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NONLINEAR PRICING IN OLIGOPOLY: HOW BRAND PREFERENCES SHAPE MARKET OUTCOMES

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JEL Classification: D82

Keywords: Competition, price discrimination, asymmetric information, preference correlation, price dispersion

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Nonlinear Pricing in Oligopoly: How Brand Preferences Shape Market Outcomes*

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1 Introduction

Since as early as 1849, with the pioneering work of Jules Dupuit, economists have investigated the effects of second-degree price discrimination on pricing and quality provision. Anticipating the modern treatments of Mussa and Rosen (1978) and Maskin and Riley (1984), Dupuit noted that profit maximization by a monopolist leads to under-provision of quality at the bottom of the product line. Intuitively, under-provision serves to prevent profit dissipation, whereby consumers with high valuations per quality purchase low-quality products, therefore making it possible to set high prices at the top of the product line.

This clarity of insight is missing in oligopolistic settings, where at least two firms compete for consumers by offering menus of products. One critical feature of these markets is that consumers's preferences over product characteristics are often correlated with their propensity to switch brands. The latter possibility has been recognized for long, and plays a key role in the empirical literature that estimates demand employing discrete-choice models with random coefficients (in the tradition of Berry, Levinsohn and Pakes 1995). In such models, consumers choose among different options from competing product lines by weighing their respective price and quality dimensions. To produce more flexible estimates, it is often assumed that consumer preferences over price/quality attributes are random, while depending on demographics such as income, age, family size, etc (see, for instance, Nevo 2001). These studies often find that consumers' price sensitivity and taste for quality are correlated, which implies that consumer segments (along the product line) systematically differ on their propensities to switch brands (e.g., in response to price discounts).[1]

While there is no reason expect this correlation to be always positive or negative (across markets), it is intuitive that common factors determine both the consumers' tastes for quality and brand loyalty. To illustrate, suppose income is the main factor behind one's tastes for quality (e.g., high earners like premium products) as well as behind brand switching (e.g., those with higher marginal utility of money are more likely to react to price discounts). In this case, consumers with stronger tastes for quality (high earners) are less likely to switch brands (for having a lower marginal utility of money). This is consistent with the empirical findings of Kaplan and Menzio (2015) and Kaplan et al (2019), who show that consumers with higher incomes engage less intensively on search activities, being more likely to remain brand-loyal, while caring more about product quality.[2] This is also consistent with Petrin (2002), who finds, in the context of minivans, that those consumers with weaker brand preferences assign less value to quality attributes such as horsepower or vehicle size.

[1] This is particularly relevant for structural empirical work investigating product design in oligopolistic settings. Examples include Gandhi et al. (2008), Chu (2010) and Fan (2013).

[2] Anecdotal evidence suggests that a related story applies to airline markets. For instance, the most loyal customers of Air France are likely to be business travelers who simultaneously exhibit higher tastes for premium features (extra leg space, access to the pre-boarding lounge, etc), as well as a lower propensity to switch airlines (be it because the employer pays the ticket, so they are less price sensitive, or because of the convexity of frequent flier rewards).

By contrast, Crawford, Scherbakov and Shum (2019), in the context of cable TV, find that consumers with stronger tastes for quality are easier to poach through price discounts than those who exhibit weaker tastes for quality.[3] One possible explanation to this finding is that consumers who purchase premium packages (defined as containing more channels) are those who spend more time watching TV, earn lower incomes,[4] and therefore exhibit a higher marginal utility of money (being therefore more price-sensitive).[5] The authors then show that the quality of premium packages is set inefficiently high by cable companies, which contrasts with the received wisdom from the monopolistic screening literature (and also from oligopolistic models, as reviewed below).

All in all, the applied literature, as well as casual observations, suggest that correlation in consumer preferences is not only empirically relevant, but also consequential for pricing and product design. Yet, theory is mostly silent about how comovements between consumers' tastes for quality and brand loyalty affect market outcomes under competition. We try to fill this gap.

Model and Results

We embed the canonical Mussa and Rosen (1978) model of price discrimination into a Hotelling framework, with two firms located at the extremes of a liner city. Crucially, the transportation cost faced by a consumer (which determines her propensity to switch brands) is assumed to vary with her taste for quality. Accordingly, "high types" (i.e., those consumers with a high valuation for quality) may be more or less loyal to the their preferred brand than "low types." As we shall see, this degree of flexibility is crucial to explain the diversity of market outcomes observed under competition.

Firms simultaneously offer menus of price-quality pairs designed to screen consumers' unobserved tastes for quality. Firms are blind to consumer brand preferences (i.e., their location in the Hotelling line), reflecting the anonymity of past market transactions, or privacy regulation.

To fix ideas, we first revisit the case of a "balanced duopoly," where the intensity of consumers' brand preferences is independent of their tastes for quality. In line with the seminal works of Armstrong and Vickers (2001) and Rochet and Stole (2002), we find that quality provision is efficient in equilibrium provided consumers have mild brand preferences. We however depart from these contributions, rather following Bénabou and Tirole (2016), on how to model the consumers' comparison between purchasing inside or outside goods. Crucially, in our model, variations in the intensity of brand preferences do not affect participation decisions (as the relative value of one's preferred brand vis-à-vis the outside option is held constant), enabling us to cover the whole spectrum of competitive

[3] A similar pattern is found by Durrmeyer (2020), who studies the automobile market in France. She finds that consumers with stronger preferences for "green" attributes (favoring cars which fuels emit less CO_2) are the most sensitive to automobile prices.

[4] There is indeed robust empirical evidence showing that the time spent watching TV is negatively correlated with household income. See for instance Nielsen (2015).

[5] Another possibility is suggested by the behavioral literature on rational inattention (see Gabaix 2014 and the references therein). This literature argues that consumers who spend more money on a product (premium cable TV) tend to be more attentive to its price, and therefore more likely to switch brands in response to price differences.

intensity. Accordingly, we show that, as consumers become more loyal to their preferred brands, the equilibrium approaches the monopolistic outcome of Mussa and Rosen (1978).[6]

In the general case where brand loyalty is type-specific, our analysis delivers four main insights. First, relative to its efficient level, equilibrium menus over-provide quality at the top of the product line if the propensity of low-type consumers to switch brands is small relative to that of high types. Intuitively, under this form of preference correlation, firms enjoy more market power among low types, who then obtain lower payoffs in equilibrium. To avoid profit dissipation (stemming from low types selecting the premium product, which profit margin is smaller), firms then inefficiently raise the quality of the premium product. This is is consistent with the aforementioned contribution of Crawford, Scherbakov and Shum (2019), who estimate "low types" to be less prone to switch cable companies, while finding that cable companies design premium packages of inefficiently high quality.[7]

Conversely, equilibrium menus under-provide quality at the bottom of the product line if the propensity of high-type consumers to switch brands is small relative to that of low types. The intuition is the mirror image of that from the previous case: Here, firms enjoy more market power among high types, who then obtain lower payoffs in equilibrium. To avoid profit dissipation (stemming now from high types selecting the baseline product), firms then inefficiently reduce the quality of the baseline product. This prediction is consistent with McManus (2007), who finds that (oligopolistic) coffee shops distort product sizes for "sweet espresso" drinks, choosing inefficiently small servings except at the largest cup size.[8] It is also consistent with anecdotal evidence from airline services: lack of comfort is common in economy class seats, which are typically foregone by business travelers exhibiting more brand loyalty than cheapskate tourists.

Second, we show that asymmetric information about one's tastes for quality may either benefit or hurt consumers, depending on the correlation between preferences for quality and brand loyalty. To understand the novelty of this finding, let us reconsider the monopolist benchmark of Mussa and Rosen (1978). Because of the self-selection constraints inherent to price discrimination, all types (weakly) benefit from privately knowing their tastes for quality (i.e., informational rents are necessarily non-negative). This conclusion holds true under competition if the consumers' brand loyalty is independent of their tastes for quality. The reason is that low types obtain zero payoffs whenever incentive constraints bind, which implies they are indifferent between the cases of complete and asymmetric information about their preferences. In turn, high types gain even more from asymmetric information as one moves from monopoly to duopoly. The reason is that asymmet-

[6] In contrast to our paper, Bénabou and Tirole (2016) study competition in linear contracts in a common-value environment.

[7] See also Crawford (2012) for an earlier discussion on how to measure quality distortions in empirical models of differentiated product demand.

[8] McManus (2007) posits in his structural model that price sensitivity is constant, being therefore orthogonal to (random) preferences over product attributes. It is likely though that consumers of "sweet espresso drinks," which are the most differentiated across shops, have strong brand preferences, the more so for those who consume more coffee.

ric information magnifies competition, as relinquishing more utility to high types relaxes incentive constraints, increasing the efficiency of low-type contracts.

By contrast, informational rents are negative to high types (but positive to low types) if brand loyalty is higher among low types. The reason is that, under this form of preference correlation, asymmetric information alleviates competition for high types, as relinquishing more utility to these consumers tightens incentive constraints, decreasing the efficiency of premium products. Conversely, increasing the indirect utility of low types alleviates the upward distortion at the top of the product line, which intensifies competition for high-type consumers. As a result, private information about one's tastes benefits low types but hurts high types.

On the other hand, informational rents are positive to high types (but negative to low types) if brand loyalty is higher among high types. The reason is that, under this form of preference correlation, asymmetric information alleviates competition for low types, as relinquishing more utility to these consumers tightens incentive constraints, decreasing the efficiency of baseline products. Conversely, increasing the indirect utility of high types alleviates the downward distortion at the bottom, which intensifies competition for these consumers. Therefore, private information about one's tastes benefits high types but hurts low types. Under either form of preference correlation, these results infirm the received wisdom according to which consumers are better off under second- rather than third-degree price discrimination (where, due to complete information, pricing by firms is not constrained by consumer self-selection) - see, for instance, Varian (2006). In other words, informational rents may well be negative under competition.

Third, we develop a number of comparative statics on pricing and quality provision that eluded previous analysis. For instance, we find that, as low types (resp., high types) become more prone to switch brands, quality provision increases (resp., decreases) along the product line. Moreover, welfare decreases (resp., increases) as low types become more prone to switch brands if quality provision is excessive at the top (resp., deficient at the bottom) of the product line. Relatedly, we also show that the price charged to low types is non-monotone in the brand loyalty of high-type consumers. The latter implications are testable, and further differentiate our model from other theories of price discrimination under competition (more on this below).

Fourth, we show that pure-strategy equilibria fail to exist whenever brand loyalty is sufficiently different across consumers types, which implies equilibria are necessarily in mixed strategies. This non-existence result is driven by the interplay between self-selection constraints and the fact that different types exhibit different propensities to switch brands. Accordingly, our theory identifies a new rationale for price/quality dispersion in private-value settings, unlike previous literature that relates dispersion to search or informational frictions (as in Varian 1980 or Burdett and Judd 1983).[9]

[9]We cautiously interpret this result as consistent with the fact that many oligopolistic markets practicing second-degree price discrimination are "unstable," in that product features and prices are constantly revised by competing

Importantly, we also characterize mixed-strategy equilibria, producing new (testable) predictions about the distribution of market offers.

Paper Outline

The rest of the paper is organized as follows. Section 2 sets up the model. Section 3 derives some preliminary results, and revisits the benchmark where the intensity of brand preferences is independent of consumers' tastes for quality. To introduce the main ideas behind our results accessibly, sections 4 and 5 consider two polar instances of our model. Section 4 studies the *bottom-of-barrel duopoly*, where low types exhibit varying degrees of brand loyalty, while high types see firms as homogenous. This terminology reflects the fact that there is perfect competition for those consumers with the highest potential to boost firms' profits (namely, the high types). In turn, section 5 studies the opposite polar case, the *cream-skimming duopoly*, where low types see firms as homogenous, but high types are loyal to their preferred brands. There is "cream to be skimmed" in that firms enjoy market power among those consumers with the largest profit potential. Section 6 considers the general case where there is imperfect competition for all types of consumers. Section 7 describes important extensions to our baseline model, including the case of a continuum of types. Section 8 collects the empirical implications, and concludes. Proofs are in the Appendix at the end of the document.

We conclude this introduction by briefly reviewing the pertinent literature.

1.1 Related Literature

This article primarily contributes to the literature that studies imperfect competition in nonlinear pricing schedules (see Stole 2007 for a comprehensive survey).

In one strand of this literature, Stole (1991), Ivaldi and Martimort (1994) and Martimort and Stole (2009) study duopolistic competition in nonlinear price schedules when consumers can purchase from more than one firm. Calzolari and Denicolò (2013) evaluate the welfare impact of exclusive contracts (whereby firms offer discounts to consumers who buy nothing from the competitor), and market-share discounts (i.e., discounts that depend on the seller's share of a consumer's total purchases).[10] These papers speak to markets where goods are divisible and/or exhibit some degree of complementarity, whereas our analysis is relevant for markets where purchases are exclusive (e.g., most markets for durable goods).

As mentioned above, our work is more closely related to Rochet and Stole (1997, 2002) and Armstrong and Vickers (2001). Several differences between our paper and these classic contributions

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firms (e.g., air travel). Some of these markets do not suffer from severe informational frictions, but do seem to exhibit heterogeneity in consumers' brand loyalty.

[10]Relatedly, Calzolari and Denicolò (2015) analyze the effect of exclusive dealing when firms are asymmetric and consumers' valuations for the product are private information.

stand out: First, these works focus on the case where the propensity to switch brands is independent of one's tastes for quality. By contrast, our focus is on the more challenging case where preferences co-move. Second, these papers adopt a "standard" Hotelling framework, in which consuming the outside option does not require incurring the transportation cost. As explained in more detail below, this specification conflates changes in the degree of competition across firms with changes in the attractiveness of the outside option. Bypassing this limitation renders our model more tractable, while permitting a clear interpretation of comparative statics.[11,12]

Another closely related paper is Ellison (2005), who examines a competitive price discrimination framework related to ours, while assuming that consumers with higher valuations for quality have stronger brand preferences. Crucially, Ellison takes qualities as exogenous, therefore focusing solely on the equilibrium choice of prices. The emphasis of his work is in comparing two settings; one of complete information about prices, the other where consumers fail to observe the price of "upgrading" the product before getting to the store.[13] By contrast, the simultaneous choice of price and quality is at the heart of the present paper, which also considers the opposite preference correlation pattern, investigates the possibility of price/quality dispersion, among many other aspects not present in Ellison's contribution.

In turn, Bonatti (2011) develops a model of nonlinear pricing with competition where consumers' tastes for quality are brand-specific. In this setting, conditional on choosing a given brand, high-type consumers are more brand loyal than low types, thus requiring larger discounts to switch brands. Importantly, this comovement is hard-wired to the structure of heterogeneity in Bonatti's contribution. In our model, by contrast, consumers' tastes for quality do not vary across brands, which allows us to *exogenously* change the brand loyalty of each type of consumer. Bonatti finds that quality levels are distorted downwards in equilibrium, which also occurs in our model when high-type consumers are less prone to switch brands.

More recently, Chade and Swinkels (2019) propose a model where vertically differentiated firms compete to screen consumers with private information about their willingness to pay for quality. Firms are differentiated in their ability to produce different quality levels, which leads to segmentation in equilibrium. As in here, they note the possibility of non-existence of pure-strategy equilibrium. We see both models as offering complementary contributions: Whereas firms are asymmetric in their ability to serve different consumer types in Chade and Swinkels (2019), the asymmetry in the current paper rather pertains to the brand loyalty of different consumer types.

[11]This limitation is naturally absent in models where consumers exhibit no brand tastes, such as Champsaur and Rochet (1989). In their model, firms are able to commit to a range of qualities before choosing prices, which may generate market power. Our analysis holds unchanged under this alternative timing assumption.

[12]Other models exhibiting the demand specification of Rochet and Stole (2002) employ numerical solution methods. Examples include Borenstein (1985), Borenstein and Rose (1994), Wilson (1993), and Yang and Ye (2008). See also Stole (1995) for the case where firms compete observing consumers' brand preferences, but not their tastes for quality.

[13]See also Verboven (1999).

Finally, Dessein (2003, 2004) studies competition between telecommunication networks for users with heterogenous calling patterns who self-select into their preferred calling plans. In his 2003 contribution, Dessein provides general conditions under which access charges do not affect profits. Dessein (2004) revisits this question in a setting where heavy and light users perceive the substitutability of the competing networks differently. He then shows that access charges (more often) affect profits, and provides sufficient conditions for the complete-information outcome to violate incentive constraints. We complement Dessein's contribution by developing a complete equilibrium analysis.

Our counter-intuitive comparative statics also relate to classic contributions in price theory, where discrimination (in the form of menus) is absent. For instance, Dorfman and Steiner (1954) show that, as firms' market power goes down (as measured by an increase in the elasticity of substitution), quality provision can either increase or decrease, as so does welfare.[14] More recently, Chen and Riordan (2008) show, in the context of a random utility model, that duopoly may lead to higher prices than monopoly, as increasing product variety may reduce the price-elasticity of demand. By contrast, our results are driven by the interplay between asymmetric information (manifested in the incentive constraints that shape the firms' decisions) and competition, as captured by the varying degrees of consumers' brand loyalty.

2 Model

There is a unit-mass continuum of consumers with single-unit demands for a vertically differentiated good. Consumers are heterogeneous in their tastes for quality, denoted by θ, and their tastes for brands, denoted by x. For each consumer, θ is a draw from a distribution with binary support $\{\theta_l, \theta_h\} \subset \mathbb{R}_{++}$, where $\Delta\theta \equiv \theta_h - \theta_l > 0$, and associated probabilities p_l and p_h (with $p_l, p_h > 0$ and $p_l = 1 - p_h$).[15] As is the Hotelling model, x is uniformly distributed over the unit segment $[0, 1]$, and independent of θ.[16] The pair (θ, x) is private information of each consumer, and independently drawn across consumers. For convenience, we abuse terminology and refer to the quality taste θ as the consumer's *type*.[17]

There are two firms associated with the two ends of the unit segment, indexed by $j \in \{a, b\}$. We assume that each firm's offer consists of a menu of quality-price pairs.[18] We let (q_k^j, y_k^j) be the quality-price pair designed by firm j for consumers whose taste for quality is θ_k, where $k \in \{l, h\}$.

[14] See Dranove and Satterthwaite (2000) for a modern treatment of this seminal contribution.

[15] See subsection 7.1 for the case of a continuum of types.

[16] See subsection 7.2 for other discrete-choice specifications.

[17] This terminology reflects the fact that firms cannot screen consumers' tastes for brands.

[18] We thus rule out stochastic as well as reciprocal mechanisms (where the offer of a firm may depend on that of its competitor). Given our restriction to menus of price-quality pairs, it is without loss of generality to suppose firms' menus have the same cardinality as the support of consumers' tastes for quality (which is two in the baseline model).

A menu is then denoted by $m^j \equiv ((q_k^j, y_k^j) : k \in \{l, h\})$. Our choice of labels implies that type-k consumers (i.e., those with taste for quality θ_k) prefer the contract (q_k^j, y_k^j) to the contract designed to the other consumer type. This leads to the following incentive-compatibility constraints:

$$IC_k : \qquad u_k^j \equiv \theta_k q_k^j - y_k^j \geq \theta_k q_{\hat{k}}^j - y_{\hat{k}}^j \quad \text{where} \quad k, \hat{k} \in \{l, h\}, k \neq \hat{k}.$$

We refer to u_k^j as type-k's indirect utility under firm j's menu.

A consumer with brand taste $x \in [0, 1]$ and taste for quality θ_k prefers purchasing from firm a rather than firm b if and only if

$$u_k^a - t_k x \geq u_k^b - t_k(1 - x),$$

where the *brand loyalty parameter* $t_k \geq 0$ captures the intensity of brand preferences by type-k consumers. The comparison between t_l and t_h determines whether consumers with low or high tastes for quality are more prone to switch brands in response to changes in firms' offers.

Following Bénabou and Tirole (2016), we assume that consumers brand tastes do not affect the comparison between one's preferred brand and the outside option. Namely, a consumer with brand taste $x \in [0, 1]$ and taste for quality θ_k prefers the contract offered by firm a relative to the outside option if and only if

$$u_k^a - t_k x \geq t_k \max\{-x, -(1 - x)\}, \tag{1}$$

and analogously for firm b. Intuitively, consumers have the option to choose among the products of non-strategic fringe suppliers, or non-market substitutes, that mimic the characteristics of each firm j, therefore inheriting their respective taste shocks. The max operator in the right-hand side of (1) means that consumers pick their preferred outside good. This specification allows us to vary the intensity of brand preferences (across consumer types), without affecting the relative value of non-participation. Indeed, condition (1) boils down to $u_k^a \geq 0$ for $x \leq \frac{1}{2}$, and implies that, whenever both firms make positive sales, the relevant margin for pricing is always the competitive one (substitution towards the competing firm), never the participation one (substitution towards the outside option).[19]

In light of the above, it is without loss of generality to restrict attention to *implementable* menus, which are those that satisfy, for each k, constraint IC_k and the standard individual rationality constraint IR_k: $u_k^j \geq 0$. For a given profile of menus (m^a, m^b), the demand for firm a's contract to type-k's consumers is then:[20]

$$D_k^a(m^a, m^b) = p_k I\left(\frac{1}{2} + \frac{u_k^a - u_k^b}{2t_k}\right), \quad \text{where} \quad I(x) \equiv \min\{\max\{0, x\}, 1\}.$$

Firms incur a per-unit cost $\varphi(q)$ for providing a good of quality q. The cost function $\varphi(\cdot)$ is twice continuously differentiable, strictly increasing and strictly convex. It also satisfies $\varphi(0) = \varphi'(0) = 0$

[19] By contrast, in the standard Hotelling framework, changes in t_k simultaneously affect substitution and participation. Disentangling these two effects helps interpretation and tractability of the model, as we discuss in the next section.

[20] The min/max operators in the definition of function $I(\cdot)$ reflect the fact that type-k demand is between 0 and p_k.

and $\lim_{q\to\infty}\varphi'(q)=\infty$, which guarantees that the efficient qualities,

$$q_k^e \equiv \arg\max_q \ \theta_k q - \varphi(q),$$

exist and are strictly positive for both consumer types, as so are the efficient surplus $S_k^e \equiv \theta_k q_k^e - \varphi(q_k^e)$.

Therefore, the profit by firm $j\in\{a,b\}$ per sale of contract (q_k^j, y_k^j) equals $y_k^j - \varphi(q_k^j)$, and its total profit under the menu profile (m^a,m^b) is

$$\sum_{k\in\{l,h\}} D_k^j(m^a,m^b)\left(y_k^j - \varphi(q_k^j)\right).$$

Firms simultaneously post menus, after which each consumer chooses her preferred contract across firms' menus. A (possibly mixed) strategy by each firm is a distribution over implementable menus σ^j. A symmetric equilibrium (for short, equilibrium), possibly in mixed strategies, is a distribution over implementable menus σ^* that is a best response to itself.

3 Preliminaries

3.1 A change of variables

Similarly to Armstrong and Vickers (2001), we find it convenient to formulate the firms' problems in terms of indirect utilities, rather than price-quality pairs. To this end, the next lemma answers the following question: which menu m maximizes a firm's profit conditional on delivering the indirect utility profile (u_l, u_h)? We drop the superscript j to lighten notation.

Lemma 0. ***[Incentive Compatibility]*** *Consider an equilibrium menu $m=\{(q_l,y_l),(q_h,y_h)\}$, and let (u_l,u_h) be its profile of indirect utilities. Then the menu's qualities are given by*

$$q_l(u_l,u_h)=\begin{cases}\frac{u_h-u_l}{\Delta\theta} & \text{if} \quad u_h-u_l<q_l^e\Delta\theta\\ q_l^e & \text{if} \quad u_h-u_l\geq q_l^e\Delta\theta\end{cases} \quad \text{and} \quad q_h(u_l,u_h)=\begin{cases}\frac{u_h-u_l}{\Delta\theta} & \text{if} \quad u_h-u_l>q_h^e\Delta\theta\\ q_h^e & \text{if} \quad u_h-u_l\leq q_h^e\Delta\theta.\end{cases}$$

Given (u_l,u_h), one can determine via Lemma 0 the equilibrium quality levels (q_l,q_h), and hence also the prices (y_l,y_h) of any equilibrium menu. It is therefore convenient to abuse notation and identify each menu to its indirect-utility profile: $m=(u_l,u_h)$. The surplus generated by each contract $k\in\{l,h\}$ in menu $m=(u_l,u_h)$ is then given by

$$S_k(u_l,u_h)\equiv q_k(u_l,u_h)\,\theta_k - \varphi\left(q_k(u_l,u_h)\right).$$

As previous literature has shown, in the *monopolistic (or Mussa-Rosen) menu*, the individual rationality constraint for low-valuation consumers (IR_l) binds ($u_l=0$), as so does the incentive constraint for high-valuation consumers (IC_h). In light of this, the high-type indirect utility at the monopolist's solution, denoted by u_h^∞, is such that

$$u_h^\infty = \arg\max_{u_h\geq 0}\ \left\{p_l S_l(0,u_h)+p_h\left(S_h^e-u_h\right)\right\}.$$

High types obtain no rents ($u_h^\infty = 0$) and low types are not served ($q_l^\infty = 0$) in case $\theta_l - \frac{p_h}{p_l}\Delta\theta \leq 0$. Otherwise, $u_h^\infty > 0$ is implicitly given by

$$\frac{p_l}{p_h}\frac{\partial S_l}{\partial u_h}(0, u_h^\infty) - 1 = 0, \tag{2}$$

in which case low-type consumers are offered a positive but inefficiently low quality level. We denote this menu by $m^\infty \equiv (0, u_h^\infty)$. By contrast, consumers appropriate the entire efficient surplus in the *perfectly competitive (or Bertrand) menu* $m^0 \equiv (S_l^e, S_h^e)$, where quality provision is efficient to consumers of all valuations and firms derive zero profits from each contract in the menu.

3.2 Balanced Duopoly

Before analyzing the full-fledged model, we shall first revisit the case of a "balanced duopoly," where brand loyalty is invariant to type. This is the subject of the next proposition, where we let $\bar{\eta} \equiv S_h^e - q_l^e \Delta\theta$.

Proposition 0. ***(Equilibrium: Balanced Duopoly)*** *Suppose the intensity of brand preferences is the same across types, and let $t \equiv t_l = t_h$. Then there exists a unique pure-strategy equilibrium, which is such that:*

(a) If $t \in [0, \bar{\eta}]$, $u_k^* = \max\{S_k^e - t, 0\}$ for $k \in \{l, h\}$. Quality provision is efficient.

(b) If $t > \bar{\eta}$, $u_l^* = 0$ and, whenever positive, u_h^* is implicitly given by

$$\frac{S_h^e - u_h^*}{t} + \left(\frac{p_l}{p_h}\frac{\partial S_l}{\partial u_h}(0, u_h^*) - 1\right) = 0. \tag{3}$$

Moreover, u_h^* is decreasing in t, and converges to the monopolistic level u_h^∞ as t grows unbounded. Quality is efficiently (resp., under-) provided to high-type (resp., low-type) consumers.

When t is small (i.e., $t \leq S_l^e$), the equilibrium is close to the perfectly competitive outcome, where neither individual rationality or incentive constraints bind. Accordingly, quality is efficiently provided to consumers of all valuations, and firms' markups are constant across the product line (and equal to t). This outcome coincides type-by-type with that of a Hotelling model where consumer valuations are observable. Moreover, it can be implemented by the "cost-plus-fee" tariff $T(q) \equiv t + \varphi(q)$, as first observed by Armstrong and Vickers (2001) and Rochet and Stole (2002).

For t larger than S_l^e, the individual rationality constraint is binding for low-valuation consumers, whose surplus is fully extracted by firms. As long as $t \leq \bar{\eta}$, this is the only binding constraint, and quality provision remains efficient to both consumer types.[21] Otherwise, the incentive constraint of high-valuation consumers also binds, and equilibrium is characterized by equation (3). As the

[21] Indeed, $\bar{\eta} > S_l^e$, as implied by the convexity of the cost function φ. This implies that constraint IR_l binds "before" (i.e, for smaller t's) IC_h in the balanced-duopoly case.

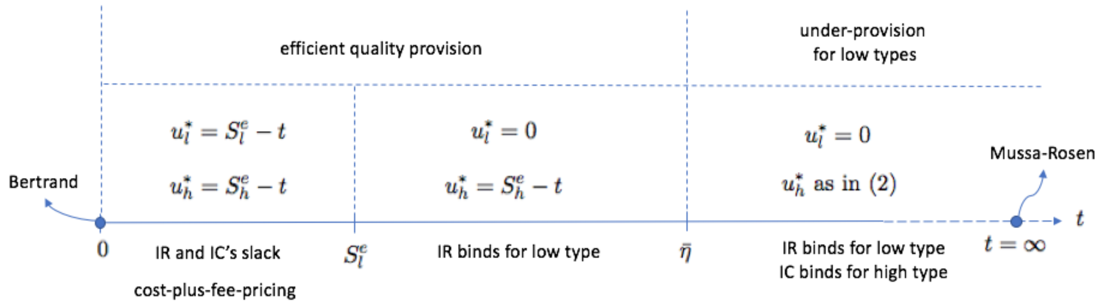

Figure 1: Equilibrium in a balanced duopoly.

intensity of brand preferences increases (i.e., t grows large), firms are able to extract more rents from high-valuation consumers, which requires decreasing the low-type quality away from its efficient level. In the limit as $t \to \infty$, equilibrium converges to the monopolistic outcome. This is can be readily seen from equation (3): Its first term vanishes as t grows unbounded, making the equilibrium condition coincide with the monopolist's optimality condition in equation (2).

Remark 1. ***(Previous literature)*** *In the Hotelling specification considered by Rochet and Stole (2002), for t in a neighborhood above $\frac{2}{3}S_l^e$, firms are local monopolists for low-type consumers (who have uniformly distributed reservation utilities), but compete under full market coverage for high types. As a result, variations in t affect the intensity of brand preferences of the latter (whose relevant comparison is across firms' contracts), while affecting consumer participation for the former (whose relevant comparison is between the closest firm and the outside option). As t grows large, the volume of sales to consumers (of any type) shrinks to zero. In the current model, by contrast, the relative value of the outside option is not affected by changes in t, which therefore can be identified with the intensity of brand preferences. As a result, firms are never local monopolists, being always in competition for both consumer types. This explains why, in the current model, equilibrium approaches the monopolist outcome as consumers become more brand loyal.*

4 Bottom-of-barrel duopoly

A bottom-of-barrel duopoly bears its name due to the fact that brand preferences are stronger among those consumers who have the lowest willingness to pay for quality (and therefore the lowest potential for profits). To capture this possibility in the starkest manner, we assume that high-valuation consumers see firms as perfect substitutes, i.e., $t_h = 0$. In turn, the intensity of brand preferences among low-valuation consumers is unconstrained, as t_l is allowed to take any positive value. To streamline the exposition, we assume that:

Assumption 1. $\eta_h \equiv S_h^e - q_h^e \Delta\theta > 0$.

When the cost function has the power form, $\varphi(q) = \frac{1}{a}q^a$, Assumption 1 is satisfied if and only if $a\theta_l > \theta_h$. Intuitively, this assumption requires consumer types to be sufficiently close so that the self-selection constraints affect equilibrium outcomes. If this condition is violated, we are left with the less interesting case where the incentive constraint IC_l never binds, and equilibrium is always efficient and in pure strategies.[22]

Before describing the equilibrium, it is convenient to define the *zero-profit h-type utility* $\mathring{u}_h$ as the implicit solution to

$$S_h(0, \mathring{u}_h) - \mathring{u}_h = 0. \tag{4}$$

In words, $\mathring{u}_h$ is the highest indirect utility that firms can relinquish to high-type consumers while obtaining zero profit from the high-type contract and fully extracting rents from low types. By virtue of Assumption 1, $\mathring{u}_h > q_h^e \Delta\theta$, which implies that the high-type quality of the menu $(0, \mathring{u}_h)$ is distorted upwards: $q_h(0, \mathring{u}_h) > q_h^e$. More broadly, Assumption 1 guarantees that the incentive constraint IC_l binds whenever low-valuation consumers obtain a sufficiently low indirect utility in equilibrium.

The next proposition clarifies when a pure-strategy equilibrium exists.

Proposition 1. ***(Pure-Strategy Equilibrium)*** *Suppose there is perfect competition for high types ($t_h = 0$), but imperfect for low types ($t_l > 0$). Then:*

(a) If $t_l \in (0, S_l^e - \eta_h]$, there is a unique pure-strategy equilibrium, with $u_l^* = S_l^e - t_l$ and $u_h^* = S_h^e$. Quality provision is efficient.

(b) No pure-strategy equilibrium exists if $t_l \in (S_l^e - \eta_h, \tilde{t}_l)$, where the threshold:

$$\tilde{t}_l \equiv \inf \left\{ t_l : \frac{\partial S_h}{\partial u_l}(0, \mathring{u}_h) < \frac{1}{2}\frac{p_l}{p_h}\left(1 - \frac{S_l^e}{t_l}\right) \right\},$$

and $\tilde{t}_l \equiv \infty$ if the inequality inside brackets is violated for all $t_l > 0$.

(c) If $t_l \in [\tilde{t}_l, \infty)$, there is a unique pure-strategy equilibrium, with $u_l^* = 0$ and $u_h^* = \mathring{u}_h$. Quality is efficiently (resp., over-) provided to low-type (resp., high-type) consumers.

As illustrated in Figure 2, Proposition 1 identifies three regions. When t_l is small, the equilibrium is *as if* there was complete information about agents' valuations, as no incentive constraint binds. Accordingly, quality is efficiently provided, and agents obtain the efficient surplus discounted by firms' profits t_k (as in a complete-information Hotelling model). Because $t_l > t_h = 0$, firms obtain zero profits from high types, but a positive profit from low types.

When t_l exceeds the threshold $S_l^e - \eta_h$, the "complete-information" equilibrium described above can no longer be sustained, as constraint IC$_l$ would be violated. This means that no pure-strategy equilibrium exhibits efficient quality provision. Perfect competition, however, implies that firms

[22]Namely, the equilibrium is such that $u_l^* = \max\{S_l^e - t_l, 0\}$ and $u_h^* = S_h^e$.

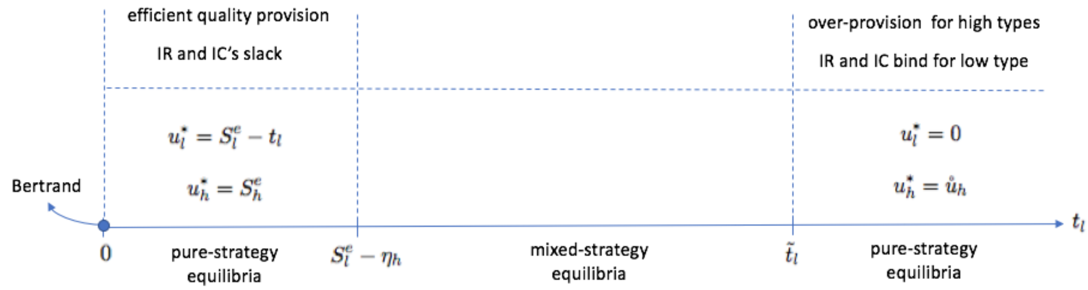

Figure 2: Equilibrium in a bottom-of-barrel duopoly.

obtain zero profit from high-type consumers. As a consequence, firms can ignore the existence of high-type consumers and respond *as if* they were playing a Hotelling competition game where only low types are present. These two observations imply that, if an equilibrium in pure strategies exists, it has to satisfy

$$u_l^* = S_l^e - t_l \quad \text{and} \quad S_h(u_l^*, u_l^*) - u_h^* = 0. \tag{5}$$

Crucially, Assumption 1 implies that the high-type quality is above its efficient level in this putative equilibrium.

There is always a profitable deviation to this putative equilibrium provided $t_l \in (S_l^e - \eta_h, S_l^e]$. It works as follows: the deviating firm grants a small discount $\delta > 0$ to low types, which relaxes the IC_l constraint. This enables the firm to reduce the quality provided to high types, therefore increasing the efficiency from their respective contracts (recall there was over-provision in the putative equilibrium). Because there is perfect competition among high types, the deviating firm can then adjust prices to slightly undercut its rival, conquering the whole high-type market and appropriating the correspondent efficiency gain. That this deviation is profitable comes from the fact that the profit gain among high types is of first-order magnitude, while the discount δ to low types entails only a second-order profit loss.[23] We refer to this strategy as the *relax-and-undercut* deviation, as it involves relaxing incentive compatibility to enable undercutting the rival firm.

For $t_l > S_l^e$, the IR_l constraint necessarily binds. By the same reasoning above, if an equilibrium in pure strategies exists, it has to be such that $(u_l^*, u_h^*) = (0, \mathring{u}_h)$, where $\mathring{u}_h$ is zero-profit h-type utility from equation (4). The relax-and-undercut deviation above now produces a first-order profit gain among high-types at the expense of a *first-order profit loss* among low types. The loss is now first-order because the putative equilibrium utility $u_l^* = 0$ is at the corner dictated by the IR_l constraint (therefore exhibiting a non-zero shadow cost). The race between these two effects is resolved in favor of deviating if and only if t_l is below the threshold $\tilde{t}_l$. Intuitively, if brand preferences are mild (in the sense that $t_l < \tilde{t}_l$), the business-stealing effect from discounting the low-type price is sufficiently

[23]To see why, note that $u_l^* = S_l^e - t_l$ in an interior optimum, therefore being a local maximand.

large to render the relax-and-undercut deviation profitable. This is always the case when $\tilde{t}_l = \infty$.[24]

By contrast, when $\tilde{t}_l < \infty$, sufficiently intense brand preferences (in the sense that $t_l \geq \tilde{t}_l$) bring the pure-strategy equilibrium back to existence. The reason is that, when competition for low types is mild, discounting the low-type price attracts too few extra customers, rendering the relax-and-undercut deviation unprofitable. The putative equilibrium $(u_l^*, u_h^*) = (0, \mathring{u}_h)$ is then an equilibrium. Because IC$_l$ binds, high types are provided inefficiently high quality, while appropriating the full (inefficient) surplus produced by their contract. Low types endure full rent extraction, as $u_l^* = 0$, and are offered their efficient quality level.

In sum, Proposition 1 reveals that, whenever the incentive constraint matters in a bottom-barrel duopoly, a pure-strategy equilibrium exists (if at all) only when the intensity of brand preferences is sufficiently different across consumer types. Else, the equilibrium exhibits dispersion of offers, whereby firms randomize their choice of menus. In particular, the non-existence of pure-strategy equilibria may be interpreted as a sign of market instability, as no pair of menus exists such that firms can stick to their offers across time while best responding each other. Crucially, this result is solely due to the interplay between asymmetric market power and self-selection constraints, which contrasts with the received wisdom according to which, in private-value settings, dispersion of offers stems from search/informational frictions faced by consumers (as in Varian 1980 and Burdett and Judd 1983).

Dispersion of Offers. We now describe a mixed-strategy equilibrium when $t_l \in (S_l^e - \eta_h, \tilde{t}_l)$. This equilibrium exhibits the following property: across any two menus offered in equilibrium, the indirect utilities offered to low- and high-type consumers co-move.[25] This is the subject of the next definition, first proposed by Garrett et al (2019):

Definition 1. *[**Ordered Equilibrium**] A mixed-strategy equilibrium is said to be ordered if, for any two menus $\mathcal{M} = (u_l, u_h)$ and $\mathcal{M}' = (u_l', u_h')$ offered in equilibrium, $u_l < u_l'$ if and only if $u_h < u_h'$. In this case, the menu (u_l', u_h') is said to be more generous than the menu (u_l, u_h).*

Because indirect utilities co-increase, every menu $\mathcal{M} = (u_l, u_h)$ offered in an ordered equilibrium can be described by a *support function* $\mathcal{U}_l$, strictly increasing and bijective, such that $u_l = \mathcal{U}_l(u_h)$. Moreover, we can describe firms' randomization solely in terms of F_h^*, the marginal cdf over the indirect utilities offered to type-h consumers. This is so because, for any equilibrium menu, $F_l^*(u_l) = F_h^*(u_h)$, where F_l^* is the marginal cdf over type-l indirect utilities. As such, the joint distribution over menus, denoted by F^*, has support over the graph of the support function $\mathcal{U}_l$, which (counter-) domain is denoted by Υ_h (Υ_l).

[24]Whether $\tilde{t}_l$ is finite or not depends on parameters. For instance, if the cost is quadratic, $\varphi(q) = \frac{1}{2}q^2$, $\tilde{t}_l = \infty$ if $\kappa_l \equiv 1 - 2\frac{p_h}{p_l}\left(\frac{2\theta_l - \theta_h}{\Delta\theta}\right) \leq 0$, but equals $\tilde{t}_l = \frac{S_l^*}{\kappa_l} < \infty$ if $\kappa_l > 0$.

[25]In fact, we can establish a stronger statement; namely, that all mixed-strategy equilibria (if more than one exists) are ordered.

Denoting by $\mathbb{E}_{F_l^*}[\tilde{u}_l]$ the mean of u_l as induced by the cdf F_l^*, let us define

$$\mathring{u}_l \equiv \frac{S_l^e - t_l + \mathbb{E}_{F_l^*}[\tilde{u}_l]}{2}.$$

The next proposition characterizes a mixed-strategy equilibrium.

Proposition 2. ***(Mixed-Strategy Equilibrium)*** *Suppose there is perfect competition for high types ($t_h = 0$), but imperfect for low types ($t_l > 0$). If $S_l^e - \eta_h < t_l < \tilde{t}_l$, there exists a mixed-strategy equilibrium, which is ordered. In this equilibrium, the support of indirect utilities is an interval, $\Upsilon_k = [\underline{u}_k, \bar{u}_k]$, and the support function $\mathcal{U}_l(\cdot)$ and cdf F_h^* of high-type's indirect utilities jointly satisfy*

$$\mathcal{U}_l(u_h) - \underline{u}_l = 2\left(\frac{S_h(\mathcal{U}_l(u_h), u_h) - u_h}{\frac{\partial S_h}{\partial u_l}(\mathcal{U}_l(u_h), u_h)}\right)\left(\frac{\mathcal{U}_l(u_h) - \mathring{u}_l}{\mathcal{U}_l(u_h) + \underline{u}_l - 2\mathring{u}_l}\right) \qquad \forall\ u_h \in [\underline{u}_h, \bar{u}_h],$$

and

$$F_h^*(u_h) = \frac{p_l}{p_h}\left(\frac{\mathcal{U}_l(u_h) - \mathring{u}_l}{t_l}\right)\left(\frac{\partial S_h}{\partial u_l}(\mathcal{U}_l(u_h), u_h)\right)^{-1} \qquad \forall\ u_h \in [\underline{u}_h, \bar{u}_h],$$

with boundary conditions

$$\underline{u}_l = \max\{\mathring{u}_l, 0\}, \qquad \underline{u}_h = S_h(\underline{u}_l, \underline{u}_h), \qquad \bar{u}_l = \mathcal{U}_l(\bar{u}_h), \qquad \textit{and} \qquad \bar{u}_h = (F_h^*)^{-1}(1).$$

Moreover, F_h^* is absolutely continuous at any $u_h \in (\underline{u}_h, \bar{u}_h]$. When t_l is sufficiently large, in which case $\mathring{u}_l < 0$, F_h^* exhibits a mass point at $\underline{u}_h$ (i.e., $F_h^*(\underline{u}_h) > 0$).

The characterization of Proposition 2 clarifies how equilibrium transitions from being in pure, then mixed, and then (possibly) again in pure-strategies as t_l increases. Namely, it identifies two varieties of mixed-strategy equilibria. In the first variety, the IR$_l$ constraint is slack at all equilibrium menus, or, equivalently, the least generous menu offers a positive indirect utility to low-valuation consumers: $\underline{u}_l > 0$. This occurs when t_l is sufficiently small, in which case the cdf F_h^* has no mass points. By contrast, for t_l sufficiently large, the IR$_l$ constraint binds in the least generous menu $(\underline{u}_l, \underline{u}_h) = (0, \mathring{u}_h)$, which is then a mass point of the mixed strategy F^*. As revealed by Proposition 1, whenever $\tilde{t}_l$ is finite, the probability of this mass point is one (i.e., a pure strategy equilibrium resumes existing) if t_l is large enough (namely, larger than $\tilde{t}_l$).

Equilibrium Properties. As firms' market power is higher among low-type consumers, the constraint IC$_l$ binds in all equilibrium menus, which therefore exhibit over-provision of quality in the high-type contract. This observation is key to understand the ordered property of equilibrium, which is intimately related to the relax-and-undercut deviation described above. Intuitively, firms differentiate themselves according to how big is the "discount" they give to low-type consumers. Those firms who grant the largest discounts are able to undertake the greatest reductions in the quality provided to high types, therefore obtaining the largest welfare gains. The larger is the surplus produced by the h-type contract, the larger is the incentive to relinquish more indirect utility to high types, so as to expand demand. As a consequence, the firms who offer the highest indirect utilities to low types (i.e., those who "discount" more) are also the ones that offer the highest indirect utilities to high types, i.e., menus are ordered.

Moreover, the quality of high-type contracts is smaller, while the efficiency is larger, the more generous is the firm. The latter property implies that firms offering more generous menus obtain a larger profit share from consumers with high types, and lower from those with low types. The following corollary summarizes this discussion.

Corollary 1. ***(Mixed-Strategy Equilibrium: Properties)*** *Consider the mixed-strategy equilibrium of Proposition 2. All equilibrium menus exhibit over-provision of quality at the top and efficient provision at the bottom of the product line. More generous menus exhibit lower qualities at the top (less distortion), and generate a higher (resp., lower) share of their profit from high-type (resp., low-type) consumers.*

Dispersion of offers has also interesting implications regarding the effects of private information on consumers' payoffs (vis-à-vis the complete-information benchmark). First, high-type consumers obtain *negative* informational rents (i.e., they would like to provide verifiable information about their tastes for quality, if that was possible). The reasons are twofold: First, firms obtain positive profits from selling to high types, even though there is perfect competition for such consumers; second, the contract tailored to these consumers is inefficient (due to over-provision of quality). As a result, the rent left to high types (difference between surplus and profit) is necessarily smaller than under complete information. Conversely, firms obtain lower profits among low types than under complete information, as $\underline{u}_l > \max\{0, S_l^e - t_l\}$, even though quality is efficiently provided to such consumers. As a result, private information is beneficial to low-type consumers, i.e., informational rents are positive.[26] These insights generalize well beyond the bottom-barrel case, as described in Section 6.

5 Cream-skimming duopoly

Just like in some markets low-valuation consumers have stronger brand preferences than those with high-valuations (e.g., cable TV), in others the reverse pattern is verified (e.g., air travel). Similarly to the bottom-barrel case, this section considers the starkest version of this asymmetry, assuming there is perfect competition for low types, $t_l = 0$, but imperfect competition for high types, $t_h > 0$.

[26] The distribution of market offers from Proposition 2 sharply differs from that obtained in Garrett et al (2019), where dispersion is due to informational frictions. First, the patterns of quality distortions are reversed (over-provision at the top of the product line, rather than under-provision at the bottom). Second, informational rents can be negative in our model, whereas they are always positive in Garrett et al (2019). Third, in contrast to our theory, the latter paper predicts that the distribution of menus never exhibits mass points.

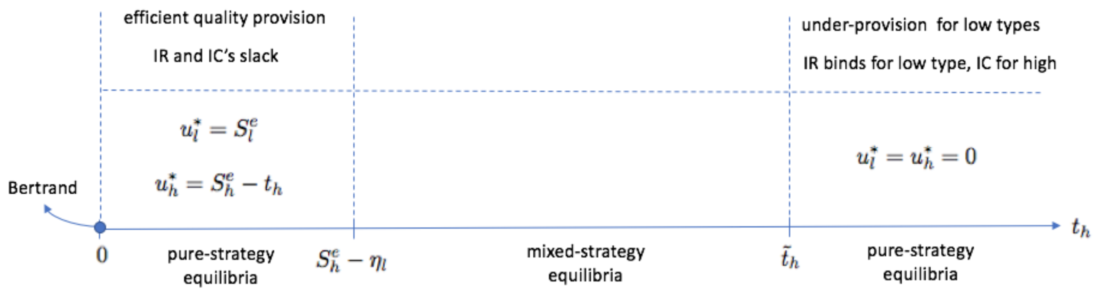

Figure 3: Equilibrium in a cream-skimming duopoly.

Because firms' market power is stronger among those consumers who have the highest willingness to pay, we refer to this case as the cream-skimming duopoly.

The analysis that follows is parallel to that of the last section. We adopt analogous notation by defining $\eta_l \equiv S_l^e + q_l^e \Delta\theta$, and observing that $S_h^e - \eta_l > 0$ (due to the convexity of the cost function). The next proposition is the counterpart to Proposition 1 in the context of a cream-skimming duopoly.

Proposition 3. ***(Pure-Strategy Equilibrium)*** *Suppose there is perfect competition for low types ($t_l = 0$), but imperfect for high types ($t_h > 0$). Then:*

(a) If $t_h \in [0, S_h^e - \eta_l]$, there is a unique pure-strategy equilibrium, with $u_l^* = S_l^e$ and $u_h^* = S_h^e - t_h$. Quality provision is efficient.

(b) No pure-strategy equilibrium exists if $t_h \in (S_h^e - \eta_l, \tilde{t}_h)$, where the threshold:

$$\tilde{t}_h \equiv \inf \left\{ t_h : \frac{\partial S_l}{\partial u_h}(0,0) < \frac{1}{2}\frac{p_h}{p_l}\left(1 - \frac{S_h^*}{t_h}\right) \right\},$$

and $\tilde{t}_h \equiv \infty$ if the inequality inside brackets is violated for all $t_h > 0$.

(c) If $t_h \in [\tilde{t}_h, \infty)$, there is a unique pure-strategy equilibrium, with $u_l^* = u_h^* = 0$. Quality is efficiently provided to high-type consumers, while low-types are not served ($q_l^* = 0$).

Proposition 3 is illustrated in Figure 4. When t_h is small, the equilibrium is in pure strategies and coincides with that under complete information. As t_h exceeds the threshold $S_h^e - \eta_l$, the incentive constraint IC_h starts to bind. Because competition is perfect, any pure-strategy equilibrium has to generate zero profit from sales to low-type consumers. This is turn implies that the equilibrium outcome in the high-type market has to be *as if* no other consumer type existed, which leads to the putative equilibrium

$$S_l(u_l^*, u_l^*) - u_l^* = 0 \quad \text{and} \quad u_h^* = S_h^e - t_h. \tag{6}$$

Crucially, that $t_h > S_h^e - \eta_l$ implies that the low-type quality is below its efficient level in this putative equilibrium.

The following deviation, analogous to the relax-and-undercut deviation considered in the last section, increases profit whenever $t_h \in (S_h^e - \eta_l, S_h^e]$. It consists on granting a small discount to high-valuation consumers, which relaxes the IC_h constraint. This enables the deviating firm to increase the quality provided to low types, therefore increasing the efficiency from their respective contracts (recall there was under-provision in the putative equilibrium). Because there is perfect competition among low types, the deviating firm can then adjust prices to slightly undercut its rival, conquering the whole low-type market and appropriating the correspondent efficiency gain. As in the bottom-of-barrel case, this deviation trades off a first-order gain (among low types, whose quality was inefficient) with a second-order loss (on the profits collected from high types).

For $t_h > S_h^e$, the IR_h constraint necessarily binds, and, by the same reasoning above, the pure-strategy equilibrium (if it exists) is such that $(u_l^*, u_h^*) = (0, 0)$. Accordingly, low types are not served, while high types get efficient quality but no rents. The relax-and-undercut deviation continues to work provided t_h is not too high (in the sense that $t_h < \tilde{t}_h$). This guarantees that the profit loss from giving a discount to high types is not too large relative to the profit gain from serving the whole low-type market (left unserved in the putative equilibrium).

By contrast, if brand preferences are sufficiently intense among high types (in the sense that $t_h \geq \tilde{t}_h$), the relax-and-undercut deviation is no longer profitable (a result of the business-stealing effect among high types being too small). The putative equilibrium is then an equilibrium, and low types are excluded from the market.[27] This occurs notwithstanding low types being "up for grabs" (they see firms as homogenous) and exhibiting a positive willingness-to-pay for quality. It is precisely because serving low types dissipates profits from (the very profitable) high types that exclusion at the bottom occurs, similarly to what happens under monopoly.

The question of whether firms can sustain in equilibrium the full extraction of high-type rents (which requires not serving low types) is reminiscent of recurring price/quality cycles in airline markets. Anecdotal evidence suggests that, in some routes served by multiple airlines, prices are high for extended periods of time, which effectively shuns cheap-stake tourists who would only buy low-priced tickets. One of the competing airlines then changes its strategy, trying to absorb this latent demand by introducing a "low-cost" alternative (of substantively lower quality), while decreasing regular fares. In some cases, such low-cost alternatives prove unprofitable, and the high-price outcome is restored. The fast-moving nature of these markets, where product features and prices are typically "unstable" and short-lasting, echoes the non-existence of pure strategy equilibria described in Proposition 3.

Dispersion of Offers. Similarly to the bottom-barrel case, a mixed-strategy equilibrium exists (i.e., offers are dispersed) whenever no pure-strategy equilibrium can be found. The structure of the

[27] As in the bottom-of-barrel case, whether $\tilde{t}_h$ is finite depends on parameters. If the cost is quadratic, $\tilde{t}_h = \infty$ if $\kappa_h \equiv 1 - 2\frac{p_l}{p_h}\frac{\theta_l}{\Delta\theta} \leq 0$, but equals $\tilde{t}_h = \frac{S_h^*}{\kappa_h} < \infty$ if $\kappa_h > 0$.

mixed-strategy equilibrium is the mirror-image of its bottom-barrel counterpart. As firms' market power is higher among high-type consumers, it is the constraint IC_h that binds (as opposed to IC_l), which implies there is under-provision of quality in the low-type contract of all equilibrium menus. Reflecting the relax-and-undercut strategy, firms differentiate themselves according to how big is the "discount" they give to high-type consumers, which enables them to provide higher quality to low-type consumers (increasing surplus). This raises incentives to poach low types, which explains why the equilibrium is again ordered.

Moreover, the quality and efficiency of low-type contracts increase as menus become more generous. As a result, the share of total profits obtained from low types also increases with the menu's generosity. Relative to complete information, equilibrium profits are lower for high types, but higher (indeed, positive) for low types (which market is perfectly competitive). Private information renders low types worse-off, but high types better-off. These predictions are the reverse of what happens in the bottom-barrel case. For brevity, we leave the complete description of this equilibrium to the Online Appendix.

6 Mid-barrel duopolies: The intermediate cases

The last two sections focused on the two extreme (but simpler) cases where there is perfect competition for one consumer type, but imperfect for the other. We characterized equilibria, unveiled the patterns of over- or under- provision of quality, and showed that, relative to complete information, asymmetric information always hurts *some* consumer type while benefits the other (which one is hurt depends on the profile of brand preferences). In this section, we show that our main insights are robust to environments where competition is imperfect for both consumer types. Completing the equilibrium characterization also enables us to develop comparative statics away from the extreme cases explored above.

6.1 Distortions

To describe the patterns of quality provision, let us define $\Lambda(t_l, t_h) \equiv t_h + \max\{S_l^e - t_l, 0\}$, and recall that $\eta_h < \bar{\eta}$.[28] Consider the following regions, illustrated in Figure 4:

$$E \equiv \left\{(t_l, t_h) \in \mathbb{R}^2_{++} : \eta_h \leq \Lambda(t_l, t_h) \leq \bar{\eta}\right\},$$

$$D_+ \equiv \left\{(t_l, t_h) \in \mathbb{R}^2_{++} : \Lambda(t_l, t_h) < \eta_h\right\}, \quad \text{and} \quad D_- \equiv \left\{(t_l, t_h) \in \mathbb{R}^2_{++} : \Lambda(t_l, t_h) > \bar{\eta}\right\}.$$

As established in the next proposition, equilibrium quality provision is efficient in region E. By contrast, quality is distorted upwards at the top (resp., downwards at the bottom) of the product line in region D_+ (resp., D_-):

[28] Indeed, recall from from Sections 3 and 4 that $\eta_h = S_h^e - q_h^e \Delta\theta < S_h^e - q_l^e \Delta\theta = \bar{\eta}$.

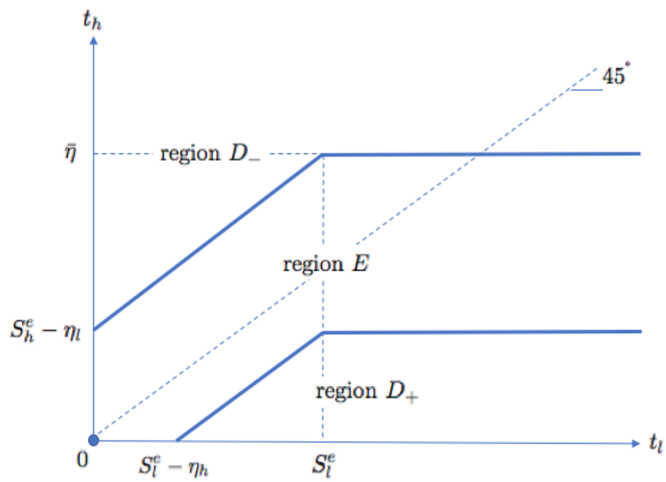

Figure 4: Equilibrium distortions and brand substitutability across types.

Proposition 4. ***(Distortions)*** *Suppose* $(t_l, t_h) \in \mathbb{R}^2_{++}$. *Then:*[29]

(a) If $(t_l, t_h) \in E$, *quality provision is efficient to both types. The unique equilibrium is in pure strategies, with* $u_k^* = \max\{S_k^e - t_k, 0\}$, $k \in \{l, h\}$.

(b) For $(t_l, t_h) \in D_+$, *there is over-provision of quality to high-type consumers in equilibrium.*

(c) For $(t_l, t_h) \in D_-$, *there is under-provision of quality to low-type consumers in equilibrium.*

Recall from Proposition 0(a) that IC's are slack in the "diagonal" (i.e., when $t_l = t_h$) provided the intensity of brand preferences is low (in the sense that $t_l = t_h < S_l^e$). The equilibrium is then identical to that under complete information. Proposition 4(a) shows that the complete-information outcome is an equilibrium provided brand preferences are not "too different" across consumer types (in which case IC's remain slack). The same holds true over the horizontal band in region E (in which $t_l \geq S_l^e$). The reason is that raising t_l, while keeping t_h constant, does not affect firm's incentives (as u_l^* is already at zero). So the balanced-duopoly equilibrium remains (the unique) equilibrium, which is efficient.

It is worth noting that, in region E, although equilibrium is efficient away from the 45-degree line, markups are not constant across contracts. This reveals that the cost-plus-fee pricing prediction of Rochet and Stole (2002) is a knife-edge consequence of assuming that $t_l = t_h$.

Let us now consider Claim (b), which reveals that, in region D_+, over-provision of quality at the top prevails in equilibrium, be it in pure or mixed strategies. This claim generalizes the insights from the bottom-of-barrel duopoly to instances where competition is imperfect for high-type consumers. Namely, it reveals that, fixing $t_h > 0$, over-provision of quality occurs in equilibrium if

[29] Notice that $(t_l, t_h) \in \mathbb{R}^2_{++}$ implies firms payoffs are continuous in actions. Therefore, a symmetric Nash equilibrium (in either pure or mixed strategies) is guaranteed to exist (by the usual arguments dating from Nash 1950).

and only if t_l is sufficiently high. The logic of this result is similar to that of Section 3: Any putative equilibrium menu not exhibiting over-provision at the top is shown to either violate constraint IC_l, or to be suboptimal, provided the profile of brand preferences (t_l, t_h) is in region D_+. Intuitively, low-types are significantly more brand-loyal than high types, constituting the market segment with higher potential for profits. To prevent profit dissipation, whereby low types migrate to the high-type contract, firms set the quality of the "premium" product inefficiently high, which renders such product less attractive to low types.

Lastly, Claim (c) asserts that holding $t_l > 0$ fixed, under-provision of quality occurs in equilibrium if and only if t_h is sufficiently high. Again, this is true be the equilibrium in pure or mixed strategies. The logic of this result is familiar: Any putative equilibrium menu not exhibiting under-provision at the bottom is shown to either violate constraint IC_h, or to be suboptimal, provided the profile of brand preferences (t_l, t_h) is in region D_-. Intuitively, low-types are significantly less brand-loyal than high types, who constitute the market segment with higher profit potential. This implies that quality has to be under-provided at the bottom of the product line to prevent high types from purchasing the low-quality good (which exhibits a smaller profit margin).

Having signed the equilibrium distortions at any profile of brand preferences (t_l, t_h), we will now study other properties of equilibria. We start with those in pure strategies.

6.2 Pure-strategy equilibrium

The following characterization paves the way for the comparative statics on prices and qualities developed below. It also reveals that, whenever incentive constraints bind under competition, informational rents are always positive for some consumer type, but negative for the other.

Proposition 5. ***(Informational Rents)*** *Let (u_l^*, u_h^*) be a pure-strategy equilibrium. Then no other pure-strategy equilibrium exists. Moreover:*

(a) If $(t_l, t_h) \in E$, consumers obtain the same payoffs as under complete information.

(b) If $(t_l, t_h) \in D_+$, relative to the complete information benchmark, high types lose, while low types gain from asymmetric information. Moreover, the equilibrium profile jointly satisfies

$$S_h(u_l^*, u_h^*) - u_h^* + t_h \left(\frac{\partial S_h}{\partial u_h}(u_l^*, u_h^*) - 1 \right) = 0 \quad \text{and} \quad \frac{S_l^e - u_l^*}{t_l} + \left(\frac{p_h}{p_l} \frac{\partial S_h}{\partial u_l}(u_l^*, u_h^*) - 1 \right) \leq 0,$$

where the second condition is an (in)equality if $u_l^* > 0$ ($u_l^* = 0$).

(c) If $(t_l, t_h) \in D_-$, relative to the complete information benchmark, high types gain, while low types lose from asymmetric information. Moreover, the equilibrium profile jointly satisfies

$$\frac{S_h^e - u_h^*}{t_h} + \left(\frac{p_l}{p_h} \frac{\partial S_l}{\partial u_h}(u_l^*, u_h^*) - 1 \right) \leq 0 \quad \text{and} \quad S_l(u_l^*, u_h^*) - u_l^* + t_l \left(\frac{\partial S_l}{\partial u_l}(u_l^*, u_h^*) - 1 \right) \leq 0,$$

where the first condition is an (in)equality if $u_h^* > 0$ *(*$u_h^* = 0$*), while the second condition is an (in)equality if* $u_l^* > 0$ *(*$u_l^* = 0$*).*[30]

In region E, the equilibrium is just like under complete information, as so are consumers' payoffs.

The situation is different when incentive constraints bind. For instance, in region D_+, consider the simpler case where $u_l^* = 0$. The equilibrium condition from Claim (b) reveals that u_h^* is determined by

$$\underbrace{\frac{S_h(0,u_h^*) - u_h^*}{t_h}}_{\text{poaching gain}} - \underbrace{1}_{\substack{\text{mark-up}\\ \text{loss per sale}}} + \underbrace{\left(\frac{\partial S_h}{\partial u_h}(0,u_h^*)\right)}_{\substack{\text{efficiency}\\ \text{loss per sale}}} = 0. \tag{7}$$

Intuitively, when choosing how much utility to leave to high types, firms balance the gains from poaching consumers away from the competitor, which is the first term in (7), with the per sale loss from reducing the price, which is the second term, compounded with the efficiency loss from tightening the incentive constraint, which is the last term. The latter is absent in the complete information benchmark, what explains why informational rents are negative for high-type consumers:

$$u_h^* = S_h(u_l^*, u_h^*) - t_h + t_h \frac{\partial S_h}{\partial u_h}(u_l^*, u_h^*) \leq S_h^e - t_h.$$

Intuitively, firms have less incentives to increase the high-type payoff relative to the complete information benchmark. The reason is that IC_l binds, so raising u_h tightens this constraint, thus decreasing the efficiency of the high-type contract. The opposite applies to low types, for which firms are more compelled to provide rents (so as to relax this constraint).

The same logic explains why, in region D_-, relative to the complete information benchmark, high types gain, while low types lose from asymmetric information. Intuitively, firms have an extra incentive to increase high-type payoffs relative to the complete information benchmark. Namely, on top of the usual poaching gains and mark-up losses, increasing u_h relaxes the binding constraint IC_h, thus increasing the efficiency of the low-type contract. The opposite applies to low types, for which firms are less compelled to provide rents (so as not to tighten this constraint).

Interestingly, the latter effect is shrouded in the balanced duopoly case, where IC_h binds only when $u_l^* = 0$ (as, note from Figure 4, $t_l > S_l^e$ whenever the 45-degree line belongs to region D_-). In this case, low types obtain the same payoff as under complete information (zero), while high-types obtain positive informational rents.

Proposition 5 derives the necessary conditions that the (unique) pure-strategy equilibrium satisfies, *if* it exists. The next proposition describes precisely when this occurs (beyond region E, where it always exists).

[30]Because incentive compatibility requires that $u_h \geq u_l$, if the first condition is an inequality, so is the second, in which case $u_h^* = u_l^* = 0$.

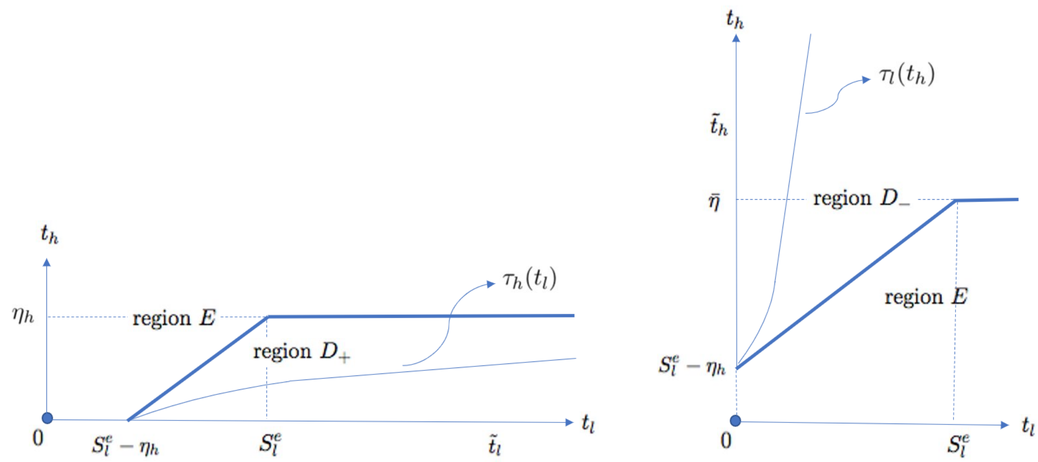

Figure 5: Existence of pure-strategy equilibrium in regions D_+ (left panel) and D_- (right panel), when $\tilde{t}_l, \tilde{t}_h = \infty$.

Proposition 6. ***(Characterization)*** *Assume the cost function φ is quadratic. Then:*

(a) Region D_+: There exists a continuous function $\tau_h : (S_l^e - \eta_h, \infty) \to [0, \eta_h)$ such that a pure-strategy equilibrium exists at $(t_l, t_h) \in D_+$ if and only if $t_h \geq \tau_h(t_l)$.[31]

(b) Region D_-: There exists a continuous function $\tau_l : (S_h^e - \eta_l, \infty) \to [0, \eta_l)$ such that a pure-strategy equilibrium exists at $(t_l, t_h) \in D_-$ if and only if $t_l \geq \tau_l(t_h)$.[32]

In light of the bottom-barrel and cream-skimming scenarios, it should not come as a surprise that a pure-strategy equilibrium might fail to exist in regions D_+ and D_-. Indeed, Proposition 6 reveals that no pure-strategy equilibrium exists "close" to the non-existence regions of the bottom-of-barrel and cream-skimming cases, but that existence is assured "close" to region E. These ideas are made precise by the existence of a threshold $\tau_h(t_l)$ in region D_+ (resp., $\tau_l(t_h)$ in region D_-) such that a pure-strategy equilibrium exists if and only if t_h (resp., t_l) exceeds this threshold. See Figure 5 for an illustration.

On a more technical level, the non-existence of a pure-strategy equilibrium relies on the fact that best responses are not quasi-concave. For instance, consider the region D_+ and fix some $t_l \in (S_l^e - \eta_h, \tilde{t}_l)$. The proof of Proposition 6 first reveals that the best response is locally quasi-concave at the putative equilibrium of Proposition 5 if and only if t_h is high enough. Otherwise, the putative equilibrium is a saddle point of the best response, thus exhibiting a local profitable deviation. This is the ultimate reason for why no pure-strategy equilibrium exists in the bottom-barrel case when $t_l \in (S_l^e - \eta_h, \tilde{t}_l)$.

Even when the best response is locally quasi-concave at the putative equilibrium of Proposition 5, we have to compare the putative equilibrium profit with that of all non-local deviations (due to

[31] Obviously, whenever $\tilde{t}_l < \infty$, the threshold $\tau_h(t_l) = 0$ for all $t_l \geq \tilde{t}_l$.
[32] Obviously, whenever $\tilde{t}_h < \infty$, the threshold $\tau_l(t_h) = 0$ for all $t_h \geq \tilde{t}_h$.

failure of global quasi-concavity). In line with the analysis of the bottom-barrel case, we then show that the best non-local deviation is the already familiar relax-and-undercut strategy.[33] Under this strategy, the deviating firm corners the high-type market, and raises the utility of low-type consumers to relax the incentive constraint and increase the efficiency of the high-type contract. This deviation improves upon the putative equilibrium profit if and only if t_h is small (as, otherwise, cornering the high-type market requires relinquishing too much rents to consumers). We then obtain the threshold structure of Proposition 6.

Ultimately, Proposition 6 shows that the main ideas behind the bottom-barrel and cream-skimming scenarios are robust to environments where competition is imperfect for both consumer types. In particular, the relax-and-undercut deviation preserves a central role in determining whether a pure-strategy equilibrium exists.

6.3 Comparative Statics

We are now ready to derive comparative statics.

Proposition 7. ***(Comparative Statics on Quality and Payoffs)*** *Consider a neighborhood around $(t_l, t_h) \in \mathbb{R}^2_+$ where the pure-strategy equilibrium exists. Then, for $k \in \{l, h\}$,*

$$\frac{\partial u_k^*}{\partial t_h}, \frac{\partial u_k^*}{\partial t_l} < 0, \qquad \textit{and} \qquad \frac{\partial q_k^*}{\partial t_h} \leq 0 \leq \frac{\partial q_k^*}{\partial t_l},$$

with strict inequality for $k = h$ (resp., $k = l$) if $(t_l, t_h) \in D_+$ (resp., $(t_l, t_h) \in D_-$ and $q_l^* > 0$).

Not surprisingly, the equilibrium indirect utility of both consumer types strictly decreases as brand preferences (of either type) become more intense. More interesting, perhaps, is the effect on qualities when some incentive constraint binds (otherwise, quality levels are efficient). In this case, as high types develop more intense brand preferences, equilibrium quality levels go down. The reason is the following: changes in t_h directly affect competition for high types, but only indirectly for low types (through incentive constraints). As a result, an increase in t_h decreases u_h^* *faster* than u_l^*, what implies that the quality of the inefficient contract decreases. When constraint IC_l is binding (as in region D_+), it is the high-type quality that is inefficient. As such, q_h^* goes down (reducing the distortion) as t_h goes up, whereas q_l^* remains constant at its efficient level. In turn, when constraint IC_h is binding (as in region D_-), it is the low-type quality that is inefficient. As such, q_l^* goes down (magnifying the distortion) as t_h goes up, whereas q_h^* remains constant at its efficient level. Therefore, variations in t_h can either increase or decrease equilibrium welfare, depending on whether the preference profile (t_l, t_h) belongs to regions D_+ or D_-. When (t_l, t_h) lies in the efficient region E, variations in t_h have no effect on equilibrium qualities.

[33] To prove this claim we relied on φ being quadratic. While we believe the result to be true more generally, its proof is elusive.

Mutatis mutandis, the same logic explains the effect of t_l on equilibrium quality levels. Because an increase in t_l decreases u_l^* *faster* than u_h^*, the quality of the inefficient contract (if positive) shall increase. Accordingly, an increase in t_l strictly increases q_h^* when $(t_l, t_h) \in D_+$ (magnifying the distortion), but strictly increases q_l^* when $(t_l, t_h) \in D_-$ and $q_l^* > 0$ (reducing the distortion).

The key take-away of Proposition 7 is therefore that, under self-selection constraints, competition and welfare are often misaligned, in that more competitive markets (in the sense that consumers are less brand-loyal) often produce lower welfare. At the heart of the matter lies the idea that, under asymmetric information, contract offers are interdependent across consumer segments. This interdependency renders competition welfare-decreasing whenever it tightens incentive constraints.

The comparative statics on prices is explored in the following result.

Proposition 8. ***(Comparative Statics on Prices)*** *Consider a neighborhood around $(t_l, t_h) \in \mathbb{R}_+^2$ where the pure-strategy equilibrium exists, and denote by (y_l^*, y_h^*) the equilibrium price profile. Then:*

(a) If $(t_l, t_h) \in E$, then $y_k^* = \min\{t_k, S_k^e\}$.

(b) If $(t_l, t_h) \in D_+$, then

$$\frac{\partial y_l^*}{\partial t_l}, \frac{\partial y_l^*}{\partial t_h}, \frac{\partial y_h^*}{\partial t_l} \geq 0,$$

with strict inequality if and only if the constraint IR_l is slack. Moreover, y_h^* is decreasing in t_h if IR_l binds, but is quasi-convex in t_h if IR_l is slack and $\varphi'''(q) \leq 0$.

(c) If $(t_l, t_h) \in D_-$, then

$$\frac{\partial y_h^*}{\partial t_h} > 0, \quad \text{and} \quad \frac{\partial y_l^*}{\partial t_l}, \frac{\partial y_h^*}{\partial t_l} \geq 0,$$

with strict inequality if and only if the constraint IR_l is slack. Moreover, y_l^* is decreasing in t_h if IR_l binds, but is quasi-convex in t_h if IR_l is slack and $\varphi'''(q) \leq 0$.

Prices always increase with the intensity of brand preferences when the quality of the product is set efficiently. This familiar intuition explains why prices increase with (t_l, t_h) in region E, and why the baseline price y_l^* (resp., premium price y_h^*) increase with (t_l, t_h) in region D_+ (resp., D_-).

The analysis is more subtle when changes in the intensity of brand preferences jointly affect quality provision and utility levels. This occurs, for instance, with the premium product when $(t_l, t_h) \in D_+$. As t_l increases, high types are worse-off (u_h^* decreases), whereas the quality of the premium product increases (q_h^* increases), so y_h^* has to increase as well. By contrast, as t_h increases, high-type payoffs decrease, as so does the quality of the premium product. The latter effect dominates when IR$_l$ binds, which implies the premium product becomes cheaper as high types become more brand loyal. When IR$_l$ is slack, this pattern is more nuanced, as the premium price y_h^* is U-shaped in the brand loyalty parameter t_h.

A similar logic explains why, in region D_-, the baseline product may become cheaper as high types becomes more brand-loyal. When IR$_l$ binds, the baseline price goes down with t_h because the quality of the baseline product falls more than the equilibrium payoff of low types. When IR_l is slack, the race between quality and payoff changes leads to U-shaped pattern.

More broadly, Proposition 8 reveals that, under self-selection constraints, variations in the level of prices are a misleading indicator of the degree of competition in the market. This is consistent with the ambiguous relationship found in the empirical literature between the degree of competition and the level of prices in markets characterized by self-selection. For instance, Chu (2010) documents that cable companies in the US reacted to new competition by satellite television by raising both price and quality (as determined by the available channels), with consumers benefiting overall from the higher-priced offerings.

6.4 Mixed-strategy equilibria

In the mid-barrel case, a closed-form characterization of mixed-strategy equilibria is difficult to obtain, as the support of equilibrium menus (and its cardinality) are bound to change for different preference profiles (t_l, t_h). Yet, we can show that the main properties identified in Sections 4 and 5 remain valid when competition is imperfect for both consumer types. Namely, mixed-strategy equilibria are ordered, and, as already established by Proposition 4, all equilibrium menus offer a premium (resp., baseline) good of inefficiently high (resp., low) quality provided $(t_l, t_h) \in D_+$ (resp., D_-). The next proposition summarizes this discussion.

Proposition 9. ***(Mixed-Strategy Equilibria)*** *Consider $(t_l, t_h) \in D_+ \cup D_-$ such that no pure-strategy equilibrium exists. Then at least one mixed-strategy equilibrium exists, and any such equilibrium is ordered.*

Beyond predicting price/quality dispersion, Proposition 9 delivers one key testable implication. Namely, the gross utilities offered by firms are similarly ranked across the product line (as follows from the ordered property of equilibria).

7 Discussion and Extensions

7.1 Continuum of Types

This section illustrates how the results from the binary-type model extend to the case where types are uniformly distributed over some interval $\left[\underline{\theta}, \bar{\theta}\right] \subseteq \mathbb{R}_+$ and the cost function is quadratic: $c(q) = \frac{1}{2}q^2$. As before, we let the intensity of brand preferences change with one's preferences for quality, as described by the brand loyalty schedule $t(\theta)$. To facilitate comparison with the binary type model of the previous sections, we assume that $t(\theta)$ is affine:

$$t(\theta) = \underline{t} + (\bar{t} - \underline{t}) \left(\frac{\theta - \underline{\theta}}{\bar{\theta} - \underline{\theta}} \right), \tag{8}$$

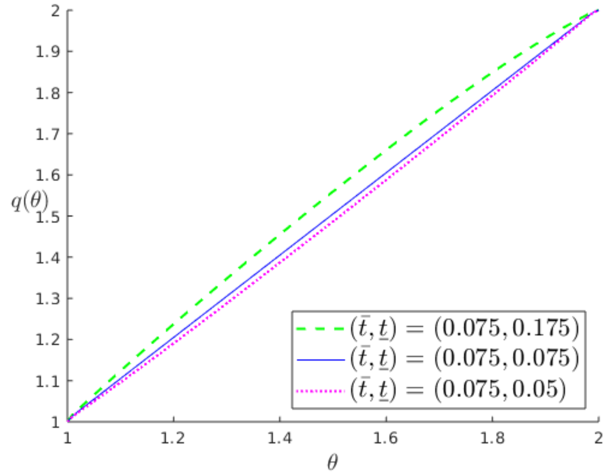

Figure 6: Equilibrium quality schedules for $\left[\underline{\theta}, \bar{\theta}\right] = [1, 2]$: The dashed (resp., dotted) line assumes that brand loyalty is decreasing (resp., increasing) in preferences for quality, whereas the full line, which is the efficient schedule, assumes it is constant ($\Delta t = 0$).

where $\underline{t}, \bar{t} \geq 0$. Note that $t(\bar{\theta}) = \bar{t}$ and $t(\underline{\theta}) = \underline{t}$, which explains the notation. If $\Delta t \equiv \bar{t} - \underline{t} < 0$, brand loyalty is decreasing in preferences for quality, and increasing if $\Delta t > 0$.

Rochet and Stole (2002) showed that, when $\Delta t = 0$ and $t \equiv \bar{t} = \underline{t}$ is sufficiently small, the equilibrium is in pure strategies, and quality provision is efficient to all types ($q^*(\theta) = \theta$). Allowing for correlation between brand loyalty and brand preferences, our model also admits a pure-strategy equilibrium (assuming $|\Delta t| \neq 0$ is small). Moreover, when correlation is positive, i.e., $\Delta t > 0$ (resp., negative, i.e., $\Delta t < 0$), almost every quality is distorted downwards (resp., upwards), which is in line with Proposition 4. Figure 6 numerically illustrates this finding for $\left[\underline{\theta}, \bar{\theta}\right] = [1, 2]$.

Furthermore, when $\Delta t > 0$ (resp., $\Delta t < 0$) low types are worse-off (resp., better-off) under asymmetric information, while high types are better-off (resp., worse-off). Intuitively, when $\Delta t > 0$, competition for low types is hindered by the fact that high types have strong brand loyalty. To prevent profit dissipation (due to high types selecting low-quality contracts), firms then provide less utility to low types (relative to the complete information outcome), leading to negative informational rents. Conversely, when $\Delta t < 0$, competition for high types is hindered by the fact that low types have strong brand loyalty, which explains why informational rents are negative for the former but positive for the latter. These conclusions generalize Proposition 5, established under binary types.

Our next result characterizes the equilibrium, and collect the findings discussed above.

Proposition 10. ***(Pure-strategy equilibrium: Continuum of types)*** *For every $\underline{t} \in (0, \underline{\tau})$, there exists $\varepsilon > 0$ such that, for all $|\Delta t| \in (0, \varepsilon)$, there exists a pure-strategy equilibrium in which the indirect utility schedule $u^*(\theta)$ satisfies, for all $\theta \in (\underline{\theta}, \overline{\theta})$, the following differential equation:*

$$\ddot{u}(\theta) = 2 - \left(\frac{1}{t(\theta)}\right)\left(\theta \dot{u}(\theta) - \frac{(\dot{u}(\theta))^2}{2} - u(\theta)\right) \quad \textit{subject to} \quad \dot{u}(\theta) = \theta \;\; \textit{for} \;\; \theta \in \{\underline{\theta}, \overline{\theta}\}. \qquad (9)$$

(a) If $\Delta t > 0$, every interior quality involves downward distortions: $q^*(\theta) < \theta$. Moreover, relative to complete information, there are types $\theta_1, \theta_2 \in (\underline{\theta}, \overline{\theta})$ such that every type $\theta < \theta_1$ is worse-off, whereas every type $\theta > \theta_2$ is better-off.

(b) If $\Delta t < 0$, every interior quality involves upward distortions: $q^*(\theta) > \theta$. Moreover, relative to complete information, there are types $\theta_1, \theta_2 \in (\underline{\theta}, \overline{\theta})$ such that every type $\theta < \theta_1$ is better-off, whereas every type $\theta > \theta_2$ is worse-off.

The next corollary employs the characterization of Proposition 10 to perform comparative statics on both the magnitude of brand loyalty, as well as on the correlation between brand loyalty and brand preferences. To do so, it is convenient to write the brand loyalty schedule as $t(\theta) = \alpha + \beta\theta$. In this parametrization, changes in the parameter α correspond to uniform shifts on brand loyalty (across consumer types), while β captures the correlation between θ and $t(\theta)$.[34]

Corollary 2. ***(Comparative Statics)*** *Consider the pure-strategy equilibrium of Proposition 10, and adopt the parametrization $t(\theta) = \alpha + \beta\theta$. Then, an increase in α reduces the indirect utility of every type, while an increase in β reduces the quality provision for every interior type.*

The effect of increasing α on equilibrium indirect utilities is expected. More interesting, perhaps, is that, as β increases, rendering high types more brand loyal vis-à-vis low types, quality provision decreases along the product line. These findings confirm that the comparative statics in Proposition 7 are robust to a continuum-type setting.

With a continuum of types, as in the binary-type case, a symmetric pure-strategy Nash equilibrium may fail to exist, as there is no guarantee that the firms' best responses are globally quasi-concave. We investigate this issue by numerically computing the putative pure-strategy equilibrium, and then searching for profitable incentive-compatible deviations.[35] Assuming $[\underline{\theta}, \bar{\theta}] = [1, 2]$, and parametrizing the brand loyalty schedule by the profile $(\underline{t}, \bar{t})$, Figure 7 identifies the regions where a pure-strategy equilibrium does (not) exist. The results are remarkably parallel to those under binary types (illustrated in Figure 5): There is always a pure-strategy equilibrium close to the "diagonal" (where $\bar{t} = \underline{t}$), whereas non-existence obtains when brand loyalty is sufficiently different across "low" and "high" types (i.e., for $\underline{t}$ small and $\bar{t}$ large, or vice-versa). This reveals that dispersion of offers is a robust feature of competitive models involving self-selection and heterogeneous brand loyalty.

7.2 Other Discrete-Choice Models

For tractability, we introduced horizontal differentiation following the Hotelling/Bénabou-Tirole framework, where demands, whenever interior, are linear in utilities. A more general formulation is

[34] There is a one-to-one relationship between the (α, β) parametrization and the one based on the brand-loyalty profile $(\underline{t}, \bar{t})$ - see the proof of Proposition 10 for details. Naturally, $\beta > 0$ if and only if $\Delta t > 0$.

[35] See the Online Appendix for details about our numerical procedure.

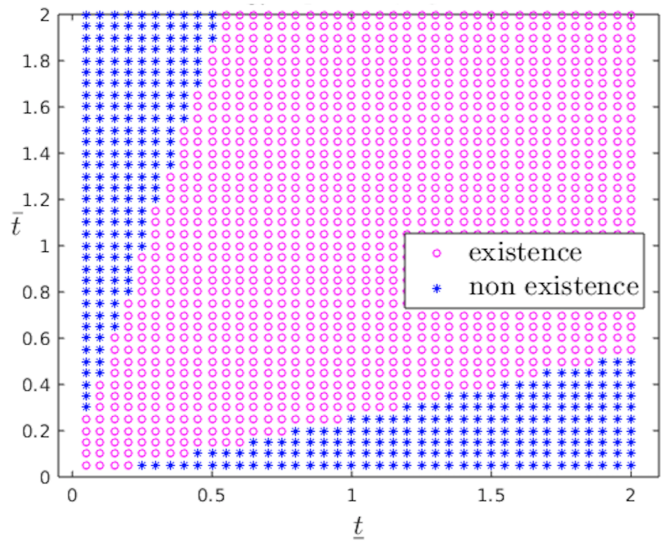

Figure 7: Brand-loyalty profiles $(\underline{t}, \bar{t})$ under which a pure-strategy equilibrium does (not) exist.

to assume that individual i, with taste for quality $k \in \{h, l\}$, obtains the utility

$$U^i_{kj} \equiv \theta_k q_{kj} - y_{kj} + t_k \varepsilon^i_j$$

whenever she purchases a good of quality q_{kj} from firm j at the price y_{kj}. The taste shock ε^i_j reflects the consumer's preferences for seller j's products, and the parameter t_k captures the intensity of brand preferences of type-k consumers. As before, $t_h = 0$ corresponds to the "bottom-of-barrel" duopoly, while $t_l = 0$ corresponds to the "cream-skimming" duopoly.

The Bénabou-Tirole's treatment of outside goods then calls for defining the utility of non-participation as $U^i_{k0} = u_0 + t_k \max_j\{\varepsilon^i_j\}$, where u_0 is a constant (typically normalized to zero). As in the baseline model, the interpretation is that consumers have access to a "generic" substitute to each firm's product line, and choose their preferred generic as the outside option. This specification allows us to vary the intensity of brand preferences (across consumer types), without affecting the relative value of non-participation.[36]

The baseline model from the previous sections corresponds to assuming that ε^i_a is uniformly distributed over $[0, 1]$, and that horizontal tastes are perfectly negatively correlated: $\varepsilon^i_a = 1 - \varepsilon^i_b$ (which is the Hotelling framework). Another natural possibility is to adopt a logit framework, where ε^i_{kj} are iid draws from a Gumbel distribution with scale parameter normalized to one.[37] Aside from

[36]Equivalently, and perhaps closer to empirical specifications, one could introduce heterogeneous price sensitivities, while assuming that brand shocks are identically distributed across types. Namely, let individual i, with taste for quality $k \in \{h, l\}$, obtain the utility $U^i_{kj} \equiv \alpha_k q_{kj} - \beta_k y_{kj} + \varepsilon^i_j$ when purchasing a good of quality q_{kj} from firm j at the price y_{kj}. Here, β_k is type-k's price sensitivity. Similarly, the utility of non-participation is set to be $U^i_{k0} = \max_j\{\varepsilon^i_j\}$. By multiplying type-$k$ utilities by $\frac{1}{\beta_k}$ and defining $\theta_k \equiv \frac{\alpha_k}{\beta_k}$ and $t_k \equiv \frac{1}{\beta_k}$, we obtain the specification in the text.

[37]In this case, the demand of firm a for its k-type product takes the form $D^a_k = p_k \exp\left(\frac{u^a_k - u^b_k}{t_k}\right)\left[1 + \exp\left(\frac{u^a_k - u^b_k}{t_k}\right)\right]^{-1}$,

the non-linearity of demands, which renders a closed-form characterization elusive, both specifications are similar, and lead to parallel conclusions. In particular, the relax-and-undercut deviation (which breaks down pure-strategy equilibria when consumer types sufficiently differ in their brand loyalty) works the same way as in sections 4 to 6. Numerical analysis reveals that the patterns of distortions, informational rents and comparative statics (in Propositions 4 to 8) are also robust to this alternative demand specification.

It is also possible to introduce random outside options (in addition to brand tastes). To do so, one needs to define the utility of non-participation as being $\tilde{U}^i_{k0} = \tilde{u}^i_0 + t_k \max_j\{\varepsilon^i_j\}$, where $\tilde{u}^i_0$ is the utility of outside expenses by consumer i (assumed heterogenous across consumers and unobserved by firms). While we expect our main insights to hold in this more general setting, analytical results are elusive.

7.3 More than Two Firms

For simplicity, the baseline assumed a duopolistic market structure. It is straightforward to extend our analysis to for more than two competing firms. One natural possibility is to use the Spokes model of Chen and Riordan (2007), where each consumer only considers two (randomly selected) firms out of all firms in the market (and makes purchasing decisions à la Hotelling). This extension can be developed isomorphically to the results on the baseline model. It is also possible to employ other discrete-choice specifications (as described in subsection 7.2) to analyze oligopolistic competition (e.g., logit demands). We expect these more general formulations to preserve our main qualitative insights, while requiring numerical techniques that our simple model conveniently bypasses.

8 Conclusions

We study oligopolistic competition by firms engaging in second-degree price discrimination. Crucially, we allow consumers with different tastes for quality to exhibit varying propensities to switch brands, which reflects a key feature of empirical models estimating demand for differentiated goods. Our analysis delivers five main take-aways:

(a) We show how patterns of quality provision relate to co-movements between consumers' tastes for quality and brand loyalty. Specifically, quality provision is inefficiently low at the bottom (high at the top) of the product line if the propensity of low-type consumers to switch brands is small (large) relative to that of high types.

(b) Informational rents may well be negative under competition. In fact, they are always negative for some type, while positive for the other, provided consumers obtain more than their reser-

where u^j_k is the gross indirect utility of good k from firm j.

vation utility in equilibrium. This is unlike the monopoly case, or the oligopoly case under uniform brand loyalty, where informational rents are always (weakly) positive.

(c) Competition and welfare are often misaligned: More competition (in the sense that consumers are less brand-loyal) is welfare-decreasing whenever it tightens incentive constraints. This is the case when low types become more prone to switch brands and quality provision is inefficiently high at the top of the product line.

(d) Variations in the level of prices are a misleading indicator of the degree of competition in the market. In particular, more competition can either decrease or increase prices, depending on the race between quality and payoff changes. This is the case, for instance, when high types become more prone to switch brands and quality provision is inefficiently low at the bottom of the product line.

(e) Dispersion of offers (due to the non-existence of pure-strategy equilibria) often occurs when different types exhibit (sufficiently) different propensities to switch brands. In this case, firms randomize over ordered menus, where indirect utilities co-move across types. This is unlike previous literature that relates dispersion in private-value settings to search or informational frictions.

Our analysis can be extended in many fruitful directions. One pertains to dynamic models of competition where consumers (with recurring consumption needs) are heterogeneous on their switching costs, while exhibiting different tastes for product characteristics (which is the case in insurance markets, for instance). Another pertains to models of competition under non-exclusive agency (as in Calzolari and Denicolò 2013), where agents differ in their tastes for quality/quantity, but also on how they perceive the complementarity/substitutability between different sellers. We expect the ideas and techniques of our paper to be useful in exploring these interesting research avenues.

9 Appendix

Proof of Proposition 0. We will only verify that the strategies described in the Proposition constitute an equilibrium. Uniqueness follow from Proposition 5.

Case 1: $t \in [0, S_l^e]$.

The best response of firm a solves

$$\max_{u_l^a, u_h^a} \left\{ p_l \left(\frac{1}{2} + \frac{u_l^a - (S_l^e - t)}{2t} \right) (S_l^e - u_l^a) + p_h \left(\frac{1}{2} + \frac{u_h^a - (S_h^e - t)}{2t} \right) (S_h^e - u_h^a) \right\}$$

subject to $u_l^a, u_h^a \geq 0$. The first-order conditions with respect to u_k^a, $k \in \{l, h\}$, are

$$\frac{1}{2t} (S_k^e - u_k^a) - \left(\frac{1}{2} + \frac{u_k^a - (S_k^e - t)}{2t} \right) = 0. \tag{10}$$

The expression above is strictly decreasing in u_k^a, which implies the firm's problem is quasi-concave. The unique solution to (10) is given by $\tilde{u}_k = S_k^e - t$ for $k \in \{l, h\}$. The argument is symmetric for firm b.

Case 2: $t \in [S_l^e, S_h^e - q_l^e \Delta\theta]$.

The best response of firm a solves

$$\max_{u_l^a, u_h^a} \left\{ p_l \left(\frac{1}{2} + \frac{u_l^a}{2t} \right) (S_l^e - u_l^a) + p_h \left(\frac{1}{2} + \frac{u_h^a - (S_h^e - t)}{2t} \right) (S_h^e - u_h^a) \right\}$$

subject to $u_l^a, u_h^a \geq 0$. The derivative of the objective function with respect to u_h^a is again given by (10), and that for u_l^a is given by

$$\frac{1}{2t} (S_l^e - u_l^a) - \left(\frac{1}{2} + \frac{u_l^a}{2t} \right). \tag{11}$$

Because both derivatives are decreasing in their respective arguments, we conclude the objective is quasi-concave. The unique optimum is such that $u_l^a = 0$ and $u_h^a = S_h^e - t$. To see why, let us evaluate (11) at $u_l^a = 0$ to obtain $\frac{S_l^e}{2t} - \frac{1}{2} \leq 0 \iff S_l^e \leq t$. The optimal u_l^a is then at the corner: $u_l^a = 0$. That $u_h^a = S_h^e - t$ directly follows from (10).

Case 3: $t > S_h^e - q_l^e \Delta\theta$.

Consider first the instance where $\theta_l - \frac{p_h}{p_l} \Delta\theta > 0$, which guarantees that $u_h^m > 0$. The best response of firm a maximizes the objective

$$\hat{\Pi}^a(u_l, u_h) \equiv p_l \left(\frac{1}{2} + \frac{u_l}{2t} \right) (S_l(u_l, u_h) - u_l) + p_h \left(\frac{1}{2} + \frac{u_h - \tilde{u}_h}{2t} \right) (S_h^e - u_h)$$

subject to $u_l, u_h \geq 0$, where $\tilde{u}_h$ solves (3). Let us consider first the range $\mathcal{D} \equiv \{(u_l, u_h) : u_l \leq u_h \leq q_l^e \Delta\theta\}$. It is straightforward to compute that

$$\frac{\partial \hat{\Pi}^a}{\partial u_l}(u_l, u_h) = \frac{p_l}{2t} (S_l(u_l, u_h) - u_l) + p_l \left(\frac{1}{2} + \frac{u_l}{2t} \right) \left(\frac{\partial S_l}{\partial u_l}(u_l, u_h) - 1 \right)$$

and

$$\frac{\partial^2 \hat{\Pi}^a}{\partial u_l^2}(u_l, u_h) = \frac{p_l}{t}\left(\frac{\partial S_l}{\partial u_l}(u_l, u_h) - 1\right) + p_l\left(\frac{1}{2} + \frac{u_l}{2t}\right)\frac{\partial^2 S_l}{\partial u_l^2}(u_l, u_h).$$

Also notice that, for all $(u_l, u_h) \in \mathcal{D}$,

$$\frac{\partial S_l}{\partial u_h}(u_l, u_h) = -\frac{\partial S_l}{\partial u_l}(u_l, u_h) = \frac{\theta_l}{\Delta\theta} - \varphi'\left(\frac{u_h - u_l}{\Delta\theta}\right)\frac{1}{\Delta\theta}$$

and

$$\frac{\partial^2 S_l}{\partial u_h^2}(u_l, u_h) = \frac{\partial^2 S_l}{\partial u_l^2}(u_l, u_h) = -\frac{\partial^2 S_l}{\partial u_l \partial u_h}(u_l, u_h) = -\varphi''\left(\frac{u_h - u_l}{\Delta\theta}\right)\left(\frac{1}{\Delta\theta}\right)^2.$$

Therefore, because φ is convex and $(u_l, u_h) \in \mathcal{D}$,

$$\frac{\partial S_l}{\partial u_l}(u_l, u_h) \leq -\frac{\theta_l}{\Delta\theta} + \varphi'\left(q_l^e\right)\frac{1}{\Delta\theta} = 0.$$

Coupled with the fact that $\frac{\partial^2 S_l}{\partial u_l^2} < 0$, this implies that

$$\frac{\partial^2 \hat{\Pi}^a}{\partial u_l^2}(u_l, u_h) = \frac{p_l}{t}\left(\frac{\partial S_l}{\partial u_l}(u_l, u_h) - 1\right) + p_l\left(\frac{1}{2} + \frac{u_l}{2t}\right)\frac{\partial^2 S_l}{\partial u_l^2}(u_l, u_h) < 0$$

for all $(u_l, u_h) \in \mathcal{D}$.

Now notice that

$$\begin{aligned}
\frac{\partial \hat{\Pi}^a}{\partial u_l}(0, u_h) &= \frac{p_l}{2t}S_l(0, u_h) + \frac{p_l}{2}\left(\frac{\partial S_l}{\partial u_l}(0, u_h) - 1\right) \\
&\leq \frac{p_l}{2}\left(\frac{S_l^e}{t} - 1 + \frac{\partial S_l}{\partial u_l}(0, u_h)\right) < \frac{\partial S_l}{\partial u_l}(0, u_h) \leq 0,
\end{aligned}$$

where the penultimate inequality follows from the fact that $t > S_h^e - q_l^e \Delta\theta > S_l^e$. Because $\frac{\partial \hat{\Pi}^a}{\partial u_l}(0, u_h) < 0$ and $\frac{\partial^2 \hat{\Pi}^a}{\partial u_l^2} < 0$ for all $(u_l, u_h) \in \mathcal{D}$, we conclude that firm a's best response entails $u_l = 0$.

We can therefore simplify firm a's problem as that of choosing $u_h \geq 0$ to maximize

$$\hat{\Pi}^a(0, u_h) \equiv \frac{p_l}{2}S_l(0, u_h) + p_h\left(\frac{1}{2} + \frac{u_h - \tilde{u}_h}{2t}\right)\left(S_h^e - u_h\right).$$

It is straightforward to compute that

$$\frac{\partial \hat{\Pi}^a}{\partial u_h}(0, u_h) = \frac{p_l}{2}\frac{\partial S_l}{\partial u_h}(0, u_h) + \frac{p_h}{2t}\left(S_h^e - u_h\right) - p_h\left(\frac{1}{2} + \frac{u_h - \tilde{u}_h}{2t}\right)$$

and

$$\frac{\partial^2 \hat{\Pi}^a}{\partial u_h^2}(u_l, u_h) = \frac{p_l}{2}\frac{\partial^2 S_l}{\partial u_h^2}(0, u_h) - \frac{p_h}{t} < 0,$$

since $\frac{\partial^2 S_l}{\partial u_h^2} < 0$ for all $(u_l, u_h) \in \mathcal{D}$. The unique pure equilibrium then solves

$$\frac{p_l}{2}\frac{\partial S_l}{\partial u_h}(0, u_h) + \frac{p_h}{2t}\left(S_h^e - u_h\right) - \frac{p_h}{2} = 0,$$

which becomes (3) after rearranging.

Writing the solution to (3) as a function of t, $\tilde{u}_h(t)$, and totally differentiating (3) with respect to t leads to

$$-\tilde{u}_h'(t) + \frac{p_l}{p_h}\frac{\partial S_l}{\partial u_h}(0, \tilde{u}_h(t)) - 1 + t\frac{p_l}{p_h}\frac{\partial^2 S_l}{\partial u_h^2}(0, \tilde{u}_h(t))\tilde{u}_h'(t) = 0.$$

Isolating $\tilde{u}_h'(t)$ leads to

$$\tilde{u}_h'(t) = \frac{\frac{p_l}{p_h}\frac{\partial S_l}{\partial u_h}(0, u_h) - 1}{1 - t\frac{p_l}{p_h}\frac{\partial^2 S_l}{\partial u_h^2}(u_l, u_h)} = \frac{\frac{\theta_l}{\Delta\theta} - \varphi'\left(\frac{\tilde{u}_h(t)}{\Delta\theta}\right)\frac{1}{\Delta\theta} - 1}{1 + t\varphi''\left(\frac{\tilde{u}_h(t)}{\Delta\theta}\right)\left(\frac{1}{\Delta\theta}\right)^2}.$$

The denominator is always strictly positive, whereas the numerator is strictly negative if and only if $\tilde{u}_h(t) > u_h^m$. Because $\frac{\partial S_l}{\partial u_h}(0, u_h) \downarrow 0$ as $u_h \downarrow q_l^e\Delta\theta$, follows that $\tilde{u}_h(t) \uparrow q_l^e\Delta\theta$ as $t \downarrow S_h^e - q_l^e\Delta\theta$. Because $q_l^e\Delta\theta > u_h^m$, this implies that $\tilde{u}_h(t)$ is strictly decreasing in a neighborhood of $t = S_h^e - q_l^e\Delta\theta$. As a result, $\tilde{u}_h(t) < q_l^e\Delta\theta$ for all $t > S_h^e - q_l^e\Delta\theta$. Since $q_l^e\Delta\theta < S_h^e$, equation (3) implies that

$$\frac{p_l}{p_h}\frac{\partial S_l}{\partial u_h}(0, \tilde{u}_h(t)) - 1 > 0 \quad \Longleftrightarrow \quad \tilde{u}_h(t) > u_h^m$$

for all for all $t > S_h^e - q_l^e\Delta\theta$. In sum, we showed that $\tilde{u}_h(t)$ is bounded and strictly decreasing over the interval $t > S_h^e - q_l^e\Delta\theta$: $\tilde{u}_h(t) \in [u_h^m, q_l^e\Delta\theta]$.

Finally, equation (3) also implies that $\tilde{u}_h(t)$ satisfies

$$\frac{S_h^* - \tilde{u}_h(t)}{t} + \frac{p_l}{p_h}\frac{\partial S_l}{\partial u_h}(0, \tilde{u}_h(t)) - 1 = 0$$

for all $t > S_h^e - q_l^e\Delta\theta$. Continuity then implies that

$$\lim_{t\to\infty}\left\{\frac{S_h^e - \tilde{u}_h(t)}{t} + \frac{p_l}{p_h}\frac{\partial S_l}{\partial u_h}(0, \tilde{u}_h(t)) - 1\right\} = 0.$$

Because $\tilde{u}_h(t)$ is bounded for all $t > S_h^e - q_l^e\Delta\theta$, we obtain $\lim_{t\to\infty}\left\{\frac{S_h^e - \tilde{u}_h(t)}{t}\right\} = 0$. As a result,

$$\lim_{t\to\infty}\left\{\frac{p_l}{p_h}\frac{\partial S_l}{\partial u_h}(0, \tilde{u}_h(t)) - 1\right\} = \frac{p_l}{p_h}\frac{\partial S_l}{\partial u_h}\left(0, \lim_{t\to\infty}\tilde{u}_h(t)\right) - 1 = 0.$$

The last equality implies that $\lim_{t\to\infty}\tilde{u}_h(t) = u_h^m$, as we wanted to show.

Consider now the instance where $\theta_l - \frac{p_h}{p_l}\Delta\theta \leq 0$, which implies that $u_h^m = 0$. In this case,

$$\frac{p_l}{p_h}\frac{\partial S_l}{\partial u_h}(0, 0) - 1 < 0$$

and there exists a finite $\bar{t}$ such that

$$S_h^e + \bar{t}\left(\frac{p_l}{p_h}\frac{\partial S_l}{\partial u_h}(0, 0) - 1\right) = 0.$$

In the unique symmetric equilibrium, $\tilde{u}_h$ is given by equation (3) if $t < \bar{t}$, and $\tilde{u}_h = 0$ if $t \geq \bar{t}$. The latter follows from the fact that $\frac{\partial^2 \hat{\Pi}^a}{\partial u_h^2} < 0$ for all $(u_l, u_h) \in \mathcal{D}$ and that

$$S_h^e + t\left(\frac{p_l}{p_h}\frac{\partial S_l}{\partial u_h}(0,0) - 1\right) \leq 0$$

whenever $t \geq \bar{t}$. Q.E.D.

Proof of Proposition 1.

Proof of Claim (a). Suppose $t_l \in [0, S_l^e - \eta_h]$. Proposition 11 shows that firms obtain zero profits from high-type contracts. In the putative equilibrium where constraints IR_k and IC_k are slack, for $k \in \{l, h\}$, we posit that $u_l^* = S_l^e - t_l$ and $u_h^* = S_h^e$.

The best response of firm a then solves

$$\max_{u_l^a, u_h^a}\left\{p_l\left(\frac{1}{2} + \frac{u_l^a - (S_l^e - t_l)}{2t_l}\right)(S_l^e - u_l^a) + p_h \hat{D}_h^a(u_h^a)(S_h^e - u_h^a)\right\},$$

where

$$\hat{D}_h^a(u_h^a) = \begin{cases} 0 & \text{if} \quad u_h^a < S_h^e \\ \frac{1}{2} & \text{if} \quad u_h^a = S_h^e \\ 1 & \text{if} \quad u_h^a > S_h^e. \end{cases}$$

This problem has a solution $u_l^* = S_l^e - t_l$ and $u_h^* = S_h^e$. Note that IC_l is slack, as

$$u_h^* - u_l^* = S_h^e - (S_l^e - t_l) \leq q_h^e \Delta\theta$$

by assumption, and IR_l is slack, as $t_l < S_l^e$. This shows that $(S_l^e - t_l, S_h^e)$ is an equilibrium. Uniqueness follows by Proposition 11.

Proof of Claim (b). As shown in the proof of Proposition 11, any putative equilibrium satisfies $u_l^* = \max\{S_l^e - t_l, 0\}$ and u_h^* solves $u_h^* = S_h(u_l^*, u_h^*)$. We will show that this is not an equilibrium when $t_l \in (S_l^e - \eta_h, S_l^e)$. For that, we will show that the firm has a profitable deviation with the following structure: the firm relinquishes an $\varepsilon \approx 0$ more utility to high types (conquering the entire type-h market), and chooses $\hat{u}_l$ to solve

$$\max_{u_l}\left\{p_l\left(\frac{1}{2} + \frac{u_l - (S_l^e - t_l)}{2t_l}\right)(S_l^e - u_l) + p_h\left(S_h(u_l, \ddot{u}_h(u_l^*)) - \ddot{u}_h(u_l^*)\right)\right\} \qquad \text{s.t.} \qquad u_l \geq S_l^e - t_l.$$

This problem is concave. Therefore, the deviation described above generates a strict improvement if and only if the first-order condition evaluated at $u_l = S_l^e - t_l$ is strictly positive:

$$\left[\frac{p_l}{2t_l}(S_l^e - u_l) - p_l\left(\frac{1}{2} + \frac{u_l - (S_l^e - t_l)}{2t_l}\right) + p_h\frac{\partial S_h}{\partial u_l}(u_l, \ddot{u}_h(u_l^*))\right]_{u_l = S_l^e - t_l} = p_h\frac{\partial S_h}{\partial u_l}(S_l^e - t_l, \ddot{u}_h(u_l^*)) > 0,$$

which is true by virtue of the fact that $t_l > S_l^e - \eta_h$. This shows that there exists no pure-strategy equilibrium when $t_l \in (S_l^e - \eta_h, S_l^e)$.

Let us now consider the case where $t_l \geq S_l^e$. In this case, the only remaining putative equilibrium takes the form $u_l^* = 0$ and $u_h^* = \ddot{u}_h(0) \equiv \mathring{u}_h$. Consider the deviation to this putative equilibrium with the following structure: the firm relinquishes an $\varepsilon \approx 0$ more utility to high types (conquering the entire type-h market), and chooses $\hat{u}_l$ to solve

$$\max_{u_l} \left\{ p_l \left(\frac{1}{2} + \frac{u_l}{2t_l} \right) (S_l^e - u_l) + p_h \left(S_h(u_l, \mathring{u}_h) - \mathring{u}_h \right) \right\} \quad \text{s.t.} \quad u_l \geq 0.$$

This problem is concave. Therefore, the deviation described above generates a strict improvement if and only if the first-order condition evaluated at $u_l = 0$ is strictly positive:

$$\left[\frac{p_l}{2t_l} (S_l^e - u_l) - p_l \left(\frac{1}{2} + \frac{u_l}{2t_l} \right) + p_h \frac{\partial S_h}{\partial u_l}(u_l, \mathring{u}_h) \right]_{u_l = 0} = \frac{p_l}{2} \left(\frac{S_l^e}{t_l} - 1 \right) + p_h \frac{\partial S_h}{\partial u_l}(0, \mathring{u}_h) > 0,$$

which, by the definition of $\tilde{t}_l$, is equivalent to $t_l < \tilde{t}_l$. This establishes that there exists no pure-strategy equilibrium whenever $S_l^e - \eta_h < t_l < \tilde{t}_l$.

Proof of Claim (c). Finally, let us show that $u_l^* = 0$ and $u_h^* = \ddot{u}_h(0) \equiv \mathring{u}_h$ is a pure-strategy equilibrium when $t_l \geq \tilde{t}_l$. Proposition 11 then shows that this is the unique pure strategy equilibrium. We start arguing that the firm cannot improve by offering a menu in which the utility of the high type is smaller than $\mathring{u}_h$. This is because in any such menu the firm would attract no high types, while the firm would not profit from low types as the menu $(0, \mathring{u}_h)$ maximizes the firm's utility when it considers only low types.

Moreover, since the firm obtains zero profits from high types, a similar argument establishes that the firm cannot increase its profits by offering a contract that gives the same utility to high types. Therefore it remains to argue that the firm cannot profit by offering menus of the kind $(\triangle u_l, \mathring{u}_h + \triangle u_h)$, where $(\triangle u_l, \triangle u_h) \in \mathbb{R}_+ \times \mathbb{R}_+$. In light of the previous argument, we consider the following upper bound to the firms utility in which the constraint IC_h is ignored and it is assumed that the firm attracts all high types whenever such consumers yield positive profits:

$$G(\triangle u_l, \triangle u_h) \equiv \left\{ \begin{array}{c} p_l \left(\frac{1}{2} + \frac{\triangle u_l}{2t_l} \right) (S_l^e - \triangle u_l) \\ + p_h \max \left\{ S_h(\triangle u_l, \mathring{u}_h + \triangle u_h) - \mathring{u}_h - \triangle u_h, 0 \right\} \end{array} \right\}.$$

Therefore, if we show that $(0,0) \in \text{argmax}_{(\triangle u_l, \triangle u_h) \in \mathbb{R}_+ \times \mathbb{R}_+} G(\triangle u_l, \triangle u_h)$ then we will have concluded that $(0, \mathring{u}_h)$ is an equilibrium. First notice that any deviation $(\triangle u_l, \triangle u_h) \in \mathbb{R}_+ \times \mathbb{R}_+$ in which $\triangle u_h \geq \triangle u_l$ is weakly dominated by $(\triangle u_l, \triangle u_h) = (0,0)$ because it decreases the profits obtained from both types. Henceforth we can restrict attention to deviations $(\triangle u_l, \triangle u_h) \in \mathbb{R}_{++} \times \mathbb{R}_+$in which $\triangle u_h < \triangle u_l$. Again, since $\triangle u_h \to S_h(\triangle u_l, \mathring{u}_h + \triangle u_h) - \mathring{u}_h - \triangle u_h$ is strictly decreasing in $\triangle u_h$, a sufficient condition for the absence of profitable deviation is that $G(0,0) \geq G(\triangle u_l, 0)$ for all $\triangle u_l > 0$. Therefore, since $\triangle u_l \to G(\triangle u_l, 0)$ is concave, a sufficient condition is

$$\partial_{\triangle u_l +} G(\triangle u_l, \triangle u_h) \mid_{(\triangle u_l, \triangle u_h) = (0,0)} = \frac{p_l}{2} \left(\frac{S_l^e}{t_l} - 1 \right) + p_h \frac{\partial S_h}{\partial u_l}(0, \mathring{u}_h) \leq 0,$$

which holds by assumption. Q.E.D.

Proof of Proposition 2. Take a symmetric ordered equilibrium F^*. For $j = l, h$, let $\underline{u}_j$ be the infimum of F_j^*, and let $\bar{u}_j$ be supremum the support of F_j^*. Let $SuppF_j^*$ be the support of F_j^*.

At any point of continuity of F_h^*, the profit of the menu (u_l, u_h) is given by

$$\Pi(u_l, u_h) := \mathbb{E}_{F_l^*}\left[p_l I\left(\frac{1}{2} + \frac{u_l - \tilde{u}_l}{2t_l}\right)(S_l(u_l, u_h) - u_l)\right] + p_h F_h^*(u_h)(S_h(u_l, u_h) - u_h)$$

Since profits are supermodular in (u_l, u_h), it is easy to show that $(\underline{u}_l, \underline{u}_h)$ is optimal, hence $\Pi(\underline{u}_l, \underline{u}_h) = \Pi^* \equiv \sup_{(u_l, u_h)} \Pi(u_l, u_h)$. The equilibrium characterization is established in steps 1-15 below. The first step shows that each obtains a profit equal to the profit from the best contract designed only to low types in any mixed-strategy equilibrium. The second and third steps imply that firms obtain zero profits from high types from the less generous contract for these types. The fourth and fifth steps show that the difference in the indirect utilities given to low types from two different equilibrium contracts is bounded above by t_l and hence competition for low types always happens in the intensive margin: $I\left(\frac{1}{2} + \frac{u_l - \tilde{u}_l}{2t_l}\right) = \left(\frac{1}{2} + \frac{u_l - \tilde{u}_l}{2t_l}\right)$ for every equilibrium utilities u_l and $\tilde{u}_l$. Step 6 shows that the support of the distribution of indirect utilities for the high type has no atom. Steps 1-6 imply that we can write the firm's problem as

$$\Pi(u_l, u_h) := p_l\left(\frac{1}{2} + \frac{u_l - \tilde{u}_l}{2t_l}\right)(S_l(u_l, u_h) - u_l) + p_h F_h^*(u_h)(S_h(u_l, u_h) - u_h)$$

(not only for points of continuity of F_h^*). Step 7 establishes that the constraint IC$_l$ for every equilibrium menu (u_l, u_h) in which $u_l > \underline{u}_l$. Step 8 shows that the support of F_l^* has no atom at any $u_l > \underline{u}_l$. Steps 9-10 show that the support of F_k^* is an interval for $k = l, h$. Step 11 and 13 provide necessary and sufficient conditions for the equilibrium to have an atom at the lowest generous contract $(\underline{u}_l, \underline{u}_h)$. Step 12 shows that the support function $u_l \to \mathcal{U}_h(u_l)$ as well as the marginals F_k^* are Lipschitz continuous (and hence absolutely continuous) on any interval $[\underline{u}_k + \varepsilon, \bar{u}_l]$. Steps 14 and 15 use these properties to obtain the conditions stated in the proposition.

Step 1: $\underline{u}_l \in \operatorname{argmax} \mathbb{E}_{F_l^*}\left[p_l I\left(\frac{1}{2} + \frac{u_l - \tilde{u}_l}{2t_l}\right)(S_l^e - u_l)\right]$.

Assume towards a contradiction that $\underline{u}_l \notin \operatorname{argmax} \mathbb{E}_{F_l^*}\left[p_l I\left(\frac{1}{2} + \frac{u_l - \tilde{u}_l}{2t_l}\right)(S_l^e - u_l)\right]$.

Notice first that

$$\begin{gathered}\sup_{u_l} \mathbb{E}_{F_l^*}\left[p_l I\left(\tfrac{1}{2} + \tfrac{u_l - \tilde{u}_l}{2t_l}\right)(S_l^e - u_l)\right] \\ \leq \mathbb{E}_{F_l^*}\left[p_l I\left(\tfrac{1}{2} + \tfrac{\underline{u}_l - \tilde{u}_l}{2t_l}\right)(S_l(\underline{u}_l, \underline{u}_h) - \underline{u}_l)\right] + p_h F_h(\underline{u}_h)(S_h(\underline{u}_l, \underline{u}_h) - \underline{u}_h) = \Pi^*.\end{gathered}$$

Using this, the contradiction assumption and $\Pi(\underline{u}_l, \underline{u}_h) = \Pi^*$ we get

$$p_h F_h(\underline{u}_h)(S_h(\underline{u}_l, \underline{u}_h) - \underline{u}_h) > 0,$$

and hence F_h has an atom at $\underline{u}_h$, which implies that, for ε sufficiently small

$$\begin{aligned} &\mathbb{E}_{F_l^*}\left[p_l I\left(\tfrac{1}{2}+\tfrac{\underline{u}_l-\tilde{u}_l}{2t_l}\right)\left(S_l\left(\underline{u}_l,\underline{u}_h\right)-\underline{u}_l\right)\right]+p_h F_h\left(\underline{u}_h\right)\left(S_h\left(\underline{u}_l,\underline{u}_h\right)-\underline{u}_h\right). \\ < &\ \mathbb{E}_{F_l^*}\left[p_l I\left(\tfrac{1}{2}+\tfrac{\underline{u}_l-\tilde{u}_l}{2t_l}\right)\left(S_l\left(\underline{u}_l,\underline{u}_h\right)-\underline{u}_l\right)\right]+p_h F_h\left(\underline{u}_h+\varepsilon\right)\left(S_h\left(\underline{u}_l,\underline{u}_h+\varepsilon\right)-\underline{u}_h-\varepsilon\right), \end{aligned}$$

a contradiction.

Step 2: $F_h\left(\underline{u}_h\right)\left(S_h\left(\underline{u}_l,\underline{u}_h\right)-\underline{u}_h\right)=0$.

Trivially, $F_h\left(\underline{u}_h\right)\left(S_h\left(\underline{u}_l,\underline{u}_h\right)-\underline{u}_h\right)\geq 0$, hence if $F_h\left(\underline{u}_h\right)\left(S_h\left(\underline{u}_l,\underline{u}_h\right)-\underline{u}_h\right)>0$ then F_h must have an atom at $\underline{u}_h$ which, using the argument presented in Step 1, leads to a contradiction. .

Step 3: $\left(S_h\left(\underline{u}_l,\underline{u}_h\right)-\underline{u}_h\right)=0$.

Assume towards a contradiction that $\left(S_h\left(\underline{u}_l,\underline{u}_h\right)-\underline{u}_h\right)>0$. Then by Step 2, $F_h\left(\underline{u}_h\right)=0$. Moreover, there exists a (small) $\varepsilon>0$ such that $\left(S_h\left(\underline{u}_l,\underline{u}_h+\varepsilon\right)-\underline{u}_h-\varepsilon\right)>0$, implying that $\left(\underline{u}_l,\underline{u}_h+\varepsilon\right)$ is a profitable deviation.

Step 4: *In any (non-degenerate) mixed-strategy equilibrium we have* $\underline{u}_l\geq S_l^e-t_l$.

If $S_l^e-t_l<0$ the claim is obvious. Hence assume that $S_l^e-t_l\geq 0$. Notice that the steps above imply that

$$\Pi\left(\underline{u}_l,\underline{u}_h\right)=\Pi^*=\max_{u_l}\mathbb{E}\left[p_l I\left(\frac{1}{2}+\frac{u_l-\tilde{u}_l}{2t_l}\right)\left(S_l\left(u_l,u_h\right)-u_l\right)\right]$$

The first-order condition relatively to $\underline{u}_l$ can be simplified to:

$$-\int_{\underline{u}_l}^{\underline{u}_l+t_l}\left(\frac{1}{2}+\frac{\underline{u}_l-\tilde{u}_l}{2t_l}\right)dF_l^*\left(\tilde{u}_l\right)+\int_{\underline{u}_l}^{\underline{u}_l+t_l}\left(\frac{S_l^e-\underline{u}_l}{2t_l}\right)dF_l^*\left(\tilde{u}_l\right)\leq 0,$$

with equality if $\underline{u}_l>0$. If the inequality above holds for $\underline{u}_l=0$ we must have

$$\begin{aligned} 0 &\geq -\textstyle\int_{\underline{u}_l}^{\underline{u}_l+t_l}\left(\frac{1}{2}+\frac{-\tilde{u}_l}{2t_l}\right)dF_l^*\left(\tilde{u}_l\right)+\int_{\underline{u}_l}^{\underline{u}_l+t_l}\left(\frac{S_l^e}{2t_l}\right)dF_l^*\left(\tilde{u}_l\right) \\ &= \textstyle\int_{\underline{u}_l}^{\underline{u}_l+t_l}\left(\frac{\tilde{u}_l}{2t_l}\right)dF_l^*\left(\tilde{u}_l\right)+\int_{\underline{u}_l}^{\underline{u}_l+t_l}\left(\frac{S_l^e-t_l}{2t_l}\right)dF_l^*\left(\tilde{u}_l\right)>0, \end{aligned}$$

which is an absurd. Hence the first-order condition above holds with equality, implying

$$-\int_{\underline{u}_l}^{\underline{u}_l+t_l}\left(t_l+\underline{u}_l-\tilde{u}_l\right)dF_l^*\left(\tilde{u}_l\right)+\int_{\underline{u}_l}^{\underline{u}_l+t_l}\left(S_l^e-\underline{u}_l\right)dF_l^*\left(\tilde{u}_l\right)=0,$$

and hence

$$\underline{u}_l=\left(S_l^e-t_l\right)+\frac{\int_{\underline{u}_l}^{\underline{u}_l+t_l}\left(\tilde{u}_l-\underline{u}_l\right)dF_l^*\left(\tilde{u}_l\right)}{F_l^*\left(\underline{u}_l+t_l\right)}\geq\left(S_l^e-t_l\right). \tag{12}$$

Step 5: *In any (non-degenerate) mixed-strategy equilibrium we have* $\bar{u}_l-\underline{u}_l\leq t_l$ *and hence for all* $u_l,\tilde{u}_l$ *in the support of* F_l^* *we have* $I\left(\frac{1}{2}+\frac{u_l-\tilde{u}_l}{2t_l}\right)=\left(\frac{1}{2}+\frac{u_l-\tilde{u}_l}{2t_l}\right)$. *Thus for any point of continuity of* F_h^* *we can write the firm's problem for any point of continuity of* F_h *as:*

$$\Pi\left(u_l,u_h\right):=p_l\left(\frac{1}{2}+\frac{u_l-\mathbb{E}_{F_j^*}\left(\tilde{u}_l\right)}{2t_l}\right)\left(S_l\left(u_l,u_h\right)-u_l\right)+p_h F_h\left(u_h\right)\left(S_h\left(u_l,u_h\right)-u_h\right).$$

Follows immediately from Step 4.

Step 6: *The support of F_h^* has no atom at any $u_h > \underline{u}_h$.*

Assume towards a contradiction that F_h^* has an atom at some $u_h > \underline{u}_h : F_h(u_{h+}) - F_h(u_{h-}) > 0$. A standard undercutting argument implies that $F_h(u_h)(S_h(u_l, u_h) - u_h) = 0$ for any optimal menu (u_l, u_h). Hence $F_h(u_h) > 0$ implies $S_h(u_l, u_h) - u_h = 0$. Thus take an optimal menu (u_l, u_h) and notice that

$$\begin{aligned}\Pi(u_l, u_h) =& p_l\left(\tfrac{1}{2} + \tfrac{u_l - \mathbb{E}_{F_j^*}(\tilde{u}_l)}{2t_l}\right)(S_l(u_l, u_h) - u_l) + p_h F_h(u_h)(S_h(u_l, u_h) - u_h) \\ &= \Pi(u_l, u_h) = p_l\left(\tfrac{1}{2} + \tfrac{u_l - \mathbb{E}_{F_j^*}(\tilde{u}_l)}{2t_l}\right)(S_l(u_l, u_h) - u_l) \\ &= \max_{u_l} p_l\left(\tfrac{1}{2} + \tfrac{u_l - \mathbb{E}_{F_j^*}(\tilde{u}_l)}{2t_l}\right)(S_l(u_l, u_h) - u_l) = \Pi(\underline{u}_l, \underline{u}_h),\end{aligned}$$

which implies that $u_l = \underline{u}_l$, and thus $(\underline{u}_l, u_h)$ and $(\underline{u}_l, \underline{u}_h)$ are optimal. Using this and Step 3 we see that $S_h(\underline{u}_l, \underline{u}_h) - \underline{u}_h = S_h(\underline{u}_l, u_h) - u_h$. However, since $\left(\frac{\partial}{\partial u_h}\right)(S_h(\underline{u}_l, \tilde{u}_h) - \tilde{u}_h) \leq -1$,we conclude that $u_h = \underline{u}_h$, which is a contradiction.

Step 7: *The constraint IC_l binds for every optimal menu (u_l, u_h) in which $u_l > \underline{u}_l$.*

Otherwise, the f.o.c. w.r.t. u_l yields

$$\begin{aligned}u_l &= \tfrac{1}{2}\left[-p_l\left(\tfrac{1}{2} + \tfrac{-\mathbb{E}_{F_j^*}(\tilde{u}_l)}{2t_l}\right) + p_l\left(\tfrac{S_l(u_l, u_h)}{2t_l}\right) + p_l\left(\tfrac{\partial S_l\left(u_l, u_h^1\right)}{\partial u_l}\right)\right] \\ &\leq \tfrac{1}{2}\left[-p_l\left(\tfrac{1}{2} + \tfrac{-\mathbb{E}_{F_j^*}(\tilde{u}_l)}{2t_l}\right) + p_l\left(\tfrac{S_l^e}{2t_l}\right)\right] \leq \underline{u}_l,\end{aligned}$$

where the first equality has to hold because u_l has to be strictly positive, while the second uses Step 1. But thus contradiction $u_l > \underline{u}_l$.

Step 8: *The support of F_l^* has no atom at any $u_l > \underline{u}_l$.*

In light of Step 7, there are u_h^1, u_h^2, with $u_h^1 < u_h^2$ for which $\left(u_l, u_h^1\right)$ and $\left(u_l, u_h^2\right)$ are optimal. The following first-order conditions must then hold

$$\begin{aligned}-p_l\left(\tfrac{1}{2} + \tfrac{u_l - \mathbb{E}_{F_j^*}(\tilde{u}_l)}{2t_l}\right) + p_l\left(\tfrac{S_l^e - u_l}{2t_l}\right) + p_l\left(\tfrac{\partial S_l\left(u_l, u_h^1\right)}{\partial u_l}\right) + p_h F_h\left(u_h^1\right)\left(\tfrac{\partial S_h\left(u_l, u_h^1\right)}{\partial u_l}\right) = 0 \\ -p_l\left(\tfrac{1}{2} + \tfrac{u_l - \mathbb{E}_{F_j^*}(\tilde{u}_l)}{2t_l}\right) + p_l\left(\tfrac{S_l^e - u_l}{2t_l}\right) + p_h F_h\left(u_h^2\right)\left(\tfrac{\partial S_h\left(u_l, u_h^2\right)}{\partial u_l}\right) = 0,\end{aligned}$$

implying $p_h F_h\left(u_h^2\right)\left(\frac{\partial S_h\left(u_l, u_h^2\right)}{\partial u_l}\right) = p_h F_h\left(u_h^1\right)\left(\frac{\partial S_h\left(u_l, u_h^1\right)}{\partial u_l}\right)$. But notice that Step 7 and $u_h^1 < u_h^2$ imply $\frac{\partial S_h\left(u_l, u_h^2\right)}{\partial u_l} > \left(\frac{\partial S_h\left(u_l, u_h^1\right)}{\partial u_l}\right)$, hence $p_h F_h\left(u_h^2\right)\left(\frac{\partial S_h\left(u_l, u_h^2\right)}{\partial u_l}\right) > p_h F_h\left(u_h^1\right)\left(\frac{\partial S_h\left(u_l, u_h^1\right)}{\partial u_l}\right)$, which is a contradiction.

Step 9: *The support of F_h^* is an interval.*

Assume towards a contradiction that there exists $u_h^a < u_h^b$, that $u_h^a, u_h^b \in \text{supp}F_h^*$, but that $\left(u_h^a, u_h^b\right) \bigcap suppF_h^* = \emptyset$. In this case, there exists let u_l^a be the supremum over all utilities such that (u_l^a, u_h^a) is optimal. Let u_l^b be the infimum over all utilities such that $\left(u_l^b, u_h^b\right)$ is optimal. If $u_l^a = u_l^b$

the argument in Step 8 above leads to a contradiction. Hence we must have $u_l^a < u_l^b$. But then Step 7 and $F_h(u_h^1) = F_h(u_h^2)$ imply $\Pi(u_l^b, u_h^b - \varepsilon) > \Pi(u_l^b, u_h^b)$ for ε sufficiently small, a contradiction.

Step 10: *The support of F_l^* is an interval.*

The argument given in the steps above imply that we only have to deal with one case: there are two optimal menus (u_l^a, u_h) and (u_l^b, u_h) with $u_l^a \neq u_l^b$. But then fixing u_h, the payoff function is strictly concave, and hence $\Pi(u_l^a, u_h) = \Pi(u_l^b, u_h)$ implies $\Pi(u_l^a, u_h) < \Pi\left(\frac{u_l^a + u_l^b}{2}, u_h\right)$, contradicting the putative optimality of (u_l^a, u_h).

Step 11: *F_l^* (resp., F_h^*) have an atom at $\underline{u}_l$ (resp., $\underline{u}_h$) if and only if*

$$-p_l\left(\frac{1}{2} - \frac{\mathbb{E}_{F_j^*}(\tilde{u}_l)}{2t_l}\right) + p_l\left(\frac{S_l^e}{2t_l}\right) < 0,$$

in which case $\underline{u}_l = 0$, $F_h^*(\underline{u}_h) = F_l^*(0) > 0$ which satisfy

$$-p_l\left(\frac{1}{2} - \frac{\mathbb{E}_{F_j^*}(\tilde{u}_l)}{2t_l}\right) + p_l\left(\frac{S_l^e}{2t_l}\right) + p_h F_h^*(\mathring{u}_h(0))\left(\frac{\partial S_h(0, \mathring{u}_h(0))}{\partial u_l}\right) = 0,$$

where $\mathring{u}_h(u_l)$ is the unique solution to $S_h(u_l, \mathring{u}_h(u_l)) - \mathring{u}_h(u_l) = 0$.

First notice that if $\underline{u}_l > 0$ then Step 1 implies

$$-p_l\left(\frac{1}{2} - \frac{\underline{u}_l - \mathbb{E}_{F_j^*}(\tilde{u}_l)}{2t_l}\right) + p_l\left(\frac{S_l^e - \underline{u}_l}{2t_l}\right) = 0.$$

On the other hand, the fact that for $k = l, h$, $F_k^*(u_k)$ is strictly increasing (hence almost everywhere differentiable) imply

$$-p_l\left(\frac{1}{2} + \frac{\underline{u}_l - \mathbb{E}_{F_j^*}(\tilde{u}_l)}{2t_l}\right) + p_l\left(\frac{S_l^e - \underline{u}_l}{2t_l}\right) + p_h F_h^*(\mathring{u}_h(\underline{u}_l))\left(\frac{\partial S_h(\underline{u}_l, \mathring{u}_h(\underline{u}_l))}{\partial u_l}\right) = 0. \quad (13)$$

Therefore the last two equations imply $F_h^*(\mathring{u}_h(\underline{u}_l)) = 0$,and hence since the problem satisfies the increasing-difference properties in (u_l, u_h), $F_l^*(\underline{u}_l) = 0$. Conversely, if

$$-p_l\left(\frac{1}{2} - \frac{\mathbb{E}_{F_j^*}(\tilde{u}_l)}{2t_l}\right) + p_l\left(\frac{S_l^e}{2t_l}\right) < 0,$$

then

$$0 = \operatorname{argmax}\left(\frac{1}{2} + \frac{u_l - \mathbb{E}_{F_j^*}(\tilde{u}_l)}{2t_l}\right)(S_l^e - u_l),$$

together with (13), imply the desired equation.

Step 12: *The functions $\mathcal{U}_h(u_l), F_l^*(u_l)$ and $F_h^*(u_h)$ are Lipschitz continuous (hence absolutely continuous) on any interval $[\underline{u}_l + \varepsilon, \bar{u}_l]$.*

The strict-difference property of the profit function $\Pi(u_l, u_h)$ and the fact F_k^* has no atom for every $u_k > \underline{u}_k$ imply that there exists a strictly increasing function $\mathcal{U}_h : [\underline{u}_l, \bar{u}_l] \to [\underline{u}_h, \bar{u}_h]$ such that

(u_l, u_h) lies in the support of F^* if and only if $u_l \in [\underline{u}_l, \bar{u}_l]$ and $u_h = \mathcal{U}_h(u_l)$. We start showing that the restriction of the function F_h^* to any interval $[\underline{u}_h + \varepsilon, \bar{u}_h]$ is Lipschitz continuous.

We have for every $u_l \in [\underline{u}_l + \varepsilon, \bar{u}_l]$

$$p_l\left(\frac{1}{2} + \frac{u_l - \mathbb{E}_{F_j^*}(\tilde{u}_l)}{2t_l}\right)(S_l^e - u_l) + p_h F_h(u_l)(S_h(u_l, \mathcal{U}_h(u_l)) - u_h) = p_l\left(\frac{1}{2} + \frac{\underline{u}_l - \mathbb{E}_{F_j^*}(\tilde{u}_l)}{2t_l}\right)(S_l^e - \underline{u}_l).$$

From the strict concavity of $u_l \to p_l\left(\frac{1}{2} + \frac{u_l - \mathbb{E}_{F_j^*}(\tilde{u}_l)}{2t_l}\right)(S_l^e - u_l)$ and the fact that

$$\underline{u}_l = \text{argmax}\left\{p_l\left(\frac{1}{2} + \frac{u_l - \mathbb{E}_{F_j^*}(\tilde{u}_l)}{2t_l}\right)(S_l^e - u_l)\right\},$$

it follows there is $\kappa_\varepsilon > 0$ such that

$$p_l\left(\frac{1}{2} + \frac{u_l - \mathbb{E}_{F_j^*}(\tilde{u}_l)}{2t_l}\right)(S_l^e - u_l) < p_l\left(\frac{1}{2} + \frac{\underline{u}_l - \mathbb{E}_{F_j^*}(\tilde{u}_l)}{2t_l}\right)(S_l^e - \underline{u}_l) + \kappa_\varepsilon.$$

This and $F_h^*(u_h) \leq 1$ for every u_h imply

$$S_h(u_l, \mathcal{U}_h(u_l)) - u_h > \frac{\kappa_\varepsilon}{p_h}. \tag{14}$$

Moreover, $\underline{u}_l + \varepsilon \leq u_l^1 \leq u_l^2$ imply that there are positive constants $\gamma_2 > \gamma_1 > 0$ such that

$$\gamma_1(u_l^2 - u_l^1) \leq p_l\left(\frac{1}{2} + \frac{u_l^1 - \mathbb{E}_{F_j^*}(\tilde{u}_l)}{2t_l}\right)(S_l^e - u_l^1) - p_l\left(\frac{1}{2} + \frac{u_l^2 - \mathbb{E}_{F_j^*}(\tilde{u}_l)}{2t_l}\right)(S_l^e - u_l^2) \leq \gamma_2(u_l^2 - u_l^1)$$

For η sufficiently small, the difference in profits between the available menu $(u_l + \eta, U_h(u_l) + \eta)$ and the optimal menu $(u_l, U_h(u_l))$ satisfies:

$$p_h F_h^*(\mathcal{U}_h(u_l) + \eta)(S_h(u_l + \eta, \mathcal{U}_h(u_l) + \eta) - u_h - \eta) - p_h F_h^*(\mathcal{U}_h(u_l))(S_h(u_l, \mathcal{U}_h(u_l)) - u_h) \leq \gamma_2 \eta,$$

which, since $S_h(u_l + \eta, \mathcal{U}_h(u_l) + \eta) = S_h(u_l, \mathcal{U}_h(u_l))$, implies

$$p_h(F_h^*(\mathcal{U}_h(u_l) + \eta) - F_h^*(\mathcal{U}_h(u_l)))(S_h(u_l, \mathcal{U}_h(u_l)) - u_h) \leq \eta(\gamma_2 + 1).$$

Using (14) we get

$$(F_h^*(\mathcal{U}_h(u_l) + \eta) - F_h^*(\mathcal{U}_h(u_l))) \leq \frac{\eta(\gamma_2 + 1)}{\kappa_\varepsilon} \equiv \lambda\eta.$$

Next we show that there exists a constant ν such that $\mathcal{U}_h(u_l + \eta) - \mathcal{U}_h(u_l) \leq \eta\nu$.

Notice that the following first-order conditions are necessary at almost every $(u_l, u_l + \eta) \in [\underline{u}_l + \varepsilon, \bar{u}_l]^2$:

$$-p_l \frac{\partial}{\partial u_l}\left[p_l\left(\frac{1}{2} + \frac{u_l + \eta - \mathbb{E}_{F_j^*}(\tilde{u}_l)}{2t_l}\right)(S_l^e - u_l - \eta)\right] = p_h F_h^*(\mathcal{U}_h(u_l + \eta))\left(\frac{\partial S_h(u_l + \eta, \mathcal{U}_h(u_l + \eta))}{\partial u_l}\right) \tag{15}$$

$$-p_l \frac{\partial}{\partial u_l}\left[p_l\left(\frac{1}{2}+\frac{u_l-\mathbb{E}_{F_j^*}\left(\tilde{u}_l\right)}{2t_l}\right)\left(S_l^e-u_l\right)\right]=p_h F_h^*\left(\mathcal{U}_h\left(u_l\right)\right)\left(\frac{\partial S_h\left(\underline{u}_l,\mathcal{U}_h\left(u_l\right)\right)}{\partial u_l}\right).$$

Since $u \rightarrow p_l \frac{\partial}{\partial u_l}\left[p_l\left(\frac{1}{2}+\frac{u-\mathbb{E}_{F_j^*}\left(\tilde{u}_l\right)}{2t_l}\right)\left(S_l^e-u\right)\right]$ is Lipschitz continuous over the range above, there exists $\varrho>0$ and such that

$$-p_l \frac{\partial}{\partial u_l}\left[p_l\left(\frac{1}{2}+\frac{u_l+\eta-\mathbb{E}_{F_j^*}\left(\tilde{u}_l\right)}{2t_l}\right)\left(S_l^e-u_l-\eta\right)\right]+p_l \frac{\partial}{\partial u_l}\left[p_l\left(\frac{1}{2}+\frac{u_l-\mathbb{E}_{F_j^*}\left(\tilde{u}_l\right)}{2t_l}\right)\left(S_l^e-u_l\right)\right] \leq \varrho\eta,$$

This and (15) imply

$$p_h F_h^*\left(\mathcal{U}_h\left(u_l+\eta\right)\right)\left(\frac{\partial S_h\left(u_l+\eta,\mathcal{U}_h\left(u_l+\eta\right)\right)}{\partial u_l}\right)-p_h F_h^*\left(\mathcal{U}_h\left(u_l\right)\right)\left(\frac{\partial S_h\left(u_l,\mathcal{U}_h\left(u_l\right)\right)}{\partial u_l}\right) \leq \varrho\eta.$$

If $\mathcal{U}_h\left(u_l+\eta\right) \leq \mathcal{U}_h\left(u_l\right)+\eta$ there is nothing to prove. Henceforth assume that $\mathcal{U}_h\left(u_l+\eta\right)>\mathcal{U}_h\left(u_l\right)+\eta$, which implies

$$0 \leq\left(\frac{\partial S_h\left(u_l+\eta,\mathcal{U}_h\left(u_l+\eta\right)\right)}{\partial u_l}\right)-\left(\frac{\partial S_h\left(u_l,\mathcal{U}_h\left(u_l\right)\right)}{\partial u_l}\right) \leq \frac{\varrho\eta}{F_h^*\left(\mathcal{U}_h\left(\underline{u}_l+\varepsilon\right)\right) p_h}:=\varphi\eta,$$

or if we let $\triangle\left(u,\mathcal{U}_h(u)\right):=\mathcal{U}_h(u)-u$ and observe that $S_h$depends only on this difference and hence can be written as $S_h\left(\triangle\left(u,\mathcal{U}_h(u)\right)\right)$ we obtain

$$0 \leq\left(\frac{\partial S_h\left(\triangle\left(u_l+\eta,\mathcal{U}_h\left(u_l+\eta\right)\right)\right)}{\partial u_l}\right)-\left(\frac{\partial S_h\left(\triangle\left(u_l,\mathcal{U}_h\left(u_l\right)\right)\right)}{\partial u_l}\right) \leq \varphi\eta,$$

Next, notice that since $\triangle\left(\cdot,\mathcal{U}_h(\cdot)\right):\left[\underline{u}_l+\varepsilon,\bar{u}_l\right]^2 \rightarrow \mathbb{R}_{++}$ lies in a compact set $\left[\triangle\theta q_h^e+\chi_1,\triangle\theta q_h^e+\chi_2\right]$ for constants $0<\chi_1<\chi_2$. This and the fact that $\left(\frac{\partial S_h(\triangle)}{\partial u_l}\right)$ is strictly increasing in $\triangle$ in the same set, implies the existence of $\psi>0$ such that

$$\left(\frac{\partial S_h\left(\triangle\left(u_l+\eta,\mathcal{U}_h\left(u_l+\eta\right)\right)\right)}{\partial u_l}\right)-\left(\frac{\partial S_h\left(\triangle\left(u_l,\mathcal{U}_h\left(u_l\right)\right)\right)}{\partial u_l}\right) \geq \psi\left[\triangle\left(u_l+\eta,\mathcal{U}_h\left(u_l+\eta\right)\right)-\triangle\left(u_l,\mathcal{U}_h\left(u_l\right)\right)\right].$$

Putting the last two inequalities together one has

$$\mathcal{U}_h\left(u_l+\eta\right)-\mathcal{U}_h\left(u_l\right) \leq \frac{\varphi\eta}{\psi}+\eta \equiv \nu\eta,$$

which shows that $\mathcal{U}_h(\cdot)$ is Lipschitz continuous. The result for F_l^* follows because

$$F_l^*\left(u_l+\eta\right)-F_l^*\left(u_l\right)=F_h^*\left(\mathcal{U}_h\left(u_l+\eta\right)\right)-F_h^*\left(\mathcal{U}_h\left(u_l\right)\right) \leq \lambda\left[\mathcal{U}_h\left(u_l+\eta\right)-\mathcal{U}_h\left(u_l\right)\right] \leq \lambda\nu\eta.$$

Step 13: *We have* $\underline{u}_l=\max\left\{\frac{S_l^e-t_l+\mathbb{E}_{F_l^*}\left[\tilde{u}_l\right]}{2},0\right\}$.

The first-order condition yields $\underline{u}_l>0$ if and only if

$$\frac{\partial\left[p_l\left(\frac{1}{2}+\frac{u_l-\mathbb{E}_{F_l^*}\left[\tilde{u}_l\right]}{2t_l}\right)\left(S_l^e-u_l\right)\right]}{\partial u_l}\Big|_{u_l=0}>0,$$

which holds if and only if $\frac{S_l^e - t_l + \mathbb{E}_{F_l^*}[\tilde{u}_l]}{2} > 0$.

Step 14: *Reformulation of the firm's problem.*

In light of the steps above, we can write the firm's problem as:

$$\max_{u_l} \left\{ p_l \left(\frac{1}{2} + \frac{u_l - \mathbb{E}_{F_l^*}[\tilde{u}_l]}{2t_l} \right) (S_l^e - u_l) + p_h F_h^*(u_h) \left(S_h(u_l, u_h) - u_h \right) \right\} \quad \text{s.t.} \quad u_h \geq u_l \geq 0.$$

We obtain the following first-order condition with respect to u_l :

$$-\frac{p_l}{t_l} \left(\mathcal{U}_l(u_h) - \left(\frac{S_l^e - t_l + \mathbb{E}_{F_l^*}[\tilde{u}_l]}{2} \right) \right) + p_h F_h^*(u_h) \frac{\partial S_h}{\partial u_l}(\mathcal{U}_l(u_h), u_h) = 0,$$

which, after rearranging, leads to

$$F_h^*(u_h) = \frac{1}{t_l} \frac{p_l}{p_h} \left(\mathcal{U}_l(u_h) - \mathring{u}_l \right) \left(\frac{\partial S_h}{\partial u_l}(\mathcal{U}_l(u_h), u_h) \right)^{-1}. \tag{16}$$

$$\mathcal{U}_l(u_h) - \underline{u}_l = 2 \left(\frac{S_h(\mathcal{U}_l(u_h), u_h) - u_h}{\frac{\partial S_h}{\partial u_l}(\mathcal{U}_l(u_h), u_h)} \right) \left(\frac{\mathcal{U}_l(u_h) - \mathring{u}_l}{\mathcal{U}_l(u_h) + \underline{u}_l - 2\mathring{u}_l} \right),$$

as in the statement of the proposition.

Step 15: *When $t_l > 2S_l^e$ we have $\mathring{u}_l < 0$, in which case F_h^* exhibits a mass point at $\underline{u}_h$ (i.e., $F_h^*(\underline{u}_h) > 0$).*

First notice that when $\mathring{u}_l < 0$, then (16) implies

$$F_h^*(\underline{u}_h) = -\frac{1}{t_l} \frac{p_l}{p_h} \mathring{u}_l \left(\frac{\partial S_h}{\partial u_l}(0, \underline{u}_h) \right)^{-1} > 0.$$

Next we show that whenever t_l is sufficiently large we must necessarily have $\mathring{u}_l < 0$. Indeed, since $\mathring{u}_l = \frac{S_l^e - t_l + \mathbb{E}_{F_l^*}[\tilde{u}_l]}{2}$ and $\mathbb{E}_{F_l^*}[\tilde{u}_l] \leq S_l^e$ as otherwise the firm would obtain a negative payoff from the low type, the result follows whenever $t_l > 2S_l^e$.Q.E.D.

Proof of Proposition 4.

Pure-Strategy Equilibria. We first analyze pure-strategy equilibria.

Case 1: Assume that $(t_l, t_h) \in E$.

Consider the problem in which both IC constraints are ignored. The best response of firm a solves

$$\max_{u_l^a, u_h^a} \left\{ p_l \left(\frac{1}{2} + \frac{u_l^a - u_l^b}{2t_l} \right) (S_l^e - u_l^a) + p_h \left(\frac{1}{2} + \frac{u_h^a - u_l^b}{2t_h} \right) (S_h^e - u_h^a) \right\}$$

subject to $u_l^a, u_h^a \geq 0$. The first-order conditions with respect to u_k^a, $k \in \{l, h\}$, are

$$\frac{1}{2t_k} (S_k^e - u_k^a) - \left(\frac{1}{2} + \frac{u_k^a - u_k^b}{2t_k} \right) \leq 0, \tag{17}$$

with equality if $u_k^a > 0$. The expression above is strictly decreasing in u_k^a, which implies the firm's problem is quasi-concave. The unique solution to (17) is given by $u_k^* = \max\{S_k^e - t_k, 0\}, k \in \{l, h\}$. The argument is symmetric for firm b. Finally notice that both IC constraints are satisfied because $(t_l, t_h) \in E$. Since there is at most one pure strategy equilibrium by Proposition 5, we conclude that all pure-strategy equilibrium must involve no distortion. The possibility of mixed-strategy equilibria in this region is ruled out in Case 3 below.

Case 2: Assume that $(t_l, t_h) \in D_+$. First consider any pure-strategy equilibrium. Notice if $q_l = q_l^e$ and $q_h = q_h^e$ for some optimal menu $\left(u_l^1, u_h^1\right)$ then the first-order necessary conditions would imply that the relaxed problem in which both IC constraints are ignored is strictly concave, which implies that it admits one solution. All equilibrium must be in pure strategies and by 1. above we must have $(t_l, t_h) \in E$, leading to a contradiction. Assume towards a contradiction that $q_l(u_l, u_h) < q_l^e$ and $q_h(u_l, u_h) = q_h^e$. In this case, the first-order condition with respect to u_h and u_l read

$$\left(\tfrac{p_h}{2t_h}\right)(S_h^e - u_h) - \tfrac{p_h}{2} + \tfrac{p_l}{2}\tfrac{\partial S_l(u_l,u_h)}{\partial u_h} \le 0,$$
$$\left(\tfrac{p_l}{2t_l}\right)(S_l(u_l, u_h) - u_l) - \tfrac{p_l}{2} + \tfrac{p_l}{2}\tfrac{\partial S_l(u_l,u_h)}{\partial u_l} \le 0.$$

First assume that $u_l = 0$,in which case

$$\Lambda(t_l, t_h) \equiv t_h + \max\{S_l^e - t_l, 0\} \ge (S_h^e - u_h) + t_h \frac{p_l}{2p_h}\frac{\partial S_l(u_l, u_h)}{\partial u_h} + u_l > S_h^e - \triangle\theta q_l^e = \bar{\eta},$$

otherwise

$$\Lambda(t_l, t_h) \equiv t_h + \max\{S_l^e - t_l, 0\} \ge (S_h^e - u_h) + t_h \frac{p_l}{2p_h}\frac{\partial S_l(u_l, u_h)}{\partial u_h} + u_l - \frac{\partial S_l(u_l, u_h)}{\partial u_l} > S_h^e - \triangle\theta q_l^e = \bar{\eta},$$

contradicting $(t_l, t_h) \in D_+$. Therefore all pure-strategy equilibria involve

$$q_l(u_l, u_h) = q_l^e \quad \text{and} \quad q_h(u_l, u_h) > q_h^e.$$

Case 3: Assume that $(t_l, t_h) \in D_-$. By an argument analogous to the one used in Case 2 above, it is easy to show that all pure-strategy equilibria involve

$$q_h(u_l, u_h) > q_l^e \quad \text{and} \quad q_l(u_l, u_h) < q_l^e.$$

Mixed Strategy Equilibria. Consider now mixed-strategy equilibria. First suppose that $(\underline{u}_l, \underline{u}_h)$ is such that $S_l(u_l, u_h) = S_l^e$. Taking a right-derivative w.r.t. u_l for the menu $(\underline{u}_l, \underline{u}_h)$ delivers

$$p_l \int_{\underline{u}_l}^{\underline{u}_l + t_l} \left(\frac{S_l^e - \underline{u}_l}{2t_l}\right) dF_l(\tilde{u}_l) - p_l \int_{\underline{u}_l}^{\underline{u}_l + t_l} \left(\frac{1}{2} + \frac{\underline{u}_l - \tilde{u}_l}{2t_l}\right) dF_l(\tilde{u}_l)$$
$$+ p_h \int_{\underline{u}_h}^{\underline{u}_h + t_h} \frac{\partial S_h(\underline{u}_l, \underline{u}_h)}{\partial u_l}\left(\frac{1}{2} + \frac{\underline{u}_h - \tilde{u}_h}{2t_h}\right) dF_h(\tilde{u}_h) \le 0,$$

which implies

$$\underline{u}_l \geq \int_{\underline{u}_l}^{\underline{u}_l+t_l} \left(S_l^e - t_l\right) dF_l\left(\tilde{u}_l\right) + \left[\frac{\int_{\underline{u}_l}^{\underline{u}_l+t_l} \left(\tilde{u}_l - \underline{u}_l\right) dF_l\left(\tilde{u}_l\right) + \frac{p_h}{p_l} t_l \int_{\underline{u}_h}^{\underline{u}_h+t_h} \frac{\partial S_h(\underline{u}_l, \underline{u}_h)}{\partial u_l} \left(1 + \frac{\underline{u}_h - \tilde{u}_h}{t_h}\right) dF_h\left(\tilde{u}_h\right)}{F_l\left(\underline{u}_l + t_l\right)}\right] \geq S_l^e - t_l,$$

and hence $\bar{u}_l \leq \underline{u}_l + t_l$. Therefore the profit from the low type reads

$$p_l\left(\frac{1}{2} + \frac{u_l - \mathbb{E}_{F_l^*}[\tilde{u}_l]}{2t_l}\right)\left(S_l\left(u_l, u_h\right) - u_l\right)$$

in which case we can take a first-order condition w.r.t. u_l at $(\underline{u}_l, \underline{u}_h)$ delivering

$$\underline{u}_l \geq \left(S_l^e - t_l\right) + \left(\mathbb{E}_{F_l^*}[\tilde{u}_l] - \underline{u}_l\right) + \frac{p_h}{p_l} t_l \int_{\underline{u}_h}^{\underline{u}_h+t_h} \frac{\partial S_h\left(\underline{u}_l, \underline{u}_h\right)}{\partial u_l}\left(1 + \frac{\underline{u}_h - \tilde{u}_h}{t_h}\right) dF_h\left(\tilde{u}_h\right).$$

This immediately imply that there exists no optimal (u_l, u_h) at which $S_l^e > S_l\left(u_l, u_h\right)$ as otherwise

$$u_l < \left(S_l\left(u_l, u_h\right) - t_l\right) + \left(\mathbb{E}_{F_l^*}[\tilde{u}_l] - u_l\right) < \underline{u}_l,$$

which would be a contradiction. Similarly if IC_l binds for $(\underline{u}_l, \underline{u}_h)$ then it must bind for every $\left(u_l^1, u_h^1\right)$ in the support as otherwise

$$\begin{aligned}\underline{u}_l \geq \tfrac{1}{2}\Big[\left(S_l^e - t_l\right) + \left(\mathbb{E}_{F_l^*}[\tilde{u}_l]\right) + \tfrac{p_h}{p_l} t_l \int_{\underline{u}_h}^{\underline{u}_h+t_h} \tfrac{\partial S_h(\underline{u}_l, \underline{u}_h)}{\partial u_l}\left(1 + \tfrac{\underline{u}_h - \tilde{u}_h}{t_h}\right) dF_h\left(\tilde{u}_h\right)\Big] \\ > \tfrac{1}{2}\Big[\left(S_l^e - t_l\right) + \left(\mathbb{E}_{F_l^*}[\tilde{u}_l]\right)\Big] \geq u_l^1,\end{aligned}$$

which is a contradiction.

It is easy to see that the profit is differentiable at $(\bar{u}_l, \bar{u}_h)$. Moreover, its derivative w.r.t. u_h satisfies

$$-F_h\left(\bar{u}_h - t_h\right) + F_h\left(\bar{u}_h - t_h\right)\frac{\partial S_h\left(\bar{u}_l, \bar{u}_h\right)}{\partial \bar{u}_h} + \int_{\bar{u}_h - t_h}^{\bar{u}_h}\left(\frac{S_h\left(\bar{u}_l, \bar{u}_h\right) - \bar{u}_h}{2t_h}\right) dF_h\left(\tilde{u}_h\right)$$

$$-\int_{\bar{u}_h - t_h}^{\bar{u}_h}\left(\frac{1}{2} + \frac{\bar{u}_h - \tilde{u}_h}{2t_h}\right) dF_h\left(\tilde{u}_h\right) = 0,$$

implying $t_h \leq S_h^e - \bar{u}_h$, with strict inequality if IC_l binds at $(\bar{u}_l, \bar{u}_h)$. We claim that IC_l binds at $(\bar{u}_l, \bar{u}_h)$ if and only if it binds at $(\underline{u}_l, \underline{u}_h)$. Indeed, if IC_l binds at $(\bar{u}_l, \bar{u}_h)$ but not at $(\underline{u}_l, \underline{u}_h)$ then the first-order condition above delivers $\bar{u}_h < S_h^e - t_h$, while by an argument absolutely analogous to the one presented above we have $\underline{u}_h \geq S_h^e - t_h$, which is a contradiction. In summary, the analysis above shows that IC_l binds at some point (u_l, u_h) in the support if and only if it binds at $(\underline{u}_l, \underline{u}_h)$ and at $(\bar{u}_l, \bar{u}_h)$.

Performing an absolutely symmetric argument for IC_h we conclude that IC_h binds at $(\bar{u}_l, \bar{u}_h)$ if and only if it binds at $(\underline{u}_l, \underline{u}_h)$ if and only if it binds at every menu (u_l, u_h) in the equilibrium support.

Case 1: IC_l binds at every menu in the equilibrium support.

In this case, $\underline{u}_l > S_l^e - t_l$ and $\bar{u}_h < S_h - t_h$ which imply

$$\Lambda(t_l, t_h) \equiv t_h + \max\{S_l^e - t_l, 0\} < (S_h^e - \bar{u}_h) + \underline{u}_l \leq S_h^e - (\bar{u}_h - \bar{u}_l) < S_h^e - \triangle\theta q_h^e.$$

Hence $(t_l, t_h) \in D_+$.

Case 2: IC_h binds at every menu in the equilibrium support.

Analogously to case 1 above, the first-order condition at $(\underline{u}_l, \underline{u}_h)$ w.r.t. u_h implies $\underline{u}_h > S_h^e - t_h$ while the first-order condition at $(\bar{u}_l, \bar{u}_h)$ w.r.t. u_l implies $S_l^e - t_l > \bar{u}_l$,hence

$$\Lambda(t_l, t_h) \equiv t_h + \max\{S_l^e - t_l, 0\} > (S_h^e - \underline{u}_h) + \bar{u}_l \geq S_h^e - (\underline{u}_h - \underline{u}_l) > S_h^e - \triangle\theta q_l^e.$$

Hence $(t_l, t_h) \in D_-$.

Case 3: No IC binds at any menu in the equilibrium support.

The first-order conditions at $(\underline{u}_l, \underline{u}_h)$ imply that $\underline{u}_k \geq S_k^e - t_k$, and hence since $\bar{u}_k \leq S_k^e$ we have $\bar{u}_k - \underline{u}_k \leq t_k$. Therefore the problem is given by

$$\max_{u_l, u_h} \left\{ p_l \left(\frac{1}{2} + \frac{u_l - \mathbb{E}_{F_l^*}[\tilde{u}_l]}{2t_l} \right) (S_l^e - u_l) + p_h \left(\frac{1}{2} + \frac{u_h - \mathbb{E}_{F_l^*}[\tilde{u}_l]}{2t_h} \right) (S_h^e - u_h) \right\}$$

This problem is strictly concave and admits a unique solution. Therefore the equilibrium is in pure-strategies and $(t_l, t_h) \in E$. Q.E.D.

Proof of Proposition 5.

We prove two facts that show that there exists at most one pure-strategy equilibrium. The statement regarding types that benefit from asymmetric information and types that are made worse off follows immediately from the necessary first-order conditions for the equilibrium.

Fact 1: *If there exists an equilibrium in which no IC constraint binds then this is the unique pure strategy equilibrium.*

It is trivial to see that there is at most one equilibrium in which no IC constraint binds and in which case this is in pure strategies. Hence we show that there is no other equilibrium in which one IC binds. Recall that $\triangle(u_l, u_h) := u_h - u_l$. Notice that this equilibrium satisfies $u_h > 0$ and

$$S_l^e - u_l^1 - t_l \leq 0 \quad (= 0 \text{ if } u_l^1 > 0) \tag{18}$$

$$S_h^e - u_h^1 - t_h = 0 \tag{19}$$

First suppose that there exists an equilibrium $\left(u_l^2, u_h^2\right)$ in which IC$_l$ binds:

$$S_l^e - u_l^2 - t_l + \left(\frac{p_h}{p_l}\right) t_l \left(\frac{\partial S_h\left(\triangle\left(u_l^2, u_h^2\right)\right)}{\partial u_l}\right) \leq 0 (= 0 \text{ if } u_l > 0) \tag{20}$$

$$S_h\left(\triangle\left(u_l^2, u_h^2\right)\right) - u_h^2 - t_h + \left(\frac{p_h}{p_l}\right) t_h \left(\frac{\partial S_h\left(\triangle\left(u_l^2, u_h^2\right)\right)}{\partial u_h}\right) = 0. \tag{21}$$

Notice that (19) and (21) imply $u_h^2 < u_h^1$, while (18) and (20) imply $u_l^2 \geq u_l^1$ implying $\triangle\left(u_l^2, u_h^2\right) < \triangle\left(u_l^1, u_h^1\right)$, a contradiction.

Next suppose that there exists an equilibrium $\left(u_l^3, u_h^3\right)$ in which IC$_h$ binds:

$$S_l^e - u_l^e - t_l + t_l \left(\frac{\partial S_l\left(\triangle\left(u_l^3, u_h^3\right)\right)}{\partial u_l}\right) \leq 0 (= 0 \text{ if } u_l^3 > 0) \tag{22}$$

$$S_h\left(\triangle\left(u_l^3, u_h^3\right)\right) - u_h^3 - t_h + \left(\frac{p_l}{h}\right) t_h \left(\frac{\partial S_l\left(\triangle\left(u_l^3, u_h^3\right)\right)}{\partial u_h}\right) \leq 0 (= 0 \text{ if } u_h^3 > 0) \tag{23}$$

Notice that (23) and (19) imply $u_h^2 > u_h^1$, while (22) and (20) imply $u_l^2 \leq u_l^1$ delivering $\triangle\left(u_l^2, u_h^2\right) > \triangle\left(u_l^1, u_h^1\right)$,a contradiction.

Fact 2: *If there exists an equilibrium in which the IC_l constraint binds then this is the unique pure strategy equilibrium.*

Let $\left(u_l^2, u_h^2\right)$ an equilibrium in which IC_l binds and $\left(u_l^3, u_h^3\right)$ an equilibrium in which IC_h binds. Using (21) and (23) we $u_h^3 > u_h^2 > 0$. On the other hand (22) and (20) imply $u_l^3 \leq u_l^2$. Therefore we have $\Delta\left(u_l^3, u_h^3\right) > \Delta\left(u_l^2, u_h^2\right)$, which is a contradiction. Q.E.D.

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Online Appendix

Omitted Proofs

Proof of Proposition 3.

Proof of Claim (a). Suppose $t_h \in [0, S_h^e - \eta_l]$. Proposition 11 shows that firms obtain zero profits from low-type contracts. Moreover, the argument in In the putative equilibrium where constraints IR_k and IC_k are slack, for $k \in \{l, h\}$, we posit that $u_l^* = S_l^e$ and $u_h^* = S_h^e - t_h$.

The best response of firm a then solves

$$\max_{u_l^a, u_h^a} \left\{ p_l \hat{D}_l^a \left(u_l^a\right) \left(S_l^e - u_l^a\right) + p_h \left(\frac{1}{2} + \frac{u_h - \left(S_h^e - t_h\right)}{2t_h}\right) \left(S_h^e - u_h^a\right) \right\},$$

where

$$\hat{D}_l^a \left(u_l^a\right) = \begin{cases} 0 & \text{if} \quad u_l^a < S_l^e \\ \frac{1}{2} & \text{if} \quad u_l^a = S_l^e \\ 1 & \text{if} \quad u_l^a > S_l^e. \end{cases}$$

This problem has a solution $u_l^* = S_l^e$ and $u_h^* = S_h^e - t_h$. Note that IC_h is slack, as

$$u_h^* - u_l^* = (S_h^e - t_h) - S_l^e \geq q_l^e \Delta\theta$$

by assumption, as $t_h \leq S_h^e - \eta_l$. This establishes the result as by Proposition 11 there exists at most one pure-strategy equilibrium.

Proof of Claim (b). Now suppose $t_h > S_h^e - \eta_l$. Assume first that there exists a pure-strategy equilibrium in which IC_h does not bind. Then since the firm makes zero profits from low types by Proposition 11, we must have $u_l = S_l^e$. On the other hand, the first-order condition with respect to u_h delivers $u_h \leq S_h^e - t_h$. Hence

$$u_h - u_l \leq S_h^e - t_h - S_l^e < q_l^e \Delta\theta,$$

which is a contradiction.

Hence since in any pure-strategy equilibrium IC_h binds and the firm obtains zero profits low types, we must have $u_l^* = S_l(u_l^*, u_h^*) < S_l^e$.

Assume first that $t_h \in (S_h^e - \eta_l, S_h^e)$. Hence the first-order condition with respect to u_h delivers

$$-\left(\frac{p_h}{2}\right) + \left(\frac{S_h^e - u_h^a}{2t_h}\right) + \frac{p_l}{2} \frac{\partial S_l(u_l^*, u_h^*)}{\partial u_h} \leq 0,$$

which implies $u_h \geq S_h^e - t_h > 0$. If $u_h > S_h^e - t_h$ then $u_h \notin \text{argmax} p_h \left(\frac{1}{2} + \frac{u_h - u_h^*}{2t_h}\right) (S_h^e - u_h)$, which in light of our finding in Proposition 11 that firm makes zero profits from low types, we conclude that there is a profitable deviation. Hence the unique candidate to a pure-strategy equilibrium is $(S_h^e - t_h, \ddot{u}_l(S_h^e - t_h))$, where the function $\ddot{u}_l(u_h)$ is defined as the unique solution in u_l to $S_l(u_l, u_h) = u_l$. Consider a deviation to this putative equilibrium with the following structure: the firm relinquishes an $\varepsilon \approx 0$ more utility to low types (conquering the entire type-l market), and chooses $\hat{u}_h$ to solve

$$\max_{u_h} \left\{ p_l \left(S_l(\ddot{u}_l(u_h^*), u_h) - \ddot{u}_l(u_h^*) \right) + p_h \left(\frac{1}{2} + \frac{u_h - (S_h^e - t_h)}{2t_h} \right) (S_h^e - u_h) \right\} \quad \text{s.t.} \quad u_h \geq S_h^e - t_h.$$

This problem is concave. Therefore, the deviation described above generates a strict improvement if and only if the first-order condition evaluated at $u_h = S_h^e - t_h$ is strictly positive:

$$\left[p_l \frac{\partial S_l}{\partial u_h}(\ddot{u}_l(u_h^*), u_h) + \frac{p_h}{2t_h}(S_h^e - u_h) - p_h \left(\frac{1}{2} + \frac{u_h - (S_h^e - t_h)}{2t_h} \right) \right]_{u_h = S_h^e - t_h} = p_l \frac{\partial S_l}{\partial u_h}(\ddot{u}_l(u_h^*), S_h^e - t_h) > 0,$$

which is true by virtue of the fact that $t_h > S_h^e - \eta_l$. This shows that there exists no pure-strategy equilibrium when $t_h \in (S_h^e - \eta_l, S_h^e)$.

Let us now consider the case where $t_h \geq S_h^e$. In this case, an analogous argument to the one used above implies that the only remaining putative equilibrium takes the form $u_h^* = 0$ and $u_l^* = \ddot{u}_l(0) \equiv 0$. Consider a deviation to this putative equilibrium with the following structure: the firm relinquishes an $\varepsilon \approx 0$ more utility to low types (conquering the entire type-l market), and chooses $\hat{u}_h$ to solve

$$\max_{u_h} \left\{ p_l S_l(0, u_h) + p_h \left(\frac{1}{2} + \frac{u_h}{2t_h} \right) (S_h^e - u_h) \right\} \quad \text{s.t.} \quad u_h \geq 0.$$

This problem is concave. Therefore, the deviation described above generates a strict improvement if and only if the first-order condition evaluated at $u_h = 0$ is strictly positive:

$$\left[p_l \frac{\partial S_l}{\partial u_h}(0, u_h) + \frac{p_h}{2t_h}(S_h^e - u_h) - p_h \left(\frac{1}{2} + \frac{u_h}{2t_h} \right) \right]_{u_h = 0} = p_l \frac{\partial S_l}{\partial u_h}(0, 0) + \frac{p_h}{2} \left(\frac{S_h^e}{t_h} - 1 \right)$$

which, by the definition of $\tilde{t}_h$, is equivalent to $t_h < \tilde{t}_h$. This establishes that there exists no pure-strategy equilibrium whenever $S_h^e - \eta_l < t_h < \tilde{t}_h$.

Proof of Claim (c). The putative equilibrium is $u_l^* = u_h^* = 0$. We will show that no firm has a profitable deviation. Consider a deviation to some utility profile $(u_l, u_h) \neq (0, 0)$. First assume $u_h > u_l$ and notice that the first-order condition w.r.t. u_h delivers

$$\begin{aligned} -p_h \left(\tfrac{1}{2} + \tfrac{u_h}{2t_h} \right) + p_h \left(\tfrac{S_h^e - u_h}{2t_h} \right) + p_l \tfrac{\partial S_l}{\partial u_h}(u_l, u_h) \\ < -p_h \left(\tfrac{1}{2} \right) + p_h \left(\tfrac{S_h^e}{2t_h} \right) + p_l \tfrac{\partial S_l}{\partial u_h}(u_l, u_l) \\ = -p_h \left(\tfrac{1}{2} \right) + p_h \left(\tfrac{S_h^e}{2t_h} \right) + p_l \tfrac{\partial S_l}{\partial u_h}(0, 0) \leq 0, \end{aligned}$$

where the last inequality used $t_h \geq \tilde{t}_h$. This shows that all such deviations are suboptimal. Hence consider deviations in which $u_h = u_l$. But in this case, $S_l(u_l, u_l) = S_l(0, 0) = 0$, which implies that no such deviation can be profitable. This completes the proof. Q.E.D.

Proof of Proposition 6. We only prove (b) as the argument for (a) is analogous. Take $(t_l, t_h) \in D_+$ and consider the putative pure-strategy equilibrium (u_l^*, u_h^*) from Proposition 5.
Consider first the case where $u_l^* > 0$. Letting $\triangle \equiv u_h - u_l$ (and $\triangle^* \equiv u_h^* - u_l^*$) it follows from the equilibrium conditions that

$$\frac{\partial \triangle}{\partial t_h} = \frac{-\frac{p_l}{t_l}\left(\frac{\partial S_h}{\partial u_h}(\triangle^*) - 1\right)}{\frac{p_l}{t_l}\left(\frac{\partial S_h}{\partial u_h}(\triangle^*) - 1\right) + \left(\frac{p_l}{t_l} + \frac{p_h}{t_h}\right) t_h \frac{\partial^2 S_h}{\partial u_h^2}(\triangle^*)} < 0,$$

$$\frac{\partial \triangle}{\partial t_h} + 1 = \frac{\left(\frac{p_l}{t_l} + \frac{p_h}{t_h}\right) t_h \frac{\partial^2 S_h}{\partial u_h^2}(\triangle^*)}{\frac{p_l}{t_l}\left(\frac{\partial S_h}{\partial u_h}(\triangle^*) - 1\right) + \left(\frac{p_l}{t_l} + \frac{p_h}{t_h}\right) t_h \frac{\partial^2 S_h}{\partial u_h^2}(\triangle^*)} > 0,$$

$$\frac{\partial u_l}{\partial t_h} = \frac{p_h \frac{\partial^2 S_h}{\partial u_h^2}(\triangle^*)\left(\frac{\partial S_h}{\partial u_h}(\triangle^*) - 1\right)}{\frac{p_l}{t_l}\left(\frac{\partial S_h}{\partial u_h}(\triangle^*) - 1\right) + \left(\frac{p_l}{t_l} + \frac{p_h}{t_h}\right) t_h \frac{\partial^2 S_h}{\partial u_h^2}(\triangle^*)} < 0,$$

$$\frac{\partial u_h}{\partial t_h} = \frac{\partial \triangle}{\partial t_h} + \frac{\partial u_l}{\partial t_h} = \frac{\left[p_h \frac{\partial^2 S_h}{\partial u_h^2}(\triangle^*) - \frac{p_l}{t_l}\right]\left(\frac{\partial S_h}{\partial u_h}(\triangle^*) - 1\right)}{\frac{p_l}{t_l}\left(\frac{\partial S_h}{\partial u_h}(\triangle^*) - 1\right) + \left(\frac{p_l}{t_l} + \frac{p_h}{t_h}\right) t_h \frac{\partial^2 S_h}{\partial u_h^2}(\triangle^*)} < 0,$$

$$\frac{\partial u_h}{\partial t_h} + 1 = \frac{\frac{\partial^2 S_h}{\partial u_h^2}(\triangle^*)\left(S_l^e - u_l^*\right)}{\frac{p_l}{t_l}\left(\frac{\partial S_h}{\partial u_h}(\triangle^*) - 1\right) + \left(\frac{p_l}{t_l} + \frac{p_h}{t_h}\right) t_h \frac{\partial^2 S_h}{\partial u_h^2}(\triangle^*)} > 0.$$

In turn, consider the case where $u_l^* = 0$. In this case, $\triangle^* = u_h^*$, and

$$\frac{\partial \triangle}{\partial t_h}(t_l, t_h) = \frac{\partial u_h^*}{\partial t_h}(t_l, t_h) = \frac{1 - \frac{\partial S_h}{\partial u_h}(\triangle^*)}{\frac{\partial S_h}{\partial u_h}(\triangle^*) - 1 + t_h \frac{\partial^2 S_h}{\partial u_h^2}(\triangle^*)} < 0,$$

$$\frac{\partial \triangle}{\partial t_h}(t_l, t_h) + 1 = \frac{\partial u_h^*}{\partial t_h}(t_l, t_h) + 1 = \frac{t_h \frac{\partial^2 S_h}{\partial u_h^2}(\triangle^*)}{\frac{\partial S_h}{\partial u_h}(\triangle^*) - 1 + t_h \frac{\partial^2 S_h}{\partial u_h^2}(\triangle^*)} > 0.$$

Obviously, in this case, $\frac{\partial u_l^*}{\partial t_h}(t_l, t_h) = 0$. The signs of these derivatives will be used below.

The best reply of each firm chooses $(u_l, \triangle)$ to maximize

$$\Pi^d(u_l, \triangle) \equiv p_l \left(\frac{1}{2} + \frac{u_l - u_l^*}{2t_l}\right)(S_l^e - u_l) + p_h I \left(\frac{1}{2} + \frac{\triangle + u_l - u_h^*}{2t_h}\right)(S_h(\triangle) - \triangle - u_l).$$

At the range where $u_h = \triangle + u_l \in (u_h^* - t_h, u_h^* + t_h)$, the Jacobian and the Hessian of the best-reply objective function exhibit the following derivatives:

$$\frac{\partial \Pi^d}{\partial u_l}(u_l, \triangle) = \frac{p_l}{2t_l}(S_l^e - u_l) - p_l \left(\frac{1}{2} + \frac{u_l - u_l^*}{2t_l}\right) - p_h \left(\frac{1}{2} + \frac{\triangle + u_l - u_h^*}{2t_h}\right) + \frac{p_h}{2t_h}(S_h(\triangle) - \triangle - u_l),$$

$$\frac{\partial \Pi^d}{\partial \delta}(u_l, \triangle) = \frac{p_h}{2t_h}(S_h(\triangle) - \delta - u_l) + p_h \left(\frac{1}{2} + \frac{\triangle + u_l - u_h^*}{2t_h}\right)\left(\frac{\partial S_h}{\partial u_h}(\triangle) - 1\right),$$

$$\frac{\partial^2 \Pi^d}{\partial u_l^2}(u_l, \triangle) = -\frac{p_l}{2t_l} - \frac{p_l}{2t_l} - \frac{p_h}{2t_h} - \frac{p_h}{2t_h} = -\left(\frac{p_l}{t_l} + \frac{p_h}{t_h}\right),$$

$$\frac{\partial^2 \Pi^d}{\partial \delta^2}(u_l, \triangle) = \frac{p_h}{t_h}\left(\frac{\partial S_h}{\partial u_h}(\triangle) - 1\right) + p_h\left(\frac{1}{2} + \frac{\triangle + u_l - u_h^*}{2t_h}\right)\left(\frac{\partial^2 S_h}{\partial u_h^2}(\triangle)\right),$$

and

$$\frac{\partial^2 \Pi^d}{\partial u_l \partial \delta}(u_l, \triangle) = -\frac{p_h}{2t_h} + \frac{p_h}{2t_h}\left(\frac{\partial S_h}{\partial u_h}(\triangle) - 1\right) = \frac{p_h}{2t_h}\left(\frac{\partial S_h}{\partial u_h}(\triangle) - 2\right).$$

Because

$$\frac{\partial^2 \Pi^d}{\partial u_l^2}(u_l, \triangle), \frac{\partial^2 \Pi^d}{\partial \triangle^2}(u_l, \triangle), \frac{\partial^2 \Pi^d}{\partial u_l \partial \triangle}(u_l, \triangle) < 0,$$

the function Π^d is coordinate-wise concave and submodular in $(u_l, \triangle)$. It is jointly weakly concave at $(u_l, \triangle)$ if and only if

$$H(u_l, \triangle) \equiv \frac{\partial^2 \Pi^d}{\partial u_l^2}\frac{\partial^2 \Pi^d}{\partial \triangle^2} - \left(\frac{\partial^2 \Pi^d}{\partial u_l \partial \triangle}\right)^2 \geq 0.$$

After some algebra, it can be shown that

$$H(u_l, \triangle) = \left(\frac{p_h}{t_h}\right)^2 \left\{-\left(\frac{p_l}{t_l}\frac{t_h}{p_h} + 1\right)\left[\left(\frac{\partial S_h}{\partial u_h}(\triangle) - 1\right) + \frac{1}{2}\left(t_h + \triangle + u_l - u_h^*\right)\left(\frac{\partial^2 S_h}{\partial u_h^2}(\triangle)\right)\right] - \frac{1}{4}\left(\frac{\partial S_h}{\partial u_h}(\triangle) - 2\right)^2\right\}.$$

At the putative equilibrium,

$$H(u_l^*, \triangle^*) = \left(\frac{p_h}{t_h}\right)^2 \Upsilon(t_l, t_h),$$

where

$$\Upsilon(t_l, t_h) \equiv \left\{-\left(\frac{p_l}{t_l}\frac{t_h}{p_h} + 1\right)\left[\left(\frac{\partial S_h}{\partial u_h}(\triangle^*) - 1\right) + \frac{t_h}{2}\left(\frac{\partial^2 S_h}{\partial u_h^2}(\triangle^*)\right)\right] - \frac{1}{4}\left(\frac{\partial S_h}{\partial u_h}(\triangle^*) - 2\right)^2\right\},$$

with the understanding that δ^* is a function of (t_l, t_h). Hence, the best-reply objective is locally weakly jointly concave at the putative equilibrium, i.e., $H(u_l^*, \triangle^*) \geq 0$, if and only if $\Upsilon(t_l, t_h) \geq 0$. Direct inspection reveals that $\Upsilon(t_l, t_h) < 0$ for t_h in an open neighborhood around zero.

Taking the partial derivative of $\Upsilon(t_l, t_h)$ with respect to t_h, we obtain:

$$\frac{\partial \Upsilon}{\partial t_h}(t_l, t_h) = -\frac{p_l}{t_l}\frac{1}{p_h}\left(\frac{\partial S_h}{\partial u_h}(\triangle^*) - 1\right)$$

$$-\left(\frac{p_l}{t_l}\frac{t_h}{p_h}\right)\left[\frac{\partial S_h^2}{\partial u_h^2}(\triangle^*)\left(\frac{\partial \delta^*}{\partial t_h} + 1\right) + \frac{t_h}{2}\frac{\partial^3 S_h}{\partial u_h^3}(\triangle^*)\frac{\partial \delta^*}{\partial t_h}\right].$$

$$-\frac{1}{2}\frac{\partial^2 S_h}{\partial u_h^2}(\triangle^*) - \frac{t_h}{2}\frac{\partial^3 S_h}{\partial u_h^3}(\triangle^*)\frac{\partial \delta^*}{\partial t_h} - \frac{1}{2}\frac{\partial \delta^*}{\partial t_h}\frac{\partial^2 S_h}{\partial u_h^2}(\triangle^*)\frac{\partial S_h}{\partial u_h}(\triangle^*) > 0.$$

The derivatives computed above for $\triangle^*$, for the cases where either $u_l^* > 0$ or $u_l^* = 0$, together with the assumption on the third derivative of φ, which guarantees that $\frac{\partial^3 S_h}{\partial u_h^3} > 0$, imply the last inequality.

Therefore, for a fixed t_l, there exists a threshold $\tau_h^1(t_l) > 0$ such that the best-reply objective is locally weakly jointly concave at the putative equilibrium, i.e., $H(u_l^*, \triangle^*) \geq 0$, if and only if $t_h \geq \tau_h^1(t_l)$. Further notice that $(t_l, \tau_h^1(t_l)) \in D_+$, as a pure-strategy equilibrium is guaranteed to exist by continuity around any (t_l, t_h) satisfying

$$\Lambda(t_l, t_h) = \eta_h,$$

i.e., in the boundary between the regions D_+ and E. Obviously, a pure-strategy equilibrium fails to exist provided $t_h < \tau_h^1(t_l)$.

For $t_h \geq \tau_h^1(t_l)$, by local joint concavity, we know there is no local deviation around the putative equilibrium. But there might exist non-local profitable deviations. To analyze the latter, note that $H(u_l, \triangle) \geq 0$ if and only if

$$u_l \geq \frac{\left(\frac{p_l}{t_l}\frac{t_h}{p_h}+1\right)\left[\left(\frac{\partial S_h}{\partial u_h}(\triangle)-1\right)+\frac{1}{2}\left(t_h+\triangle-u_h^*\right)\left(\frac{\partial^2 S_h}{\partial u_h^2}(\triangle)\right)\right]+\frac{1}{4}\left(\frac{\partial S_h}{\partial u_h}(\triangle)-2\right)^2}{-\left(\frac{\partial^2 S_h}{\partial u_h^2}(\triangle)\right)\left(\frac{p_l}{t_l}\frac{t_h}{p_h}+1\right)}.$$

The right-hand side does not depend on u_l, being an affine function of $\triangle$ when φ is quadratic as $\frac{\partial S_h}{\partial u_h}(\triangle) = \gamma_1 - \gamma_2\triangle$ for $\gamma_1 \in \mathbb{R}$ and $\gamma_2 \in \mathbb{R}_+$for $\triangle \in [q_h^e\Delta\theta, +\infty)$ and $\frac{\partial^2 S_h}{\partial u_h^2}(\triangle) = -\gamma_2$ for $\triangle$ in the same range. Therefore, Π^d is jointly concave over a convex subset of the domain $\{(u_l, \triangle) : u_l \geq 0, \triangle \in [q_h^e\Delta\theta, +\infty)\}$. Because $t_h \geq \tau_h^1(t_l)$, the putative equilibrium (u_l^*, u_h^*) belongs to this set, which implies Π^d does not possess another critical point.

This implies that the best profitable deviation cannot belong to the range

$$u_h = \triangle + u_l \in (u_h^* - t_h, u_h^* + t_h),$$

which implies it satisfies $u_h \in \{u_h^* - t_h, u_h^* + t_h\}$ (it is easy to show that no profitable deviation exist when no IC binds or where IC$_h$ binds).

Consider first the deviation that sets $u_h = u_h^* + t_h$, therefore cornering the high-type market. The optimal deviation of this kind chooses u_l to solve the following program:

$$\Pi^{d*}(t_l, t_h) \equiv \max_{u_l} \left\{ p_l \left(\frac{1}{2} + \frac{u_l - u_l^*}{2t_l}\right)(S_l^e - u_l) + p_h\left(S_h(u_l, u_h^* + t_h) - u_h^* - t_h\right)\right\}.$$

This objective is concave in u_l. The first-order condition reveals that the maximand u_l^d solves

$$\frac{p_l}{2t_l}\left(S_l^e - u_l^d\right) - p_l\left(\frac{1}{2} + \frac{u_l^d - u_l^*}{2t_l}\right) + p_h\frac{\partial S_h}{\partial u_l}(u_l^d, u_h^* + t_h) = 0,$$

which can be rewritten as

$$u_l^d = \frac{S_l^e - t_l}{2} + \frac{u_l^*}{2} + t_l\frac{p_h}{p_l}\frac{\partial S_h}{\partial u_l}(u_l^d, u_h^* + t_h). \tag{24}$$

Employing the envelope theorem, we obtain that

$$\frac{\partial \Pi^{d*}}{\partial t_h}(t_l,t_h) = -\frac{p_l}{2t_l}\frac{\partial u_l^*}{\partial t_h}(t_l,t_h)\left(S_l^e - u_l^d\right) + p_h\left(\frac{\partial S_h}{\partial u_h}(u_l, u_h^*(t_l,t_h)+t_h) - 1\right)\left[\frac{\partial u_h^*}{\partial t_h}(t_l,t_h)+1\right].$$

In turn, consider the deviation that sets $u_h = u_h^* - t_h$, therefore abandoning the high-type market. The optimal deviation of this kind chooses u_l to solve the following program:

$$\Pi^{b*}(t_l,t_h) \equiv \max_{u_l}\left\{p_l\left(\frac{1}{2}+\frac{u_l-u_l^*}{2t_l}\right)(S_l^e - u_l)\right\}.$$

We denote the maximand by u_l^b. Notice that $u_l^d > u_l^* > u_l^b$. Moreover,

$$\frac{\partial \Pi^{b*}}{\partial t_h}(t_l,t_h) = -\frac{p_l}{2t_l}\frac{\partial u_l^*}{\partial t_h}(t_l,t_h)\left(S_l^e - u_l^b\right).$$

Therefore,

$$\frac{\partial \Pi^{d*}}{\partial t_h}(t_l,t_h) - \frac{\partial \Pi^{b*}}{\partial t_h}(t_l,t_h) = \frac{p_l}{2t_l}\frac{\partial u_l^*}{\partial t_h}(t_l,t_h)\left(u_l^d - u_l^b\right) + p_h\left(\frac{\partial S_h}{\partial u_h}(u_l, u_h^*(t_l,t_h)+t_h) - 1\right)\left[\frac{\partial u_h^*}{\partial t_h}(t_l,t_h)+1\right],$$

which is negative because $u_l^d - u_l^b > 0$. Because $\Pi^{b*}(t_l,t_h) = \Pi^{d*}(t_l,t_h)$ in the locus $\{(t_l,t_h) : \Lambda(t_l,t_h) = \eta_h\}$, it follows that $\Pi^{b*}(t_l,t_h) < \Pi^{d*}(t_l,t_h)$ for all $(t_l,t_h) \in D_+$.

This establishes that the optimal deviation is such that $u_h = u_h^* + t_h$ and u_l is the maximand of $\Pi^{d*}(t_l,t_h)$. Let us then compare $\Pi^{d*}(t_l,t_h)$ with the putative equilibrium profit:

$$\Pi^*(t_l,t_h) \equiv \frac{p_l}{2}(S_l^e - u_l^*) + \frac{p_h}{2}(S_h(\triangle^*) - \triangle^* - u_l^*).$$

Partially differentiating with respect to t_h gives

$$\frac{\partial \Pi^*}{\partial t_h}(t_l,t_h) = -\frac{p_l}{2}\frac{\partial u_l^*}{\partial t_h}(t_l,t_h) + \frac{p_h}{2}\left(\frac{\partial \delta^*}{\partial t_h}(t_l,t_h)\right)\left(\frac{\partial S_h}{\partial u_h}(\triangle^*) - 1\right) - \frac{p_h}{2}\frac{\partial u_l^*}{\partial t_h}(t_l,t_h) > 0.$$

Hence,

$$\frac{\partial \Pi^*}{\partial t_h}(t_l,t_h) - \frac{\partial \Pi^{d*}}{\partial t_h}(t_l,t_h) = \frac{p_h}{2}\left(\frac{\partial \delta^*}{\partial t_h}(t_l,t_h)\right)\left(\frac{\partial S_h}{\partial u_h}(\triangle^*) - 1\right) - \frac{p_h}{2}\frac{\partial u_l^*}{\partial t_h}(t_l,t_h)$$

$$+\frac{p_l}{2t_l}\frac{\partial u_l^*}{\partial t_h}(t_l,t_h)\left(S_l^e - t_l - u_l^d\right) - \left(p_h\frac{\partial S_h}{\partial u_h}(0, u_h^*(t_l,t_h)+t_h-u_l) - p_h\right)\left[\frac{\partial u_h^*}{\partial t_h}(t_l,t_h)+1\right]$$

Plugging the expression (24) for the maximand u_l^d gives

$$\frac{\partial \Pi^*}{\partial t_h}(t_l,t_h) - \frac{\partial \Pi^{d*}}{\partial t_h}(t_l,t_h) = \frac{p_h}{2}\left(\frac{\partial \triangle^*}{\partial t_h}(t_l,t_h)\right)\left(\frac{\partial S_h}{\partial u_h}(\triangle^*) - 1\right) - \frac{p_h}{2}\frac{\partial u_l^*}{\partial t_h}(t_l,t_h)$$

$$+\frac{p_l}{4t_l}\frac{\partial u_l^*}{\partial t_h}(t_l,t_h)\left(S_l^e - t_l - u_l^* - 2t_l\frac{p_h}{p_l}\frac{\partial S_h}{\partial u_l}(u_l^d, u_h^*+t_h)\right)$$

$$-p_h\left(\frac{\partial S_h}{\partial u_h}(u_l^d, u_h^*(t_l,t_h)+t_h) - 1\right)\left[\frac{\partial u_h^*}{\partial t_h}(t_l,t_h)+1\right] > 0.$$

The derivatives computed above for $\triangle^*$, u_l^* and u_h^*, for the cases where either $u_l^* > 0$ or $u_l^* = 0$, together with the fact that $S_l^e - t_l < u_l^*$, as shown in Proposition 5, imply the last inequality.

Therefore, for a fixed t_l, there exists a threshold $\tau_h^2(t_l) > 0$ such that the putative equilibrium profit $\Pi^*(t_l, t_h)$ is greater than $\Pi^{d*}(t_l, t_h)$ if and only if $t_h \geq \tau_h^2(t_l)$. Set $\tau_h(t_l) \equiv \max\{\tau_h^1(t_l), \tau_h^2(t_l)\}$ to obtain the result. Q.E.D.

Proof of Proposition 7. Notice that the set of symmetric equilibria is upper hemicontinuous. Hence, since it is a function, it is continuous.

Consider a neighborhood U of (t_l, t_h) where an equilibrium (u_l^*, u_h^*) in which pure-strategy equilibrium exists. Moreover, assume that U is contained in E, D_+ or D_-,and hence by Proposition 4 all equilibrium (u_l^*, u_h^*) in U exhibits qualitatively the same set of distortions.

First consider the case in which $U \in E$. In this case, $u_k^* = \max\{S_k^e - t_k, 0\}, k \in \{l, h\}$, which immediately delivers the result.

Next consider an open set $U \in D_+$ where the IC_h constraint binds and IR does not. The equilibrium is given by the following equations:

$$\begin{array}{c} S_l(u_l, u_h) + t_l\left(\frac{\partial S_l(u_l,u_h)}{\partial u_l} - 1\right) = u_l \\ S_h^e - t_h + t_h\left(\frac{p_l}{p_h}\right)\frac{\partial S_l(u_l,u_h)}{\partial u_h} = u_h. \end{array}$$

Recall that $\triangle(u_l, u_h) := u_h - u_l$. To ease notation, we write only $\triangle$ below, leaving implicit its dependence on (u_l, u_h) wherever it does not lead to confusion. Consider the equation

$$G(\triangle, t_l, t_h) \equiv \left[S_h^e - t_h + t_h\left(\frac{p_l}{p_h}\right)\frac{\partial S_l(\triangle)}{\partial u_h}\right] - \left[S_l(\triangle) + t_l\left(\frac{\partial S_l(\triangle)}{\partial u_l} - 1\right)\right] - \triangle.$$

We have

$$\begin{array}{rl} \frac{\partial G(\triangle,t_l,t_h)}{\partial \triangle} & = t_h\left(\frac{p_l}{p_h}\right)\frac{\partial^2 S_l(\triangle)}{\partial \triangle^2} - \frac{\partial S_l(\triangle)}{\partial \triangle} + t_l\frac{\partial^2 S_l(\triangle)}{\partial \triangle^2} < 0 \\ \frac{\partial G(\triangle,t_l,t_h)}{\partial t_h} & = -1 + \left(\frac{p_l}{p_h}\right)\frac{\partial S_l(\triangle)}{\partial \triangle} = -\left(\frac{S_h^e - u_h}{t_h}\right) < 0 \\ \frac{\partial G(\triangle,t_l,t_h)}{\partial t_l} & = -\left(1 - \frac{\partial S_l(\triangle)}{\partial u_l}\right) = \left(\frac{S_l(u_l,u_h) - u_l}{t_l}\right) > 0 \end{array}$$

Therefore by straightforward algebra we have $\frac{\partial \triangle}{\partial t_l} > 0, \frac{\partial \triangle}{\partial t_h} < 0, \frac{\partial u_h}{\partial t_l} = t_h\left(\frac{p_l}{p_h}\right)\frac{\partial^2 S_l(u_l,u_h)}{\partial u_h^2}\frac{\partial \triangle}{\partial t_l} < 0, \frac{\partial u_h}{\partial t_h} = \frac{\partial \triangle}{\partial t_h} + \frac{\partial u_l}{\partial t_h} < 0, \frac{\partial u_l}{\partial t_h} = \left[\frac{\partial S_l(u_l,u_h)}{\partial \triangle} - t_l\left(\frac{\partial^2 S_l(u_l,u_h)}{\partial \triangle^2}\right)\right]\frac{\partial \triangle}{\partial t_h} < 0$ and $\frac{\partial u_l}{\partial t_l} = \frac{\partial u_h}{\partial t_l} - \frac{\partial \triangle}{\partial t_l} < 0$.

Next consider an open set $U \in D_+$ where both IC_h and IR bind. The equilibrium is given by the following equations:

$$\begin{array}{c} u_l = 0 \\ S_h^e - t_h + t_h\left(\frac{p_l}{p_h}\right)\frac{\partial S_l(0,u_h)}{\partial u_h} = u_h. \end{array}$$

Locally, we have $\frac{\partial \triangle}{\partial t_l} = 0, \frac{\partial \triangle}{\partial t_h} = -\frac{\left(1 - \left(\frac{p_l}{p_h}\right)\frac{\partial S_l(0,u_h)}{\partial u_h}\right)}{\left(1 - t_h\left(\frac{p_l}{p_h}\right)\frac{\partial^2 S_l(0,u_h)}{\partial u_h^2}\right)} = -\frac{\frac{1}{t_h}\left(S_h^e - u_h\right)}{\left(1 - t_h\left(\frac{p_l}{p_h}\right)\frac{\partial^2 S_l(0,u_h)}{\partial u_h^2}\right)} < 0, \frac{\partial u_h}{\partial t_l} = 0, \frac{\partial u_h}{\partial t_h} = \frac{\partial \triangle}{\partial t_h} < 0, \frac{\partial u_l}{\partial t_h} = 0$ and $\frac{\partial u_l}{\partial t_l} = 0$.

Next consider an open set $U \in D_-$ where the IC$_l$ and the IR constraints bind. The equilibrium is given by the following equations:

$$u_l = 0$$
$$u_h = S_h\left(0, u_h\right) - t_h + t_h \frac{\partial S_h(0,u_h)}{\partial u_h}$$

We have $\frac{\partial \triangle}{\partial t_l} = 0, \frac{\partial \triangle}{\partial t_h} = -\frac{\left(1 - \frac{\partial S_h(0,u_h^a)}{\partial u_h}\right)}{\left(1 - \frac{\partial S_h(0,u_h)}{\partial u_h} - \frac{\partial^2 S_h(0,u_h)}{\partial u_h^2}\right)} = -\frac{\frac{1}{t_h}\left(S_h(0,u_h) - u_h\right)}{\left(1 - \frac{\partial S_h(0,u_h)}{\partial u_h} - \frac{\partial^2 S_h(0,u_h)}{\partial u_h^2}\right)} < 0, \frac{\partial u_h}{\partial t_l} = 0, \frac{\partial u_h}{\partial t_h} = \frac{\partial \triangle}{\partial t_h} < 0, \frac{\partial u_l}{\partial t_h} = 0$ and $\frac{\partial u_l}{\partial t_l} = 0$.

Finally, consider an open set $U \in D_+$ where the only binding constraint is the constraint IC$_l$. The equilibrium is given by the following equations:

$$u_l = S_l^e + t_l \left(\frac{p_h}{p_l}\right)\left(\frac{\partial S_h(u_l,u_h)}{\partial u_l}\right) - t_l$$
$$u_h = S_h\left(u_l, u_h\right) + t_h \frac{\partial S_h(u_l,u_h)}{\partial u_h} - t_h$$

Consider the equation

$$G\left(\triangle, t_l, t_h\right) := \left[S_h\left(u_l, u_h\right) - t_h + t_h \frac{\partial S_h\left(\triangle\right)}{\partial u_h}\right] - \left[S_l^e + t_l \left(\left(\frac{p_h}{p_l}\right) \frac{\partial S_h\left(\triangle\right)}{\partial u_l} - 1\right)\right] - \triangle.$$

We have:

$$\frac{\partial G(\triangle,t_l,t_h)}{\partial \triangle} = \left[\frac{\partial S_h(u_l,u_h)}{\partial \triangle} + t_h \frac{\partial^2 S_h(\triangle)}{\partial \triangle^2} + t_l \left(\frac{p_h}{p_l}\right)\frac{\partial^2 S_h(\triangle)}{\partial \triangle^2}\right] - 1 < 0$$
$$\frac{\partial G(\triangle,t_l,t_h)}{\partial t_l} = -\left(\left(\frac{p_h}{p_l}\right)\frac{\partial S_h(\triangle)}{\partial u_l} - 1\right) = \frac{1}{t_l}\left(S_l^e - u_l\right) > 0$$
$$\frac{\partial G(\triangle,t_l,t_h)}{\partial t_h} = -\left(1 - \frac{\partial S_h(\triangle)}{\partial u_h}\right) = -\frac{1}{t_h}\left(S_h\left(u_l, u_h\right) - u_h\right) < 0$$

Therefore we have $\frac{\partial \triangle}{\partial t_l} > 0, \frac{\partial \triangle}{\partial t_h} < 0, \frac{\partial u_h}{\partial t_l} = \left(t_h \frac{\partial^2 S_h(\triangle)}{\partial u_h^2} + \frac{\partial S_h(\triangle)}{\partial u_h}\right)\frac{\partial \triangle}{\partial t_l} < 0, \frac{\partial u_h}{\partial t_h} = \frac{\partial \triangle}{\partial t_h} + \frac{\partial u_l}{\partial t_h} < 0, \frac{\partial u_l}{\partial t_h} = -t_l\left(\frac{p_h}{p_l}\right)\left(\frac{\partial S_h^2(u_l,u_h)}{\partial \triangle^2}\right)\frac{\partial \triangle}{\partial t_h} < 0$ and $\frac{\partial u_l}{\partial t_l} = \frac{\partial u_h}{\partial t_l} - \frac{\partial \triangle}{\partial t_l} < 0$. This exhausts all cases and completes the proof. Q.E.D.

Proof of Proposition 8. We proceed by analyzing five different cases.

Case 1. First consider an open set in which $U \in E$ and $t_k \neq S_k^e$ for $k \in \{l, h\}$. We have $u_k^* = \max\left\{S_k^e - t_k, 0\right\}, k \in \{l, h\}$, which immediately implies $\frac{\partial y_k}{\partial t_k} = \mathbb{I}_{\left\{S_k^e - t_k > 0\right\}}$.

Case 2. Consider an open set in which $U \in D_-$ and only the IC$_h$ constraints binds.

The equilibrium is given by the following equations:

$$S_l\left(u_l, u_h\right) + t_l\left(\frac{\partial S_l(u_l,u_h)}{\partial u_l} - 1\right) = u_l$$
$$S_h^e - t_h + t_h\left(\frac{p_l}{p_h}\right)\frac{\partial S_l(u_l,u_h)}{\partial u_h} = u_h$$

Prices are given by

$$y_l = \theta_l q_l\left(u_l, u_h\right) - u_l$$
$$y_h = \theta_h q_h^e - u_h.$$

Comparative statics are given by: $\frac{\partial y_h}{\partial t_l} = -\frac{\partial u_h}{\partial t_l} > 0$, $\frac{\partial y_h}{\partial t_h} = -\frac{\partial u_h}{\partial t_h} > 0$, $\frac{\partial y_l}{\partial t_l} = \theta_l \frac{\partial q_l(u_l,u_h)}{\partial \triangle}\frac{\partial \triangle}{\partial t_l} - \frac{\partial u_l}{\partial t_l} > 0$ and

$$\frac{\partial y_l}{\partial t_h} = \theta_l \frac{\partial q_l(u_l,u_h)}{\partial \triangle}\frac{\partial \triangle}{\partial t_h} - \frac{\partial u_l}{\partial t_h} = [\varphi'(q_l(u_l,u_h)\triangle\theta - t_l\varphi''(q_l(u_l,u_h))]\left(\frac{1}{\triangle\theta}\right)^2 \frac{\partial \triangle}{\partial t_h}.$$

We must show that if $\varphi''' \leq 0$ then y_l is quasi-convex in t_h if $\varphi'''(q) \leq 0$. For that notice that $sign\frac{\partial y_l}{\partial t_h} = sign\left[t_l\varphi''(q_l(\triangle(t_h)) - \varphi'(q_l(\triangle(t_h))\triangle\theta\right]$. Notice that $q_l \to t_l\varphi''(q_l) - \varphi'(q_l)\triangle\theta$ is decreasing in q_l and $\frac{\partial q_l}{\partial t_h} < 0$. Therefore $\frac{\partial y_l}{\partial t_h}(t_h^*) = 0$ implies $\frac{\partial y_l}{\partial t_h}(\tilde{t}_h) > 0$ for $\tilde{t}_h > t_h^*$.

$\varphi'(q)\triangle\theta - t_l\varphi''(q) = K$ which can have at most one root since $\varphi'' > 0$ and $\varphi''' \leq 0$.

Case 3. Consider an open set in which $U \in D_-$ and in which IR and IC_h bind and in which $u_h > 0$.

The equilibrium is given by:

$$S_h^e - t_h + t_h\left(\frac{p_l}{p_h}\right)\frac{\partial S_l(0,u_h)}{\partial u_h} = u_h$$

We have

$$y_h = \theta_h q_h^e - u_h.$$
$$y_l = \theta_l q_l(u_l,u_h).$$

We have the following comparative statics: $\frac{\partial y_l}{\partial t_l} = \frac{\theta_l}{\triangle\theta}\frac{\partial \triangle}{\partial t_l} = 0, \frac{\partial y_l}{\partial t_h} = \frac{\theta_l}{\triangle\theta}\frac{\partial \triangle}{\partial t_h} < 0, \frac{\partial y_h}{\partial t_l} = -\frac{\partial u_h}{\partial t_l} = 0$ and $\frac{\partial y_h}{\partial t_h} = -\frac{\partial u_h}{\partial t_h} > 0$.

Case 4. Consider an open set in which $U \in D_+$ and in which the only binding constraint is IC_l. The equilibrium is given by the following equations:

$$S_l^e + t_l\left(\frac{p_h}{p_l}\right)\left(\frac{\partial S_h(u_l,u_h)}{\partial u_l}\right) - t_l = u_l$$
$$S_h(u_l,u_h) + t_h\frac{\partial S_h(u_l,u_h)}{\partial u_h} - t_h = u_h.$$

We have

$$y_h = \theta_h q_h(u_l,u_h) - u_h.$$
$$y_l = \theta_l q_l^e - u_l.$$

We have the following comparative statics: $\frac{\partial y_l}{\partial t_l} = -\frac{\partial u_l}{\partial t_l} > 0, \frac{\partial y_l}{\partial t_h} = -\frac{\partial u_l}{\partial t_h} > 0, \frac{\partial y_h}{\partial t_l} = \left(\frac{\theta_h}{\triangle\theta}\right)\frac{\partial \triangle}{\partial t_l} - \frac{\partial u_h}{\partial t_l} > 0$ and

$$\begin{aligned}\frac{\partial y_h}{\partial t_h} &= \theta_h\frac{\partial q_h(u_l,u_h)}{\partial \triangle}\frac{\partial \triangle}{\partial t_h} - \frac{\partial u_h}{\partial t_h}\\ &= \left(\frac{\theta_h}{\triangle\theta}\right)\frac{\partial \triangle}{\partial t_h} - \left(\frac{\partial \triangle}{\partial t_h} + \frac{\partial u_l}{\partial t_h}\right)\\ &= \left(\frac{1}{\triangle\theta}\right)^2\left[\theta_l\triangle\theta + t_l\left(\frac{p_h}{p_l}\right)\left(\frac{\partial S_h^2(u_l,u_h)}{\partial q^2}\right)\right]\frac{\partial \triangle}{\partial t_h}\\ &= \frac{\partial \triangle}{\partial t_h}\left(\frac{1}{\triangle\theta}\right)^2\left[\theta_l\triangle\theta - t_l\left(\frac{p_h}{p_l}\right)\varphi''(q_h(u_l,u_h))\right].\end{aligned}$$

The desired quasiconvexity when $\varphi''' < 0$ follows because $sign\frac{\partial y_h}{\partial t_h} = sign\left[t_l\left(\frac{p_h}{p_l}\right)\varphi''(q_h) - \theta_l\triangle\theta\right]$ which is decreasing in q_h and hence since $\frac{\partial q_h}{\partial t_h} < 0$ it is increasing in t_h.

Case 5. Consider an open set in which $U \in D_+$ and in which the binding constraint are IC_l and IR. The equilibrium is given by the following equation:

$$S_h(0,u_h) + t_h \frac{\partial S_h(0,u_h)}{\partial u_h} - t_h = u_h.$$

Prices are given by:

$$y_h = \theta_h q_h(0,u_h) - u_h.$$
$$y_l = \theta_l q_l^e.$$

We have $\frac{\partial y_l}{\partial t_l} = 0, \frac{\partial y_l}{\partial t_h} = 0$ and $\frac{\partial y_h}{\partial t_l} = 0$. Finally notice that

$$\frac{\partial y_h}{\partial t_h} = \theta_h \frac{\partial q_h(u_l,u_h)}{\partial \triangle}\frac{\partial \triangle}{\partial t_h} - \frac{\partial u_h}{\partial t_h} = \left[\frac{\theta_h}{\triangle\theta} - 1\right]\frac{\partial \triangle}{\partial t_h} < 0.$$

This completes the proof. Q.E.D.

Proof of Proposition 10.

Existence of Symmetric Equilibrium when $|\beta|$ is small. Is straightforward to verify that the firm's problem is equal to the problem of Rochet and Stole (2002) when t is constant and small. It follows that there exists $\zeta_1 > 0$ such that if t is a constant belonging to $(0,\zeta_1)$ then the problem has a unique solution in which $\dot{u}(\theta) = \theta$ for every θ. Consider $t(\theta) = \alpha + \beta\theta$. A straightforward continuity argument implies that for every ζ in this range there exists $\varepsilon_1 > 0$ such that $\sup_{\theta\in[\underline{\theta},\overline{\theta}]}\|\zeta - \alpha + \beta\theta\| < \varepsilon_1$ then if $(u_{\alpha,\beta}(\theta))_{\theta\in[\underline{\theta},\overline{\theta}]}$ satisfies the Euler equation above with the same boundary conditions (which solution is continuous in α, β) then if $(\hat{u}_{\alpha,\beta}(\theta))_{\theta\in[\underline{\theta},\overline{\theta}]}$ is a best-response to this equation we must have $\left(\frac{1}{2} + \frac{\hat{u}_{\alpha,\beta}(\theta) - u_{\alpha,\beta}(\theta)}{2t(\theta)}\right) \in \left[\frac{1}{3}, \frac{2}{3}\right]$ for every θ. Without loss of generality, we make this restriction for the remainder of this proof. In light of this, we set up the optimal control problem where $y(\theta)$ is the control variable, $x(\theta)$ is the state variable and the costate variable is $p(\theta)$:

$$H(\theta,x(\theta),y(\theta),p(\theta)) = \left(\frac{1}{2} + \frac{x(\theta) - \tilde{u}(\theta)}{2t(\theta)}\right)\left(\theta y(\theta) - \frac{(y(\theta))^2}{2} - x(\theta)\right) + p(\theta)\,y(\theta),$$

Optimality Conditions are $H_y(\theta,x(\theta),y(\theta),p(\theta)) = 0$,$-\dot{p}(\theta) = H_x(\theta,x(\theta),y(\theta),p(\theta)$,$\dot{x}(\theta) = y(\theta)$ and the transversality conditions $y(\bar{\theta}) = \bar{\theta}$,$y(\underline{\theta}) = \underline{\theta}$. Solving and imposing symmetry we obtain

$$\left(\frac{1}{2}\right)(\theta - y(\theta)) = -p(\theta)$$

$$-\dot{p}(\theta) = \left(\frac{1}{2t(\theta)}\right)\left(\theta y(\theta) - \frac{(y(\theta))^2}{2} - x(\theta)\right) - \left(\frac{1}{2}\right)$$

Let $\left(x^*_{\alpha,\beta}(\theta), y^*_{\alpha,\beta}(\theta), p^*_{\alpha,\beta}(\theta)\right)$ be a solution to the system above. We claim that this is an optimal solution. For that we will use Arrow sufficiency condition. Define:

$$H(\theta,x(\theta),p^*_{\alpha,\beta}(\theta)) = \max_{\tilde{y}}\left(\frac{1}{2} + \frac{x(\theta) - \hat{u}_{\alpha,\beta}(\theta)}{2t(\theta)}\right)\left(\theta\tilde{y} - \frac{(\tilde{y})^2}{2} - x(\theta)\right) + p^*_{\alpha,\beta}(\theta)\,\tilde{y}$$

.

Letting $A(x) := \left(\frac{1}{2} + \frac{x(\theta) - \hat{u}_{\alpha,\beta}(\theta)}{2t(\theta)}\right)$,$B(x) := A(x)\theta + p^*_{\alpha,\beta}(\theta)$, we obtain

$$H(\theta, x(\theta), p^*(\theta)) = \frac{B(x)^2}{2A(x)} - x(\theta)A(x).$$

Arrow sufficiency condition will be satisfied if we guarantee that $H_{xx}(\theta, x(\theta), p^*(\theta)) \leq 0$. Since $x \to A(x) = \left(\frac{1}{2} + \frac{x(\theta) - \hat{u}_{\alpha,\beta}(\theta)}{2t(\theta)}\right)$ is positive affine, we may change variables and get $A(x) = z$, $B(x) = z\theta + p^*_{\alpha,\beta}(\theta)$ and $x(\theta) = 2zt(\theta) - t(\theta) + \hat{u}_{\alpha,\beta}(\theta)$, which implies

$$H(\theta, z, p^*(\theta)) = \frac{\left(z\theta + p^*_{\alpha,\beta}\right)^2}{2z} - \left(2zt(\theta) - t(\theta) + \hat{u}_{\alpha,\beta}(\theta)\right) z.$$

Differentiating twice with respect to z

$$\begin{aligned} H_{zz}(\theta, z, p^*(\theta)) &= \frac{p^{*2}_{\alpha,\beta}}{z^3} - 4t(\theta) \\ &= \frac{\left(\theta - y^*_{\alpha,\beta}(\theta)\right)^2}{4z^3} - 4t(\theta) \\ &< \frac{9\left(\theta - \hat{\hat{u}}_{\alpha,\beta}(\theta)\right)^2}{4} - 4(\alpha - |\beta|\theta), \end{aligned}$$

where the last line used $z = \left(\frac{1}{2} + \frac{\hat{u}_{\alpha,\beta}(\theta) - u_{\alpha,\beta}(\theta)}{2t(\theta)}\right) \in \left[\frac{1}{3}, \frac{2}{3}\right]$ and $y^*_{\alpha,\beta}(\theta) = \hat{\hat{u}}_{\alpha,\beta}(\theta)$. The solution when $(\hat{u}_{\alpha,\beta}(\theta))$ when $\beta = 0$ satisfies $\left(\theta - \hat{\hat{u}}_{\alpha,0}(\theta)\right) = 0$. By a continuity argument, there exists $\varepsilon_2 \in (0, \varepsilon_1)$ such that $sup_{\theta \in [\underline{\theta}, \overline{\theta}]} \|\zeta - \alpha + \beta\theta\| < \varepsilon_2$ implies $\max_\theta \frac{9\left(\theta - \hat{\hat{u}}_{\alpha,\beta}(\theta)\right)^2}{4} - 4(\alpha - |\beta|\theta) < 0$, which completes the argument. Moreover, a continuity argument implies that we can take $\varepsilon \in (0, \varepsilon_2)$ such that $sup_{\theta \in [\underline{\theta}, \overline{\theta}]} \|\zeta - \alpha + \beta\theta\| < \varepsilon$ implies that $\hat{\hat{u}}_{\alpha,\beta}(\theta)$ is increasing and $\hat{u}_{\alpha,\beta}(\underline{\theta}) > 0$.

Distortions when $\beta > 0$ and $\beta < 0$.

Case 1: $\beta > 0$. Let $(u(\theta))_{\theta \in [\underline{\theta}, \overline{\theta}]}$ be the solution when $t(\theta) = \alpha + \beta\theta$. We will show that $\dot{u}(\theta) < \theta$ for all $\theta \in (\underline{\theta}, \overline{\theta})$.

First we show that $\ddot{u}(\underline{\theta}) < 1$. Assume towards a contradiction that $\ddot{u}(\underline{\theta}) \geq 1$. Let $(u_\lambda(\theta))_{\theta \in [\underline{\theta}, \overline{\theta}]}$. be the solution when $t(\theta) = \alpha + \beta\underline{\theta}$ for every θ and notice that $\ddot{u}(\theta) = 1$ for every θ. Notice that

$$\begin{aligned} \frac{d\ddot{u}(\theta)}{d\theta} |_{\underline{\theta}} = &- \left(\frac{1}{(\alpha + \beta\underline{\theta})^2}\right) \left[(\underline{\theta} - \dot{u}(\underline{\theta}))\ddot{u}(\theta) + (\underline{\theta} - \dot{u}(\underline{\theta}))\right](\alpha + \beta\underline{\theta})^2 + \beta\left(\frac{1}{(\alpha + \beta\underline{\theta})^2}\right)\left(\theta\dot{u}(\theta) - \frac{(\dot{u}(\theta))^2}{2} - u(\theta)\right) \\ &= \beta\left(\frac{1}{(\alpha + \beta\underline{\theta})^2}\right)\left(\theta\dot{u}(\theta) - \frac{(\dot{u}(\theta))^2}{2} - u(\theta)\right) > 0. \end{aligned}$$

Therefore, there exists $\varepsilon > 0$ such that, for all $\theta \in (\underline{\theta}, \underline{\theta} + \varepsilon)$ we have $\ddot{u}(\theta) > 1$, implying

$$\theta\dot{u}(\theta) - \frac{(\dot{u}(\theta))^2}{2} - u(\theta) < \theta\dot{u}_\lambda(\theta) - \frac{(\dot{u}_\lambda(\theta))^2}{2} - u_\lambda(\theta).$$

Therefore $\dot{u}(\theta) > \theta$ for every θ in this set. Let θ^* be the infimum over all $\theta \geq \underline{\theta} + \varepsilon$ such that $\dot{u}(\theta) \leq \theta$. Since $\dot{u}(\overline{\theta}) = \overline{\theta}$,we conclude that θ^* is well defined. Notice that since $\dot{u}(\theta) > \theta$ for every

$\theta \in (\underline{\theta}, \theta^*)$ we have $\ddot{u}(\theta^*) \leq \ddot{u}_\lambda(\theta^*) = 1$. However, since $u(\theta^*) > u_\lambda(\theta^*)$ we must have

$$\begin{aligned}\ddot{u}(\theta^*) &= 2 - \left(\tfrac{1}{\alpha+\beta\theta^*}\right)\left(\theta^*\dot{u}(\theta^*) - \tfrac{(\dot{u}(\theta^*))^2}{2} - u(\theta^*)\right) \\ &> 2 - \left(\tfrac{1}{\alpha+\beta\underline{\theta}}\right)\left(\theta^*\dot{u}(\theta^*) - \tfrac{(\dot{u}(\theta^*))^2}{2} - u(\theta^*)\right) \\ &> 2 - \left(\tfrac{1}{\alpha+\beta\underline{\theta}}\right)\left(\theta^*\dot{u}(\theta^*) - \tfrac{(\dot{u}(\theta^*))^2}{2} - u_\lambda(\theta^*)\right) \\ &= \ddot{u_\lambda}(\theta^*),\end{aligned}$$

which is a contradiction. This shows that $\ddot{u}(\underline{\theta}) < 1$ and hence there exists $\varepsilon > 0$ such that, for all $\theta \in (\underline{\theta}, \underline{\theta} + \varepsilon)$ we have $\dot{u}(\theta) < \theta$. Assume towards a contradiction that there exists $\tilde{\theta} < \overline{\theta}$ such that $\dot{u}\left(\tilde{\theta}\right) = \tilde{\theta}$. Let $\tilde{\theta}$ be the smallest type satisfying this condition. This implies $\ddot{u}\left(\tilde{\theta}\right) \geq 1$, hence the case above applies and leads to a contradiction.

Case 2: $\beta < 0$. An argument analogous to the one presented in Case 1 above establishes that $\dot{u}(\theta) > \theta$ for every $\theta \in \left(\underline{\theta}, \overline{\theta}\right)$

Types that benefit and types that are hurt by asymmetric information.

Case 1: $\beta > 0$. As we have shown above, $\ddot{u}(\underline{\theta}) < 1$, and hence

$$\begin{aligned}u(\underline{\theta}) &= (\alpha + \beta\underline{\theta})(\ddot{u}(\underline{\theta}) - 2) + \underline{\theta}\dot{u}(\underline{\theta}) - \tfrac{(\dot{u}(\underline{\theta}))^2}{2} \\ &< (\alpha + \beta\underline{\theta})(\ddot{u}_\lambda(\underline{\theta}) - 2) + \underline{\theta}\dot{u_\lambda}(\underline{\theta}) - \tfrac{(\dot{u_\lambda}(\underline{\theta}))^2}{2} \\ &= u_\lambda(\underline{\theta}),\end{aligned}$$

where$(u_\lambda(\theta))_{\theta \in [\underline{\theta}, \overline{\theta}]}$. is the solution when $t(\theta) = \alpha + \beta\underline{\theta}$. This implies that type $\underline{\theta} < \theta_1$ is worse off under asymmetric information. By continuity there is $\theta_1$with $\underline{\theta} < \theta_1$ such that every type $\theta < \theta_1$ is worse off when there is asymmetric information. A symmetric argument shows the existence of $\theta_2 \in \left[\theta_1, \bar{\theta}\right)$ such that every type $\theta > \theta_1$ is better off under asymmetric information.

Case 2: $\beta < 0$. The argument is similar to case 1 above and omitted for brevity.

An increase in β decreases $\dot{u}(\theta)$ for every interior type.

Case 1: $\beta > 0$. Keep α constant and take two solutions $u_{\beta_1}(\theta)$ and $u_{\beta_2}(\theta)$ for $\beta_2 > \beta_1$. We claim that $\dot{u}_{\beta_1}(\theta) > \dot{u}_{\beta_2}(\theta)$ for every $\theta \in \left(\underline{\theta}, \overline{\theta}\right)$. First we claim that $\ddot{u}_{\beta_1}(\underline{\theta}) > \ddot{u}_{\beta_2}(\underline{\theta})$. Assume towards a contradiction that $\ddot{u}_{\beta_1}(\underline{\theta}) \leq \ddot{u}_{\beta_2}(\underline{\theta})$. Notice that $\ddot{u}_{\beta_1}(\underline{\theta}) = \ddot{u}_{\beta_2}(\underline{\theta})$ implies

$$\begin{aligned}&\tfrac{d\ddot{u}_{\beta_2}(\underline{\theta})}{d\theta} - \tfrac{d\ddot{u}_{\beta_1}(\underline{\theta})}{d\theta} \\ &= \beta_2\left(\tfrac{1}{(\alpha+\beta_2\underline{\theta})^2}\right)\left(\underline{\theta}\dot{u}_{\beta_2}(\underline{\theta}) - \tfrac{\left(\dot{u}_{\beta_2}(\underline{\theta})\right)^2}{2} - u_{\beta_2}(\underline{\theta})\right) - \beta_1\left(\tfrac{1}{(\alpha+\beta_1\underline{\theta})^2}\right)\left(\underline{\theta}\dot{u}_{\beta_1}(\underline{\theta}) - \tfrac{\left(\dot{u}_{\beta_1}(\underline{\theta})\right)^2}{2} - u_{\beta_1}(\underline{\theta})\right) \\ &= \left(\tfrac{\beta_2\underline{\theta}}{\alpha+\beta_2\underline{\theta}}\right)\underline{\theta}^{-1}\left(\tfrac{\underline{\theta}\dot{u}_{\beta_2}(\underline{\theta}) - \frac{\left(\dot{u}_{\beta_2}(\underline{\theta})\right)^2}{2} - u_{\beta_2}(\underline{\theta})}{\alpha+\beta_2\underline{\theta}}\right) - \left(\tfrac{\beta_1\underline{\theta}}{\alpha+\beta_1\underline{\theta}}\right)\underline{\theta}^{-1}\left(\tfrac{\underline{\theta}\dot{u}_{\beta_1}(\underline{\theta}) - \frac{\left(\dot{u}_{\beta_1}(\underline{\theta})\right)^2}{2} - u_{\beta_1}(\underline{\theta})}{\alpha+\beta_2\underline{\theta}}\right) \\ &= \underline{\theta}^{-1}\left[\left(\tfrac{\beta_2\underline{\theta}}{\alpha+\beta_2\underline{\theta}}\right) - \left(\tfrac{\beta_1\underline{\theta}}{\alpha+\beta_1\underline{\theta}}\right)\right](2 - \ddot{u}_{\beta_2}(\underline{\theta})) > 0.\end{aligned}$$

Hence if $\ddot{u}_{\beta_1}(\underline{\theta}) \leq \ddot{u}_{\beta_2}(\underline{\theta})$ then there exists $\theta_1 > \underline{\theta}$ such that $\ddot{u}_{\beta_1}(\theta) < \ddot{u}_{\beta_2}(\theta)$ and $\dot{u}_{\beta_1}(\theta) < \dot{u}_{\beta_2}(\theta)$ for all $\theta \in (\underline{\theta}, \theta_1)$. Clearly there should be $\theta_2 \in \left(\theta_1, \overline{\theta}\right)$ such that $\ddot{u}_{\beta_1}(\theta_2) = \ddot{u}_{\beta_2}(\theta_2)$, otherwise we would have $\dot{u}_{\beta_1}\left(\overline{\theta}\right) < \dot{u}_{\beta_2}\left(\overline{\theta}\right)$, which is a contradiction. Therefore let $\theta^* \in \left(\theta_1, \overline{\theta}\right)$ be the smallest element of this set such that $\ddot{u}_{\beta_1}(\theta^*) = \ddot{u}_{\beta_2}(\theta^*)$. Notice that we must have $\frac{d\ddot{u}_{\beta_1}(\theta)}{d\theta}|_{\theta^*} \geq \frac{d\ddot{u}_{\beta_2}(\theta)}{d\theta}|_{\theta^*}$ or equivalently

$$\frac{d}{d\theta}\left[\left(\frac{1}{\alpha+\beta_1\theta^*}\right)\left(\theta^*\dot{u}_{\beta_1}(\theta^*) - \frac{\left(\dot{u}_{\beta_1}(\theta^*)\right)^2}{2} - u_{\beta_1}(\theta^*)\right)\right] \leq \frac{d}{d\theta}\left[\left(\frac{1}{\alpha+\beta_2\theta^*}\right)\left(\theta^*\dot{u}_{\beta_2}(\theta^*) - \frac{\left(\dot{u}_{\beta_2}(\theta^*)\right)^2}{2} - u_{\beta_2}(\theta^*\right.\right.$$

which holds if and only if

$$\begin{aligned}&\left(\tfrac{1}{\alpha+\beta_1\theta^*}\right)\left[\theta^* - \dot{u}_{\beta_1}(\theta^*)\right]\ddot{u}_{\beta_1}(\theta^*) - \left(\tfrac{\beta_1}{(\alpha+\beta_1\theta^*)^2}\right)\left(\theta^*\dot{u}_{\beta_1}(\theta^*) - \frac{\left(\dot{u}_{\beta_1}(\theta^*)\right)^2}{2} - u_{\beta_1}(\theta^*)\right)\\ \leq &\left(\tfrac{1}{\alpha+\beta_2\theta^*}\right)\left[\theta^* - \dot{u}_{\beta_2}(\theta^*)\right]\ddot{u}_{\beta_2}(\theta^*) - \left(\tfrac{\beta_2}{(\alpha+\beta_2\theta^*)^2}\right)\left(\theta^*\dot{u}_{\beta_2}(\theta^*) - \frac{\left(\dot{u}_{\beta_2}(\theta^*)\right)^2}{2} - u_{\beta_2}(\theta^*)\right)\end{aligned}$$

But notice that $\dot{u}_{\beta_1}(\theta^*) < \dot{u}_{\beta_2}(\theta^*) < \theta^*$ implies $\theta^* - \dot{u}_{\beta_1}(\theta^*) > \theta^* - \dot{u}_{\beta_2}(\theta^*) > 0$. This, $\ddot{u}_{\beta_1}(\theta^*) = \ddot{u}_{\beta_2}(\theta^*) > 0$ and $\left(\frac{1}{\alpha+\beta_1\theta^*}\right) > \left(\frac{1}{\alpha+\beta_2\theta^*}\right)$ imply

$$\left(\frac{1}{\alpha+\beta_1\theta^*}\right)\left[\theta^* - \dot{u}_{\beta_1}(\theta^*)\right]\ddot{u}_{\beta_1}(\theta^*) > \left(\frac{1}{\alpha+\beta_2\theta^*}\right)\left[\theta^* - \dot{u}_{\beta_2}(\theta^*)\right]\ddot{u}_{\beta_2}(\theta^*).$$

On the other hand, using $\left(\frac{\theta^*\dot{u}_{\beta_i}(\theta^*) - \frac{\left(\dot{u}_{\beta_i}(\theta^*)\right)^2}{2} - u_{\beta_i}(\theta^*)}{\alpha+\beta_i\theta^*}\right) = 2 - \ddot{u}_{\beta_i}(\theta^*) > 0$, we immediately get

$$\begin{aligned}-\left(\tfrac{\beta_1}{(\alpha+\beta_1\theta^*)^2}\right)\left(\theta^*\dot{u}_{\beta_1}(\theta^*) - \frac{\left(\dot{u}_{\beta_1}(\theta^*)\right)^2}{2} - u_{\beta_1}(\theta^*)\right) + \left(\tfrac{\beta_2}{(\alpha+\beta_2\theta^*)^2}\right)\left(\theta^*\dot{u}_{\beta_2}(\theta^*) - \frac{\left(\dot{u}_{\beta_2}(\theta^*)\right)^2}{2} - u_{\beta_2}(\theta^*)\right)\\ = \tfrac{1}{\theta^*}\left[\left(\tfrac{\beta_2\theta^*}{\alpha+\beta_2\theta^*}\right) - \left(\tfrac{\beta_1\theta^*}{\alpha+\beta_1\theta^*}\right)\right]\left(2 - \ddot{u}_{\beta_1}(\theta^*)\right) > 0.\end{aligned}$$

Putting these together one gets $\frac{d\ddot{u}_{\beta_1}(\theta)}{d\theta}|_{\theta^*} < \frac{d\ddot{u}_{\beta_2}(\theta)}{d\theta}|_{\theta^*}$, a contradiction. Therefore we conclude that $\ddot{u}_{\beta_1}(\underline{\theta}) > \ddot{u}_{\beta_2}(\underline{\theta})$. This implies that there exists $\varepsilon > 0$ such that $\dot{u}_{\beta_1}(\theta) > \dot{u}_{\beta_2}(\theta)$ for all $\theta \in (\underline{\theta}, \underline{\theta}+\varepsilon)$. Assume towards a contradiction that $\dot{u}_{\beta_1}(\theta_*) = \dot{u}_{\beta_2}(\theta_*)$ for some $\theta_* < \overline{\theta}$ and let θ_* be the smallest element greater than $\underline{\theta}+\varepsilon$ satisfying this equality. We must have $\ddot{u}_{\beta_1}(\theta_*) \leq \ddot{u}_{\beta_2}(\theta_*)$. This and $\dot{u}_{\beta_1}(\theta_*) = \dot{u}_{\beta_2}(\theta_*)$ allows us to apply the first part of the proof and obtain a contradiction.

Case 2: $\beta < 0$. The proof is analogous to the case above and is omitted by brevity.

An increase in α decreases $u(\theta)$ for every type.

Keep β constant and take two solutions $u_{\alpha_1}(\theta)$ and $u_{\alpha_2}(\theta)$ for $\alpha_2 > \alpha_1$. First we show that $u_{\alpha_1}(\underline{\theta}) > u_{\alpha_2}(\underline{\theta})$. Assume towards a contradiction that $u_{\alpha_1}(\underline{\theta}) \leq u_{\alpha_2}(\underline{\theta})$ and notice that this implies $\ddot{u}_{\alpha_1}(\underline{\theta}) < \ddot{u}_{\alpha_2}(\underline{\theta})$ and hence there is $\theta_1 > \underline{\theta}$ such that for all $\theta \in (\underline{\theta}, \theta_1)$ we have $u_{\alpha_1}(\underline{\theta}) < u_{\alpha_2}(\underline{\theta})$ and $\dot{u}_{\alpha_1}(\theta) < \dot{u}_{\alpha_2}(\theta)$. Let θ_* be the smallest element θ of $[\theta_1, \overline{\theta}]$ such that $\dot{u}_{\alpha_1}(\theta) = \dot{u}_{\alpha_2}(\theta)$. We must have $\ddot{u}_{\alpha_1}(\theta) \geq \ddot{u}_{\alpha_2}(\theta)$, but then

$$\ddot{u}_{\alpha_2}(\theta_*) = 2 - \left(\tfrac{1}{\alpha_2+\beta\theta^*}\right)\left(\theta^*\dot{u}_{\alpha_2}(\theta^*) - \tfrac{(\dot{u}_{\alpha_2}(\theta^*))^2}{2} - u_{\alpha_2}(\theta^*)\right)$$
$$= 2 - \left(\tfrac{1}{\alpha_2+\beta\theta^*}\right)\left(\theta^*\dot{u}_{\alpha_1}(\theta^*) - \tfrac{(\dot{u}_{\alpha_1}(\theta^*))^2}{2} - u_{\alpha_1}(\theta^*)\right) + \left(\tfrac{1}{\alpha_2+\beta\theta^*}\right)(u_{\alpha_2}(\theta^*) - u_{\alpha_1}(\theta^*))$$
$$> 2 - \left(\tfrac{1}{\alpha_1+\beta\theta^*}\right)\left(\theta^*\dot{u}_{\alpha_1}(\theta^*) - \tfrac{(\dot{u}_{\alpha_1}(\theta^*))^2}{2} - u_{\alpha_1}(\theta^*)\right) + \left(\tfrac{1}{\alpha_2+\beta\theta^*}\right)(u_{\alpha_2}(\theta^*) - u_{\alpha_1}(\theta^*)) > \ddot{u}_{\alpha_1}(\theta_*),$$

a contradiction.

Next, suppose towards a contradiction that there exists θ such that $u_{\alpha_2}(\theta) - u_{\alpha_1}(\theta) > 0$, and then take $\theta^* \in \text{argmax}_\theta u_{\alpha_2}(\theta) - u_{\alpha_1}(\theta)$. Notice that whether $\theta^* < \bar{\theta}$ or $\theta^* = \bar{\theta}$, we must have $u_{\alpha_2}(\theta^*) - u_{\alpha_1}(\theta^*) > 0$ and $\ddot{u}_{\alpha_2}(\theta^*) - \ddot{u}_{\alpha_1}(\theta^*) \leq 0$, but then the same argument as above implies $\ddot{u}_{\alpha_2}(\theta_*) > \ddot{u}_{\alpha_1}(\theta_*)$, which leads to a contradiction. Q.E.D.

Proposition 11. ***(Only one pure-strategy equilibrium when*** $t_k = 0$ ***for some*** $k \in \{l, h\}$***).*** *There exists at most only one symmetric pure-strategy equilibrium in the model in which there is perfect competition for type k. Moreover, each firm makes zero profits from the type k.*

Proof of Proposition 11

We only prove the result for the case in which there is perfect competition for high types as the other case is analogous. We first claim that in any equilibrium (u_l, u_h) each firm makes all profits from the low type. In fact, otherwise the firm could profitably deviate to $(u_l + \varepsilon, u_h + \varepsilon)$ for some small $\varepsilon > 0$. Next notice that if IC_l binds then firm could profitably deviate by offering $(u_l, u_l + \triangle\theta q_l^*)$ .We conclude that any equilibrium must satisfy

$$(u_l, u_h) \in \arg\max_{u_l^a, u_h^a}\left\{p_l\left(\frac{1}{2} + \frac{u_l^a - u_l}{2t_l}\right)(S_l^e - u_l^a) + p_h\hat{D}_h^a(u_h^a)(S_h(u_l^a, u_h^a) - u_h)\right\}$$

which implies $u_h = S_h(u_l, u_h)$ and

$$-\left(\frac{p_l}{2}\right) + p_l\left(\frac{S_l^e - u_l}{2t_l}\right) + \left(\frac{p_h}{2}\right)\frac{\partial S_h(u_l, u_h)}{\partial u_l} = 0.$$

Since each firm obtains zero profits from high types, we must have $u_l = \max\{S_l^e - t_l, 0\}$. Hence u_h solves $S_h(\max\{S_l^e - t_l, 0\}, u_h) - u_h = 0$. Recall that $\Delta(u_l, u_h) := u_h - u_l$ and thus equilibrium requires that $S_h(\Delta(u_l, u_h)) - u_h = 0$. Observe that $\left(\frac{\partial}{\partial u_l}\right)[S_h(\Delta(u_l, u_h)) - u_h] \geq 0$,with strict inequality whenever $\Delta(u_l, u_h) > \triangle\theta q_h^*$, while $\left(\frac{\partial}{\partial u_h}\right)[S_h(\Delta(u_l, u_h)) - u_h] < 0$. This implies that whenever $\Delta(\max\{S_l^e - t_l, 0\}, S_h^*) \leq \triangle\theta q_h^*$, any pure-strategy equilibrium should be $(\max\{S_l^e - t_l, 0\}, S_h^*)$, while if $\Delta(\max\{S_l^e - t_l, 0\}, S_h^*) > \triangle\theta q_h^*$ then any pure-strategy equilibrium should be $(\max\{S_l^e - t_l, 0\}, \mathring{u}_h)$, where $\mathring{u}_h$ satisfies $S_h(\max\{S_l^e - t_l, 0\}, \mathring{u}_h) - \mathring{u}_h = 0$. Q.E.D.

Mixed-Strategy Equilibrium in the Cream-Skimming Duopoly

We now describe a mixed-strategy equilibrium that prevails when $t_h \in (S_h^e - \eta_l, \tilde{t}_h)$ and $t_l = 0$. Similarly to the bottom-of-barrel case, this equilibrium is ordered. Accordingly, the menus offered in equilibrium can be described by a support function $\mathcal{U}_h : \Upsilon_l \to \Upsilon_h$, which is strictly increasing and bijective, and maps the support of F_l^* (which is Υ_l) into the support of F_h^* (which is Υ_h). In contrast to the bottom-barrel case, however, it is convenient to set the domain of the support function as being Υ_l (rather than Υ_h), and describe the firm's randomization by means of the marginal cdf F_l^*.

Denoting by $\mathbb{E}_{F_h^*}[\tilde{u}_h]$ the mean of u_h as induced by the cdf F_h^*, let us define

$$\check{u}_h \equiv \frac{S_h^e - t_h + \mathbb{E}_{F_h^*}[\tilde{u}_h]}{2}.$$

Proposition 12. ***(Mixed-Strategy Equilibrium)*** *Suppose there is perfect competition for low types ($t_l = 0$), but imperfect for high types ($t_h > 0$). If $t_h \in (S_h^e - \eta_l, \tilde{t}_h)$, there exists a mixed-strategy equilibrium, which is ordered. In this equilibrium, the support of indirect utilities is an interval, $\Upsilon_k = [\underline{u}_k, \bar{u}_k]$, and the support function $\mathcal{U}_h(\cdot)$ and cdf F_l^* of low-type's indirect utilities jointly satisfy*

$$\mathcal{U}_h(u_l) - \underline{u}_h = 2\left(\frac{S_l(u_l, \mathcal{U}_h(u_l)) - u_l}{\frac{\partial S_l}{\partial u_h}(u_l, \mathcal{U}_h(u_l))}\right)\left(\frac{\mathcal{U}_h(u_l) - \check{u}_h}{\mathcal{U}_h(u_l) + \underline{u}_h - 2\check{u}_h}\right), \qquad \forall\ u_l \in [\underline{u}_l, \bar{u}_l],$$

and

$$F_l^*(u_l) = \frac{1}{t_h}\frac{p_h}{p_l}\left(\mathcal{U}_h(u_l) - \check{u}_h\right)\left(\frac{\partial S_l}{\partial u_h}(u_l, \mathcal{U}_h(u_l))\right)^{-1} \qquad \forall\ u_l \in [\underline{u}_l, \bar{u}_l],$$

with boundary conditions

$$\underline{u}_l = S_l(\underline{u}_l, \underline{u}_h), \qquad \underline{u}_h = \max\{\check{u}_h, 0\}, \qquad \bar{u}_l = (F_l^*)^{-1}(1), \qquad \textit{and} \qquad \bar{u}_h = \mathcal{U}_h(\bar{u}_l).$$

Moreover, F_l^* is absolutely continuous at any $u_l \in (\underline{u}_l, \bar{u}_l]$, and exhibits a mass point at $\underline{u}_l$ (i.e., $F_l^*(\underline{u}_l) > 0$) when t_h is sufficiently large.

Proof of Proposition 12. The equilibrium structure below can be derived by arguments analogous to the ones provided in the proof of Proposition 2. Hence we omit several details by brevity.

Step 1: Firms symmetrically randomize over menus (u_l, u_h) according to the distribution F^*. This distribution has support

$$\text{supp}(F^*) = \{(u_l, u_h) : u_h = \mathcal{U}_h(u_l), \underline{u}_l \le u_l \le \bar{u}_l\},$$

where the support function $\mathcal{U}_h(\cdot)$ is strictly increasing and continuous. Moreover, $\underline{u}_h = \mathcal{U}_h(\underline{u}_l)$ and $\bar{u}_h = \mathcal{U}_h(\bar{u}_l)$. The cdf F^* might possess a mass point at $(\underline{u}_l, \underline{u}_h)$, but is absolutely continuous elsewhere in its support. The marginals associated with F^* are F_l^* and F_h^*, which have respective supports $[\underline{u}_l, \bar{u}_l]$ and $[\underline{u}_h, \bar{u}_h]$. If F^* exhibits a mass point at $(\underline{u}_l, \underline{u}_h)$, then F_l^* (resp., F_h^*) exhibits a mass point at $\underline{u}_l$ (resp., $\underline{u}_h$), but is absolutely continuous elsewhere. Obviously,

$$F^*(\underline{u}_l, \underline{u}_h) = F_l^*(\underline{u}_l) = F_h^*(\underline{u}_h).$$

To construct the mixed-strategy equilibrium, denote by $\mathbb{E}_{F_h^*}[\tilde{u}_h]$ the expectation of the random variable $\tilde{u}_h$ under the law F_h^*. Analogously to Proposition 2 , the constraint IC_h always binds, any equilibrium pair (u_l, u_h) has to solve

$$\max_{u_h} \left\{ p_l F_l^*(u_l) \left(S_l(u_l, u_h) - u_l \right) + p_h \left(\frac{1}{2} + \frac{u_h - \mathbb{E}_{F_h^*}[\tilde{u}_h]}{2t_h} \right) (S_h^e - u_h) \right\} \quad \text{s.t.} \quad u_h \geq u_l \geq 0.$$

This problem is concave. Therefore, the cdf F_l^* and the support function $\mathcal{U}_h(\cdot)$ have to jointly satisfy the first-order condition:

$$p_l F_l^*(u_l) \frac{\partial S_l}{\partial u_h}(u_l, u_h) - \frac{p_h}{t_h} \left(\mathcal{U}_h(u_l) - \left(\frac{S_h^e - t_h + \mathbb{E}_{F_h^*}[\tilde{u}_h]}{2} \right) \right) = 0,$$

which, after rearranging, leads to

$$F_l^*(u_l) = \frac{1}{t_h} \frac{p_h}{p_l} \left(\mathcal{U}_h(u_l) - \left(\frac{S_h^e - t_h + \mathbb{E}_{F_h^*}[\tilde{u}_h]}{2} \right) \right) \left(\frac{\partial S_l}{\partial u_h}(u_l, u_h) \right)^{-1}. \tag{25}$$

Note that the cross-derivative of the objective is

$$p_l f_l^*(u_h) \frac{\partial S_l}{\partial u_h}(u_l, u_h) + p_l F_l^*(u_l) \frac{\partial^2 S_l}{\partial u_h \partial u_l}(u_l, u_h) > 0,$$

as both terms are positive. This reveals that the objective is supermodular; hence, if (u_l, u_h) and (u_l', u_h') are equilibrium menus, then $u_h > u_h'$ implies $u_l > u_l'$. In view of this, the support function $\mathcal{U}_l(\cdot)$ is strictly increasing.

That firms randomize requires that the following indifference condition holds across all equilibrium menus:

$$\begin{aligned} p_l F_l^*(u_l) \left(S_l(u_l, u_h) - u_l \right) + p_h \left(\frac{1}{2} + \frac{u_h - \mathbb{E}_{F_h^*}[\tilde{u}_h]}{2t_h} \right) (S_h^e - u_h) \\ = p_h \left(\frac{1}{2} + \frac{\underline{u}_h - \mathbb{E}_{F_h^*}[\tilde{u}_h]}{2t_h} \right) (S_h^e - \underline{u}_h). \end{aligned}$$

Because $\underline{u}_l = S_l(\underline{u}_l, \underline{u}_h)$ by construction, the right-hand side is the profit obtained by choosing the least generous menu in the support. Using equation (25), we can re-write this indifference condition as

$$\begin{aligned} \frac{p_h}{t_h} \left(\mathcal{U}_h(u_l) - \check{u}_h \right) \left(\frac{\partial S_l}{\partial u_h}(u_l, \mathcal{U}_h(u_l)) \right)^{-1} \left(S_l(u_l, \mathcal{U}_h(u_l)) - u_l \right) + p_h \left(\frac{1}{2} + \frac{\mathcal{U}_h(u_l) - \mathbb{E}_{F_h^*}[\tilde{u}_h]}{2t_h} \right) (S_h^e - \mathcal{U}_h(u_l)) \\ = p_h \left(\frac{1}{2} + \frac{\underline{u}_h - \mathbb{E}_{F_h^*}[\tilde{u}_h]}{2t_h} \right) (S_h^e - \underline{u}_h). \end{aligned}$$

or, equivalently, as

$$\mathcal{U}_h(u_l) - \check{u}_h$$
$$= t_h \left(\frac{\frac{\partial S_l}{\partial u_h}(u_l, \mathcal{U}_h(u_l))}{S_l(u_l, \mathcal{U}_h(u_l)) - u_l} \right) \left[\left(\frac{1}{2} + \frac{\underline{u}_h - \mathbb{E}_{F_h^*}[\tilde{u}_h]}{2t_h} \right) (S_h^e - \underline{u}_h) - \left(\frac{1}{2} + \frac{\mathcal{U}_h(u_l) - \mathbb{E}_{F_h^*}[\tilde{u}_h]}{2t_h} \right) (S_h^e - \mathcal{U}_h(u_l)) \right].$$

Tedious algebra reveals that

$$t_h \left(\frac{1}{2} + \frac{\underline{u}_h - \mathbb{E}_{F_h^*}[\tilde{u}_h]}{2t_h} \right) (S_h^e - \underline{u}_h) - t_h \left(\frac{1}{2} + \frac{\mathcal{U}_h(u_l) - \mathbb{E}_{F_h^*}[\tilde{u}_h]}{2t_h} \right) (S_h^e - \mathcal{U}_h(u_l))$$
$$= \frac{1}{2} \left[(\mathcal{U}_h(u_l) - \underline{u}_h)(\mathcal{U}_h(u_l) + \underline{u}_h - 2\check{u}_h) \right].$$

Therefore

$$\mathcal{U}_h(u_l) - \check{u}_h = \left(\frac{\frac{\partial S_l}{\partial u_h}(u_l, \mathcal{U}_h(u_l))}{S_l(u_l, \mathcal{U}_h(u_l)) - u_l} \right) \frac{1}{2} \left[(\mathcal{U}_h(u_l) - \underline{u}_h)(\mathcal{U}_h(u_l) + \underline{u}_h - 2\check{u}_h) \right],$$

which, after rearranging leads to

$$\mathcal{U}_h(u_l) - \underline{u}_h = 2 \left(\frac{S_l(u_l, \mathcal{U}_h(u_l)) - u_l}{\frac{\partial S_l}{\partial u_h}(u_l, \mathcal{U}_h(u_l))} \right) \frac{\mathcal{U}_h(u_l) - \check{u}_h}{\mathcal{U}_h(u_l) + \underline{u}_h - 2\check{u}_h},$$

as in the statement of the proposition.

Assume that if $\check{u}_h \geq 0$ and notice that since the menu $(\underline{u}_l, \underline{u}_h)$ deliverers zero profits from the low type, each firm must choose $\underline{u}_h = \check{u}_h$ and hence Equation (25) implies $F_l^*(\underline{u}_l) = 0$. On the other hand, if $\check{u}_h < 0$ each firm must set $\underline{u}_h = 0$ and hence (25) implies $F_l^*(\underline{u}_l) > 0$. That is:

$$F_l^*(\underline{u}_l) > 0 \quad \iff \quad \check{u}_h \equiv \frac{S_h^e - t_h + \mathbb{E}_{F_h^*}[\tilde{u}_h]}{2} < 0.$$

In particular, when $\underline{u}_h = 0$ we have

$$F_l^*(\underline{u}_l) = -\frac{\check{u}_h}{t_h} \frac{p_h}{p_l} \left(\frac{\partial S_l}{\partial u_h}(0,0) \right)^{-1}.$$

Finally, we choose $\bar{u}_l$ such that $F_l^*(\bar{u}_l) = 1$ so as to guarantee that F_l^* is a proper cdf. The construction above guarantees that firms are indifferent across all equilibrium menus, which best respond the mixed strategy F^*.

Step 2: F_l^* exhibits a mass point at $\underline{u}_l$ (i.e., $F_l^*(\underline{u}_l) > 0$) when t_h is sufficiently large.

Notice that since $\mathbb{E}_{F_h^*}[\tilde{u}_h] < S_h^e$, we have $\check{u}_h < 0$ when $t_h > 2S_h^e$. Q.E.D.

Continuum of Types: Numerical Procedure

This appendix (together with the Matlab codes) detail the numerical procedure employed to investigate the existence of a pure-strategy equilibrium in the continuum-type model of subsection 7.1. We first describe how the putative equilibrium is constructed, allowing for the possibility of bunching on some interval. Second, we present how the best deviation to the putative equilibrium is computed. Finally, we describe the test of equilibrium existence for a given brand loyalty schedule $t(\theta)$.

The general problem

Denoting by $u(\theta)$ the utility of type θ, and recalling that incentive compatibility requires that $\dot{u}(\theta) = q(\theta)$, one can write the profit of a firm obtained from consumers of type θ as

$$\pi(\theta, u(\theta), \dot{u}(\theta) \mid \tilde{u}(\theta)) = \left(\frac{1}{2} + \frac{u(\theta) - \tilde{u}(\theta)}{2t(\theta)}\right)(S(\theta) - u(\theta)), \quad \text{where} \quad S(\theta) \equiv \left(\theta\dot{u}(\theta) - \frac{(\dot{u}(\theta))^2}{2}\right)$$

is the surplus produced by the contract designed to type-θ consumers, and $\widetilde{u}(\theta)$ is the indirect-utility schedule offered by the competing firm.

The best-response problem of the firm consists in finding the utility profile $u(\theta)$ that maximizes

$$\Pi = \int_{\underline{\theta}}^{\bar{\theta}} \pi(\theta, u(\theta), \dot{u}(\theta) \mid \tilde{u}(\theta))\, d\theta, \tag{26}$$

where $\tilde{u}(\theta)$ is taken as given, and the control $u(\theta)$ is subject to the participation, positive-quality, and monotonicity constraints:

$$u(\theta), \dot{u}(\theta), \ddot{u}(\theta) \geq 0.$$

Accordingly, define the Hamiltonian as:

$$H(\theta, u(\theta), \dot{u}(\theta), \ddot{u}(\theta)) = \left(\frac{1}{2} + \frac{u(\theta) - \widetilde{u}(\theta)}{2t(\theta)}\right)\left(\theta\dot{u}(\theta) - \frac{\dot{u}(\theta)}{2} - u(\theta)\right) + \rho_1(\theta)\, u(\theta) + \rho_2(\theta)\, \dot{u}(\theta) + \mu(\theta)\, \ddot{u}(\theta),$$

where $\rho_1(\theta), \rho_2(\theta)$ denote respectively the co-state variables associated with $u(\theta)$ and $\dot{u}(\theta)$, while $\mu(\theta)$ is the Lagrange multiplier associated with constraint $\ddot{u}(\theta) \geq 0$. After imposing symmetry, the necessary conditions are

$$\dot{\rho_1}(\theta) = -\frac{1}{2t(\theta)}\left(\theta\dot{u}(\theta) - \frac{\dot{u}(\theta)}{2} - u(\theta)\right) + \frac{1}{2}, \tag{27}$$

$$\dot{\rho_2}(\theta) = -\frac{1}{2}(\theta - \dot{u}(\theta)) - \rho_1(\theta), \tag{28}$$

$$\mu(\theta) = -\rho_2(\theta), \mu(\theta) \geq 0, \mu(\theta)\, \ddot{u}(\theta) = 0, \tag{29}$$

and the transversality conditions require that

$$\rho_1(\underline{\theta}) = \rho_1(\overline{\theta}) = \rho_2(\underline{\theta}) = \rho_2(\overline{\theta}) = 0. \tag{30}$$

We now describe how the program is numerically solved to obtain the putative symmetric equilibrium.

The first step

(a) We first compute the relaxed problem considering that $\mu(\theta) = 0$ over the whole interval $[\underline{\theta}, \overline{\theta}]$. Differentiation of (28) with respect to θ and substitution into (27) yields:

$$\ddot{u}(\theta) = 2 - \left(\frac{1}{t(\theta)}\right)\left(\theta\dot{u}(\theta) - \frac{(\dot{u}(\theta))^2}{2} - u(\theta)\right) \tag{31}$$

with transversality conditions $\dot{u}(\underline{\theta}) = \underline{\theta}$ and $\dot{u}(\overline{\theta}) = \overline{\theta}$. This yields the solution $u_r^*(\theta)$. If the solution involves $\ddot{u}_r^*(\theta) \geq 0$ for every $\theta \in [\underline{\theta}, \overline{\theta}]$ then the first step ends. For some $t(\theta)$, however, we find that $\ddot{u}_r^*(\theta) < 0$ either in an interval at the bottom, i.e., $\theta \in [\underline{\theta}, \theta_1]$, or at the top, i.e., $\theta \in [\theta_2, \overline{\theta}]$. We briefly describe here the procedure to find the solution when bunching occurs at the bottom of the product line, i.e., over some interval $\theta \in [\underline{\theta}, \theta_c]$ (the procedure for allowing bunching at the top is symmetric).

(b) Over an interval $\theta \in [\underline{\theta}, \theta_c]$ in which $\ddot{u}(\theta) = 0$, the solution $u_c(\theta, \theta_c)$ is characterized by an affine function of the form:

$$u_c(\theta, \theta_c) = \kappa_0 + \kappa_1 \theta. \tag{32}$$

This solution has to

$$\max_{u_c(\theta,\theta_c)} \int_{\underline{\theta}}^{\theta_c} \left(\frac{1}{2} + \frac{u_c(\theta, \theta_c) - \widetilde{u}_c(\theta)}{2t(\theta)}\right) \left(\theta \dot{u}_c(\theta, \theta_c) - \frac{\dot{u}_c(\theta, \theta_c)}{2} - u_c(\theta, \theta_c)\right) d\theta.$$

Using (32), this is equivalent to finding

$$\{\kappa_0^*(\theta_c), \kappa_1^*(\theta_c)\} = \arg\max_{\kappa_0, \kappa_1} \left(-\frac{\kappa_1^2}{2} - \kappa_0\right) \int_{\underline{\theta}}^{\theta_c} \left(\frac{1}{2} + \frac{\kappa_0 + \kappa_1 \theta - \widetilde{u}_c(\theta)}{2t(\theta)}\right) d\theta,$$

where $\widetilde{u}_c(\theta)$ is given.

(c) Over the interval $\theta \in [\theta_c, \overline{\theta}]$, the solution $u_c^*(\theta, \theta_c)$ is given by the ODE (31). By continuity, the boundary conditions are $\dot{u}_c(\overline{\theta}, \theta_c) = \overline{\theta}$, $\dot{u}_c(\theta_c, \theta_c) = \kappa_1^*(\theta_c)$, and $u_c(\theta_c, \theta_c) = \kappa_0^*(\theta_c) + \theta_c \kappa_1^*(\theta_c)$.

(d) The alogirthm then computes $u_c^*(\theta, \theta_c)$ for any θ_c by imposing boundary conditions given by $\dot{u}_c(\theta_c, \theta_c) = \kappa_1^*(\theta_c)$ and $u_c(\theta_c, \theta_c) = \kappa_0^*(\theta_c) + \theta_c \kappa_1^*(\theta_c)$. Then the optimal θ_c^* is found by iteration until one converges to the transversality condition $\dot{u}_c(\overline{\theta}, \theta_c) = \overline{\theta}$. This procedure leads to $u_c^*(\theta, \theta_c^*)$.

(e) The putative symmetric Nash equilibrium is thus given by $u^*(\theta) = u_r^*(\theta)$ if there is no bunching, and $u^*(\theta) = u_c^*(\theta, \theta_c^*)$ otherwise.

(f) This yields a profit value for both firms that is given by $\Pi^* = \left(\frac{1}{2}\right) \int_{\underline{\theta}}^{\overline{\theta}} \left(\theta \dot{u}^*(\theta) - \frac{\dot{u}^*(\theta)}{2} - u^*(\theta)\right) d\theta$.

We next describe how we numerically construct the best deviation from this putative symmetric equilibrium.

The best deviation

We compute a candidate function $u_p^*(\theta)$ defined as a polynomial function of degree $n = 4$ which is given by:[38]

$$u_p^*(\theta) = \arg\max_{u_p} \int_{\underline{\theta}}^{\bar{\theta}} \left(\frac{1}{2} + \frac{u_p(\theta) - u^*(\theta)}{2t(\theta)} \right) \left(\theta \dot{u}_p(\theta) - \frac{\dot{u}_p(\theta)}{2} - u_p(\theta) \right) d\theta$$
$$\text{subject to } \dot{u}_p(\theta), \ddot{u}_p(\theta) \geq 0 \text{ for every } \theta.$$

This yields the deviation profit $\mathit{\Pi}_p^* = \int_{\underline{\theta}}^{\bar{\theta}} \left(\frac{1}{2} + \frac{u_p^*(\theta) - u^*(\theta)}{2t(\theta)} \right) \left(\theta \dot{u}_p^*(\theta) - \frac{\dot{u}_p^*(\theta)}{2} - u_p^*(\theta) \right) d\theta$.

Testing the existence of pure strategic Nash equilibrium

For a given specification of $t(\theta)$, the steps above yield values $\mathit{\Pi}^*$ and $\mathit{\Pi}_p^*$. The algorithm rejects the existence of pure strategy symmetric equilibrium if $\mathit{\Pi}_p^* > \mathit{\Pi}^*$.

[38] Note that, by the Weierstrass approximation theorem, the procedure above correctly identifies equilibria provided the deviating profile $u_d(\theta)$ is a polynomial of sufficiently large degree.