US.

Real interest rates (negative in the US) and population growth were in the bottom quartile of the distribution of historical observations just before the start of COVID-19 pandemic. By contrast, the value of the US public debt-to-GDP ratio before the pandemic of 80 percent of GDP are in the top quartile. Together, these levels of population growth and real GDP growth along with high levels of debt-to-GDP seem problematic for debt sustainability; however, very low real interest rates have kept debt servicing costs quite low.

## 2.2 Median Servicing Cost

Figure 3 plots the debt servicing cost for the US—long-term real interest rates less GDP growth. The solid line shows this measure for the US where the real interest rate is calculated using a three year moving average of inflation. The dashed red line is a five-year moving average of the solid line to smooth out business cycle fluctuations. As the figure shows, the cost of servicing the debt has frequently been negative historically and for a large part of the postwar period. Indeed, the period between 1980 and 2000 is the exception; one of the few periods where real interest rates on government debt consistently exceeded real GDP growth. In the postwar period, the fiscal cost measure displays less volatility and greater persistence than the prewar or interwar periods.

Table 2 presents statistics on the fiscal cost of servicing the debt: $r - g$. We take averages over five year periods (non-overlapping) of $r - g$ for the US and 16 other advanced economies, presenting median values and ranges.[11] As the table shows, over the full sample of advanced economies, the median value of $r - g$ is nearly zero (eight basis points). In the US, that median value ($-16$ basis points) has been negative over the past 140 years. The finding that $r - g$ is close to zero is not a function of historically extreme periods. Excluding the world wars and the interwar years (including the Great Depression) leaves the median value slightly higher for all countries and unaffected for the US. Limiting the sample to only the postwar period, net fiscal cost becomes *more* negative for both the full sample and the US.

Though the median value is negative, there remains substantial variability in the cost of servicing the debt. Table 2 provides the interquartile range of $r - g$ for both the full sample and the US. The 75th percentile is roughly one percent in the US, while the 25th percentile is substantially negative. These percentiles display the same level shift; $r - g$

[11] Five-year periods with fiscal cost above ten percent or below minus ten percent are winsorized at these levels.

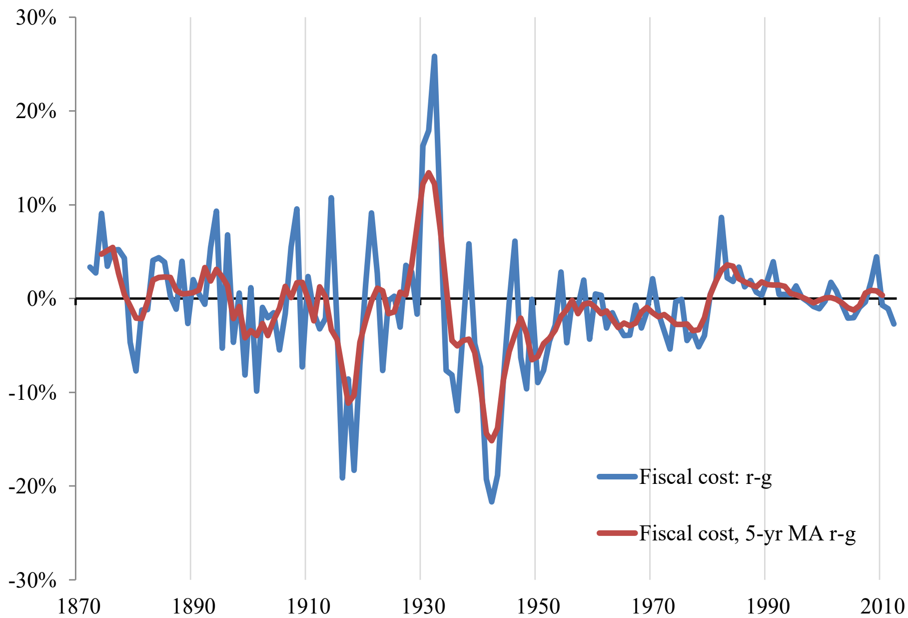


**Figure 3:** US (unit) debt servicing cost $r - g$.

is lower at each quartile than the corresponding figure for the all country sample. An interquartile range of four to five percentage points demonstrates the substantial variability in net fiscal cost.

Table 2 also shows the fraction of observations with a negative debt servicing cost or a substantially negative cost (less than negative two percent). In the all-country sample, half of the observations are negative and between 20 to 35 percent of five-year periods feature a substantially negative value for the fiscal cost depending on the time period. In the case of the US, these percentages are somewhat higher than those for the global sample. Again, the percentage of years with a negative cost for the public debt are not driven by the interwar years and the Great Depression, or the world wars. If anything, the postwar period has featured a greater percentage of years with $r - g$ negative. Quite remarkably, 70 percent of five-year periods in the US and 55 percent of five-year periods across all advanced countries show negative net fiscal cost in the postwar period.

Nevertheless, values for $r - g$ at the 75th percentile illustrate the perils of carrying a high public debt level. Sustained periods of even relatively moderate levels of debt servicing costs would require a substantial primary surplus to stabilize the debt to GDP ratio, particularly among countries at or above debt to GDP ratios of 100 percent. Here, we have just treated $r - g$ as a statistical object; in the next section, we turn to the question of debt dynamics when $r - g$ is determined endogenously.

**Table 2:** Moments of the debt servicing cost.

| | 17 Advanced Countries | | | United States | | |
|---|---|---|---|---|---|---|
| | 1870-2016 | 1870-1914, 1946-2016 | 1946-2016 | 1870-2016 | 1870-1914, 1946-2016 | 1946-2016 |
| r - g (percent) | | | | | | |
| 25th percentile | -2.7 | -1.8 | -3.0 | -2.2 | -2.2 | -2.2 |
| Median | 0.1 | 0.2 | -0.8 | -0.3 | -0.4 | -1.0 |
| 75th percentile | 2.5 | 2.4 | 1.4 | 1.7 | 1.7 | 0.9 |
| Fraction < 0 | 49 | 47 | 58 | 52 | 55 | 71 |
| Fraction < -2% | 30 | 24 | 34 | 31 | 27 | 36 |
| No. of observations | 491 | 373 | 238 | 29 | 22 | 14 |

Real interest rate is the long-term nominal interest rate less a three-year moving average of inflation rates. "Fraction < 0" is the fraction of years experssed in percent with negative debt servicing cost. "Fraction < -2%" is the fraction of years with the debt servicing cost of less than negative two percent. Statistics are based on the dataset after observations with the fiscal cost more than 10 percent or less than -10 percent are winsorized at thresholds.

# 3 Debt Sustainability without Default

In this section, we introduce a simple continuous-time general equilibrium model in which aggregate output follows an exogenous geometric Brownian motion process and fiscal policy ensures absence of default. We will first derive closed-form expressions for the real interest rate on government debt, the return on risky assets, and the distribution of the public debt-to-GDP ratio. Then we will present conditions that guarantee stationarity of the debt-to-GDP ratio. We establish that stationarity of debt-to-GDP is not cleanly related to a notion of debt sustainability that we provide below. Finally, we conclude this section with a numerical exercise that assesses the importance of different forces that we discuss.

## 3.1 Households

Time is continuous, infinite, and indexed by $t$. The economy is populated by a representative household with a continuum of identical infinitely-lived members whose measure $N_t$ grows deterministically at a constant rate $n$ with the initial value at time zero of one. The members of the household derive utility from consuming and from holding safe and liquid bonds that can only be supplied by the government. Each member of the household is endowed with a unit of traded Lucas trees that yield dividends that follow geometric Brownian motion:

$$\frac{dy_t}{y_t} = g_y dt + \sigma_y dZ_t^y, \tag{2}$$

where $g_y$ is the growth rate of per capita endowment, $\sigma_y^2$ is the volatility of shocks to the growth rate, and $dZ_t^y$ is standard Brownian motion. The initial value is $y_0$. Total output $Y_t$ equals to $N_t y_t$. We refer to $y_t$ as productivity. We draw a distinction between the growth rate of output $g$ that we have referred to earlier and the growth rate of output per capita (or productivity), $g_y$. In the model, average output growth $g$ is the sum of population growth $n$ and productivity growth $g_y$.

Financial markets are dynamically complete, i.e., agents have access to government-issued liquid bonds, safe bonds, which are non-liquid, and a risky security.[12] The safe bonds are assumed to be in zero net supply. Equity—a claim on a Lucas tree that pays consumption goods at rate $y_t$—is in positive net supply that increases at the rate of population growth.

A typical household chooses paths for consumption $\{c_t\}$, wealth $\{w_t\}$, investments in liquid, safe, and risky assets $\{b_t, s_t, x_t w_t\}$, where $x_t$ denotes a fraction of wealth invested in risky assets, to maximize

$$\mathbb{E}_0 \int_0^\infty e^{-(\rho-n)t} \left[ \frac{c_t^{1-\gamma} - 1}{1-\gamma} + y_t^{1-\gamma} u\left(\frac{b_t}{y_t}\right) \right] dt, \tag{3}$$

subject to the flow budget constraint

$$dw_t = (r_t^s s_t + r_t b_t - c_t - \tau_t - w_t n)dt + w_t x_t dr_t^x, \tag{4}$$

and subject to the wealth breakdown into safe, liquid, and risky assets

$$w_t = s_t + b_t + x_t w_t, \tag{5}$$

together with a no-Ponzi game condition that we express below. All the variables in the above optimization problem represent per member of household quantities: $c_t$ is consumption of a member of the household, $w_t$ is the financial wealth, $b_t$ is the purchases of government bonds, which are liquid and safe, $s_t$ is purchases of safe assets that do not provide liquidity services, $\tau_t$ are taxes. Note that the part of the drift in financial wealth of a member of the household $-nw_t$ captures the fact that new members dilute the household level of wealth. The returns on liquid, safe, and the risky assets over a short period of time $dt$ are $r_t dt, r_t^s dt$, and $dr_t^x$ respectively.[13]

[12] When the households are free to re-optimize their portfolios at each instant, their access to just three securities mentioned above and optimal portfolio choice is equivalent to the presence of complete markets, i.e., the access to state-contingent Arrow-Debreau securities.

[13] All equalities featuring random variables hold "a.s.", and all stochastic differential equations are assumed to have solutions.

The equity price $q_t$ is assumed to follow the process $dq_t/q_t = (\mu_t - y_t/q_t)dt + \sigma_t dZ_t^y$, where yet unknown processes $\mu_t$ and $\sigma_t$ are determined in equilibrium. The return on risky equity is

$$dr_t^x = \frac{dq_t + y_t dt}{q_t} = \mu_t dt + \sigma_t dZ_t^y, \tag{6}$$

so that $\mu_t$ and $\sigma_t$ are interpreted as the drift and volatility of the risky equity return. We next turn the flow budget constraint in equation (4) into an intertemporal budget constraint. To do this we assume (and verify in the proof of Proposition 3) that there exists a unique continuous-time stochastic discount factor $\xi_t$ that follows a diffusion process of the form

$$\frac{d\xi_t}{\xi_t} = -r_t^s dt - \frac{\mu_t - r_t^s}{\sigma_t} dZ_t^y. \tag{7}$$

with some initial value $\xi_0$.[14] The no-Ponzi game condition can be expressed using this stochastic discount factor as

$$\lim_{T\to\infty} \mathbb{E}_0 e^{nT} \xi_T w_T \geq 0. \tag{8}$$

The following lemma presents the intertemporal budget constraint of the household.

**Lemma 1.** *The flow budget constraint* (4) *together with the no-Ponzi game condition imply the intertemporal budget constraint*

$$w_0 \geq \mathbb{E}_0 \int_0^\infty [c_t + \tau_t + (r_t^s - r_t) b_t] \frac{e^{nt}\xi_t}{\xi_0} dt. \tag{9}$$

The proof is in Appendix A.1. The inequality (9) states that the value of initial wealth, comprised of the value of Lucas trees and the stock of government debt, does not exceed an expected integral of future discounted spending on consumption, taxes, and government debt holding costs. Note that the last term in the square brackets on the right-hand side of the inequality (9) is like the costs of renting a durable good, e.g., housing. However, in our model, this durable good is government bonds that provide liquidity services. The inequality reflects the imposed no-Ponzi game condition. The number of members in the household, i.e., $e^{nt}$, on the right-hand side of the inequality (9) reflects the fact that all expenditure terms are expressed per household member.

The members of the household derive utility from consumption stream $\{c_t\}$ and from holding government bonds $\{b_t\}$. Formally, the utility function (3) consists of two terms that capture the utility from consumption and utility from holding government bonds.

[14] A stochastic discount factor is usually defined as $M_{t+1} = \beta u'(c_{t+1})/u'(c_t)$ in discrete-time models. By contrast, the term stochastic discount factor is usually used to denote the following object $\xi_t = e^{-\rho t} u'(c_t)$ in continuous-time models. This will also be the case here. As the proof of Proposition 3 shows, the stochastic discount factor defined as $\xi_t = e^{-\rho t} u'(c_t)$ satisfied equation (7).

The assumption that households have non-pecuniary preferences over government debt is a reduced form way to represent special features of government debt such as safety and liquidity. This assumption has been used, for example, to improve asset-pricing properties of the standard finance models (Krishnamurthy and Vissing-Jorgensen, 2012), explain business cycles (Fisher, 2015), solve for optimal government debt maturity (Greenwood, Hanson, and Stein, 2015), resolve New Keynesian puzzles (Michaillat and Saez, 2018), and fit the consumption response to income shocks (Auclert, Rognlie, and Straub, 2018). In our setting, with a representative household, these preferences introduce a deviation from the Ricardian equivalence. As a result, changes in the supply of government bonds affect the interest rate paid on these bonds, and, hence, the cost of servicing the public debt.

The preferences over holding government bonds, the second term in equation (3), are additive over time. The instantaneous utility over liquid bonds depends on the exogenous endowment and liquid debt holdings relative to this endowment. This specific form of utility will result in demand for liquid debt over aggregate output that is a function of liquidity yield only. We choose the following structural form for tractability of the model:

$$u\left(\frac{b_t}{y_t}\right) = \frac{b_t}{y_t}\left[\alpha_u + \beta_u - \beta_u \log\left(\frac{b_t}{y_t}\right)\right], \tag{10}$$

where $\alpha_u$ are $\beta_u$ are non-negative real numbers. Equation (10) implies that the marginal preferences from holding government debt, i.e., $u'(b_t/y_t) = \alpha_u - \beta_u \log(b_t/y_t)$, is decreasing in debt holdings. The fact that the marginal utility turns negative after a certain level of debt-to-GDP can be thought of as a reduced form way of capturing the idea that people may worry about government debt when it is too large. Finally, it is worth emphasizing that most of our results do not depend on government debt providing liquidity services; that is, most of our findings hold when we set coefficients $\alpha_u$ and $\beta_u$ to zero.

## 3.2 Government

The government issues instantaneous riskless debt. The flow budget constraint is

$$dB_t = r_t B_t dt + D_t dt, \tag{11}$$

where $B_t$ is total outstanding government debt and $D_t \equiv N_t\left(g_t - \tau_t\right)$ is the drift of the primary deficit of the government with $g_t$ and $\tau_t$ representing per capita government spending and revenues. The primary deficit drift $D_t$ follows a fiscal rule of the form

$$\frac{D_t}{Y_t} = \alpha_D \frac{B_t}{Y_t} - \beta_D \frac{B_t}{Y_t} \log\left(\frac{B_t}{Y_t}\right), \tag{12}$$

where $\alpha_D$ is a real number and $\beta_D$ is a non-negative real number. This fiscal rule has two important properties. First, the deficit reacts proportionally to the level of debt as captured by the first term in equation (12). Second, when $\beta_D$ is strictly positive, the government reacts to increases in the debt-to-GDP ratio by strongly (i.e., more than one-for-one) reducing the primary deficit-to-GDP ratio. The presence of each of these two terms has distinct implications for debt dynamics and sustainability that we characterize below. Equation (12) represents a relatively strong response of fiscal policy to variations in debt-to-GDP ratio; in Section 4, we investigate weaker fiscal rules that are characterized by a maximum primary surplus.

The fiscal rule (12) only specifies the behavior of the primary fiscal deficit. To complete the description of fiscal policy, we assume that the government engages in wasteful spending which is a constant fraction of total output:

$$g_t = \gamma_G y_t. \tag{13}$$

Public debt dynamics in equation (11) can be re-expressed in terms of the log debt-to-GDP ratio denoted as $\widehat{B}_t$. We summarize this useful intermediate result in the next lemma.

**Lemma 2.** *Given the productivity process* (2) *and the fiscal policy* (11)-(13)*, the log debt-to-GDP ratio follows*

$$d\widehat{B}_t = (r_t - g_y - n + \alpha_D - \beta_D \widehat{B}_t + \sigma_y^2/2)dt + \sigma_y dZ_t^y. \tag{14}$$

See Appendix A.2 for details. Equation (14) states that the drift of the log of debt-to-GDP ratio depends positively on the interest rate, the deficit-to-debt ratio $\alpha_D - \beta_D \widehat{B}_t$, and the volatility of productivity process; and negatively on the growth rate of total output $g_y + n$. A convenient property of expression (14) is that the diffusion is constant, which we use to derive a closed-form solution.

### 3.3 Equilibrium Characterization

In this section, we first define equilibrium and then characterize its properties.

**Definition 1** (Equilibrium)**.** An equilibrium is a collection of interest rates $\{r_t dt, r_t^s dt, dr_t^x\}$ and allocations $\{c_t, w_t, b_t, s_t, x_t, g_t, \tau_t\}$ such that households solve (3)-(5), (8), the government acts according to (11)-(13), and the markets clear: (i) $N_t b_t = B_t$ (government bonds), $N_t s_t = 0$ (safe bonds), $c_t + g_t = y_t$ (goods).

A combination of the no-Ponzi game condition (8) and the household transversality condition $\lim_{T\to\infty} \mathbb{E}_0 e^{nT} \xi_T w_T \leq 0$ (a necessary condition of optimal behavior), imply

$$\lim_{T\to\infty} \mathbb{E}_0 e^{nT} \xi_T w_T = 0. \tag{15}$$

In equilibrium, total financial wealth of the household is a sum of the value of Lucas trees and the government debt, i.e., $w_t = q_t + b_t$. Because $q_t$ and $b_t$ are always non-negative, the condition (15) necessarily implies that in equilibrium:

$$\lim_{T\to\infty} \mathbb{E}_0 e^{nT} \xi_T b_T = 0. \tag{16}$$

We will call the limit in equation (16) *the government transversality condition* (TVC).

The household intertemporal budget constraint (9), equation (15), and the market clearing conditions yield

$$b_0 = \mathbb{E}_0 \int_0^\infty [\tau_t - g_t + (r_t^s - r_t) b_t] e^{nt} \frac{\xi_t}{\xi_0} dt, \tag{17}$$

which we call *the intertemporal budget constraint of the government*. Equations (16) and (17) place equilibrium restrictions on the set of feasible fiscal policies. Intuitively, fiscal policy must satisfy these equations if the households purchase government debt in equilibrium.

**Definition 2** (Sustainable fiscal policy)**.** Government fiscal policy is sustainable if it satisfies equations (16) and (17).

Following the consumption asset pricing literature, we derive the returns on the assets in the model. We illustrate these steps in Appendix C and summarize asset returns in the following lemma.

**Lemma 3** (Asset pricing)**.** *The equilibrium interest rate on safe assets and liquid government bonds are:*

$$r^s = \rho + \gamma g_y - \frac{\gamma(\gamma+1)}{2}\sigma_y^2, \tag{18}$$

$$r_t = r^s - \alpha_u + \beta_u \widehat{B}_t, \tag{19}$$

*and the risky asset return is characterized by the drift* $\mu_t = r^s + \gamma\sigma_y^2$ *and the diffusion* $\sigma_t = \sigma_y$.

The safe real interest rate is constant while the return on government bonds varies with the debt-to-GDP ratio so long as $\beta_u$ is not equal zero. This lemma shows how candidate explanations for secular stagnation affect the safe and liquid rates. A lower rate of trend output growth (equivalently, productivity growth) or an increase in the variance of the output process lower the safe rate of return. The elasticity of the real interest rate with respect to trend growth is governed solely by the intertemporal elasticity of substitution $1/\gamma$—for a coefficient smaller than unity, the real interest rate responds more than one-for-one with a change in trend growth. A rise in volatility of endowment process lowers the safe interest rate and raises the risk premium with the strength of the effect rising with the coefficient of risk aversion $\gamma$. Importantly, neither the risk premium nor the safe interest rate depend on population growth. Lower population growth—modeled as a slower rate of birth of new members of the representative household—leaves the real interest rate unaffected. Equation (19) shows that as the debt-to-GDP ratio rises, the marginal benefit of liquid wealth decreases, raising the rate of return on government debt.

### 3.4 Dynamics of the Debt-to-GDP Ratio

Using the law of motion for the debt-to-GDP ratio in Lemma 2 together with the expression for the return on liquid government bonds in Lemma 3, we have:

**Proposition 1.** *In equilibrium, the log debt-to-GDP ratio follows an Ornstein-Uhlenbeck process:*

$$d\widehat{B}_t = \left(\alpha - \beta\widehat{B}_t\right) dt + \sigma_y dZ_t^y,$$

*where* $\alpha \equiv \alpha_D - \alpha_u + \rho + (\gamma - 1)g - n - [\gamma(\gamma+1) - 1]\,\sigma_y^2/2$ *and* $\beta \equiv \beta_D - \beta_u$.

The parameter $\alpha$ is a sufficient statistic for all of the forces that increase the drift of the debt-to-GDP ratio and that are independent of the level of debt-to-GDP, while $\beta$ is a net effect of the forces that reduce the drift and that are proportional to the level of the debt-to-GDP ratio.

As a geometric Brownian motion, the output process has an ever increasing variance as the time horizon expands. However, the log debt-to-GDP ratio is mean-reverting when $\beta > 0$ with finite stationary variance. The law of motion is an Ornstein-Uhlenbeck (OU) process, which is the continuous-time analogue of an auto-regressive process of order one, i.e., an AR(1) process. As with an AR(1) process, it is straightforward to derive the evolution of the distribution of the log of debt-to-GDP ratio as well as its stationary distribution by solving the Kolmogorov forward equation (see, e.g., Stokey, 2009). The following lemma presents the stationary distribution.

**Proposition 2** (Stationary debt-to-GDP)**.** *If $\beta > 0$, then in equilibrium, the stationary distribution of the log debt-to-GDP ratio exists and it is a normal distribution with mean $\alpha/\beta$ and variance $\sigma_y^2/2\beta$. In levels, the stationary distribution of the debt-to-GDP ratio is log-normal with mean* $\exp[(\alpha + \sigma_y^2)/\beta]$ *and variance* $[\exp(\sigma_y^2/2\beta) - 1]\exp(2\alpha/\beta + \sigma_y^2/2\beta)$.

Proposition 2 provides a sufficient condition for existence of a stationary distribution and expresses the key parameters of the distributions in terms of the parameters $\alpha$ and $\beta$ from the law of motion for public debt and the underlying volatility of the endowment shocks.

Propositions 1 and 2 allow us to characterize how the distribution of the debt-to-GDP ratio changes with the possible explanations for secular stagnation. A decline in population growth $n$ increases $\alpha$ shifting the mean of the log debt to GDP ratio to the right while leaving the variance unchanged. In levels, both the mean and variance of the debt-to-GDP ratio increase. Slower population growth has a direct effect on GDP growth while leaving the real interest rate unchanged.[15] The cost of servicing a given stock of public debt is spread over a smaller population, raising the debt-to-GDP ratio.

By contrast, a decline in the trend productivity growth $g_y$ can decrease $\alpha$, thereby lowering the mean debt-to-GDP ratio. Whether a decline in productivity growth raises or lowers parameter $\alpha$ depends crucially on the intertemporal elasticity of substitution (IES) $1/\gamma$. If this parameter is less than one, a decline in productivity growth lowers both the mean and variance of the debt-to-GDP ratio in levels. An extensive literature has attempted to measure the elasticity of substitution by examining how household's consumption growth responds to changes in the real interest rate faced by these households. The IES is commonly assumed to be less than one in macroeconomics literature, with some estimates suggesting that it is in fact substantially less than one (e.g., Hall, 1988; Campbell, 1999). Paradoxically, a lower trend growth rate $g_y$ actually shifts the debt-to-GDP ratio lower by exerting a stronger downward pressure on real interest rates relative to its direct effects on growth.

Lower real interest rates could also be due to an increase in the volatility of shocks in the output process. Greater volatility raises the risk premium and lowers real interest rates. This puts downward pressure on the debt-to-GDP ratio. However, higher volatility of shocks to output also tends to increase the debt-to-GDP ratio because the "Jensen's inequality effect"—the debt-to-GDP ratio is a convex function of output. Overall, an increase in volatility will decrease the mean debt-to-GDP ratio only if its effect through the interest rate dominates the Jensen's inequality effect. A higher coefficient of risk aversion and a lower endogenous response to debt via fiscal policy (lower $\beta_D$) make this more likely. Similarly, in principle, higher volatility may raise or lower the variance of the debt to GDP ratio depending on the strength of the effect on the real interest rate.

Fiscal policy and the liquidity parameters also effect the stationary distribution of the

[15] The finding that population growth does not affect the interest rates is specific to this model. As Eggertsson, Mehrotra, and Robbins (2018) show, in a quantitative life-cycle model, slower population growth generally lowers the real interest rate. However, this effect in unlikely to be strong enough in standard quantitative life-cycle models (Carvalho, Ferrero, and Nechio, 2016) to overturn a rightward shift in debt-to-GDP distribution.

debt-to-GDP ratio. A stronger fiscal response to changes in public debt is necessary to ensure that a stationary distribution exists—as the fiscal response weakens, i.e., $\beta_D$ approaches $\beta_u$ from above, both the mean and variance of the debt-to-GDP ratio grow unboundedly. The constant component of the liquidity premium $\alpha_u$ lowers the real interest rate on government debt, decreasing the mean and variance of the debt-to-GDP ratio in much the same way as lower productivity growth when the IES is less than one.

So far, we consider a situation in which the fiscal authority chooses a rule that ensures mean-reverting dynamics of the of debt-to-GDP ratio by guaranteeing $\beta$ to be positive. However, we can characterize the behavior of the debt-to-GDP ratio when $\beta$ is not positive. It is clear that when $\beta$ is strictly negative, the interest rate on government debt increases faster than the primary surplus and the transversality condition of the government, equation (16), is not satisfied, making fiscal policy unsustainable in the sense of Definition 2. The easiest way to see why the transversality condition fails is to note that mean of the *log* of debt-to-GDP ratio increases exponentially, as can be seen by solving the differential equation in Proposition 1. At the same time, the *log* of the discount factor decreases only *linearly* with time according to equation (7). The following proposition summarizes this observation.

**Proposition 3** (Unsustainable fiscal policy)**.** *When $\beta < 0$, the fiscal policy defined by equations* (12)-(13) *is unsustainable in the sense of Definition* 2*.*

When $\beta$ is zero, the *level* of debt-to-GDP either shrinks to zero when $\alpha$ is below zero or grows to infinity when $\alpha$ is greater than zero. In the latter case, the debt-to-GDP is exponential in time. Importantly, the transversality condition (16) can fail in the former case, but still hold in the latter case. The following lemma summarizes the necessary and sufficient conditions for debt sustainability in this case.

**Proposition 4** (Non-stationary debt-to-GDP)**.** *When $\beta = 0$, the fiscal policy defined by equations* (12)-(13) *is sustainable if and only if $\alpha_u - \alpha_D > 0$.*

The proof is in Appendix A.4. Crucially, Proposition 4 shows that debt sustainability is *not* tightly linked to the debt dynamics in an environment with no default. First, even when the debt-to-GDP ratio grows without limit over time, i.e., $\alpha$ is positive, debt is sustainable when $\alpha_u - \alpha_D$ is above zero. The reason why this scenario is sustainable is that the growth rate of debt is lower than the growth rate of the discount factor so that the transversality condition of the government is satisfied. In this case, the intertemporal budget constraint of the government is satisfied automatically. A key parameter that ensures sustainability is the sum of the sensitivity of the primary fiscal surplus to the stock of government debt $-\alpha_D$ and the the constant part of the liquidity yield $\alpha_u$. Note that it is sufficient for this object to be positive; it does not have to be large. Furthermore, if the government surplus does not respond to the level of debt at all, i.e., $\alpha_D$ is zero, then it is still possible for the debt to be sustainable because of the remaining term $\alpha_u$, which reflects profits that the government collects due of its ability to issue liquid debt that households value for their non-pecuniary return. This profit is analogous to seigniorage that the central bank receives on printing money that provides liquidity services.

At the same time, even if the debt-to-GDP ratio is shrinking over time on average, ($\alpha$ is negative), it is not necessarily true that debt is sustainable. For example, it is possible that $\alpha$ is negative (because, for example, the interest rate is low due to high volatility of shocks to output growth $\sigma_y^2$) and the key parameter that governs the sustainability of debt $\alpha_u - \alpha_D$ is also negative, making debt unsustainable in the sense of violating the transversality condition of the government.

## 3.5 Extensions

We present two extensions of the model. First, we analyze a modified fiscal rule that allows for a lower bound on the debt-to-GDP ratio. Then, we add disaster shocks to the endowment process. The model with these extensions still admits a closed-form solution for stationary debt-to-GDP ratio when it exists allowing us to generalize our results.

**Lower-reflecting barrier.** When $\alpha$ is negative, the debt-to-GDP ratio is trending to zero. If we impose a lower reflecting barrier (a minimum level of public debt), it is straightforward to show that the debt-to-GDP ratio admits a stationary distribution:

**Proposition 5** (Lower reflecting barrier). *If $\beta = 0$, $\alpha < 0$, and $\widehat{B}_t \geq \widehat{B}_{min}$, then there is a stationary distribution of log debt-to-GDP ratio which is a negative exponential with rate parameter $\xi \equiv -2\alpha/\sigma_y^2$. In levels, the stationary distribution of the debt-to-GDP ratio is a Pareto distribution with shape parameter $\xi$. Moreover, the transversality condition (16) holds if and only if $\alpha_u - \alpha_D > 0$.*

That the stationary distribution of the log debt-to-GDP ratio is an exponential distribution is a standard result in the analysis of continuous-time stochastic processes (e.g., Dixit, 1994). Given that the debt-to-GDP ratio has a Pareto distribution, its mean and variance depends crucially on the shape parameter. The mean and variance of a Pareto distribution only exist when the shape parameter is larger than two. Importantly, a higher negative drift $\alpha$ or a lower volatility shocks to output raise the shape parameter, lowering both the mean and variance. Again, sustainability is not closely tied to the existence of a stationary distribution. Debt may be sustainable even if the mean and/or variance do not exist because the Pareto tail is too fat.

Comparative statics in this case are similar to what was shown in Proposition 2. Specifically, lower population growth lowers the shape parameter thereby increasing the mean and variance of the debt-to-GDP ratio. By contrast, when the elasticity of intertemporal substitution is less than unity, lower productivity growth raises the shape parameter and lowers the mean and variance of debt. The effect of higher volatility of output shocks on the mean and variance of the debt-to-GDP ratio depends again on the relative strengths of its effect on the drift (via lower interest rates) versus the Jensen's inequality effect and the direct effects on the shape parameter (since $\sigma_y$ appears in the denominator).

**Disasters.** Anticipating our analysis in Section 4, we discuss the consequences of adding rare disaster shocks to the above setup with a lower reflecting barrier. Now the output process is a jump-diffusion. Formally, we assume the output process, which was previously defined by equation (2), is

$$\frac{dy_t}{y_t} = g_y dt + \sigma_y dZ_t^y + \left(e^{-Z_t} - 1\right) dJ_t, \tag{20}$$

where $J_t$ is a Poisson process with constant intensity $\lambda > 0$. $Z_t$ are positive, independent, and identically distributed random variables that describe an instantaneous change in log output when a disaster occurs.

Following Wachter (2013), we compute the interest rate when the endowment experiences jumps. This allows us to derive the law of motion and the corresponding stationary distribution of the debt-to-GDP ratio. We present all details in Appendix A.6 and summarize the results in the following proposition:

**Proposition 6** (Disasters)**.** *If $\beta = 0$ and there is a lower reflecting barrier $\widehat{B}_{min}$, then the law of motion of the log of the debt-to-GDP ratio for $\widehat{B}_t > \widehat{B}_{min}$ is*

$$d\widehat{B}_t = \widetilde{\alpha} dt - \sigma_y dZ_t^y + Z_t dJ_t,$$

*where $\widetilde{\alpha} \equiv \alpha - \lambda(\mathbb{E}_Z[e^{\gamma Z}] - 1)$. If $\widetilde{\alpha} < -\lambda \mathbb{E}_Z[Z]$, then there is a stationary distribution of the log debt-to-GDP ratio which is exponential with the rate parameter $\xi$ that solves*

$$\widetilde{\alpha}\xi + \frac{\sigma_y^2}{2}\xi^2 = \lambda(1 - \mathbb{E}_Z[e^{\xi Z}]). \tag{21}$$

*In levels, the stationary distribution of the debt-to-GDP ratio is a Pareto distribution with shape parameter $\xi$. Moreover, when $\widetilde{\alpha} \geq -\lambda \mathbb{E}_Z[Z]$, the transversality condition (16) holds if and only if $\alpha_u - \alpha_D > 0$; when $\widetilde{\alpha} < -\lambda \mathbb{E}_Z[Z]$, it holds if $\xi > 1$.*

The shape parameter $\xi$ solves equation (21), which depends on the disaster intensity

| Parameter | Description | Value | Source/target |
|---|---|---|---|
| $g_y$ | Productivity growth rate | 0.02 | US data |
| $\sigma_y$ | Output per capita std | 0.025 | US data |
| $n$ | Population growth rate | 0.0115 | US data |
| $1/\theta$ | Intertemporal elasticity of substitution | 0.75 | Standard in macro lit. |
| $\alpha_u$ | Liquidity yield independent of debt | 0.0052 | AAA bonds & gov. yield |
| $\beta_u$ | Semi elasticity of liquidity yield | 0.0028 | AAA bonds & gov. yield |
| $\sigma_B$ | Public bonds growth rate shocks std | 0.45 | $cor(dy_t/y_t, r_t) = -0.056$ |

**Table 3:** Parameters common across the three calibrations.

$\lambda$, the distribution of disasters, the drift $\alpha$, and the diffusion $\sigma_y$ of the debt-to-GDP ratio. Thus our model can accommodate an output process where rare disasters lead to intermittent jumps in the debt-to-GDP ratio.

## 3.6 Calibration

In this section, we quantify effects of various secular stagnation forces on the stationary distribution of the debt-to-GDP ratio. There are three distinct calibrations corresponding to the model in Sections 3.1-3.4 and its two extensions in Section 3.5. Considering three calibrations, allows us to analyze the effects of different fiscal policies and different kinds of shocks. In each of these calibrations, we will use a richer version of the model just presented. In particular, we use Epstein-Zin-Weil (EZW) preferences to separate the IES coefficient and the coefficient of relative risk aversion, and we add shocks to the fiscal policy rule to dampen the covariance between the interest rates and productivity growth.

We proceed under the assumption that fiscal policy takes a particular form in (12) extended with shocks so that the law of motion of public debt is

$$dB_t = (r_t B_t + D_t)\, dt + B_t \sigma_B dZ_t^B$$

**Calibrated parameters.** The parameters in Table 3 are fixed across the three calibrations. We set the intertemporal elasticity of substitution to $1/\theta$ to 0.75. The average growth rate of annual GDP per capita since 1950 of 0.02 is used to set $g_y$, while annual postwar population growth of 0.0115 determines $n$. The diffusion term on output growth $\sigma_y$ is 0.025 to match the standard deviation of annual output growth in the US since 1950. The difference between AAA corporate debt and the US 10-year Treasury yield and its elasticity to the debt-to-GDP ratio pins down both $\alpha_u$ and $\beta_u$ (Krishnamurthy and Vissing-Jorgensen, 2012).[16] As mentioned above, we introduce a fiscal policy shock $\sigma_B$ to match the correlation of the real interest rate and output growth for the 17 countries in our dataset since

[16] A regression of the AAA-10 year Treasury spread on the log debt-to-GDP ratio determines $\beta_u$ and $\alpha_u$, which is the constant from this regression plus $\beta_u$.

| Parameter | Description | Cal. 1 | Cal. 2 | Cal. 3 | Source/target |
|---|---|---|---|---|---|
| $\gamma$ | CRRA | 83 | 83 | 3.47 | Equity premium |
| $\rho$ | Subjective disc. factor | 0.07 | 0.07 | 0.06 | Mean 10-year Treasury yield |
| $\beta_d$ | Fiscal rule parameter | 0.16 | $-\beta_u$ | $-\beta_u$ | Debt-to-GDP distribution |
| $\alpha_d$ | Fiscal rule parameter | 0.01 | -0.02 | -0.03 | Debt-to-GDP distribution |
| $\min(B_t/Y_t)$ | Reflecting boundary | - | 0.24 | 0.18 | Debt-to-GDP distribution |
| $\lambda$ | Disaster's arrival rate | - | - | 0.02 | Barro (2006) |
| $\bar{z}$ | Mean log disaster size | - | - | 0.23 | Barro (2006) |

**Table 4:** Calibration-specific parameters. Columns Cal. 1, Cal. 2, and Cal. 3 refer to our calibrations 1-3.

1950. This correlation is $-0.055$, meaning that our model requires the relative variance of the fiscal policy shocks to be higher than that for productivity growth to dampen the correlation between $r_t$ and $dy_t/y_t$.[17]

The remaining parameters, presented in Table 4, differ across the three calibrations. Calibration 1 assumes that all shocks are Brownian and the fiscal policy strongly reacts to the level of debt-to-GDP ratio so that the stationary distribution is log-normal as described in Proposition 2. Calibration 2 changes a fiscal response relative to calibration 1 by introducing a lower reflecting barrier and by setting $\beta_D - \beta_u$ to zero for levels of debt-to-GDP above the reflecting barrier as in the first extension in Section 3.5. Finally, calibration 3 adds disaster shocks to calibration 2, which allows us to lower the coefficient of relative risk aversion required to match the equity premium.

We now describe calibration-specific parameters in detail. The average postwar real return on long-term government bonds of 0.025 in the US is used to pin down the rate of time preference $\rho$. The equity premium of 0.052 in the US is used to determine the coefficient of relative risk aversion given the size of the shocks.[18] In calibration 1, the fiscal response parameters $\alpha_D$ and $\beta_D$ are set to match the mean log debt-to-GDP ratio in the US postwar period and match the variance of the log debt-to-GDP ratio across the full set of 17 countries in the Jordà, Schularick, and Taylor (2016) data set. We use the full set of 17 countries given the high degree persistence in the debt-to-GDP ratio and absent strong evidence on how systematically fiscal policy responds to the debt-to-GDP ratio. In calibrations 2 and 3, the same empirical moments for the debt-to-GDP ratio determine $\alpha_D$ and the position of a lower reflecting barrier $\min(B_t/Y_t)$. $\beta_D$ is just set to offset the effect of liquidity parameter $\beta_u$ so that $\beta$ is 0 in calibrations 2 and 3. Finally, in calibration 3, we assume that productivity follows the stochastic process with disasters in equation (20).

[17] An alternative way to dampen correlation between the interest rate $r_t$ and the growth rate of productivity without setting a high variance of instantaneous shocks to the fiscal rule $\sigma_B$ is to assume that the fiscal rule parameter $\alpha_D$ is not constant but rather a mean-reverting Ornstein-Uhlenbeck process with a sufficiently high persistence.

[18] Both the bonds return and equity premium are from Jorda, Knoll, Kuvshinov, Schularick, and Taylor (2018).

| Variable | Description | Baseline | Ex. 1: $n \downarrow$ | Ex. 2: $g \downarrow$ | Ex. 3: $\mathbb{E}[\mu_t] - r^s \uparrow$ |
|---|---|---|---|---|---|
| **Panel A: calibration 1** | | | | | |
| $\mathbb{E}[B_t/Y_t]$ | Mean of debt-to-GDP | 0.75 | 0.77 | 0.73 | 0.65 |
| $std(B_t/Y_t)$ | Std of debt-to-GDP | 0.72 | 0.74 | 0.70 | 0.62 |
| $r^s$ | Safe rate | 0.032 | 0.032 | 0.014 | 0.009 |
| $\mathbb{E}[r_t]$ | Average liquid rate | 0.025 | 0.025 | 0.007 | 0.002 |
| $\mathbb{E}[\mu_t] - r^s$ | Equity risk premium | 0.052 | 0.052 | 0.052 | 0.072 |
| **Panel B: calibration 2** | | | | | |
| $\mathbb{E}[B_t/Y_t]$ | Mean of debt-to-GDP | 1.25 | 1.48 | 1.09 | 0.76 |
| $std(B_t/Y_t)$ | Std of debt-to-GDP | - | - | - | - |
| $r^s$ | Safe rate | 0.032 | 0.032 | 0.014 | 0.009 |
| $\mathbb{E}[r_t]$ | Average liquid rate | 0.025 | 0.025 | 0.007 | 0.002 |
| $\mathbb{E}[\mu_t] - r^s$ | Equity risk premium | 0.052 | 0.052 | 0.052 | 0.072 |
| **Panel C: calibration 3** | | | | | |
| $\mathbb{E}[B_t/Y_t]$ | Mean of debt-to-GDP | 0.93 | 1.10 | 0.82 | 0.86 |
| $std(B_t/Y_t)$ | Std of debt-to-GDP | - | - | - | - |
| $r^s$ | Safe rate | 0.032 | 0.032 | 0.014 | 0.009 |
| $\mathbb{E}[r_t]$ | Average liquid rate | 0.025 | 0.025 | 0.006 | 0.001 |
| $\mathbb{E}[\mu_t] - r^s$ | Equity risk premium | 0.052 | 0.052 | 0.052 | 0.072 |

**Table 5: Moments generated under Calibrations 1-3**. Column "Baseline" shows the moments under the baseline version of a calibration, while columns "Ex. 1," "Ex. 2," and "Ex. 3" shows the moments for the three experiments that correspond to a decline in the population growth rate (Ex. 1), a decline in productivity growth rate (Ex. 2), and an increase in equity premium (Ex. 3).

Following Barro and Jin (2011) and Rebelo, Wang, and Yang (2018), we assume that sizes of disasters (a log change in productivity) are distributed exponentially with the lowest value of zero and the mean of $\bar{z} > 0$. We set the values of $\lambda$ and $\bar{z}$ to their empirical counterparts in Barro (2006). Importantly, due to the presence of disasters, calibration 3 requires a coefficient of relative risk aversion of around 4 to match the equity premium relative to its value of 84 in calibrations 1 and 2.

**Results.** Table 5 presents several moments from our three calibrations. "Baseline" shows the moment for baseline parameters where the three calibrations are split into the three panels. Columns "Ex. 1," "Ex. 2," and "Ex. 3" contain the moments of three comparative statics. We see an important difference across the three calibrations in the baseline column. The stationary distribution of debt-to-GDP ratio is log-normal in calibration 1 and Pareto with the shape parameter between one and two in calibrations 2 and 3. As a result, the second moment of the stationary distribution does not exist in the letter two calibrations.

In column Ex. 1, we show the effect of a decline in population growth rate to 0.7 percent—a projected US population growth over the next decade. Slower population growth directly lowers GDP growth while leaving the real interest rate unchanged thereby worsening debt dynamics. All rates of return are left unaffected by changes in population growth, so the effects on the debt-to-GDP ratio come solely through the effects on GDP. Calibration 1 shows a modest increase in the mean of the debt-to-GDP ratio relative to calibrations 2 and 3. This is because a change in population growth affects a tail of Pareto distribution with a stronger effect on the means in calibrations 2 and 3, while this effect is limited under calibration 1 when the stationary distribution is log-normal. Quantitatively, the decline in population growth raises the mean debt-to-GDP ratio by 2, 23, and 17 percentage points in calibrations 1, 2, and 3, respectively.

Column Ex. 2 in the table presents the effects of a decline in productivity growth to 0.7 percent—in line with the post-2008 productivity growth. Slower productivity growth is beneficial for debt sustainability, shifting the debt-to-GDP ratio downward as the fall in $r_t$ outpaces the decline in GDP growth when the IES is below one. Under calibration 1, the mean debt to GDP ratio falls by 2 percentage points. A decline in productivity reduces the government interest rate to 0.7 percent. As before, the quantitative effects on the debt-to-GDP ratio are modest under calibration 1. Calibrations 2 and 3 feature 16 and 11 percentage points decline in the mean debt-to-GDP ratio. This again results from the fact that the mean is quite sensitive to the changes in the tail of the Pareto distribution.

Finally, column "Ex. 3" presents the model moments after an increase in volatility of GDP such that the equity premium rises by 2 percentage points. In calibrations 1 and 2, we achieve this by increasing the value of $\sigma_y$ from 0.025 to 0.029, while in calibration 3, we change the arrival rate of disasters $\lambda$ from 0.02 to 0.028. A rise in volatility shifts the debt-to-GDP distribution leftward in all of the cases. The real rate of return on government debt falls to 0.2 percent under calibrations 1 and 2, and to 0.1 percent in calibration 3. The mean debt to GDP ratio falls by 10, 49, and 7 percentage points in the three calibrations, respectively. Overall, our findings show that the indirect effects of output volatility on the interest rate and consequently the drift of the debt-to-GDP ratio dominate the direct effects of spreading out the distribution of output.

# 4 Debt Sustainability with Default

In this section, we analyze debt dynamics in an environment where the primary surplus is bounded above. In this case, public debt becomes risky, carrying a sovereign default premium. We highlight two main results in this section. First, there is a threshold level of debt-to-GDP ratio, which we call a tipping point, after which the interest rate on debt is high enough that the debt-to-GDP steadily rises toward default. Second, the tipping point and other debt thresholds defined below react to secular stagnation forces in counterintuitive ways.

## 4.1 Model

The model is nearly identical to the previous section, so we only highlight the differences. Household preferences are the same as in equation (3) except that we dispense with non-pecuniary benefits of holding government bonds to focus solely on the riskiness of public debt as the level of debt varies. Second, productivity grows stochastically over time according to equation (20) that allows for jumps. We assume that the Brownian motion term is absent, i.e., $\sigma_y = 0$, and that after a single realization of disaster, there are no further disasters or any other sources of uncertainty. These assumptions allow us to solve the model in closed form. Third, the government flow budget constraint follows equation (11) as before. However, we assume that the deficit $D_t$ takes the form:

$$D_t = -Y_t s\left(\frac{B_t}{Y_t}\right). \tag{22}$$

Importantly, unlike in Section 3, we assume that the surplus function $s(\cdot)$ is bounded above by a positive maximum primary fiscal surplus $\bar{s}$. For example, this limit would emerge in a production economy with distortionary labor taxation only; the government would not be able to raise additional revenue beyond the peak of the Laffer curve. Ghosh, Kim, Mendoza, Ostry, and Qureshi (2013) label this property of the surplus function *fiscal fatigue*. The solid curve in Figure 4 presents a typical sigmoid function shape of a surplus function with such a property. The primary surplus is low or even negative at low levels of debt-to-GDP ratio and rises with the debt-to-GDP ratio, but never exceeds $\bar{s}$. The dashed line demonstrates a step-function approximation of the surplus function. It is positive and equals $\bar{s}$ for positive values of debt-to-GDP, and zero when debt is zero, and it equals $\underline{s}$, which we assume to be large and negative, when debt is negative. Formally,

$$s\left(B_t/Y_t\right) = \begin{cases} \bar{s}, & B_t/Y_t > 0, \\ 0, & B_t/Y_t = 0, \\ \underline{s}, & B_t/Y_t < 0. \end{cases}$$

We will use this step-function to obtain closed-form solutions. However, the qualitative results that we highlight below hold for the solid function in Figure 4 as well, which can be verified by redrawing the phase diagrams in Figures 5 and 6 below for the sigmoid surplus function.

Combining the flow budget constraint of the government and the endowment process,

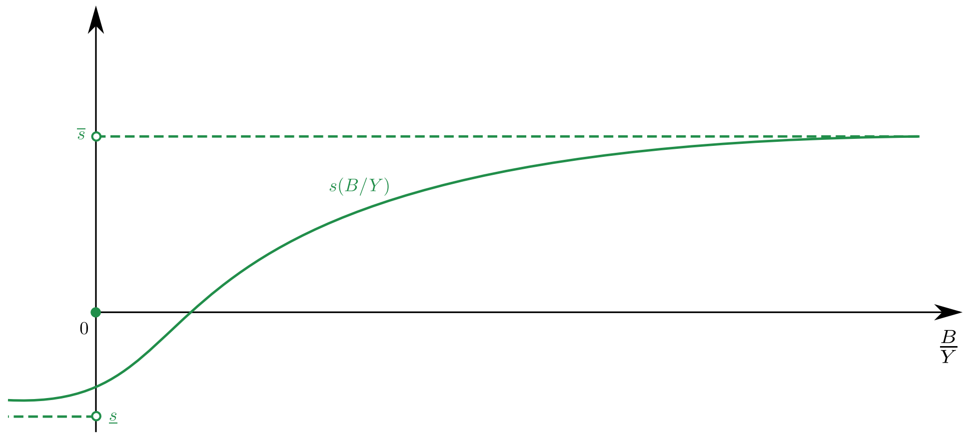


**Figure 4:** Primary fiscal surplus with fiscal fatigue (solid line) and its piece-wise approximation (dashed line).

we get the law of motion for the debt-to-GDP ratio:

$$d\left(\frac{B_t}{Y_t}\right) = \left[(r_t - g_y - n)\frac{B_t}{Y_t} - s\left(\frac{B_t}{Y_t}\right)\right]dt + \frac{B_t}{Y_t}(e^{Z_t} - 1)dJ_t. \tag{23}$$

The first term on the right-hand side is the standard law of motion of debt-to-GDP ratio in a deterministic setting. The second term incorporates the effect of uncertainty; when a rare disaster decreases output by $Z_t$ in log terms, the debt-to-GDP ratio jumps by $Z_t$ in log terms.

## 4.2 Equilibrium Characterization

The structure of uncertainty assumed above allows us to solve the model by backward induction. In particular, we first solve the model after uncertainty has realized. We then describe the evolution of the economy before disaster shock hits the economy.

**After the disaster.** The interest rate on a risk-free security absent uncertainty is $r = \rho + \gamma g_y$, which is a typical Euler equation in continuous time deterministic models. To understand the dynamics of the debt-to-GDP ratio after the resolution of uncertainty, we plot the two forces affecting the debt in Figure 5. The upward-sloping straight black line shows the difference between the interest rate and GDP growth rate times the debt-to-GDP ratio, while the green step-function is the primary surplus over GDP. Dynamic efficiency requires that the black line is upward sloping. If this is not the case, the present value of output is unbounded.

This differential equation admits two steady states: one is stable at zero debt-to-GDP and one is unstable, which we label *the fiscal limit* and denote as $\mathcal{B}_{FL}$, where the black and

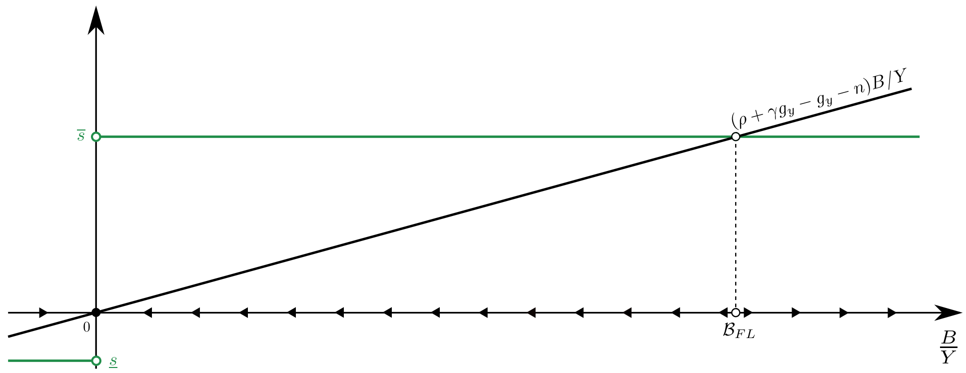


**Figure 5:** Debt-to-GDP dynamics after the resolution of uncertainty.

green lines intersect at a positive debt-to-GDP ratio.[19] The arrows on the horizontal axis show the dynamics of the debt-to-GDP ratio on either side of the fiscal limit. It is evident that debt-to-GDP grows without bound to the right of unstable steady state. Specifically, at a sufficiently high level of debt-to-GDP ratio where the presence of constant surplus does not materially matter anymore, debt-to-GDP grows exponentially. Importantly, this path does not satisfy the transversality condition of the government in equation (16). To see this, it is enough to observe that the government TVC in the no uncertainty case becomes

$$\lim_{T\to\infty} \mathbb{E}_0[e^{nT}\xi_T b_T] = (1-\gamma_G)^{-\gamma} \lim_{T\to\infty} e^{-(\rho-n)T} y_T^{1-\gamma} \frac{B_T}{Y_T}.$$

If the government starts with a positive level of debt, the last equation is not equal to zero because the product of the exponent, GDP per capita, which grows at rate of $g_y$, and the debt-to-GDP ratio, which grows at rate $r - g_y - n$ when debt is sufficiently high, is constant over time.

**Before the disaster.** Before the shock occurs, the interest rate on a riskless asset must reflect the fact that the discount factor jumps by $\gamma Z$ percent whenever the disaster occurs. Moreover, if the shock is large enough and the debt-to-GDP level jumps over the fiscal limit $\mathcal{B}_{FL}$, the government defaults. For simplicity, we assume a complete default, but it is straightforward to relax this assumption as in Yue (2010). The following Lemma presents the expressions for the interest rates on completely safe debt and defaultable government debt.

[19]There is a third steady state with a negative value of debt-to-GDP ratio if we do not impose that $\underline{s} = -\infty$. The presence of this steady state is, however, inconsequential for our analysis because debt-to-GDP can never become negative with only positive disasters and a positive initial level of debt.

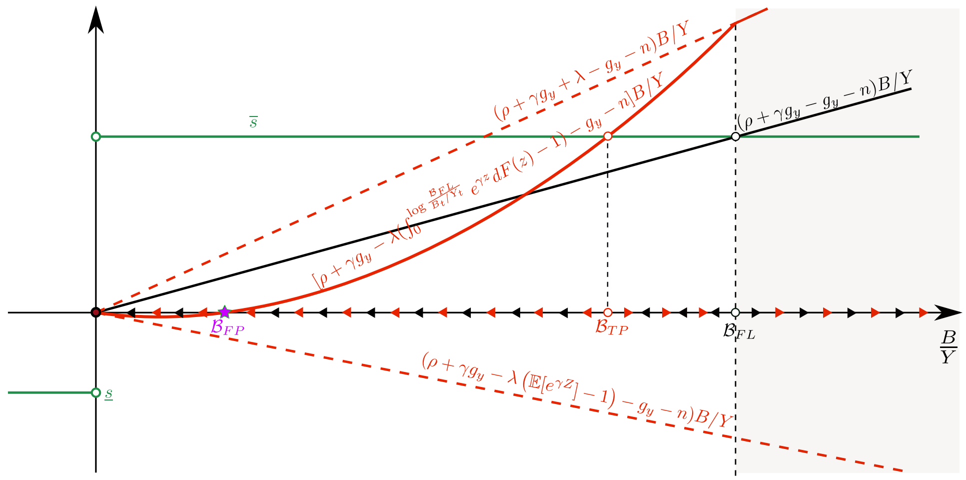


**Figure 6:** Debt-to-GDP dynamics before (red arrows) and after (black arrows) resolution of uncertainty.

**Lemma 4.** *The riskless interest and the interest rate on public debt are*

$$r_t^s = \rho + \gamma g_y - \lambda(\mathbb{E}_Z[e^{\gamma Z}] - 1), \tag{24}$$

$$r_t = \rho + \gamma g_y - \lambda \left\{ \mathbb{E}_Z \left[ e^{\gamma z} \mathbb{I} \left( \frac{B_t}{Y_t} e^Z \leq \mathcal{B}_{FL} \right) \right] - 1 \right\}. \tag{25}$$

The proof is in Appendix A.7. Equation (24) states that, in the presence of disaster shocks, households willingness to save is higher which reduces the safe interest rate. The interest rate falls when the intensity of disasters $\lambda$ is higher, risk aversion $\gamma$ is greater, or the distribution of disasters has a fatter right tail.

Equation (25) shows that government debt features an endogenous credit spread. Formally, the random variable inside the expectations operator is positive only when shocks do not trigger default, unlike the expression in equation (24). The interest rate on government debt depends on the arrival rate of disasters $\lambda$, the probability of crossing the fiscal limit $\mathcal{B}_{FL}$, and the output decline conditional on a disaster. The default premium rises as the debt-to-GDP ratio approaches the fiscal limit. Notice that a rise in the arrival rate of disasters $\lambda$ has an ambiguous effect on the interest rate on government debt. On the one hand, higher disaster risk increases the likelihood of default. On the other hand, elevated disaster risk reduces the safe interest rate $r_t^s$. However, when the debt-to-GDP ratio is infinitely close to the fiscal limit, then the interest rate $r_t$ unambiguously grows with $\lambda$ and equation (25) reduces to:

$$r_t = \rho + \lambda + \gamma g_y. \tag{26}$$

Figure 6 extends the diagram in Figure 5 by adding the dynamics of the debt-to-GDP ratio before disaster shock arrival. There are three new curves. The two red dashed lines show the debt-servicing cost, the interest rate net of GDP growth rate multiplied by the debt-to-GDP ratio, in two cases. First, a downward sloping line corresponds to riskless debt that pays interest rate $r_t^s$, where we assumed that the risk of disasters is sufficiently high to make $r - g$ negative. Second, the upward sloping line shows the case of the debt that always defaults when a disaster arrives. The actual debt-servicing cost curve, a solid red line, lies between these two extremes.

The unstable steady state of this system—a point in which the solid red debt-servicing cost curve intersects the green horizontal surplus line—is $\mathcal{B}_{TP}$, which we label the *tipping point*. The tipping point is an unstable steady state, but debt-to-GDP levels to the right of this point satisfy the government transversality condition. Therefore, an economy between the tipping point and the fiscal limit is in equilibrium, in contrast to the region above the fiscal limit.[20] Hence if the economy is pushed to the right of this tipping point, the debt-to-GDP ratio rises toward the fiscal limit and the government eventually defaults even before the disaster shock occurs.

The debt-servicing cost changes its sign at the point $\mathcal{B}_{FP}$ shown in Figure 6, which we label the *flipping point*. Although this point does not change the dynamics of the debt-to-GDP qualitatively, it has received a considerable attention by policymakers because it can potentially be an early warning signal about a looming debt crisis.

**Multiplicity of equilibria.** We close the characterization of equilibrium by noting that the sovereign debt literature (see Aguiar and Amador, 2014 for a recent review) highlights a possibility of equilibria multiplicity in models with public debt. Two prominent examples are rollover crises models that emphasize the contemporaneous interaction between the interest rate and a decision to default (e.g., Cole and Kehoe, 2000) and sovereign default models in the tradition of Calvo (1988) that focus on the link between the interest rate today and a choice to default tomorrow. The model presented in this section also features rollover crises that imply a possibility of default at any level of debt. However, we focus on a more gradual debt dynamics generated by a possibility of future default, which is also an approach taken in Ghosh, Kim, Mendoza, Ostry, and Qureshi (2013) and Lorenzoni and Werning (2019). Formally, we assume that the government does not default if it can roll over its debt at a finite interest rate. At the same time, our setup does not have a Calvo-style multiplicity because we assume that the government first chooses the amount of bonds it sells to investors and then the investors determine the interest rate on

[20]The reason why economy is still in equilibrium between the tipping point and the fiscal limit is that there is a chance that a sufficiently small disaster occurs that does not to trigger default before the public debt reaches the fiscal limit. Once uncertainty disappears after the shock, the interest rate on government debt falls relative to the high interest rate before the shock.

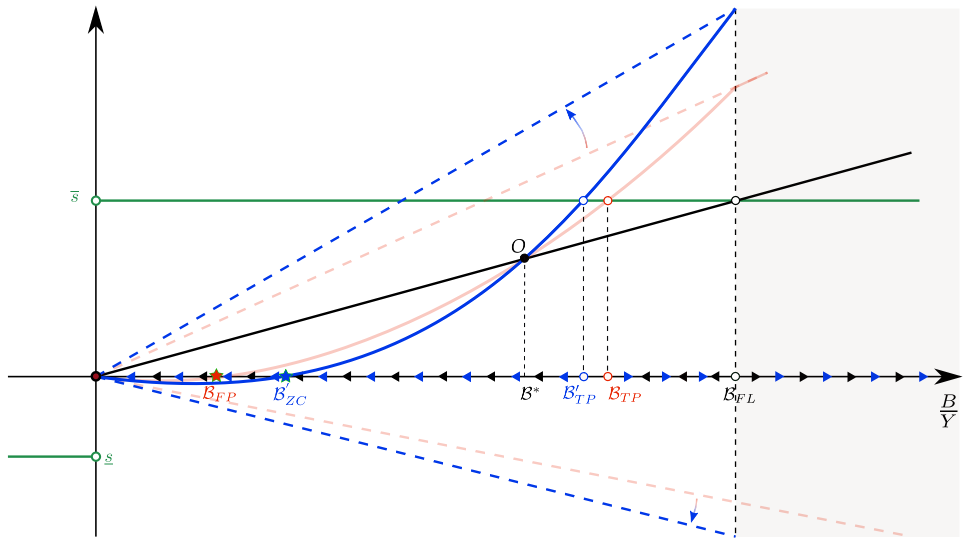


**Figure 7:** Debt-to-GDP dynamics before the resolution of uncertainty under different arrival rate of disasters $\lambda$. The light red solid and dashed lines correspond to a low value of $\lambda$, while the blue solid and dashed lines correspond to high levels of $\lambda$. To avoid cluttering the diagram, we omitted explicit labels that are similar to those in Figure 6.

these bonds (Eaton and Gersovitz, 1981).

### 4.3 Secular Stagnation Forces

**A higher arrival rate of disasters.** Figure 7 illustrates the dynamics of the debt-to-GDP ratio with a higher probability of disasters $\lambda$. The figure copies the curves presented in Figure 6 in pale red and adds the comparative statics of higher $\lambda$ in blue.

**Proposition 7** (Disasters arrival rate)**.** *A higher disaster intensity $\lambda$ does not change the fiscal limit $\mathcal{B}_{FL}$ but unambiguously moves the tipping point $\mathcal{B}_{TP}$ to the left and the flipping point $\mathcal{B}_{FP}$ to the right.*

The proof is straightforward. A higher $\lambda$ has two opposing effects on the government bond interest rate. The interest rate is a weighted average of the riskless rate and the interest rate on debt that always defaults conditional on a disaster. With higher $\lambda$, the riskless interest rate declines since households face a riskier world. The corresponding downward-sloping debt-servicing cost rotates clockwise with $\lambda$ as shown with dashed blue line in Figure 7. At the same time, debt that always defaults conditional on disaster is riskier with higher $\lambda$, raising interest rates. The debt-servicing cost curve for this type of debt rotates counterclockwise with $\lambda$. These two forces carry an ambiguous effect on the government bonds interest rate. It turns out that the interest rate declines for low levels of debt and increases for high levels of debt, as presented with the solid blue line in the figure. To see this, note that, there is a fixed point $O$ on the debt-servicing curve, which does not move when the disaster intensity changes. The fixed point $O$ has two important properties: (i) it lies on the black line representing the debt servicing cost absent risk; (ii) the debt-to-GDP ratio in this point is lower than the fiscal limit.[21] Point $O$ is the point where the effect of a decline in the safe interest rate exactly balances the higher probability of default due to a higher arrival rate of disasters. As a result, the blue solid debt-servicing cost is always higher to the right of point $O$ relative to its analogue under a lower $\lambda$. The last observation implies that the new tipping point $\mathcal{B}_{TP}^{'}$ is lower than the original one $\mathcal{B}_{TP}$.

At the same time, the flipping point $\mathcal{B}_{FP}$, where the debt-servicing cost changes its sign, shifts to the right because the debt servicing cost curve goes down for the levels of debt-to-GDP between zero and $\mathcal{B}^*$. This result shows how periods of elevated disaster risk can simultaneously cause $r < g$, while lowering the tipping point level of debt.

**An increase in disaster sizes.** Next we turn our attention to disaster sizes and, more specifically, to the effects of a rightward shift in their distribution in the first-order stochastic dominance sense.

This change has two opposing effects on the debt servicing cost curve. First, it reduces the safe interest rate in equation (24) via the household's stochastic discount factor.[22] Second, it increases the probability of default conditional on a disaster and, as a result, raises the sovereign default premium. Each of these two forces can dominate at any level of debt, which is in contrast to the previous example where a change in $\lambda$ had an unambiguous effects on the interest rate for the level of debt-to-GDP above or below $\mathcal{B}^*$.

To illustrate this point, consider a special case when the distribution $F$ is exponential with the mean of $\bar{z}$ that satisfies $\bar{z}\gamma < 1$.[23] Figure 8 presents three examples in which the net effect of the above two forces is either negative for all levels of debt, positive for all levels of debt, or negative for some and positive for the other levels of debt. The left panel presents the case in which the households are risk neutral, i.e., $\gamma = 0$. In this case, the effect of larger disasters on the safe interest rate is absent and, hence, a higher probability of default conditional on a disaster dominates. As can be seen from Figure 8, for all levels

[21] The fact that the point $O$ lies on the black line can be deduced by observing that the only situation in which the interest rate on government debt does no depend on the intensity $\lambda$ is when the term in brackets of equation (25) is zero, implying that the interest rate is $\rho + \gamma g_y$. The equation that defines the debt-to-GDP level $\mathcal{B}^*$ that corresponds to this fixed point is

$$\int_0^{\log \frac{\mathcal{B}_{FL}}{\mathcal{B}^*}} e^{\gamma z} dF(z) = 1.$$

Clearly $\mathcal{B}^* < \mathcal{B}_{FL}$, otherwise the left-hand side of this equation is negative.

[22] If the distribution $\widetilde{F}(z)$ first-order stochastically dominates the distribution $F(z)$, then for any increasing function, including $D(z) = e^{\gamma z}$, we have that $\mathbb{E}_F g(z) < \mathbb{E}_{\widetilde{F}} g(z)$.

[23] Without this parameters restriction, which states that either the risk aversion is small enough or the average disaster is not too large, the demand for safe assets is infinite.

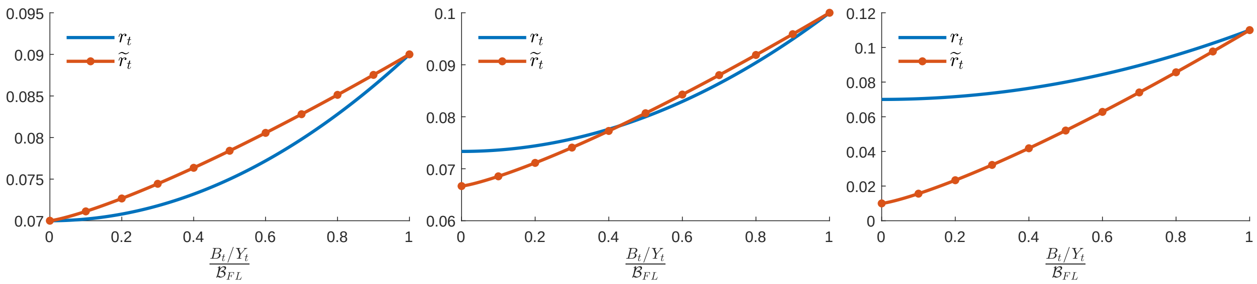


**Figure 8:** The three plots of this figure illustrate how the public bonds interest rate changes from $r_t$ to $\widetilde{r}_t$ when the distribution of disasters, represented by the exponential distribution with the density of $f(z) = \overline{z}^{-1}\exp(-z/\overline{z})$, changes its mean from $\overline{z}$ to $\widetilde{\overline{z}} > \overline{z}$. The left panel presents the case of $\gamma = 0$. The middle panel shows the case for $\gamma = 0.5$. The right panel plots the interest rate for $\gamma = 1$. All the other parameters are identical across the three panels.

of the debt-to-GDP ratio between 0 and the fiscal limit, the interest rate $\widetilde{r}_t$ under larger disasters rises relative to $r_t$ . By contrast, the right panel of Figure 8 draws the change in the interest rate in the case when the risk aversion is relatively high, such as $\gamma = 1$. In this case, a safe interest rate decline is the dominant force that pushes down the public debt interest rate for all debt-to-GDP levels considered. Next, the middle panel of Figure 8, presents an intermediate case with the coefficient of relative risk aversion $\gamma = 0.5$. In this scenario, there is a cutoff value of the debt-to-GDP ratio below which the decline in the safe interest rate dominates and above which a higher probability of disasters dominates. Finally, we note that this interest rate behavior directly translates into the behavior of the debt-servicing costs. As a result, in general, it is impossible to determine the direction of a change in the tipping and flipping points.

**A productivity growth decline.** A decline in productivity growth $g_y$ has unambiguous effects on the debt-to-GDP thresholds. To understand the intuition, note first that after uncertainty has resolved, the lower growth rate lowers the interest rate on bonds. As a result, the debt-servicing cost after uncertainty resolution, represented by the gray line Figure 9, turns clockwise to a new position indicated by the black line (under the assumption of a low IES, i.e., $\gamma^{-1} < 1$). As a result, the fiscal limit increases to $\mathcal{B}'_{FL}$.

Before disaster arrival, the safe interest rate and the interest rate on debt that always defaults also decline. Moreover, higher debt limit reduces probability of default. The last two observations imply that the debt servicing cost, the solid blue curve, is below the light red curve. Hence the tipping and flipping points move to the right.

**A population growth decline.** In our environment, a decline in the population growth rate is a mirror image of a decline in the growth rate of productivity that we presented in Figure 9. Lower population growth directly reduces GDP growth but leaves the riskless

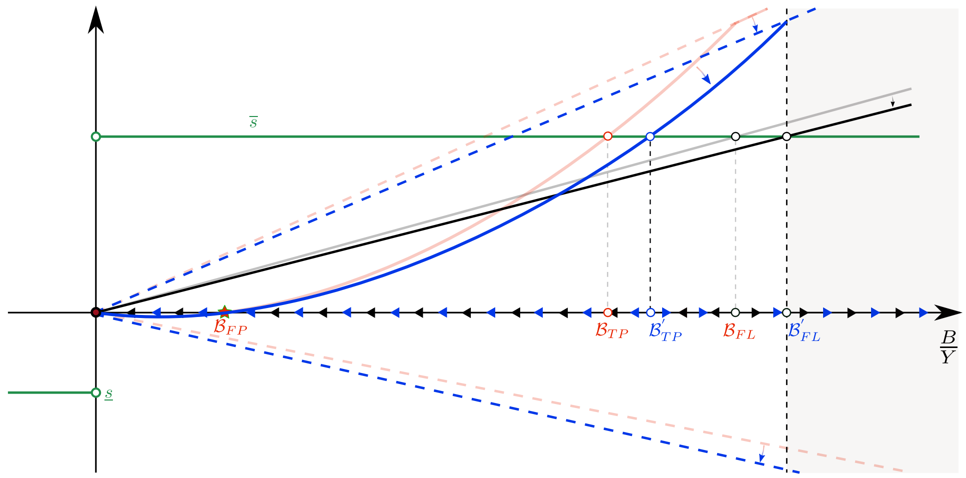


**Figure 9:** Debt-to-GDP dynamics before and after the resolution of uncertainty under different productivity growth rates $g_y$. The light red solid and dashed lines and the gray line correspond to a high value of $g_y$, while the blue solid and dashed lines as well as the black line correspond to low levels of $g_y$.

interest rates unchanged. Consequently, the fiscal limit, the tipping and flipping points move to the left.

## 4.4 Calibration

We now provide quantitative magnitudes for the effects that we have discussed in this section. To do this, we keep the IES parameter $1/\theta$ and the average size of disasters $\bar{z}$ the values used in calibration 3 presented in Section 3.6. We choose the remaining parameters to reflect the recent state of the US economy in 2019, right before the COVID-19 pandemic. The growth rate of real GDP per capita $g_y$ is set to 0.017, population growth $n$ is 0.005. We use a higher value for the arrival rate of disasters $\lambda$ of 0.065 based on the structural estimate in Farhi and Gourio (2018).[24] The subjective discount factor $\rho$ is set to match the average yield on 10-year inflation-protected Treasury bonds of 0.004. When computing $\rho$, we take into account that government bonds are defaultable in this version of the model. Specifically, we match the interest rate at a debt-to-GDP ratio of 80 percent. As a result, the parameter $\rho$ varies with the maximum fiscal surplus in the calibration. We set the coefficient of relative risk aversion $\gamma$ to 2.74 to match the equity risk premium of 5.5 percent. Faced with some uncertainty about the maximum fiscal surplus, we experiment with three possible values: 2 percent (a low value), 5 percent (an intermediate value), and 10 percent (a high value).

Table 6 reports values of the three debt-to-GDP thresholds that we discussed in this

[24] It is difficult to have a precise estimate of a time-varying probability of disasters. However, given that the first two decades of the twenty first century have already brought us two large disasters, the arrival rate might have become higher.

| Variable | **Baseline** | **Ex. 1**: $n \uparrow$ | **Ex. 2**: $g \uparrow$ | **Ex. 3**: $\mathbb{E}[\mu_t] - r^s \downarrow$ |
|---|---|---|---|---|
| **Panel A:** $\bar{s} = 0.02$ | | | | |
| $\mathcal{B}_{FP}$ | 0.99 | 2.63 | 0.91 | 0.99 |
| $\mathcal{B}_{TP}$ | 1.15 | 2.79 | 1.07 | 1.15 |
| $\mathcal{B}_{FL}$ | 1.84 | 4.61 | 1.71 | 1.84 |
| **Panel B:** $\bar{s} = 0.05$ | | | | |
| $\mathcal{B}_{FP}$ | 1.06 | 1.59 | 1.01 | 1.04 |
| $\mathcal{B}_{TP}$ | 1.44 | 1.98 | 1.39 | 1.45 |
| $\mathcal{B}_{FL}$ | 2.22 | 3.12 | 2.14 | 2.22 |
| **Panel C:** $\bar{s} = 0.1$ | | | | |
| $\mathcal{B}_{FP}$ | 1.18 | 1.59 | 1.13 | 1.14 |
| $\mathcal{B}_{TP}$ | 1.95 | 2.36 | 1.91 | 1.96 |
| $\mathcal{B}_{FL}$ | 2.90 | 3.58 | 2.83 | 2.90 |

**Table 6: Thresholds**. Column "Baseline" shows the cutoffs under the baseline version of a calibration, while columns "Ex. 1," "Ex. 2," and "Ex. 3" show the cutoffs for the three experiments that correspond to an increase in the population growth rate (Ex. 1), an increase in the productivity growth rate (Ex. 2), and a decline in equity premium through lower probability of disasters $\lambda$ (Ex. 3). The values of subjective discount factor $\rho$ are 0.010 in Panel A, 0.022 in Panel B, and 0.034 in Panel C.

section under the baseline calibration and the three comparative statics experiments. These experiments are the opposite of those in Section 3.6. Specifically, we investigate how the fiscal thresholds change if one parameter reverts from its baseline (2019) value to its post-WWII average value.

Consider first the middle Panel B where the maximum fiscal surplus to GDP ratio is 5 percent. In the baseline calibration, the fiscal limit is 222 percent of GDP, a number nearly three times the current value of US public debt. The tipping point is at 144 percent, above the 90 percent level considered unsustainable in Reinhart and Rogoff (2009). The flipping point at which $r > g$ is at 106 percent of GDP. An increase in the population growth (column Ex. 1) leads to a substantial increase in all of the three cutoffs, while an increase in the growth rate of productivity, column Ex. 2, somewhat reduces these thresholds slightly. A decline in the arrival rate of disasters, column Ex. 3, does not change the fiscal limit, however, it slightly lowers the flipping point and somewhat increases the tipping point. The small changes in Experiment 3 reflect the fact that that equity premium only falls from 5.5 percent to 5.2 percent.

Panels A and C demonstrate a considerable sensitivity of the thresholds to the maximum fiscal surplus. For example, if the government can set a surplus-to-GDP ratio of 10 percent, it can sustain the level of debt of nearly 300 percent. Moreover, even with a conservative maximum surplus of 2 percent of GDP, the fiscal limit is more than twice the current level of the US debt to GDP ratio, suggesting appreciable remaining fiscal space.

# 5 Conclusion

Among advanced economies, interest rates on government debt frequently fall below GDP growth and exhibit substantial variability. We use a continuous time asset-pricing framework that captures the fact that on average $r < g$, and we study how debt dynamics respond to drivers of low interest rates.

In our framework, lower trend growth lowers the drift of the debt-to-GDP ratio when the IES is above one. Higher risk premium has theoretically ambiguous effects on the drift of the debt-to-GDP ratio but in a version of the model calibrated to the US data, it reduces the drift of the debt-to-GDP ratio. Moreover, we show that negative servicing costs are neither necessary nor sufficient to ensure debt sustainability. In particular, when debt servicing costs are negative, the debt-to-GDP ratio may be non-explosive in the sense of having a well-defined stationary distribution, but that may not satisfy the sustainability criteria. Conversely, debt sustainability criteria may be satisfied even though the debt-to-GDP ratio does not converge to a well-defined stationary distribution.

Finally, we consider the possibility of government default if the primary surplus is bounded above. In this environment, an endogenous debt limit emerges, and the interest rate on government debt rises as debt approaches the limit. Lower trend growth raises the debt limit if the IES is greater than one and lowers interest rate at any debt-to-GDP ratios. However, higher disaster risk has competing effects by lowering the debt limit (and potentially raising default premia) while making government debt more attractive as the risk free asset. Our extension with default provides a well-defined notion of a tipping point level of debt beyond which default does not immediately occur but happens in finite time.

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# Appendix

# A Proofs

## A.1 Proof of Lemma 1

First, take the household flow budget constraint

$$dw_t = (r_t^s s_t + r_t b_t - c_t - \tau_t - n w_t)dt + w_t x_t dr_t^x,$$

together with $s_t + b_t + x_t w_t = w_t$.

Second, consider the process $\xi_t$ such that

$$\frac{d\xi_t}{\xi_t} = -r_t^s dt - \frac{\mu_t - r_t^s}{\sigma_t} dZ_t^y.$$

Third, compute $d(\xi_t e^{nt} w_t)$ to get

$$d(\xi_t e^{nt} w_t) = e^{nt}\xi_t \left[ \left( r_t - r_t^s \right) b_t - c_t - \tau_t \right] dt + \left( x_t - \frac{\mu_t - r_t^s}{\sigma_t^2} \right) e^{nt} \sigma_t \xi_t w_t dZ_t^y.$$

Fourth, integrate the above process forward and take expectations

$$\xi_T e^{nT} w_T - \xi_0 w_0 = \int_0^T e^{nt}\xi_t \left[ \left( r_t - r_t^s \right) b_t - c_t - \tau_t \right] dt + \int_0^T \left( x_t - \frac{\mu_t - r_t^s}{\sigma_t^2} \right) e^{nt} \sigma_t \xi_t w_t dZ_t^y,$$

take expectations

$$\mathbb{E}_0 \xi_T e^{nT} w_T - \xi_0 w_0 = \mathbb{E}_0 \int_0^T e^{nt}\xi_t \left[ \left( r_t - r_t^s \right) b_t - c_t - \tau_t \right] dt,$$

and, finally, take the limit

$$\xi_0 w_0 = \lim_{T \to \infty} \mathbb{E}_0 \xi_T e^{nT} w_T + \mathbb{E}_0 \int_0^\infty e^{nt}\xi_t \left[ \left( r_t^s - r_t \right) b_t + c_t + \tau_t \right] dt,$$

where in the last equation we assumed that limit and expectations are interchangeable.

Assuming that $\xi_T$ is the stochastic discount factor, the no-Ponzi game condition is $\lim_{T\to\infty} \mathbb{E}_0 \xi_T e^{nT} w_T \geq 0$, which results in

$$\xi_0 w_0 \geq \mathbb{E}_0 \int_0^\infty e^{nt}\xi_t \left[ \left( r_t^s - r_t \right) b_t + c_t + \tau_t \right] dt.$$

## A.2 Proof of Lemma 2

Log GDP is

$$
\begin{aligned}
d\log y_t &= \frac{dy_t}{y_t} - \frac{1}{2}\left(\frac{dy_t}{y_t}\right)^2 \\
&= g_y dt + \sigma_y dZ_t^y - \frac{1}{2}\left(g_y dt + \sigma_y dZ_t^y\right)^2 \\
&= \left(g_y - \frac{\sigma_y^2}{2}\right) dt + \sigma_y dZ_t^y.
\end{aligned}
$$

Log debt is

$$
\begin{aligned}
d\log B_t &= \frac{(r_t B_t + G_t - T_t)\, dt + B_t \sigma_B dZ_t^B}{B_t} - \frac{1}{2}\left[\frac{(r_t B_t + G_t - T_t)\, dt + B_t \sigma_B dZ_t^B}{B_t}\right]^2 \\
&= \left(r_t + \frac{G_t - T_t}{B_t}\right) dt + \sigma_B dZ_t^B - \frac{1}{2}\left[\left(r_t + \frac{G_t - T_t}{B_t}\right) dt + \sigma_B dZ_t^B\right]^2 \\
&= \left(r_t + \frac{G_t - T_t}{B_t} - \frac{1}{2}\sigma_B^2\right) dt + \sigma_B dZ_t^B.
\end{aligned}
$$

The low of motion of $\widehat{B}_t \equiv \log\left[B_t/(N_t y_t)\right]$ is

$$
d\widehat{B}_t = \left(r_t - g_y - n + \alpha_D + \frac{\sigma_y^2 - \sigma_B^2}{2} - \beta_D \widehat{B}_t\right) dt + \sigma_B dZ_t^B - \sigma_y dZ_t^y.
$$

## A.3 Proof of Lemma 3

**Step 0: preliminaries.** We evaluate a part of the value function that depends on the stream of consumption

$$
V_t = \mathbb{E}_t \int_t^\infty e^{-(\rho-n)(u-t)} \frac{c_u^{1-\gamma}}{1-\gamma} du.
$$

Note that

$$
\frac{dc_t}{c_t} = g_y dt + \sigma_y dZ_t^y
$$

implies

$$
d\log c_t^{1-\gamma} = (1-\gamma)\left(g_y - \frac{\sigma_y^2}{2}\right) dt + (1-\gamma)\sigma_y dZ_t^y.
$$

Hence

$$
c_u^{1-\gamma} = c_t^{1-\gamma} \exp\left\{(1-\gamma)\left(g_y - \frac{\sigma_y^2}{2}\right)(u-t) + (1-\gamma)\sigma_y Z_{u-t}^y\right\}
$$

and

$$
\mathbb{E}_t \int_t^T e^{-(\rho-n)(u-t)} c_u^{1-\gamma} du = \frac{1 - e^{\left[-\rho+n+(1-\gamma)\left(g_y - \frac{\gamma\sigma_y^2}{2}\right)\right](T-t)}}{\rho - n - (1-\gamma)\left(g_y - \frac{\gamma\sigma_y^2}{2}\right)} c_t^{1-\gamma}.
$$

As long as $\rho - n - (1-\gamma)(g_y - \gamma\sigma_y^2/2) > 0$, we have

$$V_t = \frac{c_t^{1-\gamma}}{\rho - n - (1-\gamma)\left(g_y - \frac{\gamma\sigma_y^2}{2}\right)}.$$

Note that the condition $\rho - n - (1-\gamma)(g_y - \gamma\sigma_y^2/2) > 0$ is necessary for the household problem to have a solution. We will later show how it relates to the standard transversality conditions.

**Step 1: household problem.** The household problem when the budget constraint takes the intertemporal form is

$$\begin{aligned}
&\max_{c_t, b_t} \mathbb{E}_0 \int_0^\infty e^{-(\rho-n)t} \left[\frac{c_t^{1-\gamma}-1}{1-\gamma} + y_t^{1-\gamma} u\left(\frac{b_t}{y_t}\right)\right] dt \\
&s.t.: w_0 \geq \mathbb{E}_0 \int_0^\infty \left[c_t + \tau_t + (r_t^s - r_t) b_t\right] \frac{e^{nt}\xi_t}{\xi_0} dt,
\end{aligned}$$

Following the arguments in **?**, one can show that the solution to this problem is equivalent to the solution of the original household problem.

The Lagrangian of this problem is

$$\mathcal{L}_0 = \mathbb{E}_0 \int_0^\infty e^{-(\rho-n)t} \left[\frac{c_t^{1-\gamma}-1}{1-\gamma} + y_t^{1-\gamma} u\left(\frac{b_t}{y_t}\right)\right] - \kappa \left[\mathbb{E}_0 \int_0^\infty \left[c_t + \tau_t + (r_t^s - r_t) b_t\right] \frac{e^{nt}\xi_t}{\xi_0} dt - w_0\right].$$

Note that $\mathcal{L}_0$ is a functional such that $\mathcal{L}_0 : \mathbb{L} \times \mathbb{L} \to \mathbb{R}$, where $\mathbb{L}$ is a space of square integrable progressively measurable processes with values in $\mathbb{R}$.

**Step 2: optimal choices.** The first order conditions for this optimization take the following form

$$\begin{aligned}
\partial c_t &: e^{-\rho t} c_t^{-\gamma} = \kappa \xi_t, \\
\partial b_t &: e^{-\rho t} y_t^{-\gamma} u'\left(\frac{b_t}{y_t}\right) = \kappa \xi_t \left(r_t^s - r_t\right),
\end{aligned}$$

together with the transversality condition

$$\lim_{T\to\infty} \mathbb{E}_0 e^{nT} \xi_T w_T \leq 0.$$

Together the transversality condition and no-Ponzi game condition imply that

$$\lim_{T\to\infty} \mathbb{E}_0 e^{nT} \xi_T w_T = 0.$$

Individual optimality condition with respect to consumption can be solved to get consumption

$$c_t = \left(e^{\rho t} \xi_t \kappa\right)^{-1/\gamma},$$

and with respect to liquid bonds for bonds

$$\frac{b_t}{y_t} = (u')^{-1}\left[y_t^{\gamma} c_t^{-\gamma}(r_t^s - r_t)\right] = (u')^{-1}\left[y_t^{\gamma} e^{\rho t}\kappa\xi_t(r_t^s - r_t)\right].$$

Substituting out $\kappa\xi_t$ from the first order conditions, we obtain

$$y_t^{-\gamma} u'\left(\frac{b_t}{y_t}\right) = c_t^{-\gamma}\left(r_t^s - r_t\right).$$

It can be rewritten as

$$r_t^s - r_t = y_t^{-\gamma} c_t^{\gamma} u'\left(\frac{b_t}{y_t}\right).$$

In equilibrium, we will have $r_t^s - r_t = (1-\gamma_G)^{\gamma} u'(b_t/y_t)$.

Note that the Lagrange multiplier $\kappa$ solves the intertemporal budget constraint after substituting out for optimal consumption and bond holdings:

$$w_0 = \mathbb{E}_0 \int_0^{\infty} \left\{ (e^{\rho t}\kappa\xi_t)^{-\frac{1}{\gamma}} + \tau_t + (r_t^s - r_t) y_t \left(u'\right)^{-1}\left[y_t^{\gamma} e^{\rho t}\kappa\xi_t(r_t^s - r_t)\right] \right\} e^{nt}\frac{\xi_t}{\xi_0} dt.$$

**Step 3: SDF.** First, we apply Ito's lemma to the FOC wrt $c_t$ to get the law of motion of $\xi_t$:

$$\begin{aligned}
\kappa d\xi_t &= d\left(e^{-\rho t}\right) c_t^{-\gamma} + e^{-\rho t} d\left(c_t^{-\gamma}\right) + d\left(e^{-\rho t}\right) d\left(c_t^{-\gamma}\right) \\
&= -\rho e^{-\rho t} c_t^{-\gamma} dt - \gamma e^{-\rho t} c_t^{-\gamma-1} dc_t + \frac{\gamma(\gamma+1)}{2} e^{-\rho t} c_t^{-\gamma-2} dc_t^2.
\end{aligned}$$

Divide both sides by $\kappa\xi_t$

$$\frac{d\xi_t}{\xi_t} = -\rho dt - \gamma\frac{dc_t}{c_t} + \frac{\gamma(\gamma+1)}{2}\left(\frac{dc_t}{c_t}\right)^2.$$

Next, we use the goods market clearing condition $y_t = c_t + G_t$ to note that $(1-\gamma_G) y_t = c_t$ and $dy_t/y_t = dc_t/c_t$. As a result,

$$\begin{aligned}
\frac{d\xi_t}{\xi_t} &= -\rho dt - \gamma\frac{dy_t}{y_t} + \frac{\gamma(\gamma+1)}{2}\left(\frac{dy_t}{y_t}\right)^2 \\
&= -\rho dt - \gamma(g_y dt + \sigma_y dZ_t^y) + \frac{\gamma(\gamma+1)}{2}\sigma_y^2 dt \\
&= -\left[\rho + \gamma g_y - \frac{\gamma(\gamma+1)}{2}\sigma_y^2\right] dt - \gamma\sigma_y dZ_t^y.
\end{aligned} \tag{A.1}$$

**Step 4: safe rate.** No arbitrage implies that the price $p_t$ of any security that pays dividends $d_s$ to its holder equals

$$p_t = \frac{1}{\xi_t}\mathbb{E}_t \int_t^{\infty} \xi_s d_s ds. \tag{A.2}$$

The differential version of this equation is

$$0 = \xi_t d_t dt + \mathbb{E}_t\left[d(\xi_t p_t)\right]. \tag{A.3}$$

The safe bond is a security with the price of 1 and the dividend $r_t^s$. As a result,

$$
\begin{aligned}
0 &= \xi_t r_t^s dt + \mathbb{E}_t d\xi_t, \\
r_t^s &= -\frac{1}{dt}\mathbb{E}_t\left(\frac{d\xi_t}{\xi_t}\right) = \rho + \gamma g_y - \frac{\gamma(\gamma+1)}{2}\sigma_y^2.
\end{aligned}
$$

This confirms the guess of the drift in equation (7).

**Step 5: equity price.** The value of Lucas trees $q_t$ to the household that growth at rate $n$ and collects $y_t e^{nt}$ in dividends is

$$
q_t = \frac{1}{\xi_t}\mathbb{E}_t \int_t^\infty \xi_s e^{nt} y_s ds.
$$

We can compute the last integral explicitly. First note that $f_t \equiv \xi_t e^{nt} y_t = e^{-(\rho-n)t} y_t^{1-\gamma}$ follows the geometric Brownian motion

$$
\frac{df_t}{f_t} = -\left[\rho - n + (\gamma-1)g_y - \frac{(\gamma-1)\gamma}{2}\sigma_y^2\right] dt - (\gamma-1)\sigma_y dZ_t^y,
$$

which has the following solution

$$
f_s = f_t \exp\left\{-\left[\rho - n + (\gamma-1)g_y - \frac{(\gamma-1)\sigma_y^2}{2}\right](s-t) - (\gamma-1)\sigma_y(Z_s^y - Z_t^y)\right\}.
$$

Thus, we obtain

$$
\begin{aligned}
q_t &= \frac{1}{\xi_t}\mathbb{E}_t \int_t^\infty f_s ds \\
&= \frac{y_t}{\rho - n + (\gamma-1)g_y - \frac{\gamma(\gamma-1)\sigma_y^2}{2}} \\
&= \frac{y_t}{r^s - g_y - n + \gamma\sigma_y^2}.
\end{aligned}
$$

The last formula is a version of Gordon's growth formula with risk. It implies that $dq_t/q_t = dy_t/y_t$, which, in turn, yields

$$
\begin{aligned}
\sigma_t &= \sigma_y, \\
\mu_t &= g_y + \frac{y_t}{q_t} = r^s + \gamma\sigma_y^2.
\end{aligned}
$$

The last finding allows us to verify the diffusion part of the guess in equation 7. Specifically,

$$
\frac{\mu_t - r_t^s}{\sigma_t} = \gamma\sigma_y,
$$

which is the same turn as we obtained in equation (A.1).

## A.4 Proof of Proposition 4

In this proof, we first derive the conditions for which the transversality condition hold. Then, we show that under this condition, the intertemporal budget constraint of the government is automatically satisfied.

**TVC.** First, note that

$$\begin{aligned}\mathbb{E}_t e^{n(s-t)} \frac{\xi_s}{\xi_t} b_s &= \mathbb{E}_t e^{n(s-t)} e^{-\rho(s-t)} \left(\frac{y_s}{y_t}\right)^{-\gamma} b_s \\ &= e^{-(1-\gamma)\log y_t + (\rho-n)t} \mathbb{E}_t e^{-(\rho-n)s + (1-\gamma)\log y_s + \widehat{B}_s}\end{aligned}$$

The law of motion of the term in the exponent of the last equation is

$$d\left[-(\rho-n)s + (1-\gamma)\log y_s + \widehat{B}_s\right] = \left[\alpha - \rho + n + (1-\gamma)\left(g_y - \frac{\sigma_y^2}{2}\right)\right] ds + \sigma_B dZ_s^B - \gamma\sigma_y dZ_s^y.$$

As a result $-(\rho-n)(s-t) + (1-\gamma)(\log y_s - \log y_t) + \widehat{B}_s - \widehat{B}_t$ is distributed normally with mean of $[\alpha - \rho + n + (1-\gamma)(g - \sigma_y^2/2)](s-t)$ and the variance of $(\sigma_B^2 + \gamma^2\sigma_y^2)(s-t)$. Hence, we obtain

$$\begin{aligned}\mathbb{E}_t e^{n(s-t)} \frac{\xi_s}{\xi_t} b_s &= e^{\widehat{B}_t} \mathbb{E}_t e^{-(\rho-n)(s-t) + (1-\gamma)(\log y_s - \log y_t) + \widehat{B}_s - \widehat{B}_t} \\ &= e^{\widehat{B}_t} e^{\left[\alpha - \rho + n + (1-\gamma)\left(g_y - \frac{\sigma_y^2}{2}\right) + \frac{1}{2}\left(\sigma_B^2 + \gamma^2\sigma_y^2\right)\right](s-t)}\end{aligned}$$

The TVC holds when

$$\alpha - \rho + n + (1-\gamma)\left(g_y - \frac{\sigma_y^2}{2}\right) + \frac{1}{2}\left(\sigma_B^2 + \gamma^2\sigma_y^2\right) < 0,$$

which, after substituting out $\alpha$, reduces to

$$\alpha_D - \alpha_u < 0.$$

**The intertemporal budget constraint of the government.** The iBC of the government is

$$\begin{aligned}b_0 &= \mathbb{E}_0 \int_0^\infty [\tau_t - g_t + (r_t^s - r_t) b_t] e^{nt} \frac{\xi_t}{\xi_0} dt \\ &= \mathbb{E}_0 \int_0^\infty [\alpha_u - \alpha_D + \beta \log(b_t/y_t)] b_t e^{nt} \frac{\xi_t}{\xi_0} dt.\end{aligned}$$

After imposing $\beta = 0$, the budget constraint becomes

$$\begin{aligned}b_0 &= (\alpha_u - \alpha_D)\mathbb{E}_0 \int_0^\infty b_t e^{nt} \frac{\xi_t}{\xi_0} dt \\ &= b_0(\alpha_u - \alpha_D)\mathbb{E}_0 \int_0^\infty e^{\widehat{B}_t - \widehat{B}_0 + nt - \rho t + (1-\gamma)(\log y_t - \log y_0)} dt.\end{aligned}$$

Note that

$$\widehat{B}_t - \widehat{B}_0 = \alpha t + \sigma_{\widehat{B}} \left( Z_t^{\widehat{B}} - Z_0^{\widehat{B}} \right),$$

$$\log y_t - \log y_0 = \left( g_y - \frac{\sigma_y^2}{2} \right) t + \sigma_y \left( Z_t^y - Z_0^y \right).$$

or

$$\begin{aligned} &\widehat{B}_t - \widehat{B}_0 - (\rho - n) t + (1-\gamma) (\log y_t - \log y_0) \\ &= \left[ \alpha - \rho + n + (1-\gamma) \left( g_y - \frac{\sigma_y^2}{2} \right) \right] t + \sigma_B \left( Z_t^B - Z_0^B \right) - \gamma \sigma_y \left( Z_t^y - Z_0^y \right). \end{aligned}$$

As a result,

$$\begin{aligned} \mathbb{E}_0 \int_0^\infty e^{\widehat{B}_t + nt - \rho t - \gamma(\log c_t - \log c_0)} dt &= e^{\widehat{B}_0} \mathbb{E}_0 \int_0^\infty e^{\left[ \alpha - \rho + n + (1-\gamma) \left( g_y - \frac{\sigma_y^2}{2} \right) \right] t + \sigma_B \left( Z_t^B - Z_0^B \right) - \gamma \sigma_y \left( Z_t^y - Z_0^y \right)} dt \\ &= e^{\widehat{B}_0} \int_0^\infty e^{\left[ \alpha - \rho + n + (1-\gamma) \left( g_y - \frac{\sigma_y^2}{2} \right) + \frac{\sigma_B^2}{2} + \frac{\gamma^2 \sigma_y^2}{2} \right] t} dt \\ &= e^{\widehat{B}_0} \frac{\lim_{t \to \infty} e^{\left[ \alpha - \rho + n + (1-\gamma) \left( g_y - \frac{\sigma_y^2}{2} \right) + \frac{\sigma_B^2}{2} + \frac{\gamma^2 \sigma_y^2}{2} \right] t} - 1}{\alpha - \rho + n + (1-\gamma) \left( g_y - \frac{\sigma_y^2}{2} \right) + \frac{\sigma_B^2}{2} + \frac{\gamma^2 \sigma_y^2}{2}} \end{aligned}$$

Replace $\alpha$ in the following expression

$$\alpha - \rho + n + (1-\gamma) \left( g_y - \frac{\sigma_y^2}{2} \right) + \frac{\sigma_B^2}{2} + \frac{\gamma^2 \sigma_y^2}{2} = \alpha_D - \alpha_u.$$

As a result, we get

$$\mathbb{E}_0 \int_0^\infty e^{\widehat{B}_t + nt - \rho t - \gamma(\log c_t - \log c_0)} dt = e^{\widehat{B}_0} \frac{1}{\alpha_u - \alpha_D}.$$

Plug this in the original equation

$$b_0 = (\alpha_u - \alpha_D) b_0 \frac{1}{\alpha_u - \alpha_D}.$$

The last equation holds for any initial value $b_0$ and for any $\alpha_u - \alpha_D > 0$.

## A.5 Proof of Proposition 5

In this proof, we only derive the conditions for which the TVC is satisfied. Specifically, we will show that a necessary and sufficient condition for $\lim_{T \to \infty} \mathbb{E}_t e^{nT} \xi_T b_T = 0$ is $\alpha_u - \alpha_D > 0$. The challenge in proving this result stems from the fact that the log of debt-to-GDP ratio is not just Brownian motion with drift, but a reflected Brownian motion with drift.

To slightly simplify the notation (but without any loss of generality), we assume that $\widehat{B}_0 = 0$, $\widehat{B}_{min} = 0$ and $\gamma = 1$. In the end of the proof, we comment on the consequences that dropping these assumption.

First, note that

$$\mathbb{E}_t e^{nT}\xi_T b_T = \mathbb{E}_t e^{nT} e^{-\rho T}\frac{b_T}{(1-\gamma_G)y_T} = \frac{1}{1-\gamma_G}\mathbb{E}_t e^{-(\rho-n)T} e^{\widehat{B}_T}.$$

Second, we use the observation, which is straightforward to prove (see, for example, Harrison, 1985) using the so-called reflection principle from probability theory, that the cdf of the reflected Brownian motion with negative drift and a single (lower) reflecting barrier is

$$\mathbb{P}\left(\widehat{B}_t \le x|\widehat{B}_0 = 0\right) = \Phi\left(\frac{x-\alpha t}{\sigma_{\widehat{B}}\sqrt{t}}\right) - e^{\frac{2\alpha x}{\sigma^2_{\widehat{B}}}}\Phi\left(\frac{-x-\alpha t}{\sigma_{\widehat{B}}\sqrt{t}}\right) \equiv F(x), \tag{A.4}$$

where $\Phi(\cdot)$ is the cdf of the standard normal distribution. Equation (A.4) implies that the PDF of the $\widehat{B}_t$ is

$$f(x) = \frac{1}{\sigma_{\widehat{B}}\sqrt{t}}\phi\left(\frac{x-\alpha t}{\sigma_{\widehat{B}}\sqrt{t}}\right) + \frac{1}{\sigma\sqrt{t}}e^{\frac{2\alpha x}{\sigma^2_{\widehat{B}}}}\phi\left(\frac{-x-\alpha t}{\sigma_{\widehat{B}}\sqrt{t}}\right) - \frac{2\alpha}{\sigma^2_{\widehat{B}}}e^{\frac{2\alpha x}{\sigma^2_{\widehat{B}}}}\Phi\left(\frac{-x-\alpha t}{\sigma_{\widehat{B}}\sqrt{t}}\right). \tag{A.5}$$

When $2\alpha/\sigma^2_{\widehat{B}} + 1 \neq 0$ (we will consider the special case when $2\alpha/\sigma^2_{\widehat{B}} + 1 = 0$ separately), we have

$$\begin{aligned}
\mathbb{E}_0 e^{\widehat{B}_t} &= \int_0^\infty e^x\left[\frac{1}{\sigma_{\widehat{B}}\sqrt{t}}\phi\left(\frac{x-\alpha t}{\sigma_{\widehat{B}}\sqrt{t}}\right) + \frac{1}{\sigma_{\widehat{B}}\sqrt{t}}e^{\frac{2\alpha x}{\sigma^2_{\widehat{B}}}}\phi\left(\frac{-x-\alpha t}{\sigma_{\widehat{B}}\sqrt{t}}\right) - \frac{2\alpha}{\sigma^2_{\widehat{B}}}e^{\frac{2\alpha x}{\sigma^2_{\widehat{B}}}}\Phi\left(\frac{-x-\alpha t}{\sigma_{\widehat{B}}\sqrt{t}}\right)\right]dx \\
&= \underbrace{\int_0^\infty e^x \frac{1}{\sigma_{\widehat{B}}\sqrt{t}}\phi\left(\frac{x-\alpha t}{\sigma_{\widehat{B}}\sqrt{t}}\right)dx}_{A_1} + \underbrace{\int_0^\infty e^{\left(\frac{2\alpha}{\sigma^2_{\widehat{B}}}+1\right)x}\frac{1}{\sigma_{\widehat{B}}\sqrt{t}}\phi\left(\frac{-x-\alpha t}{\sigma_{\widehat{B}}\sqrt{t}}\right)dx}_{A_2} \\
&\quad \underbrace{+\frac{\frac{-2\alpha}{\sigma^2_{\widehat{B}}}}{\frac{2\alpha}{\sigma^2_{\widehat{B}}}+1}\left[-\Phi\left(\frac{-\alpha t}{\sigma_{\widehat{B}}\sqrt{t}}\right) + \frac{1}{\sigma_{\widehat{B}}\sqrt{t}}\int_0^\infty e^{\left(\frac{2\alpha}{\sigma^2_{\widehat{B}}}+1\right)x}\phi\left(\frac{-x-\alpha t}{\sigma_{\widehat{B}}\sqrt{t}}\right)dx\right]}_{A_3}.
\end{aligned} \tag{A.6}$$

To compute $A_1$, $A_2$ and $A_3$, note that

$$e^{ax}\phi(bx+c) = e^{-\frac{c^2-\left(c-\frac{a}{b}\right)^2}{2}}\phi\left(bx+c-\frac{a}{b}\right).$$

As a result,

$$A_1 = e^{\frac{\sigma^2_{\widehat{B}}\left(\frac{2\alpha}{\sigma^2_{\widehat{B}}}+1\right)}{2}t}\Phi\left(\frac{\sigma^2_{\widehat{B}}+\alpha}{\sigma_{\widehat{B}}}\sqrt{t}\right), \tag{A.7}$$

$$A_2 = e^{\frac{\sigma^2_{\widehat{B}}\left(\frac{2\alpha}{\sigma^2_{\widehat{B}}}+1\right)}{2}t}\Phi\left(\frac{\sigma^2_{\widehat{B}}+\alpha}{\sigma_{\widehat{B}}}\sqrt{t}\right) = A_1, \tag{A.8}$$

$$A_3 = \frac{\frac{-2\alpha}{\sigma^2_{\widehat{B}}}}{\frac{2\alpha}{\sigma^2_{\widehat{B}}}+1}\left\{-\Phi\left(\frac{-\alpha}{\sigma_{\widehat{B}}}\sqrt{t}\right) + e^{\frac{\sigma^2_{\widehat{B}}\left(\frac{2\alpha}{\sigma^2_{\widehat{B}}}+1\right)}{2}t}\Phi\left(\frac{\sigma^2_{\widehat{B}}+\alpha}{\sigma_{\widehat{B}}}\sqrt{t}\right)\right\}. \tag{A.9}$$

Plugging equations (A.7)-(A.9) into (A.6) and rearranging, we get

$$\begin{aligned}
\mathbb{E}_0 e^{\widehat{B}_t} =& e^{\frac{\sigma_{\widehat{B}}^2\left(\frac{2\alpha}{\sigma_{\widehat{B}}^2}+1\right)}{2}t}\Phi\left(\frac{\sigma_{\widehat{B}}^2+\alpha}{\sigma_{\widehat{B}}}\sqrt{t}\right)+e^{\frac{\sigma_{\widehat{B}}^2\left(\frac{2\alpha}{\sigma_{\widehat{B}}^2}+1\right)}{2}t}\Phi\left(\frac{\sigma_{\widehat{B}}^2+\alpha}{\sigma_{\widehat{B}}}\sqrt{t}\right)\\
&+\frac{\frac{-2\alpha}{\sigma_{\widehat{B}}^2}}{\frac{2\alpha}{\sigma_{\widehat{B}}^2}+1}\left\{-\Phi\left(\frac{-\alpha}{\sigma_{\widehat{B}}}\sqrt{t}\right)+e^{\frac{\sigma_{\widehat{B}}^2\left(\frac{2\alpha}{\sigma_{\widehat{B}}^2}+1\right)}{2}t}\Phi\left(\frac{\sigma_{\widehat{B}}^2+\alpha}{\sigma_{\widehat{B}}}\sqrt{t}\right)\right\}\\
=& 2\frac{\frac{\alpha}{\sigma_{\widehat{B}}^2}+1}{\frac{2\alpha}{\sigma_{\widehat{B}}^2}+1}e^{\frac{\sigma_{\widehat{B}}^2\left(\frac{2\alpha}{\sigma_{\widehat{B}}^2}+1\right)}{2}t}\Phi\left[\sigma_{\widehat{B}}\left(1+\frac{\alpha}{\sigma_{\widehat{B}}^2}\right)\sqrt{t}\right]-\frac{\frac{-2\alpha}{\sigma_{\widehat{B}}^2}}{\frac{2\alpha}{\sigma_{\widehat{B}}^2}+1}\Phi\left(\frac{-\alpha}{\sigma_{\widehat{B}}}\sqrt{t}\right). \qquad \text{(A.10)}
\end{aligned}$$

When $2\alpha/\sigma_{\widehat{B}}^2+1>0$, then $\alpha/\sigma_{\widehat{B}}^2+1>-\alpha/\sigma_{\widehat{B}}^2>0$, so that the first term in equation (A.10) is positive. Moreover, when again $2\alpha/\sigma_{\widehat{B}}^2+1>0$, the first term on the last line always goes to infinity when $t$ tends to infinity dwarfing the second term in equation (A.10). At the same time, when $2\alpha/\sigma_{\widehat{B}}^2+1<0$, then as $t\to\infty$ the first term in equation (A.10) disappears leaving only the second term to be non-negligible. We now use these properties to compute the transversality condition.

Now, we can compute the limit

$$\begin{aligned}
\lim_{t\to\infty} e^{-(\rho-n)t}\mathbb{E}_0 e^{\widehat{B}_t} &= \lim_{t\to\infty}\left\{2\frac{\frac{\alpha}{\sigma_{\widehat{B}}^2}+1}{\frac{2\alpha}{\sigma_{\widehat{B}}^2}+1}e^{\left[\frac{\sigma_{\widehat{B}}^2\left(\frac{2\alpha}{\sigma_{\widehat{B}}^2}+1\right)}{2}-\rho+n\right]t}\Phi\left[\sigma_{\widehat{B}}\left(1+\frac{\alpha}{\sigma_{\widehat{B}}^2}\right)\sqrt{t}\right]-e^{-(\rho-n)t}\frac{\frac{-2\alpha}{\sigma_{\widehat{B}}^2}}{\frac{2\alpha}{\sigma_{\widehat{B}}^2}+1}\Phi\left(\frac{-\alpha}{\sigma_{\widehat{B}}}\sqrt{t}\right)\right\}\\
&= \lim_{t\to\infty}\left\{2\frac{\frac{\alpha}{\sigma_{\widehat{B}}^2}+1}{\frac{2\alpha}{\sigma_{\widehat{B}}^2}+1}e^{\left[\frac{\sigma_{\widehat{B}}^2\left(\frac{2\alpha}{\sigma_{\widehat{B}}^2}+1\right)}{2}-\rho+n\right]t}\Phi\left[\sigma_{\widehat{B}}\left(1+\frac{\alpha}{\sigma_{\widehat{B}}^2}\right)\sqrt{t}\right]\right\}\\
&= 2\frac{\frac{\alpha}{\sigma_{\widehat{B}}^2}+1}{\frac{2\alpha}{\sigma_{\widehat{B}}^2}+1}\lim_{t\to\infty} e^{\left[\frac{\sigma_{\widehat{B}}^2\left(\frac{2\alpha}{\sigma_{\widehat{B}}^2}+1\right)}{2}-\rho+n\right]t}
\end{aligned}$$

where the second equality took into account the fact that $\lim_{t\to\infty} e^{-(\rho-n)t}\Phi\left(\frac{-\alpha}{\sigma_{\widehat{B}}}\sqrt{t}\right)=0$. Finally, we obtain that

$$\lim_{t\to\infty} e^{-(\rho-n)t}\mathbb{E}_0 e^{\widehat{B}_t} = \begin{cases} 0, & \frac{\sigma_{\widehat{B}}^2}{2}<-\alpha+\rho-n,\\ 2\frac{\frac{\alpha}{\sigma_{\widehat{B}}^2}+1}{\frac{2\alpha}{\sigma_{\widehat{B}}^2}+1}, & \frac{\sigma_{\widehat{B}}^2}{2}=-\alpha+\rho-n,\\ +\infty, & \frac{\sigma_{\widehat{B}}^2}{2}>-\alpha+\rho-n. \end{cases}$$

After noting that

$$\alpha - \rho + n + \frac{\sigma_{\widehat{B}}^2}{2} = \alpha_D - \alpha_u,$$

we can write

$$\lim_{t\to\infty} e^{-(\rho-n)t}\mathbb{E}_0 e^{\widehat{B}_t} = \begin{cases} 0, & \alpha_u - \alpha_D > 0, \\ 2\frac{\frac{\alpha}{\sigma_{\widehat{B}}^2}+1}{\frac{2\alpha}{\sigma_{\widehat{B}}^2}+1}, & \alpha_u - \alpha_D = 0, \\ +\infty, & \alpha_u - \alpha_D < 0. \end{cases}$$

**Remark 1.** So far, we have considered the case when $2\alpha/\sigma_{\widehat{B}}^2 + 1 \neq 0$. When, instead, $2\alpha/\sigma_{\widehat{B}}^2 + 1 = 0$, the above calculations simplify considerably. The key difference starts from equation (A.10), which will not feature $2\alpha/\sigma_{\widehat{B}}^2 + 1$ in the denominator anymore.

**Remark 2.** While the assumptions that $\widehat{B}_0 = \widehat{B}_{min} = 0$ are clearly non-consequential. The assumption of $\gamma = 1$ can look suspicious. In fact, all of the calculation go through. The only non-standard feature is that we need to deal with a joint distribution of correlated Brownian motion (i.e., $\log y_T$) and reflected Brownian motion (i.e., $\widehat{B}_T$) because we have to compute the following object

$$\mathbb{E}_t e^{nT}\xi_T b_T = \mathbb{E}_t e^{-(\rho-n)T}\left[(1-\gamma_G)y_T\right]^{-\gamma} b_T = (1-\gamma_G)^{-\gamma} e^{-(\rho-n)T}\mathbb{E}_t e^{(1-\gamma)\log y_T + \widehat{B}_T}.$$

This is straightforward but needs some care.

## A.6 Proof of Proposition 6

**The law of motion.** Taking the difference between the low of motion of the log of debt and the log of output, i.e.,

$$d\log B_t = \left(r_t + \frac{G_t - T_t}{B_t}\right)dt,$$
$$d\log y_t = \left(g_y - \frac{\sigma_y^2}{2}\right)dt + \sigma_y dZ_t^y - Z_t dJ_t,$$

We obtain the law of motion of the log of debt-to-GDP ratio

$$d\widehat{B}_t = \left(r_t - g_y - n + \alpha_D + \frac{\sigma_y^2}{2} - \beta_D\widehat{B}_t\right)dt - \sigma_y dZ_t^y + Z_t dJ_t.$$

The interest rates are

$$r^s = \rho + \gamma g_y - \frac{\gamma(\gamma+1)}{2}\sigma_y^2 - \lambda(\mathbb{E}_Z e^{\gamma Z} - 1),$$
$$r_t = r^s - \alpha_u + \beta_u\widehat{B}_t.$$

As a result

$$d\widehat{B}_t = \left(\widetilde{\alpha} - \beta\widehat{B}_t\right)dt - \sigma_y dZ_t^y + Z_t dJ_t.$$

where

$$\widetilde{\alpha} = \rho + \gamma g_y - \frac{\gamma(\gamma+1)}{2}\sigma_y^2 - \alpha_u - g_y - n + \alpha_D + \frac{\sigma_y^2}{2} - \lambda(\mathbb{E}_Z e^{\gamma Z} - 1) = \alpha - \lambda(\mathbb{E}_Z e^{\gamma Z} - 1).$$

which is similar to the definition of $\alpha$ before but that takes into account rare disasters.

**Stationary distribution.** Taking into account the fact that $\beta = 0$, we can write the Kolmogorov forward equation for the density function $f = f(\widehat{B}, t)$ in the region $\widehat{B} > \widehat{B}_{min}$ as

$$\frac{\partial f}{\partial t} = -\widetilde{\alpha}\frac{\partial f}{\partial \widehat{B}} + \frac{\sigma_y^2}{2} \cdot \frac{\partial^2 f}{\partial \widehat{B}^2} + \lambda \left\{ \mathbb{E}\left[ f(\widehat{B} - Z, t) \right] - f(\widehat{B}, t) \right\}. \tag{A.11}$$

The stationary distribution satisfied $\partial f / \partial t = 0$. We guess and verify that

$$f(\widehat{B}) = \overline{f} \cdot e^{-\xi \widehat{B}},$$

where $\overline{f}$ and $\widehat{B}$ are constants to be determined. Plugging this guess in equation A.11 and taking into account that $\partial f / \partial t = 0$, we obtain the implicit equation that determines the rate parameter $\xi$

$$\widetilde{\alpha}\xi + \frac{\sigma_y^2}{2}\xi^2 = \lambda\left(1 - \mathbb{E}\left[e^{\xi Z}\right]\right). \tag{A.12}$$

This equation has one obvious solution of $\xi = 0$, which immediately implies that $f(\widehat{B}) = 0$ for all $\widehat{B} > \widehat{B}_{min}$. This must not be the case for the reflected process. As a result, we discard this solution. It is easy to see by plotting the left- and the right-hand sides of equation A.12 as functions of $\xi$, that the remaining solution of the equation is positive when $\widetilde{\alpha} + \lambda\mathbb{E}[Z] < 0$ and it is negative in the opposite case of $\widetilde{\alpha} + \lambda\mathbb{E}[Z] > 0$. In the knife edge case of $\lambda\mathbb{E}[Z] = -\widetilde{\alpha}$, there is only one solution of $\xi = 0$. The stationary distribution exists only for negative $\xi$.

Constant $\overline{f}$ is determined by requiring that

$$\int_{\widehat{B}_{min}}^{\infty} f(\widehat{B}) d\widehat{B} = 1.$$

As a result,

$$\overline{f} = \xi e^{\xi \widehat{B}_{min}},$$

and

$$f(\widehat{B}) = \xi e^{-\xi\left(\widehat{B} - \widehat{B}_{min}\right)}.$$

**Transversality condition.** To prove that the transversality condition holds if and only if $\alpha_u - \alpha_D > 0$, we consider two separate cases.

First, consider the case when the stationary distribution does not exist. Specifically, suppose that $\widetilde{\alpha} + \lambda\mathbb{E}[Z] > 0$. In this case, the debt-to-GDP ratio increases unboundedly on average. As a result, we can ignore the influence of the lower reflecting barrier. As a result, the expectation in the transversality condition

$$\mathbb{E}_t e^{nT} \xi_T b_T = \frac{1}{(1-\gamma_G)^2} \mathbb{E}_t e^{-(\rho-n)T} e^{\widehat{B}_T + (1-\gamma)\log y_T}.$$

Next, express $\widehat{B}_T + (1-\gamma)\log y_T$ as

$$\begin{aligned}
d\widehat{B}_t + (1-\gamma)d\log y_t &= \widetilde{\alpha}dt - \sigma_y dZ_t^y + Z_t dJ_t + (1-\gamma)\left[\left(g_y - \frac{\sigma_y^2}{2}\right)dt + \sigma_y dZ_t^y - Z_t dJ_t\right] \\
&= \left[\widetilde{\alpha}dt + (1-\gamma)\left(g_y - \frac{\sigma_y^2}{2}\right)\right]dt - \gamma\sigma_y dZ_t^y + \gamma Z_t dJ_t \\
&= \left[\rho - \frac{\gamma^2}{2}\sigma_y^2 - \alpha_u - n + \alpha_D - \lambda(\mathbb{E}_Z e^{\gamma Z} - 1)\right]dt - \gamma\sigma_y dZ_t^y + \gamma Z_t dJ_t.
\end{aligned}$$

Integrate the last equation

$$\begin{aligned}
\widehat{B}_T + (1-\gamma)\log y_T - [\widehat{B}_t + (1-\gamma)\log y_t] &= \left[\rho - \frac{\gamma^2}{2}\sigma_y^2 - \alpha_u - n + \alpha_D - \lambda(\mathbb{E}_Z e^{\gamma Z} - 1)\right](T-t) \\
&\quad - \gamma\sigma_y\left(Z_T^y - Z_t^y\right) + \gamma\sum_{k=1}^{n_{t,T}} Z_{t_k},
\end{aligned}$$

where $n_{t,T}$ is a (random) number of Poisson event arrivals between $t$ and $T$. As a result,

$$\begin{aligned}
&\mathbb{E}_t e^{-(\rho-n)T} e^{\widehat{B}_T + (1-\gamma)\log y_T} \\
=& e^{-(\rho-n)t + \widehat{B}_t + (1-\gamma)\log y_t}\mathbb{E}_t e^{[-\gamma^2\sigma_y^2/2 - \alpha_u + \alpha_D - \lambda(\mathbb{E}_Z e^{\gamma Z} - 1)](T-t) - \gamma\sigma_y\left(Z_T^y - Z_t^y\right) + \gamma\sum_{k=1}^{n_{t,T}} Z_{t_k}} \\
=& e^{-(\rho-n)t + \widehat{B}_t + (1-\gamma)\log y_t}\mathbb{E}_t e^{\left[-\frac{\gamma^2}{2}\sigma_y^2 - \alpha_u + \alpha_D - \lambda(\mathbb{E}_Z e^{\gamma Z} - 1)\right](T-t) + \frac{\gamma^2\sigma_y^2}{2}(T-t) + (T-t)\lambda\mathbb{E}_t[e^{\lambda Z_t} - 1]} \\
=& e^{-(\rho-n)t + \widehat{B}_t + (1-\gamma)\log y_t}\mathbb{E}_t e^{[-\alpha_u + \alpha_D](T-t)}.
\end{aligned}$$

It is clear from the last expression that the transversality condition holds if and only if $-\alpha_u + \alpha_D < 0$.

Consider now the second case in which the stationary distribution exists, that is, $\widetilde{\alpha} + \lambda\mathbb{E}[Z] < 0$. For simplicity of exposition, we consider the case of $\gamma = 1$, which allows us to write

$$\begin{aligned}
\mathbb{E}_t e^{nT}\xi_T b_T &= \frac{1}{(1-\gamma_G)^2}\mathbb{E}_t e^{-(\rho-n)T} e^{\widehat{B}_T} \\
&= \frac{1}{(1-\gamma_G)^2} e^{-(\rho-n)T}\int_{\widehat{B}_{\min}}^{\infty} e^{\widehat{B}}\zeta e^{-\zeta\widehat{B}} d\widehat{B} \\
&= \frac{1}{(1-\gamma_G)^2(1-\zeta)} e^{-(\rho-n)T}\zeta\int_{\widehat{B}_{\min}}^{\infty} e^{(1-\zeta)\widehat{B}} d(1-\zeta)\widehat{B} \\
&= \frac{1}{(1-\gamma_G)^2(\zeta-1)} e^{-(\rho-n)T}\zeta e^{(1-\zeta)\widehat{B}_{\min}}
\end{aligned}$$

where the last inequality is only valid under $\zeta > 1$. As a result, the TVC holds if $\zeta > 1$ and $\rho > n$.

## A.7 Proof of Lemma 4

We lay out a heuristic proof here.

**Safe interest rate.** The discount factor is

$$m_{t,t+dt} = e^{-\rho dt}\frac{c_{t+dt}^{-\gamma}}{c_t^{-\gamma}} = \begin{cases} e^{-\rho dt}e^{-\gamma g_y dt}, & \text{no disaster,} \\ e^{-\rho dt}e^{-\gamma g_y dt}e^{-\gamma(-Z)}, & \text{disaster,} \end{cases}$$
$$= \begin{cases} e^{-(\rho+\gamma g_y)dt}, & \text{no disaster,} \\ e^{-(\rho+\gamma g_y)dt+\gamma Z}, & \text{disaster,} \end{cases}$$

the return is

$$R_{t,t+dt} = e^{r_t^s dt}.$$

The Euler equation is

$$1 = \mathbb{E}_t\left(m_{t,t+dt}R_{t,t+dt}\right).$$

Use the values of the SDF and the return

$$1 = (1-\lambda dt)e^{-(\rho+\gamma g_y)dt}e^{r_t^s dt} + \lambda dt \mathbb{E}_Z e^{-(\rho+\gamma g_y)dt+\gamma Z}e^{r_t^s dt},$$

and simplify

$$e^{(\rho+\gamma g_y - r_t^s)dt} = 1 + \lambda dt\left(\mathbb{E}_Z e^{\gamma Z} - 1\right),$$
$$r_t^s = \rho + \gamma g_y - \lambda\left(\mathbb{E}_Z e^{\gamma Z} - 1\right).$$

Note that this last formula is just a special case of a more general formula in Proposition 8 in Appendix B.

**Defaultable interest rate.** The return is

$$R_{t,t+dt} = \begin{cases} e^{r_t dt}, & \text{no default,} \\ 0 & \text{default.} \end{cases}$$

The Euler equation is

$$1 = \mathbb{E}_t\left(m_{t,t+dt}R_{t,t+dt}\right).$$

Use the values of the SDF and the return

$$1 = (1-\lambda dt)e^{-(\rho+\gamma g_y)dt}e^{r_t dt} + \lambda dt e^{-(\rho+\gamma g_y)dt}e^{r_t dt}\mathbb{P}\left(Z < \log\left(\frac{\mathcal{B}_{FL}}{B_t/Y_t}\right)\right)\mathbb{E}\left[e^{\gamma Z} | Z < \log\left(\frac{\mathcal{B}_{FL}}{B_t/Y_t}\right)\right],$$

and simplify

$$
\begin{aligned}
e^{\left(\rho+\gamma g_y-r_t\right)dt} =& (1-\lambda dt)+\lambda dt\mathbb{P}\left(Z<\log\left(\frac{\mathcal{B}_{FL}}{B_t/Y_t}\right)\right)\mathbb{E}\left[e^{\gamma Z}|Z<\log\left(\frac{\mathcal{B}_{FL}}{B_t/Y_t}\right)\right],\\
\left(\rho+\gamma g_y-r_t\right) =& \lambda\left\{\mathbb{P}\left(Z<\log\left(\frac{\mathcal{B}_{FL}}{B_t/Y_t}\right)\right)\mathbb{E}_{Z|Z<\log\left(\frac{\mathcal{B}_{FL}}{B_t/Y_t}\right)}[e^{\gamma Z}]-1\right\},\\
r_t =& \rho+\gamma g_y-\lambda\left\{\mathbb{P}\left(Z<\log\left(\frac{\mathcal{B}_{FL}}{B_t/Y_t}\right)\right)\mathbb{E}_{Z|Z<\log\left(\frac{\mathcal{B}_{FL}}{B_t/Y_t}\right)}[e^{\gamma Z}]-1\right\}.
\end{aligned}
$$

In the case of the exponential distribution of random variable $Z$, i.e., a change in log output, with the probability distribution function

$$
f_Z(z)=\begin{cases}\xi e^{-\xi z}, & z\geq 0,\\ 0, & z<0,\end{cases}
$$

we have

$$
\begin{aligned}
r_t &= \rho+\gamma g_y-\lambda\left[\int_0^{\log\frac{\mathcal{B}_{FL}}{B_t/Y_t}} e^{\gamma z}dF_Z(z)-1\right]\\
&= \rho+\gamma g_y-\lambda\left[\xi\int_0^{\log\frac{\mathcal{B}_{FL}}{B_t/Y_t}} e^{(\gamma-\xi)z}dz-1\right]\\
&= \rho+\gamma g_y-\lambda\left[\frac{\xi}{\gamma-\xi}e^{(\gamma-\xi)z}\Big|_0^{\log\frac{\mathcal{B}_{FL}}{B_t/Y_t}}-1\right]\\
&= \rho+\gamma g_y-\lambda\left\{\frac{\xi}{\xi-\gamma}\left[1-\left(\frac{\mathcal{B}_{FL}}{B_t/Y_t}\right)^{\gamma-\xi}\right]-1\right\}.
\end{aligned}
$$

# B A Model with EZW Preferences

This section of the Appendix provides the details omitted in Section 3.6 and 4.4 by first describing the recursive preferences and then by stating some results that we prove in an online appendix C.

## B.1 A no-Disaster Case

In a model without disasters, a typical household maximizes the following preferences

$$
W_t = V_t+\mathbb{E}_t\int_t^{\infty}\pi_{t,s}y_s u\left(\frac{b_s}{y_s}\right)ds, \tag{B.1}
$$

where

$$
\begin{aligned}
V_t &= \mathbb{E}_t \int_t^{\infty} f(c_s, V_s) ds, \\
f(c_s, V_s) &= \frac{[(1-\gamma)V_s]^{\frac{\theta-\gamma}{1-\gamma}}}{1-\theta} \{c_s^{1-\theta} - (\rho - n)[(1-\gamma)V_s]^{\frac{1-\theta}{1-\gamma}}\}, \\
\pi_{t,s} &= e^{\left\{\frac{\theta-\gamma}{1-\gamma}\left[\rho-n-(1-\theta)\left(g-\frac{\gamma\sigma_y^2}{2}\right)\right]-(\rho-n)(1-\gamma)\right\}\frac{s-t}{1-\theta}} \left[\rho - n - (1-\theta)\left(g_y - \frac{\gamma\sigma_y^2}{2}\right)\right]^{-\frac{\theta-\gamma}{1-\theta}} C_s^{-\gamma}.
\end{aligned} \tag{B.2}
$$

Formally, the utility function (3) consists of two terms that capture the utility from consumption and utility from holding government bonds. We assume that the utility from consumption is represented by the Epstein-Zin-Weil preferences with subjective discount factor $\rho$, the coefficient of relative risk aversion $\gamma$, and the intertemporal elasticity of substitution $1/\theta$. One advantage of using these preferences is that they allow for separation of the coefficient of relative risk aversion (CRRA) and the intertemporal elasticity of substitution (IES) that will be convenient in our calibration. We use the continuous-time formulation of these preferences introduced by Duffie and Epstein (1992). When $\gamma = \theta$, the preferences in (B.1) reduce to the preferences we used in the main text and that are given by equation (3).

With the process (B.2) entering the preferences for public debt (3), the demand for liquid bonds does not depend on current consumption of the household in equilibrium, i.e., the wealth effect on demand for government bonds is zero in equilibrium.

By repeating steps is the proof of Lemma 2, which can be found in the Appendix A.2, we can write the following law of motion for the log of debt-to-GDP ratio.

$$
d\widehat{B}_t = \left( r_t - g_y - n + \alpha_D + \frac{\sigma_y^2 - \sigma_B^2}{2} - \beta_D \widehat{B}_t \right) dt + \sigma_{\hat{B}} dZ_t^{\hat{B}},
$$

where $dZ_t^{\hat{B}} \equiv (\sigma_B/\sigma_{\widehat{B}})dZ_t^B - (\sigma_y/\sigma_{\widehat{B}})dZ_t^y$ and $\sigma_{\widehat{B}}^2 \equiv \sigma_B^2 + \sigma_y^2$. Note that we added disasters in this expression.

Asset market clearing conditions combined with optimal choices by households gives the asset pricing equations summarized in the next proposition.

**Proposition 8.** *In equilibrium, the interest rate on safe assets and liquid government bonds are:*

$$
\begin{aligned}
r^s &= \rho + \theta g_y - \frac{\gamma(\theta+1)}{2}\sigma_y^2, \\
r_t &= r^s - \alpha_u + \beta_u + \beta_u \widehat{B}_t,
\end{aligned}
$$

*and the drift and diffusion terms for the return on the risky asset are given by:*

$$
\begin{aligned}
\mu_t &= r^s + \gamma\sigma_y^2, \\
\sigma_t &= \sigma_y.
\end{aligned}
$$

The proof is in Appendix C. Proposition 8 states that the only difference in the asset pricing in this extended model compared to the model in Section 3 is the explicit presence of the IES parameter in the safe interest rate

## B.2 A Case with Disasters

Adding disasters is straightforward. Equation B.2 has to be modified to take into account the fact that the household faces not only Brownian but also disaster risk. The law of motion of debt to GDP becomes

$$d\widehat{B}_t = \left( r_t - g_y - n + \alpha_D + \frac{\sigma_y^2 - \sigma_B^2}{2} - \beta_D \widehat{B}_t \right) dt + \sigma_{\hat{B}} dZ_t^{\hat{B}} + Z_t dJ_t, \tag{B.3}$$

We present the extension of Proposition 8 to the disaster case.

**Proposition 9.** *In equilibrium, the interest rate on safe assets and liquid government bonds are:*

$$\begin{aligned} r^s &= \rho + \theta g_y - \frac{\gamma(\theta+1)}{2}\sigma_y^2 + \lambda \mathbb{E}\left[\frac{\theta-\gamma}{1-\gamma}(e^{-(1-\gamma)Z} - 1) - (e^{\gamma Z} - 1)\right], \\ r_t &= r^s - \alpha_u + \beta_u + \beta_u \widehat{B}_t, \end{aligned}$$

*and the drift and diffusion terms for the return on the risky asset are given by:*

$$\begin{aligned} \mu_t &= r^s + \gamma \sigma_y^2 + \lambda \mathbb{E}_Z\left[(e^{\gamma Z} - 1)(1 - e^{-Z})\right], \\ \sigma_t &= \sigma_y. \end{aligned}$$

It is straightforward to extend the proof of Proposition 8 to the case with disasters by following, for example, Tsai and Wachter (2015).

As a result, the law of motion of the log of public debt-to-GDP ratio is

$$d\widehat{B}_t = \left(\overline{\alpha} - \beta \widehat{B}_t\right) dt + \sigma_{\hat{B}} dZ_t^{\hat{B}} + Z_t dJ_t.$$

where

$$\overline{\alpha} \equiv \rho + \gamma g_y - \{\sigma_B^2 + [\gamma(\theta+1) - 1]\,\sigma_y^2\}/2 - \alpha_u - g_y - n + \alpha_D + \lambda \mathbb{E}_Z\left[\frac{\theta-\gamma}{1-\gamma}\left(e^{-(1-\gamma)Z} - 1\right) - (e^{\gamma Z} - 1)\right].$$

which is similar to the definition of $\widetilde{\alpha}$ in Proposition 6 but that takes into account the fact that the IES and CRRA are not equal each other.

Note that the stationary distribution of $\widehat{B}_t$ when there is a lower reflecting barrier $\widehat{B}_{\min}$ and $\beta = 0$ is again exponential with the rate parameter that solves

$$\overline{\alpha}\xi + \frac{\sigma_{\hat{B}}^2}{2}\xi^2 = \lambda(1 - \mathbb{E}_Z[e^{\xi Z}]).$$

Assume that government defaults when the debt jumps over the debt limit as in Section 4. The household needs to be compensated for this risk. The next proposition presents the interest rate paid on government debt absent default.

**Proposition 10.** *Conditional on no default, public debt pays*

$$r_t = r^s + \lambda \mathbb{E}\left[e^{\gamma Z_t} \mathbb{I}\left(Z > \log\left(\frac{\mathcal{B}_{FL}}{B_t/Y_t}\right)\right)\right].$$

The proof of this result uses Proposition 1 from Tsai and Wachter (2015). As a result,

$$r_t = \rho + \theta g_y - \frac{\gamma(\theta+1)}{2}\sigma_y^2 + \lambda\frac{\theta-\gamma}{1-\gamma}\mathbb{E}\left[e^{-(1-\gamma)Z} - 1\right] - \lambda\mathbb{E}\left[e^{\gamma Z}\mathbb{I}\left(Z < \log\left(\frac{\mathcal{B}_{FL}}{B_t/Y_t}\right)\right) - 1\right].$$

When we assume that $Z$ has an exponential distribution with the pdf $f_Z(z) = \bar{z}^{-1}e^{-z/\bar{z}}$ for $z \geq 0$, we get

$$r_t = \rho + \theta g_y - \frac{\gamma(\theta+1)}{2}\sigma_y^2 - \lambda\frac{\theta-\gamma}{1+\bar{z}-\bar{z}\gamma}\bar{z} - \lambda\left(\frac{1}{1-\bar{z}\gamma}\left(1 - \left(\frac{B_t/Y_t}{\mathcal{B}_{FL}}\right)^{\frac{1-\bar{z}\gamma}{\bar{z}}}\right) - 1\right).$$

where I used the fact that $\lim_{z\to\infty} e^{(\gamma-1/\bar{z}-1)z} = \lim_{z\to\infty} e^{(\gamma-1/\bar{z})z} = 0$, which can only happen when $\gamma < \xi < \xi + 1$, where the second inequality holds automatically. Moreover, the equity premium is

$$\mu_t - r^s = \gamma\sigma_y^2 + \lambda\frac{1+\bar{z}+1-\bar{z}\gamma}{(1-\gamma\bar{z})(1+\bar{z})(1-\gamma\bar{z}+\bar{z})}\gamma\bar{z}^2.$$

and the safe rate is

$$r^s = \rho + \theta g_y - \frac{\gamma(\theta+1)}{2}\sigma_y^2 - \lambda\bar{z}\left(\frac{\theta-\gamma}{1+\bar{z}-\bar{z}\gamma} + \frac{\gamma}{1-\bar{z}\gamma}\right).$$

# Online Appendix

## C Proof of Proposition 8

**Step 0: preliminaries.** First, the partial derivatives of function

$$f(c,V)=\frac{[(1-\gamma)V]^{\frac{\theta-\gamma}{1-\gamma}}}{1-\theta}\left[c_t^{1-\theta}-(\rho-n)\left((1-\gamma)V\right)^{\frac{1-\theta}{1-\gamma}}\right]$$

are

$$\begin{aligned}
f_1(c,V)&=c_t^{-\theta}\left[(1-\gamma)V_t\right]^{\frac{\theta-\gamma}{1-\gamma}},\\
f_2(c,V)&=\frac{\theta-\gamma}{1-\gamma}\cdot\frac{f_t}{V_t}-\rho+n.
\end{aligned}$$

**Second**, we now evaluate the value of $V_t$ when the consumption follows a geometric Brownian motion process. Formally, we solve the follow system of equations

$$\begin{aligned}
V_t&=\mathbb{E}_t\left[\int_t^\infty f(c_u,V_u)du\right],\\
f(c,V)&=\frac{[(1-\gamma)V]^{\frac{\theta-\gamma}{1-\gamma}}}{1-\theta}\left[c^{1-\theta}-(\rho-n)\left((1-\gamma)V\right)^{\frac{1-\theta}{1-\gamma}}\right],\\
\frac{dc_t}{c_t}&=g_y dt+\sigma_y dZ_t^y.
\end{aligned}$$

We guess the solution of the form

$$V_t=vc_t^{1-\gamma},$$

where $v$ is a positive constant. We plug this guess

$$\begin{aligned}
f(c_t,V_t)&=\frac{\left[(1-\gamma)vc_t^{1-\gamma}\right]^{\frac{\theta-\gamma}{1-\gamma}}}{1-\theta}\left[c_t^{1-\theta}-(\rho-n)\left((1-\gamma)vc_t^{1-\gamma}\right)^{\frac{1-\theta}{1-\gamma}}\right]\\
&=\frac{[(1-\gamma)v]^{\frac{\theta-\gamma}{1-\gamma}}}{1-\theta}\left[1-(\rho-n)\left((1-\gamma)v\right)^{\frac{1-\theta}{1-\gamma}}\right]c_t^{1-\gamma}.
\end{aligned}$$

As a result

$$\begin{aligned}
V_t&=\mathbb{E}_t\left[\int_t^T f(c_u,V_u)du+V_T\right]\\
&=\frac{[(1-\gamma)v]^{\frac{\theta-\gamma}{1-\gamma}}}{1-\theta}\left[1-(\rho-n)\left((1-\gamma)v\right)^{\frac{1-\theta}{1-\gamma}}\right]\mathbb{E}_t\int_t^T c_u^{1-\gamma}du+v\mathbb{E}_t c_T^{1-\gamma}.
\end{aligned}$$

To compute the last expectations note that

$$d\log c_t=\frac{dc_t}{c_t}-\frac{1}{2}\left(\frac{dc_t}{c_t}\right)^2=\left(g_y-\frac{\sigma_y^2}{2}\right)dt+\sigma_y dZ_t^y.$$

As a result,

$$
\begin{aligned}
d \log c_t^{1-\gamma} &= (1-\gamma)\left(g_y - \frac{\sigma_y^2}{2}\right) dt + (1-\gamma)\sigma_y dZ_t^y, \\
\log c_u^{1-\gamma} - \log c_t^{1-\gamma} &= (1-\gamma)\left(g_y - \frac{\sigma_y^2}{2}\right)(u-t) + (1-\gamma)\sigma_y Z_{u-t}^y, \\
c_u^{1-\gamma} &= c_t^{1-\gamma} \exp\left\{(1-\gamma)\left(g_y - \frac{\sigma_y^2}{2}\right)(u-t) + (1-\gamma)\sigma_y Z_{u-t}^y\right\}, \\
\mathbb{E}_t c_u^{1-\gamma} &= c_t^{1-\gamma} e^{(1-\gamma)\left(g - \frac{\gamma\sigma_y^2}{2}\right)(u-t)}, \\
\mathbb{E}_t \int_t^T c_u^{1-\gamma} du &= \int_t^T \mathbb{E}_t c_u^{1-\gamma} du \\
&= \frac{c_t^{1-\gamma}}{(1-\gamma)\left(g - \frac{\gamma\sigma_y^2}{2}\right)}\left[e^{(1-\gamma)\left(g_y - \frac{\gamma\sigma_y^2}{2}\right)(T-t)} - 1\right].
\end{aligned}
$$

This implies

$$
\begin{aligned}
V_t =& \frac{[(1-\gamma)v]^{\frac{\theta-\gamma}{1-\gamma}}}{1-\theta}\left[1 - (\rho - n)\left((1-\gamma)v\right)^{\frac{1-\theta}{1-\gamma}}\right] \mathbb{E}_t \int_t^T c_u^{1-\gamma} du + v \mathbb{E}_t c_T^{1-\gamma} \\
=& e^{(1-\gamma)\left(g_y - \frac{\gamma\sigma_y^2}{2}\right)(T-t)} c_t^{1-\gamma} \left\{ \frac{\frac{[(1-\gamma)v]^{\frac{\theta-\gamma}{1-\gamma}}}{1-\theta}\left[1 - (\rho-n)\left((1-\gamma)v\right)^{\frac{1-\theta}{1-\gamma}}\right]}{(1-\gamma)\left(g_y - \frac{\gamma\sigma_y^2}{2}\right)} + v \right\} \\
& - \frac{[(1-\gamma)v]^{\frac{\theta-\gamma}{1-\gamma}}}{1-\theta}\left[1 - (\rho-n)\left((1-\gamma)v\right)^{\frac{1-\theta}{1-\gamma}}\right] \frac{c_t^{1-\gamma}}{(1-\gamma)\left(g_y - \frac{\gamma\sigma_y^2}{2}\right)}.
\end{aligned}
$$

The term with $T - t$ must be equal to zero for the conjecture to be correct

$$
\begin{aligned}
\frac{[(1-\gamma)v]^{\frac{\theta-\gamma}{1-\gamma}}}{1-\theta} &= (\rho - n)\frac{(1-\gamma)v}{1-\theta} - v(1-\gamma)\left(g_y - \frac{\gamma\sigma_y^2}{2}\right), \\
v &= \frac{1}{1-\gamma}\left[\rho - n - (1-\theta)g_y + (1-\theta)\frac{\gamma\sigma_y^2}{2}\right]^{-\frac{1-\gamma}{1-\theta}}.
\end{aligned}
$$

As a result, we obtain

$$
V_t = \frac{c_t^{1-\gamma}}{1-\gamma}\left[\rho - n - (1-\theta)\left(g_y - \frac{\gamma\sigma_y^2}{2}\right)\right]^{-\frac{1-\gamma}{1-\theta}}.
$$

**Third**, we will later show that the discount factor in this economy is given by

$$
\frac{\xi_s}{\xi_t} = e^{-n(s-t)} e^{\int_0^s f_2(c_\tau, V_\tau) d\tau} f_1(c_s, V_s).
$$

We next compute the equilibrium stochastic discount factor multiplied by population increase and show that it equals $\pi_{t,s}$.

$$\frac{\xi_s}{\xi_0}e^{ns} = e^{\int_0^s f_2(c_\tau,V_\tau)d\tau} f_1(c_s,V_s) = e^{\int_0^s \left[\frac{\theta-\gamma}{1-\gamma}\cdot\frac{f_t}{V_t}-\rho+n\right]d\tau} c_s^{-\theta}\left[(1-\gamma)V_s\right]^{\frac{\theta-\gamma}{1-\gamma}}$$

Consider the argument of the exponent first

$$\begin{aligned}
e^{\int_0^s f_2(c_\tau,V_\tau)d\tau} &= e^{\int_0^s \left[\frac{\theta-\gamma}{1-\gamma}\cdot\frac{1}{1-\theta}\left[(1-\gamma)V_u\right]^{-\frac{1-\theta}{1-\gamma}}c_u^{1-\theta}-(\rho-n)\frac{1-\gamma}{1-\theta}\right]du} \\
&= e^{\int_0^s \left[\frac{\theta-\gamma}{1-\gamma}\cdot\frac{1}{1-\theta}\left[(1-\gamma)vc_u^{1-\gamma}\right]^{-\frac{1-\theta}{1-\gamma}}c_u^{1-\theta}-(\rho-n)\frac{1-\gamma}{1-\theta}\right]du} \\
&= e^{\left\{\frac{\theta-\gamma}{1-\gamma}\left[\rho-n-(1-\theta)\left(g_y-\frac{\gamma\sigma_y^2}{2}\right)\right]-(\rho-n)(1-\gamma)\right\}\frac{s}{1-\theta}}.
\end{aligned}$$

As a result,

$$\begin{aligned}
\frac{\xi_s}{\xi_0}e^{ns} &= e^{\left\{\frac{\theta-\gamma}{1-\gamma}\left[\rho-n-(1-\theta)\left(g_y-\frac{\gamma\sigma_y^2}{2}\right)\right]-(\rho-n)(1-\gamma)\right\}\frac{s}{1-\theta}} c_s^{-\theta}\left[(1-\gamma)vc_s^{1-\gamma}\right]^{\frac{\theta-\gamma}{1-\gamma}} \\
&= e^{\left\{\frac{\theta-\gamma}{1-\gamma}\left[\rho-n-(1-\theta)\left(g_y-\frac{\gamma\sigma_y^2}{2}\right)\right]-(\rho-n)(1-\gamma)\right\}\frac{s}{1-\theta}} c_s^{-\gamma}\left[\rho-n-(1-\theta)\left(g_y-\frac{\gamma\sigma_y^2}{2}\right)\right]^{-\frac{\theta-\gamma}{1-\theta}}.
\end{aligned}$$

The last expression equals $\pi_{0,t}$ introduced in the text.

Note that in the case of the CRRA utility, $\pi_s$ has the following familiar look

$$\pi_{0,s} = e^{-(\rho-n)s}c_s^{-\gamma}.$$

**Step 1: Intertemporal Budget Constraint.**

$$\begin{aligned}
&\max_{\{c_t,w_t,,x_t,b_t,s_t\}} W_0, \\
&s.t.: dw_t = (r_t^s s_t + r_t b_t - c_t - T_t - nw_t)dt + w_t x_t dr_t^x, \\
&\qquad s_t + b_t + x_t w_t = w_t,
\end{aligned}$$

Rewrite the problem by substituting out $dr_t^x$ and $s_t$ as follows

$$\begin{aligned}
&\max_{c_t,w_t,\phi_t,x_t,b_t} W_0, \\
&s.t.: \frac{d\left(e^{nt}w_t\right)+e^{nt}\left[c_t+T_t+\left(r_t^s-r_t\right)b_t\right]dt}{e^{nt}w_t} = \left[r_t^s + x_t(\mu_t - r_t^s)\right]dt + x_t\sigma_t dZ_t^y + \phi_t\sigma_t^\phi dZ_t^B,
\end{aligned}$$

Let the discount factor be $\xi_t$, which exists and is unique under the complete markets assumption, and must satisfy

$$\frac{d\xi_t}{\xi_t} = -r_t^s dt - \kappa_t^x dZ_t^y. \tag{C.1}$$

where $\kappa_t \equiv (\mu_t - r_t^s)/\sigma_t$. Note that $\xi_t$ is the per member of the household discount factor. Under such interpretation, optimally invested wealth must satisfy

$$w_t = \mathbb{E}_t \int_t^\infty [c_s + T_s + (r_s^s - r_s^B) b_s] e^{n(s-t)} \frac{\xi_s}{\xi_t} ds.$$

As a result, the household problem is

$$\max_{c_t, b_t} W_0(a_0; \widehat{B}_0),$$
$$s.t. : w_0 = \mathbb{E}_0 \int_0^\infty [c_t + T_t + (r_t^s - r_t) b_t] \frac{e^{nt}\xi_t}{\xi_0} dt,$$

where we omitted $x_t$ and $w_t$ from maximization arguments because we assume that the wealth is optimally allocated across assets safe and risky assets. The Lagrangian of this problem is

$$\mathcal{L}_0 = W_0 - \kappa \left[ \mathbb{E}_0 \int_0^\infty [c_t + T_t + (r_t^s - r_t) b_t] \frac{e^{nt}\xi_t}{\xi_0} dt - w_0 \right].$$

Note that $\mathcal{L}_0$ is a functional such that $\mathcal{L}_0 : \mathbb{L} \times \mathbb{L} \to \mathbb{R}$, where $\mathbb{L}$ is a space of square integrable progressively measurable processes with values in $\mathbb{R}$.

**Step 2: First Order Conditions.** The first order conditions for this optimization take the following form, where we use notation of Duffie and Skiadas (1994),

$$\nabla \mathcal{L}_0(c, \widetilde{c}) = 0, \ \forall \widetilde{c},$$
$$\nabla \mathcal{L}_0(b, \widetilde{b}) = 0, \ \forall \widetilde{b}.$$

The last two equations state that *the Gateaux derivative* of the Lagrangian with respect to consumption and bond holdings processes are zeros in any direction $\widetilde{c}$ (in case of consumption) and $\widetilde{b}$ (in case of liquid bonds). We next compute these derivatives explicitly. We start with $\nabla V_0(c, \widetilde{c})$ and $\nabla V_0(b, \widetilde{b})$.

$$\nabla W_0(c, \widetilde{c}) = \mathbb{E}_0 \int_0^\infty e^{\int_0^s f_2(c_\tau, V_\tau) d\tau} f_1(c_s, V_s) \widetilde{c}_s ds,$$
$$\nabla W_0(b, \widetilde{b}) = \mathbb{E}_t \int_t^\infty (1-\theta) y_s^{1-\theta} \frac{\widetilde{b}_s}{y_s} u' \left( \frac{b_s}{y_s} \right) ds,$$

As a result, the derivative of the Lagrangian with respect to consumption process is

$$\begin{aligned} 0 &= \nabla \mathcal{L}_0(c, \widetilde{c}) \\ &= \mathbb{E}_0 \int_0^\infty e^{\int_0^s f_2(c_\tau, V_\tau) d\tau} f_1(c_s, V_s) \widetilde{c}_s ds - \kappa \mathbb{E}_0 \int_0^\infty \widetilde{c}_t \frac{e^{nt}\xi_t}{\xi_0} dt \\ &= \mathbb{E}_0 \int_0^\infty \left( e^{\int_0^s f_2(c_\tau, V_\tau) d\tau} f_1(c_s, V_s) - \frac{\kappa e^{ns}\xi_s}{\xi_0} \right) \widetilde{c}_s ds. \end{aligned}$$

Because, the last equation has to hold for any $\widetilde{c}$, we must have that

$$e^{\int_0^s f_2(c_\tau, V_\tau) d\tau} f_1(c_s, V_s) = \frac{\kappa e^{ns}\xi_s}{\xi_0}.$$

Taking the ratio of this equation at times $t$ and $s$ and using explicit expression for partial derivative $f_1$, we obtain

$$e^{\int_s^t f_2(c_\tau, V_\tau)d\tau} \left(\frac{c_t}{c_s}\right)^{-\theta} \left(\frac{V_t}{V_s}\right)^{\frac{\theta-\gamma}{1-\gamma}} = e^{n(t-s)} \frac{\xi_t}{\xi_s}. \tag{C.2}$$

Analogously, the optimality wrt to liquid debt is

$$\pi_{t,s} u'\left(\frac{b_s}{y_s}\right) = \left(r_s^s - r_s^b\right) \frac{\kappa e^{ns} \xi_s}{\xi_0}. \tag{C.3}$$

Diving the last two equations, we obtain

$$r_s^b = r_s^s - \frac{\pi_{t,s} u'\left(\frac{b_s}{y_s}\right)}{e^{\int_t^s f_2(c_\tau, V_\tau)d\tau} f_1(c_s, V_s)}.$$

In equilibrium, we have

$$\begin{aligned} r_s^b &= r_s^s - \frac{\pi_{t,s} u'\left(\frac{b_s}{y_s}\right)}{e^{\int_t^s f_2(c_\tau, V_\tau)d\tau} f_1(c_s, V_s)} \\ &= r_t^s - u'\left(\frac{b_t}{y_t}\right) \end{aligned} \tag{C.4}$$

Finally, when households optimize, it must be true that

$$V_a = f_1. \tag{C.5}$$

**Step 3: Stochastic Discount Factor.** First, we want to compute the law of motion of $\xi_t$ (here we assume $\kappa \equiv \kappa/\xi_0$). We will apply the Ito's lemma to the FOC wrt to $c$. To do it, we separately compute several stochastic differentials

$$\begin{aligned} df_1(c_t, V_t) =& d\left\{\omega c_t^{-\theta} \left[(1-\gamma)V_t\right]^{\frac{\theta-\gamma}{1-\gamma}}\right\} \\ =& \omega\left(\left[(1-\gamma)V_t\right]^{\frac{\theta-\gamma}{1-\gamma}} d\left\{c_t^{-\theta}\right\} + c_t^{-\theta} d\left\{\left[(1-\gamma)V_t\right]^{\frac{\theta-\gamma}{1-\gamma}}\right\} + d\left\{c_t^{-\theta}\right\} d\left[(1-\gamma)V_t\right]^{\frac{\theta-\gamma}{1-\gamma}}\right) \\ =& \omega\Bigg(\left[(1-\gamma)V_t\right]^{\frac{\theta-\gamma}{1-\gamma}} \left(-\theta c_t^{-\theta}\right) \left[\frac{dc_t}{c_t} - \frac{1+\theta}{2} \cdot \frac{dc_t^2}{c_t^2}\right] \\ &+ c_t^{-\theta}(\theta-\gamma) \left[(1-\gamma)V_t\right]^{\frac{\theta-\gamma}{1-\gamma}} \left[\frac{dV_t}{(1-\gamma)V_t} + \frac{1}{2}(\theta-1)\frac{dV_t^2}{\left[(1-\gamma)V_t\right]^2}\right] \\ &+ \left(-\theta c_t^{-\theta}\right) \left[\frac{dc_t}{c_t} - \frac{1+\theta}{2} \cdot \frac{dc_t^2}{c_t^2}\right] (\theta-\gamma) \left[(1-\gamma)V_t\right]^{\frac{\theta-\gamma}{1-\gamma}} \left[\frac{dV_t}{(1-\gamma)V_t} + \frac{1}{2}(\theta-1)\frac{dV_t^2}{\left[(1-\gamma)V_t\right]^2}\right]\Bigg) \\ =& f_1(c_t, b_t, V_t)\Bigg(-\theta\left[\frac{dc_t}{c_t} - \frac{1+\theta}{2} \cdot \frac{dc_t^2}{c_t^2}\right] + \frac{\theta-\gamma}{1-\gamma}\left[\frac{d\left[(1-\gamma)V_t\right]}{(1-\gamma)V_t} + \frac{1}{2} \cdot \frac{\theta-1}{1-\gamma} \cdot \frac{\left(d\left[(1-\gamma)V_t\right]\right)^2}{\left[(1-\gamma)V_t\right]^2}\right] \\ &- \theta\frac{\theta-\gamma}{1-\gamma} \cdot \frac{dc_t}{c_t} \cdot \frac{d(1-\gamma)V_t}{(1-\gamma)V_t}\Bigg). \end{aligned}$$

Note that the preferences have the following differential representation

$$\frac{dV_t}{V_t} = -\frac{f(c_t, V_t)}{V_t}dt + \sigma_{V,y}dZ_t^y, \tag{C.6}$$

where $\sigma_V$ can be time varying. As a result, (also taking into account that $c_t = (1-\gamma_G)y_t$)

$$\begin{aligned}\frac{df_1}{f_1} =& -\theta\left[\frac{dc_t}{c_t} - \frac{1+\theta}{2}\cdot\frac{dc_t^2}{c_t^2}\right] + \frac{\theta-\gamma}{1-\gamma}\left[\frac{d\left[(1-\gamma)V_t\right]}{(1-\gamma)V_t} + \frac{1}{2}\cdot\frac{\theta-1}{1-\gamma}\cdot\frac{\left(d\left[(1-\gamma)V_t\right]\right)^2}{\left[(1-\gamma)V_t\right]^2}\right] - \theta\frac{\theta-\gamma}{1-\gamma}\cdot\frac{dc_t}{c_t}\cdot\frac{d\left[(1-\gamma)V_t\right]}{(1-\gamma)V_t} \\ =& \left[-\theta\left(g_y - \frac{1+\theta}{2}\sigma_y^2 + \frac{\theta-\gamma}{1-\gamma}\sigma_y\sigma_{V,y}\right) + \frac{\theta-\gamma}{1-\gamma}\left(-\frac{f_t}{V_t} + \frac{\sigma_{V,y}^2}{2}\cdot\frac{\theta-1}{1-\gamma}\right)\right]dt + \left(\frac{\theta-\gamma}{1-\gamma}\sigma_{V,y} - \theta\sigma_y\right)dZ_t^y.\end{aligned}$$

Next

$$\begin{aligned}de^{\int_0^t f_2(c_\tau,V_\tau)d\tau} &= e^{\int_0^t f_3(c_\tau,V_\tau)d\tau}d\int_0^t f_2(c_\tau,V_\tau)d\tau + \frac{1}{2}e^{\int_0^t f_2(c_\tau,V_\tau)d\tau}\left[d\int_0^t f_2(c_\tau,V_\tau)d\tau\right]^2 \\ &= e^{\int_0^t f_2(c_\tau,V_\tau)d\tau}f_2(c_t,V_t)dt.\end{aligned}$$

Note that the last expression implies that $de^{\int_0^t f_2(c_\tau,V_\tau)d\tau}df_1(c_t,V_t) = 0$. As a result,

$$\begin{aligned}e^{nt}\kappa d\xi_t + \kappa\xi_t e^{nt}ndt &= de^{\int_0^t f_2(c_\tau,V_\tau)d\tau}f_1(c_t,V_t) + e^{\int_0^t f_2(c_\tau,V_\tau)d\tau}df_1(c_t,V_t) + de^{\int_0^t f_2(c_\tau,V_\tau)d\tau}df_1(c_t,V_t) \\ &= \kappa\xi_t e^{nt}\left[f_2(c_t,V_t)dt + \frac{df_1(c_t,V_t)}{f_1(c_t,V_t)}\right].\end{aligned}$$

Collecting previous results, we obtain

$$\begin{aligned}\frac{d\xi_t}{\xi_t} =& -ndt + f_1(c_t,V_t)dt + \frac{df_1(c_t,V_t)}{f_1(c_t,V_t)} \\ =& -\left[\rho + \theta g_y - \theta\frac{1+\theta}{2}\sigma_y^2 + \frac{\theta-\gamma}{1-\gamma}\left(\theta\sigma_y\sigma_{V,y} - \frac{\theta-1}{1-\gamma}\cdot\frac{\sigma_{V,y}^2+\sigma_{V,B}^2}{2}\right)\right]dt \\ &- \left(\theta\sigma_y - \frac{\theta-\gamma}{1-\gamma}\sigma_{V,y}\right)dZ_t^y.\end{aligned} \tag{C.7}$$

**Step 4: Riskless Rate.** No arbitrage implies that the price $p_t$ of any security that pays dividends $d_s$ to its holder equals

$$p_t = \frac{1}{\xi_t}\mathbb{E}_t\int_t^\infty \xi_s d_s ds. \tag{C.8}$$

The differential version of this equation is

$$0 = \xi_t d_t dt + \mathbb{E}_t\left[d(\xi_t p_t)\right]. \tag{C.9}$$

The safe bond is a security with the price of 1 and the dividend $r_t^s$. As a result,

$$\begin{aligned}0 &= \xi_t r_t^s dt + \mathbb{E}_t d\xi_t, \\ r_t^s &= -\frac{1}{dt}\mathbb{E}_t\left(\frac{d\xi_t}{\xi_t}\right) = \rho + \theta g_y - \frac{\theta(\theta+1)}{2}\sigma_y^2 + \frac{\theta-\gamma}{1-\gamma}\left(\theta\sigma_y\sigma_{V,y} - \frac{\theta-1}{1-\gamma}\cdot\frac{\sigma_{V,y}^2+\sigma_{V,B}^2}{2}\right).\end{aligned}$$

First, we have two formulas for $dV_t$ in equations (**??**) and (C.6). By equating the diffusion terms in these two formulas, we get

$$\frac{1-\gamma}{\gamma}\kappa_t^x dZ_t^y = \sigma_{V,y} dZ_t^y.$$

Because the last expression has to hold for all realizations of shocks, it must be true that

$$\sigma_{V,y} = \frac{1-\gamma}{\gamma}\kappa_t^x. \tag{C.10}$$

Second, we have two expressions for $d\xi_t$ in equations (C.1) and (C.7)

$$-\kappa_t^x dZ_t^y = -\left(\theta\sigma_y - \frac{\theta-\gamma}{1-\gamma}\sigma_{V,y}\right) dZ_t^y,$$

Again, because the last expression has to hold for all realizations of shocks, we obtain

$$\kappa_t^x = \theta\sigma_y - \frac{\theta-\gamma}{1-\gamma}\sigma_{V,y}, \tag{C.11}$$

Equations (C.10) and (C.11) lead to

$$\sigma_{V,y} = (1-\gamma)\sigma_y.$$

As a result, the riskless rate is

$$\begin{aligned} r_t^s &= \rho + \theta g_y - \frac{\theta(\theta+1)}{2}\sigma_y^2 + \frac{\theta-\gamma}{1-\gamma}\left(\theta\frac{\sigma_y}{\sigma_{V,y}} - \frac{\theta-1}{2(1-\gamma)}\right)\sigma_{V,y}^2 \\ &= \rho + \theta g_y - \gamma\frac{(\theta+1)}{2}\sigma_y^2. \end{aligned}$$

**Step 5: the risky asset price.**

$$\begin{aligned} q_t =& \mathbb{E}_t \int_t^\infty \frac{\xi_s}{\xi_t} y_s ds \\ =& \mathbb{E}_t \int_t^\infty \frac{\overline{\pi}_s e^{-ns}}{\overline{\pi}_t e^{-nt}} y_s ds \\ =& c_t^\gamma \mathbb{E}_t \int_t^\infty e^{\left\{\frac{\theta-\gamma}{1-\gamma}\left[\rho-n-(1-\theta)\left(g_y-\frac{\gamma\sigma_y^2}{2}\right)\right]-(\rho-n)(1-\gamma)\right\}\frac{s-t}{1-\theta}} c_s^{-\gamma}\left[\rho-n-(1-\theta)\left(g_y-\frac{\gamma\sigma_y^2}{2}\right)\right]^{-\frac{\theta-\gamma}{1-\theta}} e^{-n(s-t)} y_s ds \\ =& y_t\left[\rho-n-(1-\theta)\left(g_y-\frac{\gamma\sigma_y^2}{2}\right)\right]^{-\frac{\theta-\gamma}{1-\theta}} \\ & \cdot \int_t^\infty e^{\left\{\frac{\theta-\gamma}{(1-\theta)(1-\gamma)}\left[\rho-n-(1-\theta)\left(g_y-\frac{\gamma\sigma_y^2}{2}\right)\right]-\frac{(\rho-n)(1-\gamma)}{1-\theta}-n+(1-\gamma)\left(g_y-\frac{\gamma\sigma_y^2}{2}\right)\right\}(s-t)} ds. \end{aligned}$$