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Abstract

This paper develops a multi-sector endogenous growth model that includes an innovation network, which captures intrasectoral as well as heterogeneous intersectoral knowledge flows. We analyze the importance of sectors (nodes) and directed knowledge linkages (edges) in the innovation network by their contribution to the growth of knowledge in this economy. We show that the growth rate of knowledge is equal to the spectral radius of the innovation network. We also demonstrate that a sector's importance to growth (``key sectors'') is related to its positions in both the downstream and upstream technology network. Finally, the importance of a knowledge linkage is characterized by both the upstream centrality of its source sector, the downstream centrality of its target sector and the strength of knowledge flows from the source sector to the target sector.

JEL Classification: D85, E2, O4

Keywords: innovation networks, Endogenous Growth, Key players

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Key Sectors in Endogeneous Growth

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This paper develops a multi-sector endogenous growth model that includes an innovation network, which captures intrasectoral as well as heterogeneous intersectoral knowledge flows. We analyze the importance of sectors (nodes) and directed knowledge linkages (edges) in the innovation network by their contribution to the growth of knowledge in this economy. We show that the growth rate of knowledge is equal to the spectral radius of the innovation network. We also demonstrate that a sector's importance to growth ("key sectors") is related to its positions in both the downstream and upstream technology network. Finally, the importance of a knowledge linkage is characterized by both the upstream centrality of its source sector, the downstream centrality of its target sector and the strength of knowledge flows from the source sector to the target sector.

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Key words: Endogenous growth, innovation network, knowledge spillovers, key sectors, key links, social optimum.

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We are like dwarfs sitting on the shoulders of giants. We see more, and things that are more distant, than they did, not because our sight is superior or because we are taller than they, but because they raise us up, and by their great stature add to ours.

— John of Salisbury

1. Introduction

Knowledge spillovers have been the driving engine of technological advancements. A pronounced feature of technology development nowadays is the cross-disciplinary application of knowledge. For instance, the development of new techniques such as X-ray diffraction and chromatography in chemistry have been applied to the analysis of metabolic pathways of the cell in biology. At the same time, studies on molecular biology, in turn, promote a better understanding of the chemistry of the cell, which then benefits research in medicine. Technological advancements in one area depend on breakthroughs in other areas. For example, Acemoglu et al. (2016b) empirically show that patent growth in upstream technology fields has strong predictive power on future downstream innovation given the preexisting network structure measured by intersectoral patent citations. Despite this empirical finding, several important questions remain: How does the structure of the innovation network relate to the behavior of the economy? How can we measure the importance of a sector given that each sector contributes to not only their own but also to others' development? What is the implication of the interdependence of different sectors' technological progress for policy design? This paper provides one of the first attempts to answer these questions.

We develop an endogenous growth model a la Romer (1990) with multiple sectors in the economy. Individual firms specialize in one sector, and try to apply preexisting knowledge from their own sectors as well as other sectors to conduct innovation. Consequently, aggregate knowledge flows from one sector to another depending on the strength of intersectoral knowledge spillovers and the resources devoted to innovation, which are endogenously determined in equilibrium. In this setup, the whole economy can be perceived as an innovation network with each sector representing a node and intersectoral knowledge flows representing directed edges.

The first theoretical finding of our paper shows the equivalence between the growth rate of the total knowledge stock and the spectral radius of the innovation network. This finding provides a parsimonious way for us to analyze the network economy because we can relate the impact of any change in the network structure to changes in the spectral radius. We apply this idea to examine the importance of nodes and edges in the network. The importance of a(an) node(edge) is measured by the reduction of the spectral radius if a(an) node(edge) is removed from the network. We find that a node's importance is related to both its centrality in the upstream innovation network and its centrality in the downstream innovation network. Higher centrality, which implies more central positions, in either the downstream network or the upstream network for a sector indicates that this sector is of great importance in the network. The intuition is that each sector serves as a knowledge distributor in the upstream innovation network and a knowledge consumer in the downstream innovation network. A sector with high centrality in the upstream network produces knowledge flows to other important sectors. On the other hand, a sector with high downstream centrality functions like a hub of knowledge applications, which absorbs and consumes knowledge from other sectors. The removal of either type of sectors will severely impair the production of knowledge, thus leading to a substantial slowdown of knowledge growth. We show that the importance of an edge is determined by the source sector's centrality in the upstream network, the target sector's centrality in the downstream network and the knowledge flows from the source sector to the target sector.

The complex interaction of sectoral innovation gives rise to the challenge of policy design. To see this, we solve a social planning problem where a planner optimally allocate labor between innovation and production within a sector and across sectors. The decentralized equilibrium of the baseline model features a constant ratio of the number of innovation workers to the number of production workers, which is the same for every sector. In contrast, this ratio is sector-specific in the centralized economy, which depends on the product of the upstream centrality and the downstream centrality, and a sector's output share. Consequently, optimal policies involve sector-specific subsidies (taxes).

Our paper contributes to several strands of literature. First, it is related to the large literature on endogenous growth.[1] Most of the previous studies on endogenous growth have focused on a representative sector and ignored the potential heterogeneous intersectoral knowledge spillovers. The only exception is Cai and Li (2019). Their paper explicitly models intersectoral knowledge linkages and uses the model to explain stylized facts such as research firms' choices of entry into different industries and heterogeneous research intensities across sectors. Our paper has a distinct focus. We aim to disentangle the connection between the structural property of each sector in the innovation network and their importance in the process of knowledge accumulations. This is, to the best of our knowledge, the first study that tries to investigate this relationship in the literature.

Our paper is also related to the large literature on social and economic networks.[2] Among them, Ballester et al. (2006) propose a measure, based on a player's centrality and her contribution to

[1]See, for example, Aghion and Howitt (1992), Aghion et al. (1997), Grossman and Helpman (1991), Kortum (1997), Klette and Kortum (2004), Romer (1990), Jones (1995), Acemoglu et al. (2018), Cai and Li (2019) and Akcigit and Kerr (2018). For an overview, see Acemoglu (2009).

[2]For overviews, see Jackson (2008), Jackson and Zenou (2015) and Jackson et al. (2017).

the centrality of the others, to capture the key players in a network. We obtain a similar result since we also demonstrate that the centrality of a sector is related to its importance in the network. However, our paper examines an innovation network whose structure is endogenously determined by the equilibrium allocation of innovation workers across sectors, while their paper treats the network structure as given and builds their analysis upon this premise.

Finally, our paper is related to the growing theoretical and empirical literature on the role of production networks in macroeconomics.[3] The network is described through input–output linkages and this literature has developed the theoretical foundations for the role of input–output linkages as a shock propagation channel and as a mechanism for transforming microeconomic shocks into macroeconomic fluctuations. The seminal paper by Acemoglu et al. (2012) characterizes the conditions under which input–output linkages in the economy can generate sizable aggregate fluctuations from purely idiosyncratic shocks. Some of these results have been tested empirically at the firm level (Barrot and Sauvagnat (2016), Carvalho et al. (2020)) and the industry level (Acemoglu et al. (2016a)). More recently, a small but growing literature has focused on developing a joint theory of production and endogenous network formation so that, after the shocks, firms can respond to changes in economic conditions by altering their trading partners (see e.g., Atalay et al. (2011) and Acemoglu and Azar (2020)). Our paper is related to this literature since we also endogeneize the network structure in a macroeconomic framework. However, there are many differences. First, instead of an input–output network, we have an innovation network where we study knowledge spillovers between and within sectors. Second, we have a dynamic model as we examine how the network structure affects the growth of the economy. Third, we determine the key sectors by studying the economic consequences in terms of growth rate if these sectors would disappear. We believe that we are among the first to introduce an explicit network analysis in an endogeneous growth model.

The rest of the paper is organized as follows. Section 2 presents the basic economic environment and characterize the equilibrium. Section 3 analyzes the main theoretical results by determining the growth of of the innovation network, the key sectors and the key technological linkages for growth. In Section 4, we present some numerical results. Finally, Section 5 cncludes. All proofs can be found in the Appendix.

[3]For overviews of this literature, see Carvalho (2014) and Carvalho and Tahbaz-Salehi (2019).

2. Model

2.1. Preferences and Final Good Production

There is a representative household who consumes a final consumption good $C(t)$ at each period of time t and has logarithmic preferences given by:

$$U = \int_0^\infty e^{-\rho t} \ln C(t) dt, \quad \rho > 0, \tag{1}$$

where ρ is the discount rate. The household is made of a continuum of individuals of measure one. Each member is endowed with one unit of labor that is supplied inelastically either to produce intermediate goods or to conduct R&D. The division of labor will be discussed in detail below. Individuals have access to a risk-free bond with interest rate $r(t)$, subject to the following lifetime household budget constraint

$$\int_0^\infty q(t)C(t)dt = x(0) + \int_0^\infty q(t)w(t)dt, \tag{2}$$

where $q(t) \equiv exp(-\int_0^t r(s)ds)$ is the intertemporal interest rate, $x(0)$ is the initial wealth that this household holds, and $w(t)$ is the wage rate that the household earns. The representative household chooses $C(t)$ that maximizes (1) under the budget constraint (2). This leads to the standard consumption Euler equation:

$$\frac{\dot{C}(t)}{C(t)} = r(t) - \rho. \tag{3}$$

The economy consists of N technological sectors denoted by $\mathcal{N} = \{1, 2, \cdots N\}$. The final good $C(t)$ consumed by the representative household at time t is produced by a producer and is an aggregate of the N sectoral goods according to the following CES function:

$$Y(t) = \left[\sum_{k=1}^{N} \alpha_k Y_k(t)^{\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}, \tag{4}$$

where α_k refers to the relative importance of sector k in the production of final good, and σ is the elasticity of substitution across sectors. $Y_k(t)$ represents a composite of different varieties produced by sector k at time t. The final good producer maximizes the following profit function by choosing the quantity of the sectoral good $Y_k(t)$, that is

$$Y(t) - \sum_{k=1}^{N} P_k(t) Y_k(t), \tag{5}$$

where $P_k(t)$ is the price index for sector k. As it is standard, from this maximization problem, we obtain an inverse demand function in sector k, which is equal to:

$$Y_k(t) = \left[\frac{\alpha_k}{P_k(t)} \right]^{\sigma} Y(t). \tag{6}$$

In each sector k, there is a monopolistic competition between a continumm of firms. We refer to firm ki the firm i that produces a variety i in sector k. The composite good in sector k is given by

$$Y_k(t) = \left[\int_0^{A_k(t)} y_{ki}(t)^{\frac{\eta-1}{\eta}} di \right]^{\frac{\eta}{\eta-1}}, \quad \forall k \in \mathcal{N}, \tag{7}$$

where $y_{ki}(t)$ is the quantity of variety i demanded in sector k at time t and $\eta > 1$ is the elasticity of substitution within sector k.[4] When $\eta \to \infty$, varieties are perfect substitutes. We follow Romer (1990) to interpret $A_k(t)$ as the number of designs for different varieties in sector k at time t, which can grow over time without bound. Conceptually, $A_k(t)$ captures the *knowledge stock* in sector k, the dynamics of which will be discussed in the next section.

Firms produce with labor only. Each firm/individual ki has access to a linear production technology, given by:

$$y_{ki}(t) = l_{ki}(t), \quad \forall i \in A_k(t), \, \forall k \in \mathcal{N} \tag{8}$$

where $l_{ki}(t)$ is the labor input that each firm ki needs to produce $y_{ki}(t)$ at time t. Firm ki needs one unit of labor to produce one unit of its variety i. Given the wage rate $w(t)$, each firm ki, at time t, chooses the price $p_{ki}(t)$ of their product/variety that maximizes her profit:

$$\pi_{ki}(t) = p_{ki}(t) y_{ki}(t) - w(t) l_{ki}(t), \tag{9}$$

subject to the inverse demand function

$$y_{ki}(t) = \left[\frac{P_k(t)}{p_{ki}(t)} \right]^{\eta} Y_k(t), \tag{10}$$

where

$$P_k(t) = \left(\int_0^{A_k(t)} p_{ki}(t)^{1-\eta} di \right)^{\frac{1}{1-\eta}}. \tag{11}$$

The price that a firm ki charges is a constant markup over the marginal cost of production, i.e., $p_{ki}(t) = \frac{\eta}{\eta-1} w(t)$. Thus,

$$P_k(t) = \frac{\eta}{\eta - 1} w(t) [A_k(t)]^{\frac{1}{1-\eta}}. \tag{12}$$

Plugging the value of $p_{ki}(t)$ into the profit (9) yields:

$$\pi_{ki}(t) = \frac{1}{\eta - 1} w(t) l_{ki}(t). \tag{13}$$

[4] We use the subscripts k, l, m for sectors and subscripts i, j for firms/varieties.

2.2. Research and Development

Besides producing intermediate goods, individuals/firms/entrepreneurs also engage in R&D in order to expand the existing knowledge stock and, thereby, facilitate innovations. Specifically, innovators benefit from two types of knowledge: knowledge in the sector they conduct innovation and knowledge from other sectors. Put differently, an increase in knowledge in a given sector leads to *intrasectoral* as well as *intersectoral knowledge spillovers*. The average strength of knowledge *intersectoral* spillovers from sector l to sector k is denoted by s_{kl} while $s_{kk} > 0$ referred to as the average strength of *intrasectoral* knowledge spillovers for sector k. Note that $s_{kl} \geq 0, \ \forall k, l$, and in general $s_{kl} \neq s_{lk}, \forall i \neq j$, as knowledge spillovers from sector l to sector k may be different from those from sector k to sector l.

Denote by $L_k^A(t)$, the number of firms that conduct innovations in sector k at time t. Then, the evolution of the knowledge stock $A_k(t)$ in sector k is assumed to be

$$\dot{A}_k(t) = \theta_k L_k^A(t) \sum_{l=1}^{N} s_{kl} A_l(t), \quad \forall k \in \mathcal{N}, \tag{14}$$

where θ_k captures the sectoral research productivity that governs the difficulty of conducting R&D in sector k. The sectoral labor input for innovations $L_k^A(t)$ will be determined in equilibrium, and as we will show later, without population growth, $L_k^A(t)$ will be constant. In (14), $\dot{A}_k(t)$ is the growth rate of new ideas for the whole sector k. This formulation of knowledge accumulation is an extension of Romer (1990), by including the sectoral differentials and the possibility of knowledge spillovers across sectors. It is useful to rewrite equation (14) by dividing both sides by $A_k(t)$. We obtain:

$$g_k^A(t) = \theta_k L_k^A(t) \sum_{l=1}^{N} s_{kl} \frac{M_l(t)}{M_k(t)}, \quad \forall k \in \mathcal{N}, \tag{15}$$

where $g_k^A(t) := \dot{A}_k(t)/A_k(t)$, $M_k(t) := A_k(t)/A(t)$ and $A(t) := \sum_{k=1}^{N} A_k(t)$. As we will discuss later, $M_k(t)$, which is sector k's share of the total knowledge stock, is constant in equilibrium, and so is the growth rate.

Let us, now, define the *innovation network* $\mathbf{\Gamma}$, whose *weighted* and *asymmetric* adjacency $(N \times N)$ matrix is denoted by $\mathbf{\Gamma}$[5] with elements $\gamma_{kl} := \theta_k s_{kl} L_k^A$ indicate the average knowledge flows from sector l to sector k, resulting from endogenous research efforts to applying innovations from sector l to sector k. This innovation network captures both the exogenous bilateral knowledge linkages (s_{kl}) and sectoral research productivity (θ_k) and the endogenous allocation of human resources for innovations across sectors (L_k^A). Since s_{kl} may be different from s_{lk}, the network is *directed*. The

[5] For simplicity, we denote by $\mathbf{\Gamma}$ both the innovation network and its adjacency matrix.

upstream network for a sector k is defined by all sectors that produce knowledge spillovers to sector k and is captured by s_{kl}, for all $l \neq k$. . The *downstream network* for a sector k is defined by all sectors that receive knowledge spillovers from sector k and is captured by s_{lk}, for all $l \neq k$.

The structure of the innovation network $\mathbf{\Gamma}$ determines the speed of knowledge accumulation for every sector. This implies that equation (15) can be written as:

$$g_k^A(t) = \sum_{l=1}^{N} \gamma_{kl} \frac{M_l(t)}{M_k(t)}, \quad \forall k \in \mathcal{N}. \tag{16}$$

We are looking for an equilibrium featuring the balanced growth path where knowledge in all sectors grows at the same rate. Such an equilibrium cannot exist without conditions. A simple example is when the innovation network reduces to a diagonal matrix. This corresponds to an economy where sectoral growth is independent of each other. In this paper, we rule out this possibility. Instead, we want to investigate a model where all sectors have either a direct or indirect impact on all other sectors. The following definition formalizes this idea.

Definition 1. *An innovation network* Γ *is strongly connected if,* $\forall k, l$*, there exists a sequence of edges such that* $\gamma_{km_1}\gamma_{m_1m_2}\cdots\gamma_{m_nl} > 0$.

A strongly connected network ensures that intersectoral knowledge spillovers affect all sectors directly or indirectly. Indeed, even if a sector does not directly benefit from the knowledge advancement of another sector, the knowledge spillovers from the latter still pass through to the former via a third sector (or a sequence of sectors). The assumption of strongly connected innovation network is a sufficient condition to guarantee the existence of this higher-order effect, which is summarized by the following lemma.

Lemma 1. *Assume that the innovation network* Γ *is strongly connected. Then, in equilibrium, the knowledge stock of each sector grows at the same rate and is equal to the growth rate of the total knowledge stock of the economy, that is*

$$g^A = g_k^A, \quad \forall k \in \mathcal{N}. \tag{17}$$

For the rest of this paper, we maintain the assumption of a strongly connected innovation network.

To close the economy, we still need to specify the entry of new firms. In our model, any individual/entrepreneur can create a new firm by conducting R&D and designing a new blueprint for a product. A firm ki that designs a new blueprint will enjoy the monopoly rights forever. The value of a blueprint $v_{ki}(t)$ of firm ki introduced at t is thus

$$v_{ki}(t) = \int_t^{\infty} \frac{q(\tau)}{q(t)} \pi_{ki}(\tau) d\tau, \quad \tau \geq t. \tag{18}$$

where $\pi_{ki}(t)$ denotes the flow profit of firm ki (i.e., firm producing variety i in sector k) at date t and is given by (13), and $q(\tau)/q(t)$ discounts the flow profit from τ to t. Given the free mobility of labor between the production sector and the research sector, an individual/entrepreneur should be indifferent between working and innovating. Thus, for each period of time t, the free-entry condition is equal to:

$$v_{ki}(t) \sum_{l=1}^{N} \theta_k s_{kl} A_l(t) = w(t), \quad \forall k \in \mathcal{N}. \tag{19}$$

When a firm/person enters a sector k at time t producing variety i, the expected benefit of participating in innovation is equal to her lifetime discounted profit $v_{ki}(t)$ times the benefits of the intra and intersectoral spillover effects of the innovation $\sum_{l=1}^{N} \theta_k s_{kl} A_l(t)$. The last term is obtained by dividing the right-hand side of (14), which gives the growth rate of new ideas for a whole sector k, by L_k^A, in order to obtain the per person innovation rate. The free-entry condition (19) at time t captures the fact that a person/firm should be indifferent between working in the manufacturing sector and obtain a wage $w(t)$ and engaging in innovation. Note that we assume positive entry in equilibrium.

2.3. Equilibrium

Let us, now, characterize the equilibrium of this economy. We focus on the steady state where knowledge in all sectors grows at the constant rate g^A.

In equilibrium, all firms in a given sector k produce the same amount of intermediate goods, and, therefore, hire the same number of workers at any point of time t. Thus, using (8), all firms in sector k will have the same production technology, which is given by:

$$y_{ki}(t) = l_{ki}(t) = \frac{L_k^Y}{A_k(t)}. \tag{20}$$

where L_k^Y is the total labor used in sector k, and, as above, $A_k(t)$ is the knowledge stock in sector k at time t. Plugging this firm labor input into the sectoral output (7) yields

$$Y_k(t) = [A_k(t)]^{\frac{1}{\eta-1}} L_k^Y. \tag{21}$$

In equilibrium, the output growth in sector k is proportional to the growth of the knowledge stock, that is $g_k^Y = \frac{1}{\eta-1} g_k^A = \frac{1}{\eta-1} g^A$. The output growth is simply the knowledge growth adjusted for the within sector elasticity of substitution. Using (4), we can derive the growth rate of the final output, which is equal to the growth rate of the sectoral output g_k^Y, that is

$$g^Y = \frac{1}{\eta - 1} g^A. \tag{22}$$

Since the final good sector is perfectly competitive, producers make zero profit, which implies that $Y(t) = \sum_{k=1}^{N} P_k(t)Y_k(t)$. Using this relationship together with sectoral price index (12) and (21), we obtain:

$$Y(t) = \frac{\eta}{\eta - 1} w(t) L^Y, \tag{23}$$

where $L^Y = \sum_{k=1}^{N} L_k^Y$. A direct implication of this result is that the output growth is equal to the wage growth, $g^Y = g^w$.

For the goods market to clear, we need to have that the final output equals total consumption, that is $Y(t) = C(t)$. From the consumption Euler equation (3), we have

$$g^Y = g^C = r - \rho \tag{24}$$

where g^C is consumption growth. Now, we can characterize the evolution of the value of a new blueprint for a firm. First, recall that, in equilibrium, all firms within a sector employ the same amount of labor $L_k^A / A_k(t)$ at time t. Therefore, the profit of a firm ki is given by:

$$\pi_{ki}(t) = \frac{1}{(\eta - 1)} \frac{w(t) L_k^Y}{A_k(t)} \tag{25}$$

As a result, in equilibrium, the growth rate of the profit per product is equal to the difference between the growth rate of the wage and the growth rate of the sectoral knowledge stock. That is,

$$g^\pi = g^w - g^A = g^Y - g^A = \frac{1}{\eta - 1} g^A - g^A = \left(\frac{2 - \eta}{\eta - 1}\right) g^A, \quad \eta > 1 \tag{26}$$

Notice that when there is relatively low substitution between different varieties within a sector, i.e., $1 < \eta < 2$, the per-product profit growth rate is positive. On the other hand, when there is relatively high substitution between varieties, i.e., $\eta > 2$, the per-product profit experiences negative growth.

Along the balanced growth path, the per-product profit grows at the same rate g^π. Combining this with (18), the value of a new product i introduced at time t in sector k can be written as

$$v_{ki}(t) = \int_t^\infty e^{-r(\tau - t)} \pi_{ki}(t) e^{g^\pi(\tau - t)} d\tau = \frac{\pi_{ki}(t)}{r - g^\pi}, \quad \tau \geq t \tag{27}$$

where $r - g^\pi = \rho + g^A$.

It remains to determine the allocation of labor across sectors and within a sector for production and innovation. First, combining the free-entry condition (19) with (25) and (27), we obtain:

$$\frac{w(t) L_k^Y}{(\eta - 1) A_k(t)(\rho + g^A)} \sum_{l=1}^{N} \theta_k s_{kl} A_l(t) = w(t). \tag{28}$$

Taking into account the fact that $g^A / L_k^A = \sum_{l=1}^{N} \theta_k s_{kl} A_l(t) / A_k(t)$, the above equation can be written as

$$\frac{1}{\eta - 1} \frac{L_k^Y}{L_k^A} \frac{g^A}{\rho + g^A} = 1, \quad \forall k \in \mathcal{N}. \tag{29}$$

A higher growth rate of the knowledge stock g^A is associated with a higher innovation to production-labor ratio L_k^A/L_k^Y. Moreover, as varieties become more substitutable, that is, a higher η, the profit generated from a new variety declines. As a result, individuals/firms have less incentive to engage in innovations, and thus, allocate more labor for production.

Next, we characterize the division of labor among sectors for production and innovation. Equation (29) shows that the ratio of production to innovation labor is sectoral independent and is the same across sectors. Thus, the allocation of production workers across sectors is proportional to the allocation of innovation workers across sectors.

Lemma 2. *Along the balanced growth equilibrium, the ratio of labor allocation across sectors for production is equal to the ratio of labor allocation across sectors for innovation. We have:*

$$\frac{L_k^Y}{L_l^Y} = \frac{L_k^A}{L_l^A}, \quad \forall k \neq l. \tag{30}$$

In addition, it can be shown that

$$\frac{L_k^Y}{L_l^Y} = \left(\frac{\alpha_k}{\alpha_l}\right)^{\sigma} \left(\frac{M_k}{M_l}\right)^{\frac{1-\sigma}{1-\eta}}, \tag{31}$$

$$\frac{L_k^A}{L_l^A} = \frac{\sum_{m=1}^N \theta_l s_{lm} M_m/M_l}{\sum_{m=1}^N \theta_k s_{km} M_m/M_k}, \tag{32}$$

where $M_m = A_m/A$ is sector m's share of total knowledge stock, and the vector $\mathcal{M}(t) = \{M_k(t)\}_{k\in\mathcal{N}}$ are solutions to the following system of equations

$$\left(\frac{\alpha_k}{\alpha_1}\right)^{\sigma} \left(\frac{M_k}{M_1}\right)^{\frac{\eta-\sigma}{\eta-1}} = \frac{\sum_{m=1}^N \theta_k s_{km} M_m}{\sum_{m=1}^N \theta_1 s_{1m} M_m}, \quad \forall k \neq 1, \tag{33}$$

and

$$\sum_{k=1}^N M_k = 1. \tag{34}$$

We can now state the definition of equilibrium of this economy.

Definition 2. *(Balanced Growth Path Equilibrium) A balanced growth path (BGP) of this economy is an equilibrium path in which consumption and final output grow at the same rate, which is proportional to the knowledge growth rate. It consists of the following time paths $C(t)$, $Y(t)$, $\{l_{ki}(t)\}_{k\in\mathcal{N}}$, $\{y_{ki}(t)\}_{k\in\mathcal{N}}$, $\{p_{ki}(t)\}_{k\in\mathcal{N}}$, $\{\pi_{ki}(t)\}_{k\in\mathcal{N}}$, $\{v_{ki}(t)\}_{k\in\mathcal{N}}$, $\{L_k^Y(t)\}_{k\in\mathcal{N}}$, $\{L_k^A(t)\}_{k\in\mathcal{N}}$, $\{M_k(t)\}_{k\in\mathcal{N}}$, $\{A_k(t)\}_{k\in\mathcal{N}}$, $\{P_k(t)\}_{k\in\mathcal{N}}$, $\{pi_{ki}(t)\}_{k\in\mathcal{N}}$, $\{v_{ki}(t)\}_{k\in\mathcal{N}}$, g_k^A, g^A, g^Y such that:*

1. *The representative household chooses $C(t)$ that maximizes utility (1) under the budget constraint (2).*

2. *Each final good producer k chooses sectoral good $Y_k(t)$ that maximizes profit (5) subject to the production function (4) and the sectoral price index $P_k(t)$ given by (12).*

3. *Each individual firm ki chooses $l_{ki}(t)$ the amount of labor for production, $y_{ki}(t)$ the quantity of variety i produced and $p_{ik}(t)$ the price of this variety that maximize her profit (9) subject to the production technology (8) and the inverse demand (10). Equations (13) and (18) determine $\pi_{ki}(t)$ and $v_{ki}(t)$, respectively.*

4. *The allocation of labor satisfies (29), (30), (31), (32), and the aggregate labor market clearing condition $\sum_{k=1}^{N}(L_k^A + L_k^Y) = 1$.*

5. *The shares of sectoral knowledge stock $\{M_k(t)\}_{k\in\mathcal{N}}$ satisfy (33) and (34).*

6. *The sectoral knowledge stocks $\{A_k(t)\}_{k\in\mathcal{N}}$ satisfy (14).*

7. *The good market clears, that is $C(t) = Y(t)$.*

8. *The growth rate of the sectoral knowledge stock g_k^A, total knowledge stock g^A, and output g^Y satisfy (15), (17) and (22), respectively.*

3. Network growth

3.1. Growth in the innovation network

We would, now, like to characterize the relationship between the equilibrium growth rate of the knowledge stock and the structure of the innovation network. We provide a sufficient statistics that summarizes the impact of the innovation network on the knowledge accumulation. First, we can rewrite equation (14) in a matrix form

$$\dot{\mathcal{A}}(t) = \mathbf{\Gamma}\mathcal{A}(t), \tag{35}$$

where $\mathcal{A}(t) = \{A_k(t)\}_{i\in\mathcal{N}}$ denotes the vector of sectoral knowledge stocks, and $\mathbf{\Gamma}$ is the $(N \times N)$ weighted and asymmetric adjacency matrix representative of the innovation network whose elements are $\gamma_{kl} := \theta_k s_{kl} L_k^A$. Equation (35) highlights the intrinsic nature of interdependence of knowledge accumulations across sectors. Specifically, the evolution of the knowledge stock in any sector depends on the dynamics of their neighboring sectors, which themselves are dependent of the dynamics of their own neighbors, and so forth.

Proposition 1. *Given that the spectral radius of $\mathbf{\Gamma}$ is non-zero, the equilibrium growth rate of the total knowledge stock is*

$$g^A = \lambda^*, \tag{36}$$

where λ^* is the spectral radius of $\mathbf{\Gamma}$. Further, the equilibrium sectoral knowledge shares $\mathcal{M} = \{M_i\}_{i \in \mathcal{N}}$ satisfy

$$\lambda^* \mathcal{M}^T = \mathbf{\Gamma} \mathcal{M}^T, \tag{37}$$

where $\mathcal{M}^T = (M_1,\ M_2\ \cdots\ M_N)^T$ is the transpose of $\mathcal{M}$.

This proposition sheds light on how the structure of the innovation network shapes the economy. To understand the first part of this proposition, notice that (35) is a system of linear differential equations. Given that the spectral radius (or dominant eigenvalue) of $\mathbf{\Gamma}$ is non-zero, this system of equations admits a solution that expresses the knowledge stock in any sector as a function of a linear combination of exponential functions with power terms equal to the eigenvalues of $\mathbf{\Gamma}$. In the limit, when t goes to infinity, the spectral radius dominates the process of knowledge accumulations, and, thus, the growth rate converges to λ^*. This result offers an elegant and simple way of summarizing the information required from the innovation network to determine the speed of the knowledge accumulation.

The second part of this proposition reveals a profound connection between the relative size of a sector and its position in the downstream innovation network. First, notice that $\mathcal{M} = (M_1, \cdots\ M_N)$, where $M_k := A_k/A$, represents the *generalized eigenvector centrality* in the innovation network.[6] The generalized eigenvector centrality measures the importance of a node in a network (see Jackson (2008) for a definition and discussion). In particular, a node gets higher scores if it is connected with other high score nodes. In our context, sectors consume knowledge from both their own sectors and others to conduct innovation. The eigenvector centrality of a sector reflects its position in the innovation network that determines the knowledge inflows to the sector, which then determines the size of the sector.

3.2. Key sectors for growth

We have so far demonstrated how different sectors' positions in the innovation network are related to their relative sizes. We would, now, like to analyze how much a sector contributes to g^A the growth of the total knowledge stock. The way by which the contribution of a sector is measured here is by

[6]The usual eigenvector centrality is associated with the undirected adjacency network; the notion of generalized eigenvector centrality here is adjusted for our weighted directed network.

calculating the proportional reduction of the economic growth rate upon the removal of the sector. This is similar to the key player concept (Zenou, 2016). Specifically, we re-calculate the growth rate of the total knowledge stock holding everything constant except shutting down all channels of knowledge spillovers associated with this sector, be it knowledge spillovers into or from this sector. This involves changing the entries of the corresponding sector's row and column in matrix $\mathbf{\Gamma}$ into zeros. So, basically, the "key" sector is the sector whose removal leads to the highest reduction in economic growth.

In an economy with intersectoral knowledge spillovers, a sector's importance goes beyond its size. Specifically, a sectors plays two distinct roles. On the one hand, a sector absorbs and use knowledge from neighboring sectors for its own innovation, thereby serving as a *knowledge consumer*. As shown in Proposition 1, the relative size of a sector is captured by the centrality of the sector in the downstream innovation network. On the other hand, different sectors also produce knowledge that is used for innovation by other sectors. In this sense, each sector is a *knowledge producer* and its importance is captured by how much knowledge each sector produces and contributes to other sectors' innovations. These two functions together determine the total contribution of each sector to aggregate growth, as stated in the following proposition.

Proposition 2. *Define the importance of sector k as*

$$T_k = -\frac{\Delta\lambda_k^*}{\lambda^*}, \tag{38}$$

where $\Delta\lambda_k^*$ is the change of the aggregate growth rate when sector k is removed. For large enough N,

$$T_k = \frac{(\lambda^* - \gamma_{kk})V_k M_k}{\lambda^*(\mathcal{V}\mathcal{M}^T - V_k M_k)}, \tag{39}$$

where $\mathcal{V} = \{V_k\}_{k\in\mathcal{N}}$ satisfies

$$\lambda^*\mathcal{V} = \mathcal{V}\mathbf{\Gamma}, \tag{40}$$

and $\mathcal{M} = \{M_k\}_{k\in\mathcal{N}}$ is defined by (37).

This proposition puts forward the fact that a sector's position in both the *upstream* and the *downstream innovation network* matter for its contribution to the accumulation of the total knowledge stock. From (40), $\mathcal{V}$ is the vector of eigenvector centralities of individual sectors in the *upstream* innovation network. A key sector in the upstream innovation network is a sector that produces knowledge to other key sectors. In other words, a key sector in the upstream innovation network is at the origin of knowledge spillovers. We can compare this with $\mathcal{M}$, which captures the importance (in terms of centrality) of sectors in the *downstream* innovation network. As a knowledge consumer, sectors with higher centrality are the hub of knowledge application, and are more active in applying knowledge from other sectors for innovations. These two forces jointly determine the contribution of individual sectors to aggregate growth and determine which are the key sectors.

In summary, the *upstream network* for a sector k is the source of knowledge flowing into this sector while the *upstream centrality* V_k of sector k is the contribution of this sector to its downstream sectors; it is the left eigenvector centrality of the innovation network (see (40)). The former (upstream network) refers to all sectors that produce knowledge spillovers to a given sector, while the latter (upstream centrality) refers to the position of this sector in the network. Since what matters in the upstream network is outward spillovers, the centrality of a sector in upstream network is defined as its contribution to all other sectors. Similarly, the *downstream network* for a sector k is defined by all sectors that receive knowledge spillovers from sector k while the *downstream centrality* M_k of sector k is the contribution of this sector to its upstream sectors; it is the right eigenvector centrality of the innovation network (see (37)). In this way, downstream centrality captures the role of a sector as a knowledge hub. A central sector should be one that benefits from either lots of other sectors or a few important sectors. Therefore, what matters is the inflow of knowledge to this sector.

To better understand this proposition, we can decompose (39) into two parts. The first part is given by $\frac{\lambda^* - \gamma_{kk}}{\lambda^*}$ while, the second one, is $\frac{V_k M_k}{\mathcal{V}\mathcal{M}^T - V_k M_k}$. The first part reflects how *intrasectoral* knowledge flows affect the sector's contribution to aggregate growth, while the second part captures the impact of the position of a sector in the upstream and the downstream innovation networks. Ceteris paribus, sectors with higher overall centrality in the upstream and downstream innovation network are more important contributors to the knowledge accumulation in the economy. On the other hand, for two sectors with the same overall centrality, sectors that are more self-contained (higher γ_{kk}) are less important in the network. This seems counter-intuitive. Shouldn't a sector with higher within sector knowledge flows also contributes more to the overall knowledge accumulation, holding everything else equal? This intuition is not correct because the contribution of the intrasectoral knowledge flows to growth has already been endogenized by the sector's centrality. In fact, both the upstream centrality and the downstream centrality are functions of γ_{kk}. Given a sector's overall centrality, higher γ_{kk} implies that a sector is more isolated in the network, thus contributes less to knowledge accumulation.

It is worth noting that, if the innovation matrix $\mathbf{\Gamma}$ was symmetric, that is $\gamma_{kl} = \gamma_{lk}, \forall k \neq l$, then the two types of centrality would be the same. In that case, each sector's downstream centrality would be equal to its upstream centrality. However, this is unlikely to be the case in general. Heterogeneity of bilateral knowledge spillovers is frequently documented by the past studies.[7]

[7]For example, semiconductor is commonly thought as an important technology that underlines the development of

3.3. Key technological linkages for growth

We, now, characterize the importance of a directed technological linkage.

Proposition 3. *Define the importance of a directed technological linkage from sector k to l by*

$$T_{kl} \equiv -\frac{\Delta\lambda^*_{kl}}{\lambda^*}, \tag{41}$$

where $\Delta\lambda_{kl}$ is the change in the aggregate growth rate when the edge kl is removed. For large enough N,

$$T_{kl} = \frac{\gamma_{kl} V_k M_l}{\lambda^* \mathcal{V}\mathcal{M}^T} \tag{42}$$

The importance of a specific technological linkage depends on the positions of the source sector and the target sector associated with this linkage as well as the strength of this technological linkage. Specifically, the contribution of a technological linkage to the knowledge accumulation is determined by three factors: the source sector's centrality in the *upstream* innovation network, V_k, which determines the origin of knowledge outflows; the target sector's centrality in the *downstream* innovation network, M_l; the strength of knowledge spillovers from the source sector to the target sector, $\gamma_{kl} \equiv \theta_k s_{kl} L^A_k$. The technological linkages associated with either higher source sector centrality or higher target sector centrality or stronger knowledge spillovers are the most important edges in the innovation network, and affect the rate of knowledge accumulation the most.

Proposition 3 provides a way to think about individual sectors' outward and inward knowledge spillovers. Indeed, by using the fact that $\sum_{l=1}^N \gamma_{kl} M_l = \lambda^* M_k$, we can see that the contribution of the inward knowledge spillovers to sector k to the overall knowledge accumulation is $\sum_{l\neq k} T_{kl} = \frac{(\lambda^*-\gamma_{kk})V_k M_k}{\lambda^* \mathcal{V}\mathcal{M}^T}$. Note that we do not include sector k in the summation to highlight that this formula captures the *intersectoral* spillovers. Similarly, combining the equivalence condition $\sum_{l=1}^N V_l \gamma_{lk} = \lambda^* V_k$ with equation (42) yields $\sum_{l\neq k} T_{lk} = \frac{(\lambda^*-\gamma_{kk})V_k M_k}{\lambda^* \mathcal{V}\mathcal{M}^T}$. This is the contribution of sector k's total outward knowledge spillovers to overall growth. Note that the total *inward-knowledge* spillovers to a sector and the total *outward-knowledge* spillovers from the sector contribute the same to the growth of the knowledge stock. It turns out that a sector's upstream centrality and downstream centrality are two sufficient statistics that summarize all the information required to determine a sector's outward-knowledge spillovers and inward-knowledge spillovers.

3.4. Changes in the network structure

We, now, explore how changes in the network structure affect the downstream and upstream centrality of sectors. In particular, we investigate the impact of an increase in s_{kl} on the different cen-

information technology, and not vice versa.

tralities. This can be interpreted as some random scientific discovery that increases the applicability of knowledge in a sector to innovation in another sector.

Proposition 4. *Given that $\eta > 1$ and $0 < \sigma < 1$, an increase in the strength of knowledge spillovers from sector l to sector k, that is $s_{kl}^{'} > s_{kl}$, leads to:*

(a) $M_k^{'} > M_k$ and $M_k^{'}/M_k > M_m^{'}/M_m$, $\forall m \neq k$;

(b) $L_k^{A'}/L_m^{A'} < L_k^A/L_m^A$, $\forall m \neq k$;

(c) $V_l^{'} > V_l$ and $V_l^{'}/V_l > V_m^{'}/V_m$, $\forall m \neq l$;

(d) $g^{A'} > g^A$ and $L^{A'}/L^{Y'} > L^A/L^Y$.

The first part of this proposition demonstrates heterogeneous changes of sectors' positions induced by a change of the network structure. Specifically, the strengthening of knowledge spillovers from sector l to sector k makes the target sector a more central one in the downstream network. Recall that the downstream centrality of a sector is equal to its share of the total knowledge stock, the higher centrality of sector k implies that this sector benefits from stronger knowledge spillovers.

Part (a) of this proposition highlights that growth in sector k's share is most pronounced among all sectors. At the same time, innovation labor reallocates relatively from sector k to other sectors, as shown in part (b). This result is more subtle than it appears at first glance. The increase in the knowledge spillovers from sector l to sector k leads to an increase in the growth rate of sector k, which results in an increase in sector k's knowledge stock share. A higher growth rate of sector k promotes the competition among incumbents in this sector, thus reducing the lifetime profit of a new patent. However, since sector k becomes more efficient in utilizing knowledge and conducting innovation due to higher capacity in absorbing knowledge from sector l, the arrival rate of new patents in sector k increases. The first effect dominates the second, and, therefore, the expected value of a new patent in sector k decreases. Consequently, entrants have less incentive to innovate in this sector. The free-entry condition compels entrepreneurs to innovate more intensively in other sectors. During this process, sector k generates additional knowledge spillovers to its neighboring sectors, which spill over knowledge to other sectors further away. In the new equilibrium, all sectors restore the same growth rate.

Part (c) constitutes the upstream counterpart of part (a). Sector l, the source sector of s_{kl}, experiences the largest percentage increase in upstream centrality, leading to the fact that the importance of sector l as an origin of knowledge spillovers rises. The last part of Proposition 4 states that a strengthening of knowledge spillovers from one sector to another increases the speed of knowledge accumulation, and reallocates labor from production to innovation. Indeed, a higher s_{kl} increases the productivity of sector k in innovation, which, then, produces higher intersectoral knowledge spillovers, and benefit all the other sectors in the network. Due to the higher productivity in innovation, to maintain the free-entry condition, labor has to flow from the production sector to the innovation sector. This reduces the expected value of innovation and makes sure that people are indifferent between working in the production sector and engaging in innovation.

3.5. Social optimal allocation of labor

The previous section characterizes the decentralized equilibrium of our multisectoral growth model. As demonstrated by Romer (1990), the model presented here may feature too little human capital devoted to research due to the externality created from the knowledge spillovers and the monopoly power in the intermediate good sector. In addition, the misallocation of human capital across sectors generates another channel that makes the equilibrium allocation of labor not optimal. Let us, now, determine the socially optimal allocation of labor.

The basic setup here is the same to what we had before, except for the fact that there is no private market for the intermediate goods. The social planner maximizes the representative household's lifetime utility subject to the output production, the knowledge production function and the resource constraint in the labor market. The planner's problem can, thus, be written as:

$$\max \int_0^\infty e^{-\rho t} \log C(t) dt,$$

subject to

$$\begin{aligned}
Y(t) &= \left[\sum_{k=1}^{N} \alpha_k \left(\int_0^{A_k(t)} y_{ki}^{\frac{\eta-1}{\eta}} di \right)^{\frac{\eta}{\eta-1}\frac{\sigma-1}{\sigma}} \right]^{\frac{\sigma}{\sigma-1}}, \\
\dot{\mathcal{A}}(t) &= \mathbf{\Gamma} \mathcal{A}(t), \\
L &= \sum_{k=1}^{N} \left(L_k^A + L_k^Y \right), \\
C(t) &= Y(t).
\end{aligned}$$

This optimization problem involves two steps. First, given the total sectoral labor input, the social planner determines the optimal allocation of labor between production and innovation *within* a sector. Second, the allocation of labor *across* sectors for innovation is determined. Given the results of the first step, the division of labor for innovation across sectors thus reveals the allocation of production labor across sectors. Denote the downstream centrality and the upstream centrality of this problem as $\widetilde{\mathcal{M}} = \{\widetilde{M}_k\}_{k\in\mathcal{N}}$ and $\widetilde{\mathcal{V}} = \{\widetilde{V}_k\}_{k\in\mathcal{N}}$, respectively.

Proposition 5. *For the social planner problem, the optimal labor allocation between innovation and production within a sector satisfies*

$$\frac{L_k^A}{L_k^Y} = \frac{\lambda_s^* G_k}{\rho(\eta-1)\alpha_k Q_k^{\frac{\sigma-1}{\sigma}}}\,, \quad \forall k \in \mathcal{N}, \tag{43}$$

where $\lambda_s^* = g^{A'}$ is the new growth rate of the knowledge stock, $Q_k = Y_k/Y$ is the equilibrium output share of sector k, and $G_k = \frac{\widetilde{V}_k\widetilde{M}_k}{\widetilde{\mathcal{V}}\widetilde{\mathcal{M}}^T}$. The optimal labor allocation of innovation across sectors satisfies

$$\frac{L_k^A}{L_l^A} = \frac{\widetilde{V}_k\widetilde{M}_k}{\widetilde{V}_l\widetilde{M}_l}\,, \quad \forall k \neq l. \tag{44}$$

The optimal labor allocation for innovation across sectors is such that the marginal contribution of an additional unit of labor devoted to innovation to growth should be the same for all sectors. In this economy, such an outcome is achieved when the ratio of the number of innovation workers between two sectors is equal to the ratio of their corresponding centrality products. The ratio of the number of innovation workers to the number of production workers for a given sector is positively related with this sector's relative positions in both the upstream technology network and the downstream technology network, while inversely related with this sector's output share.

The allocation of labor chosen by the social planner deviates from that in the decentralized economy. To see this, compare equations (43) and (29). In the decentralized equilibrium, the free-entry condition forces the ratio of the number of innovation workers to the number of production workers to be the same across sectors, as shown in (29). This is, in general, not the case for the socially optimal outcome. The social planner weighs the cost of reducing one unit of production labor against the benefit of increasing one unit of innovation labor. The optimum is achieved when the marginal cost is equal to the marginal benefit. Under this situation, the innovation to production labor ratio is sector dependent. Higher centrality or lower output share implies a higher innovation to production labor ratio for a sector. Therefore, to achieve the first-best allocation of labor in a decentralized economy, one needs to impose sector-specific subsidies (or taxes) that take into account the sector's positions in the innovation network.

4. Numerical examples

In this section, we implement some numerical exercises to illustrate the key properties of our model. In order to focus on the impact of the different network structures on each sector's centrality, we set all sector-specific parameters to be the same and equal to 1 in all sectors, that is, $\alpha_k = 1$ and $\theta_k = 1$, $\forall k \in \mathcal{N}$. In addition, we assume that the intrasectoral elasticity of substitution to be equal to $\eta = 1.5$, and the intersectoral elasticity of substitution to be equal to $\sigma = 0.5$. Finally, we set the discount rate to be $\rho = 0.1$. For simplicity, we set $s_{kl} = 1$ if there exists knowledge spillovers from sector l to sector k, and 0 otherwise, so that the network is unweighted and the adjacency matrix only consists of 1 and 0.[8] Consider the following two innovation networks $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$:

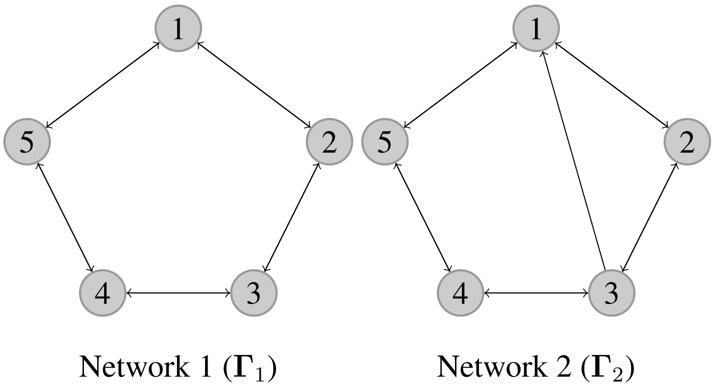

FIGURE 1

Network 1 is an undirected symmetric network, where each sector is connected to two neighboring sectors. In this case, all sectors are equally important, and thus have the same downstream and upstream centrality and, consequently, have the same contribution to the knowledge accumulation. On the contrary, network 2 is directed, asymmetric and denser. The only difference with network 1 is that there is a direct link from sector 3 to sector 1, thereby indicating upstream knowledge spillovers for sector 1 and downstream knowledge spillovers for sector 3. This change of the network structure substantially alters each sector's positions in both the downstream and upstream network.

Table 1 displays the key outcome variables for each sector/node of these two networks. Compared to network 1, when we add the link between sectors 3 and 1, that is $s_{13} = 1$, we see that, in the first and second row of network 2, the target sector, sector 1, experiences an increase in the downstream centrality M_k, from 0.2 to 0.2257, that is an increase of 11.38%, while all other sectors experience a decrease in their downstream centrality M_k. However, sectors 2 and 5 experience a lower decrease than sectors 3 and 4 because the former have a direct link to sector sector 1, the most central node in the downstream network, while, the latter have only an indirect link (distance 2) to sector 1. Consider, now the upstream network. Adding $s_{13} = 1$ leads to the largest increase in the upstream centrality V_k for the source sector, sector 3. Now, sectors 1 ane 5 are the ones that experience the highest decrease in centrality because they are the farthest away (distance 2) from sector 3. Notice that, in the upstream

[8]These choices of parameter value will not affect the qualitative results of our simulation.

network, despite the fact that sectors 2 and 4 have symmetric positions, their upstream centrality is different. This is because there are more innovation workers reallocated into sector 4 than sector 2 (see the fifth row of Table 1), which makes sector 4 a more important source of knowledge spillovers than sector 2. Consequently, sector 4 displays higher upstream centrality than sector 2.

In row 3, we determine the key sector T_k of each network, as given by (39), in both the upstream and downstream network. In network 2, we see that the key sector is sector 3. Even though it has the lowest centrality M_k in the downstream network (but not a big difference compared to sectors 2, 4, 5), it is the key sector because it has the highest centrality V_k in the upstream network since sector 3 is the one that produces the highest knowledge to other central sectors. In other words, sector 3 is the most important contributor to the knowledge accumulation of this economy.

TABLE 1

	Network 1					Network 2				
node	1	2	3	4	5	1	2	3	4	5
M_k	0.2	0.2	0.2	0.2	0.2	0.2257	0.1964	0.1907	0.1907	0.1964
V_k	0.2	0.2	0.2	0.2	0.2	0.1615	0.1946	0.2642	0.2109	0.1687
T_k	0.1667	0.1667	0.1667	0.1667	0.1667	0.1623	0.1621	0.2281	0.1703	0.1362
g^A	0.3667	0.3667	0.3667	0.3667	0.3667	0.3886	0.3886	0.3886	0.3886	0.3886
L_k^A	0.1222	0.1222	0.1222	0.1222	0.1222	0.1084	0.1245	0.1283	0.1283	0.1245
L_k^Y	0.0778	0.0778	0.0778	0.0778	0.0778	0.0681	0.0783	0.0806	0.0806	0.0783

Finally, when the network becomes denser (i.e., more connected), the spectral radius increases, which implies a higher growth rate g^A of knowledge stock (row 4). At the same time, labor is reallocated from more central sectors in the downstream network to less central ones, and from production sector to innovation sector (rows 5 and 6), as shown in Proposition 4.

5. Conclusion

This paper investigates a multi-sectoral endogenous growth model with both intrasectoral and intersectoral knowledge spillovers. Individual firms specialize in a particular technology sector and apply knowledge from both their own sector as well as other sectors to conduct innovation. Consequently, the accumulation of knowledge in every sector is endogenously determined by pairwise knowledge spillovers, which constitutes the innovation network in this paper. There is a deep connection between the structure of the innovation network and the behavior of the economy. In particular, we show that the spectral radius of the innovation network determines the growth rate of the total knowledge stock. Moreover, the positions of sectors in the innovation network determine their contribution to the knowledge accumulation in the economy. Specifically, a sector is more important for knowledge growth if it occupies more central positions in either the downstream innovation network or the upstream innovation network. We also demonstrate that the importance of a directed edge in the innovation network is determined by the source sector's centrality in the upstream innovation network, the target sector's centrality in the downstream network and the strength of knowledge spillovers from the source sector to the target sector. Finally, we show that the equilibrium is not socially optimal and, in order to restore the first best, one needs to impose sector-specific subsidies (or taxes) that take into account the sector's centrality in the innovation network.

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APPENDIX

A. Proof of all results

Proof of Lemma 1. Suppose that, in equilibrium, sectors diverge in their growth rates. Let's denote the fastest growing sector by sector m. For any other sector k that receives direct knowledge spillovers from sector m, using (16), we have:

$$g_k^A(t) = \sum_{l \neq m} \gamma_{kl} \frac{M_l(t)}{M_k(t)} + \gamma_{km} \frac{M_m(t)}{M_k(t)}, \quad \forall \gamma_{km} \neq 0.$$

If sector m grows faster than any other sector, then $\lim_{t\to\infty} M_m(t) \to 1$ and $\lim_{t\to\infty} M_k(t) \to 0$. Thus, $\lim_{t\to\infty} M_m(t)/M_k(t) \to \infty$. However, this implies that $g_k^A(t)$ explodes, and will eventually exceed the growth rate of sector m, which contradicts our initial assumption. Therefore, these sectors must grow at the same rate as sector m.

For any sector, say l, which does not directly receive knowledge spillovers from sector m, the strongly connected innovation network γ guarantees the existence of a sequence of sectors such that at least one of these sectors directly receives knowledge spillovers from sector m, and another sector produces knowledge spillovers to sector l. Then, we can apply the same argument as above to show that sector l will also grows at the same rate as sector m.

This completes the proof. □

Proof of Lemma 2. Rewrite (29) as

$$\frac{L_k^Y}{L_k^A} = \frac{\rho + g^A}{g^A}(\eta - 1), \quad \forall k \in \mathcal{N}.$$

Note that the above equality applies to all sectors. Therefore, we have:

$$\frac{L_k^Y}{L_k^A} = \frac{L_l^Y}{L_l^A},$$

which lead to (30).

Next, recall that $Y_k(t) = \left[\frac{\alpha_k}{P_k(t)}\right]^\sigma Y(t)$. Combining this equation with (21) leads to:

$$[A_k(t)]^{\frac{1}{\eta-1}} L_k^Y = \left[\frac{\alpha_k}{P_k(t)}\right]^\sigma Y(t).$$

The labor ratio of sector k over sector l for production is, thus, given by:

$$\frac{L_k^Y}{L_l^Y} = \left(\frac{\alpha_k}{\alpha_l}\right)^\sigma \left(\frac{A_k(t)}{A_l(t)}\right)^{\frac{1}{1-\eta}} \left[\frac{P_l(t)}{P_k(t)}\right]^\sigma.$$

Notice that

$$P_k(t) = \left(\int_0^{A_k(t)} [p_{ki}(t)]^{1-\eta} di \right)^{\frac{1}{1-\eta}} = \left(\int_0^{A_k(t)} \left(\frac{\eta}{\eta - 1} w(t) \right)^{1-\eta} \right)^{\frac{1}{1-\eta}} = \frac{\eta}{\eta - 1} w(t) [A_k(t)]^{\frac{1}{1-\eta}},$$

which is (12). Therefore, $P_l(t)/P_k(t) = [A_l(t)/A_k(t)]^{\frac{1}{1-\eta}}$. Substituting this ratio back to the labor-ratio equation yields:

$$\frac{L_k^Y}{L_l^Y} = \left(\frac{\alpha_k}{\alpha_l}\right)^{\sigma} \left(\frac{A_k(t)}{A_l(t)}\right)^{\frac{1-\sigma}{1-\eta}} = \left(\frac{\alpha_k}{\alpha_l}\right)^{\sigma} \left(\frac{A_k(t)/A(t)}{A_l(t)/A(t)}\right)^{\frac{1-\sigma}{1-\eta}} = \left(\frac{\alpha_k}{\alpha_l}\right)^{\sigma} \left(\frac{M_k(t)}{M_l(t)}\right)^{\frac{1-\sigma}{1-\eta}}$$

The last equality is equation (31) given in the lemma.

Now, if we use the fact that sectors grow at the same rate, we have

$$L_k^A(t)\theta_k \sum_{m=1}^{N} s_{km} \frac{M_m(t)}{M_k(t)} = L_l^A(t)\theta_l \sum_{m=1}^{N} s_{lm} \frac{M_m(t)}{M_l(t)}.$$

By rearranging this equation, we obtain (32).

Now, we can use (30), (31) and (32), and fix $l = 1$ to obtain (33). The last equation (34) in the lemma is true by construction. Equations (31) and (32) form a system of N equations with N unknowns $\{M_k\}_{k\in\mathcal{N}}$. The solutions, in turn, determine the equilibrium level of labor division across sectors for production and innovation. □

Proof of Proposition 1. Recall that the accumulation of knowledge stocks across sectors are determined by (35), that is

$$\dot{\mathcal{A}}(t) = \mathbf{\Gamma} \mathcal{A}(t),$$

By the fundamental existence and uniqueness theorem of Picard and Lindelöf of differential equations (also known as the CauchyLipschitz theorem; see e.g. Coddington and Levinson (1955)), which gives a set of conditions under which an initial value problem has a unique solution, the above system admits the following solution:

$$\mathcal{A}(t) = \mathcal{A}(0) e^{\mathbf{\Gamma} t}, \quad \text{given } \mathcal{A}(0).$$

Given that the dominant eigenvalue (spectral radius) of the innovation network $\mathbf{\Gamma}$ is positive, we can decompose $e^{\mathbf{\Gamma} t}$ as follows

$$e^{\mathbf{\Gamma} t} = \mathbf{V} e^{\mathbf{T} t} \mathbf{V}^{-1},$$

where

$$\mathbf{T} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_N \end{bmatrix}, \quad \mathbf{V} = (\mathbf{V}_1, \mathbf{V}_2, \cdots, \mathbf{V}_N)$$

where $\{\lambda_k\}_{k\in\mathcal{N}}$ are the eigenvalues of the matrix $\mathbf{\Gamma}$, and $\{\mathbf{V}_k\}_{i\in\mathcal{N}}$ are the corresponding eigenvectors. According to the Perron-Frobenius theorem, there is a dominant eigenvalue λ^* such that $\lambda^* > \lambda_k$, $\forall k \in \mathcal{N}$. Therefore, we can express the knowledge stock of sector k at time t as

$$A_k(t) = \sum_{l=1}^{N} c_l e^{\lambda_l t} = e^{\lambda^* t} \sum_{l=1}^{N} c_l e^{(\lambda_l - \lambda^*)t},$$

where c_l is a function of elements from $\mathbf{V}$, $\mathbf{V}^{-1}$ and $\mathcal{A}(0)$. As $t \to \infty$, $A_k(t) = e^{\lambda^* t} \sum_{l=1}^{N} c_l$. The growth rate of sector k is thus equal to $g_k^A = \lambda^*$, $\forall k \in \mathcal{N}$. Since all sectors grow at the same rate, so does the total knowledge. This completes the first part of the proof.

To prove the second part of the proposition, we divide both sides of (35) by the total knowledge stock $A(t)$ and expand the system of equations to obtain:

$$\begin{bmatrix} \dot{A}_1(t)/A(t) \\ \vdots \\ \dot{A}_N(t)/A(t) \end{bmatrix} = \begin{bmatrix} \gamma_{11} & \cdots & \gamma_{1N} \\ \vdots & \ddots & \vdots \\ \gamma_{N1} & \cdots & \gamma_{NN} \end{bmatrix} \begin{bmatrix} M_1(t) \\ \vdots \\ M_N(t) \end{bmatrix},$$

which can be manipulated as follows

$$\begin{bmatrix} \frac{\dot{A}_1(t)}{A_1(t)} \frac{A_1(t)}{A(t)} \\ \vdots \\ \frac{\dot{A}_N(t)}{A_N(t)} \frac{A_N(t)}{A(t)} \end{bmatrix} = \begin{bmatrix} \gamma_{11} & \cdots & \gamma_{1N} \\ \vdots & \ddots & \vdots \\ \gamma_{N1} & \cdots & \gamma_{NN} \end{bmatrix} \begin{bmatrix} M_1(t) \\ \vdots \\ M_N(t) \end{bmatrix}.$$

In the equilibrium, $g^A = g_k^A \equiv \dot{A}_k(t)/A_i(t)$, $\forall k$. Therefore, the above system of equations can be written as

$$\begin{bmatrix} g^A M_1 \\ \vdots \\ g^A M_N \end{bmatrix} = \begin{bmatrix} \gamma_{11} & \cdots & \gamma_{1N} \\ \vdots & \ddots & \vdots \\ \gamma_{N1} & \cdots & \gamma_{NN} \end{bmatrix} \begin{bmatrix} M_1 \\ \vdots \\ M_N \end{bmatrix}.$$

This can be expressed in a compact form as

$$\lambda^* \mathcal{M} = \mathbf{\Gamma} \mathcal{M}.$$

This completes the proof. □

Proof of Proposition 2 . The proof of this proposition and that of Proposition 3 is based on Restrepo et al. (2006). Assume that, following the removal of a sector k, the perturbation of the innovation matrix $\mathbf{\Gamma}$, the associated dominant eigenvalue λ^*, and the right eigenvector centrality $\mathcal{M}$ are denoted by $\Delta\mathbf{\Gamma}$, $\Delta\lambda^*$ and $\Delta\mathcal{M}$, respectively. Instead of (37), we have:

$$(\mathbf{\Gamma} + \Delta\mathbf{\Gamma})(\mathcal{M}^T + \Delta\mathcal{M}^T) = (\lambda^* + \Delta\lambda^*)(\mathcal{M}^T + \Delta\mathcal{M}^T).$$

where

$$\Delta\mathbf{\Gamma} = \begin{bmatrix} 0 & \cdots & \Delta\gamma_{1k} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ \Delta\gamma_{k1} & \cdots & \Delta\gamma_{kk} & \cdots & \Delta\gamma_{kN} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & \Delta\gamma_{N1} & \cdots & 0 \end{bmatrix} \qquad \begin{array}{rcl} \Delta\gamma_{kl} & = & -\gamma_{kl} \quad , \quad l \neq k \\ \Delta\gamma_{lk} & = & -\gamma_{lk} \quad , \quad l \neq k \\ \Delta\gamma_{kk} & = & -\gamma_{kk} \end{array}$$

and

$$\Delta\mathcal{M}^T = \begin{bmatrix} \Delta M_1 & \cdots & \Delta M_i & \cdots & \Delta M_N \end{bmatrix}^T,$$

where $\Delta M_k = -M_k$, and, for large enough N, $\Delta M_k \to 0$.

If we left multiply the above equation by $\mathcal{V}$, we obtain:

$$\mathcal{V}(\mathbf{\Gamma} + \Delta\mathbf{\Gamma})(\mathcal{M}^T + \Delta\mathcal{M}^T) = \mathcal{V}(\lambda + \Delta\lambda^*)(\mathcal{M}^T + \Delta\mathcal{M}^T).$$

We can now expand this equation and rearrange terms to obtain:

$$\mathcal{V}(\mathbf{\Gamma}\mathcal{M}^T - \lambda^*\mathcal{M}^T) + (\mathcal{V}\mathbf{\Gamma} - \mathcal{V}\lambda^*)\Delta M^T + \mathcal{V}\Delta\mathbf{\Gamma}\mathcal{M}^T + \mathcal{V}\Delta\mathbf{\Gamma}\Delta M^T = \mathcal{V}\mathcal{M}^T\Delta\lambda^* + \mathcal{V}\Delta\mathcal{M}^T\Delta\lambda^*.$$

Note that, by definition, $\mathbf{\Gamma}\mathcal{M}^T - \lambda^*\mathcal{M}^T = 0$ and $\mathcal{V}\mathbf{\Gamma} - \mathcal{V}\lambda^* = 0$. Thus, this equation reduces to:

$$\mathcal{V}\Delta\mathbf{\Gamma}\mathcal{M}^T + \mathcal{V}\Delta\mathbf{\Gamma}\Delta M^T = (\mathcal{V}\mathcal{M}^T + \mathcal{V}\Delta\mathcal{M}^T)\Delta\lambda^*$$

With some algebra, it can be shown that

$$\begin{aligned} \mathcal{V}\Delta\mathbf{\Gamma}\mathcal{M}^T &= -V_k\sum_{l=1}^{N}\gamma_{kl}M_l - M_k\sum_{l=1}^{N}\gamma_{lk}V_l + \gamma_{kk}M_kV_k, \\ &= -2\lambda^*V_kM_k + \gamma_{kk}V_kM_k, \end{aligned}$$

and, for large enough N,

$$\mathcal{V}\Delta\mathbf{\Gamma}\Delta M^T \approx \lambda^*V_kM_k.$$

By combining the above two equations, we obtain:

$$\mathcal{V}\Delta\mathbf{\Gamma}\mathcal{M}^T + \mathcal{V}\Delta\mathbf{\Gamma}\Delta M^T = -(\lambda^* - \gamma_{kk})V_kM_k.$$

Lastly, we have:

$$\mathcal{V}\mathcal{M}^T + \mathcal{V}\Delta\mathcal{M}^T \approx \mathcal{V}\mathcal{M}^T - V_kM_k.$$

Therefore, we can express the importance of a sector k as

$$T_k \equiv -\frac{\Delta\lambda_k^*}{\lambda^*} = \frac{(\lambda^* - \gamma_{kk})V_kM_k}{\lambda^*(\mathcal{V}\mathcal{M}^T - V_kM_k)},.$$

which is equation (39) in the theorem. □

Proof of Proposition 3. Following the proof of Proposition 2 , we define the perturbation of the innovation matrix $\mathbf{\Gamma}$, the associated dominant eigenvalue λ^*, and the right eigenvector centrality $\mathcal{M}$, upon the removal of an edge kl, as $\Delta\mathbf{\Gamma}$, $\Delta\lambda^*_{kl}$ and $\Delta\mathcal{M}$, respectively. We have

$$(\mathbf{\Gamma} + \Delta\mathbf{\Gamma})(\mathcal{M}^T + \Delta\mathcal{M}^T) = (\lambda^* + \Delta\lambda^*_{kl})(\mathcal{M}^T + \Delta\mathcal{M}^T),$$

where $[\Delta\mathbf{\Gamma}]_{ss'} = -\gamma_{kl}$, if $s = k$ and $s' = l$, and 0 otherwise. For large enough N, ΔM^T captures the small perturbation of the right eigenvector centrality so that $\Delta M_k \to 0$. Left multiply this equation and after some algebra, we obtain:

$$\Delta\lambda^*_{kl} = \mathbf{V}\Delta\mathbf{\Gamma}\mathcal{M}^T/\mathcal{V}\mathcal{M}^T.$$

It can be shown that

$$\mathbf{V}\Delta\mathbf{\Gamma}\mathcal{M}^T = -\gamma_{kl}V_k M_l,$$

which can be substituted back to the previous equation to get

$$T_{kl} \equiv -\frac{\Delta\lambda^*_{kl}}{\lambda^*} = \frac{\gamma_{kl}V_k M_l}{\lambda^*\mathcal{V}^T\mathcal{M}}.$$

This completes the proof. $\square$

Proof of Proposition 4.

Part (a): We use proof by contradiction to prove this part. The share of each sector's knowledge stock $\{M_m\}_{m\in\mathcal{N}}$ is a function of s_{kl}, thus a change in s_{kl} induces changes of $\{M_m\}_{m\in\mathcal{N}}$. This implies that $\exists\, O \subset \mathcal{N}$ such that $M_o^{'} > M_o\ \forall o \in O$, since $\sum_{m=1}^N M_m = \sum_{m=1}^N M_m^{'} = 1$. Let $\tilde{o} \in O$ be such that $M_{\tilde{o}}^{'}/M_{\tilde{o}} \geq M_m^{'}/M_m\ \forall m \in \mathcal{N}$. Note that the inequality holds for at least one m. Similarly, $\exists h \subset \mathcal{N}$ such that $M_h^{'} < M_h$. Let $\tilde{h} \in H$ be such that $M_{\tilde{h}}^{'}/M_{\tilde{h}} \leq M_m^{'}/M_m\ \forall m \in \mathcal{N}$. Again the inequality holds for at least one m.

Suppose that $k \neq \tilde{o}$. Recall that $L_{\tilde{h}}^Y/L_{\tilde{o}}^Y = (\alpha_{\tilde{h}}/\alpha_{\tilde{o}})^\sigma(M_{\tilde{h}}/M_{\tilde{o}})^{\frac{1-\sigma}{1-\eta}}$, since $M_{\tilde{h}}^{'} < M_{\tilde{h}}$, and $M_{\tilde{o}}^{'} > M_{\tilde{o}}$, given $M_{\tilde{h}}^{'}$, $M_{\tilde{h}}$, $M_{\tilde{o}}^{'}$, $M_{\tilde{o}} \in (0,1)$, we have $M_{\tilde{h}}^{'}/M_{\tilde{o}}^{'} < M_k/M_{\tilde{o}}$. Since $0 < \sigma < 1$, $\eta > 1$, the above result implies that $(M_{\tilde{h}}^{'}/M_{\tilde{o}}^{'})^{\frac{1-\sigma}{1-\eta}} > (M_{\tilde{h}}/M_{\tilde{o}})^{\frac{1-\sigma}{1-\eta}}$, which then implies that $L_{\tilde{h}}^{Y'}/L_{\tilde{o}}^{Y'} > L_{\tilde{h}}^Y/L_{\tilde{o}}^Y$.

Now, recall that in the new equilibrium $g_{\tilde{h}}^{A'} = g_{\tilde{o}}^{A'}$, which implies that

$$\frac{L_{\tilde{h}}^{A'}}{L_{\tilde{o}}^{A'}} = \frac{\sum_{m=1}^N \theta_{\tilde{o}} s_{\tilde{o}m} M_m^{'}/M_{\tilde{o}}^{'}}{\sum_{m=1}^N \theta_{\tilde{h}} s_{\tilde{h}m} M_m^{'}/M_{\tilde{h}}^{'}}.$$

Since $M_{\tilde{o}}^{'}/M_{\tilde{o}} \geq M_m^{'}/M_m\ \forall m \in \mathcal{N}$, and the inequality holds for at least one m, we have $M_m^{'}/M_{\tilde{o}}^{'} \leq M_m/M_o\ \forall m \in \mathcal{N}$, which implies that

$$\sum_{m=1}^N \theta_{\tilde{o}} s_{\tilde{o}m} M_m^{'}/M_{\tilde{o}}^{'} < \sum_{m=1}^N \theta_{\tilde{o}} s_{\tilde{o}m} M_m/M_{\tilde{o}}.$$

Similarly, we can infer from $M_{\tilde{h}}^{'}/M_{\tilde{h}} \leq M_m^{'}/M_m \; \forall m \in \mathcal{N}$ that

$$\sum_{m=1}^{N} \theta_{\tilde{h}} s_{\tilde{h}m} M_m^{'}/M_{\tilde{h}}^{'} > \sum_{m=1}^{N} \theta_{\tilde{h}} s_{\tilde{h}m} M_m/M_{\tilde{h}}.$$

Combining the above two results leads to $L_{\tilde{h}}^{A'}/L_{\tilde{o}}^{A'} < L_{\tilde{h}}^{A}/L_{\tilde{o}}^{A}$, which implies that $L_{\tilde{h}}^{A'}/L_{\tilde{o}}^{A'} < L_{\tilde{h}}^{A}/L_{\tilde{o}}^{A} = L_{\tilde{h}}^{Y}/L_{\tilde{o}}^{Y} < L_{\tilde{h}}^{Y'}/L_{\tilde{o}}^{Y'}$. A contradiction with the free entry condition that states that $L_{\tilde{h}}^{A'}/L_{\tilde{o}}^{A'} = L_{\tilde{h}}^{Y'}/L_{\tilde{o}}^{Y'}$.

To see why it can only be the case that $k = \tilde{o}$, note that

$$\frac{L_{\tilde{h}}^{A'}}{L_k^{A'}} = \frac{\sum_{m \neq l} \theta_k s_{km} M_m^{'}/M_k^{'} + \theta_k s_{kl}^{'} M_l^{'}/M_k^{'}}{\sum_{m=1}^{N} \theta_{\tilde{h}} s_{\tilde{h}m} M_m^{'}/M_{\tilde{h}}^{'}}.$$

To ensure that $L_{\tilde{h}}^{A'}/L_{\tilde{o}}^{A'} > L_{\tilde{h}}^{A}/L_{\tilde{o}}^{A}$, we require that

$$\sum_{m=1}^{N} \theta_{\tilde{o}} s_{\tilde{o}m} M_m^{'}/M_{\tilde{o}}^{'} > \sum_{m=1}^{N} \theta_{\tilde{o}} s_{\tilde{o}m} M_m/M_{\tilde{o}}.$$

The only way this can happen is that the increase in s_{kl} dominates the decrease in $M_m/M_k \; \forall m \neq k$, so that overall

$$\sum_{m \neq l} \theta_k s_{km} M_m^{'}/M_k^{'} + \theta_k s_{kl}^{'} M_l^{'}/M_k^{'} > \sum_{m=1}^{N} \theta_k s_{kl} M_l/M_k.$$

Part (b): From part (a), we know that when $s_{kl}^{'} > s_{kl}$, $M_k^{'}/M_k > M_m^{'}/M_m \; \forall m \neq k$. By the free-entry conditions in the original and the new equilibrium, we have:

$$\frac{L_k^{A'}}{L_m^{A'}} = \frac{L_k^{Y'}}{L_m^{Y'}} = \left(\frac{\alpha_k}{\alpha_m}\right)^{\sigma} \left(\frac{M_k^{'}}{M_m^{'}}\right)^{\frac{1-\sigma}{1-\eta}} < \left(\frac{\alpha_k}{\alpha_m}\right)^{\sigma} \left(\frac{M_k}{M_m}\right)^{\frac{1-\sigma}{1-\eta}} = \frac{L_k^{Y}}{L_m^{Y}} = \frac{L_k^{A}}{L_m^{A}}, \; \forall m \neq k$$

where the inequality follows from $M_k^{'} > M_k$ and $M_l^{'} < M_l$. This finishes the proof of this part.

Part (c): Let $\tilde{h} \in H$ be such that $M_{\tilde{h}}^{'}/M_{\tilde{h}} \leq M_m^{'}/M_m, \forall m \in \mathcal{N}$. Again the inequality holds for at least one m. Then, we have:

$$g_{\tilde{h}}^{A} = \sum_{m=1}^{N} \theta_{\tilde{h}} L_{\tilde{h}}^{A} s_{\tilde{h}m} M_m/M_{\tilde{h}}.$$

Let $\bar{o}$ be such that $V_{\bar{o}}^{'}/V_{\bar{o}} \geq V_m^{'}/V_m, \forall m \in \mathcal{N}$, with the inequality holds for at least one m. Suppose that $\bar{o} \neq l$, then:

$$g_{\tilde{o}}^{A} = \sum_{m=1}^{N} \theta_m s_{m\bar{o}} L_m^{A} V_m/V_{\bar{o}}$$

In equilibrium, we have $g_{\tilde{h}}^{A} = g_{\tilde{o}}^{A}$, which implies that

$$\sum_{m=1}^{N} \theta_{\tilde{h}} L_{\tilde{h}}^{A} s_{\tilde{h}m} M_m/M_{\tilde{h}} = \sum_{m=1}^{N} \theta_m s_{m\bar{o}} L_m^{A} V_m/V_{\bar{o}}.$$

This equation can be rearranged into

$$\sum_{m=1}^{N} s_{\tilde{h}m}\frac{M_m}{M_{\tilde{h}}} = \sum_{m=1}^{N}\frac{\theta_m}{\theta_{\tilde{h}}} s_{m\bar{o}}\frac{L_m^A}{L_{\tilde{h}}^A}\frac{V_m}{V_{\bar{o}}}.$$

Since $M_{\tilde{h}}^{'}/M_{\tilde{h}} \leq M_m^{'}/M_m, \forall m \in \mathcal{N}$ with the inequality holds for at least one m. We have:

$$\sum_{m=1}^{N} s_{\tilde{h}m}\frac{M_m^{'}}{M_{\tilde{h}}^{'}} > \sum_{m=1}^{N} s_{\tilde{h}m}\frac{M_m}{M_{\tilde{h}}}$$

Since $V_{\bar{o}}^{'}/V_{\bar{o}} \geq V_m^{'}/V_m \ \forall m \in \mathcal{N}$ with the inequality holds for at least one m, we have $V_m^{'}/V_{\bar{o}}^{'} \leq V_m/V_{\bar{o}}$. From part (b), we have shown that $L_m^{A'}/L_{\tilde{h}}^{A'} \leq L_m^A/L_{\tilde{h}}$. By combining this with the previous result, we obtain:

$$\sum_{m=1}^{N}\frac{\theta_m}{\theta_{\tilde{h}}} s_{m\bar{o}}\frac{L_m^A}{L_{\tilde{h}}^A}\frac{V_m}{V_{\bar{o}}} > \sum_{m=1}^{N}\frac{\theta_m}{\theta_{\tilde{h}}} s_{m\bar{o}}\frac{L_m^{A'}}{L_{\tilde{h}}^{A'}}\frac{V_m^{'}}{V_{\bar{o}}^{'}}.$$

The above results imply that

$$\sum_{m=1}^{N} s_{\tilde{h}m}\frac{M_m^{'}}{M_{\tilde{h}}^{'}} > \sum_{m=1}^{N}\frac{\theta_m}{\theta_{\tilde{h}}} s_{m\bar{o}}\frac{L_m^{A'}}{L_{\tilde{h}}^{A'}}\frac{V_m^{'}}{V_{\bar{o}}^{'}}.$$

Note that in the new equilibrium, $g_{\tilde{h}}^{A'} = g_{\bar{o}}^{A'}$, which implies that

$$\sum_{m=1}^{N} s_{\tilde{h}m}\frac{M_m^{'}}{M_{\tilde{h}}^{'}} = \sum_{m=1}^{N}\frac{\theta_m}{\theta_{\tilde{h}}} s_{m\bar{o}}\frac{L_m^{A'}}{l_{\tilde{h}}^{A'}}\frac{V_m^{'}}{V_{\bar{o}}^{'}},$$

a contradiction with the previous result.

Part (d): Suppose that $g^{A'} < g^A$. Since we have shown in part (a) that $M_k^{'} > M_k$ and $M_l^{'} < M_l$, $\exists D \subset \mathcal{N}$ such that $M_d^{'} < M_d, \forall d \in D$. Let $\tilde{d} \in D$ be such that

$$\frac{\sum_{m=1}^{N} s_{\tilde{d}m}M_m^{'}/M_{\tilde{d}}^{'}}{\sum_{m=1}^{N} s_{\tilde{d}m}M_m/M_{\tilde{d}}} \geq \frac{\sum_{m=1}^{N} s_{lm}M_m^{'}/M_l^{'}}{\sum_{m=1}^{N} s_{lm}M_m/M_l}, \forall l \in \mathcal{N}.$$

Note that $g_{\tilde{d}}^{A'} = g_m^{A'}$ and $g_{\tilde{d}}^A = g_m^A \ \forall m \in \mathcal{N}$, we have $L_{\tilde{d}}^{A'}/L_{\tilde{d}}^A \leq L_m^{A'}/L_m^A \ \forall l \in \mathcal{N}$. From the proof in part (a), we know that $M_k^{'} > M_k$, and since $\tilde{d} \in D$, $M_{\tilde{d}}^{'} < M_{\tilde{d}}$, which implies that $M_{\tilde{d}}^{'}/M_{\tilde{d}} < M_k^{'}/M_k$, which can be rearranged to get $M_{\tilde{d}}^{'}/M_k^{'} < M_{\tilde{d}}/M_k$.

Recall that

$$\frac{L_{\tilde{d}}^{A'}}{L_k^{A'}} = \left(\frac{\alpha_{\tilde{d}}}{\alpha_k}\right)^{\sigma}\left(\frac{M_{\tilde{d}}^{'}}{M_k^{'}}\right)^{\frac{1-\sigma}{1-\eta}} > \left(\frac{\alpha_{\tilde{d}}}{\alpha_k}\right)^{\sigma}\left(\frac{M_{\tilde{d}}}{M_k}\right)^{\frac{1-\sigma}{1-\eta}} = \frac{L_{\tilde{d}}^A}{L_k^A}.$$

This result contradicts the previous result that $L_{\tilde{d}}^{A'}/L_{\tilde{d}}^A \leq L_m^{A'}/L_m^A, \forall l \in \mathcal{N}$. Therefore, $g^{A'} > g^A$.

Next, we show that $L^{A'}/L^{Y'} > L^A/L^Y$. From the free entry condition, we have:

$$\frac{1}{\eta-1}\frac{L_k^Y}{L_k^A}\frac{g^A}{\rho+g^A} = 1, \quad \forall k \in \mathcal{N}.$$

It is easy to see that L_k^Y/L_k^A is constant for all sectors, which implies that L^Y/L^A should be constant as well and should be inversely related to g^A. Since we have shown that $g^{A'} > g^A$, $L_k^{Y'}/L_k^{A'} < L_k^Y/L_k^A$ follows. □

Proof of Proposition 5 . The social planner aims to maximize the representative household's lifetime utility along the balanced growth path

$$
\begin{aligned}
U &= \int_0^\infty e^{-\rho t} \log C(t) dt \\
&= \int_0^\infty e^{-\rho t} \log \left(C(0) e^{g^{C'} t} \right) dt \\
&= \int_0^\infty e^{-\rho t} \left(\log C(0) + g^{C'} t \right) dt \\
&= \int_0^\infty e^{-\rho t} \left(\log C(0) + \frac{g^{A'}}{\eta - 1} t \right) dt \\
&= \frac{\log C(0)}{\rho} + \frac{g^{A'}}{\rho^2 (\eta - 1)}.
\end{aligned}
$$

The second equality follows the fact that consumption grows at the constant rate $g^{C'}$ in equilibrium, and the forth equality follows from (22) and (24). In addition,

$$
\begin{aligned}
\log C(0) &= \log Y(0) \\
&= \frac{\sigma}{\sigma - 1} \log \left(\sum_{k=1}^N \alpha_k Y_k(0)^{\frac{\sigma-1}{\sigma}} \right) \\
&= \frac{\sigma}{\sigma - 1} \log \left[\sum_{k=1}^N \alpha_k \left(A_k(0)^{\frac{1}{\eta-1}} L_k^Y \right)^{\frac{\sigma-1}{\sigma}} \right]
\end{aligned}
$$

The social planner maximizes the representative household's lifetime utility through two steps. First, for a given level of sectoral labor $L_k = L_k^A + L_k^Y$, the social planner equals the marginal utility of an additional unit of production labor and the marginal utility of an additional unit of innovation labor as follows

$$
\frac{\partial \log C(0)}{\partial L_k^Y} = \frac{1}{\rho(\eta - 1)} \frac{\partial g^{A'}}{\partial L_k^A}.
$$

Use the fact that $C(0) = Y(0)$, we have

$$
\begin{aligned}
\frac{\partial \log C(0)}{\partial L_k^Y} &= \frac{\alpha_k \left(A_k(0)^{\frac{1}{\eta-1}} L_k^Y \right)^{\frac{\sigma-1}{\sigma}}}{\sum_{k=1}^N \alpha_k \left(A_k(0)^{\frac{1}{\eta-1}} L_k^Y \right)^{\frac{\sigma-1}{\sigma}} L_k^Y} \\
&= \frac{\alpha_k Y_k(0)^{\frac{\sigma-1}{\sigma}}}{Y(0)^{\frac{\sigma-1}{\sigma}}} \frac{1}{L_k^Y} \\
&= \frac{\alpha_k Q_k^{\frac{\sigma-1}{\sigma}}}{L_k^Y}.
\end{aligned}
$$

The last equality uses the fact that along the balanced growth path, the sectoral output share is constant, i.e. $Q_k(0) = Q_k$.

Next, denote the new innovation matrix as $\widetilde{\mathbf{\Gamma}}$. Then, recall that

$$\widetilde{\mathbf{\Gamma}}\widetilde{\mathcal{M}}^T = \lambda_s^* \widetilde{\mathcal{M}}^T,$$

where $\lambda_s^* = g^{A'}$. Take the partial derivative with respect to L_k^A from both sides of the above equation to obtain:

$$\frac{\partial \widetilde{\mathbf{\Gamma}}}{\partial L_k^A}\widetilde{\mathcal{M}}^T + \widetilde{\mathbf{\Gamma}}\frac{\partial \widetilde{\mathcal{M}}^T}{\partial L_k^A} = \frac{\partial \lambda_s^*}{\partial L_k^A}\widetilde{\mathcal{M}}^T + \lambda_s^* \frac{\partial \widetilde{\mathcal{M}}^T}{\partial L_k^A},$$

where

$$\frac{\partial \widetilde{\mathbf{\Gamma}}}{\partial L_k^A} = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ \theta_k s_{k1} & \cdots & \theta_k s_{kN} \\ 0 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \end{bmatrix}.$$

Left multiply both sides of the above equation by $\widetilde{V}$ to get

$$\widetilde{V}\frac{\partial \widetilde{\mathbf{\Gamma}}}{\partial L_k^A}\widetilde{\mathcal{M}}^T + \widetilde{V}\widetilde{\mathbf{\Gamma}}\frac{\partial \widetilde{\mathcal{M}}^T}{\partial L_k^A} = \frac{\partial \lambda_s^*}{\partial L_k^A}\widetilde{V}\widetilde{\mathcal{M}}^T + \lambda_s^* \widetilde{V}\frac{\partial \widetilde{\mathcal{M}}^T}{\partial L_k^A}$$

By simplifying and rearranging, we obtain:

$$\begin{aligned} \frac{\partial \lambda_s^*}{\partial L_k^A} &= \frac{\widetilde{V}\frac{\partial \widetilde{\mathbf{\Gamma}}}{\partial L_k^A}\widetilde{\mathcal{M}}^T}{\widetilde{\mathcal{V}}\widetilde{\mathcal{M}}^T} \\ &= \frac{\widetilde{V}_k \sum_{l=1}^N \theta_k s_{kl}\widetilde{M}_l}{\widetilde{\mathcal{V}}\widetilde{\mathcal{M}}^T} \\ &= \frac{\widetilde{V}_k \sum_{l=1}^N \widetilde{\gamma}_{kl}\widetilde{M}_l / L_k^A}{\widetilde{\mathcal{V}}\widetilde{\mathcal{M}}^T} \\ &= \frac{\lambda_s^* \widetilde{V}_k \widetilde{M}_k}{\mathcal{V}\mathcal{M}^T L_k^A}. \end{aligned}$$

Therefore, we have

$$\frac{\alpha_k Q_k^{\frac{\sigma-1}{\sigma}}}{L_k^Y} = \frac{1}{\rho(\eta-1)}\frac{\lambda_s^* \widetilde{V}_k \widetilde{M}_k}{\mathcal{V}\mathcal{M}^T L_k^A}.$$

By rearranging the above equation, we obtain (43).

The social planner then decides the optimal allocation of innovation workers across sectors to ensure that the marginal contribution of one additional unit of innovation labor to growth is the same across sectors. This requires that

$$\frac{\partial \lambda_s^*}{\partial L_k^A} = \frac{\partial \lambda_s^*}{\partial L_l^A}, \quad \forall k \neq l.$$

By plugging in what we have derived before and rearranging, we obtain:

$$\frac{L_k^A}{L_l^A} = \frac{\widetilde{V}_k \widetilde{M}_k}{\widetilde{V}_l \widetilde{M}_l}\,, \quad \forall k \neq l,$$

which is (44). $\square$